

Quantization of Projective Homogeneous Spaces and Duality Principle

N. Ciccoli[‡], R. Fiorese[‡], F. Gavarini^{# 1}

[‡] *Dipartimento di Matematica, Università di Perugia*
Via Vanvitelli 1, I-06123 Perugia, Italy
e-mail: ciccoli@dipmat.unipg.it

[‡] *Dipartimento di Matematica, Università di Bologna*
Piazza di Porta San Donato, 5 — I-40127 Bologna, Italy
e-mail: fiorese@dm.unibo.it

[#] *Dipartimento di Matematica, Università di Roma “Tor Vergata”*
Via della ricerca scientifica 1 — I-00133 Roma, Italy
e-mail: gavarini@mat.uniroma2.it

Abstract

We introduce a general recipe to construct quantum projective homogeneous spaces, with a particular interest for the examples of the quantum Grassmannians and the quantum generalized flag varieties. Using this construction, we extend the quantum duality principle to quantum projective homogeneous spaces.

Contents

1	Introduction	2
2	The classical setting	4
2.1	The affine case	4
2.2	The projective case	6

¹N. Ciccoli was partially supported by TOK “*Non commutative geometry and quantum groups*”, no. MKTD-CT-2004-509794, University of Warsaw — R. Fiorese and F. Gavarini were partially supported by the European RTN network “*LIEGRITS - Flags, Quivers and Invariant Theory in Lie Representation Theory*”, no. MRTN-CT 2003-505078, and by the Italian research program PRIN 2005 “*Moduli and Lie Theories*”.

MSC 2000: Primary 20G42, 14M17; Secondary 17B37, 81R50.

Keywords: Quantum Homogeneous Spaces.

3	Quantum bundles and quantum homogeneous spaces	11
3.1	Quantum groups	11
3.2	Quantum subgroups and quantum coisotropic subgroups . . .	12
3.3	Quantum line bundles	14
4	The Quantum Duality Principle (QDP)	24
4.1	The QDP philosophy	24
4.2	The QDP functor	26
5	Examples: Quantum Grassmannians and quantum flag varieties	35
5.1	The quantum Grassmannian as quantum projective homogeneous space	35
5.2	QDP for quantum Grassmannians	43
5.3	Quantum generalized flag varieties for simple groups as quantum projective homogeneous spaces	48
5.4	The coisotropic case	52

1 Introduction

A projective variety can be described via its homogeneous graded coordinate ring. This ring is not an invariant associated to the variety, but depends on a chosen embedding of the variety into some projective space. Different embeddings will, in general, produce non isomorphic graded rings.

When a projective variety is homogeneous, i.e. endowed with a transitive action of an (affine) algebraic group on it, it can be realized as quotient of affine algebraic groups G/H . In this case a projective embedding can be obtained via sections of a line bundle on G/H , uniquely given once a character of H is specified.

If one approaches a quantization of this picture in the context of quantum groups the problem immediately arising is that standard quantum groups have a very limited set of quantum subgroups. This explains why usually the preferred approach goes through representation theoretic techniques.

An explanation of the lack of quantum subgroups, together with a way to circumvent this problem, is suggested by considering the semiclassical picture, i.e. in the context of algebraic Poisson groups. In such setting

algebraic Poisson subgroups are quite rare too; however there is no need of an algebraic Poisson subgroup to cook up a Poisson quotient. The existence of a surjective Poisson map $G \rightarrow G/H$ is guaranteed simply by requiring H to be a coisotropic subgroup of G . This condition can be expressed by saying that the defining ideal of H , in the function algebra of G , is required to be a Poisson subalgebra rather than a Poisson ideal, as required for Poisson subgroups.

Let $\mathcal{O}_q(G)$ be a quantization of the affine algebraic Poisson groups G . At the quantum level, a quantization $\mathcal{O}_q(H)$ of its *coisotropic subgroup* H can be defined through conditions on the projection $\pi: \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(H)$. We will see this in full detail in the sections 2, 3.

Our first aim is to build a quantum deformation $\mathcal{O}_q(G/H)$ of the projective variety G/H , i.e. of its graded ring $\mathcal{O}(G/H)$, subject to the following requirements:

(1) there exists a one dimensional corepresentation of the quantum coisotropic subgroup $\mathcal{O}_q(H)$ which is a deformation of the corepresentation of $\mathcal{O}(H)$ corresponding to the character of H which defines the line bundle giving the projective embedding of G/H ;

(2) a quantum analogue $\mathcal{O}_q(G/H)$ to $\mathcal{O}(G/H)$ is defined as the subset — inside $\mathcal{O}_q(G)$ — of “semi-invariant functions” with respect to the given corepresentation of $\mathcal{O}_q(H)$;

(3) the subset $\mathcal{O}_q(G/H)$ is a graded subalgebra of $\mathcal{O}_q(G)$;

(4) the graded subalgebra $\mathcal{O}_q(G/H)$ is a graded left coideal of $\mathcal{O}_q(G)$, so the coproduct in $\mathcal{O}_q(G)$ induces a (left) $\mathcal{O}_q(G)$ -coaction on $\mathcal{O}_q(G/H)$, and the latter can be thought of as a *quantum homogeneous space*.

(5) the semiclassical limit of $\mathcal{O}_q(G/H)$ is $\mathcal{O}(G/H)$ — embedded into $\mathcal{O}(G)$ — as a graded subalgebra, left coideal and graded Poisson subalgebra.

In other words, a quantum deformation of a projective homogeneous space, embedded into some projective space, consists of the deformation of the graded algebra associated to the embedding, in such a way that the action of the group on the homogeneous space is also naturally quantized.

We will work out the details of the construction for the case of the Grassmannian and its Plücker embedding, that is when G is the special linear group and $H = P$ is a maximal parabolic subgroup and we will sketch it in the more general case of quantum flag varieties of simple Lie groups.

Our main motivation to develop this point of view is to adapt to projective homogeneous spaces, the correspondence introduced by Ciccoli and Gavarini [5] for coinvariant subalgebras. This recipe allows to associate functorially to a quantum quasi-affine homogeneous space another quantum homogeneous space, through a generalization of the *quantum duality principle* (QDP), defined by Drinfeld for quantum groups. A part of the arguments in [5] does not directly apply to *projective* homogeneous spaces, since it is based on the realization of the ring of the homogeneous space as the set of coinvariant functions inside the ring of the quantum group acting on it. But this is possible — as in the classical case — if and only if the homogeneous space is quasi-affine, which is not the case of projective varieties. The coordinate ring of the homogeneous space is replaced by a graded ring inside the quantum group ring, consisting of semi-coinvariants with respect to a one-dimensional representation, which can be seen as a deformation of the line bundle that classically determines the projective embedding. The definitions introduced in section 3 will allow us to define a quantum duality functor and obtain the QDP construction in this more general setting. In the last chapter we will discuss applications to quantum flag manifolds.

2 The classical setting

In this section we recall some Poisson geometry (see Ref. [20] for details).

2.1 The affine case

Let \mathbb{k} be a fixed field of characteristic 0. When doing algebraic geometry construction, we tacitly assume that \mathbb{k} be algebraically closed. However, this assumption is *not* needed for our quantum constructions.

Let G be an affine algebraic group over \mathbb{k} . We denote by $\mathcal{O}(G)$ the algebra of regular functions of G , in short its “function algebra”, which is naturally a Hopf algebra (over \mathbb{k}). We denote $\mathfrak{g} := \text{Lie}(G)$ the tangent Lie algebra of G , and similarly $\mathfrak{h} := \text{Lie}(H)$ for any closed algebraic subgroup H of G .

Assume that G is a Poisson group: this means that $\mathcal{O}(G)$ is a *Poisson* Hopf algebra, i.e. we have a Poisson bracket $\{ , \}: \mathcal{O}(G) \otimes \mathcal{O}(G) \rightarrow \mathcal{O}(G)$ which is compatible with the Hopf algebra structure. Moreover, \mathfrak{g} is a Lie bialgebra, for some Lie cobracket $\delta: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, and the same holds for its dual space \mathfrak{g}^* , these two Lie bialgebras structures being dual to each

other. Indeed, the notion of Lie bialgebra is the infinitesimal counterpart of the notion of Poisson group. Since the dual \mathfrak{g}^* of \mathfrak{g} is itself a Lie bialgebra, it follows that any connected algebraic group G^* with $\mathfrak{g}^* = \text{Lie}(G^*)$, is a Poisson group on its own, called (*Poisson*) *dual* to G . We are going to see the example of the Poisson group GL_n treated in detail in section 5.2.

Definition 2.1. A (closed) subgroup H of G is called *coisotropic* if its defining ideal $I(H)$ is a Poisson subalgebra of $\mathcal{O}(G)$. Also, H is called a *Poisson subgroup* if the embedding $H \hookrightarrow G$ is a Poisson map; this is equivalent to require $I(H)$ to be a Poisson *ideal*. Hence a Poisson subgroup is coisotropic.

The following equivalent conditions give an infinitesimal characterization for a *connected* subgroup H to be coisotropic (see [24]):

Proposition 2.2. *Let G be an algebraic group and H a (closed) subgroup of the Poisson group G . Then the following are equivalent:*

- (C-i) H is a coisotropic subgroup of G ;
- (C-ii) $\delta(\mathfrak{h}) \subseteq \mathfrak{h} \wedge \mathfrak{g}$, that is \mathfrak{h} is (a Lie subalgebra and) a Lie coideal of \mathfrak{g} ;
- (C-iii) the orthogonal space \mathfrak{h}^\perp is (a Lie coideal and) a Lie subalgebra of \mathfrak{g}^* .

Remark 2.3. Note that, thanks to these characterizations, the infinitesimal counterpart of the notion of coisotropic subgroup is that of Lie subalgebra Lie coideal. The latter notion is self-dual. In fact, let G^* be any connected Poisson group dual to G . If H is coisotropic in G , then any connected subgroup of G^* , with tangent Lie algebra \mathfrak{h}^\perp , is in turn a coisotropic subgroup of G^* , called “complementary dual” to H and denoted by H^\perp .

We now want to describe the notion of *Poisson quotient*.

Definition 2.4. Let M be a Poisson affine variety, i.e. an affine variety whose function algebra $\mathcal{O}(M)$ is a Poisson algebra. Then M is a *Poisson homogeneous G -space* if there is a (regular) transitive action $\phi : G \times M \rightarrow M$ which is a Poisson map with respect to the product Poisson structure on $G \times M$. We say that a Poisson homogeneous G -space M is a *Poisson quotient* if there is a coisotropic closed Lie subgroup H_M of G such that $G/H_M \simeq M$ and the projection $p_M : G \rightarrow G/H_M \simeq M$ is a Poisson map.

The following is a characterization of Poisson quotients (cf. [29]).

Proposition 2.5. *Let $\phi : G \times M \rightarrow M$ be a homogeneous action of G on M . Then the following are equivalent:*

- (PQ-i) *there exists $\bar{m} \in M$ whose stabilizer $G_{\bar{m}}$ is a coisotropic subgroup of G ;*
- (PQ-ii) *there exists $\bar{m} \in M$ such that $\phi_{\bar{m}} : G \rightarrow M, g \mapsto \phi(g, \bar{m})$, is a Poisson map, that is to say M is a Poisson quotient;*
- (PQ-iii) *there is $\bar{m} \in M$ such that $\{\bar{m}\}$ is a symplectic leaf of M .*

For any $\bar{m} \in M$ with stabilizer $G_{\bar{m}}$ one has $M \sim G/G_{\bar{m}}$ as affine G -varieties. As M is affine, this is equivalent to $\mathcal{O}(M) \cong \mathcal{O}(G/G_{\bar{m}})$. Finally, $\mathcal{O}(G/G_{\bar{m}}) \cong \mathcal{O}(G)^{G_{\bar{m}}}$, the subalgebra of $G_{\bar{m}}$ -invariants in $\mathcal{O}(G)$. The same holds with $G_{\bar{m}}$ replaced by any subgroup H whose coset space G/H is affine. We have then an additional characterization of (affine) Poisson quotients.

Proposition 2.6. *If M is as above, then M is a Poisson quotient if and only if there exists $\bar{m} \in M$ such that $\mathcal{O}(G)^{G_{\bar{m}}}$ is a Poisson subalgebra of $\mathcal{O}(G)$.*

In particular, if H is a subgroup of G , and G/H is affine then the following are equivalent:

1. H is coisotropic,
2. G/H is a Poisson quotient,
3. $\mathcal{O}(G/H) = \mathcal{O}(G)^H$ is a Poisson subalgebra of $\mathcal{O}(G)$.

2.2 The projective case

We are now interested in the case when the homogeneous G -variety G/H is *projective*, i.e. H is *parabolic*. To describe it in algebraic terms (the setting we need for quantum deformations), we require a specific realization, namely an embedding into a projective space.

Given a representation ρ of H on some vector space V , we can construct a vector bundle associated to it, namely

$$\mathcal{V} := G \times_H V = G \times V / \simeq, \quad (gh, v) \simeq (g, h^{-1}v), \quad \forall h \in H, g \in G, v \in V.$$

The space of global sections of this bundle is identified with the induced module (see, e.g., [18] for more details)

$$H^0(G/H, \mathcal{V}) = \text{Ind}_H^G(V) = \{f : G \rightarrow V \mid f \text{ is regular, } f(gh) = h^{-1}.f(g)\}.$$

Definition 2.7. Let $\chi: H \rightarrow \mathbb{k}^*$ be a character of H , i.e. a one dimensional representation of H on $L \cong \mathbb{k}$. Then $L^{\otimes n}$ is again a one dimensional representation of H with character χ^n . Let $\mathcal{L}^n := G \otimes_H L^{\otimes n}$. Define

$$\mathcal{O}(G/H)_n := H^0(G/H, \mathcal{L}^n)$$

$$\mathcal{O}(G/H) := \bigoplus_{n \geq 0} \mathcal{O}(G/H)_n \subseteq \mathcal{O}(G) \quad .$$

Then $\mathcal{O}(G/H)$ is a subalgebra of $\mathcal{O}(G)$, whose elements are called *semi-invariants*. Note that now the notation $\mathcal{O}(G/H)$ has not the same meaning as when G/H is affine.

Assume now the bundle \mathcal{L} to be very ample. In the present context, this is the same (cf. [18], §II.7) as saying that \mathcal{L} is generated by a set of global sections $f_0, f_1, \dots, f_N \in \mathcal{O}(G/H)_1$ — in particular, the algebra $\mathcal{O}(G/H)$ is *graded, generated in degree 1* (by the f_i 's). Then $\mathcal{O}(G/H)$ is the homogeneous coordinate ring of the projective variety G/H , with respect to the embedding given via the global sections of \mathcal{L} (see [17], p. 176).

We want to reformulate this classical construction in purely Hopf algebraic terms, more suited to the quantum setting we shall presently deal with.

Remark 2.8. In algebraic terms, saying that $\chi: H \rightarrow \mathbb{k}^*$ is a character is the same as saying that it is a group-like element in the coalgebra $\mathcal{O}(H)$. The same holds for all powers χ^n ($n \in \mathbb{N}$). In fact if χ is group-like, then the same is true for all its powers χ^n since $\mathcal{O}(H)$ is a Hopf algebra. As the χ^n 's are group-like, if they are pairwise different they also are linearly independent, which ensures that the sum $\sum_{n \in \mathbb{N}} \mathcal{O}(G/H)_n$ — inside $\mathcal{O}(G)$ — is a direct one.

Moreover, once the embedding is given, each summand $\mathcal{O}(G/H)_n$ can be described in purely Hopf algebraic terms as

$$\begin{aligned} \mathcal{O}(G/H)_n &:= \{f \in \mathcal{O}(G) \mid f(gh) = \chi^n(h^{-1})f(g)\} = \\ &= \left\{ f \in \mathcal{O}(G) \mid ((id \otimes \pi) \circ \Delta)(f) = f \otimes S(\chi^n) \right\} \end{aligned}$$

with $\pi: \mathcal{O}(G) \rightarrow \mathcal{O}(H)$ the standard projection, S the antipode of $\mathcal{O}(H)$.

To simplify notation, we set $\lambda := S(\chi)$ — the character of H which maps $h \in H$ to $\lambda(h) = \chi(h^{-1})$ — and we set $\Delta_\pi := (id \otimes \pi) \circ \Delta$, so that

$$\mathcal{O}(G/H)_n = \left\{ f \in \mathcal{O}(G) \mid \Delta_\pi(f) = f \otimes \lambda^n \right\} \quad . \quad (2.1)$$

Proposition 2.9. *Let G/H be embedded into some projective space via some very ample line bundle. Then there exists a $t \in \mathcal{O}(G)$ such that*

$$\Delta_\pi(t) := ((id \otimes \pi) \circ \Delta)(t) = t \otimes \pi(t) \quad (2.2)$$

$$\pi(t^m) \neq \pi(t^n) \quad \forall m \neq n \in \mathbb{N} \quad (2.3)$$

$$\mathcal{O}(G/H)_n = \left\{ f \in \mathcal{O}(G) \mid \Delta_\pi(f) = f \otimes \pi(t^n) \right\} \quad (2.4)$$

$$\mathcal{O}(G/H) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}(G/H)_n \quad (2.5)$$

and $\mathcal{O}(G/H)$ is generated in degree 1, namely by $\mathcal{O}(G/H)_1$.

Proof. If $f \in \mathcal{O}(G/H)_n$, then

$$\pi(f) = \pi\left(\sum_{(f)} \epsilon(f_{(1)}) f_{(2)}\right) = (\epsilon \otimes \pi)(\Delta(f)) = (\epsilon \otimes id)(\Delta_\pi(f)) = \epsilon(f) \lambda^n$$

Now, by assumption there exists a non-zero global section of the line bundle on G/H , i.e. a regular function $t \in \mathcal{O}(G/H)_1 \setminus \{0\}$ on G and $\epsilon(t) \neq 0$. By the above (for $n = 1$), up to dividing out by $\epsilon(t)$, we can assume that $\pi(t) = \lambda$. The result follows immediately. \square

Notice that while $\lambda = \pi(t)$ is group-like, t instead is something less, yet still has an “almost group-like property”, given by (2.2). This element t and its quantization will turn to be crucial for the quantum setting.

Remark 2.10. We point out that $\mathcal{O}(G/H)$ is a unital subalgebra, and also a (left) coideal of $\mathcal{O}(G)$; the latter reflects the fact that G/H is a (left) G -space. Thus, the restriction of the comultiplication of $\mathcal{O}(G)$, namely

$$\Delta|_{\mathcal{O}(G/H)} : \mathcal{O}(G/H) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G/H) \quad ,$$

is a coaction of $\mathcal{O}(G)$ on $\mathcal{O}(G/H)$, which makes $\mathcal{O}(G/H)$ into an $\mathcal{O}(G)$ -comodule algebra, in the sense of [26], §4.1. Moreover $\mathcal{O}(G/H)$ is graded and the coaction $\Delta|_{\mathcal{O}(G/H)}$ is also graded w.r.t. the trivial grading on $\mathcal{O}(G)$, so that each $\mathcal{O}(G/H)_n$ is indeed a coideal of $\mathcal{O}(G)$ as well.

As to Poisson geometrical properties in this projective setup, the following characterization — which might be used to define the notion of *Poisson quotient* structure for the projective G -space G/H — holds:

Proposition 2.11. *Let G be a Poisson algebraic group, H a closed parabolic subgroup, and $t \in \mathcal{O}(G)$ as in Remark 2.8. The following are equivalent:*

(a) $\{I(H), I(H)\} \subseteq I(H)$ — that is, H is coisotropic — and in addition $\{t, \mathcal{O}(G/H)\} \subseteq I(H)$;

(b) $\{\mathcal{O}(G/H)_r, \mathcal{O}(G/H)_s\} \subseteq \mathcal{O}(G/H)_{r+s}$ for all $r, s \in \mathbb{N}$ — that is, $\mathcal{O}(G/H)$ is a graded Poisson subalgebra of $\mathcal{O}(G)$.

Proof. To simplify notation, we set $I_{H,n} := I(H) \cap \mathcal{O}(G/H)_n$ for $n \in \mathbb{N}$.

