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MATHEMATICS

Generalized Lévy Laplacians and Cesàro Means

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Generalized Lévy Laplacians were introduced in [2] under the name of exotic Lévy Laplacians in relation to the construction from [1] describing a relationship between the classical Lévy Laplacians (introduced by Paul Lévy) and the Yang–Mills equations; the original idea of the construction is due to I.A. Aref'eva.

According to the considerations of [2, 7], a second-order homogeneous linear differential operator acting on number-valued functions defined on the locally convex space (LCS) E is determined by a linear functional S on a vector subspace of the space $\mathcal{L}(E, E^*)$ (the notation is given below). Moreover, the image of a function f under such a differential operator D_S is a function $D_S f(\cdot)$ (not necessarily everywhere defined), whose value at a point $x \in E$ is $D_S f(x) = S(f''(x))$. The generalized Lévy Laplacian is defined as follows. Suppose that an LCS E is a dense vector subspace of a (real) separable Hilbert space for which the embedding $E \to H$ is continuous (so that $E \subset H \subset E^*$ is a rigged Hilbert space) and a functional S is determined by a functional $\mathcal{A}$ on a linear subspace of the space $\mathbb{C}^\infty$ of infinite sequences of complex numbers and by an orthonormal basis $\{e_n\} \subset E$ by means of the equality $S(F) = \mathcal{A}((Fe_n, e_n))$ where $(\cdot, \cdot)$ is an extension of inner product in H. Moreover, $\mathcal{A}(g) = \lim\limits_{n \to \infty} \frac{1}{n^p} \sum\limits_1^n g(n)$, where $p \ge 0$; it is natural to refer to such a functional as the generalized Cesàro mean of order p. For $p = 0$, this definition coincides with that of the Volterra–Gross Laplacian and for $p = 1$, of a the classical Lévy Laplacian.

In [6], for $p = 1$, these authors defined an auxiliary Hilbert space with inner product generated by the functional $\mathcal{A}$ so that the classical Lévy Laplacian reduced to the Volterra–Gross Laplacian. In [4], these constructions were generalized to arbitrary p by using white noise analysis of T. Hida. In recent paper [5], a different generalization of the Lévy Laplacian was used; the corresponding operator was called the nonclassical Lévy Laplacian and defined as the composition of the classical Lévy Laplacian and a linear transformation of the domain of the function to which this operator was applied (for special cases, this definition was given in [2, 3]). In the same paper [5], it was shown that all results on the classical Lévy Laplacian, including its relationship to the theory of quantum random processes considered in [3], can be extended to the nonclassical Lévy Laplacian thus defined.

In this paper, we show that the composition of the generalized Lévy Laplacian of order p (which is generated by the corresponding Cesàro mean) and a certain linear transformation of the domain of the function to which it is applied is proportional to the generalized Lévy Laplacian of smaller or larger order; this makes it possible to transform the generalized Lévy Laplacian of any order into the classical Lévy Laplacian. The construction is based on an expression of generalized Cesàro means in terms of ordinary ones, which is also suggested in this paper. By virtue of results of [5], this implies that all results on the classical Lévy Laplacian have analogues for the generalized Lévy Laplacian. In particular, we consider a relationship between the generalized Lévy Laplacian and quantum random processes.

1. PRELIMINARIES AND GENERAL REMARKS

For real LCSs E and G, $\mathcal{L}(E, G)$ denotes the space of all continuous linear mappings from E to G; $\mathcal{L}(E) = \mathcal{L}(E, E)$. A mapping F from an open subset V of E to the space G is said to be Hadamard differentiable at a point $x \in E$ if the space $\mathcal{L}(E, G)$ contains an element $F'(x)$, which is called the derivative of the mapping F at the point x, such that, for the mapping $r_x\colon E \to G$ defined by $r_x(h) = F(x + h) - F(x) - F'(x)h$, any convergent sequence (h_n) in E, and any number sequence $(t_n) \subset \mathbf{R}^1\backslash\{0,\}$ converging to zero, we have $t_n^{-1} r_x(t_n h_n) \to 0$.

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A mapping $F: E \to G$ is said to be differentiable if it is differentiable at each point of its domain; in this case, $F': E \to \mathcal{L}(E, G)\ x \mapsto F'(x)$ is the derivative of F.

Higher-order derivatives are defined by induction; the spaces $\mathcal{L}(E, G)$ and $\mathcal{L}(E, \mathcal{L}(E, G))$ are assumed to be endowed with the topology of convergence on sequentially compact subsets; in particular, if $G' = \mathbf{R}^1$, then $F'(x) \in E^* = \mathcal{L}(E, R^1)$ and $F''(x) \in \mathcal{L}(E, E^*)$. For $g_1, g_2 \in E^*$, the symbol $g_1 \otimes g_2$ denotes the element of the space $\mathcal{L}(E, E^*)$ defined by $(g_1 \otimes g_2)(x) = g_1 g_2(x)$.

