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ON THE CHARACTERISTIC FUNCTION OF RANDOM VARIABLES ASSOCIATED WITH BOSON LIE ALGEBRAS

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Abstract. We compute the characteristic function of random variables defined as self-adjoint linear combinations of the Schrödinger algebra generators. We consider the extension of the method to infinite dimensional Lie algebras. We show that, unlike the second order (Schrödinger) case, in the third order (cube of a Gaussian random variable) case the method leads to a nonlinear infinite system of ODEs whose form is explicitly determined.

1. Introduction

It is known (see [5] and the bibliography therein) that, while the usual Heisenberg algebra and $sl(2;\mathbb{R})$ admit separately a continuum limit with respect to the Fock representation, the corresponding statement for the Lie algebra generated by these two, i.e. the Schrödinger algebra is false.

Stated otherwise: there is a standard way to construct a net of C^*–algebras each of which, intuitively speaking, is associated to all possible complex valued step functions defined in terms of a fixed finite partition of $\mathbb{R}$ into disjoint intervals (see [6]), but there are obstructions to the *natural extension* of the Fock representation to this net of C^*–algebras: these obstructions are called no-go theorems. The detailed analysis of these theorems is a deep problem relating the theory of representations of $*$–Lie algebras with the theory of infinitely divisible processes.

To formulate these connections in a mathematically satisfactory way is a problem, which requires the explicit knowledge of all the vacuum characteristic functions of the self-adjoint operators associated to the Fock representation of a given sub-algebra of the full oscillator algebra.

For the first order case these characteristic functions have been known for a long time and correspond to the standard Gaussian and Poisson distributions on $\mathbb{R}$.

For the full second order case they have been identified in [7] with the three remaining (i.e. in addition to Gaussian and Poisson) Meixner classes.

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For the Schrödinger algebra they were not known and it is intuitively obvious that they should form a family of characteristic functions interpolating among all the five Meixner classes: this is because the generators of the Schrödinger algebra are obtained by taking the union of the generators of the Heisenberg algebra with those of $sl(2;\mathbb{R})$.

It is also intuitively clear that the class of these characteristic functions cannot be reduced to those corresponding to the Meixner distributions because these are known to be infinitely divisible while, from the no-go theorem, one can deduce that some of the characteristic functions associated to the Schrödinger algebra cannot be infinitely divisible.

Our goal is to describe this latter class of characteristic functions. This will provide a deeper insight into the no-go theorems (which up to now have been proved by trial and error, showing that the scalar product, canonically associated to the Fock representation, is not positive definite) as well as a deeper understanding of the quantum decomposition of some classes of infinitely divisible random variables.

The first step towards achieving this goal is to calculate these characteristic functions. This is done by solving some systems of Riccati equations. This leads, in particular, to a representation of the Meixner characteristic functions in a unified form that we have not found in the literature (where there are many explicit unified forms for some classes of the Meixner distributions). In the last part of the paper we show that, by applying the same method to hermitian operators involving the cube of creators and annihilators, one obtains an infinite chain of coupled Riccati equations whose explicit solution at the moment is not known.

2. The Full Oscillator Algebra

Definition 2.1. If a and $a^\dagger$ are a Boson pair, i.e.

$$[a,a^\dagger]:=a\,a^\dagger-a^\dagger\,a=1$$

then

(i) the Heisenberg algebra $\mathcal{H}$ is the Lie algebra generated by $\{a,a^\dagger,1\}$

(ii) the $sl(2)$ algebra is the Lie algebra generated by

$$\{{a^\dagger}^2,a^2,a^\dagger\,a+\frac{1}{2}\}$$

(iii) the oscillator algebra $\mathcal{O}$ is the Lie algebra generated by

$$\{a,a^\dagger,a^\dagger\,a+\frac{1}{2},1\}$$

(iv) the Schrödinger algebra $\mathcal{S}$ is the Lie algebra generated by

$$\{a,a^\dagger,{a^\dagger}^2,a^2,a^\dagger\,a+\frac{1}{2},1\}$$

(v) the universal enveloping Heisenberg algebra $\mathcal{U}$ (containing $\mathcal{H}$, $sl(2)$, $\mathcal{O}$ and $\mathcal{S}$ as sub-algebras) is the Lie algebra generated by

$$\{{a^\dagger}^n\,a^k\ ;\ n,k=0,1,2,...\}$$

Defining duality by $(a)^* = a^\dagger$, we can view $\mathcal{H}$, $sl(2)$, $\mathcal{O}$, $\mathcal{S}$ and $\mathcal{U}$ as $*$-Lie algebras. In particular, $\mathcal{H}$, $sl(2)$, $\mathcal{O}$ and $\mathcal{S}$ are $*$-Lie sub-algebras of $\mathcal{U}$.