(a) \implies (b): First of all, note that (2.1) can be reformulated as

$$\mathcal{O}(G/H)_n = \left\{ f \in \mathcal{O}(G) \mid \Delta(f) \in f \otimes t^n + \mathcal{O}(G) \otimes I(H) \right\} \quad (2.6)$$

Second, by Remark 2.10(a), each $\mathcal{O}(G/H)_n$ is a coideal of $\mathcal{O}(G)$, that is $\Delta(\mathcal{O}(G/H)_n) \subseteq \mathcal{O}(G) \otimes \mathcal{O}(G/H)_n$. This along with (2.6) gives

$$\Delta(f) \in f \otimes t^n + \mathcal{O}(G) \otimes I_{H,n} \quad \forall f \in \mathcal{O}(G/H)_n. \quad (2.7)$$

Then, for any $f \in \mathcal{O}(G/H)_r$ and $\ell \in \mathcal{O}(G/H)_s$, we have

$$\begin{aligned} \Delta(\{f, \ell\}) &= \{\Delta(f), \Delta(\ell)\} \in \left\{ f \otimes t^r + \mathcal{O}(G) \otimes I_{H,r}, \ell \otimes t^s + \mathcal{O}(G) \otimes I_{H,s} \right\} = \\ &= \{f \otimes t^r, \ell \otimes t^s\} + \left\{ f \otimes t^r, \mathcal{O}(G) \otimes I_{H,s} \right\} + \\ &\quad + \left\{ \mathcal{O}(G) \otimes I_{H,r}, \ell \otimes t^s \right\} + \left\{ \mathcal{O}(G) \otimes I_{H,r}, \mathcal{O}(G) \otimes I_{H,s} \right\} \subseteq \\ &\subseteq \{f, \ell\} \otimes t^{r+s} + f \ell \otimes \{t^r, t^s\} + \left\{ f, \mathcal{O}(G) \right\} \otimes t^r I_{H,s} + \\ &\quad + f \mathcal{O}(G) \otimes \left\{ t^r, I_{H,s} \right\} + \left\{ \mathcal{O}(G), \ell \right\} \otimes I_{H,r} t^s + \mathcal{O}(G) \ell \otimes \left\{ I_{H,r}, t^s \right\} + \\ &\quad + \left\{ \mathcal{O}(G), \mathcal{O}(G) \right\} \otimes I_{H,r} I_{H,s} + \mathcal{O}(G) \mathcal{O}(G) \otimes \left\{ I_{H,r}, I_{H,s} \right\} \subseteq \\ &\subseteq \{f, \ell\} \otimes t^{r+s} + \mathcal{O}(G) \otimes I_{H,r+s} \end{aligned}$$

thanks to (2.7) and to (a). Thus $\{f, \ell\} \in \mathcal{O}(G/H)_{r+s}$, by (2.6) again.

(b) \implies (a): By assumption we have $t \in \mathcal{O}(G/H)_1$, hence $\{t, f\} \in \mathcal{O}(G/H)_{1+n}$ for all $f \in \mathcal{O}(G/H)_n$ by (b). In particular, this gives

$$\{t, f\}(h) = t^{1+n}(h) \{t, f\}(1_G) = 0 \quad \forall h \in H$$

because any Poisson group structure is zero at the identity. Eventually, this yields $\{t, \mathcal{O}(G/H)\} \subseteq I(H)$, q.e.d.

To prove that $\{I(H), I(H)\} \subseteq I(H)$, we need some additional tools.

First, let $\mathcal{O}(G)_{1_G}$ be the localization of $\mathcal{O}(G)$ at $J := \text{Ker}(\epsilon_{\mathcal{O}(G)})$ — a maximal ideal in $\mathcal{O}(G)$. This is the stalk at the point 1_G of the structure sheaf of G , and the Poisson bracket of $\mathcal{O}(G)$ canonically (and uniquely) extends — for $f_1, f_2 \in \mathcal{O}(G)$, $y_1, y_2 \in J$, $n_1, n_2 \in \mathbb{N}$ — via the identity

$$\begin{aligned} \{f_1 y_1^{-n_1}, f_2 y_2^{-n_2}\} &= \{f_1, f_2\} y_1^{-n_1} y_2^{-n_2} - n_1 f_1 y_1^{-n_1-1} \{y_1, f_2\} y_2^{-n_2} - \\ &\quad - n_2 f_2 \{f_1, f_2\} y_1^{-n_1} y_2^{-n_2-1} + n_1 n_2 f_1 f_2 \{y_1, y_2\} y_1^{n_1-1} y_2^{n_2-1} \end{aligned}$$

to the local algebra $\mathcal{O}(G)_{1_G}$. The counit ϵ of $\mathcal{O}(G)$ uniquely extends to an algebra morphism from $\mathcal{O}(G)_{1_G}$ to \mathbb{k} , again denoted by ϵ , whose kernel is J_{1_G} , the localization of J inside $\mathcal{O}(G)_{1_G}$. Finally, we denote by $I(H)_{1_G}$ the localization, inside $\mathcal{O}(G)_{1_G}$, of the ideal $I(H)$ of $\mathcal{O}(G)$.

Second, let $X_t := \{t = 0\}$ be the zero locus in G/H defined by the vanishing of the divisor t , and let $\Gamma := (G/H) \setminus X_t$. This is an affine open dense subset of G/H , whose algebra of regular functions is the graded localisation of $\mathcal{O}(G/H)$ by the multiplicative subset $\{t^n\}_{n \in \mathbb{N}}$, that is $\mathcal{O}(G/H)_{[t]} := \bigoplus_{n \in \mathbb{N}} t^{-n} \mathcal{O}(G/H)_n$. Note that $\mathcal{O}(G/H)_{[t]}$ naturally embeds into $\mathcal{O}(G)_{1_G}$, because $t \in J^{G/H} := \mathcal{O}(G/H) \setminus J$. Again, the Poisson bracket of $\mathcal{O}(G/H)$, induced by that of $\mathcal{O}(G)$, uniquely extends to $\mathcal{O}(G/H)_{[t]}$, and so the latter is a *graded* Poisson subalgebra of $\mathcal{O}(G)_{1_G}$; thus X_t is an affine Poisson variety. Also, ϵ induces an algebra morphism from $\mathcal{O}(G/H)_{[t]}$ to \mathbb{k} , whose kernel we denote by $J_{[t]}^{G/H}$.

Third, let $\mathcal{O}(G/H)_{\overline{1_G}}$ be the localization of $\mathcal{O}(G/H)_{[t]}$ at $J_{[t]}^{G/H}$: by construction, this is the stalk at $\overline{1_G} = 1_G H$ of the structure sheaf of G/H , and the Poisson bracket of $\mathcal{O}(G/H)_{[t]}$ uniquely extends to $\mathcal{O}(G/H)_{\overline{1_G}}$.

Now, the maximal ideal J_{1_G} in the local algebra $\mathcal{O}(G)_{1_G}$ can be generated by a local system of parameters on G at the point 1_G , say $\{y_1, \dots, y_n\}$, with $n := \dim(G)$. As H is a closed subgroup of G , we can choose this system of parameters in such a way that:

- (1) if $h := \dim(H)$, the image inside $\mathcal{O}(H)_{1_H} \cong \mathcal{O}(G)_{1_G}/I(H)_{1_G}$ of $\{y_1, \dots, y_h\}$ is a local system of parameters on H at the point $1_H = 1_G$;
- (2) $y_h = t - 1$;

(3) $\{y_{h+1}, \dots, y_n\}$ is a local system of parameters on X_t at 1_G ; in particular, it generates in $\mathcal{O}(G/H)_{\overline{1_G}}$ the ideal $J_{\overline{1_G}}^{G/H} := \text{Ker}(\epsilon) \cap \mathcal{O}(G/H)_{\overline{1_G}}$.

As a direct consequence of the above assumptions, the elements y_{h+1}, \dots, y_n generate the ideal $I(H)_{1_G}$ inside $\mathcal{O}(G)_{1_G}$. Moreover, we have $y_i = f_i \ell_i^{-1}$ for some $f_i \in \mathcal{O}(G/H)$, $\ell_i \in \mathcal{O}(G/H) \setminus J^{G/H}$ (for $i = h+1, \dots, n$). Then

$$\begin{aligned} \{y_i, y_j\} &= \{f_i \ell_i^{-1}, f_j \ell_j^{-1}\} = \{f_i, f_j\} \ell_i^{-1} \ell_j^{-1} - \{f_i, \ell_j\} \ell_i^{-1} f_j \ell_j^{-2} - \\ &\quad - f_i \{ \ell_i, f_j \} \ell_i^{-2} \ell_j^{-1} + f_i f_j \{ \ell_i^{-1}, \ell_j^{-1} \} \ell_i^{-2} \ell_j^{-2} \end{aligned}$$

which — by assumption (b), that yields $\{\mathcal{O}(G/H), \mathcal{O}(G/H)\} \subseteq \mathcal{O}(G/H)$ — gives $\{y_i, y_j\} \in (y_{h+1}, \dots, y_n) = I(H)_{1_G}$, for all i, j . This together with Leibnitz' rule implies that $\{k_i y_i, k_j y_j\} \in (y_{h+1}, \dots, y_n) = I(H)_{1_G}$ for any $k_i, k_j \in \mathcal{O}(G)_{1_G}$ (with $i, j = h+1, \dots, n$); in turn, $I(H)_{1_G} = (y_{h+1}, \dots, y_n)$ satisfies $\{I(H)_{1_G}, I(H)_{1_G}\} \subseteq I(H)_{1_G}$.

Eventually, as $I(H) = \mathcal{O}(G) \cap I(H)_{1_G}$, the above results give also

$$\{I(H), I(H)\} \subseteq \mathcal{O}(G) \cap \{I(H)_{1_G}, I(H)_{1_G}\} \subseteq \mathcal{O}(G) \cap I(H)_{1_G} = I(H) \quad \square$$

3 Quantum bundles and quantum homogeneous spaces

3.1 Quantum groups

We want to translate all the framework of section 2 into the quantum setup. The first step is to introduce quantum groups, in the form of quantum (or “quantized”) function algebras, as follows.

Let G be an algebraic Poisson group, $\mathcal{O}(G)$ its function algebra.

Definition 3.1. By *quantization* of $\mathcal{O}(G)$, we mean a Hopf algebra $\mathcal{O}_q(G)$ over the ground ring $\mathbb{k}_q := \mathbb{k}[q, q^{-1}]$, where q is an indeterminate, such that:

(a) the *specialization* of $\mathcal{O}_q(G)$ at $q = 1$, that is $\mathcal{O}_q(G) / (q-1) \mathcal{O}_q(G)$, is isomorphic to $\mathcal{O}(G)$ as a Poisson Hopf algebra;

(b) $\mathcal{O}_q(G)$ is torsion-free, as a \mathbb{k}_q -module;

(c) if $I_G := (q-1) \mathcal{O}_q(G) + \text{Ker}(\epsilon_{\mathcal{O}_q(G)})$, then $\bigcap_{n \geq 0} I_G^n = \bigcap_{n \geq 0} (q-1)^n \mathcal{O}_q(G)$.

We call $\mathcal{O}_q(G)$ *quantum* (or *quantized*) *function algebra* over G , or *quantum*

deformation of G , or even simply *quantum group*. It is standard terminology to say that the *Poisson Hopf algebra* $\mathcal{O}(G)$ is the *semiclassical limit* of $\mathcal{O}_q(G)$.

Similarly, we say that a \mathbb{k}_q -algebra $\mathcal{O}_q(X)$ is a *quantization* of the commutative \mathbb{k} -algebra $\mathcal{O}(X)$ if it is torsion-free and $\mathcal{O}_q(X)/(q-1)\mathcal{O}_q(X) \cong \mathcal{O}(X)$. Then $\mathcal{O}(X)$ is also a *Poisson algebra*, called *semiclassical limit* of $\mathcal{O}_q(X)$.

Remark 3.2.

(1) The technical requirement in (c) corresponds, in the context of formal deformations, to require that the algebra is separated; we also point out that it is satisfied by all quantum function algebras usually considered in literature. In any case, it will not be necessary till §5. Moreover, both (b) and (c) above are automatically satisfied when $\mathcal{O}_q(G)$ is free as a \mathbb{k}_q -module.

(2) The classical algebra $\mathcal{O}(G)$ inherits from $\mathcal{O}_q(G)$ a Poisson bracket, given as follows: if $x, y \in \mathcal{O}_q(G)/(q-1)\mathcal{O}_q(G) \cong \mathcal{O}(G)$, then

$$\{x, y\} := \frac{x' y' - y' x'}{q-1} \pmod{(q-1)\mathcal{O}_q(G)}$$

for any lifts $x', y' \in \mathcal{O}_q(G)$ of x and y respectively. One checks that this bracket is well-defined, and makes $\mathcal{O}(G)$ into a Poisson Hopf algebra, so that G is a Poisson group. But G already had, by assumption, a Poisson group structure; then, the requirement in (a) above that $\mathcal{O}_q(G)/(q-1)\mathcal{O}_q(G) \cong \mathcal{O}(G)$ as *Poisson Hopf algebra* amounts to say, in particular, that the two Poisson group structures of G are isomorphic.

On the other hand, if we start without asking G to have a Poisson group structure, then the previous analysis tells that if a quantization $\mathcal{O}_q(G)$ exists, then it automatically endows $\mathcal{O}(G)$ with a Poisson algebra structure. And similarly for a quantization $\mathcal{O}_q(X)$ of a commutative algebra $\mathcal{O}(X)$.

3.2 Quantum subgroups and quantum coisotropic subgroups

Our second step is to introduce the notions of *quantum coisotropic subgroup* and of *quantum subgroup*, the former being weaker than the latter.

Definition 3.3. By *quantum coisotropic subgroup* of $\mathcal{O}_q(G)$ we mean a \mathbb{k}_q -coalgebra $\mathcal{O}_q(H)$, along with a projection $\pi : \mathcal{O}_q(G) \twoheadrightarrow \mathcal{O}_q(H)$, such that

- (a) $\mathcal{O}_q(H)$ is torsion-free, as a \mathbb{k}_q -module;
- (b) π is a \mathbb{k}_q -coalgebra (epi)morphism;
- (c) π is an $\mathcal{O}_q(G)$ -module (epi)morphism, where $\mathcal{O}_q(H)$ has the $\mathcal{O}_q(G)$ -module structure induced by π , that is $f \cdot \pi(g) = \pi(fg)$.

If, in addition, $\mathcal{O}_q(H)$ is a Hopf algebra and π is a Hopf algebra morphism, and for $I_H := (q-1)\mathcal{O}_q(H) + \text{Ker}(\epsilon_{\mathcal{O}_q(H)})$ we have

$$(e) \quad \bigcap_{n \geq 0} I_H^n = \bigcap_{n \geq 0} (q-1)^n \mathcal{O}_q(H)$$

then we say that $\mathcal{O}_q(H)$ is a *quantum subgroup* of G .

For later use, we introduce also the notation $I_q(H) := \text{Ker}(\pi)$.

Remark 3.4.

(1) $I_q(H) := \text{Ker}(\pi)$ satisfies $I_q(H) \cap (q-1)\mathcal{O}_q(G) = (q-1)I_q(H)$. So the specialization of $I_q(H)$ at $q = 1$, i.e. $I_1(H) := I_q(H)/(q-1)I_q(H)$, coincides with the image of $I_q(H)$ under the specialization $\mathcal{O}_q(G)/(q-1)\mathcal{O}_q(G) \cong \mathcal{O}(G)$ of $\mathcal{O}_q(G)$, which is $I_q(H)/(I_q(H) \cap (q-1)\mathcal{O}_q(G))$.

(2) Conditions (b) and (c) imply that $I_q(H)$ is a two-sided coideal and a left ideal of $\mathcal{O}_q(G)$. Then, by (1), the specialization $I_1(H)$ is a coideal and a (two-sided) ideal in the commutative ring $\mathcal{O}(G)$. Moreover, $I_1(H)$ equals the kernel of $\pi_1: \mathcal{O}(G) = \mathcal{O}_1(G) \longrightarrow \mathcal{O}_1(H)$, the specialization of π at $q = 1$, where $\mathcal{O}_1(H) := \mathcal{O}_q(H)/(q-1)\mathcal{O}_q(H)$ is the specialization of $\mathcal{O}_q(H)$. So $\mathcal{O}_1(H)$ admits the unique quotient Hopf algebra structure such that π_1 is the canonical Hopf algebra epimorphism. In particular, $\mathcal{O}_1(H)$ is the function algebra $\mathcal{O}(H)$ of some closed algebraic subgroup H of G , and $I_1(H) = \text{Ker}(\pi_1: \mathcal{O}(G) \longrightarrow \mathcal{O}(H)) = I(H)$, whence the terminology and notation.

In the Hopf algebra language, conditions (b) and (c) are expressed by saying that $\mathcal{O}_q(H)$ is an $\mathcal{O}_q(G)$ -module coalgebra, that is a coalgebra and $\mathcal{O}_q(G)$ -module such that $\Delta_{\mathcal{O}_q(H)}$ and $\epsilon_{\mathcal{O}_q(H)}$ are $\mathcal{O}_q(G)$ -module morphisms.

(3) Assumptions at the quantum level imply properties for specializations. So, the semiclassical specialization of a quantum coisotropic subgroup is (the function algebra of) a *coisotropic* subgroup, because $I_1(H) = \text{Ker}(\pi_1)$ is a Poisson subalgebra of $\mathcal{O}(G)$. On the other hand, the specialization of a quantum subgroup instead is (the function algebra of) a *Poisson* subgroup.

At the semiclassical level there are many examples of coisotropic subgroups, among which only a few are Poisson subgroups. This is a key motivation to focus on the more general setting of quantum coisotropic subgroups.

(4) A quantum coisotropic subgroup $\mathcal{O}_q(H)$ is by no means a “quantum group”, in the sense of Definition 3.1, unless it is a quantum subgroup.

3.3 Quantum line bundles

We now want to carry across to the quantum setting the notion of embedding $G/H \hookrightarrow \mathbb{P}^N$ associated to a line bundle \mathcal{L} that we assume to be very ample. The idea is to transfer to this framework the description (2.4) of $\mathcal{O}(G/H)_n$ given in terms of an element $t \in \mathcal{O}(G)$ as in Remark 2.8 and Proposition 2.9. Thus, the starting point will be a quantization of such an element t that we will call a pre-quantum section.

Given G and H as in §2.2, we assume that quantizations of them be given, i.e. we are given $\mathcal{O}_q(G)$, $\mathcal{O}_q(H)$ and $\pi : \mathcal{O}_q(G) \twoheadrightarrow \mathcal{O}_q(H)$ as in Definitions 3.1, 3.3. To simplify notation, hereafter we shall also write $\bar{\ell} := \pi(\ell) \in \mathcal{O}_q(H)$ for every $\ell \in \mathcal{O}_q(G)$.

Moreover, we assume that an element $t \in \mathcal{O}(G)$ as in Remark 2.8, and the corresponding closed embedding $G/H \hookrightarrow \mathbb{P}^N$, be given as in Proposition 2.9 (so, in particular, t is a section of the line bundle \mathcal{L} on G/H).

We define a quantization of the latter setup as follows:

Definition 3.5. We define *pre-quantization of t* , or *pre-quantum section of the line bundle \mathcal{L} on G/H (given by t)*, any $d \in \mathcal{O}_q(G)$ such that

- (a) $\Delta_\pi(d) = d \otimes \pi(d)$, i.e. $\Delta(d) \in (d \otimes d + \mathcal{O}_q(G) \otimes I_q(H))$
- (b) $d \bmod (q-1)\mathcal{O}_q(G) = t \quad (\in \mathcal{O}(G))$

with respect to the identification $\mathcal{O}_q(G)/(q-1)\mathcal{O}_q(G) \cong \mathcal{O}(G)$.

Remark 3.6.

(a) Given a pre-quantum section d , property (a) in Definition 3.5 implies that $\pi(d) = \bar{d}$ is a group-like element in $\mathcal{O}_q(H)$. Therefore, it defines a one-dimensional corepresentation of $\mathcal{O}_q(H)$, namely

$$\rho_d : \mathbb{k}_q \longrightarrow \mathbb{k}_q \otimes_{\mathbb{k}_q} \mathcal{O}_q(H) \quad , \quad 1 \mapsto 1 \otimes \bar{d}$$

which gives back, modulo $(q-1)$, the one-dimensional representation of $\mathcal{O}(H)$

$$\rho_\lambda : \mathbb{k} \longrightarrow \mathbb{k} \otimes_{\mathbb{k}} \mathcal{O}(H) \quad , \quad 1 \mapsto 1 \otimes \lambda$$

corresponding to the character $\lambda = \pi(t)$ of the group H we started from.

(b) In the classical setup, having the character λ is equivalent to having a Hopf algebra morphism $\mathbb{k}[x, x^{-1}] \rightarrow \mathcal{O}(H)$, given by $x^z \mapsto \lambda^z$ ($z \in \mathbb{Z}$). Indeed, this occurs because the powers λ^z do exist in $\mathcal{O}(H)$, and are group-like because λ is. In fact, each one of them corresponds to a one-dimensional corepresentation, namely the z -th tensor power of ρ_λ

$$\rho_\lambda^{\otimes z} = \rho_{\lambda^z} : \mathbb{k} \longrightarrow \mathbb{k} \otimes_{\mathbb{k}} \mathcal{O}(H) \ , \quad 1 \mapsto 1 \otimes \lambda^z$$

On the other hand, in the quantum setup there is no natural analogue, since the powers \bar{d}^z are not even defined in $\mathcal{O}_q(H)$ — which is not an algebra! — nor we can assume (would we define them in some way) that they are group-like. This means that we miss somehow the “tensor powers” of ρ_d . In [1] one can find an example of a countable family of group-like elements in a quantum coisotropic subgroups, which are not obtained by projecting powers of the same element, but quantize a classical character.