Let S be a linear functional on the vector subspace dom$S$ of $\mathcal{L}(E, E^*)$ nonnegative in the sense that, for each $g \in E^*$, $S(g \otimes g) \geq 0$.

Definition 1. The Laplacian determined by S or the S-Laplacian, is the mapping Δ_S from some subspace domΔ_S of the space of number- (real- or complex-) valued functions on E to the space of all number-valued functions on E defined as follows: $F \in$ domΔ_S if and only if $F''(x) \in$ dom$S$ for any $x \in E$, and, in this case, $(\Delta_S F)(x) = S(F''(x))$.

Certainly, the function $\Delta_S F$ can be assumed to be defined only on some subset of the space E.

Let H_0 be a vector subspace of E^* such that, for any $g_1, g_2 \in H_0$, we have $g_1 \otimes g_2 \in$ dom$S$; then, the bilinear functional b on $H_0 \times H_0$ defined by $b_S(g_1, g_2) = S(g_1 \otimes g_2)$ is a semi-inner product; if $b(g, g) > 0$ for any nonzero $g \in H_0$, then b is an inner product (of course, H_0 may be incomplete with respect to the norm determined by this inner product). In what follows, we use the symbol $\langle \cdot, \cdot \rangle_S$ instead of $b_S(\cdot, \cdot)$.

Example 1. Suppose that E is a Hilbert space, $C \in \mathcal{L}(E)$, and S^C is the functional defined as follows: domS^C is the set of those $A \in \mathcal{L}(E)$ for which the product AC is a nuclear operator, and if $A \in$ domS^C, then $S^C(A) = \mathrm{tr} AC$.

For $C \geq 0$, Δ_{S^C} is called the Volterra (–Gross) Laplacian generated by C. Such operators were considered by Volterra himself (under the assumption that E is a function space); 60 years after that, L. Gross and Yu. Daletskii considered the general case (however, they assumed C to be a nuclear operator).

Proposition 1. *If* $F''(x) \in H_0 \otimes H_0$, *then* $\Delta_S F(x) = \mathrm{tr}_S F''(x)$, *where* tr *is the trace on* $\mathcal{L}(H_0)$ *generated by the inner product* $\langle \cdot, \cdot \rangle_S$.

This proposition follows from the definition of the inner product $\langle \cdot, \cdot \rangle_S$.

This means that the operator Δ_S can be reduced to the Volterra Laplacian.

In the next section, we introduce the generalized Cesàro means; then (in the third section), we apply them to construct functionals on the operator space which determine generalized Lévy Laplacians.

2. GENERALIZED CESÀRO MEANS

In this section, we define generalized Cesàro means and show how to reduce calculating them to calculating usual Cesàro means. Generalized Cesàro means are linear functionals on vector subspaces of the space $\mathbb{C}^\infty$. The elements of $\mathbb{C}^\infty$ are identified with functions of positive integer argument taking values in $\mathbb{C}$.

For each $p \geq 0$, consider the functional A_p defined by

$$\mathrm{dom} A_p = \left\{ a \in \mathbb{C}^\infty;\ \exists \lim_{n \to \infty} \frac{1}{n^p} \sum_{k=1}^{n} a(k) \right\};$$

if $a \in$ domA_p, then

$$A_p(a) = \lim_{n \to \infty} \frac{1}{n^p} \sum_{k=1}^{n} a(k).$$

Definition 2. The functional A_p is called the generalized Cesàro mean of order p (for $p = 1$, this is the usual Cesàro mean).

Let N be the operator on $\mathbb{C}^\infty$ defined as follows: if $a \in \mathbb{C}^\infty$, then $(Na)(n) = na(n)$ (so that $(N^p a)(n) = n^p a(n)$ for each $p \in \mathbb{R}$).

Lemma 1. *Suppose that* $p \geq 0$ *and* $a \in \mathbb{C}^\infty$. *If* $\frac{a(n)}{n^p} \to 0$ *as* $n \to \infty$, *then* $\frac{1}{n^{p+1}} \sum_{1}^{n} a(k) \to 0$ *as* $n \to \infty$; *if* $\frac{a(n)}{n^{p+1}} \to 0$ *as* $n \to \infty$, *then* $\frac{1}{n^p} \sum_{1}^{n} \frac{a(k)}{k^2} \to 0$ *as* $n \to \infty$.