Remark 2.2. The reason for the additive term $\frac{1}{2}$, in the generators of $sl(2)$, $\mathcal{O}$ and $\mathcal{S}$, is explained in the remark following Proposition 2.4.

Lemma 2.3. *If a and $a^\dagger$ are a Boson pair then*

$$[a^\dagger\, a, a^\dagger] = a^\dagger\, ;\ [a, a^\dagger\, a] = a$$

$$[a^2, {a^\dagger}^2] = 2 + 4\, a^\dagger\, a\, ;\ [a^2, a^\dagger\, a] = 2\, a^2\, ;\ [a^\dagger\, a, {a^\dagger}^2] = 2\, {a^\dagger}^2$$

and

$$[a^2, a^\dagger] = 2\, a\, ;\ [a, {a^\dagger}^2] = 2\, a^\dagger$$

Proof. This is a well known fact that can be checked without difficulty. □

Proposition 2.4. *(Commutation relations in $\mathcal{S}$) Using the notation $S_0^1 = a^\dagger$, $S_1^0 = a$, $S_0^2 = {a^\dagger}^2$, $S_2^0 = a^2$, $S_1^1 = a^\dagger\, a + \frac{1}{2}$, $S_0^0 = 1$ we can write the commutation relations among the generators of $\mathcal{S}$ as*

$$[S_k^n, S_K^N] = (k\, N - K\, n)\ S_{k+K-1}^{n+N-1} \tag{2.1}$$

with $(S_k^n)^* = S_n^k$, for all $n, k, N, K \in \{0, 1, 2\}$ with $n + k \le 2$ and $N + K \le 2$.

Proof. The proof follows by using Lemma 2.3. □

Remark 2.5. The concise formula (2.1) for the commutation relations among the generators of $\mathcal{S}$, in particular the inclusion of $[S_2^0, S_0^2] = 4\, S_1^1$, was the reason behind the inclusion of the additive $\frac{1}{2}$ term in the definition of the generators of $\mathcal{S}$.

To describe the commutation relations in $\mathcal{U}$, using the notation $\binom{y,z}{x} = \binom{y}{x}\, z^{(x)}$, $\epsilon_{n,k} = 1 - \delta_{n,k}$ where $\delta_{n,k}$ is Kronecker's delta, $x^{(y)} = x(x-1)\cdots(x-y+1)$ for $x \ge y$, $x^{(y)} = 0$ for $x < y$, $x^{(0)} = 1$, we define

$$\theta_L(N, K; n, k) = H(L-1)\ \left(\epsilon_{K,0}\, \epsilon_{n,0} \binom{K, n}{L} - \epsilon_{k,0}\, \epsilon_{N,0} \binom{k, N}{L} \right) \tag{2.2}$$

where $H(x)$ is the Heaviside function (i.e., $H(x) = 1$ if $x \ge 0$ and $H(x) = 0$ otherwise). Notice that if L exceeds $(K \wedge n) \vee (k \wedge N)$ then $\theta_L(N, K; n, k) = 0$.

Proposition 2.6. *(Commutation relations in $\mathcal{U}$) Using the notation $B_y^x = {a^\dagger}^x\, a^y$, for all integers $n, k, N, K \ge 0$*

$$[B_K^N, B_k^n] = \sum_{L \ge 1} \theta_L(N, K; n, k)\, B_{K+k-L}^{N+n-L}$$

where $\theta_L(N, K; n, k)$ is as in (2.2).

Proof. The proof follows from a discretization of Lemma 2.3 of [5] (i.e. by eliminating the time indices t and s and replacing the Dirac delta function by 1) and is a consequence of the General Leibniz Rule, Proposition 2.2.2. of [11] . □

Definition 2.7. (The Heisenberg Fock space) The Heisenberg Fock space $\mathcal{F}$ is the Hilbert space completion of the linear span $\mathcal{E}$ of the set of exponential vectors $\{y(\lambda) = e^{\lambda\, a^\dagger}\,\Phi\,;\, \lambda \in \mathbb{C}\}$, where Φ is the vacuum vector such that $a\,\Phi = 0$ and $||\Phi|| = 1$, with respect to the inner product

$$\langle y(\lambda), y(\mu)\rangle = e^{\bar{\lambda}\,\mu}$$

$\mathcal{H}$, $\mathcal{O}$, $sl(2)$, $\mathcal{S}$ and $\mathcal{U}$ can all be represented as operators acting on the exponential vectors domain $\mathcal{E}$ of the Heisenberg Fock space $\mathcal{F}$ according to

$$a^{\dagger^n}\, a^k\, y(\lambda) = \lambda^k\, \frac{\partial^n}{\partial\, \epsilon^n}|_{\epsilon=0}\, y(\lambda+\epsilon)\ \ ;\ \ n,k = 0,1,2,...$$

If $s \in \mathbb{R}$ and X is a self-adjoint operator on the Heisenberg Fock space $\mathcal{F}$ then, through Bochner's theorem, $\langle \Phi, e^{i\,s\,X}\,\Phi\rangle$ can be viewed as the characteristic function (i.e. the Fourier transform of the corresponding probability measure) of a classical random variable whose quantum version is X.

