Semilinear elliptic equations with dependence on the gradient via mountain–pass techniques

Djairo De Figueiredo[(*)]
Mario Girardi[(**)])
Michele Matzeu[(***)]

Abstract. A class of semilinear elliptic equations with dependence on the gradient is considered. The existence of a positive and a negative solution is stated through an iterative method based on mountain pass techniques.

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Introduction. In this paper we consider the solvability of the Dirichlet problem for semilinear elliptic equations with the nonlinearity depending also on the gradient of the solution, namely

$$(1)\qquad \begin{cases} -\Delta u = f(x, u, \nabla u) & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$. This type of equations has not been extensively studied as the case when there is not the presence of the gradient. The obvious reason is that, contrary to the latter case, equation (1) is not variational. So the well developed critical point theory is of no avail for, at least, a direct attack to problem (1) above. For that matter, there have been several works using topological degree. Thus, in this case, appropriate hypotheses are required on the nonlinearity in order to get a priori bounds on the eventual solutions, as well as to obtain sub–and supersolutions. See, for instance, [1], [3], [4], [5], [6], [7], [8], [9], [10]. Our approach here is new and completely distinct of the previous works. Indeed, the technique used in this paper consists of associating with problem (1) a family of semilinear elliptic problems with no dependence on the gradient of the solution. Namely, for each $w \in H_0^1(\Omega)$, we consider the problem

$$(2)\qquad \begin{cases} -\Delta u = f(x, u, Dw) & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

(*) IMECC–UNICAMP–Caixa Postal 6065 – Campinas, S.P. Brazil
(**)) Univ. Roma Tre, Dip. Mat., Largo S. Leonardo Murialdo 1, 00146 Roma, Italy
(***) Univ. Roma Tor Vergata, Dip. Mat., Viale della Ricerca Scientifica, 00133 Roma, Italy.
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Now problem (2) is variational and we can treat it by Variational Methods. In this paper we want to stress the role of mountain–pass techniques to deal with problem (1). So we assign hypotheses on f in such a way that problem (2) can be treated by the mountain–pass theorem by Ambrosetti and Rabinowitz (see [2]).

Our first set of assumptions on the nonlinearity f is the following

(f_0) $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ is locally Lipschitz continuous

(f_1) $\lim_{t\to 0} \frac{f(x,t,\xi)}{t} = 0$ uniformly for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$

(f_2) There exist constants $a_1 > 0$ and $p \in \left(1, \frac{N+2}{N-2}\right)$ such that

$$|f(x,t,\xi)| \leq a_1(1 + |t|^p) \; \forall x \in \overline{\Omega}, \; t \in \mathbb{R}, \; \xi \in \mathbb{R}^N$$

(f_3) There exist constant $\vartheta > 2$ and $t_0 > 0$ such that

$$0 < \theta F(x,t,\xi) \leq t f(x,t,\xi) \; \forall x \in \overline{\Omega}, \; |t| \geq t_0, \; \xi \in \mathbb{R}^N$$

where

$$F(x,t,\xi) = \int_0^t f(x,s,\xi) ds$$

(f_4) There exist constants a_2, $a_3 > 0$ such that

$$F(x,t,\xi) \geq a_2|t|^\vartheta - a_3 \; \forall x \in \overline{\Omega}, \; t \in \mathbb{R}, \; \xi \in \mathbb{R}^N$$

Remark 1. From (f_2) and (f_3) it follows that $\vartheta \leq p + 1$.

Remark 2. An example of a function satisfying the above hypotheses is given by

$$f(x,t,\xi) = b_1|t|^{p-1}tg(\xi)$$

where $b_1 > 0$ and g is an L^∞–function such that $0 < b_2 \leq g(\zeta)$ for some cosntant b_2.

Our first result concerns the solvability of (2) in $H_0^1(\Omega)$, and obtaining bounds on their solutions. The norm in $H_0^1(\Omega)$ is the usual one $\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$.

THEOREM 1. *Suppose that* (f_0), . . . , (f_4) *holds. Then there exist positive constnts* c_1 *and* c_2 *such that, for each* $w \in H_0^1(\Omega)$, *problem (2) has one solution* u_w *such that*

$$c_1 \leq \|u_w\| \leq c_2$$

Moreover, under the above hypotheses, (2) has a positive and a negative solution.

