On the density-density critical indices in interacting Fermi systems

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Abstract

The behaviour of correlation functions of $d = 1$ interacting fermionic systems is determined by a small number of critical indices. We prove that one of them is exactly zero. As a consequence, the behavior of the Fourier transform of the density-density correlation at zero momentum is qualitatively unaffected by the interaction, contrary to what happens at $\pm 2\tilde{p}_F$, if $\tilde{p}_F$ is the Fermi momentum. The result is obtained by implementing Ward identities in a Renormalization Group approach.

1 Introduction and main results

1.1 Motivations and results

If $a^\pm_x$, $x = -[\frac{L-1}{2}], ..., [\frac{L}{2}]$, is a set of fermionic creation and annihilation operators, we consider the Hamiltonian

$$H = \sum_{x=-[\frac{L-1}{2}]}^{[\frac{L}{2}]} \Big\{ \frac{1}{2}(a^+_{x+1} - a^+_x)(a^-_{x+1} - a^-_x) - \mu a^+_x a^-_x + \lambda (a^+_x a^-_x - \frac{1}{2})(a^+_{x+1} a^-_{x+1} - \frac{1}{2}) \Big\}, \quad (1)$$

describing a system of spinless fermions in $d = 1$ with chemical potential μ, a nearest-neighbor interaction and periodic boundary conditions. The space-time *density-density correlation function* at temperature β^{-1} is given by

$$\Omega_{L,\beta}(\mathbf{x}) =< a^+_{\mathbf{x}} a^-_{\mathbf{x}} a^+_{\mathbf{0}} a^-_{\mathbf{0}} >_{L,\beta} - < a^+_{\mathbf{x}} a^-_{\mathbf{x}} >_{L,\beta} < a^+_{\mathbf{0}} a^-_{\mathbf{0}} >_{L,\beta}, \tag{2}$$

where $\mathbf{x} = (x, x_0)$, $a^{\pm}_{\mathbf{x}} = e^{Hx_0} a^{\pm}_x e^{-Hx_0}$ and $< . >_{L,\beta}= Tr[e^{-\beta H}.]/Tr[e^{-\beta H}]$ denotes the expectation in the grand canonical ensemble. We shall use also the notation $\Omega(\mathbf{x}) \equiv \lim_{L,\beta\to\infty} \Omega_{L,\beta}(\mathbf{x})$.

If the fermions are non interacting ($\lambda = 0$), one can easily check that, if $|\mathbf{x}| \geq 1$, $\cos p_F = 1 - \mu$, $v_0 = \sin p_F > 0$,

$$\begin{aligned}
&\Omega(\mathbf{x}) = \cos(2p_F x)\Omega^a_0(\mathbf{x}) + \Omega^b_0(\mathbf{x}) + \Omega^c_0(\mathbf{x}),\\
&\Omega^a_0(\mathbf{x}) = \frac{1}{2\pi^2[x^2 + (v_0 x_0)^2]},\\
&\Omega^b_0(\mathbf{x}) = \frac{1}{2\pi^2[x^2 + (v_0 x_0)^2]}\frac{x_0^2 - (x/v_0)^2}{x^2 + (v_0 x_0)^2},\\
&|\Omega^c_0(\mathbf{x})| \leq \frac{1}{1 + |\mathbf{x}|^{2+\vartheta}},
\end{aligned} \tag{3}$$

for some positive constant $\vartheta < 1$.

The interaction has two main effects: the period of the oscillating term $\cos(2p_F x)\Omega^a_0(\mathbf{x})$ changes and the large distance asymptotic decay is modified by *critical indices*. It was indeed proved in [BM] by a Renormalization Group analysis that, for λ small enough and $|\mathbf{x}| \geq 1$,

$$\begin{aligned}
&\Omega(\mathbf{x}) = \cos(2\tilde p_F x)\Omega^a(\mathbf{x}) + \Omega^b(\mathbf{x}) + \Omega^c(\mathbf{x}),\\
&\Omega^a(\mathbf{x}) = \frac{1 + \lambda B_1(\mathbf{x})}{2\pi^2[x^2 + (v_0^* x_0)^2]^{1+\eta_a}},\\
&\Omega^b(\mathbf{x}) = \frac{1}{2\pi^2[x^2 + (v_0^* x_0)^2]^{1+\eta_b}}\Big\{\frac{x_0^2 - (x/v_0^*)^2}{x^2 + (v_0^* x_0)^2} + \lambda B_2(\mathbf{x})\Big\},\\
&|B_i(\mathbf{x})| \leq C, \qquad |\Omega^c(\mathbf{x})| \leq \frac{1}{1 + |\mathbf{x}|^{2+\vartheta}},
\end{aligned} \tag{4}$$