Proof. Suppose that $\frac{a(n)}{n^p} \to 0$ as $n \to \infty$. Then, for any $\varepsilon > 0$, there exists a positive integer n_0 such that if $n > n_0$, then $\left| \frac{a(n)}{n^p} \right| < \varepsilon$, or, equivalently, $|a(n)| < n^p \cdot \varepsilon$. But

$$\left| \sum_{1}^{n} a(k) \right| \leq \left| \sum_{1}^{n_0} a(k) + \sum_{n_0}^{n} a(k) \right| \leq \left| \sum_{1}^{n_0} a(k) \right| + \left| \sum_{n_0}^{n} a(k) \right| \leq \left| \sum_{1}^{n_0} a(k) \right| + \max_{n > n_0} |a(n)| \cdot n < \left| \sum_{1}^{n_0} a(k) \right| + n^{p+1} \cdot \varepsilon.$$

Hence, there exists an $n_1 > n_0$ such that if $n > n_1$, then $\frac{1}{n^{p+1}} \sum_{1}^{n} a(k) < \varepsilon$. This proves the first assertion of the lemma; the proof of the second is similar.

Theorem 1. *Let $p > 0$. If $a \in \mathrm{dom}A_p$, then $Na \in \mathrm{dom}A_{p+1}$ and*

$$A_{p+1}(Na) = \frac{p}{p+1}A_p(a).$$

Theorem 2. *Let $p > 0$. If $c \in \mathrm{dom}A_{p+1}$, then $N^{-1}c \in \mathrm{dom}A_p$ and*

$$\frac{p+1}{p}A_{p+1}(c) = A_p(N^{-1}c).$$

Proof of Theorem 1. Suppose that $g \in \mathrm{dom}A_p$, where $p > 0$. This implies the existence of $\lim\limits_{t\to\infty}\frac{1}{n^p}\sum_1^n g(n)(= A_p(g))$. Applying the Abel transform [8], we obtain

$$\frac{1}{n^{p+1}}\sum_1^n kg(k) = \frac{1}{n^{p+1}}\left(\left(\sum_1^n g(k)\right)\cdot n - \sum_{k=1}^{n-1}\sum_{r=1}^{k} g(r)\right) = \frac{1}{n^p}\sum_1^n g(k) - \frac{1}{n^{p+1}}\sum_{k=1}^{n-1}\sum_{r=1}^{k} g(r).$$

The existence of the limit $\lim\limits_{n\to\infty}\frac{1}{n^p}\sum_1^n g(k)$ means that $\frac{1}{n^p}\left[\sum_1^n g(k) - n^p \cdot A_p(g)\right] \to 0$ as $n \to \infty$; therefore, according to the first part of Lemma 1,

$$\lim_{n\to\infty}\frac{1}{n^{p+1}}\sum_1^n kg(k) = A_{p+1}(Ng) = \lim_{n\to\infty}\frac{1}{n^p}\sum_1^n g(k) - \lim_{n\to\infty}\frac{1}{n^{p+1}}\sum_1^{n-1}k^p\cdot A_p(g) = A_p(g) - \frac{1}{p+1}A_p(g)$$

$$= \frac{p}{p+1}A_g(p)$$

(we use the equality $\lim\limits_{n\to\infty}\frac{1}{n^{p+1}}\sum_1^n k^p = \frac{1}{p+1}$, which can be proved by, e.g., passing to its integral counterpart and taking into account the equality $\lim\limits_{n\to\infty}\frac{1}{n^{p+1}}\sum_1^n k^p = \lim\limits_{t\to\infty}\frac{1}{t^{p+1}}\int_1^t s^p ds$). This proves Theorem 1.

Proof of Theorem 2. Suppose that $f \in \mathrm{dom}A_{p+1}$, where $p > 0$. This implies the existence of $\lim\limits_{n\to\infty}\frac{1}{n^{p+1}}\sum_1^n f(k) = A_{p+1}(f)$; i.e., $\frac{1}{n^{p+1}}\left[\sum_1^n f(k) - n^{p+1}\times A_{p+1}(f)\right] \to 0$ as $t \to \infty$. Again using the Abel transform and applying the second assertion of Lemma 1, we obtain

$$A_p(N^{-1}f) = \frac{1}{n^p}\sum_1^n\frac{f(k)}{k} = \frac{1}{n^p}\left(\left(\sum_1^n f(k)\right)\cdot\frac{1}{n} + \sum_1^{n-1}\sum_{s=1}^{k} f(s)\cdot\frac{1}{k(k+1)}\right) = \frac{1}{n^{p+1}}\sum_1^n f(k)$$

$$+ \frac{1}{n^p}\sum_1^{n-1}\left(\sum_{s=1}^{k} f(s)\right)\frac{1}{k(k+1)}dt = \lim_{n\to\infty}\frac{1}{n^{p+1}}$$

$$\times\sum_1^n f(k) + \lim_{n\to\infty}\frac{1}{n^p}\sum_1^{n-1}\frac{k^{p+1}}{k(k+1)}A_p(f)d\tau$$

$$= A_{p+1}(f) + \frac{1}{p}A_{p+1}(f) = \frac{p+1}{p}A_{p+1}(f),$$

which completes the proof of the theorem.

