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A REMARK ON TRACE PROPERTIES OF K-CYCLES

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ABSTRACT. In this paper we discuss trace properties of d^+-summable K-cycles considered by A. Connes in [6]. More precisely we give a proof of a trace theorem on the algebra $\mathcal{A}$ of a K-cycle stated in [6], namely we show that a natural functional on $\mathcal{A}$ is a trace functional. Then we discuss whether this functional gives a trace on the whole universal graded differential algebra $\Omega(\mathcal{A})$. On the one hand we prove that the regularity conditions on K-cycles considered in [6] imply the trace property on $\Omega(\mathcal{A})$. On the other hand, by constructing an explicit counterexample, we remark that the sole K-cycle assumption is not sufficient for such a property to hold.

KEYWORDS: *Noncommutative geometry, Dirac operator, traces on K-cycles, singular traces.*

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INTRODUCTION

A major role in A. Connes' non commutative geometry is played by the concept of d^+-summable K-cycle $(\mathcal{A}, D, \mathcal{H})$. Such an object generalizes, in an operator-theoretical framework, classical Riemannian geometry on a compact space. There, given a smooth, closed, compact, Riemannian spin-manifold (M, g), one has the Riemannian measure $\mathrm{d}m_g$ over M and the Hilbert space $\mathcal{H} = L^2(M, S, \mathrm{d}m_g)$ of the square-integrable sections of the spinor bundle S on M. On $\mathcal{H}$ there is a natural action of the $*$-algebra $\mathcal{A}$ of C^∞-functions (by pointwise multiplication of sections) and the action of the Dirac operator $D = \partial_M$, associated with a fixed Riemannian connection over S. (In order to simplify the discussion, we shall assume $\mathrm{Ker}(\partial_M) = \{0\}$.) A key relation among these objects is that D implements a $B(\mathcal{H})$-valued derivation on $\mathcal{A}$, i.e. the commutator $[D, f]$ between

D and any element $f \in \mathcal{A}$ seen as an operator on $\mathcal{H}$ is a bounded operator (given by the Clifford multiplication by the external derivative of f). In this operator-theoretical framework the dimension of the manifold M can be detected looking at the eigenvalue distribution of the Laplacian operator D^2. In fact, the celebrated Weyl's theorem implies $\sup \frac{1}{\log n}\sigma_n(|D|^{-d}) < \infty$ (where d is the dimension of M and $\sigma_n(T)$ denotes the sum of the first n eigenvalues $\mu_k(|T|)$ of the compact operator $|T|$). In the language of the normed ideals this can be expressed saying that $|D|^{-d}$ belongs to the dual $\mathcal{L}^{1^+}(\mathcal{H})$ of the Macaev ideal. The trace-theorem of A. Connes ([5]) allows to reconstruct, from the data $(\mathcal{A}, D, \mathcal{H})$, the Riemannian measure of M:

$$\int_M f \, dm_g = c(d) \cdot \tau_\omega(f|D|^{-d})$$

where τ_ω is the Dixmier trace (a non normal trace over $B(\mathcal{H})$ which is finite precisely on $\mathcal{L}^{1^+}(\mathcal{H})$; see below) and $c(d)$ is a constant depending only on d.

A. Connes stated that for any K-cycle $(\mathcal{A}, D, \mathcal{H})$ the state $\varphi(a) := \tau_\omega(|D|^{-d}a)$, defined on $B(\mathcal{H})$ gives a hypertrace on $\mathcal{A}$ ([6], Chapter IV.2, Proposition 15). He also discussed some consequences of this result on the hermitian structures and the action functional associated with the K-cycle ([6], Chapter VI.1, Proposition 5). To our knowledge, the proof of the previous statement has not been published. On the contrary, in some published papers such result appeared as an assumption (see e.g. [11]). Therefore we think it is worthwhile to give a proof of it (Theorem 1.3 of the next section).