Remark 3. If we are looking only for positive solutions, we need assumptions (f_3) and (f_4) only for positive t.

Our main result concerns the solvability of equation (1). For that matter we need a further assumption:

(f_5) The function f satisfies the following local Lipschitz conditions:

$$|f(x,t',\xi) - f(x,t'',\xi)| \leq L_1|t' - t''| \; \forall x \in \overline{\Omega}, \; t', t'' \in [0, \rho_1], \; |\xi| \leq \rho_2$$

where ρ_1 and ρ_2 depend explicitely on $p, N, \vartheta, a_1, a_2, a_3$ given in the previous hypotheses.

$$|f(x,t,\xi') - f(x,t,\xi'')| \le L_2|\xi' - \xi''| \; \forall\, x \in \overline{\Omega},\; t \in [0,\rho_1] \;\; |\xi'|, |\xi''| \le \rho_2$$

THEOREM 2. *Assume conditions* $(f_0), \ldots, (f_5)$ *holds. Then problem* (1) *has a positive and a negative solution provided*

$$\lambda_1 L_1 + \lambda_1^{1/2} L_2 < 1 \ ,$$

where λ_1 *is the first eigenvalue of* $-\Delta$. *Moreover the solutions obtained are of class* C^2.

The proof of Theorem 1 is given in § 1 using mountain–pass techniques and the proof of Theorem 2 is given in § 2 using iteration methods.

We point out that some analogous techniques can be applied in order to deal with the case that the laplacian is replaced by the p–laplacian (with $p > 2$), with appropriate conditions on the nonlinear term f.

Finally, we remark that a solution of (1) may be obtained without the assumption of condition (f_5), provided the solution u_w of (2), obtained in Lemma 3, is *unique* for each $w \in H_0^1(\Omega)$. In this case the mapping, which associates the unique u_w with each $w \in H_0^1(\Omega)$ is compact, and we can apply Schauder Fixed Point theorem. However, such a uniqueness question is a very hard problem, and there are not many results available in the literature. And in fact, we know of no result for general non–autonomous equation. We can deal here with a very special case. Namely the nonlinearity presented in Remark 2 above, provided the function g is near a constant, p is near 1 and the domain Ω is convex in R^N. For that matter one uses arguments from [4]; this well be discussed elsewhere.

§ **1. Proof of Theorem 1**. As usual, a weak solution of a probem as in (2), which is variational, is obtained as a critical point of an associated functional $I_w : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$I_w(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega F(x, v, \nabla w) \tag{11}$$

The proof of Theorem 1 is broken in several lemmas. We prove that the functional I_w has the geometry of the mountain–pass theorem, that it satisfies the Palais–Smale condition and finally that the obtained solutions have the uniform bounds stated in the theorem.

LEMMA 1. *Let* $w \in H_0^1(\Omega)$. *Then there exist positive numbers* ρ *and* α, *which are independent of* w, *such that*

$$I_w(v) \ge \alpha \;\forall\, v \in H_0^1(\Omega) : \|v\| = \rho \tag{12}$$

Proof. It follows from (f_1) and (f_2) that, given $\varepsilon > 0$, there exists a positive constant k_ε, independent of w, such that

$$|F(x,t,\xi)| \leq \frac{1}{2}\,\varepsilon t^2 + k_\varepsilon |t|^{p+1}$$

So, using Poincaré inequality and Sobolev embedding theorem, we estimate

$$I_w(v) \geq \frac{1}{2}\left(1 - \frac{\varepsilon}{\lambda_1}\right)\|u\|^2 - \tilde{k}_\varepsilon \|u\|^{p+1}$$

where $\tilde{k}_\varepsilon$ is a constant independent of w. Since $p > 1$, the result follows. □

LEMMA 2. *Let $w \in H_0^1(\Omega)$. Fix $v_0 \in H_0^1(\Omega)$, with $\|v_0\| = 1$. Then there is a $T > 0$, independent of w, such that*

$$I_w(tv_0) \leq 0 \quad \textit{for all} \quad t \geq T \tag{13}$$

Proof. It follows from (f_4) that

$$I_w(tv_0) = \frac{1}{2}\,t^2 - \int_\Omega F(x, tv_0, \nabla w) \leq \frac{1}{2}\,t^2 - a_2|t|^\vartheta \int_\Omega |v_0|^\vartheta - a_3|\Omega|$$

Again by Sobolev embedding ($\vartheta \leq p+1$) see Remark 1 we obtain

$$I_w(tv_0) \leq \frac{1}{2}\,t^2 - a_2|t|^\vartheta (S_\vartheta)^\vartheta - a_3|\Omega|$$

where S_ϑ is the constant of the embedding of $H_0^1(\Omega)$ into $L^\vartheta(\Omega)$. Since $\vartheta > 2$, we obtain independent of v_0 and also of w, such that (13) holds.