where C is a positive constant, η_a, η_b are critical indices expressed by convergent series in λ, $v_0^* = v_0 + \delta^*$ and $\tilde p_F(\lambda, p_F) = p_F + \lambda f(\lambda, p_F)$ with δ^*, f analytic in λ and $|\delta^*| \leq C|\lambda|$, $|f(\lambda, p_F)| \leq C$; note that $f(\lambda, \frac{\pi}{2}) = 0$, by symmetry reasons.

By an explicit computation of the lowest order of the convergent series for η_a one obtains that $\eta_a = -a_1\lambda + O(\lambda^2)$, where $a_1 > 0$ is a non vanishing constant. The lowest order contributions to η_b are instead *vanishing*, in agreement with the conjecture (see for instance [Sp]) that η_b is *exactly zero*. The aim of this paper is to prove such conjecture.

Theorem 1.1 *There exists a positive constant λ_0 such that, if $|\lambda| \le \lambda_0$, the density-density correlation function (2) can be written as in (4) with the critical index η_b identically vanishing.*

The vanishing of the critical index η_b has many interesting consequences. For instance, see [BM], if $\lambda = 0$ the Fourier transform $\hat\Omega(k)$ of $\Omega(x,0)$ has three cusps, at $k = 0$ and $k = \pm 2p_F$, *i.e.* $\partial_k \Omega(k)$ has a first order discontinuity at $k = 0$ and $k = \pm 2p_F$. The vanishing of $\eta_b = 0$ implies that $\hat\Omega(k)$ has still a cusp at $k = 0$ even if $\lambda \neq 0$; in fact it was proved in [BM] that, if $\eta_b = 0$, the possible logarithmic singularity of $\partial\hat\Omega(k)$ at $k = 0$ is changed by a parity cancellation into a first order discontinuity with jump $1 + O(\lambda)$; this is remarkable because, generally, the qualitative behaviour close to critical points is deeply changed by the interaction; for instance $\partial_k \hat\Omega(k)$ at $k = \pm 2\tilde p_F$ in the $\lambda \neq 0$ case is continuous for $\lambda < 0$, while it diverges as $|k - (\pm 2\tilde p_F)|^{2\eta_a}$ for $\lambda > 0$.

Note finally that the model (1) is equivalent to the XXZ spin-chain with magnetic field $h = \mu - 1$, as one can show by a *Schwinger-Dyson* transformation [LSM], with (2) representing the spin-spin correlation function along the third axis. Moreover our proof that $\eta_b = 0$ could be easily extended to a large class of models; for instance one can replace the nearest neighbor interaction with a non nearest neighbor one, or the lattice with a continuum, or to consider the anisotropic *XYZ spin chain*, see [BM]. We remember finally that there are remarkable relations, based on exact solutions, between properties of quantum spin chains and bidimensional classical statistical mechanics models; for instance the spin-spin correlation function of the XYZ spin chain is believed to be equal to the correlation between two vertical arrows in the same row in the *eigth vertex model*, see [B] and [JKM], if a suitable identification of the parameters is done. Hence our results could be relevant also for such problems. Another application is for models of vicinal surfaces, see [Sp].

1.2 Remarks

In [BM] we derived a convergent expansion for the critical index η_b; each order is obtained by summing up a certain numbers of terms, and $\eta_b = 0$ means that there is a cancellation at all orders between such terms. While one can easily check from such expansion that this is the case at the second order, to prove directly that such cancellation occurs at all orders looks to us essentially impossible. We proceed instead in a different way and our proof is conceptually divided in two main steps.

For the first step we refer to [BM], where the proof that $\eta_b = 0$ is reduced to a special property (see (20) below) of the Schwinger functions of a model (which we will call *reference model*), describing fermions with a linear "relativistic" dispersion relation and allowed momenta restricted by infrared and ultraviolet cut-offs. This result, which is resumed in Theorem 2.2 below, gives further ground to the remarkable observation of Tomonaga [T], according to which the model (1) is essentially equivalent, as far as the low energy behaviour is considered, to a system of interacting massless relativistic fermions.