The functional φ gives rise to a state on the whole universal graded differential algebra $\Omega(\mathcal{A})$. In the commutative case this is the trace on the Clifford bundle of the manifold M, hence it is natural to ask whether the trace property for such a state holds for the algebra $\Omega(\mathcal{A})$ associated with any K-cycle.

The preceding result does not hold in full generality; indeed, Section 2 is devoted to the construction of a counterexample of such a statement. However, we shall prove the trace property on $\Omega(\mathcal{A})$ under natural regularity conditions (Theorem 1.7). These conditions have been proposed by Connes ([6], p. 546; cf. also [6], Theorem 8, p. 308) as the analogue of differentiability conditions for functions and differential forms in the non commutative setting.

1. TRACES CONSTRUCTED BY K-CYCLES

We start recalling the definition of K-cycle given by A. Connes ([6]). Let $B(\mathcal{H})$ be the space of bounded linear operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$ and consider the ideal

$$\mathcal{L}^{1^+} := \left\{ a \in B(\mathcal{H}) : a \text{ is compact and } \left(\frac{\sigma_n(a)}{\log n}\right)_{n\in\mathbb{N}} \text{ is a bounded sequence} \right\}$$

where $\sigma_n(a) = \sum_{k=0}^{n-1} \mu_k(a)$ and $\{\mu_k(a)\}_k$ is the sequence of singular values of a counted with multiplicities and arranged in a decreasing order. In the following, an operator a is said d^+-summable (or $a \in \mathcal{L}^{d^+}$), $d > 0$, if $|a|^d \in \mathcal{L}^{1^+}$.

DEFINITION 1.1. A *K-cycle* is a triple $(\mathcal{A}, D, \mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{A}$ is a $*$-subalgebra of $B(\mathcal{H})$, D is an unbounded selfadjoint operator on $\mathcal{H}$ such that D has compact resolvent and $[D, a]$ is bounded for every $a \in \mathcal{A}$. A K-cycle is called *d^+-summable* if $|D|^{-1}$ is d^+-summable.

REMARK 1.2. In the previous definition $[D, a]$ being bounded means that one of the following three equivalent conditions is satisfied (see e.g. [4], Proposition 3.2.55):

(i) there exists a core $\mathcal{D}$ for D (indeed the whole domain of D) such that, for each $a \in \mathcal{A}$, the sesquilinear form

$$q(x, y) := (Dx, ay) - (a^*x, Dy)$$

is bounded on $\mathcal{D} \times \mathcal{D}$;

(ii) $\mathcal{A}$ is contained in the domain of the derivation $\mathrm{i}[D, \cdot]$, which is, by definition, the infinitesimal generator of the one-parameter group of automorphisms implemented by $\mathrm{e}^{\mathrm{i}tD}$;

(iii) For any $a \in \mathcal{A}$, $a\xi \in \mathcal{D}(D)$ for any ξ in the domain $\mathcal{D}(D)$ of D and $[D, a]$ is norm bounded on $\mathcal{D}(D)$.

An important object in non commutative geometry is a particular type of non normal traces on $B(\mathcal{H})$ which where introduced by Dixmier in [9] (see also [1], [12] for further developments). These traces are parametrized by two objects; one is the domain of the trace, which is described in terms of the asymptotic behavior of the sequence $\sigma_n(\cdot)$, the second is a generalized limit procedure, more precisely a dilation invariant state on bounded sequences on $\mathbb{N}$. By a *Dixmier trace* is generally meant a trace which "sums" logarithmic divergences, i.e. whose domain is the ideal $\mathcal{L}^{1^+}$.

We shall denote by τ_ω the Dixmier trace corresponding to the state ω (we shall not choose any particular state ω, but see also [6] for a possible canonical choice).