However, when $\mathcal{O}_q(H)$ is a quantum subgroup instead, it is a Hopf algebra, hence the group-like \bar{d} is invertible, and all powers \bar{d}^z exist, and are group-like in $\mathcal{O}_q(H)$. So we do have all “tensor power corepresentations”

$$\rho_d^{\otimes z} : \mathbb{k}_q \longrightarrow \mathbb{k}_q \otimes_{\mathbb{k}_q} \mathcal{O}_q(H) \ , \quad 1 \mapsto 1 \otimes \bar{d}^z$$

which in turn means that *having \bar{d} is equivalent to having a Hopf \mathbb{k}_q -algebra morphism $\mathbb{k}_q[x, x^{-1}] \rightarrow \mathcal{O}_q(H)$* , given by $x^z \mapsto \bar{d}^z$ ($z \in \mathbb{Z}$).

Moreover, notice also that $\bar{d}^z = d^z$ for all $z \geq 0$, so that $\rho_d^{\otimes z}$ for $z \geq 0$ can be directly recovered from the element d^z in $\mathcal{O}_q(G)$; thus in the end we can handle everything working with the elements $d^n \in \mathcal{O}_q(G)$, $n \in \mathbb{N}$.

Definition 3.7. Let $d \in \mathcal{O}_q(G)$ be a pre-quantum section on G/H .

(a) We call *d -semi-invariants of degree n* the elements of the set

$$\begin{aligned} \mathcal{O}_q(G/H)_n &:= \left\{ \ell \in \mathcal{O}_q(G) \mid \Delta_\pi(\ell) = \ell \otimes \pi(d^n) \right\} = \\ &= \left\{ \ell \in \mathcal{O}_q(G) \mid \Delta(\ell) \in (\ell \otimes d^n + \mathcal{O}_q(G) \otimes I_q(H)) \right\} \end{aligned}$$

(b) We call *d -semi-invariants* the elements of the set

$$\mathcal{O}_q(G/H) := \sum_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n$$

It is clear that each $\mathcal{O}_q(G/H)_n$ is a \mathbb{k}_q -submodule of $\mathcal{O}_q(G)$, hence the same holds for $\mathcal{O}_q(G/H)$. We shall now see some further properties of these modules, which eventually will tell us that — under suitable, additional assumptions — we can take $\mathcal{O}_q(G/H)$ as a quantization of $\mathcal{O}(G/H)$.

Lemma 3.8. *Let $d \in \mathcal{O}_q(G)$ be a pre-quantum section on G/H . Then*

- (a) $d \in \mathcal{O}_q(G/H)_1$, i.e. d is semi-invariant of degree 1.
- (b) for any $n \in \mathbb{N}$, and any $\ell \in \mathcal{O}_q(G/H)_n$, we have $\bar{\ell} = \epsilon(\ell) \bar{d}^n$.
- (c) the map $\pi : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(H)$ restricts to a \mathbb{k}_q -module epimorphism

$$\pi' : \mathcal{O}_q(G/H) \longrightarrow \text{Span}_{\mathbb{k}_q} \left(\{ \bar{d}^n \}_{n \in \mathbb{N}} \right).$$

Proof. The only statement which needs a proof here is (b), which quickly follows applying $(\epsilon \otimes id)$ to both sides of $\Delta_\pi(\ell) = \ell \otimes \bar{d}^n$. \square

Remark 3.9. The semi-invariants have a good arithmetic property, which ensures that the specialization of $\mathcal{O}_q(G/H)_n \subset \mathcal{O}_q(G)$ (and of $\mathcal{O}_q(G/H)$) at $q = 1$ will be consistent with that of $\mathcal{O}_q(G)$ itself. Namely, given $n \in \mathbb{N}$, since $\mathcal{O}_q(G/H)_n$ is defined by \mathbb{k}_q -linear conditions we find at once

$$\mathcal{O}_q(G/H)_n \cap c \mathcal{O}_q(G) = c \mathcal{O}_q(G/H)_n \quad \forall c \in \mathbb{k}_q$$

and in particular, $\mathcal{O}_q(G/H)_n \cap (q-1) \mathcal{O}_q(G) = (q-1) \mathcal{O}_q(G/H)_n$.

Next result shows that each $\mathcal{O}_q(G/H)_n$ is a left coideal of $\mathcal{O}_q(G)$, hence it bears a structure of left $\mathcal{O}_q(G)$ -comodule.

Proposition 3.10. *Every $\mathcal{O}_q(G/H)_n$ is a left coideal of $\mathcal{O}_q(G)$, that is*

$$\Delta \left(\mathcal{O}_q(G/H)_n \right) \subseteq \mathcal{O}_q(G) \otimes \mathcal{O}_q(G/H)_n \quad \forall n \in \mathbb{N}$$

so that $\Delta|_{\mathcal{O}_q(G/H)_n}$ makes $\mathcal{O}_q(G/H)_n$ into a left $\mathcal{O}_q(G)$ -comodule. Therefore, $\mathcal{O}_q(G/H)$ is a left coideal of $\mathcal{O}_q(G)$, hence a left $\mathcal{O}_q(G)$ -comodule.

Proof. Let $\mathcal{O} := \mathcal{O}_q(G)$, $\mathcal{O}_n := \mathcal{O}_q(G/H)_n$, and set $\mathcal{O}' := \mathbb{k}(q) \otimes_{\mathbb{k}_q} \mathcal{O}$, $\mathcal{O}'_n := \mathbb{k}(q) \otimes_{\mathbb{k}_q} \mathcal{O}_n$. Then $\mathcal{O} \otimes_{\mathbb{k}_q} \mathcal{O}$ naturally embeds into $\mathcal{O}' \otimes_{\mathbb{k}(q)} \mathcal{O}'$, because \mathcal{O} and \mathcal{O}'_n are torsion free as \mathbb{k}_q -modules. Using this embedding, given any $\ell \in \mathcal{O}_q(G/H)_n$ we can always write $\Delta(\ell) = \sum_i g'_i \otimes h'_i$ for some $g'_i, h'_i \in \mathcal{O}'$ such that the g'_i 's are all linearly independent, and similarly for the h'_i 's. Then (taking a common denominator) there exists $c \in \mathbb{k}_q$ such that $\Delta(\ell) = c^{-1} \sum_i g_i \otimes h_i$, with the g_i 's in $\mathcal{O}_q(G)$ being linearly independent, and the h_i 's in $\mathcal{O}_q(G)$ which are linearly independent too.

We shall now prove that

$$\sum_i g_i \otimes \sum_{(h_i)} (h_i)_{(1)} \otimes \overline{(h_i)_{(2)}} = \sum_i g_i \otimes h_i \otimes \overline{d^n} . \quad (3.1)$$

Indeed, the left-hand side in (3.1) is just the image of $c\ell$ via the map

$$\left(id \otimes ((id \otimes \pi) \circ \Delta) \right) \circ \Delta = (id \otimes id \otimes \pi) \circ (id \otimes \Delta) \circ \Delta$$

By coassociativity of Δ , the latter map coincides with

$$(id \otimes id \otimes \pi) \circ (id \otimes \Delta) \circ \Delta = (\Delta \otimes \pi) \circ \Delta = (\Delta \otimes id) \circ (id \otimes \pi) \circ \Delta = (\Delta \otimes id) \circ \Delta_\pi$$

and now the last map applied to $c\ell$ gives

$$(\Delta \otimes id)(\Delta_\pi(c\ell)) \stackrel{\circledast}{=} (\Delta \otimes id)(c\ell \otimes \overline{d^n}) = \sum_i g_i \otimes h_i \otimes \overline{d^n}$$

where $\stackrel{\circledast}{=}$ follows from the assumption $\ell \in \mathcal{O}_q(G/H)_n$, which implies $c\ell \in \mathcal{O}_q(G/H)_n$ as well. This eventually gives the right-hand side of (3.1), q.e.d.

Now, because of the linear independence of the h_i 's, the identity (3.1) implies that all the h_i 's satisfy $\sum_{(h_i)} (h_i)_{(1)} \otimes \overline{(h_i)_{(2)}} = h_i \otimes \overline{d^n}$, which means exactly $h_i \in \mathcal{O}_q(G/H)_n$, for every index i . Thus we have

$$\Delta(c\ell) = \sum_i g_i \otimes h_i , \quad g_i \in \mathcal{O}_q(G), h_i \in \mathcal{O}_q(G/H)_n ,$$

the g_i 's, resp. the h_i 's, being linearly independent, and also

$$\Delta(c\ell) \in c\mathcal{O}_q(G) \otimes \mathcal{O}_q(G) .$$

These two conditions imply $h_i \in \mathcal{O}_q(G/H)_n \cap c\mathcal{O}_q(G) = c\mathcal{O}_q(G/H)_n$ thanks to Remark 3.9. Therefore $\Delta(\ell) \in \mathcal{O}_q(G) \otimes \mathcal{O}_q(G/H)_n$. Finally, the claim for $\mathcal{O}_q(G/H)$ follows from that for the $\mathcal{O}_q(G/H)_n$'s. \square

The above construction provides us with a reasonable candidate for a quantum analogue of $\mathcal{O}(G/H)$, namely the space of the d -semi-invariants $\mathcal{O}_q(G/H)$, which we proved has many important properties. Nevertheless, we still would like $\mathcal{O}_q(G/H)$ to verify three more key properties, namely:

- (a) $\mathcal{O}_q(G/H)$ is a subalgebra of $\mathcal{O}_q(G)$;
- (b) $\mathcal{O}_q(G/H)$ is a graded object, its n -th (for all $n \in \mathbb{N}$) graded summand being $\mathcal{O}_q(G/H)_n$;
- (c) the grading is compatible with all other structures, so $\mathcal{O}_q(G/H)$ is a graded $\mathcal{O}_q(G)$ -comodule algebra (when $\mathcal{O}_q(G)$ is given the trivial grading).

Indeed, we are still missing these properties so far. In order to have them, an additional property must be required to the pre-quantum section d we started from. This is provided by the following result.

Proposition 3.11. *Set $I_q(H) := \text{Ker}(\pi)$, and let d be a pre-quantum section on G/H . Then the following properties are equivalent:*

- (a) $\mathcal{O}_q(G/H)_r \cdot \mathcal{O}_q(G/H)_s \subseteq \mathcal{O}_q(G/H)_{r+s}$ for all $r, s \in \mathbb{N}$, hence, in particular, $\mathcal{O}_q(G/H)$ is a \mathbb{k}_q -subalgebra of $\mathcal{O}_q(G)$;
- (b) $\overline{[d, f]} = \bar{0}$ in $\mathcal{O}_q(H)$, for all $f \in \mathcal{O}_q(G/H)$;
- (c) $[d, \mathcal{O}_q(G/H)] \subseteq I_q(H)$.

Proof. (b) \implies (a): For any $r, s \in \mathbb{N}$, pick $f \in \mathcal{O}_q(G/H)_r, g \in \mathcal{O}_q(G/H)_s$. Then — by Proposition 3.10 — for $\Delta(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}$ and $\Delta(g) = \sum_{(g)} g_{(1)} \otimes g_{(2)}$ we have $f_{(2)} \in \mathcal{O}_q(G/H)_r, g_{(2)} \in \mathcal{O}_q(G/H)_s$. This, along with assumption (b) to get the equality $\stackrel{\circledast}{=}$, yields the chain of identities

$$\begin{aligned}
\Delta_\pi(fg) &= \sum_{(f),(g)} f_{(1)} g_{(1)} \otimes \overline{f_{(2)} g_{(2)}} = \sum_{(f),(g)} f_{(1)} g_{(1)} \otimes f_{(2)} \cdot \overline{g_{(2)}} = \\
&= \sum_{(f),(g)} (f_{(1)} \otimes f_{(2)}) \cdot (g_{(1)} \otimes \overline{g_{(2)}}) = \left(\sum_{(f)} f_{(1)} \otimes f_{(2)} \right) \cdot \left(\sum_{(g)} g_{(1)} \otimes \overline{g_{(2)}} \right) = \\
&= \left(\sum_{(f)} f_{(1)} \otimes f_{(2)} \right) \cdot (g \otimes \overline{d^s}) = \sum_{(f)} f_{(1)} g \otimes f_{(2)} \cdot \overline{d^s} = \\
&= \sum_{(f)} f_{(1)} g \otimes \overline{f_{(2)} d^s} \stackrel{\circledast}{=} \sum_{(f)} f_{(1)} g \otimes \overline{d^s f_{(2)}} = \\
&= \sum_{(f)} f_{(1)} g \otimes (d^s f_{(2)}) \cdot \bar{1} = \left((1 \otimes d^s) \cdot \sum_{(f)} f_{(1)} \otimes f_{(2)} \right) \cdot (g \otimes \bar{1}) =
\end{aligned}$$

$$\begin{aligned}
&= \left((1 \otimes d^s) \Delta(f) \right) \cdot (g \otimes \bar{1}) = \left((1 \otimes d^s) (f \otimes d^r + \varphi \otimes \eta) \right) \cdot (g \otimes \bar{1}) = \\
&= (1 \otimes d^s) \cdot (f g \otimes \bar{d}^r + \varphi g \otimes \bar{\eta}) = f g \otimes \overline{d^{s+r}} = f g \otimes \pi(d^{s+r})
\end{aligned}$$

for some suitable $\varphi \in \mathcal{O}_q(G)$, $\eta \in I_q(H)$, with notation $(x \otimes y) \cdot (a \otimes \bar{b}) := (x a) \otimes (y \cdot \bar{b})$ referring to the action of $\mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$ onto $\mathcal{O}_q(G) \otimes \mathcal{O}_q(H)$ induced by the action of $\mathcal{O}_q(G)$ onto $\mathcal{O}_q(H)$, via π , and onto itself, via left regular representation. So $f g$ is also d -semi-invariant, of degree $r + s$, q.e.d.

(a) \implies (b): Assume that (a) holds. Then for $f \in \mathcal{O}_q(G/H)_n$ we have $d f, f d \in \mathcal{O}_q(G/H)_{n+1}$, and so $[d, f] \in \mathcal{O}_q(G/H)_{n+1}$. Then the identity

$$\overline{[d, f]} = \epsilon([d, f]) \overline{d^{\partial(f)+1}}$$

holds, by Proposition 3.8. But clearly $\epsilon([d, f]) = 0$, hence $\overline{[d, f]} = \bar{0}$, that is $[d, f] \in I_q(H)$. The outcome is $[d, \mathcal{O}_q(G/H)] \subseteq I_q(H)$, q.e.d.

(b) \iff (c): This is just a matter of rephrasing. \square

Definition 3.12. We call *quantization of t* , or *quantum section* (of the line bundle \mathcal{L}) on G/H , any pre-quantum section d of G/H (cf. Definition 3.5) which satisfies any one of the equivalent conditions in Proposition 3.11.

The following result gives a criterion to detect quantum sections, and shows that for quantum subgroups they are just pre-quantum sections.

Proposition 3.13.

(a) Let d be a pre-quantum section on G/H .

If $[d, I_q(H)] \subseteq I_q(H)$, then d is a quantum section.

(b) Let $\mathcal{O}_q(H)$ be a quantum subgroup.

Then any pre-quantum section on G/H is a quantum section.

Proof. (a) Pick any $f \in \mathcal{O}_q(G/H)_r$, $g \in \mathcal{O}_q(G/H)_s$. Definition 3.7 gives

$$\begin{aligned}
\Delta(f) &= f \otimes d^r + f_1 \otimes \phi_1, & f_1 &\in \mathcal{O}_q(G), \phi_1 \in I_q(H) \\
\Delta(g) &= g \otimes d^s + g_1 \otimes \gamma_1, & g_1 &\in \mathcal{O}_q(G), \gamma_1 \in I_q(H)
\end{aligned}$$

Therefore, for the product $f g$ we have

$$\begin{aligned}
\Delta(f g) &= \Delta(f) \Delta(g) = (f \otimes d^r + f_1 \otimes \phi_1) (g \otimes d^s + g_1 \otimes \gamma_1) = \\
&= f g \otimes d^{r+s} + f g_1 \otimes d^r \gamma_1 + f_1 g \otimes \phi_1 d^s + f_1 g_1 \otimes \phi_1 \gamma_1
\end{aligned}$$

Now, $d^r \gamma_1, \phi_1 \gamma_1 \in I_q(H)$ because $I_q(H)$ is a (left) $\mathcal{O}_q(G)$ -submodule, and

$$\phi_1 d^s = d^s \phi_1 + [\phi_1, d^s] \in \left(I_q(H) + I_q(H) \right) = I_q(H)$$

because, in addition, $[d, I_q(H)] \subseteq I_q(H)$, by assumption (d). Thus

$$\Delta(fg) \in \left(fg \otimes d^{r+s} + \mathcal{O}_q(G) \otimes I_q(H) \right)$$

which means exactly $fg \in \mathcal{O}_q(G/H)_{r+s}$, by Definition 3.7 again. Thus condition (a) of Proposition 3.11 holds, hence we conclude by Definition 3.12.

(2) If $\mathcal{O}_q(H)$ is a quantum subgroup, then $I_q(H)$ is a two-sided ideal. Therefore, $[d, I_q(H)] \subseteq I_q(H)$, hence by (1) we get the claim. \square

The following result records yet another feature of quantum sections:

Lemma 3.14. *Let d be a quantum section on G/H .*

Then $d^n \in \mathcal{O}_q(G/H)_n$, and $\overline{d^n}$ is group-like in $\mathcal{O}_q(H)$, for all $n \in \mathbb{N}$. Moreover, $\text{Span}_{\mathbb{k}_q}(\{\overline{d^n}\}_{n \in \mathbb{N}})$ is a \mathbb{k}_q -subcoalgebra of $\mathcal{O}_q(H)$, and

$$\text{Span}_{\mathbb{k}_q}(\{\overline{d^n}\}_{n \in \mathbb{N}}) = \bigoplus_{n \in \mathbb{N}} \mathbb{k}_q \overline{d^n} \quad .$$

Proof. By Definition 3.12 and condition (a) in Proposition 3.11 we have $d^n \in \mathcal{O}_q(G/H)_n$, for all $n \in \mathbb{N}$. This means $\Delta(d^n) \in (d^n \otimes d^n + \mathcal{O}_q(G) \otimes I_q(H))$, whence — as $\pi : \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(H)$ is a coalgebra morphism — we get

$$\Delta(\overline{d^n}) = \Delta(\pi(d^n)) = (\pi \otimes \pi)(\Delta(d^n)) = \pi(d^n) \otimes \pi(d^n) = \overline{d^n} \otimes \overline{d^n}$$

thus the $\overline{d^n}$'s are group-like, and different from each other because the $t^n = d^n|_{q=1}$ are. Finally, this implies that $\text{Span}_{\mathbb{k}_q}(\{\overline{d^n}\}_{n \in \mathbb{N}})$ is a \mathbb{k}_q -subcoalgebra of $\mathcal{O}_q(H)$, and also that the $\overline{d^n}$'s are linearly independent, which eventually gives $\text{Span}_{\mathbb{k}_q}(\{\overline{d^n}\}_{n \in \mathbb{N}}) = \bigoplus_{n \in \mathbb{N}} \mathbb{k}_q \overline{d^n}$. \square

Gathering all together the previous results, we can now show that semi-invariants built out of a quantum section satisfy all properties we look for:

Theorem 3.15. *Let d be a quantum section on G/H . Then*

(a) $\mathcal{O}_q(G/H)$ is a graded \mathbb{k}_q -module, its n -th graded summand ($n \in \mathbb{N}$) being $\mathcal{O}_q(G/H)_n$;

(b) $\mathcal{O}_q(G/H)$ is a subalgebra of $\mathcal{O}_q(G)$;

(c) the grading in (a) is compatible with all other structures of $\mathcal{O}_q(G/H)$, so that $\mathcal{O}_q(G/H)$ is a graded $\mathcal{O}_q(G)$ -comodule algebra, where we take on $\mathcal{O}_q(G)$ the trivial grading;

(d) for every $c \in \mathbb{k}_q$, we have $\mathcal{O}_q(G/H) \cap c\mathcal{O}_q(G) = c\mathcal{O}_q(G/H)$. In particular, $\mathcal{O}_q(G/H) \cap (q-1)\mathcal{O}_q(G) = (q-1)\mathcal{O}_q(G/H)$.

Proof. (a) We must simply prove that the sum $\sum_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n$ is direct, so that $\mathcal{O}_q(G/H) := \sum_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n$. But this is an easy consequence of Lemma 3.14.

Indeed, let $\sum_{n \in \mathbb{N}} c_n f_n = 0$ a linear dependence relation, with $f_n \in \mathcal{O}_q(G/H)_n$ and $c_n \in \mathbb{k}_q$ (almost all zero) for every $n \in \mathbb{N}$. Applying Δ_π to this relation we get $\sum_{n \in \mathbb{N}} c_n f_n \otimes \bar{d}^n = 0$. But the \bar{d}^n 's, by Lemma 3.14, are linearly independent; thus $c_n = 0$ for all n , q.e.d.

(b) This follows directly from Definition 3.12 and Proposition 3.11.

(c) This follows (again) from Definition 3.12 and Proposition 3.11, from Proposition 3.10, and from the Hopf algebra axioms.