In the second step we deduce such property of the reference model by using a suitable *Ward identity*, which is obtained through a local gauge transformation. Usually in relativistic quantum field theory Ward identities are relations between correlation functions; the Ward identity we find is instead a relation between correlation functions *and* some other extra terms, which we call "correction" as they would be formally zero if the cut-offs were removed. The extra terms do not vanish when the infrared cut-off is removed. The property that we need is reduced to suitable bounds (see Theorems 2.1 and 2.3), proved by using convergent expansions for all terms appearing in the Ward identity.

We conclude the introduction with a technical note. With respect to previous applications of Wilsonian Renormalization Group to $d = 1$ interacting fermionic theories, like [BG] or [BGPS], we are able here to rigorously implement in this scheme the method of Ward identities (based on local gauge transformations) to produce non trivial results. In the physical literature there are many claims on the vanishing of η_b, see for instance [DL], [ES], [DM], and our results convert such ideas into a rigorous proof. Finally, note that there are many examples of QFT models in which Ward Identities are implemented in a mathematical way, perturbatively (see for instance [FHRW], [KK]) or non perturbatively (see for instance [BFS] or [MSR]). However such works consider the application of Ward Identities to *relativistic* QFT; hence corrections to formal exact Ward Identities are possibly found as a consequence of the cut-offs imposed to regularize the theory, but they are vanishing when the cut-offs are removed. The main novelty of our paper is that we try to implement the method of Ward identities in the *not relativistic* model (1), where there is no reason why a Ward Identity involving only correlation functions should be valid. The corrections are not vanishing and the technical problem is to get for such terms bounds good enough to prove that $\eta_b = 0$.

2 Ward Identities

2.1 The reference model

The reference model is not Hamiltonian and is defined in terms of *Grassmann variables*. Given the interval $[0, L]$, the inverse temperature β and the (large) integer N, we introduce in $\Lambda = [0, L] \times [0, \beta]$ a lattice Λ_N, whose sites are given by the *space-time points* $\mathbf{x} = (x, x_0) = (na, n_0 a_0)$, $a = L/N$, $a_0 = \beta/N$, $n, n_0 = 0, 1, \ldots, N-1$. We also consider the set $\mathcal{D}$ of *space-time momenta* $\mathbf{k} = (k, k_0)$, with $k = \frac{2\pi}{L}(n + \frac{1}{2})$ and $k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})$, $n, n_0 = 0, 1, \ldots, N-1$. With each $\mathbf{k} \in \mathcal{D}$ we associate four Grassmanian variables $\hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega}$, $\sigma, \omega \in \{+, -\}$. The lattice Λ_N is introduced only for technical reasons so that the number of Grassmann variables is finite, and eventually the limit $N \to \infty$ is taken (and it is trivial, see [BM], §2.1). If γ is a fixed number greater than 1 and h is a negative integer, we define the function $[C_{h,0}]^{-1}(\mathbf{k})$ as a strictly positive smooth function acting as a cut-off for momenta $|\mathbf{k}| \geq \gamma$ (ultraviolet region) and $|\mathbf{k}| \leq \gamma^{h-1}$ (infrared region) and having value 1 in the intermediate region $\gamma^h \leq |\mathbf{k}| \leq 1$. The infrared cut-off γ^h is not fixed, because we are interested in the dependence on h of the reference model. The exact definition of $[C_{h,0}]^{-1}(\mathbf{k})$ is the following one. We introduce a positive function $\chi_0 \in C^\infty(\mathbb{R}_+)$ such that

$$\chi_0(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \,, \\ 0 & \text{if } t \geq \gamma_0 \,, \quad 1 < \gamma_0 \leq \gamma \end{cases} \tag{5}$$

and we define, for any integer $j \leq 0$,

$$f_j(\mathbf{k}) = \chi_0(\gamma^{-j}|\mathbf{k}|) - \chi_0(\gamma^{-j+1}|\mathbf{k}|) \,. \tag{6}$$