THEOREM 1.3. *Let $(\mathcal{A}, D, \mathcal{H})$ be a d^+-summable K-cycle. Then the functional*

$$\varphi(a) := \tau_\omega(|D|^{-d}a), \quad \forall\, a \in B(\mathcal{H})$$

is a hypertrace on $\mathcal{A}$, i.e.

$$\varphi(ab) = \varphi(ba), \quad \forall\, a \in \mathcal{A},\ b \in \mathcal{B}(\mathcal{H}).$$

Our proof relies on the following lemmas.

LEMMA 1.4. *Hölder inequality holds for τ_ω on $B(\mathcal{H})$.*

Proof. Let $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. It is sufficient to consider a, b compact operators. We have

$$\begin{aligned}\sigma_N(ab) &= \sum_{k=0}^{N-1} \mu_k(ab) \leqslant \sum_{k=0}^{N-1} \mu_k(a)\mu_k(b) \\ &\leqslant \left(\sum_{k=0}^{N-1} \mu_k(a)^p\right)^{\frac{1}{p}} \left(\sum_{k=0}^{N-1} \mu_k(b)^q\right)^{\frac{1}{q}} = (\sigma_N(|a|^p))^{\frac{1}{p}} (\sigma_N(|b|^q))^{\frac{1}{q}}\end{aligned}$$

by the Weyl inequality, Hölder inequality for $\mathbf{R}^N$ and the spectral theorem. Then, dividing by $\log N$ and applying the state ω to the previous inequality, we get, by definition of τ_ω,

$$\begin{aligned}\tau_\omega(|ab|) &= \omega\left(\frac{\sigma_N(ab)}{\log N}\right) \\ &\leqslant \omega\left(\left(\frac{\sigma_N(|a|^p)}{\log N}\right)^{\frac{1}{p}} \left(\frac{\sigma_N(|b|^q)}{\log N}\right)^{\frac{1}{q}}\right) \\ &\leqslant \omega\left(\frac{\sigma_N(|a|^p)}{\log N}\right)^{\frac{1}{p}} \omega\left(\frac{\sigma_N(|b|^q)}{\log N}\right)^{\frac{1}{q}} = \tau_\omega(|a|^p)^{\frac{1}{p}} \tau_\omega(|b|^q)^{\frac{1}{q}}\end{aligned}$$

where we used Hölder inequality for states on abelian C^*-algebras.

The case $p = 1$, $q = \infty$, a compact and b bounded can be proven by the same methods. ∎

The following lemma is well known, see e.g. [8], [13]. For the sake of completeness, we give a proof of the statement.

LEMMA 1.5. *Let a be a bounded operator and D a selfadjoint operator with bounded inverse on $\mathcal{H}$ such that $[D, a]$ is bounded. Then, for any $0 < r < 1$, the operator $[|D|^r, a]$ is bounded and the following inequality holds:*

$$\|[|D|^r, a]\| \leqslant C\|[D, a]\|,$$

where $C > 0$ does not depend on a and $\|\cdot\|$ denotes the usual operator norm on $B(\mathcal{H})$.

Proof. First we show that the following property on the domain of a derivation holds (cf. [3]): let D be an unbounded selfadjoint operator with bounded inverse, let a be in the domain of the derivation $[D, \cdot]$ and let g be a C^1 function on $\mathbb{R}$ such that $\widehat{g'}$, the Fourier transform of g', is in $L^1(\mathbb{R}, \mathrm{d}x)$. Then a is in the domain of the derivation $[g(D), \cdot]$ and

$$\|[g(D), a]\| \leqslant \|\widehat{g'}\|_1 \|[D, a]\|. \tag{1.1}$$

Let $\mathcal{D}$ be the space of vectors in $\mathcal{H}$ having compact support with respect to the spectral measure of D. $\mathcal{D}$ is a common core for D and $g(D)$ and, for each $x, y \in \mathcal{D}$, the following formula holds (cf. again [3]):

$$(g(D)x, ay) - (x, ag(D)y)$$
$$= \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}p\, p\widehat{g}(p) \int_0^1 \mathrm{d}t\, \left((D\mathrm{e}^{-\mathrm{i}tpD}x, a\mathrm{e}^{\mathrm{i}(1-t)pD}y) - (x, \mathrm{e}^{\mathrm{i}tpD}aD\mathrm{e}^{\mathrm{i}(1-t)pD}y)\right).$$

By a straightforward estimate on the previous equality and the equivalence stated in the remark following Definition 1.1, we get that $[g(D), a]$ is bounded and (1.1) holds.