Remark 1. Of course, to prove Theorem 2, it suffices to show that $N^{-1}f \in A_p$ and apply Theorem 1.

Corollary 1. *If $g \in \mathrm{dom}A_1$ and $p > 0$, then $N^p g \in \mathrm{dom}A_{p+1}$ and*

$$A_{p+1}(N^p g) = \frac{1}{p+1}A_1(g).$$

Corollary 2. *If $g \in \mathrm{dom}A_{p+1}$ and $p > 0$, then $N^{-p}g \in \mathrm{dom}A_1$ and*

$$A_1(N^{-p}g) = (p+1)A_p(g).$$

3. GENERALIZED LÉVY LAPLACIANS

Let E be a locally convex subspace of a separable Hilbert space H; this means that E is a dense linear subspace of H and the canonical embedding $E \subset H$ is continuous; in this case, $H^* \subset E^*$, and H^* is a dense vector subspace in E^*. We assume that H^* is endowed with the topology determined by its Hilbert norm and E^* is endowed with any topology compatible with the duality between E^* and E. We identify H with H^*; for $x \in E$ and $g \in E^*$, we set $(g, x) = g(x)$. Let $(e_n) \subset E$ be an orthonormal basis in H.

Definition 3. The generalized Cesàro trace of order $p > 0$ is the functional C_p on the space $\mathcal{L}(E, E^*)$ defined as follows: if $D \in \mathcal{L}(E, E^*)$, then $C_p(D) = A_p((De_n, e_n))$.

Definition 4. The generalized Lévy Laplacian of order $p > 0$ is the operator Δ_{C_p} (of course, this definition depends on the choice of the basis).

In what follows, we write Δ^p instead of Δ_{C_p}. Let $\mathcal{N}$ be an operator from $\mathcal{L}(E, E^*)$ defined by $\mathcal{N}e_k = ke_k$. For each number-valued function f on E, by $f \circ \mathcal{N}^q$ we denote the composition of the corresponding mappings.

Theorem 3. *If* $p \geq 1$, *then* $\Delta^p f = \frac{1}{p}\Delta^1\left(f \circ \mathcal{N}^{-\frac{p-1}{2}}\right)$.

This theorem follows from Corollary 1. Thus, we have expressed the generalized Lévy Laplacian in terms of the classical Lévy Laplacian.

Remark 2. It can be shown (see [3, 5]) that the nonclassical Lévy Laplacians considered in [1] coincide with compositions of the classical Lévy Laplacian and operators of the form $\mathcal{N}^q$, where $q < 0$; thus, the preceding theorem describes a relationship between them and the generalized Lévy Laplacians (this relationship was mentioned in [2]).

4. GENERALIZED LAPLACIANS AND QUANTUM RANDOM PROCESSES

A quantum random process is a function defined on a part of the real line and taking values in an operator space. In this section, the exposition is partly formal, and we do not fix any particular space of this kind.

Let $H = L_2(0, \pi)$, and let E be the space of infinitely differentiable functions on the interval $[0, \pi]$ vanishing at zero endowed with the topology of uniform convergence of functions and their derivatives of any order; take the orthonormal basis with elements determined by the equalities $e_n(t) = \sqrt{\frac{2}{\pi}}\sin t$. For each $h \in E^*$, by $b(h)$ we denote the operator of differentiation in the direction h (see [5]), which acts on the function space on E. Thus, for $z \in C_2(E)$ and $x \in E^*$, we have $b(h)(z)(x) = z'(x)h$. If $h = \delta_t$, then we write $b(t)$ instead of $b(h)$. The quantity $\int_0^\pi b(t)^2 dt^2 \equiv \int_0^\pi b(t)b(t)dt^2$ is defined by

$$\int_0^\pi b(t)^2 dt^2 = \lim_{\epsilon \to 0} \int_{\{|t-s| < \epsilon;\ t, s \in [0, \pi]\}} b(t)b(s)dt\, ds$$

(the integral on the right-hand side is the integral of an operator-valued function); the other notations are similar. For each negative integer n, we have $b^{(n)}(t) = \int_0^t b^{(n+1)}(\tau)d\tau$ [5].

Theorem 4. *For any positive integer p,*

$$\Delta^p = \frac{1}{p}\int_0^\pi b^{(1-p)}(t)b^{(1-p)}(t)dt^2.$$

This theorem follows from Theorem 3 and results obtained in [3, 5].

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