The differential method (see [11], [10] and Lemma 3.3 below) for the computation of $\langle \Phi, e^{i\,s\,X}\,\Phi\rangle$ relies on splitting the exponential $e^{s\,X}$ into a product of exponentials of the Lie algebra generators. In the case of a Fock representation, where the Lie algebra generators are divided into creation (raising) operators, annihilation (lowering) operators that kill the vacuum vector Φ, and conservation (number) operators having Φ as an eigenvector, the exponential $e^{s\,X}\,\Phi$ is split into a product of exponentials of creation operators.

The coefficients of the creation operators appearing in the product exponentials are, in general, functions of s satisfying certain ordinary differential equations (ODEs) and vanishing at zero. In the case of a finite dimensional Lie algebra (see section 3) the ODEs can be solved explicitly and the characteristic function of the random variable in consideration can be explicitly determined.

For infinite dimensional Lie algebras such as, for example, the sub-algebra of $\mathcal{U}$ generated by a^3 and $a^{\dagger^3}$, the situation is different. As illustrated in section 5, the ODEs form an infinite non-linear system which, in general, is hard (if at all possible) to solve explicitly.

3. Schrödinger Stochastic Processes

In this section, using the notation of Proposition 2.4, we compute the characteristic function $\langle \Phi, e^{i\,s\,V}\,\Phi\rangle$, $s \in \mathbb{R}$, of the quantum random variable

$$V = a\,S_0^1 + \bar{a}\,S_1^0 + b\,S_0^2 + \bar{b}\,S_2^0 + \lambda\;S_0^0 + \nu\,S_1^1$$

where $a, b \in \mathbb{C}$ with $b \neq 0$ (the case $b = 0$ is discussed in the remarks following Proposition 3.4) and $\lambda, \nu \in \mathbb{R}$ with $4\,|b|^2 - \nu^2 \neq 0$.

The above form of V is the extention of the forms considered in [2]-[4] to the full Schrödinger algebra. The splitting lemmas for the Heisenberg and $sl(2)$ Lie algebras can be found in [11]. An analytic proof of the splitting lemma for $sl(2)$ can be found in [8].

Remark 3.1. Here, and in what follows, the terms *splitting* and *disentanglement* refer to the expression of the exponential of a linear combination of operators as a product of exponentials.

Lemma 3.2. *For all $a, b \in \mathbb{C}$,*

(i)

$$S_2^0\, e^{a\, S_0^2} = 4\, a^2\, S_0^2\, e^{a\, S_0^2} + 4\, a\, e^{a\, S_0^2}\, S_1^1 + e^{a\, S_0^2}\, S_2^0$$

(ii)

$$S_2^0\, e^{b\, S_0^1}\, \Phi = b^2\, e^{b\, S_0^1}\, \Phi$$

(iii)

$$S_1^0\, e^{a\, S_0^2} = 2\, a\, S_0^1\, e^{a\, S_0^2} + e^{a\, S_0^2}\, S_1^0$$

(iv)

$$S_1^1\, e^{a\, S_0^1} = a\, S_0^1\, e^{a\, S_0^1} + e^{a\, S_0^1}\, S_1^1$$

(v)

$$S_1^1\, e^{a\, S_0^1}\, \Phi = \left(a\, S_0^1 + \frac{1}{2}\right)\, e^{a\, S_0^1}\, \Phi$$

(vi)

$$S_1^1\, e^{b\, S_0^2} = 2\, b\, S_0^2\, e^{b\, S_0^2} + e^{b\, S_0^2}\, S_1^1$$

(vii)

$$S_1^1\, e^{b\, S_0^2}\, \Phi = \left(2\, b\, S_0^2 + \frac{1}{2}\right)\, e^{b\, S_0^2}\, \Phi$$

Proof. The proof of (i) through (iii) can be found in [2]. The proof of (iv) and (vi) follows, respectively, from Propositions 2.4.2 and 3.2.1 of [11] with, in the notation of [11] , $D = S_1^0$, $X = S_0^1$, $X\, D = S_1^1 - \frac{1}{2}$, $H = t\, 1$, and $f(X) = e^{a\, X}$ for (iv), and $S_0^2 = 2\, R$, $S_2^0 = 2\, \Delta$, $S_1^1 = \rho$, and $f(R) = e^{b\, R}$ for (vi). Finally, (v) and (vii) follow from the fact that $S_1^1\, \Phi = \frac{1}{2}$. □

The following lemma is the extention of the splitting lemmas of [1]-[3] to the full Schrödinger algebra.