Finally let us choose $g \in C^\infty(\mathbb{R})$ such that $g(x) = |x|^r$ when $|x| \geqslant \mathrm{dist}(0, \sigma(D))$. For such a g one has $g(D) = |D|^r$.

The lemma is proved if we show that the Fourier transform of g' is in $L^1(\mathbb{R}, \mathrm{d}x)$.

Since g' is the sum of an homogeneous function of degree $r-1$ plus a function with compact support, its Fourier transform is the sum of an homogeneous function of degree $-r$ plus an analytic function, therefore is locally summable since $r < 1$. But g' is also C^∞, therefore its Fourier transform is a function of rapid decay. ∎

Prooof of Theorem 1.3. First let us notice that, making use of the trace property of τ_ω, the statement of the theorem is equivalent to prove

$$\tau_\omega\left(\left|[|D|^{-d},a]\right|\right)=0 \quad \forall a\in\mathcal{A}. \tag{1.2}$$

Then we recall that, if H is a selfadjoint operator with bounded inverse and $[H,a]$ is bounded, for any $k\in\mathbf{N}$ the following identity holds:

$$[a,H^{-k}]=\sum_{j=1}^{k}H^{-j}[H,a]H^{-k-1+j}. \tag{1.3}$$

Now choose $r\in(0,1)$ such that $k:=\frac{d}{r}$ is a natural number. Then, applying identity (1.3) with $H=|D|^r$, Hölder inequality, and Lemma 1.5 we get

$$\begin{aligned}\tau_\omega\left(|[a,|D|^{-d}]|\right)&=\tau_\omega\left(|[a,(|D|^r)^{-k}]|\right)\\&\leqslant\sum_{i=1}^{k}\tau_\omega\left(\left||D|^{-rj}[a,|D|^r]\,|D|^{r(-k-1+j)}\right|\right)\\&\leqslant\|[a,|D|^r]\|\sum_{j=1}^{k}\left(\tau_\omega\left(|D|^{-rjp_j}\right)\right)^{\frac{1}{p_j}}\left(\tau_\omega\left(|D|^{r(-k-1+j)q_j}\right)\right)^{\frac{1}{q_j}}\\&\leqslant C\|[a,D]\|\sum_{j=1}^{k}\left(\tau_\omega\left(|D|^{-rjp_j}\right)\right)^{\frac{1}{p_j}}\left(\tau_\omega\left(|D|^{r(-k-1+j)q_j}\right)\right)^{\frac{1}{q_j}}.\end{aligned}$$

With a suitable choice of p_j and q_j, e.g. $p_j=\frac{2d}{r(2j-1)}$, $q_j=\frac{2d}{r(2k+1-2j)}$, both the exponents of $|D|$ in the last term are strictly smaller than $-d$, therefore the Dixmier traces vanish, which proves the theorem. ∎

We refer to [6] for conditions on the non triviality of φ on $\mathcal{A}$.

As we pointed out in the introduction, one is interested in proving that the state φ constructed above gives a trace not only on the algebra $\mathcal{A}$, but also on the universal graded differential algebra $\Omega(\mathcal{A})$ on $\mathcal{A}$ via the formula

$$\tau(a_0\mathrm{d}a_1\cdots\mathrm{d}a_n):=\mathrm{i}^n\varphi(a_0[D,a_1]\cdots[D,a_n]). \tag{1.4}$$

This result does not hold in full generality, as it is shown by the counterexample described in the next section. Here we will discuss regularity conditions which ensure that the trace property holds.