(d) This easily follows from the identity $\mathcal{O}_q(G/H) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n$, given by claim (a), and from Remark 3.9. \square

Corollary 3.16. *Let d be a quantum section on G/H (in the sense of Definition 3.12). Then the restriction of $\pi : \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(H)$ yields an epimorphism of graded $\mathcal{O}_q(G)$ -module coalgebras*

$$\pi' : \mathcal{O}_q(G/H) \longrightarrow \text{Span}_{\mathbb{k}_q} \left(\{ \bar{d}^n \}_{n \in \mathbb{N}} \right) = \bigoplus_{n \in \mathbb{N}} \mathbb{k}_q \bar{d}^n .$$

Proof. By Lemma 3.8 we know that π' is a well-defined epimorphism of graded \mathbb{k}_q -modules. The rest follows from $\mathcal{O}_q(G/H)$ being a subalgebra of $\mathcal{O}_q(G)$, and π being a morphism of $\mathcal{O}_q(G)$ -module coalgebras. \square

Last aspect to take into account is the behavior of $\mathcal{O}_q(G/H)$ under specialization at $q = 1$. The last part of claim (d) in Theorem 3.15 ensures

that such specialization is consistent with that of $\mathcal{O}_q(G)$: in other words, the embedding $\mathcal{O}_q(G/H) \hookrightarrow \mathcal{O}_q(G)$ gets down under specialization to an embedding $\mathcal{O}_1(G/H) := \mathcal{O}_q(G/H) / (q-1)\mathcal{O}_q(G/H) \hookrightarrow \mathcal{O}(G)$. The next result tells us something about the specialized space $\mathcal{O}_1(G/H)$ itself.

Proposition 3.17. $\mathcal{O}_1(G/H) := \mathcal{O}_q(G/H) / (q-1)\mathcal{O}_q(G/H)$ is a graded Poisson subalgebra of $\mathcal{O}(G/H) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}(G/H)_n$ and a graded left coideal of $\mathcal{O}(G)$, with $\mathcal{O}_1(G/H)_n := \mathcal{O}_q(G/H)_n / (q-1)\mathcal{O}_q(G/H)_n$ as n -th graded summand ($n \in \mathbb{N}$). In particular, it is a left $\mathcal{O}(G)$ -comodule algebra.

Proof. For all $n \in \mathbb{N}$, the restriction to $\mathcal{O}_q(G/H)_n$ of the specialization map

$$p_1 : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(G) / (q-1)\mathcal{O}_q(G) \cong \mathcal{O}(G)$$

has kernel $\mathcal{O}_q(G/H)_n \cap (q-1)\mathcal{O}_q(G) = (q-1)\mathcal{O}_q(G/H)_n$, by Remark 3.9. This in turn ensures also that the restriction of p_1 to $\mathcal{O}_q(G/H) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n$ has kernel $\bigoplus_{n \in \mathbb{N}} (q-1)\mathcal{O}_q(G/H)_n$, so its image is $\mathcal{O}_q(G/H) / (q-1)\mathcal{O}_q(G/H)$, i.e. just the specialization of $\mathcal{O}_q(G/H)$. So

$$\mathcal{O}_1(G/H) := \mathcal{O}_q(G/H) / (q-1)\mathcal{O}_q(G/H) = p_1\left(\mathcal{O}_q(G/H)\right)$$

where the right-hand side is a subalgebra of $p_1\left(\mathcal{O}_q(G)\right) = \mathcal{O}(G)$. Moreover, we have also that the specialization maps preserves the grading, namely

$$p_1\left(\mathcal{O}_q(G/H)\right) = p_1\left(\bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n\right) = \bigoplus_{n \in \mathbb{N}} p_1\left(\mathcal{O}_q(G/H)_n\right)$$

so that $\mathcal{O}_1(G/H)$ is graded, with n -th graded summand

$$p_1\left(\mathcal{O}_q(G/H)_n\right) = \mathcal{O}_q(G/H)_n / (q-1)\mathcal{O}_q(G/H)_n =: \mathcal{O}_1(G/H)_n$$

Now Theorem 3.15 implies at once that $\mathcal{O}_1(G/H)$ is a graded subalgebra left coideal inside $p_1\left(\mathcal{O}_q(G)\right) = \mathcal{O}(G)$, hence a graded (left) $\mathcal{O}(G)$ -comodule algebra. In addition, the identity

$$\mathcal{O}_q(G/H) \cap (q-1)\mathcal{O}_q(G) = (q-1)\mathcal{O}_q(G/H)$$

implies also that the Poisson bracket defined in $\mathcal{O}(G/H)$ starting from its quantization $\mathcal{O}_q(G/H)$ — see Remark 3.2 — coincides with the restriction to $\mathcal{O}(G/H)$ of the Poisson bracket similarly induced on $\mathcal{O}(G)$ from $\mathcal{O}_q(G)$. Therefore, $\mathcal{O}_1(G/H)$ is also a Poisson subalgebra of $\mathcal{O}(G)$.

We are only left to prove that the embedding of $\mathcal{O}_1(G/H)$ into $\mathcal{O}(G)$ maps $\mathcal{O}_1(G/H)$ into $\mathcal{O}(G/H)$ respecting the grading on either side, that is

$$\mathcal{O}_1(G/H)_n \subseteq \mathcal{O}(G/H)_n \quad \forall n \in \mathbb{N} \quad (3.2)$$

Now, the left-hand side of (3.2) is $\mathcal{O}_q(G/H)_n / (q-1)\mathcal{O}_q(G/H)_n$, with

$$\mathcal{O}_q(G/H)_n := \left\{ \ell \in \mathcal{O}_q(G) \mid \Delta_\pi(\ell) = \ell \otimes \pi(d^n) \right\}$$

(cf. Definition 3.7), while the right-hand side, by (2.4), is

$$\mathcal{O}(G/H)_n = \left\{ f \in \mathcal{O}(G) \mid \Delta_\pi(f) = f \otimes \pi(t^n) \right\}$$

But all specialization maps commute with the coproducts Δ and with the (quantum and classical) maps π , and the specialization of each d^n is nothing but t^n . Therefore, we conclude that (3.2) holds. \square

Finally, we can define our “quantum projective homogeneous spaces”.

Definition 3.18. Let G be an algebraic Poisson group, H a coisotropic subgroup, as in §2.2, and let $\mathcal{O}_q(G)$, $\mathcal{O}_q(H)$ and $\pi : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(H)$ be given (cf. Definitions 3.1, 3.3). Let d be a quantum section on G/H (see Definition 3.12, Proposition 2.9), in particular

$$(a) \quad \Delta_\pi(d) = d \otimes \pi(d)$$

(b) $d \equiv t \pmod{(q-1)}$, where t is a non-zero section of the very ample line bundle on G/H giving the embedding into some projective space.

Then given

$$\begin{aligned} \mathcal{O}_q(G/H)_n &:= \left\{ \ell \in \mathcal{O}_q(G) \mid \Delta_\pi(\ell) = \ell \otimes \pi(d^n) \right\} = \\ &= \left\{ \ell \in \mathcal{O}_q(G) \mid \Delta(\ell) \in (\ell \otimes d^n + \mathcal{O}_q(G) \otimes I_q(H)) \right\} \end{aligned}$$

we say that

$$\mathcal{O}_q(G/H) := \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/H)_n$$

is a quantization of $\mathcal{O}(G/H)$ if

$$\mathcal{O}_q(G/H) / (q-1) \mathcal{O}_q(G/H) \cong \mathcal{O}(G/H)$$

as graded $\mathcal{O}(G)$ -module algebras and as Poisson algebras over \mathbb{k} . We will then refer to $\mathcal{O}_q(G/H)$ as *quantum projective homogeneous space*.

In particular, we have seen that any such $\mathcal{O}_q(G/H)$ has the following properties (Theorem 3.15):

- (I) it is a graded subalgebra of $\mathcal{O}_q(G)$;
- (II) It is a left coideal of $\mathcal{O}_q(G)$, hence a left $\mathcal{O}_q(G)$ -comodule via

$$\Delta|_{\mathcal{O}_q(G/H)} : \mathcal{O}_q(G/H) \longrightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q(G/H)$$

Remark 3.19. As a matter of fact, the only additional property required in Definition 3.18 is that the embedding of $\mathcal{O}_q(G/H) / (q-1) \mathcal{O}_q(G/H)$ into $\mathcal{O}(G/H)$ provided by Proposition 3.17 be *onto*. But actually, as both these are *graded* algebras, and $\mathcal{O}(G/H)$ is generated in degree one, *this is equivalent to require (only) that is onto the embedding*

$$\mathcal{O}_q(G/H)_1 / (q-1) \mathcal{O}_q(G/H)_1 \hookrightarrow \mathcal{O}(G/H)_1$$

This requirement might be seen as the quantum analogue of the requirement — at the semiclassical level, see §2.2 — of having enough global sections of the line bundle \mathcal{L} on G/H so to have an embedding of G/H into \mathbb{P}^N .

In section 5 we show that quantum Grassmannians and quantum generalized flag varieties are examples of quantum projective homogeneous space.

4 The Quantum Duality Principle (QDP)

4.1 The QDP philosophy

The quantum duality principle (QDP) is a two-fold recipe which allows to obtain a quantum group dual, in an appropriate sense, to a given one.

In [5] Ciccoli and Gavarini extended this principle to quantum formal homogeneous spaces. Their result goes as follows.

Given a Poisson group G , we consider pairs $(H, G/H)$ where H is a coisotropic subgroup of G and G/H is the corresponding homogeneous space.

At a local level, i.e. in the setup of formal geometry, any such pair can be described in algebraic terms by any one of the following four objects:

$$U(\mathfrak{h}), \quad U(\mathfrak{g})\mathfrak{h}, \quad I(H), \quad \mathcal{O}(G/H) \quad (4.1)$$

where hereafter $\mathfrak{h} := \text{Lie}(H)$ and $\mathfrak{g} := \text{Lie}(G)$, and the notation is standard, but for $\mathcal{O}(G/H)$ which here denotes the algebra of regular functions on the formal algebraic variety G/H . The main result in [5] is a four-fold functorial recipe which, from a quantization of each object in (4.1), constructs a quantization of one of the four object of the similar quadruple which describes the “dual” pair $(H^\perp, G^*/H^\perp)$, where “dual” refers to Poisson duality.

If we try to do the same at a global level (cf. [6]), i.e. not restricting ourselves to the framework of formal geometry, then something changes when handling the algebra $\mathcal{O}(G/H)$. Namely, the latter is meaningful — that is, it permits to get back the pair $(H, G/H)$ — only if G/H is a quasi-affine variety. This is the case, in particular, if G/H is affine, and instead it is not if G/H is projective. Therefore, in the latter case one describes G/H taking, instead of the algebra of regular functions, the algebra of (algebraic) sections of a line bundle on G/H realizing an embedding in a projective space, i.e. what is denoted $\mathcal{O}(G/H)$ in §2. Once this is settled, one can consider a quantization $\mathcal{O}_q(G/H)$ and try to cook up a suitable analogue of the recipe of [5], [6].

With this program in mind, we want to associate to any quantum homogeneous G -space $\mathcal{O}_q(G/H)$ — in the sense of Definition 3.18 — a (local) quantization $U_q(\mathfrak{h}^\perp)$ of the dual G^* -space (actually, of the dual coisotropic subgroup), right in the spirit of the QDP.

Warning: In order to make our statements simpler, from now on we take as ground ring the local ring

$$\mathbb{k}'_q := (\mathbb{k}_q)_{(q-1)} = \text{localization of } \mathbb{k}_q \text{ at the ideal generated by } (q-1).$$

Therefore, hereafter we shall tacitly extend scalars from \mathbb{k}_q to \mathbb{k}'_q for all \mathbb{k}_q -modules and \mathbb{k}_q -algebras we have considered so far, with no further mention.

Let G be an affine algebraic group and H a closed coisotropic parabolic subgroup, i.e. G/H is a projective homogeneous space. Let $\mathcal{O}_q(G)$, $\mathcal{O}_q(H)$ and $\mathcal{O}_q(G/H)$ be quantum deformations of $\mathcal{O}(G)$, $\mathcal{O}(H)$ and $\mathcal{O}(G/H)$ as

defined in Section 3. In particular, $\mathcal{O}_q(G/H)$ is built out of a specific quantum section d on G/H . Also, $\pi(d) = \bar{d} \in \mathcal{O}_q(H)$ is (non-zero) group-like, hence $\epsilon(d) = \epsilon(\bar{d}) = 1$, and d specializes to $d|_{q=1} = t \in \mathcal{O}(G/H) \subseteq \mathcal{O}(G)$.

In addition, we make the following assumption:

d is an Ore element in the algebra $\mathcal{O}_q(G)$.

This property will allow us to enlarge the algebras $\mathcal{O}_q(G)$ and $\mathcal{O}_q(G/H)$ by the formal inverse d^{-1} . Geometrically, it corresponds to ask to have — besides a quantization of $\mathcal{O}(G/H)$ — a quantization of a Zariski neighbourhood of the identity element; more precisely, it means that we have a quantization of the function algebra $\mathcal{O}(X_t)$ of the affine variety X_t , the complement in G/H of the divisor defined by the function $t = d|_{q=1}$. This property is satisfied in the examples of the Grassmannian and the flag varieties (cf. §5), with a suitable choice of d .

Let us define

$$\mathcal{O}_q^{loc}(G/H) := \mathcal{O}_q(G/H)[d^{-1}]_{proj} \subseteq \mathcal{O}_q(G)[d^{-1}]$$

where the localization is a *projective localization*, i.e. we take the elements of degree zero in the ring $\mathcal{O}_q(G/H)[d^{-1}]$, where d^{-1} is given degree -1 .

4.2 The QDP functor

We now recall the definition of the functor $\mathcal{O}_q(G) \mapsto \mathcal{O}_q(G)^\vee$, which sends quantized function algebras (of Poisson groups) to quantized universal enveloping algebras (of Lie bialgebras). More precisely, $\mathcal{O}_q(G)^\vee$ is a quantization of $U(\mathfrak{g}^*)$, where \mathfrak{g}^* is the Lie bialgebra dual to \mathfrak{g} . For more details and proofs, we refer the reader to [14], Theorem 2.2 and Theorem 4.7.

Remark: the overall assumption in [14] for G is to be connected; nevertheless, this condition is not needed in the proof of Theorem 4.7 therein.

Definition 4.1. Let $J_G := \text{Ker}(\epsilon : \mathcal{O}_q(G) \rightarrow \mathbb{k}'_q)$ be the augmentation ideal of $\mathcal{O}_q(G)$. Also, let $I_G := J_G + (q-1)\mathcal{O}_q(G)$. We define

$$\mathcal{O}_q(G)^\vee := \sum_{n \geq 0} (q-1)^{-n} I_G^n = \sum_{n \geq 0} ((q-1)^{-1} I_G)^n = \bigcup_{n \geq 0} ((q-1)^{-1} I_G)^n .$$

This is a well defined \mathbb{k}'_q -subalgebra of $\mathbb{k}(q) \otimes_{\mathbb{k}'_q} \mathcal{O}_q(G)$. Notice also that

$$\mathcal{O}_q(G)^\vee = \sum_{n \geq 0} (q-1)^{-n} J_G^n = \sum_{n \geq 0} ((q-1)^{-1} J_G)^n \quad .$$

The results of [14] — in particular, Theorem 4.7 therein — tell us that $\mathcal{O}_q(G)^\vee$ is a quantization of $U(\mathfrak{g}^*)$, that is $\mathcal{O}_q(G)^\vee / (q-1) \mathcal{O}_q(G)^\vee \cong U(\mathfrak{g}^*)$ as co-Poisson Hopf algebras. Our idea is to “restrict to $\mathcal{O}_q(G/H)$ ”, somehow, the definition of $\mathcal{O}_q(G)^\vee$, so as to define $\mathcal{O}_q(G/H)^\vee$. To begin with, let

$$J_{G/H} := \text{Ker}(\epsilon|_{\mathcal{O}_q(G/H)}) \quad .$$

Notice that ϵ extends uniquely to $\mathcal{O}_q(G)[d^{-1}]$, so we can define also

$$J_{G/H}^{loc} := \text{Ker}(\epsilon|_{\mathcal{O}_q^{loc}(G/H)}) \quad .$$

Definition 4.2. We define

$$\mathcal{O}_q(G/H)^\vee := \sum_{n \geq 0} (q-1)^{-n} (J_{G/H}^{loc})^n = \sum_{n \geq 0} ((q-1)^{-1} J_{G/H}^{loc})^n \quad ,$$

the unital \mathbb{k}'_q -subalgebra of $\mathbb{k}(q) \otimes_{\mathbb{k}'_q} \mathcal{O}_q^{loc}(G/H)$ generated by $(q-1)^{-1} J_{G/H}^{loc}$, or — what amounts to be the same — by $(q-1)^{-1} I_{G/H}^{loc}$, where by definition we set $I_{G/H}^{loc} := J_{G/H}^{loc} + (q-1) \mathcal{O}_q^{loc}(G/H)$.

Indeed, one can check that the construction $\mathcal{O}_q(G/H) \mapsto \mathcal{O}_q(G/H)^\vee$ is functorial, in a natural sense (see Remark 4.12 later on).

In order to study $\mathcal{O}_q(G/H)^\vee$, we need a rather explicit description of it. In turn, this requires a description of $\mathcal{O}_q(G)^\vee$, which we take from [14].

Let J_1 be the augmentation ideal of $\mathcal{O}(G)$, namely

$$J_1 := \text{Ker}(\epsilon : \mathcal{O}(G) \longrightarrow \mathbb{k}) = J_G \text{ mod } (q-1) \mathcal{O}_q(G)$$

so that $J_1 / J_1^2 = \mathfrak{g}^*$, the cotangent Lie bialgebra of G . Let $\{y_1, \dots, y_n\}$ be a subset of J_1 whose image in the local ring of G at e (the unit element of the group G) is a local system of parameters; in particular, $n = \dim(G)$. Define $\{j_1, \dots, j_n\}$ as a pull-back of $\{y_1, \dots, y_n\}$ to J_G .

Theorem 4.3. (see [14], Lemma 4.1)

(a) The set of ordered monomials $\left\{ j^{\underline{e}} := \prod_{s=1}^n j_s^{e_s} \mid \underline{e} := (e_1, \dots, e_n) \in \mathbb{N}^n \right\}$ is a \mathbb{k}'_q -pseudobasis (or topological basis) of $\widetilde{\mathcal{O}}_q(G)$, the I_G -adic completion of $\mathcal{O}_q(G)$. In other words, each element of $\widetilde{\mathcal{O}}_q(G)$ has a unique expansion as a formal infinite \mathbb{k}'_q -linear combination of the $j^{\underline{e}}$'s. In particular, $\widetilde{\mathcal{O}}_q(G)$ is generated — as a topological \mathbb{k}'_q -algebra — by $\{j_1, \dots, j_n\}$.

(b) The $(q-1)$ -adic completion $\widehat{\mathcal{O}}_q(G)^\vee$ of $\mathcal{O}_q(G)^\vee$ admits the set of ordered monomials $\left\{ (q-1)^{-(e_1+\dots+e_n)} \prod_{s=1}^n j_s^{e_s} \mid (e_1, \dots, e_n) \in \mathbb{N}^n \right\}$ as a \mathbb{k}'_q -pseudobasis. In particular, $\widehat{\mathcal{O}}_q(G)^\vee$ is generated — as a topological \mathbb{k}'_q -algebra — by $\{j_s^\vee := (q-1)^{-1}j_s \mid s = 1, \dots, n\}$.

The description of $\mathcal{O}_q(G/H)^\vee$ goes much along the same lines.

Recall that $t \in \mathcal{O}(G/H)$ is the specialization of $d \in \mathcal{O}_q(G/H)$. We then consider X_t , the open subvariety of G/H where $t \neq 0$. On this variety, choose functions l_1, \dots, l_{n-h} — where $h = \dim(H)$ — such that the set

$$\left\{ x_s := l_s \bmod (q-1)\mathcal{O}_q(G/H) \mid s = 1, \dots, n-h \right\}$$

yields, in the localization of $\mathcal{O}(X_t)$ at $eH \in X_t (\subseteq G/H)$, a local system of parameters at eH .

Theorem 4.4.

(a) The set of ordered monomials

$$\left\{ \prod_{s=1}^{n-h} l_s^{e_s} \mid (e_1, \dots, e_{n-h}) \in \mathbb{N}^{n-h} \right\}$$

is a \mathbb{k}'_q -pseudobasis of $\widetilde{\mathcal{O}}_q^{loc}(G/H)$, the latter being the $I_{G/H}^{loc}$ -adic completion of $\mathcal{O}_q^{loc}(G/H)$, where $I_{G/H}^{loc} := J_{G/H}^{loc} + (q-1)\mathcal{O}_q^{loc}(G/H)$.

In particular, $\widetilde{\mathcal{O}}_q^{loc}(G/H)$ is (topologically) generated by $\{l_1, \dots, l_{n-h}\}$.

