More Appendices for paper "Robust trajectory tracking for a class of hybrid systems: an internal model principle approach" by S. Galeani, L. Menini and A. Potini,

to appear in IEEE Transactions on Automatic Control

July 07, 2011

Abstract

This report contains the proofs of Lemmas 3-7.

C. Proof of Lemma 3: the tube $\mathcal{X}_{\kappa,\varepsilon_0}$ around $\bar{\mathbf{x}}_{\bar{\imath}_k}(\cdot)$.

By periodicity, it is enough to consider the case $h = 0$ so that $k = \kappa$. Since the only switching surface hit by the motion $\bar{\mathbf{x}}_{\bar{\imath}_\kappa}(\cdot)$ in the interval $[\bar{t}_\kappa, \bar{t}_{\kappa+1})$ is $\mathcal{C}_{\bar{\jmath}_\kappa\bar{\imath}_\kappa}$, denoting by $\mathrm{dist}(\bar{\mathbf{x}}_{\bar{\imath}_\kappa}(t), \mathcal{C}_{l\bar{\imath}_\kappa})$ the distance between $\bar{\mathbf{x}}_{\bar{\imath}_\kappa}(t)$ and $\mathcal{C}_{l\bar{\imath}_\kappa}$, it follows that $\varepsilon_\kappa := \inf_{\substack{t\in[\bar{t}_\kappa,\bar{t}_{\kappa+1})\\ l\neq\bar{\jmath}_\kappa}} \mathrm{dist}(\bar{\mathbf{x}}_{\bar{\imath}_\kappa}(t), \mathcal{C}_{l\bar{\imath}_\kappa})$ is strictly positive. Defining $\varepsilon_0^\star = \min_{\kappa\in\mathcal{N}}\{\varepsilon_\kappa\}$, it follows that for any $\varepsilon_0 \in (0, \varepsilon_0^\star)$ all points contained in the set $\mathcal{X}_{\kappa,\varepsilon_0}$ in (33) can belong at most to the switching surface $\mathcal{C}_{\bar{\jmath}_\kappa\bar{\imath}_\kappa}$.

D. Proof of Lemma 4: ensuring that reference and actual switching times are pairwise close.

Since $\varepsilon_1 \leq \varepsilon_0$, by Lemma 3 the switching event at time t_{k+1} can only happen when the switching surface $\mathcal{C}_{\bar{\jmath}_k\bar{\imath}_k}$ is hit. In the case $t_{k+1} = \bar{t}_{k+1}$, there is nothing to prove. Hence, the following computations consider the two cases $t_{k+1} > \bar{t}_{k+1}$ and $t_{k+1} < \bar{t}_{k+1}$. In both cases, the proof requires to compute the solutions between $\bar{t}_{k+1}$ and $\bar{t}_{k+1} \pm \omega$, and to use the constants ω and M_ω computed in the procedure in Subsection VI-A to show that if a switching event does not occur in the considered interval then a contradiction arise.

Consider the case $t_{k+1} > \bar{t}_{k+1}$ first. Since $t_{k+1} \not\leq \bar{t}_{k+1}$, it follows that $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\mathbf{x}_{\bar{\imath}_k}(t) - b_{\bar{\jmath}_k\bar{\imath}_k} < 0$, $\forall t \in [t_k, \bar{t}_{k+1}]$. By contradiction, assume that there is no switching time $t_{k+1} \in (\bar{t}_{k+1}, \bar{t}_{k+1}+\omega]$; this implies that the scalar function $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\mathbf{x}_{\bar{\imath}_k}(t) - b_{\bar{\jmath}_k\bar{\imath}_k}$ remains strictly negative also for all $t \in (\bar{t}_{k+1}, \bar{t}_{k+1} + \omega]$. Recalling (20) (where $\mathbf{u}_{b,\kappa+1} = \bar{\mathbf{u}}(\bar{t}_{\kappa+1}^-)$) and (17c), one has for $t > \bar{t}_{k+1}$:

$$\begin{aligned}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\mathbf{x}_{\bar{\imath}_k}(t) - b_{\bar{\jmath}_k\bar{\imath}_k} &= \mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k} e^{\mathbf{A}_{\bar{\imath}_k}(t-\bar{t}_{k+1})}\left(\bar{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-) + \tilde{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-) + \int_{\bar{t}_{k+1}}^{t} e^{\mathbf{A}_{\bar{\imath}_k}(t-\tau)}\mathbf{B}_{\bar{\imath}_k}\mathbf{u}_{b,\kappa+1}\right) - b_{\bar{\jmath}_k\bar{\imath}_k}\\ &= (\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(t) - b_{\bar{\jmath}_k\bar{\imath}_k}) + \mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k} e^{\mathbf{A}_{\bar{\imath}_k}(t-\bar{t}_{k+1})}\tilde{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\end{aligned} \tag{42}$$