As shown in [6], the condition $[D,\mathcal{A}]$ being bounded contained in the K-cycle assumption corresponds to the Lipschitz regularity for functions. According to [6], p. 546, higher regularity conditions may be given in terms of the derivation δ

corresponding to the commutator with $|D|$, namely the generator of the following one-parameter group of automorphisms of $B(\mathcal{H})$:

$$\alpha_t(a) := e^{it|D|} a e^{-it|D|}, \quad a \in B(\mathcal{H}). \tag{1.5}$$

More precisely we may introduce the subalgebras $\mathcal{A}_n$ generated by the elements $a \in \mathcal{A}$ such that a and $[D, a]$ are in the domain of δ^{n-1}. The intersection of such algebras corresponds to $\mathcal{C}^\infty$ functions.

First we notice that while Lipschitz regularity is expressed by the boundedness of the commutator with D, higher commutators with D do not describe higher regularity, because in the commutative setting $[D,[D,f]]$ is a sum of a Clifford multiplication operator and a differential operator, and therefore cannot be bounded. On the other hand, δ^n is bounded on $\mathcal{C}^\infty$ functions (or forms) in the commutative case.

This fact may be explained since the flow described in (1.5) is deeply related to the geodesic flow (cf. [10], [2], and especially [7]).

Now we show that the mentioned regularity conditions imply the trace property on $\Omega(\mathcal{A})$. First we remark that the functional τ on $\Omega(\mathcal{A})$ in formula (1.4) is a trace if and only if φ is a trace on the $*$-algebra $\langle \mathcal{A}, [D, \mathcal{A}] \rangle$ generated by $\mathcal{A}$ and the commutators of D with the elements of $\mathcal{A}$. As a consequence, the following property holds.

PROPOSITION 1.6. *The functional τ is a trace on $\Omega(\mathcal{A}_2)$.*

Proof. In fact $(\langle \mathcal{A}_2, [D, \mathcal{A}_2] \rangle, |D|, \mathcal{H})$ is a d^+-summable K-cycle, hence the assert follows from Theorem 1.3 and the previous observation. ∎

We may restate the proposition saying that τ is a trace on $\Omega(\mathcal{A})$ if, extending the usual notation for Sobolev spaces, the K-cycle is $W^{2,\infty}$ regular, namely $\mathcal{A} = \mathcal{A}_2$.

However, the result given by Proposition 1.6 is not completely analogous to its commutative counterpart. In fact, the proof of the trace property in the commutative case (cf. e.g. [11]) is based on the fact that, since the principal symbol of the product of two pseudo-differential operators is the product of the principal symbols of the factors, the order of the commutator is the sum of the orders of the factors minus 1. Therefore, on an n-dimensional manifold, the Dixmier trace vanishes on the commutator $[|D|^{-n}, \sigma]$, when σ is a section of the Clifford bundle, because it is a pseudo-differential operator of order $n-1$, and the regularity (differentiability) assumptions on σ plays no crucial role.

A natural assumption, that holds in the commutative case and allows to extend the trace property to $\Omega(\mathcal{A})$, is to ask that $\mathcal{A}_2$ is a large enough subalgebra of $\mathcal{A}$.

THEOREM 1.7. *Let* $(\mathcal{A}, D, \mathcal{H})$ *be a* d^+*-summable* K*-cycle. If* $\mathcal{A}_2$ *is a core for the derivation* $[D, \cdot]$ *on* $\mathcal{A}$*, then* τ *given by* (1.4) *is a trace on the algebra* $\Omega(\mathcal{A})$.