Lemma 3.3. *For all $s \in \mathbb{C}$*

$$e^{s\, V}\, \Phi = e^{w_1(s)\, S_0^2}\, e^{w_2(s)\, S_0^1}\, e^{w_3(s)}\, \Phi$$

where, letting

$$K = \sqrt{4\, |b|^2 - \nu^2}\ ;\ L = \arctan\left(\frac{\nu}{K}\right)\ ;\ \gamma = -\frac{\bar{a}}{2\, \bar{b}}$$

$$\alpha = \frac{a}{K} + \gamma\, \tan\, L\ ;\ \beta = -\left(\gamma\, \cos\, L + \alpha\, \sin\, L\right)$$

we have that

$$w_1(s) = \frac{K\, \tan\, (K\, s + L) - \nu}{4\, \bar{b}} \tag{3.1}$$

$$w_2(s) = \alpha\, \tan\, (K\, s + L) + \beta\, \sec\, (K\, s + L) + \gamma \tag{3.2}$$

and

$$\begin{aligned} w_3(s) = \{ & c_0 + c_1\, s + c_2\, \ln\, (\cos\, (K\, s + L)) \\ & + c_3\, \ln\, (\sec\, (K\, s + L) + \tan\, (K\, s + L)) \\ & + c_4\, \tan\, (K\, s + L) + c_5\, \sec\, (K\, s + L)\} \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
c_1 &= \lambda+\bar a\,\gamma+\bar b\,(\gamma^2-\alpha^2)\\
c_2 &= -\left(\frac{\bar a\,\alpha+2\,\gamma\,\alpha\,\bar b}{K}+\frac{1}{2}\right)\\
c_3 &= \frac{\bar a\,\beta+2\,\beta\,\gamma\,\bar b}{K}\\
c_4 &= \frac{\bar b\,(\beta^2+\alpha^2)}{K}\\
c_5 &= \frac{2\,\alpha\,\beta\,\bar b}{K}\\
c_0 &= -(c_2\ \ln\ (\cos\ L)+c_3\ \ln\ (\sec\ L+\tan\ L)\\
&\quad +c_4\ \tan\ L+c_5\ \sec\ L)
\end{aligned}$$

Proof. We will show that $w_1(s), w_2(s)$ and $w_3(s)$ satisfy the differential equations

$$\begin{aligned}
w_1'(s) &= 4\,\bar b\,w_1(s)^2+2\,\nu\,w_1(s)+b\\
w_2'(s) &= (4\,\bar b\,w_1(s)+\nu)\,w_2(s)+a+2\,\bar a\,w_1(s)\\
w_3'(s) &= \lambda+\bar a\,w_2(s)+2\,\bar b\,w_1(s)+\bar b\,w_2(s)^2+\frac{\nu}{2}
\end{aligned}$$

with $w_1(0)=w_2(0)=w_3(0)=0$, whose solutions are given by (3.1), (3.2) and (3.3) respectively. We remark that the differential equations defining $w_1(s)$ and $w_2(s)$ are, respectively, of Riccati and linear type. Moreover, unlike the infinite dimensional case treated in section 5, the ODE for w_1 does not involve w_2 and w_3.

So let

$$E\,\Phi := e^{s\,(b\,S_0^2+\bar b\ S_2^0+a\,S_0^1+\bar a\,S_1^0+\lambda\,S_0^0+\nu\,S_1^1)}\,\Phi \tag{3.4}$$

Then, by the disentanglement assumption,

$$E\,\Phi = e^{w_1(s)\,S_0^2}\,e^{w_2(s)\,S_0^1}\,e^{w_3(s)}\,\Phi \tag{3.5}$$

where $w_i(0)=0$ for $i\in\{1,2,3\}$. Then, (3.5) implies that

$$\frac{\partial}{\partial s}\,E\,\Phi = \left(w_1'(s)\,S_0^2+w_2'(s)\,S_0^1+w_3'(s)\right)\,E\,\Phi \tag{3.6}$$

and, since by Lemma 3.2,

$$\begin{aligned}
S_2^0\,E\,\Phi &= (4\,w_1(s)^2\,S_0^2+4\,w_1(s)\,w_2(s)\,S_0^1\\
&\quad +2\,w_1(s)+w_2(s)^2)\,E\,\Phi
\end{aligned}\tag{3.7}$$

$$S_1^0\,E\,\Phi = \left(2\,w_1(s)\,S_0^1+w_2(s)\right)\,E\,\Phi \tag{3.8}$$

and

$$S_1^1\,E\,\Phi = \left(2\,w_1(s)\,S_0^2+w_2(s)\,S_0^1+\frac{1}{2}\right)\,E\,\Phi \tag{3.9}$$

from (3.4) we also obtain

$$\begin{aligned}
\tfrac{\partial}{\partial s}\,E\,\Phi &= ((b+4\,\bar b\,w_1(s)^2+2\,\nu\,w_1(s))\,S_0^2\\
&\quad +(a+2\,\bar a\,w_1(s)+4\,\bar b\,w_1(s)\,w_2(s)+\nu\,w_2(s))\,S_0^1\\
&\quad +\lambda+\bar a\,w_2(s)+2\,\bar b\,w_1(s)+\bar b\,w_2(s)^2+\tfrac{\nu}{2})\,E\,\Phi
\end{aligned}\tag{3.10}$$