The scalar, continuously differentiable function $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(t) - b_{\bar{\jmath}_k\bar{\imath}_k}$ is positive (since $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}) - b_{\bar{\jmath}_k\bar{\imath}_k} = 0$ and $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-) > 0$) and by (22) it is lower bounded by $\frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)|t - \bar{t}_{k+1}|$, $\forall t \in [\bar{t}_{k+1}, \bar{t}_{k+1} + \omega]$; in particular, $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1} + \omega) - b_{\bar{\jmath}_k\bar{\imath}_k} > \frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\omega$. On the other hand, in order for the right hand side of (42) to be negative the term $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}e^{\mathbf{A}_{\bar{\imath}_k}(t-\bar{t}_{k+1})}\tilde{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)$ must be negative, and by (23) it satisfies $|\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}e^{\mathbf{A}_{\bar{\imath}_k}(t-\bar{t}_{k+1})}\tilde{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)| < M_\omega\varepsilon_1\frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-) < \frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\omega$, $\forall t \in [\bar{t}_{k+1}, \bar{t}_{k+1} + \omega]$ contradicting $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\mathbf{x}_{\bar{\imath}_k}(t) - b_{\bar{\jmath}_k\bar{\imath}_k} < 0$, $\forall t \in (\bar{t}_{k+1}, \bar{t}_{k+1} + \omega]$. Since $|\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}e^{\mathbf{A}_{\bar{\imath}_k}(t_{k+1}-\bar{t}_{k+1})}\tilde{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)| = |\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}) - b_{\bar{\jmath}_k\bar{\imath}_k}|$ at t_{k+1}, then $|\tilde{t}_{k+1}| < M_\omega \left\|\tilde{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\right\|$.

Finally, consider the case $t_{k+1} < \bar{t}_{k+1}$. By hypothesis,

$$\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\mathbf{x}_{\bar{\imath}_k}(t_{k+1}^-) - b_{\bar{\jmath}_k\bar{\imath}_k} = (\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}) - b_{\bar{\jmath}_k\bar{\imath}_k}) + \mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\tilde{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}^-) = 0. \tag{43}$$

Since $(\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}) - b_{\bar{\jmath}_k\bar{\imath}_k}) < -\frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)|\tilde{t}_{k+1}| < 0$, it follows that $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\tilde{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}^-) > 0$ and $\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\tilde{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}^-) < \|\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\| \left\|\tilde{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}^-)\right\| < M_\omega\frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\varepsilon_1 < \frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\omega$; hence, (43) implies $\frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\omega > \mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\tilde{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}^-) = -(\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(t_{k+1}) - b_{\bar{\jmath}_k\bar{\imath}_k}) > \frac{1}{2}\mathbf{J}_{\bar{\jmath}_k\bar{\imath}_k}\dot{\bar{\mathbf{x}}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)|\tilde{t}_{k+1}|$ and then $t_{k+1} \in (\bar{t}_{k+1} - \omega, \bar{t}_{k+1})$. The bound $|\tilde{t}_{k+1}| < M_\omega \left\|\tilde{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-)\right\|$ follows as in the case $t_{k+1} > \bar{t}_{k+1}$.

E. Proof of Lemma 5

The proof requires to compute the motions between t_{k+1}^m and t_{k+1}^M, in order to evaluate the errors at the two instants. The two cases $t_{k+1} \geq \bar{t}_{k+1}$ and $t_{k+1} \leq \bar{t}_{k+1}$ must be considered separately due to the different definition of $\mathbf{u}_a(t)$ and $\mathbf{u}(t)$ in the two cases according to (17b) and (17c).