LEMMA 3. *Assume $(f_0), \ldots, (f_4)$. Then problem (2) has at least one solution $u_w \not\equiv 0$ for any $w \in H_0^1(\Omega)$.*

Proof. Lemmas 1 and 2 show that the functional has the mountain–pass geometry. Hypotheses (f_2) and (f_3) imply, in a standard way, that I_w satisfies the PS–condition. So, by the mountain pass theorem, a weak solution of (2), u_w, is obtained as a critical point of I_w at an inf max level. Namely

$$I'_w(u_w) = 0\ , \quad I_w(u_w) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_w(\gamma(t))$$

where $\Gamma = \{\gamma \in C^0([0,1]; H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = Tv_0\}$, for some v_0 and T as in Lemma 2. From now on we fix such a v_0 and such a T. □

LEMMA 4. *Let $w \in H_0^1(\Omega)$. There exists a positive constant c_1, independent of w, such that*

$$\|u_w\| \geq c_1 \tag{14}$$

for all solutions u_w obtained in Lemma 3.

Proof. Using the equation that u_w satisfies, namely (2), we obtain

$$\int_\Omega |\nabla u_w|^2 = \int_\Omega f(x, u_w, \nabla_w) u_w \tag{15}$$

It follows from (f_1) and (f_2) that, given $\varepsilon > 0$, there exists a positive constant c_ε, independent of w, such that

$$|f(x,t,\xi)| \leq \varepsilon |t| + c_\varepsilon |t|^p$$

Using this inequality, we estimate (15) and obtain

$$\int_\Omega |\nabla u_w|^2 \leq \varepsilon \int_\Omega |u_w|^2 + c_\varepsilon \int_\Omega |u_w|^{p+1}$$

Again by Poincaré inequality and Sobolev embedding, we obtain

$$\left(1 - \frac{\varepsilon}{\lambda_1}\right) \|u_w\|^2 \leq \tilde{c}_\varepsilon \|u_w\|^{p+1}$$

which implies (14). □

LEMMA 5. *Let $w \in H_0^1(\Omega)$. There exists a positive constant c_2, independent of w_1 such that*

$$\|u_w\| \leq c_2 \tag{16}$$

for all solutions u_w obtained in Lemma 3.

Proof. From the inf max characterization of u_w in Lemma 3, we obtain

$$I_w(u_w) \leq \max_{t \geq 0} I_w(tv_0) \tag{17}$$

with v_0 choosen in Lemma 3. We estimate $I_w(tv_0)$ using (f_4):

$$I_w(tv_0) \leq \frac{t^2}{2} - a_2 |t|^\vartheta \int_\Omega |v|^\vartheta + a_3 |\Omega| =: h(t) \ , \tag{18}$$

whose maximum is achieved at some $\bar{t}_0 > 0$ and the value $h(\bar{t}_0)$ can be taken as c_2. Clearly it is independent of w. □

Now we prove the existence of a positive solution (of course the proof of the existence of a negative one is analogous).

Proof. Proceed as in the previous lemmas replacing the nonlinearity f by $\tilde{f}$ defined as

$$\tilde{f}(x,t,\xi) = \begin{cases} f(x,t,\xi) & \text{if} \quad t \geq 0 \\ 0 \quad \text{if} \quad t < 0 \end{cases}$$

Of course $\tilde{f}$ satisfies (f_3) and (f_4) only for $t \geq 0$. For that matter, in the proof of Lemma 2, we choose $v_0 > 0$ in Ω. That is the only modification, since (PS) also holds for such $\tilde{f}$. Applying the mountain press theorem as above, we obtain a solution u_w of (2), namely

$$\begin{cases} -\Delta u_w = \tilde{f}(x, u_w, \nabla w) \quad \text{in} \quad \Omega \\ u_w = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Multiplying the equation by u_w^- and integrating by parks, we conclude that $u_w^- \equiv 0$. So u_w is positive. □