Proof. As already remarked, it is sufficient to show the trace property of φ on $\langle \mathcal{A}, [D, \mathcal{A}] \rangle$. By Proposition 1.6, φ is a trace on $\langle \mathcal{A}_2, [D, \mathcal{A}_2] \rangle$. Since $\mathcal{A}_2$ is dense in $\mathcal{A}$ in the graph norm of $[D, \cdot]$ (hence in the usual norm, too), $[D, \mathcal{A}_2]$ is norm dense in $[D, \mathcal{A}]$ and this implies the trace property on (the closure of) $\langle \mathcal{A}, [D, \mathcal{A}] \rangle$. ∎

2. AN EXAMPLE OF A K-CYCLE

Let $\mathcal{K}$ be an infinite dimensional Hilbert space and b an unbounded selfadjoint positive operator with compact inverse, such that $0 < \tau_\omega(b^{-d}) < \infty$. We consider on $\mathcal{H} = \mathbf{C}^2 \otimes \mathcal{K}$ the $*$-algebra $\mathcal{A}$ generated by the element a in $B(\mathcal{H})$, where

$$a = m_{12} \otimes b^{-1},$$

and m_{ij}, $i, j = 1, 2$, are the matrix units.

We consider also the unbounded selfadjoint operator $D \equiv \alpha \otimes b$ on $\mathcal{H}$, where $\alpha = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda, \mu \in \mathbf{R}$.

PROPOSITION 2.1. *The triple* $(\mathcal{A}, D, \mathcal{H})$ *is a* d^+*-summable* K*-cycle.*

Proof. Since $a^2 = (a^*)^2 = 0$, $\mathcal{A}$ is linearly generated by the following elements:

$$x_{11}^k \equiv (a^*a)^{k+1}, \quad x_{22}^k \equiv (aa^*)^{k+1}, \quad x_{12}^k \equiv a(a^*a)^k, \quad x_{21}^k \equiv a^*(aa^*)^k, \quad k \geqslant 0.$$

A computation shows:

$$x_{ii}^k = m_{ii} \otimes b^{-(2k+2)}, \quad i = 1, 2;$$
$$x_{ij}^k = m_{ij} \otimes b^{-(2k+1)}, \quad i \neq j.$$

Denoting by Tr_ω the Dixmier trace on $\mathcal{H}$, we have $\mathrm{Tr}_\omega(|D|^{-d}) = \mathrm{tr}(|\alpha|^{-d}) \cdot \tau_\omega(b^{-d}) < \infty$ and moreover

$$[D, x_{ij}^k] = (i - j)(\lambda - \mu) m_{ij} \otimes b^{-2k}, \quad i, j = 1, 2$$

hence $[D, \mathcal{A}] \subseteq B(\mathcal{H})$. ∎

Theorem 1.3 implies that $\varphi(\cdot) \equiv \mathrm{Tr}_\omega(\cdot |D|^{-d})$ is a trace on $\mathcal{A}$. The following proposition completely characterizes the trace property on $\Omega(\mathcal{A})$ in terms of λ and μ.

PROPOSITION 2.2. *The functional φ on $\Omega(\mathcal{A})$ is a trace if and only if $|\lambda| = |\mu|$.*

Proof. As explained in the preceding section, this is equivalent to show that φ is a trace on $\langle \mathcal{A}, [D, \mathcal{A}] \rangle$.

If $|\lambda| = |\mu|$, the state φ writes as $\varphi(\cdot) = |\lambda|^{-d} \mathrm{tr}(\cdot) \otimes \tau_\omega(\cdot b^{-d})$. Since $\langle \mathcal{A}, [D, \mathcal{A}] \rangle$ is contained in $M_2 \otimes C(\sigma(b^{-1}))$ the trace property of φ follows.

Conversely, let us assume that φ is a trace on $\langle \mathcal{A}, [D, \mathcal{A}] \rangle$. Since we have

$$[[D, a], [D, a^*]] = -(\lambda - \mu)^2 (m_{11} - m_{22}) \otimes \mathbf{1},$$

it follows that

$$0 = \varphi([[D, a], [D, a^*]]) = -(\lambda - \mu)^2 (|\lambda|^{-d} - |\mu|^{-d}) \tau_\omega(b^{-d})$$

which proves the assert. ∎

It is instructive to verify by direct arguments that, when $|\lambda| \neq |\mu|$, $\mathcal{A}_2$ is not dense in $\mathcal{A}$ in the graph norm of the derivation $[D, \cdot]$. More precisely, motivated by Theorem 1.7, we compute the norm closures of $\mathcal{A}$ and $\mathcal{A}_2$ and their closures in the graph norm of the derivation $[D, \cdot]$, showing that while the first two closures coincide, the latter do not.