From (3.6) and (3.10), by equating coefficients of S_0^2, S_0^1 and 1, we obtain (3.1), (3.2) and (3.3) thus completing the proof. □

Proposition 3.4. *(Characteristic Function of V) In the notation of Lemma 3.3, for all $s \in \mathbb{R}$*

$$\begin{aligned}\langle \Phi, e^{i\,s\,V}\,\Phi\rangle &= \exp\left(\{c_0 + i\,c_1\,s + c_2\,\ln\,(\cos\,(K\,i\,s + L))\right.\\ &\quad + c_3\,\ln\,(\sec\,(K\,i\,s + L) + \tan\,(K\,i\,s + L))\\ &\quad \left.+ c_4\,\tan\,(K\,i\,s + L) + c_5\,\sec\,(K\,i\,s + L)\}\right)\end{aligned}$$

Proof. Using the fact that for any $z \in \mathbb{C}$

$$e^{z\,S_2^0}\,\Phi = e^{z\,S_1^0}\,\Phi = \Phi$$

by Lemma 3.3 we have

$$\begin{aligned}\langle \Phi, e^{s\,V}\,\Phi\rangle &= \langle \Phi, e^{w_1(s)\,S_0^2}\,e^{w_2(s)\,S_0^1}\,e^{w_3(s)}\,\Phi\rangle\\ &= \langle e^{\bar{w}_2(s)\,S_1^0}\,e^{\bar{w}_1(s)\,S_2^0}\,\Phi, e^{w_3(s)}\,\Phi\rangle\\ &= \langle \Phi, e^{w_3(s)}\,\Phi\rangle\\ &= e^{w_3(s)}\,\langle \Phi, \Phi\rangle\\ &= e^{w_3(s)}\\ &= \exp\left(\{c_0 + c_1\,s + c_2\,\ln\,(\cos\,(K\,s + L))\right.\\ &\quad + c_3\,\ln\,(\sec\,(K\,s + L) + \tan\,(K\,s + L))\\ &\quad \left.+ c_4\,\tan\,(K\,s + L) + c_5\,\sec\,(K\,s + L)\}\right)\end{aligned}$$

and the formula for the characteristic function of V is obtained by replacing s by $i\,s$ where $s \in \mathbb{R}$. □

Remark 3.5. In the well known (see [13] and [11]) Gaussian case $a = 1, b = \lambda = \nu = 0$, the differential equations for $w_1(s), w_2(s)$ and $w_3(s)$ in the proof of Lemma 3.3 are greatly simplified and yield

$$w_1(s) = 0\ ;\ w_2(s) = s\ ;\ w_3(s) = \frac{s^2}{2}$$

which implies that

$$\langle \Phi, e^{i\,s\,V}\,\Phi\rangle = \langle \Phi, e^{i\,s\,(S_0^1 + S_1^0)}\,\Phi\rangle = e^{-\frac{s^2}{2}} \tag{3.11}$$

i.e. $V = S_0^1 + S_1^0$ is a Gaussian random variable.

Remark 3.6. In the case when $a = 0$ we find that $\alpha = \beta = \gamma = 0$, $c_1 = \lambda$, $c_2 = -\frac{1}{2}$, $c_3 = c_4 = c_5 = 0$, $c_0 = \frac{1}{2}\,\ln\,(\cos\,L)$ and so

$$w_3(s) = \frac{1}{2}\,\ln\,(\cos\,L) - \frac{1}{2}\,\ln\,(\cos\,(K\,s + L)) + \lambda\,s$$

Therefore, by Proposition 3.4, assuming that $\cos\,L \neq 0$, for all $s \in \mathbb{C}$ such that $\cos\,(i\,K\,s + L) \neq 0$, the characteristic function of $V = b\,S_0^2 + \bar{b}\,S_2^0 + \lambda\,S_0^0 + \nu\,S_1^1$ is

$$\langle \Phi, e^{i\,s\,V}\,\Phi\rangle = (\cos\,L)^{1/2}\,\left(\sec\,(i\,K\,s + L)\right)^{1/2}\,e^{i\,\lambda\,s}$$

which, in the case $\lambda = 0$, is the characteristic function of a continuous binomial random variable (see [11]).