(b) The $(q-1)$ -adic completion $\widehat{\mathcal{O}}_q(G/H)^\vee$ of $\mathcal{O}_q(G/H)^\vee$ admits as a \mathbb{k}'_q -pseudobasis the set of ordered monomials

$$\left\{ (q-1)^{-(e_1+\dots+e_{n-h})} \prod_{s=1}^{n-h} l_s^{e_s} \mid (e_1, \dots, e_{n-h}) \in \mathbb{N}^{n-h} \right\} .$$

In particular, $\widehat{\mathcal{O}}_q(G/H)^\vee$ is (topologically) generated by the set

$$\left\{ l_s^\vee := (q-1)^{-1} l_s \mid s = 1, \dots, n-h \right\} .$$

Proof. The argument follows the proof of Theorem 4.9 in [15]. In this theorem is treated the general setting of a quantization $\mathcal{O}_q(V)$ of any Poisson affine variety V with a distinguished point on it, given by a character χ of $\mathcal{O}_q(V)$, the kernel of χ playing the role of J_G above. Here we apply all this to $V = X_t (\subseteq G/H)$ with $\epsilon|_{\mathcal{O}(X_t)}$ as character on $\mathcal{O}(X_t) = \mathcal{O}_q^{loc}(G/H)$. \square

The next lemma plays a crucial role in the construction of the QDP.

Lemma 4.5. *The quantum section $d \in \mathcal{O}_q(G/H)$ enjoys the following properties:*

- (a) d is invertible in $\widetilde{\mathcal{O}}_q^{loc}(G/H)$, with $d^{-1} = \sum_{k=0}^{+\infty} (1-d)^k$;
- (b) d is invertible in $\widehat{\mathcal{O}}_q(G/H)^\vee$, with $d^{-1} = \sum_{k=0}^{+\infty} (q-1)^k \left(\frac{(1-d)}{(q-1)} \right)^k$.

Proof. Observe that $\epsilon(d) = 1$ implies that

$$d = 1 - (1-d) \in (1 + J_{G/H}^{loc}) \subseteq (1 + I_{G/H}^{loc}) \quad ;$$

this gives the invertibility in $\widetilde{\mathcal{O}}_q^{loc}(G/H)$.

Similarly since $J_{G/H}^{loc} \subseteq (q-1)\mathcal{O}_q(G/H)^\vee$, the identity

$$d = 1 + (d-1) \in (1 + J_{G/H}^{loc}) \subseteq (1 + (q-1)\mathcal{O}_q(G/H)^\vee)$$

also ensures that d is invertible in $\widehat{\mathcal{O}}_q(G/H)^\vee$.

In both cases, the explicit formula for d^{-1} follows by taking the limit of the geometric series, namely $(1-x)^{-1} = \sum_{k=0}^{+\infty} x^k$, applied to $x = 1-d$. \square

Proposition 4.6. *There are natural embeddings*

$$\widetilde{\mathcal{O}}_q(G/H) \hookrightarrow \widetilde{\mathcal{O}}_q(G) \quad , \quad \widehat{\mathcal{O}}_q(G/H) \hookrightarrow \widehat{\mathcal{O}}_q(G)$$

which both are extensions of the embedding $\mathcal{O}_q(G/H) \hookrightarrow \mathcal{O}_q(G)$. Moreover, via these embeddings the pseudobases for the (topological) algebras on G/H identify with subsets of the corresponding ones for the (topological) algebras on G .

Proof. By construction, we have $l_k = j_s/d^{c_s}$ for some $j_s \in J_{G/H}$, $c_s \in \mathbb{N}$ ($s = 1, \dots, n-h$). Since d is invertible (in both cases), the previous analysis tells us that we can replace the l_k 's with the j_k 's ($k = 1, \dots, n-h$) in the descriptions of $\widetilde{\mathcal{O}}_q(G/H)^\vee$ and $\widehat{\mathcal{O}}_q(G/H)^\vee$ given above (i.e. in the \mathbb{k}'_q -pseudobases and as topological generators). But then, as

$$\{j_s \mid s = 1, \dots, n-h\} \subseteq J_{G/H} \subseteq \mathcal{O}_q(G/H) \subseteq \mathcal{O}_q(G)$$

we can always complete $\{j_s \mid s = 1, \dots, n-h\}$ to a set $\{j_r \mid r = 1, \dots, n\}$ such that $\{y_r := j_r \bmod (q-1)\mathcal{O}_q(G) \mid r = 1, \dots, n\}$ yield, in the localization of $\mathcal{O}(G)$ at $e \in G$, a local system of parameters at e . Thus using the latter we can describe $\widetilde{\mathcal{O}}_q(G)$ and $\widehat{\mathcal{O}}_q(G)$ as explained above. \square

From now on we shall use these embeddings to identify $\widetilde{\mathcal{O}}_q(G/H)$ and $\widehat{\mathcal{O}}_q(G/H)$ with a subalgebra of $\widetilde{\mathcal{O}}_q(G)$ and of $\widehat{\mathcal{O}}_q(G)$ respectively.

Lemma 4.7. $\mathcal{O}_q(G/H)^\vee \cap (q-1)\mathcal{O}_q(G)^\vee = (q-1)\mathcal{O}_q(G/H)^\vee$

Proof. Let us choose a subset $\{j_1, \dots, j_n\}$ in $J_{G/H}^{loc}$ as explained above for the description of $\mathcal{O}_q(G/H)^\vee$. Then, mapping $\mathcal{O}_q(G)^\vee$ and $\mathcal{O}_q(G/H)^\vee$ into their $(q-1)$ -adic completions, and exploiting the descriptions of the latter via pseudobases given above, we easily get the claim. \square

Next result is that $\mathcal{O}_q(G/H)^\vee$ is a quantization of $U(\mathfrak{h}^\perp)$:

Theorem 4.8. $\mathcal{O}_q(G/H)^\vee$ is a quantization of $U(\mathfrak{h}^\perp)$ as a \mathbb{k} -algebra — subalgebra of $U(\mathfrak{g}^*)$ — where $\mathfrak{h} = \text{Lie}(H)$, $\mathfrak{g} = \text{Lie}(G)$.

Proof. By assumption, H is coisotropic in G . Therefore, $\mathfrak{h} = \text{Lie}(H)$ is a Lie coideal (and subalgebra) of $\mathfrak{g} = \text{Lie}(G)$, and \mathfrak{h}^\perp is a Lie subalgebra (and coideal) of \mathfrak{g}^* (see Proposition 2.2). Thus the claim does make sense.

In order to prove the statement, we proceed much like in the proof of the fact that $\mathcal{O}_q(G)^\vee / (q-1)\mathcal{O}_q(G)^\vee \cong U(\mathfrak{g}^*)$ — cf. [14], Theorem 4.7. The arguments being the same, we briefly recall them.

Fix again a special subset $\{j_1, \dots, j_n\}$ of J_G as we did in the proof of Proposition 4.6, in particular with $j_1, \dots, j_{n-h} \in J_{G/H}$. Also, set notation:

$$\mathcal{O}_1(G)^\vee := \mathcal{O}_q(G)^\vee / (q-1)\mathcal{O}_q(G)^\vee, \quad J_G^\vee := (q-1)^{-1}J_G \subseteq \mathcal{O}_q(G)^\vee$$

$$j^\vee := (q-1)^{-1}j \quad \forall j \in J_G, \quad \mathfrak{t} := J_G^\vee \bmod (q-1)\mathcal{O}_q(G)^\vee.$$

Taking into account that the specializations at $q=1$ of any \mathbb{k}'_q -module and of its $(q-1)$ -adic completion are the same, the above discussion gives that

$$\left\{ \prod_{s=1}^n (j_s^\vee)^{e_s} \bmod (q-1)\mathcal{O}_q(G)^\vee \mid (e_1, \dots, e_n) \in \mathbb{N}^n \right\}$$

is a \mathbb{k} -basis of $\mathcal{O}_1(G)^\vee$. Similarly, $\{j_1^\vee, \dots, j_n^\vee\}$ is a \mathbb{k} -basis of \mathfrak{t} .

Now, $j_\mu j_\nu - j_\nu j_\mu \in (q-1)J_G$ (for $\mu, \nu \in \{1, \dots, n\}$) implies that

$$j_\mu j_\nu - j_\nu j_\mu = (q-1) \sum_{s=1}^n c_s j_s + (q-1)^2 \gamma_1 + (q-1) \gamma_2$$

for some $c_s \in \mathbb{k}'_q$, $\gamma_1 \in J_G$ and $\gamma_2 \in J_G^2$. Therefore

$$\begin{aligned} [j_\mu^\vee, j_\nu^\vee] &:= j_\mu^\vee j_\nu^\vee - j_\nu^\vee j_\mu^\vee = \sum_{s=1}^n c_s j_s^\vee + \gamma_1 + (q-1) \gamma_2^\vee \equiv \\ &\equiv \sum_{s=1}^n c_s j_s^\vee \bmod (q-1)\mathcal{O}_q(G)^\vee \end{aligned}$$

where $\gamma_2^\vee := (q-1)^{-2} \gamma_2 \in (q-1)^{-2} (J_G^\vee)^2 \subseteq \mathcal{O}_q(G)^\vee$; thus \mathfrak{t} is a Lie subalgebra of $\mathcal{O}_1(G)^\vee$. But then we have $\mathcal{O}_1(G)^\vee \cong U(\mathfrak{t})$ as Hopf algebras, by the above description of $\mathcal{O}_1(G)^\vee$ and PBW theorem.

Next, the specialization map $p^\vee: \mathcal{O}_q(G)^\vee \longrightarrow \mathcal{O}_1(G)^\vee = U(\mathfrak{t})$ restricts to $\eta: J_G^\vee \longrightarrow \mathfrak{t} := J_G^\vee \bmod (q-1)\mathcal{O}_q(G)^\vee$. Moreover, multiplication by $(q-1)^{-1}$ yields a \mathbb{k}'_q -module isomorphism $\mu: J_G \xrightarrow{\cong} J_G^\vee$. Consider the natural projection map $\rho: J_1 \longrightarrow J_1/J_1^2 = \mathfrak{g}^*$, and let $\nu: \mathfrak{g}^* \hookrightarrow J_1$ be a section of ρ . The specialization map $p: \mathcal{O}_q(G) \longrightarrow \mathcal{O}(G)$ restricts to $p': J_G \longrightarrow J_1$, and we fix a section $\gamma: J_1 \hookrightarrow J_G$ of p' . Then the composition map $\sigma := \eta \circ \mu \circ \gamma \circ \nu: \mathfrak{g}^* \longrightarrow \mathfrak{t}$ is a well-defined Lie bialgebra isomorphism, independent of the choice of ν and γ .

So far we did not exploit our special choice of the subset $\{j_1, \dots, j_n\}$: we do it now to prove that $\mathfrak{t} = \mathfrak{h}^\perp$. In fact, the analysis above to prove that $\sigma: \mathfrak{g}^* \cong \mathfrak{t}$ shows also that the unital subalgebra

$$\mathcal{O}_1(G/H)^\vee := \mathcal{O}_q(G/H)^\vee \bmod (q-1)\mathcal{O}_q(G)^\vee$$

of $U(\mathfrak{g}^*)$ is generated by $\eta(\{j_1^\vee, \dots, j_{n-h}^\vee\})$, and

$$\begin{aligned} \eta(\{j_1^\vee, \dots, j_{n-h}^\vee\}) &= (\eta \circ \mu)(\{j_1, \dots, j_{n-h}\}) = (\eta \circ \mu \circ \gamma)(\{y_1, \dots, y_{n-h}\}) = \\ &= (\eta \circ \mu \circ \gamma \circ \nu)(\{\bar{y}_1, \dots, \bar{y}_{n-h}\}) = \sigma(\{\bar{y}_1, \dots, \bar{y}_{n-h}\}) \end{aligned}$$

where $\bar{y}_s := y_s \bmod J_1^2$ ($s = 1, \dots, n-h$). Therefore $\mathcal{O}_1(G/H)^\vee$ is the subalgebra of $U(\mathfrak{g}^*)$ generated by the \mathbb{k} -span of $\{\bar{y}_1, \dots, \bar{y}_{n-h}\}$.

Finally, the \mathbb{k} -span of $\{\bar{y}_1, \dots, \bar{y}_{n-h}\}$ coincides with the subspace \mathfrak{h}^\perp of \mathfrak{g}^* . Indeed, as $\mathcal{O}(G/H)$ is the algebra of semi-invariant functions on G , every y_s is a $(H-)$ semi-invariant function on G : but it also vanishes at $e \in H$, hence by H -semi-invariance it vanishes on all of H . When mapping y_s to $\bar{y}_s \in J_1$, then, it is mapped into \mathfrak{h}^\perp . Thus the whole \mathbb{k} -span of $\{\bar{y}_1, \dots, \bar{y}_{n-h}\}$ is contained in \mathfrak{h}^\perp , hence coincides with it by dimension equality.

The outcome is that $\mathcal{O}_1(G/H)^\vee$ is the subalgebra of $U(\mathfrak{g}^*)$ generated by \mathfrak{h}^\perp , which is a Lie subalgebra of \mathfrak{g}^* , so $\mathcal{O}_1(G/H)^\vee = U(\mathfrak{h}^\perp)$. \square

We now wish to explore the nature of $\widehat{\mathcal{O}}_q(G/H)^\vee$ as a “quantum homogeneous space”. We start with an important observation on the extensions of the comultiplication Δ in $\mathcal{O}_q(G)$ to the new algebras we have defined.

Remark 4.9. Let $\Delta : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$ be the comultiplication in $\mathcal{O}_q(G)$. Then Δ can be uniquely (and canonically) extended to a coassociative morphism of topological algebras

$$\tilde{\Delta} : \widetilde{\mathcal{O}}_q(G) \longrightarrow \widetilde{\mathcal{O}}_q(G) \widetilde{\otimes} \widetilde{\mathcal{O}}_q(G)$$

where again $\widetilde{\mathcal{O}}_q(G)$ is the I_G -adic completion of $\mathcal{O}_q(G)$, and $\widetilde{\otimes}$ is the $I_{G \otimes}$ -adic completion of $\mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$, with $I_{G \otimes} := I_G \otimes \mathcal{O}_q(G) + \mathcal{O}_q(G) \otimes I_G$. Even more, such a $\tilde{\Delta}$ actually restricts to a coassociative algebra morphism (we use the same symbol to denote it):

$$\tilde{\Delta} : \mathcal{O}_q(G)[d^{-1}] \longrightarrow \mathcal{O}_q(G)[d^{-1}] \widetilde{\otimes} \mathcal{O}_q(G)[d^{-1}] \quad .$$

In fact, as d is a quantum section we have (see Definition 3.12)

$$\Delta(d) = d \otimes d + \sum_i h_i \otimes k_i \quad , \quad \text{for some } h_i \in \mathcal{O}_q(G), k_i \in I_q(H) \quad . \quad (4.2)$$

Since d is Ore, we can re-write $\tilde{\Delta}(d) = \Delta(d)(d^{-1} \otimes d^{-1})(d \otimes d)$, which reads

$$\tilde{\Delta}(d) = (1 \otimes 1 + \sum_i h_i d^{-1} \otimes k_i d^{-1})(d \otimes d) \quad .$$

This in turn implies

$$\begin{aligned} \tilde{\Delta}(d^{-1}) &= \tilde{\Delta}(d)^{-1} = (d \otimes d)^{-1} (1 \otimes 1 + \sum_i h_i d^{-1} \otimes k_i d^{-1})^{-1} = \\ &= (d^{-1} \otimes d^{-1}) \sum_{n=0}^{+\infty} (-1)^n \left(\sum_i h_i d^{-1} \otimes k_i d^{-1} \right)^n \end{aligned} \quad (4.3)$$

where the bottom term does belong to $\mathcal{O}_q(G)[d^{-1}] \tilde{\otimes} \mathcal{O}_q(G)[d^{-1}]$, as expected, because $k_i \in I_q(H) \subset I_G$ (for every i), hence the last formal series above is convergent in the $I_{G \otimes}$ -adic topology.

Let us now turn our attention to the algebra $\mathcal{O}_q(G)^\vee$ and its $(q-1)$ -adic completion $\widehat{\mathcal{O}}_q(G)^\vee$. By construction (cf. [14]), the coproduct of $\mathcal{O}_q(G)^\vee$, hence of $\widehat{\mathcal{O}}_q(G)^\vee$ too, is induced by the coproduct Δ of $\mathcal{O}_q(G)$. Note that the coproduct $\widehat{\Delta}$ of $\widehat{\mathcal{O}}_q(G)^\vee$ takes values in the *topological* tensor product $\widehat{\mathcal{O}}_q(G)^\vee \widehat{\otimes} \widehat{\mathcal{O}}_q(G)^\vee$, which by definition is the $(q-1)$ -adic completion of the *algebraic* tensor product $\widehat{\mathcal{O}}_q(G)^\vee \otimes \widehat{\mathcal{O}}_q(G)^\vee$ — and coincides, moreover, with the $(q-1)$ -adic completion of $\mathcal{O}_q(G)^\vee \otimes \mathcal{O}_q(G)^\vee$.

We are ready to move another key step.

Proposition 4.10. $\widehat{\mathcal{O}}_q(G/H)^\vee$ is a left coideal of $\widehat{\mathcal{O}}_q(G)^\vee$.

Proof. We want to show that the coproduct $\widehat{\Delta}$ maps $\widehat{\mathcal{O}}_q(G/H)^\vee$ into the topological tensor product $\widehat{\mathcal{O}}_q(G)^\vee \widehat{\otimes} \widehat{\mathcal{O}}_q(G/H)^\vee$.

We first observe that

$$\tilde{\Delta}(d^{-1}) \in \mathcal{O}_q(G)[d^{-1}] \tilde{\otimes} \mathcal{O}_q(G/H)[d^{-1}] .$$

This is because $\mathcal{O}_q(G/H)$ is a left coideal of $\mathcal{O}_q(G)$ — cf. Proposition 3.10, Theorem 3.15 — hence we have that the elements k_i occurring in formula (4.2) can be taken to belong to $\mathcal{O}_q(G/H)$.

Even more precisely, as the $\mathcal{O}_q(G)$ -coaction on $\mathcal{O}_q(G/H)$ via Δ is graded (by Theorem 3.15(c)), all the k_i 's have degree 1, like d itself. Thus, the series occurring in (4.3) in fact belongs to $\mathcal{O}_q(G)[d^{-1}] \tilde{\otimes} \mathcal{O}_q^{loc}(G/H)$. To sum up,

$$\tilde{\Delta}(d^{-1}) = (d^{-1} \otimes d^{-1}) \cdot \delta_\otimes , \quad \text{with } \delta_\otimes \in \mathcal{O}_q(G)[d^{-1}] \tilde{\otimes} \mathcal{O}_q^{loc}(G/H) . \quad (4.4)$$

Since the coaction $\Delta : \mathcal{O}_q(G/H) \rightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q(G/H)$ is grading-preserving and product-preserving, the definitions of $\mathcal{O}_q^{loc}(G/H)$ and $\mathcal{O}_q^{loc}(G)$ and (4.4) together yield

$$\tilde{\Delta}\left(\mathcal{O}_q^{loc}(G/H)\right) \subseteq \mathcal{O}_q(G)[d^{-1}] \tilde{\otimes} \mathcal{O}_q^{loc}(G/H) . \quad (4.5)$$

Now, we described above the completions of the algebras A and A^\vee — for $A \in \{\mathcal{O}_q(G/H), \mathcal{O}_q(G)\}$ — w.r.t. the I_G -adic or the $(q-1)$ -adic topology. Using that, or an entirely similar analysis, we see also that

$$\mathcal{O}_q(G)[d^{-1}] \tilde{\otimes} \mathcal{O}_q^{loc}(G/H) \subseteq \widehat{\mathcal{O}}_q(G)^\vee \widehat{\otimes} \widehat{\mathcal{O}}_q(G/H)^\vee .$$

In short, this is because $I_G \subseteq (q-1)\mathcal{O}_q(G)^\vee$ and $I_{G/H} \subseteq (q-1)\mathcal{O}_q(G/H)^\vee$. Also, it is easily seen that

$$\tilde{\Delta}(I_{G/H}^{loc}) \subseteq \mathcal{O}_q(G)[d^{-1}] \tilde{\otimes} I_{G/H}^{loc} + I_G \tilde{\otimes} \mathcal{O}_q^{loc}(G/H) \quad ;$$

this along with (4.5) immediately implies

$$\widehat{\Delta}((q-1)^{-1}I_{G/H}^{loc}) \subseteq \mathcal{O}_q(G)[d^{-1}] \widehat{\otimes} (q-1)^{-1}I_{G/H}^{loc} + (q-1)^{-1}I_G \widehat{\otimes} \mathcal{O}_q^{loc}(G/H)$$

which in turn yields, by the very definition of $\mathcal{O}_q(G)^\vee$ and $\mathcal{O}_q(G/H)^\vee$,

$$\widehat{\Delta}(\mathcal{O}_q(G/H)^\vee) \subseteq \mathcal{O}_q(G)^\vee \widehat{\otimes} \mathcal{O}_q(G/H)^\vee .$$

Finally, taking $(q-1)$ -adic completions on both sides, and also noting that $\mathcal{O}_q(G)^\vee \widehat{\otimes} \mathcal{O}_q(G/H)^\vee = \widehat{\mathcal{O}}_q(G)^\vee \widehat{\otimes} \widehat{\mathcal{O}}_q(G/H)^\vee$, we get

$$\widehat{\Delta}(\widehat{\mathcal{O}}_q(G/H)^\vee) \subseteq \mathcal{O}_q(G)^\vee \widehat{\otimes} \widehat{\mathcal{O}}_q(G/H)^\vee \quad . \quad \square$$

In the end, we get the main result of this section:

Theorem 4.11. $\widehat{\mathcal{O}}_q(G/H)^\vee$ is a quantization of $U(\mathfrak{h}^\perp)$ as a subalgebra and left coideal of $U(\mathfrak{g}^*)$. In other words, $\widehat{\mathcal{O}}_q(G/H)^\vee$ is an infinitesimal quantization of the coisotropic subgroup H^\perp of G^* .