Then we define $[C_{h,0}(\mathbf{k})]^{-1} = \sum_{j=h}^{0} f_j(\mathbf{k})$. If $\tilde{\mathcal{D}} = \{\mathbf{k} \in \mathcal{D} : [C_{h,0}(\mathbf{k})]^{-1} \neq 0\}$, we define the functional integration $\int D\psi^{[h,0]}$ as the linear functional on the Grassmann algebra generated by the variables $\hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega}$, such that, given a monomial $Q(\hat{\psi})$ in the variables $\hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega}$, its value is 0, except in the case $Q(\hat{\psi}) = \prod_{\mathbf{k} \in \tilde{\mathcal{D}}, \omega = \pm} \hat{\psi}^{[h,0]-}_{\mathbf{k},\omega} \hat{\psi}^{[h,0]+}_{\mathbf{k},\omega}$, up to a permutation of the variables. In this case the value of the functional is determined, by using the anticommuting properties of the variables, by $\int D\psi^{[h,0]} Q(\hat{\psi}) = 1$. We also define the *Grassmanian field* on the lattice Λ_N as

$$\psi^{[h,0]\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k} \in \mathcal{D}} e^{i\sigma\mathbf{k}\mathbf{x}} \hat{\psi}^{[h,0]\sigma}_{\mathbf{k},\omega} \,, \quad \mathbf{x} \in \Lambda_N \,. \tag{7}$$

Note that $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$ is antiperiodic both in time and space variables.

The *Schwinger functions* of the reference model are

$$S(\mathbf{x}_1,\sigma_1,\omega_1;...;\mathbf{x}_s,\sigma_s,\omega_s) = \frac{\int P(d\psi^{[h,0]})e^{-V(\psi^{[h,0]})}\prod_{i=1}^s \psi^{[h,0]\sigma_i}_{\mathbf{x}_i,\omega_i}}{\int P(d\psi^{[h,0]})e^{-V(\psi^{[h,0]})}}\;, \tag{8}$$

where

$$V(\psi^{[h,0]}) = \lambda\int d\mathbf{x}\; \psi^{[h,0]+}_{\mathbf{x},+}\psi^{[h,0]-}_{\mathbf{x},+}\psi^{[h,0]+}_{\mathbf{x},-}\psi^{[h,0]-}_{\mathbf{x},-}\;, \tag{9}$$

$\int d\mathbf{x}$ is a shorthand for "$a\ a_0\ \sum_{\mathbf{x}\in\Lambda_N}$" and

$$P(d\psi^{[h,0]}) = \mathcal{N}^{-1}D\psi^{[h,0]}\;\cdot \tag{10}$$
$$\cdot\ \exp\left\{-\frac{1}{L\beta}\sum_{\omega=\pm1}\sum_{\mathbf{k}\in\tilde{\mathcal{D}}} C_{h,0}(\mathbf{k})(-ik_0+\omega k)\hat\psi^{[h,0]+}_{\mathbf{k},\omega}\hat\psi^{[h,0]-}_{\mathbf{k},\omega}\right\}\;,$$

with $\mathcal{N} = \prod_{\mathbf{k}\in\tilde{\mathcal{D}}}[(L\beta)^{-2}(-k_0^2-k^2)C_{h,0}(\mathbf{k})^2]$.

We also define the *connected Schwinger functions* as the functional derivatives of the *Generating functional*

$$\mathcal{W}(\phi,J) = \log\int P(d\psi)e^{-V(\psi)+\sum_\omega\int d\mathbf{x}\left[J_{\mathbf{x},\omega}\psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}+\phi^+_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}+\psi^{[h,0]+}_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}\right]} \tag{11}$$

with respect to the *external field variables* $\phi^\sigma_{\mathbf{x},\omega}$ and $J_{\mathbf{x},\omega}$, $\mathbf{x}\in\Lambda_N$, $\omega=\pm1$. The variables $\phi^\sigma_{\mathbf{x},\omega}$ are antiperiodic in x_0 and x and anticommuting with themselves and $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$, while the variables $J_{\mathbf{x},\omega}$ are periodic and commuting with themselves and all the other variables. We shall need in particular the following connected Schwinger functions:

$$G^{2,1}_\omega(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{\partial}{\partial J_{\mathbf{x},\omega}}\frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}}\mathcal{W}(\phi,J)|_{\phi=J=0}\;, \tag{12}$$

$$G^2_\omega(\mathbf{y},\mathbf{z}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}}\mathcal{W}(\phi,J)|_{\phi=J=0}\;. \tag{13}$$

They will be pictorially represented as in Fig. 1.