Remark 4. (*On the regularity of the solution of* (2)). In Lemma 3 we have obtained a weak solution u_w of (2) for each given $w \in H_0^1(\Omega)$. since $p < \frac{N+2}{N-2}$, a standard bootstrap argument, using the L^p–regularity theory, shows that u_w is, in fact, in $C^{0,\alpha}(\overline{\Omega})$. Further regularity cannot be obtained if w is an arbitrary function in $H_0^1(\Omega)$. However, if w is C^1, using the Schauder regularity theory, we show that u_w is in $C^{2,\alpha}$. Recall that f is assumed to be locally Lipschitz continuous in all variables. As a consequence of the Sobolev embedding theorems and Lemma 5 we conclude with the following

LEMMA 7. *Let $w \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$. Then there exist positive constants p_1 and p_2, independent of w, such that the solution u_w obtained in Lemma 3 satisfies*

$$\|u_w\|_{C^0} \leq \rho_1 \qquad \|\nabla u_w\|_{C^0} \leq \rho_2$$

§ **2. Proof of Theorem 2**. The idea of the proof consists of using Theorem 1 in an interactive way, as follows. We construct a sequence $\{u_n\} \subset H_0^1(\Omega)$ as solutions of

$$(21)_n \qquad \begin{cases} -\Delta u_n = f(x, u_n, \nabla u_{n-1}) \quad \text{in} \quad \Omega \\ u_n = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

obtained by the mountain pass theorem in Theorem 1, starting with an arbitrary $u_0 \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$.

By Remark 4, we see that

$$\|u_n\|_{C^0} \leq \rho_1 \quad \text{and} \quad \|\nabla u_n\|_{C^0} \leq \rho_2$$

On the other hand, using $(21)_n$ and $(21)_{n+1}$, we obtain

$$\int_\Omega \nabla u_{n+1}(\nabla u_{n+1} - \nabla u_n) = \int_\Omega f(x, u_{n+1}, \nabla u_n)(u_{n+1} - u_n)$$

$$\int_\Omega \nabla u_n(\nabla u_{n+1} - \nabla u_n) = \int_\Omega f(x, u_n, \nabla u_{n-1})(u_{n+1} - u_n)$$

which gives

$$\|u_{n+1} - u_n\|^2 = \int_\Omega [f(x, u_{n+1}, \nabla u_n) - f(x, u_n, \nabla u_n)](u_{n+1} - u_n) + \\ + \int_\Omega [f(x, u_n, \nabla u_n) - f(x, u_n, \nabla u_{n-1})](u_{n+1} - u_n)$$

We can then estimate the integrals above using hypothesis (f_5):

$$\|u_{n+1} - u_n\|^2 \leq L_1 \int_\Omega |u_{n+1} - u_n|^2 + L_2 \int_\Omega |\nabla u_n - \nabla u_{n-1}|\ |u_{n+1} - u_n| \tag{22}$$

Next, using Cauchy–Schartz and Poincaré inequalities, we estimate further (22):

$$\|u_{n+1} - u_n\|^2 \leq L_1 \lambda_1^{-1} \|u_{n+1} - u_n\|^2 + L_2 \lambda_1^{-1/2} \|u_{n+1} - u_n\|\ \|u_n - u_{n-1}\|$$

from which it follows

$$\|u_{n+1} - u_n\| \leq \frac{L_2 \lambda_1^{-1/2}}{1 - L_1 \lambda_1^{-1}} \|u_n - u_{n-1}\| =: k \|u_n - u_{n-1}\|$$

Since the coefficient k is less than 1, then it follows that the sequence $\{u_n\}$ strongly converges in H_0^1 to some function $u \in H_0^1$, as it easily follows proving that $\{u_n\}$ is a Cauchy sequence in H_0^1.

Since $\|u_n\| \geq c_1$ for all n (see Lemma 4), it follows that $u \not\equiv 0$. In this way we obtain a nontrivial solution of (1). As for the positivity of u, we can argue as in lemma 6 of § 1, by replacing f by $\tilde{f}$. □

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