Proof. Just collect the previous results. First we have

$$\widehat{\mathcal{O}}_q(G/H)^\vee \cap (q-1)\widehat{\mathcal{O}}_q(G)^\vee = (q-1)\widehat{\mathcal{O}}_q(G/H)^\vee$$

as an easy consequence of Lemma 4.7. Then, by Theorem 4.8 and by the fact that $\widehat{\mathcal{O}}_q(G/H)^\vee \Big|_{q=1} = \mathcal{O}_q(G/H)^\vee \Big|_{q=1}$, we have that the specialization of $\widehat{\mathcal{O}}_q(G/H)^\vee$ is $U(\mathfrak{h}^\perp)$. Moreover, Proposition 4.10 proves that the subalgebra $\widehat{\mathcal{O}}_q(G/H)^\vee$ of $\widehat{\mathcal{O}}_q(G)^\vee$ is also a left coideal. Therefore, $\widehat{\mathcal{O}}_q(G/H)^\vee$ is an infinitesimal quantization of H^\perp , in the standard sense. \square

Remark 4.12. The construction of $\widehat{\mathcal{O}}_q(G/H)^\vee$ is functorial, in the following sense. For a fixed $\mathcal{O}_q(G)$, every $\mathcal{O}_q(G/H)$ is uniquely characterized by the pair (π_H, d_H) given by the projection $\pi_H: \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(H)$ and the quantum section $d_H \in \mathcal{O}_q(G)$. The natural notion of morphism among such pairs, say $(\pi_{H'}, d_{H'}) \longrightarrow (\pi_{H''}, d_{H''})$, can be cast into the form a Hopf algebra endomorphism ϕ of $\mathcal{O}_q(G)$ such that $\phi(Ker(\pi_{H'})) \subseteq Ker(\pi_{H''})$ — or $\phi(I_q(H')) \subseteq I_q(H'')$ — and $\phi(d_{H'}) = d_{H''}$. Then, one defines $()^\vee$ on morphisms by scalar extension followed by restriction; proving the functoriality is a matter of bookkeeping. More in general, one might decide not to fix $\mathcal{O}_q(G)$, nor even G . Then morphisms $\phi: \mathcal{O}_q(G') \longrightarrow \mathcal{O}_q(G'')$ take the place of the endomorphisms of (the single) $\mathcal{O}_q(G)$ in the recipe above, yet $()^\vee$ is defined again on morphisms via scalar extension and restriction — and one has to exploit the functoriality of $()^\vee$ over quantum groups $\mathcal{O}_q(G)$.

5 Examples: Quantum Grassmannians and quantum flag varieties

In this section we want to examine in detail some examples of quantum homogeneous spaces and apply the quantum duality principle recipe to them. We start with the quantum Grassmannian.

5.1 The quantum Grassmannian as quantum projective homogenous space

Let us recall the classical setting.

Let $G = GL_n(\mathbb{k})$ and let $H = P$ a (standard) maximal parabolic subgroup, say

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid A \in GL_r(\mathbb{k}), B \in Mat_{r, n-r}(\mathbb{k}), D \in GL_{n-r}(\mathbb{k}) \right\} .$$

We want first to give a very ample line bundle on the homogeneous space G/P — the Grassmann variety — that realizes the classical Plücker embedding into a projective space, following the recipe in §2.2.

Let $I = (i_1, \dots, i_r)$, $1 \leq i_1 < \dots < i_r \leq n$. Define

$$d^I : g = (x_{ij}) \mapsto d^I(g) := \sum_{\sigma \in \mathcal{S}_r} (-1)^{\ell(\sigma)} x_{i_1, \sigma(1)} \cdots x_{i_r, \sigma(r)} \quad (5.1)$$

as the function corresponding to the determinant of the minor of a matrix $g = (x_{ij}) \in GL_n(\mathbb{k})$ obtained by taking rows i_1, \dots, i_r and columns $1, \dots, r$. Then $d^I \in \mathcal{O}(GL_n(\mathbb{k}))$ for all I , i.e. these are regular functions on $GL_n(\mathbb{k})$.

If $I_0 := (1, \dots, r)$, then d^{I_0} restricts to a map (with same name)

$$d^{I_0} : P \longrightarrow \mathbb{k}^\times, \quad M := \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto d^{I_0}(M) = \det(A)$$

which is a character of P . One checks that the line bundle \mathcal{L} associated to d^{I_0} is very ample, and it provides an embedding of G/P into a projective space, following the recipe in §2.2. Algebraically, this means that the graded algebra $\mathcal{O}(G/P)$ is realized as embedded into $\mathcal{O}(G)$ as

$$\mathcal{O}(G/P) = \bigoplus_{n \geq 0} \mathcal{O}(G/P)_n = \bigoplus_{n \geq 0} H^0(G/P, \mathcal{L}^{\otimes n}) \quad .$$

In particular, one can easily verify, for any set I of r rows, that

$$d^I(gp) = d^{I_0}(p) d^I(g) \quad \forall g \in GL_n(\mathbb{k}), p \in P$$

i.e. d^I is d^{I_0} -semi-invariant of degree 1. In addition, one proves that the d^I 's form a \mathbb{k} -basis of the space $\mathcal{O}(G/P)_1$ of semi-invariants of degree 1 (cf. [23]).

On the other hand, the spaces $\mathcal{O}(G/P)_n = H^0(G/P, \mathcal{L}^{\otimes n})$ are in one-to-one correspondence — up to twisting by any integral power of \det (i.e., by any character of $GL_n(\mathbb{k})$) — with the irreducible representations of $GL_n(\mathbb{k})$.

We will now see that this picture extends to the quantum setup.

Let $\mathcal{O}_q(M_n)$ be the unital associative algebra over $\mathbb{k}_q = \mathbb{k}[q, q^{-1}]$ with generators x_{ij} (for $1 \leq i, j \leq n$) and relations

$$\begin{aligned} x_{ij} x_{ik} &= q x_{ik} x_{ij}, & x_{ji} x_{ki} &= q x_{ki} x_{ji} & \forall j < k, \quad \forall i \\ x_{ij} x_{kl} &= x_{kl} x_{ij} & & \forall i < k, j > l \text{ or } i > k, j < l \\ x_{ij} x_{kl} - x_{kl} x_{ij} &= (q - q^{-1}) x_{kj} x_{il} & \forall i < k, j < l. \end{aligned}$$

(the so-called ‘‘Manin relations’’). This algebra bears also a structure of \mathbb{k}_q -bialgebra, whose coproduct and counit are given by

$$\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}, \quad \epsilon(x_{ij}) = \delta_{ij} \quad \forall i, j.$$

Define the “quantum determinant” (of order n) det_q as

$$det_q := \sum_{\sigma \in \mathcal{S}_n} (-q)^{\ell(\sigma)} x_{1, \sigma(1)} \cdots x_{n, \sigma(n)} \in \mathcal{O}_q(M_n) \quad .$$

One proves that det_q belongs to the center of $\mathcal{O}(M_n)$, and it is group-like, i.e. $\Delta(det_q) = det_q \otimes det_q$ and $\epsilon(det_q) = 1$.

More in general, for any $1 \leq r \leq n$ and for any choice of r -tuples of *increasing* indices $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$, we define the “quantum determinant of the minor (I, J) ”, i.e. of the minor (of the matrix with entries the x_{ij} ’s) whose sets of rows and columns are I and J , namely

$$D_J^I := \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)} x_{i_1, j_{\sigma(1)}} \cdots x_{i_r, j_{\sigma(r)}} \quad . \quad (5.2)$$

These satisfy (cf. [21], §9.2.2) the following quantum analogue of well-known classical identities (e.g., the first one is analogous to Binet theorem):

$$\Delta(D_J^I) = \sum_K D_K^I \otimes D_J^K \quad , \quad \epsilon(D_J^I) = \delta_{I, J} \quad . \quad (5.3)$$

Since det_q is central in $\mathcal{O}_q(M_n)$, it is a Ore element as well, and we can consider the enlarged algebra $\mathcal{O}_q(GL_n) := \mathcal{O}_q(M_n)[det_q^{-1}]$ obtained from $\mathcal{O}_q(M_n)$ by formally inverting det_q . Then — see [21] again — the bialgebra structure of $\mathcal{O}_q(M_n)$ uniquely extends to $\mathcal{O}_q(GL_n)$; even more, the latter then is a *Hopf algebra* indeed. In fact, by construction $\mathcal{O}_q(GL_n)$ is a *quantum group*, namely a quantization of $GL_n(\mathbb{k})$, in the sense of Definition 3.1.

We shall again denote by x_{ij} the images in $\mathcal{O}_q(GL_n)$ of the generators x_{ij} of $\mathcal{O}_q(M_n)$. Similarly, we shall again denote by D_J^I the images in $\mathcal{O}_q(GL_n)$ of the “quantum minors” of $\mathcal{O}_q(M_n)$: then they again enjoy (5.2) and (5.3). Letting $J_0 := (1, \dots, r) =: I_0$, hereafter we shall set $D^I := D_{J_0}^I$.

The specialization (at $q = 1$) of any quantum minor D_J^I is the corresponding classical minor d_J^I (on the same sets of rows and columns); in particular, every D^I specializes to d^I — see (5.1) — and, among them, D^{I_0} to d^{I_0} .

We define the *quantum parabolic subgroup* $\mathcal{O}_q(P)$ as the quotient algebra

$$\mathcal{O}_q(P) := \mathcal{O}_q(GL_n) \Big/ \left(\{ x_{ij} \mid r+1 \leq i \leq n; 1 \leq j \leq r \} \right) \quad .$$

One can easily check that this $\mathcal{O}_q(P)$ is in fact a *Hopf algebra* quotient. Thus the natural projection map $\pi : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(P)$ is a Hopf algebra

epimorphism. Therefore, $\mathcal{O}_q(P)$ is a quantum Poisson subgroup of $\mathcal{O}_q(G) = \mathcal{O}_q(GL_n(\mathbb{k}))$, in the sense of Definition 3.3, whose specialization is $\mathcal{O}(P)$.

We are now in a position to appreciate the first important fact — in the present setting — about quantum minors:

Lemma 5.1. *The quantum minor D^{I_0} is a quantum section of the line bundle on G/P given by d^{I_0} , in the sense of Definition 3.12.*

Proof. Using the first identity in (5.3) one gets

$$\Delta_\pi(D^{I_0}) = ((id \otimes \pi) \circ \Delta)(D^{I_0}) = (id \otimes \pi)\left(\sum_K D_K^{I_0} \otimes D_{J_0}^K\right) = \sum_K D_K^{I_0} \otimes \overline{D_{J_0}^K}$$

and then from this

$$\Delta_\pi(D^{I_0}) = D_{I_0}^{I_0} \otimes \overline{D_{J_0}^{I_0}} = D^{I_0} \otimes \overline{D^{I_0}}$$

because $\overline{D_{J_0}^K} := \pi(D_{J_0}^K) = \delta_{K, I_0} \overline{D_{J_0}^{I_0}}$, by definition of π , and $D_{I_0}^{I_0} = D_{J_0}^{I_0} = D^{I_0}$. Therefore (Definition 3.5) D^{I_0} is a pre-quantum section; but $\mathcal{O}_q(P)$ is a quantum subgroup, so (Proposition 3.13) D^{I_0} is a quantum section. \square

Using D^{I_0} , we can perform the construction of the algebra $\mathcal{O}_q(G/P)$ of D^{I_0} -semi-invariants (or simply *semi-invariants*), as in §3. First we have

Lemma 5.2. *The quantum minors D^I are all semi-invariants of degree 1, that is to say $D^I \in \mathcal{O}_q(G/P)_1$ for every set of rows $I = (i_1, \dots, i_r)$.*

Proof. Arguing as in the proof of Lemma 5.1, we prove the claim by

$$\begin{aligned} \Delta_\pi(D^I) &= ((id \otimes \pi) \circ \Delta)(D^I) = (id \otimes \pi)\left(\sum_K D_K^I \otimes D_{J_0}^K\right) = \\ &= \sum_K D_K^I \otimes \overline{D_{J_0}^K} = D_{I_0}^I \otimes \overline{D_{J_0}^{I_0}} = D^I \otimes \overline{D^{I_0}} \quad . \quad \square \end{aligned}$$

Roughly speaking, the outcome of this last result is that the line bundle on G/P given by d^{I_0} has enough “quantum sections” to provide a “quantum projective embedding”. To be precise, the following holds:

Corollary 5.3. *The space $\mathcal{O}_q(G/P)$ of all D^{I_0} -semi-invariants is a quantization of $\mathcal{O}(G/P)$, in the sense of Definition 3.18.*

Proof. By construction, every quantum minor D^I specializes to the corresponding classical minor d^I . By §5.1, the latter form a basis of $\mathcal{O}(G/P)_1$.

This along with Lemma 5.2 proves that the natural embedding

$$\mathcal{O}_q(G/P)_1 / (q-1) \mathcal{O}_q(G/P)_1 \hookrightarrow \mathcal{O}(G/P)_1$$

is onto. But then, as noticed in Remark 3.19, this is enough to conclude. \square

Actually, we can prove the following, much more precise result:

Proposition 5.4. *The algebra $\mathcal{O}_q(G/P)$ is generated by the D^I 's.*

Proof. By Lemma 5.2, the D^I 's belong to $\mathcal{O}_q(G/P)$. Therefore, we are only left to prove that, conversely, every semi-invariant is contained in the \mathbb{k} -subalgebra of $\mathcal{O}_q(G)$ generated by the D^I 's.

To this end, Theorems 1.2 and 1.3 in [16] give us immediately the result if we take $\mathbb{k}(q)$ as ground ring instead of $\mathbb{k}_q := \mathbb{k}[q, q^{-1}]$. Then Lemmas 3.9, 3.10, 3.11 in [12] give us our result. We now see that in detail.

We start by rewriting the Proposition 1.1 in [16] in our notation:

Let $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ be a complex of \mathbb{k}_q -modules, such that C is torsion free. Suppose there are \mathbb{k}_q -module decompositions $A = \bigoplus_j A_j$, $B = \bigoplus_j B_j$, $C = \bigoplus_j C_j$ such that B_j is finitely generated, and the maps ϕ and ψ respect the decomposition, that is $\phi(A_j) \subseteq B_j$ and $\psi(B_j) \subseteq C_j$. Then if the sequence $\overline{A} \rightarrow \overline{B} \rightarrow \overline{C}$ obtained by reduction modulo $(q-1)$ is exact, then so is also

$$\mathbb{k}(q) \otimes_{\mathbb{k}_q} A \xrightarrow{\phi} \mathbb{k}(q) \otimes_{\mathbb{k}_q} B \xrightarrow{\psi} \mathbb{k}(q) \otimes_{\mathbb{k}_q} C$$

Let's apply this result to our special situation.

The subalgebra $A := \mathbb{k}_q[D^I]$ generated in $\mathcal{O}_q(SL_n)$ by quantum determinants is a \mathbb{k}_q -graded module (by the degree). This fact is non trivial and depends on the explicit form of this algebra in terms of generators and relations, see [9]. We have that an element $a \in \mathcal{O}_q(SL_n)$ is in A_j iff

$$\sum_{(a)} a_{(1)} \otimes \bar{a}_{(2)} = a \otimes \bar{d}^j, \quad \text{where } \Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

where \bar{x} denotes reduction of x modulo $I_q(P)$ (see notation in section 3.3).

So we can set up maps

$$A \xrightarrow{\phi} \mathcal{O}_q(SL_n) \xrightarrow{\psi} \mathcal{O}_q(SL_n) \otimes \mathcal{O}_q(P)$$

where ϕ is the inclusion and ψ is defined by

$$\psi(a) = \sum_{(a)} a_{(1)} \otimes \bar{a}_{(2)} - a \otimes \bar{d}^j \quad \forall a \in A_j .$$

One can check that all the hypothesis of the previous result, for $B := \mathcal{O}_q(SL_n)$ and $C := \mathcal{O}_q(SL_n) \otimes \mathcal{O}_q(P)$, are satisfied, hence we obtain that $\mathbb{k}(q) \otimes_{\mathbb{k}} A \cong \text{Ker}(id \otimes \psi)$. In other words, the semi-invariants coincide with the subalgebra generated by the quantum determinants over the ring $\mathbb{k}(q)$.

We now obtain the result over \mathbb{k}_q by Lemma 3.11 in [10], namely

$$\text{If } wX \in \mathbb{k}_q[D^I], \quad w \in \mathbb{k}_q, \quad w \neq 0, \quad \text{then } X \in \mathbb{k}_q[D^I]. \quad \square$$

Remark 5.5. Thus, using our own recipe, we have constructed the quantum homogeneous space $\mathcal{O}_q(G/P)$. It is immediate to see that this is the same as the deformation of the algebra of the classical Grassmannian, along with its classical Plücker embedding, as it is described in [9] or in [28].

Finally, for the $\mathcal{O}_q(G)$ -comodule structure of the space of semi-invariants of degree 1, we have also the following analogue of a classical result:

Proposition 5.6. $\mathcal{O}_q(G/P)_1 \cong \bigwedge_q(\mathbb{k}_q^n)$ as left $\mathcal{O}_q(G)$ -comodules.

Proof. This is a direct calculation. Let's sketch it. By all the previous analysis, we already know that $\mathcal{O}_q(G/P)_1$ has basis the set of all the D^I 's, and the left $\mathcal{O}_q(G)$ -coaction on $\mathcal{O}_q(G/P)_1$ is given by

$$D^I \mapsto \sum_K D_K^I \otimes D^K .$$

Now consider the coaction of $\mathcal{O}_q(G)$ on $\bigwedge_q(\mathbb{k}_q^n)$, given by

$$\begin{aligned} \xi_{i_1} \cdots \xi_{i_r} &\mapsto \sum_{k_1, \dots, k_r} g_{i_1 k_1} \cdots g_{i_r k_r} \otimes \xi_{k_1} \cdots \xi_{k_r} = \\ &= \sum_{\sigma} (-q)^{\ell(\sigma)} g_{i_1 k_1} \cdots g_{i_r k_r} \otimes \xi_{k_1^{\sigma}} \cdots \xi_{k_r^{\sigma}} = \sum_K D_K^I \otimes \xi_{k_1^{\sigma}} \cdots \xi_{k_r^{\sigma}} \end{aligned}$$

where σ is the permutation reordering k_1, \dots, k_r and $K = (k_1^{\sigma}, \dots, k_r^{\sigma})$ are the same indices, but reordered. Now the result follows. \square

Remark 5.7. Similar arguments can be used to prove that any quantum flag variety is a “quantum projective homogeneous space” in the sense of Definition 3.18 (for details about quantum flag varieties, we refer to [11]).

For the flag of type (m_1, \dots, m_s) , the quantum section d to start with is

$$d := D^{(m_1)} \dots D^{(m_n)}$$

where the $D^{(m_j)}$'s are the principal quantum minors of size m_j .

The proofs of all results go over exactly as in the Grassmannian case.

We now turn to the construction of the quantum big cell ring, that will be crucial for the explicit construction of the QDP functor.

Definition 5.8. Let $I_0 = (1 \dots r)$, $D_0 := D^{I_0}$. Define

$$\mathcal{O}_q(G)[D_0^{-1}] := \mathcal{O}_q(G)[T] / (TD_0 - 1, D_0T - 1)$$

Moreover, we define the *big cell ring* $\mathcal{O}_q^{loc}(G/P)$ to be the \mathbb{k}_q -subalgebra of $\mathcal{O}_q(G)[D_0^{-1}]$ generated by the elements

$$t_{ij} := (-q)^{r-j} \Delta_{ij} D_0^{-1} \quad \forall i, j : 1 \leq j \leq r < i \leq n$$

where $\Delta_{ij} := D^{1 \dots \widehat{j} \dots r i}$, for all i, j as above (see [10] for more details).

As in the commutative setting, we have the following result:

Proposition 5.9. $\mathcal{O}_q^{loc}(G/P) \cong \mathcal{O}_q(G/P)[D_0^{-1}]_{proj}$, where the right-hand side is the degree-zero component of $\mathcal{O}_q(G/P)[T] / (TD_0 - 1, D_0T - 1)$.