Consider the case $t_{k+1} \geq \bar{t}_{k+1}$ first, so that $\tilde{t}_{k+1} \in [0, \omega)$. According to (20) and (17c),

$$\mathbf{x}_{\bar{\jmath}_k}(t_{k+1}) = \mathbf{\Gamma}_{\bar{\jmath}_k\bar{\imath}_k}\left[e^{\mathbf{A}_{\bar{\imath}_k}\tilde{t}_{k+1}}\left(\bar{\mathbf{x}}_{\bar{\jmath}_k}(\bar{t}_{k+1}^-) + \tilde{\mathbf{x}}_{\bar{\jmath}_k}(\bar{t}_{k+1}^-)\right) + \int_0^{\tilde{t}_{k+1}} e^{\mathbf{A}_{\bar{\imath}_k}(\tilde{t}_{k+1}-\tau)}\mathbf{B}_{\bar{\imath}_k}\mathbf{u}_{b,\kappa+1}d\tau\right] + \gamma_{\bar{\jmath}_k\bar{\imath}_k},$$

$$\bar{\mathbf{x}}_{\bar{\jmath}_k}(t_{k+1}) = e^{\mathbf{A}_{\bar{\jmath}_k}\tilde{t}_{k+1}}\left(\mathbf{\Gamma}_{\bar{\jmath}_k\bar{\imath}_k}\bar{\mathbf{x}}_{\bar{\imath}_k}(\bar{t}_{k+1}^-) + \gamma_{\bar{\jmath}_k\bar{\imath}_k}\right) + \int_0^{\tilde{t}_{k+1}} e^{\mathbf{A}_{\bar{\jmath}_k}(\tilde{t}_{k+1}-\tau)}\mathbf{B}_{\bar{\jmath}_k}\bar{\mathbf{u}}(\bar{t}_{k+1} + \tau)d\tau,$$

with $\bar{\mathbf{u}}(\bar{t}_{k+1}+\tau) = \bar{\mathbf{u}}(\bar{t}^-_{k+1}) = \mathbf{u}_{b,\kappa+1}$ due to (20), whereas by (17a) and (17b), $\mathbf{x}_a(t_{k+1}) = e^{\mathbf{A}_{\bar{\jmath}_k}\tilde{t}_{k+1}}(\bar{x}_a(\bar{t}_{k+1})+\tilde{\mathbf{\Lambda}}_{\kappa+1}(h-1))$, $\bar{\mathbf{x}}_a(t_{k+1}) = e^{\mathbf{A}_{\bar{\jmath}_k}\tilde{t}_{k+1}}\bar{x}_a(\bar{t}_{k+1})$. Recalling (21), it follows that $\tilde{\mathbf{x}}_{\bar{\jmath}_k}(t_{k+1}) = \mathbf{\Gamma}_{\bar{\jmath}_\kappa\bar{\imath}_k}e^{\mathbf{A}_{\bar{\imath}_k}\tilde{t}_{k+1}}\tilde{\mathbf{x}}_{\bar{\jmath}_k}(\bar{t}^-_{k+1})+\mathbf{f}_{1,k}(\tilde{t}_{k+1})$, $\tilde{\mathbf{x}}_a(t_{k+1}) = e^{\mathbf{A}_{\bar{\jmath}_k}\tilde{t}_{k+1}}\tilde{\mathbf{\Lambda}}_{\kappa+1}(h-1)$. By the definitions in Subsection VI-A, $\left\|\tilde{\mathbf{x}}_{\bar{\jmath}_k}(t^M_{k+1})\right\| < M_x\left\|\tilde{\mathbf{x}}_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\|+M_{\tilde{t}}|\tilde{t}_{k+1}| < (M_{\tilde{t}}M_\omega+M_x)\left\|\tilde{\mathbf{x}}_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\| \leq M_g\left\|\tilde{\mathbf{x}}^e_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\|$, $\|\tilde{\mathbf{x}}_a(t_{k+1})\| < M_a\left\|\tilde{\mathbf{\Lambda}}_{\kappa+1}(h-1)\right\| < M_g\left\|\tilde{\mathbf{\Lambda}}_{\kappa+1}(h-1)\right\|$. Finally, since $\mathbf{\Lambda}_\kappa(h) = e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k)}\mathbf{x}_a(\bar{t}^-_{k+1}) = e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k)}\left(\bar{\mathbf{x}}_a(\bar{t}^-_{k+1})+\tilde{\mathbf{x}}_a(\bar{t}^-_{k+1})\right) = \bar{\mathbf{x}}_a(\bar{t}_{k+N})+e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k)}\tilde{\mathbf{x}}_a(t^{m-}_{k+1})$ it follows that $\tilde{\mathbf{\Lambda}}_\kappa(h) = e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k)}\tilde{\mathbf{x}}_a(t^{m-}_{k+1})$ and hence