Remark 3.7. If $b=0$ then the ODEs for w_1, w_2, w_3 in the proof of Lemma 3.3 take the form

$$\begin{aligned} w_1'(s) &= 2\,\nu\, w_1(s) \\ w_2'(s) &= \nu\, w_2(s) + a + 2\,\bar{a}\, w_1(s) \\ w_3'(s) &= \lambda + \bar{a}\, w_2(s) + \frac{\nu}{2} \end{aligned}$$

with $w_i(0)=0$ for $i \in \{1,2,3\}$. Therefore, for $\nu \neq 0$

$$\begin{aligned} w_1(s) &= 0 \\ w_2(s) &= \frac{a}{\nu}\,(e^{\nu\, s}-1) \\ w_3(s) &= \frac{|a|^2}{\nu^2}\,(e^{\nu\, s}-1) + \left(\lambda + \frac{\nu}{2} + \frac{|a|^2}{\nu}\right) s \end{aligned}$$

Therefore, the characteristic function of $V = a\,S_0^1 + \bar{a}\,S_1^0 + \lambda\; S_0^0 + \nu\,S_1^1$ is

$$\langle \Phi, e^{i\,s\,V}\,\Phi\rangle = e^{\frac{|a|^2}{\nu^2}\,(e^{i\,\nu\,s}-1)+i\,\left(\lambda+\frac{\nu}{2}+\frac{|a|^2}{\nu}\right)\,s}$$

which, for $\lambda + \frac{\nu}{2} + \frac{|a|^2}{\nu} = 0$, reduces to the characteristic function of a Poisson random variable (see [13] and also Proposition 5.2.3 of [11]).

4. Random Variables in $\mathcal{U}$: The General Scheme

In analogy with section 3, using the notation $B_k^n = {a^\dagger}^n\, a^k$, we consider quantum random variables of the form

$$W = \sum_{n,k\geq 0} \left(c_{n,k}\,B_k^n + \bar{c}_{n,k}\,B_n^k\right)$$

where $c_{n,k} \in \mathbb{C}$.

Extending the framework and terminology of [11] and [12] to the infinite dimensional case, the group element (in terms of coordinates of the first kind)

$$e^{s\,W} = e^{s\,\sum_{n,k\geq 0}\left(c_{n,k}\,B_k^n + \bar{c}_{n,k}\,B_n^k\right)}$$

can be put, through an appropriate splitting lemma, in the form of coordinates of the second kind

$$e^{s\,W} = \prod_{n,k\geq 0} e^{f_{n,k}(s)\,B_k^n}\,e^{\bar{f}_{n,k}(s)\,B_n^k}$$

for some functions $f_{n,k}(s)$. Since $B_k^n\,\Phi = 0$ for all $k \neq 0$, after several commutations, we find that

$$e^{s\,W}\Phi = \prod_{n\geq 0} e^{w_n(s)\,B_0^n}\,\Phi$$

for some functions $w_n(s)$ (we may call them the vacuum coordinates of the second kind). Therefore, as in Proposition 3.4,

$$\langle \Phi, e^{i\,s\,W}\,\Phi\rangle = e^{w_0(i\,s)}$$

For finite-dimensional Lie algebras (see, for example, Lemma 3.3) solving the differential equations that define the w_n's is relatively easy. As shown in the next section, in the case of an infinite dimensional Lie algebra the situation is much more complex.

5. Example: The Cube of a Gaussian Random Variable

To illustrate the method described in section 4, we will consider the characteristic function $\langle \Phi, e^{i\,s\,W}\,\Phi\rangle$ of the quantum random variable

$$W = (a + a^\dagger)^3 = (B_1^0 + B_0^1)^3$$

By (3.11) W is the cube of a Gaussian random variable. Using $a\,a^\dagger = 1 + a^\dagger\,a$ we find that

$$(a + a^\dagger)^2 = a^2 + {a^\dagger}^2 + 2\,a^\dagger\,a + 1$$

and

$$(a + a^\dagger)^3 = a^3 + 3\,a^\dagger + 3\,{a^\dagger}^2\,a + 3\,a + 3\,a^\dagger\,a^2 + {a^\dagger}^3$$

Thus

$$W = a^3 + 3\,a^\dagger + 3\,{a^\dagger}^2\,a + 3\,a + 3\,a^\dagger\,a^2 + {a^\dagger}^3$$

Lemma 5.1. *For all analytic functions f and for all $w_i \in \mathbb{C}$, $i = 1, 2, 3, 4$*

$$[a, f(a^\dagger)] = f'(a^\dagger) \tag{5.1}$$

$$[f(a), a^\dagger] = f'(a) \tag{5.2}$$

Moreover, if $a\,\Phi = 0$ then for $n \geq 1$

$$a^n\,f(a^\dagger)\,\Phi = f^{(n)}(a^\dagger)\,\Phi \tag{5.3}$$

where $f^{(n)}$ denotes the n-th derivative of f.