First we notice that the maps

$$p \to m_{ii} \otimes p(b^{-1}), \quad i = 1, 2$$

are isomorphisms from the algebra of even polynomials of degree $\geqslant 2$ on $\sigma(b^{-1})$ to the diagonal blocks of $\mathcal{A}$ (which coincide with the diagonal blocks of $\mathcal{A}_2$). Therefore the diagonal blocks of the norm closure of $\mathcal{A}$ coincide with the corresponding diagonal blocks of the norm closure of $\mathcal{A}_2$ and are isomorphic to the norm closure of the algebra of even polynomials of degree $\geqslant 2$ on $\sigma(b^{-1})$ which is $C_0(\sigma(b^{-1}))$, the algebra of continuous functions on the spectrum of b^{-1} vanishing at 0.

In a similar way the maps

$$p \to m_{ij} \otimes p(b^{-1}), \quad i \neq j, \ i, j = 1, 2$$

give isomorphisms from the vector space of odd polynomials of degree $\geqslant 1$ on $\sigma(b^{-1})$ to the antidiagonal blocks of $\mathcal{A}$ and isomorphisms from the vector space of odd polynomials of degree $\geqslant 3$ on $\sigma(b^{-1})$ to the antidiagonal blocks of $\mathcal{A}_2$. Since the norm closure of the vector space of odd polynomials of degree $\geqslant p > 0$ on $\sigma(b^{-1})$ is $C_0(\sigma(b^{-1}))$, the antidiagonal blocks of the norm closure of $\mathcal{A}$ coincide with the corresponding antidiagonal blocks of the norm closure of $\mathcal{A}_2$ and are isomorphic to $C_0(\sigma(b^{-1}))$. Finally, we get $\overline{\mathcal{A}} = \overline{\mathcal{A}}_2 \simeq M_2 \otimes C_0(\sigma(b^{-1}))$.

Now we consider the closures of $\mathcal{A}$ and $\mathcal{A}_2$ in the graph norm of $[D, \cdot]$. Since the commutator with D is zero on diagonal elements and coincides up to a constant with the multiplication by $\mathbf{1} \otimes b$ on the antidiagonal elements, the diagonal part of the closures of $\mathcal{A}$ and of $\mathcal{A}_2$ in the graph norm are equal to the corresponding norm closures and therefore coincide and, since b^{-1} is bounded, the graph norm on the antidiagonal elements $m_{ij} \otimes p(b^{-1})$ is equivalent to the norm

$$||m_{ij} \otimes p(b^{-1})|| := ||bp(b^{-1})||.$$

Then the maps

$$p \to m_{ij} \otimes b^{-1}p(b^{-1}), \quad i \neq j, \ i, j = 1, 2$$

give isometric isomorphisms between the vector space of even polynomials on the spectrum of b^{-1} and the antidiagonal blocks of $(\mathcal{A}, ||\cdot||)$ and isometric isomorphisms between the vector space of even polynomials of degree $\geqslant 2$ on the spectrum of b^{-1} and the antidiagonal blocks of $(\mathcal{A}_2, ||\cdot||)$.

As a consequence, an antidiagonal block of the graph norm closure of $\mathcal{A}$ is isomorphic to the space of continuous functions on $\sigma(b^{-1})$ vanishing at 0 and with finite derivative in 0, while an antidiagonal block of the graph norm closure of $\mathcal{A}_2$ is isomorphic to the space of continuous functions on $\sigma(b^{-1})$ vanishing at 0 and with null derivative in 0.

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