Proof. In the classical setting, the analogous result is proved by this argument: one uses the so-called “straightening relations” to get rid of the extra minors (see, for example, [8], §2). Here the argument works essentially the same, using the *quantum straightening* (or *Plücker*) *relations* (see [9], §4, [28], formula (3.2)(c) and Note I, Note II). \square

Remark 5.10. As before, we have that

$$\mathcal{O}_q^{loc}(G/P) \cap (q-1)\mathcal{O}_q^{loc}(G) = (q-1)\mathcal{O}_q^{loc}(G/P)$$

This can be easily deduced from Remark 3.9, taking into account Proposition 5.9. As a consequence, the map

$$\mathcal{O}_q^{loc}(G/P) / (q-1)\mathcal{O}_q^{loc}(G/P) \longrightarrow \mathcal{O}_q^{loc}(G) / (q-1)\mathcal{O}_q^{loc}(G)$$

is *injective*, so that the specialization map

$$\pi_{G/P}^{loc} : \mathcal{O}_q^{loc}(G/P) \longrightarrow \mathcal{O}_q^{loc}(G/P) / (q-1)\mathcal{O}_q^{loc}(G/P)$$

coincides with the restriction of the specialization map

$$\pi_G^{loc} : \mathcal{O}_q^{loc}(G) \longrightarrow \mathcal{O}_q^{loc}(G) / (q-1)\mathcal{O}_q^{loc}(G) .$$

The following proposition gives a description of the algebra $\mathcal{O}_q^{loc}(G/P)$:

Proposition 5.11. *The big cell ring is isomorphic to a matrix algebra*

$$\begin{array}{ccc} \mathcal{O}_q^{loc}(G/P) & \longrightarrow & \mathcal{O}_q(M_{(n-r) \times r}) \\ t_{ij} & \mapsto & x_{ij} \end{array} \quad \forall 1 \leq j \leq r < i \leq n$$

i.e. the generators t_{ij} 's satisfy the Manin relations.

Proof. See [10], Proposition 1.9. □

Remark 5.12. The Grassmannian GL_n/P can also be realized as a similar quotient of SL_n by a suitable parabolic P' (corresponding to P , say). Then one can also perform all related quantum constructions — the previous and the later ones — using SL_n instead of GL_n , and modifying each step as needed. To begin with, one considers

$$\mathcal{O}_q(SL_n) := \mathcal{O}_q(GL_n) / (\det_q - 1) \cong \mathcal{O}_q(M_n) / (\det_q - 1)$$

— where $(\det_q - 1)$ is the (two-sided) ideal generated by $\det_q - 1$ — which is again a Hopf algebra, for the quotient structure from either $\mathcal{O}_q(GL_n)$ or

$\mathcal{O}_q(M_n)$. This is a quantization of $SL_n(\mathbb{k})$, in the sense of Definition 3.1, for which we can consider again quantum minors and a corresponding $\mathcal{O}_q(P)$ as before. Then all this can be used to give an alternative definition of $\mathcal{O}_q(G/P) = \mathcal{O}_q(SL_n/P')$ and of all was considered above. Similarly, all constructions and results of section 5.2 hereafter can be carried on using $\mathcal{O}_q(SL_n)$ — and its related gadgets — instead of $\mathcal{O}_q(GL_n)$.

Finally, similar considerations hold as well for the quantum flag varieties mentioned in Remark 5.7.

5.2 QDP for quantum Grassmannians

We now turn to the quantum duality principle applied explicitly to the quantum homogeneous spaces constructed above. We start with Grassmannians.

Let us first explicitly describe the Poisson structure of the algebraic group GL_n . Starting from $\mathcal{O}_q(GL_n)$, as usual the classical algebra $\mathcal{O}(GL_n)$ inherits from the former a Poisson bracket, which makes it into a Poisson Hopf algebra, so that GL_n becomes a Poisson group (see Remark 3.2(2)). We want to describe now this Poisson bracket. Recall that

$$\mathcal{O}(GL_n) = \mathbb{k}[\{\bar{x}_{ij}\}_{i,j=1,\dots,n}][det^{-1}] = \mathbb{k}[\{\bar{x}_{ij}\}_{i,j=1,\dots,n}][t] / (t det - 1)$$

where $det := det(\bar{x}_{i,j})_{i,j=1,\dots,n}$ is the usual determinant. Setting $\bar{x} = p(x)$ for $p : \mathcal{O}_q(GL_n) \rightarrow \mathcal{O}(GL_n)$, the Poisson structure is given (as usual) by

$$\{\bar{a}, \bar{b}\} := (q-1)^{-1}(ab - ba) \Big|_{q=1} \quad \forall \bar{a}, \bar{b} \in \mathcal{O}(GL_n).$$

In terms of generators, we have

$$\begin{aligned} \{\bar{x}_{ij}, \bar{x}_{ik}\} &= \bar{x}_{ij} \bar{x}_{ik} \quad \forall j < k, & \{\bar{x}_{ij}, \bar{x}_{\ell k}\} &= 0 \quad \forall i < \ell, k < j \\ \{\bar{x}_{ij}, \bar{x}_{\ell j}\} &= \bar{x}_{ij} \bar{x}_{\ell j} \quad \forall i < \ell, & \{\bar{x}_{ij}, \bar{x}_{\ell k}\} &= 2\bar{x}_{ij} \bar{x}_{\ell k} \quad \forall i < \ell, j < k \\ \{det^{-1}, \bar{x}_{ij}\} &= 0, & \{det, \bar{x}_{ij}\} &= 0 \quad \forall i, j = 1, \dots, n. \end{aligned}$$

As GL_n is a Poisson Lie group, its Lie algebra \mathfrak{gl}_n has a Lie bialgebra structure (see [3], pg. 24). To describe it, let us denote with m_{ij} the elementary matrices, which form a basis of \mathfrak{gl}_n . Define ($\forall i = 1, \dots, n-1, j = 1, \dots, n$)

$$e_i := m_{i,i+1}, \quad g_j := m_{j,j}, \quad f_i := m_{i+1,i}, \quad h_i := g_i - g_{i+1}$$

Then $\{e_i, f_i, g_j \mid i = 1, \dots, n-1, j = 1, \dots, n\}$ is a set of Lie algebra generators of \mathfrak{gl}_n , and a Lie cobracket is defined on \mathfrak{gl}_n by

$$\delta(e_i) = h_i \otimes e_i - e_i \otimes h_i, \quad \delta(g_j) = 0, \quad \delta(f_i) = h_i \otimes f_i - f_i \otimes h_i \quad \forall i, j.$$

This cobracket makes \mathfrak{gl}_n itself into a *Lie bialgebra*: this is the so-called *standard* Lie bialgebra structure on \mathfrak{gl}_n . It follows immediately that $U(\mathfrak{gl}_n)$ is a co-Poisson Hopf algebra, whose co-Poisson bracket is the (unique) extension of the Lie cobracket of \mathfrak{gl}_n while the Hopf structure is the standard one.

Similar constructions hold for the group SL_n . One simply drops the generator d^{-1} , and imposes the relation $d=1$, in the description of $\mathcal{O}(SL_n)$, and replaces the g_s 's with the h_i 's ($i = 1, \dots, n$) when describing \mathfrak{sl}_n .

Since \mathfrak{gl}_n is a Lie bialgebra, its dual space \mathfrak{gl}_n^* admits a Lie bialgebra structure, dual to the one of \mathfrak{gl}_n . Let $\{e_{ij} := m_{ij}^* \mid i, j = 1, \dots, n\}$ be the basis of \mathfrak{gl}_n^* dual to the basis of elementary matrices for \mathfrak{gl}_n . As a Lie algebra, \mathfrak{gl}_n^* can be realized as the subset of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ of all pairs

$$\left(\begin{pmatrix} -m_{11} & 0 & \cdots & 0 \\ m_{21} & -m_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \cdots & 0 \\ m_{n,1} & m_{n,2} & \cdots & -m_{n,n} \end{pmatrix}, \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} & m_{1,n} \\ 0 & m_{22} & \cdots & m_{2,n-1} & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ 0 & 0 & \cdots & 0 & m_{n,n} \end{pmatrix} \right)$$

with its natural structure of Lie subalgebra of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$. In fact, the elements e_{ij} correspond to elements in $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ in the following way:

$$e_{ij} \cong (m_{ij}, 0) \quad \forall i > j, \quad e_{ij} \cong (-m_{ij}, +m_{ij}) \quad \forall i = j, \quad e_{ij} \cong (0, m_{ij}) \quad \forall i < j.$$

Then the Lie bracket of \mathfrak{gl}_n^* is given by

$$[e_{i,j}, e_{h,k}] = \delta_{j,h} e_{i,k} - \delta_{k,i} e_{h,j}, \quad \forall i \leq j, h \leq k \quad \text{and} \quad \forall i > j, h > k$$

$$[e_{i,j}, e_{h,k}] = \delta_{k,i} e_{h,j} - \delta_{j,h} e_{i,k}, \quad \forall i = j, h > k \quad \text{and} \quad \forall i > j, h = k$$

$$[e_{i,j}, e_{h,k}] = 0, \quad \forall i < j, h > k \quad \text{and} \quad \forall i > j, h < k$$

Note that the elements $(1 \leq i \leq n-1, 1 \leq j \leq n)$

$$e_i = e_i^* = e_{i,i+1}, \quad f_i = f_i^* = e_{i+1,i}, \quad g_j = g_j^* = e_{jj}$$

are Lie algebra generators of \mathfrak{gl}_n^* . In terms of them, the Lie bracket reads

$$[e_i, f_j] = 0, \quad [g_i, e_j] = \delta_{ij} E_i, \quad [g_i, f_j] = \delta_{ij} f_j \quad \forall i, j.$$

On the other hand, the Lie cobracket structure of \mathfrak{gl}_n^* is given by

$$\delta(e_{i,j}) = \sum_{k=1}^n e_{i,k} \wedge e_{k,j} \quad \forall i, j = 1, \dots, n$$

where $x \wedge y := x \otimes y - y \otimes x$.

Finally, all these formulæ also provide a presentation of $U(\mathfrak{gl}_n^*)$ as a co-Poisson Hopf algebra.

A similar description holds for $\mathfrak{sl}_n^* = \mathfrak{gl}_n^* / Z(\mathfrak{gl}_n^*)$, where $Z(\mathfrak{gl}_n^*)$ is the centre of \mathfrak{gl}_n^* , generated by $\mathfrak{l}_n := g_1 + \dots + g_n$. The construction is immediate by looking at the embedding $\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n$.

We now turn to the construction of the QDP functor.

Observation 5.13. $\mathcal{O}_q(G)^\vee$ (see Definition 4.1 and §5.1) is generated, as a unital subalgebra of $\mathcal{O}_q(G) \otimes_{\mathbb{k}_q} \mathbb{k}(q)$, by the elements

$$\Delta_- := (q-1)^{-1} (D_q^{-1} - 1), \quad \chi_{ij} := (q-1)^{-1} (x_{ij} - \delta_{ij}) \quad \forall i, j = 1, \dots, n$$

where the x_{ij} 's are the generators of $\mathcal{O}_q(G)$. As $x_{ij} = \delta_{ij} + (q-1)\chi_{ij} \in \mathcal{O}_q(G)^\vee$, we have an obvious embedding of $\mathcal{O}_q(G)$ into $\mathcal{O}_q(G)^\vee$.

Following Definition 4.2, we define

$$\mathcal{O}_q(G/P)^\vee := \langle (q-1)^{-1} J_{G/P}^{loc} \rangle = \sum_{n=0}^{\infty} (q-1)^{-n} (J_{G/P}^{loc})^n.$$

We can provide a concrete description of $\mathcal{O}_q(G/P)^\vee$:

Proposition 5.14. *We have*

$$\mathcal{O}_q(G/P)^\vee = \mathbb{k}_q \langle \{ \mu_{ij} \}_{i=r+1, \dots, n}^{j=1, \dots, r} \rangle / I_M$$

where $\mu_{ij} := (q-1)^{-1} t_{ij}$ (for all i and j), I_M is the ideal of the Manin relations among the μ_{ij} 's, and $t_{ij} := (-q)^{r-j} \Delta_{ij} D_0^{-1}$ (for all i and j).

Proof. Trivial from definitions and Proposition 5.11. \square

We want to see explicitly what is $\mathcal{O}_q(G/P)^\vee|_{q=1}$ inside $U(\mathfrak{gl}_n^*)$. In other words, we want to understand what is the space that $\mathcal{O}_q(G/P)^\vee$ is quantizing. We check now by direct inspection that this is $U(\mathfrak{p}^\perp)$, as already prescribed by Theorem 4.8.

Proposition 5.15.

$$\mathcal{O}_q(G/P)^\vee|_{q=1} = U(\mathfrak{p}^\perp)$$

as a subalgebra of $\mathcal{O}_q(G)^\vee|_{q=1} = U(\mathfrak{gl}_n^*)$, where \mathfrak{p}^\perp is the orthogonal subspace to $\mathfrak{p} := \text{Lie}(P)$ inside \mathfrak{gl}_n^* .

Proof. Thanks to the previous discussion, it is enough to show that

$$\pi_G^\vee(\mathcal{O}_q(G/P)^\vee) = U(\mathfrak{p}^\perp) \subseteq U(\mathfrak{gl}_n^*) = \mathcal{O}_q(G)^\vee|_{q=1}.$$

To do this, we describe the isomorphism $\mathcal{O}_q(G)^\vee|_{q=1} \cong U(\mathfrak{gl}_n^*)$ (cf. [14]). According to Remark 5.13, the algebra $\mathcal{O}_q(G)^\vee$ is generated by the elements

$$\Delta_- := (q-1)^{-1}(D_q^{-1}-1), \quad \chi_{ij} := (q-1)^{-1}(x_{ij}-\delta_{ij}) \quad \forall i, j = 1, \dots, n$$

inside $\mathcal{O}_q(G) \otimes_{\mathbb{k}_q} \mathbb{k}(q)$. In terms of these generators, the isomorphism reads

$$\begin{aligned} \mathcal{O}_q(G)^\vee|_{q=1} &\longrightarrow U(\mathfrak{gl}_n^*) \\ \overline{\Delta_-} &\mapsto -(e_{1,1} + \dots + e_{n,n}), \quad \overline{\chi_{i,j}} \mapsto e_{i,j} \quad \forall i, j. \end{aligned}$$

where we used notation $\overline{X} := X \bmod (q-1)\mathcal{O}_q(G)^\vee$. Indeed, from $\overline{\chi_{i,j}} \mapsto e_{i,j}$ and $(q-1)^{-1}(D_q-1) \in \mathcal{O}_q(G)^\vee$, one gets $\overline{D_q} \mapsto 1$ and $(q-1)^{-1}(D_q-1) \mapsto e_{1,1} + \dots + e_{n,n}$. Moreover, the relation $D_q D_q^{-1} = 1$ in $\mathcal{O}_q(G)$ implies $D_q \Delta_- = -(q-1)^{-1}(D_q-1)$ in $\mathcal{O}_q(G)^\vee$, whence clearly $\overline{\Delta_-} \mapsto -(e_{1,1} + \dots + e_{n,n})$ as claimed.

In other words, the specialization $p_G^\vee : \mathcal{O}_q(G)^\vee \longrightarrow U(\mathfrak{gl}_n^*)$ is given by

$$p_G^\vee(\Delta_-) = -(e_{1,1} + \dots + e_{n,n}), \quad p_G^\vee(\chi_{i,j}) = e_{i,j} \quad \forall i, j.$$

If we look at $\widehat{\mathcal{O}_q(G)}^\vee$, things are even simpler. Since

$$D_q \in \left(1 + (q-1) \mathcal{O}_q(G)^\vee\right) \subset \left(1 + (q-1) \widehat{\mathcal{O}_q(G)}^\vee\right),$$

then $D_q^{-1} \in \left(1 + (q-1) \widehat{\mathcal{O}_q(G)}^\vee\right)$, and the generator Δ_- can be dropped. The specialization map $\widehat{p_{G/P}^\vee}$ of course is still described by formulæ as above.

Now let us compute $p_{G/P}^\vee\left(\mathcal{O}_q(G/P)^\vee\right) = \widehat{p_G^\vee}\left(\mathcal{O}_q(G/P)^\vee\right)$. Recall that $\mathcal{O}_q(G/P)^\vee$ is generated by the μ_{ij} 's, with

$$\mu_{ij} := (q-1)^{-1} t_{ij} = (q-1)^{-1} (-q)^{r-j} \Delta_{ij} D_0^{-1}$$

for $i = r+1, \dots, n$, and $j = 1, \dots, r$; thus we must compute $\widehat{p_G^\vee}(\mu_{ij})$.

By definition, for every $i \neq j$ the element $x_{ij} = (q-1)\chi_{ij}$ is mapped to 0 by $\widehat{p_G^\vee}$. Instead, for each ℓ the element $x_{\ell\ell} = 1 + (q-1)\chi_{\ell\ell}$ is mapped to 1 (by $\widehat{p_G^\vee}$ again). But then, expanding the q -determinants one easily finds that

$$\begin{aligned} \widehat{p_G^\vee}\left((q-1)^{-1} \Delta_{ij}\right) &= \left((q-1)^{-1} \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)} x_{1\sigma(1)} \cdots x_{r\sigma(r)}\right) = \\ &= \widehat{p_G^\vee}\left((q-1)^{-1} \sum_{\sigma \in \mathcal{S}_r} (-q)^{\ell(\sigma)} (\delta_{1\sigma(1)} + (q-1)\chi_{1\sigma(1)}) \cdots (\delta_{1\sigma(r)} + (q-1)\chi_{1\sigma(r)})\right) \end{aligned}$$

The only term in $(q-1)$ in the expansion of Δ_{ij} comes from the product $(1 + (q-1)\chi_{11}) \cdots (1 + (q-1)\chi_{rr}) (q-1)\chi_{ij} \equiv (q-1)\chi_{ij} \pmod{(q-1)^2 \mathcal{O}(G/P)}$

Therefore, from the previous analysis we get

$$\begin{aligned} \widehat{p_G^\vee}\left((q-1)^{-1} \Delta_{ij}\right) &= \widehat{p_G^\vee}(\chi_{i,j}) = e_{i,j} \\ \widehat{p_G^\vee}(D_0) &= \widehat{p_G^\vee}(1) = 1, \quad \widehat{p_G^\vee}(D_0^{-1}) = \widehat{p_G^\vee}(1) = 1 \end{aligned}$$

hence we conclude that $\widehat{p_G^\vee}(\mu_{ij}) = (-1)^{r-j} e_{i,j}$, for all $1 \leq j \leq r < i \leq n$.

The outcome is that $p_{G/P}^\vee\left(\mathcal{O}_q(G/P)^\vee\right) = U(\mathfrak{h})$, where

$$\mathfrak{h} := \text{Span}\left(\{e_{i,j} \mid r+1 \leq i \leq n, 1 \leq j \leq r\}\right).$$

On the other hand, from the very definitions and our description of \mathfrak{gl}_n^* one easily finds that $\mathfrak{h} = \mathfrak{p}^\perp$, for $\mathfrak{p} := \text{Lie}(P)$. The claim follows. \square

Proposition 5.16. $\widehat{\mathcal{O}}_q(G/P)^\vee$ is a left coideal of $\widehat{\mathcal{O}}_q(G)^\vee$.

Proof. This is the same as Proposition 4.10. \square

Hence for the quantum Grassmannian we have proved directly the following result:

Theorem 5.17. $\widehat{\mathcal{O}}_q(G/P)^\vee$ is a quantum homogeneous G^* -space, which is an infinitesimal quantization of the homogeneous G^* -space \mathfrak{p}^\perp .

5.3 Quantum generalized flag varieties for simple groups as quantum projective homogeneous spaces

We now turn to a more general example of quantum projective space: namely the quantization of a generalized flag variety for any simple group, following [19]. As before, we begin with a brief description of the classical setting.

Let G be a connected, simply connected, complex simple Lie group, and let \mathfrak{g} its Lie algebra. Let S be a subset of simple roots of \mathfrak{g} and let $\lambda = \sum_{\alpha_i \notin S} \omega_i$ be a weight of \mathfrak{g} , where the ω_i 's are fundamental weights.

Let $V(\lambda)$ be the highest weight representation of \mathfrak{g} (and of G) associated with the weight λ , and let v_λ be a non-zero highest weight vector of $V(\lambda)$. We have the following morphism of algebraic varieties:

$$\begin{aligned} G &\longrightarrow \mathbb{P}(V(\lambda)) \\ g &\longmapsto g \cdot v_\lambda \end{aligned}$$

This induces a projective embedding of the flag variety G/P_S into the projective space $\mathbb{P}(V(\lambda))$, where $P_S := \text{Stab}_G(v_\lambda)$ is the parabolic subgroup associated to the set S . The graded algebra of regular functions on G/P_S relative to this embedding is given by

$$\mathcal{O}(G/P_S) = \bigoplus_{n \in \mathbb{N}} V(n\lambda)^* \tag{5.4}$$

where the grading is given by $\mathcal{O}(G/P_S)_n := V(n\lambda)^*$ and the multiplication is via the Cartan multiplication (see [12] for more details).

We are now going to identify $\mathcal{O}(G/P_S)$ with a graded subalgebra of $\mathcal{O}(G)$. Indeed, the algebra $\mathcal{O}(G)$ is in (Hopf) duality with $U(\mathfrak{g})$ and it can be

thought of as the linear span inside $U(\mathfrak{g})^*$ of the functionals $c_{f,v}^\lambda : U(\mathfrak{g}) \longrightarrow \mathbb{C}$ (the so-called “matrix coefficients”) given by

$$c_{f,v}^\lambda(u) := f(u.v) \quad \forall u \in U(\mathfrak{g}), \quad f \in V(\lambda)^*, \quad v \in V(\lambda).$$

Then $\mathcal{O}(G/P_S)$ can be realized, equivalently, as the subalgebra of $\mathcal{O}(G)$ generated by the $c_{f,v}^\lambda$ ’s, for all $f \in V(\lambda)^*$.