Proof. The proof of (5.1) and (5.2) can be found in [11]. To prove (5.3) we notice that, by Proposition 2.6, for $n \geq 1$ and $k \geq 0$

$$[a^n, {a^\dagger}^k] = \sum_{l \geq 1} \binom{n}{l} k^{(l)}\ {a^\dagger}^{k-l}\ a^{n-l}$$

Thus, assuming that $f(a^\dagger) = \sum_{k\geq 0} c_k\,{a^\dagger}^k$, we have that

$$\begin{aligned} a^n\,f(a^\dagger)\,\Phi &= [a^n, f(a^\dagger)]\,\Phi \\ &= \sum_{k\geq 0} c_k\,[a^n, {a^\dagger}^k]\,\Phi \\ &= \sum_{k\geq 0} c_k \sum_{l\geq 1} \binom{n}{l} k^{(l)}\ {a^\dagger}^{k-l}\ a^{n-l}\,\Phi \\ &= \sum_{k\geq 0} c_k\,k^{(n)}\ {a^\dagger}^{k-n}\,\Phi \\ &= f^{(n)}(a^\dagger)\,\Phi \end{aligned}$$

□

Corollary 5.2. *For an analytic function g and $n \in \{1,2,3\}$,*

$$a^n\, e^{g(a^\dagger)}\,\Phi = G_n(a^\dagger)\, e^{g(a^\dagger)}\,\Phi$$

where

$$\begin{aligned} G_1(a^\dagger) &= g'(a^\dagger) \\ G_2(a^\dagger) &= g''(a^\dagger) + g'(a^\dagger)^2 \\ G_3(a^\dagger) &= g'''(a^\dagger) + 3\,g'(a^\dagger)\,g''(a^\dagger) + g'(a^\dagger)^3 \end{aligned}$$

Proof. The proof follows from Lemma 5.1 by taking $f(a^\dagger) = e^{g(a^\dagger)}$. □

Corollary 5.3. *Let $g(a^\dagger) = \sum_{k=0}^{\infty} w_k\, {a^\dagger}^k$. Then, for $n \in \{1,2,3\}$,*

$$a^n\, e^{g(a^\dagger)}\,\Phi = G_n(a^\dagger)\, e^{g(a^\dagger)}\,\Phi$$

where $G_1(a^\dagger) = \sum_{k=0}^{\infty} (k+1)\, w_{k+1}\, {a^\dagger}^k$ and

$$\begin{aligned} G_2(a^\dagger) &= \sum_{k=0}^{\infty} (k+1)\,(k+2)\, w_{k+2}\, {a^\dagger}^k \\ &\quad + \sum_{k,m=0}^{\infty} (k+1)\,(m+1)\, w_{k+1}\, w_{m+1}\, {a^\dagger}^{k+m} \end{aligned}$$

$$G_3(a^\dagger) = \sum_{k=0}^{\infty} (k+1)\,(k+2)\,(k+3)\, w_{k+3}\, {a^\dagger}^k$$
$$+3\sum_{k,m=0}^{\infty} (k+1)\,(m+1)\,(m+2)\, w_{k+1}\, w_{m+2}\, {a^\dagger}^{k+m}$$
$$+\sum_{k,m,\rho=0}^{\infty} (k+1)\,(m+1)\,(\rho+1)\, w_{k+1}\, w_{m+1}\, w_{\rho+1}\, {a^\dagger}^{k+m+\rho}$$

Proof. The proof follows directly from Corollary 5.2. □

Lemma 5.4. *Let $s \in \mathbb{C}$. Then*

$$e^{s\,W}\,\Phi = \prod_{n=0}^{\infty} e^{w_n(s)\, {a^\dagger}^n}\,\Phi = \prod_{n=0}^{\infty} e^{w_n(s)\, B_0^n}\,\Phi$$

where $w_n(0) = 0$ for all $n \geq 0$ and the w_n satisfy the nonlinear infinite system of disentanglement ODEs

$$\begin{aligned} w_0'(s) &= w_1(s)^3 + 3\,w_1(s)\,(1+2\,w_2(s)) + 6\,w_3(s) \\ w_1'(s) &= 3\,w_1(s)^2\,(1+2\,w_2(s)) + 18\,w_3(s)\,w_1(s) \\ &\quad +12\,w_2(s)^2 + 12\,w_2(s) + 24\,w_4(s) + 3 \\ w_2'(s) &= 12\,w_2(s)^2\,w_1(s) + (54\,w_3(s) + 12\,w_1(s))\,w_2(s) \\ &\quad +3\,w_1(s) + 27\,w_3(s) + 36\,w_1(s)\,w_4(s) \\ &\quad +9\,w_1(s)^2\,w_3(s) + 60\,w_5(s) \\ w_3'(s) &= 54\,w_3(s)^2 + (30\,w_1(s)\,w_2(s) + 18\,w_1(s))\,w_3(s) \\ &\quad +24\,w_4(s)\,w_2(s) + 60\,w_1(s)\,w_5(s) + 72\,w_2(s)\,w_4(s) \\ &\quad +12\,w_1(s)^2\,w_4(s) + 120\,w_6(s) + 6\,w_2(s) + 48\,w_4(s) \\ &\quad +12\,w_2(s)^2 + 1 \end{aligned}$$