$$\left\|\tilde{\mathbf{\Lambda}}_\kappa(h)\right\| < M_{\tilde{\Lambda}}\left\|\tilde{\mathbf{x}}_a(t^{m-}_{k+1})\right\| \leq M_g\left\|\tilde{\mathbf{x}}^e_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\|.$$

On the other hand, when $t_{k+1} < \bar{t}_{k+1}$, one has $\tilde{t}_{k+1} \in (-\omega, 0)$. According to (17c),

$$\mathbf{x}_{\bar{\jmath}_k}(\bar{t}_{k+1}) = e^{-\mathbf{A}_{\bar{\jmath}_k}\tilde{t}_{k+1}}\left[\mathbf{\Gamma}_{\bar{\jmath}_k\bar{\imath}_k}\left(\bar{\mathbf{x}}_{\bar{\imath}_k}(t^-_{k+1})+\tilde{\mathbf{x}}_{\bar{\imath}_k}(t^-_{k+1})\right)+\gamma_{\bar{\jmath}_k\bar{\imath}_k}\right]+\int_{\tilde{t}_{k+1}}^0 e^{-\mathbf{A}_{\bar{\jmath}_k}\tau}\mathbf{B}_{\bar{\jmath}_k}\mathbf{u}_{c,k+1}d\tau,$$

$$\bar{\mathbf{x}}_{\bar{\jmath}_k}(\bar{t}_{k+1}) = \mathbf{\Gamma}_{\bar{\jmath}_k\bar{\imath}_k}\left(e^{-\mathbf{A}_{\bar{\imath}_k}\tilde{t}_{k+1}}\bar{\mathbf{x}}_{\bar{\imath}_k}(t^-_{k+1})+\int_{\tilde{t}_{k+1}}^0 e^{-\mathbf{A}_{\bar{\imath}_k}\tau}\mathbf{B}_{\bar{\imath}_k}\bar{\mathbf{u}}(\bar{t}_{k+1}+\tau)d\tau\right)+\gamma_{\bar{\jmath}_k\bar{\imath}_k}.$$

By (21), it follows that $\tilde{\mathbf{x}}_{\bar{\jmath}_k}(\bar{t}_{k+1}) = e^{-\mathbf{A}_{\bar{\jmath}_k}\tilde{t}_{k+1}}\mathbf{\Gamma}_{\bar{\jmath}_\kappa\bar{\imath}_k}\tilde{\mathbf{x}}_{\bar{\jmath}_k}(\bar{t}^-_{k+1})+\mathbf{f}_{2,k}(\tilde{t}_{k+1})$, $\tilde{\mathbf{x}}_a(t_{k+1}) = \tilde{\mathbf{\Lambda}}_{\kappa+1}(h-1)$; by the definitions in Subsection VI-A, it follows that $\|\tilde{\mathbf{x}}_a(t_{k+1})\| = \left\|\tilde{\mathbf{\Lambda}}_{\kappa+1}(h-1)\right\| \leq M_g\left\|\tilde{\mathbf{\Lambda}}_{\kappa+1}(h-1)\right\|$, $\left\|\tilde{\mathbf{x}}_{\bar{\jmath}_k}(t^M_{k+1})\right\| < M_x\left\|\tilde{\mathbf{x}}_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\|+M_{\tilde{t}}|\tilde{t}_{k+1}| < (M_{\tilde{t}}M_\omega+M_x)\left\|\tilde{\mathbf{x}}_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\| \leq M_g\left\|\tilde{\mathbf{x}}^e_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\|$. Since

$$\begin{aligned}\mathbf{\Lambda}_\kappa(h) &= e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k)}\mathbf{x}_a(\bar{t}^-_{k+1}) = e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k)}e^{\mathbf{A}_a(\bar{t}_{k+1}-t_{k+1})}\left(\bar{\mathbf{x}}_a(t^-_{k+1})+\tilde{\mathbf{x}}_a(t^-_{k+1})\right)\\ &= \bar{\mathbf{x}}_a(\bar{t}_{k+N})+e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k+\tilde{t}_{k+1})}\tilde{\mathbf{x}}_a(t^{m-}_{k+1})\end{aligned}$$