This point of view carries over without changes to the quantum setting: it is considered in [22], [27] and by several others; a quick review of this construction can be found in [19]. The key point is that every finite dimensional representation $V(\mu)$ of $U(\mathfrak{g})$ of highest weight μ has a (standard) quantization, which is a representation for $U_q(\mathfrak{g})$, call it $V_q(\mu)$. In particular, one can again define matrix coefficient functionals $c_{f,v}^\mu$ — for every $f \in V_q(\mu)^*$ and $v \in V_q(\mu)$ — for all dominant weights μ of \mathfrak{g} . Their \mathbb{k}_q -span inside $U_q(\mathfrak{g})^*$ is, by definition, the algebra $\mathcal{O}_q(G)$, which is a quantization of $\mathcal{O}(G)$. In fact, the latter follows because $V_q(\mu)$, respectively $V_q(\mu)^*$, specializes to $V(\mu)$, respectively to $V(\mu)^*$, at $q = 1$, hence “quantum” and “classical” matrix coefficients bijectively correspond to each other — via $c_{f,v}^\mu \mapsto c_{f,v}^\mu$, say — under specialization at $q = 1$.

For later use, we point out how matrix coefficient behave under the co-product. For any dominant weight μ of \mathfrak{g} , let $\{v_1, \dots, v_r\}$ be a \mathbb{k}_q -basis of $V_q(\mu)$ — with $r = \dim(V_q(\mu))$ — and let $\{f_1, \dots, f_r\}$ be the dual basis of $V_q(\mu)^*$. Then every matrix coefficient $c_{f,v}^\mu$ has coproduct given by

$$\Delta(c_{f,v}^\mu) = \sum_{i=1}^r c_{f,v_i}^\mu \otimes c_{f_i,v}^\mu \quad (5.5)$$

(just because $U_q(\mathfrak{g})$ is acting on $V_q(\mu)$ via matrices!).

From the quantization $\mathcal{O}_q(G)$ the group G inherits a Poisson group structure — cf. Remark 3.2(2) — for which P_S is a Poisson subgroup. On the other hand, let $I_q(P_S)$ be the two-sided ideal of $\mathcal{O}_q(G)$ generated by the set of matrix coefficients $\left\{ c_{f,v}^\mu \mid \forall n \in \mathbb{N}, \mu \neq n\lambda \vee v \notin (\mathbb{k}_q \setminus \{0\}) \cdot v_{n\lambda} \right\}$. Then, using (5.5), one easily shows that $I_q(P_S)$ is a Hopf ideal of $\mathcal{O}_q(G)$; therefore the quotient \mathbb{k}_q -module and the canonical map

$$\mathcal{O}_q(P_S) := \mathcal{O}_q(G) / I_q(P_S) \quad , \quad \pi : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(P_S) \quad (5.6)$$

are respectively a quotient Hopf algebra and a Hopf algebra epimorphism. Indeed, this provides a quantization of P_S , as a Poisson subgroup of G , namely $\mathcal{O}_q(P_S)$ is a quantum subgroup of G in the sense of Definition 3.3.

In sight of the above construction of $\mathcal{O}_q(G)$ and of the classical description of $\mathcal{O}(G/P_S)$ in (5.4), we define

$$\mathcal{O}_q(G/P_S)' := \bigoplus_{n \in \mathbb{N}} V_q(n\lambda)^* = \mathbb{k}_q\text{-span of } \{c_{f, v_\lambda}^{n\lambda}\}_{f \in V_q(n\lambda)^*, n \in \mathbb{N}} \quad (5.7)$$

where v_λ is a non-zero element of weight λ in $V_q(\lambda)$. Then, as the quantum matrix coefficients specialize to the classical ones, comparing (5.7) with (5.4) we see at once that

$$\mathcal{O}_q(G/P_S)' / (q-1) \mathcal{O}_q(G/P_S)' \cong \mathcal{O}(G/P_S) \quad (5.8)$$

so that $\mathcal{O}_q(G/P_S)'$ is a quantization, as a \mathbb{k}_q -module, of $\mathcal{O}(G/P_S)$.

We are now going to show that this $\mathcal{O}_q(G/P_S)'$ is in fact a quantum homogeneous space in the sense of Definition 3.18, in particular it can be realized as the space of semi-invariants inside $\mathcal{O}_q(G)$ with respect to a suitable quantum section. Indeed, we shall find $\mathcal{O}_q(G/P_S)' = \mathcal{O}_q(G/P_S)$ where the latter is the space of all semi-invariants (for a suitable quantum section) as in Definition 3.7 and Definition 3.18.

First of all, let λ be the dominant weight fixed above, and let v_λ be the (up to a scalar factor) uniquely determined non-zero element of weight λ in $V_q(\lambda)$. Fix a \mathbb{k}_q -basis $\{v_1, \dots, v_r\}$ of $V_q(\lambda)$ with $v_1 = v_\lambda$, and let $\{f_1, \dots, f_r\}$ be the dual basis of $V_q(\mu)^*$, for which we set $f_\lambda := f_1$.

Proposition 5.18. *The element $c_{f_\lambda, v_\lambda}^\lambda$ is a quantum section in $\mathcal{O}_q(G)$.*

Proof. Thanks to Proposition 3.13(b) we only need to show that $c_{f_\lambda, v_\lambda}^\lambda$ is a pre-quantum section, with respect to the setup of (5.6), i.e.

$$\Delta_\pi(c_{f_\lambda, v_\lambda}^\lambda) = c_{f_\lambda, v_\lambda}^\lambda \otimes \pi(c_{f_\lambda, v_\lambda}^\lambda) \quad .$$

But this follows at once from the identity (5.5), applied to the bases chosen above, once we notice in addition that $c_{f_i, v_\lambda}^\lambda \in I_q(P_S)$ for all $i \neq 1$. \square

Proposition 5.19. *The space of $c_{f_\lambda, v_\lambda}^\lambda$ -semi-invariants of degree 1 inside $\mathcal{O}_q(G)$, that is $\mathcal{O}_q(G/P_S)_1 := \left\{ f \in \mathcal{O}_q(G) \mid \Delta_\pi(f) = f \otimes \pi(c_{f_\lambda, v_\lambda}^\lambda) \right\}$, is just the \mathbb{k}_q -span of $\{c_{f, v_\lambda}^\lambda \mid f \in V_q(\lambda)^*\}$. In other words,*

$$\mathcal{O}_q(G/P_S)_1 = \mathbb{k}_q\text{-span of } \{c_{f, v_\lambda}^\lambda \mid f \in V_q(\lambda)^*\} \quad .$$

Proof. This again is immediate as before. Consider any \mathbb{k}_q -linear combination of several $c_{f,v}^\mu$'s which is semi-invariant of degree 1, with respect to the quantum section $d := c_{f_\lambda, v_\lambda}^\lambda$. We can assume these $c_{f,v}^\mu$'s to be linearly independent over \mathbb{k}_q , and so the semi-invariance of their linear combination as a whole also implies the semi-invariance of each of the $c_{f,v}^\mu$'s on its own.

Now, assume that a single matrix coefficient $c_{f,v}^\mu$ is semi-invariant of degree 1 (with respect to $c_{f_\lambda, v_\lambda}^\lambda$). Then (5.5) implies at once that $\mu = \lambda$. Moreover, choosing bases $\{v_1 = v_\lambda, v_2, \dots, v_r\}$ and $\{f_1 = f_\lambda, f_2, \dots, f_r\}$ as before Proposition 5.18, the identity (5.5) also gives

$$c_{f,v}^\lambda \otimes \pi(c_{f_\lambda, v_\lambda}^\lambda) = \Delta_\pi(c_{f,v}^\lambda) = c_{f, v_\lambda}^\lambda \otimes \pi(c_{f_\lambda, v}^\lambda) + \sum_{i=2}^r c_{f, v_i}^\lambda \otimes \pi(c_{f_i, v}^\lambda)$$

This forces $c_{f_i, v}^\lambda \in \text{Ker}(\pi) = I_q(P_S)$ for all $i > 1$, so that $v \in \mathbb{k}_q \cdot v_\lambda$, say $v = \kappa v_\lambda$ for some $\kappa \in \mathbb{k}$, whence eventually

$$c_{f,v}^\lambda = c_{f, \kappa v}^\lambda = \kappa c_{f, v}^\lambda \in \mathbb{k}_q\text{-span of } \{c_{f, v_\lambda}^\lambda \mid f \in V_q(\lambda)^*\} .$$

This proves that $\mathcal{O}_q(G/P_S)_1$ is indeed contained in the prescribed \mathbb{k}_q -span. The converse is clear, just reversing the previous argument. \square

Proposition 5.20. *The space of $c_{f_\lambda, v_\lambda}^\lambda$ -semi-invariants of degree n inside $\mathcal{O}_q(G)$, that is $\mathcal{O}_q(G/P_S)_n := \left\{ f \in \mathcal{O}_q(G) \mid \Delta_\pi(f) = f \otimes \pi\left((c_{f_\lambda, v_\lambda}^\lambda)^n\right) \right\}$, is just the \mathbb{k}_q -span of $\{c_{f, v_{n\lambda}}^{n\lambda} \mid f \in V_q(n\lambda)^*\}$. In other words,*

$$\mathcal{O}_q(G/P_S)_n = \mathbb{k}_q\text{-span of } \{c_{f, v_{n\lambda}}^{n\lambda} \mid f \in V_q(n\lambda)^*\} .$$

Proof. This follows from an argument which closely mimics the one used in the proof of Proposition 5.19. One takes into account, in addition, the following two remarks:

(a) the vector $v_\lambda^{\otimes n}$ has weight $n\lambda$ inside $V_q(\lambda)^{\otimes n}$; thus it can be canonically identified with a (non-zero) highest weight vector, say $v_{n\lambda}$, in $V_q(n\lambda)$, hence it can be chosen as $v_1 := v_{n\lambda}$, the first element of a suitable \mathbb{k}_q -basis of $V_q(n\lambda)$ to be used in that argument;

(b) with notation as above, the n -th power function $(c_{f_\lambda, v_\lambda}^\lambda)^n$ inside $\mathcal{O}_q(G)$ is nothing but a matrix coefficient again, namely $(c_{f_\lambda, v_\lambda}^\lambda)^n = c_{f_{n\lambda}, v_{n\lambda}}^{n\lambda}$.

These two remarks, drafted into an argument totally similar to the one used for Proposition 5.19, eventually yield the claim. \square

We are now ready for the main result of this subsection:

Theorem 5.21. *Let $\mathcal{O}_q(G/P_S)$ be defined as in Definition 3.7, with respect to the quantum section $d := c_{f_e, v_\lambda}^\lambda \in \mathcal{O}_q(G)$. Then $\mathcal{O}_q(G/P_S)$ is a quantum projective homogeneous space, namely, it is a quantization of $\mathcal{O}(G/P_S)$, in the sense of Definition 3.18.*

Proof. This follows at once by putting together the previous results, i.e. Propositions 5.18, 5.19 and 5.20, and the specialization formula (5.8). \square

Remark 5.22.

(1) With some extra work, one can also show that $\mathcal{O}_q(G/P_S)$ is generated — as a graded algebra — in degree 1, i.e. by $\mathcal{O}_q(G/P_S)_1$.

(2) Our last remark is that in this setup of quantum generalized flag varieties one can also apply the QDP, following the general recipe of §4.

Indeed, in [19], §3.4, it is noticed that the quantum section $d := c_{f_e, v_\lambda}^\lambda$ is a Ore element in $\mathcal{O}_q(G)$. Therefore, as pointed out in §4.1, we can define the localizations

$$\mathcal{O}_q(G/P_S) \left[(c_{f_e, v_\lambda}^\lambda)^{-1} \right] \subseteq \mathcal{O}_q(G) \left[(c_{f_e, v_\lambda}^\lambda)^{-1} \right] .$$

and we can then apply the QDP — according to §4 — to this setting.

5.4 The coisotropic case

One could argue whether the generality we are working with is necessary or not. In this section we will describe how to find families of coisotropic parabolic subgroups inside semisimple algebraic groups with the standard multiplicative Poisson structure.

Such families give rise to smooth Poisson bivectors on the projective quotients which cannot be obtained as quotient by Poisson parabolic subgroups. It would be therefore interesting to investigate whether in such examples it is possible to find and quantize a graded Poisson algebra associated to them as described in section 2.

We shall start with a low-dimensional example and describe in a very explicit manner the objects we are interested in and then we will generalize such results.

Let us consider the group $SL_2(\mathbb{C})$ with the standard Poisson algebraic structure, normalized as follows: if a, b, c, d are matrix coefficients in positions $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we let $\{a, b\} = ab$, $\{a, c\} = ac$, $\{b, d\} = bd$, $\{c, d\} = cd$, $\{b, c\} = 0$, $\{a, d\} = 2ad$ (this is the opposite normalization to that in [20]). We take the standard parabolic subgroup of upper triangular matrices

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{C} \right\}$$

This is a Poisson subgroup in $SL_2(\mathbb{C})$; thus, the quotient $\mathbb{P}^1\mathbb{C} \simeq SL_2(\mathbb{C})/P$ is endowed with the (homogenous) quotient smooth Poisson bivector π_0 .

Let us now consider the following element

$$g_\varepsilon := \begin{pmatrix} \sqrt{\varepsilon} & \sqrt{1-\varepsilon} \\ -\sqrt{1-\varepsilon} & \sqrt{\varepsilon} \end{pmatrix}, \quad \varepsilon \in [0, 1]$$

and let $P_\varepsilon := g_\varepsilon P g_\varepsilon^{-1}$. Then P_ε is defined inside the group $SL_2(\mathbb{C})$ by the equation:

$$\sqrt{\varepsilon(1-\varepsilon)}(a-d) = (\varepsilon-1)b + \varepsilon c \quad .$$

The infinitesimal generators of its Lie algebra are

$$\begin{aligned} H_\varepsilon &:= g_\varepsilon H g_\varepsilon^{-1} = (2\varepsilon-1)H - 2\sqrt{\varepsilon(1-\varepsilon)}(X^+ + X^-) \quad , \\ X_\varepsilon &:= g_\varepsilon X^+ g_\varepsilon^{-1} = \sqrt{\varepsilon(1-\varepsilon)}H + \varepsilon X^+ - (1-\varepsilon)X^- \quad . \end{aligned}$$

It is then easily verified, through the infinitesimal criterion of Proposition 2.2, that P_ε is coisotropic, because

$$\delta(H_\varepsilon) = H_\varepsilon \wedge H \quad , \quad \delta(X_\varepsilon) = X_\varepsilon \wedge H \quad .$$

This means that on $\mathbb{P}^1\mathbb{C}$ there is an induced Poisson bivector π_ε as quotient $SL_2(\mathbb{C})/P_\varepsilon$. That this Poisson bivector is different from π_0 follows considering the image of the diagonal subgroup of $SL_2(\mathbb{C})$, which induces a single 0-dimensional Poisson leaf with respect to π_0 and an \mathbb{S}^1 -family of 0-dimensional leaves with respect to π_ε .

This phenomenon, as said above, is not specific only of $\mathbb{P}^1\mathbb{C}$ but can be, for example, shown to take place for all complex Grassmannians.

Let G be a complex semisimple algebraic group and let K be its real compact form. Up to a factor ι , which is inessential in what follows, the standard Poisson structure on G can be identified with the one which is automatically defined on it when it is identified with the Drinfeld's double of K . Let now H be a coisotropic subgroup of K and let us consider the subgroup HK^* of G (here $K^* = AN$ is the Manin dual of K inside G). Then $P = HK^*$ is parabolic in G , $H = P \cap K$ and $K/H \simeq G/P$ as smooth manifolds. It can be shown quite easily that the coisotropy of K implies the coisotropy of P , and furthermore, via Theorem 4.1 in [7], that K/H and G/P are also Poisson diffeomorphic. Thus in order to check whether P is coisotropic it is enough to check whether $P \cap K$ is coisotropic w.r. to the standard Poisson structure on the compact real group K . There we can rely on results in [7], where a 1-parameter family of coisotropic subgroups $H_\varepsilon \subseteq SU(n)$ was given. Such subgroups induce a 1-parameter family of homogeneous Poisson quotients on complex Grassmannians.

References

- [1] F. Bonechi, N. Ciccoli, R. Giachetti, E. Sorace and M. Tarlini, *The coisotropic subgroup structure of quantum $SL(2, \mathbb{R})$* , Journ. Geom. Phys. **37**, 190–200 (2001).
- [2] A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics **126**, Springer-Verlag, New York-Heidelberg, 1991.
- [3] V. Chari, A. Pressley, *A guide to quantum groups*, Cambridge Press, 1994.
- [4] N. Ciccoli, *Quantization of Co-isotropic Subgroups*, Lett. Math. Phys. **42** (1997), 123–138.
- [5] N. Ciccoli, F. Gavarini, *A quantum duality principle for coisotropic subgroups and Poisson quotients*, Adv. Math. **199** (2006), 104–135.
- [6] N. Ciccoli, F. Gavarini, *A global quantum duality principle for subgroups and homogeneous spaces*, work in progress.
- [7] N. Ciccoli, A. J.-L. Sheu, *Covariant Poisson structures on complex Grassmannians*, Comm. Anal. Geom., **14** (2006), 443–474.

- [8] C. De Concini, D. Eisenbud, C. Procesi, *Young Diagrams and Determinantal Varieties*, *Invent. Mathematicæ* **56** (1980), 129–165.
- [9] R. Fiorese, *Quantization of the Grassmannian manifold*, *J. Algebra* **214** (1999), 418–447.
- [10] R. Fiorese, *A deformation of the big cell inside the Grassmannian manifold $G(r, n)$* , *Rev. Math. Phys.* **11** (1999), 25–40.
- [11] R. Fiorese *Quantum deformation of the flag variety* *Comm. Algebra*, Vol. 27, n. 11 (1999).
- [12] R. Fiorese, *Quantum coinvariant theory for the quantum special linear group and Quantum Schubert varieties*, *J. Algebra* **242** (2001), 433–446.
- [13] F. Gavarini, *The quantum duality principle*, *Ann. Inst. Fourier* **52** (2002), 809–834.
- [14] F. Gavarini, *The global quantum duality principle*, *Journal für die reine und angewandte Mathematik* **612** (2007), 17–33 — see also the expanded version <http://arxiv.org/abs/math/0303019> (2003).
- [15] F. Gavarini, *The global quantum duality principle: theory, examples, and applications*, <http://arxiv.org/abs/math/0303019> (2003).
- [16] K. Goodearl, T. Lenagan, *Quantized coinvariants at transcendental q* , in “Hopf algebras in non commutative geometry and physics”, 155–165, *Lecture Notes in Pure and Appl. Math.* **239**, Dekker, 2005.
- [17] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley Interscience, 1994.
- [18] R. Hartshorne, *Algebraic Geometry*, *Graduate Texts in Mathematics* **52**, Springer-Verlag, New York-Heidelberg, 1977.
- [19] S. Kolb, *The AS-Cohen-Macaulay property for quantum flag manifolds of minuscule weight*, *J. Algebra* (2008), to appear — see also <http://arxiv.org/abs/0707.1389> (2007).
- [20] L. Korogodsky, Y. Soibelman, *Algebras of functions on quantum groups, Part I*, *Math. Surv. and Monographs* **56**, A.M.S., Providence (RI), 1998.

- [21] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.
- [22] V. Lakshmibai, N. Reshetekhin *Quantum flag and Schubert schemes*, (Amherst, MA, 1990), 145–181, Contemp. Math., 134, Amer. Math. Soc., Providence, RI, (1992).
- [23] V. Lakshmibai, C. S. Seshadri, C. Musili, *Geometry of G/P . IV. Standard monomial theory for classical types*, Proc. Indian Acad. Sci. Sect. A Math. Sci. **88** (1979), 279–362.
- [24] J. H. Lu, *Multiplicative and affine Poisson structures on Lie groups*, Ph.D. thesis University of California, Berkeley, 1990 — see also <http://hkumath.hku.hk/~jhlu/thesis.tex>.
- [25] Y. Manin, *Topics in non commutative geometry*, M. B. Porter Lectures, Princeton University Press, Princeton, 1991.
- [26] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Mathematics **82**, AMS, Providence, 1993.
- [27] Y. S. Soibelman *On the quantum flag manifold*, Func. Ana. Appl. **25**, 225-227, (1992).
- [28] E. Taft, J. Towber, *Quantum deformation of flag schemes and Grassmann schemes. I. A q -deformation of the shape-algebra for $GL(n)$* , J. Algebra **142** (1991), 1–36.
- [29] S. Zakrzewski, *Poisson homogeneous spaces*, in: J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), *Quantum groups (Karpacz, 1994)*, 629–639, PWN, Warsaw, 1995.