and for $n \geq 4$,

$$\begin{aligned} w_n'(s) &= (n+1)\,(n+2)\,(n+3)\,w_{n+3}(s) \\ &+3\,(n-1)\,w_{n-1}(s) + 3\,(n+1)^2\,w_{n+1}(s) \\ &+3 \sum_{\substack{k,m\geq 0\\ k+m=n}} (k+1)\,(m+1)\,(m+2)\,w_{k+1}(s)\,w_{m+2}(s) \\ &+ \sum_{\substack{k,m,\rho\geq 0\\ k+m+\rho=n}} (k+1)\,(m+1)\,(\rho+1)\,w_{k+1}\,w_{m+1}\,w_{\rho+1} \\ &+3 \sum_{\substack{k,m\geq 0\\ k+m=n-1}} (k+1)\,(m+1)\,w_{k+1}(s)\,w_{m+1}(s) \end{aligned}$$

Proof. As in the proof of Lemma 3.3, let

$$E\,\Phi := e^{s\,(a^3+3\,a^\dagger+3\,{a^\dagger}^2\,a+3\,a+3\,a^\dagger\,a^2+{a^\dagger}^3)}\,\Phi. \tag{5.4}$$

Then by the disentanglement assumption for $e^{s\,W}\,\Phi$, we also have that

$$E\,\Phi = \prod_{n=0}^{\infty} e^{w_n(s)\,{a^\dagger}^n}\,\Phi \tag{5.5}$$

where $w_n(0)=0$ for all $n \geq 0$. Then, (5.5) implies that

$$\frac{\partial}{\partial s}\,E\,\Phi = \sum_{n=0}^{\infty} w_n'(s)\,{a^\dagger}^n\,E\,\Phi \tag{5.6}$$

and, using Corollary 5.3, from (5.4) we also obtain

$$\begin{aligned} \frac{\partial}{\partial s}\,E\,\Phi &= (a^3+3\,a^\dagger+3\,{a^\dagger}^2\,a+3\,a+3\,a^\dagger\,a^2+{a^\dagger}^3)\,E\,\Phi \\ &= \{\sum_{k\geq 0}(k+1)\,(k+2)\,(k+3)\,w_{k+3}(s)\,{a^\dagger}^k \\ &+3\sum_{k,m\geq 0}(k+1)\,(m+1)\,(m+2)\,w_{k+1}(s)\,w_{m+2}(s)\,{a^\dagger}^{k+m} \\ &+\sum_{k,m,\rho=0}^{\infty}(k+1)\,(m+1)\,(\rho+1)\,w_{k+1}\,w_{m+1}\,w_{\rho+1}\;{a^\dagger}^{k+m+\rho} \\ &+3\,a^\dagger+3\sum_{k\geq 0}(k+1)\,w_{k+1}(s)\,{a^\dagger}^{k+2} \\ &+3\sum_{k\geq 0}(k+1)\,w_{k+1}(s)\,{a^\dagger}^k+3\sum_{k\geq 0}(k+1)\,(k+2)\,w_{k+2}(s)\,{a^\dagger}^{k+1} \\ &+3\sum_{k,m\geq 0}(k+1)\,(m+1)\,w_{k+1}(s)\,w_{m+1}(s)\,{a^\dagger}^{k+m+1} \\ &+{a^\dagger}^3\}\,E\,\Phi \end{aligned} \tag{5.7}$$

and the differential equations defining the w_n's are obtained from (5.6) and (5.7) by equating coefficients of the powers of $a^\dagger$. □

Remark 5.5. As in the finite dimensional case, w_0 is determined by straightforward integration. However, unlike the finite-dimensional (Schrödinger) case of Lemma 3.3, in the infinite-dimensional case the disentanglement ODEs are coupled, with the ODE for each w_n' depending on $w_1, ..., w_{n+3}$. The ODEs for w_1, w_2, and w_3 are of pseudo (due to coupling)-Riccati type.

Proposition 5.6. *(Characteristic function of W) For all $s \in \mathbb{R}$*

$$\langle \Phi, e^{i\,s\,W}\,\Phi\rangle = e^{w_0(i\,s)}$$

where w_0 is as in Lemma 5.4.

Proof. The proof is similar to that of Proposition 3.4. □

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