it follows that $\tilde{\mathbf{\Lambda}}_\kappa(h) = e^{-\mathbf{A}_a(\bar{t}_{k+1}-\bar{t}_k+\tilde{t}_{k+1})}\tilde{\mathbf{x}}_a(t^{m-}_{k+1})$ and then $\left\|\tilde{\mathbf{\Lambda}}_\kappa(h)\right\| < M_{\tilde{\Lambda}}\left\|\tilde{\mathbf{x}}_a(t^{m-}_{k+1})\right\| \leq M_g\left\|\tilde{\mathbf{x}}^e_{\bar{\jmath}_k}(t^{m-}_{k+1})\right\|$.

F. Proof of Lemma 6

From Lemma 1, the choice of the gains $\mathbf{K}^e_{\bar{\imath}_k}$ guarantees that the solution is inside $\mathcal{X}_{\kappa,\varepsilon_1} \subset \mathcal{X}_{\kappa,\varepsilon_0}$ (so that Lemma 3 and Lemma 4 can be applied) and that $\left\|\tilde{\mathbf{x}}^e_{\bar{\imath}_k}(t^{m-}_{k+1})\right\| < \gamma\left\|\tilde{\mathbf{x}}^e_{\bar{\imath}_k}(t^M_k)\right\|$. Applying Lemma 5 and Lemma 4 yields (34a) and (34b). Finally, (34c) can be obtained noting that $\tilde{\mathbf{y}}(t) = \mathbf{y}(t)-\bar{\mathbf{y}}(t) = \mathbf{C}^o_{\bar{\imath}_k}\tilde{\mathbf{x}}^e_{\bar{\imath}_k}$, $\forall t \in [t^M_k, t^m_{k+1})$. Using (32c), the bound (34c) follows by choosing $M_1 = bM_0 = \gamma e^{\eta\rho_0}M_0$ where $b = \gamma e^{\eta\rho_0}$ and $M_0 = \max_{k\in\mathcal{N}}\{\left\|\mathbf{C}^o_{\bar{\imath}_k}\right\|\}$.

G. Proof of Lemma 7.

Proceeding by induction on κ, it will now be shown that $\left\|\tilde{\boldsymbol{\xi}}_h\right\| < \delta_0$, $|\tilde{t}_{1+hN}| < \delta_0$ imply

$$\|\boldsymbol{\chi}_{\kappa+hN}\| \leq \left\|\begin{matrix}\tilde{t}_{1+hN}\\ \tilde{\boldsymbol{\xi}}_h\end{matrix}\right\|, \quad \kappa \in \mathcal{N}, \tag{44a}$$

$$\left\|\begin{matrix}\tilde{t}_{1+(h+1)N}\\ \tilde{\boldsymbol{\xi}}_{h+1}\end{matrix}\right\| < \alpha\left\|\begin{matrix}\tilde{t}_{1+hN}\\ \tilde{\boldsymbol{\xi}}_h\end{matrix}\right\|. \tag{44b}$$

In order to show (44a), it is enough to show that

$$|\tilde{t}_{\kappa+hN}| \leq \left\|\begin{matrix}\tilde{t}_{1+hN}\\ \tilde{\boldsymbol{\xi}}_h\end{matrix}\right\|, \ \left\|\tilde{\mathbf{\Lambda}}_{N+1}(h-1)\right\| < \left\|\begin{matrix}\tilde{t}_{1+hN}\\ \tilde{\boldsymbol{\xi}}_h\end{matrix}\right\|, \ \left\|\tilde{\mathbf{x}}_{\bar{\imath}_{\kappa+hN}}(t^M_{\kappa+hN})\right\| < \left\|\begin{matrix}\tilde{t}_{1+hN}\\ \tilde{\boldsymbol{\xi}}_h\end{matrix}\right\|, \ \kappa = 2,\ldots,N,$$

since $\tilde{\mathbf{x}}_{\bar{\imath}_{1+hN}}(t^M_{1+hN})$, $\tilde{\mathbf{\Lambda}}_\kappa(h-1)$, $\kappa \in \mathcal{N}$, are subvectors of $\tilde{\boldsymbol{\xi}}_h$ and by hypothesis $|\tilde{t}_{1+hN}| < \delta_0$. Similarly, in order to show (44b), it is enough to show that

$$\left\| \begin{matrix} \tilde{t}_{1+(h+1)N} \\ \tilde{\mathbf{x}}_{\bar{\imath}_{1+hN}}(t^M_{1+(h+1)N}) \end{matrix} \right\| < \alpha \left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\|, \quad \left\| \tilde{\mathbf{\Lambda}}_\kappa(h) \right\| < \alpha \left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\|, \quad \kappa \in \mathcal{N}.$$

Case $\kappa = 1$. Since $\tilde{\mathbf{x}}_{\bar{\imath}_{1+hN}}(t^M_{1+hN})$, $\tilde{\mathbf{\Lambda}}_1(h-1)$, $\tilde{\mathbf{\Lambda}}_2(h-1)$, are subvectors of $\tilde{\boldsymbol{\xi}}_h$, it holds that $\|\boldsymbol{\chi}_{1+hN}\| \leq \left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\|$; moreover $\left\| \tilde{\mathbf{\Lambda}}_1(h) \right\| = \left\| \tilde{\mathbf{\Lambda}}_{N+1}(h-1) \right\| < \alpha \|\boldsymbol{\chi}_{1+hN}\| < \left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\|$ by (34b).

Case $\kappa = 2, \ldots, N$. Assume that $\|\boldsymbol{\chi}_{m+hN}\| \leq \left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\|$ and $\left\| \tilde{\mathbf{\Lambda}}_m(h) \right\| < \alpha \left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\|$ have been proven for $m = 1, \ldots, \kappa - 1$, and recall that $\left\| \tilde{\mathbf{\Lambda}}_i(h-1) \right\| < \left\| \tilde{\boldsymbol{\xi}}_h \right\|$, $i = 1, \ldots, N+1$, (for $i = 1, \ldots, N$ because $\tilde{\mathbf{\Lambda}}_i(h-1)$ is a subvector of $\tilde{\boldsymbol{\xi}}_h$, and for $i = N+1$ because it was proven in the case $\kappa = 1$). The application of (34a) leads to $\left\| \begin{matrix} \tilde{t}_{\kappa+hN} \\ \tilde{\mathbf{x}}_{\bar{\imath}_{\kappa+hN}}(t^M_{\kappa+hN}) \end{matrix} \right\| < \left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\|$, which implies the required inequality for $\|\boldsymbol{\chi}_{\kappa+hN}\|$. Then, the application of (34b) implies that $\left\| \tilde{\mathbf{\Lambda}}_\kappa(h) \right\| < \alpha \|\boldsymbol{\chi}_{1+hN}\|$, which yields the required inequality for $\left\| \tilde{\mathbf{\Lambda}}_\kappa(h) \right\|$. Finally, for $\kappa = N$ the bound $\left\| \begin{matrix} \tilde{t}_{1+(h+1)N} \\ \tilde{\mathbf{x}}_{\bar{\imath}_{1+hN}}(t^M_{1+(h+1)N}) \end{matrix} \right\| < \alpha \|\boldsymbol{\chi}_{\kappa+hN}\|$ follows by (34a), thus proving (44).

Now, note that $\left\| \tilde{\boldsymbol{\xi}}_0 \right\| < \bar{\delta} < \delta_0$ by hypothesis and $|\tilde{t}_1| = 0$ by the definition in Problem 1. By induction on h, using (44b) it is then immediate to show that $\left\| \begin{matrix} \tilde{t}_{1+hN} \\ \tilde{\boldsymbol{\xi}}_h \end{matrix} \right\| < \alpha^h \left\| \begin{matrix} \tilde{t}_1 \\ \tilde{\boldsymbol{\xi}}_0 \end{matrix} \right\| = \alpha^h \left\| \tilde{\boldsymbol{\xi}}_0 \right\| < \alpha^h \bar{\delta}$, $\forall h \in \mathbb{Z}^+$, and then by (44a) it also follows that $\|\boldsymbol{\chi}_{\kappa+hN}\| < \alpha^h \bar{\delta}$, $\forall h \in \mathbb{Z}^+$, $\forall \kappa \in \mathcal{N}$.