We also need the Fourier transforms of $G^{2,1}_\omega$ and G^2_ω, defined by

$$G^2_\omega(\mathbf{x},\mathbf{y}) = \frac{1}{(L\beta)}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\hat G^2_\omega(\mathbf{k})\;, \tag{14}$$

$$G^{2,1}_\omega(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{1}{(L\beta)^2}\sum_{\mathbf{k},\mathbf{p}} e^{i\mathbf{p}\mathbf{x}}e^{-i\mathbf{k}\mathbf{y}}e^{i(\mathbf{k}-\mathbf{p})\mathbf{z}}\hat G^{2,1}_\omega(\mathbf{p},\mathbf{k})\;, \tag{15}$$

In §3 we prove the following bounds for the reference model with cut-off γ^h, which has to be of course larger than $\min\{\pi/L,\pi/\beta\}$ (otherwise the set $\mathcal{D}$ is empty).

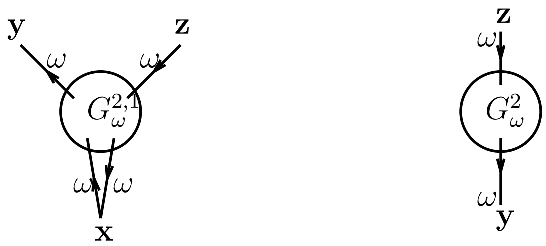

Figure 1: Graphical representation of the connected Schwinger functions $G^{2,1}_\omega$ and G^2_ω.

Theorem 2.1 *There exists a positive constant λ_0, independent of h, such that, if $|\lambda| \leq \lambda_0$, there exist two positive functions of λ, $Z^{(2)}_h$ and Z_h, and a positive constant C, independent of h, so that, uniformly in N, L, β large enough, if $\bar{k} \in \mathcal{D}$ is such that $\gamma^h \leq |\bar{\mathbf{k}}| \leq \gamma^{h+1}$*

$$\hat{G}^{2,1}_\omega(2\bar{\mathbf{k}}, -\bar{\mathbf{k}}) = -\frac{Z^{(2)}_h}{Z^2_h D_\omega(\bar{\mathbf{k}})^2}[1 + O(\lambda^2)] \, , \tag{16}$$

$$\hat{G}^2_\omega(\bar{\mathbf{k}}) = \frac{1}{Z_h D_\omega(\bar{\mathbf{k}})}[1 + O(\lambda^2)] \, , \tag{17}$$

$$C\lambda^2|h| \leq \log Z^{(2)}_h \leq 2C\lambda^2|h| \, , \qquad C|h|\lambda^2 \leq \log Z_h \leq 2C\lambda^2|h| \tag{18}$$

with $D_\omega(\mathbf{k}) = -ik_0 + \omega k$. Moreover

$$\lim_{h\to-\infty} \log \frac{Z^{(2)}_{h-1}}{Z^{(2)}_h} = \eta_2(\lambda) \, , \qquad \lim_{h\to-\infty} \log \frac{Z_{h-1}}{Z_h} = \eta(\lambda) \, , \tag{19}$$

with $\eta(\lambda) = a_2\lambda^2 + O(\lambda^3)$, and $\eta_2(\lambda) = a_2\lambda^2 + O(\lambda^3)$ where a_2 is a positive constant.

The connection between the model (1) and the reference model is given by the following theorem, which is proved in [BM] , even if it is not explicitly formulated. To be more precise, in §5.5 of [BM] we show that the condition (20), equivalent to eq. (5.35) of [BM], implies the bound (5.38) of [BM], which is equivalent to say that $\eta_b = 0$.

Theorem 2.2 *Under the same assumptions of Theorem 2.1, there exists a constant C such that, if for all negative integer h the functions $Z_h, Z^{(1)}_h$ in (16), (17) verify*

$$C\lambda^2 \leq |\frac{Z^{(2)}_h}{Z_h} - 1| \leq 2C\lambda^2 \, , \tag{20}$$

then in(4) $\eta_b(\lambda) = 0$.

Hence by Theorem 2.2 the proof of $\eta_b = 0$ is reduced to the verification of (20), to which the rest of this paper is devoted. Note that (20) is equivalent, by (18), to $\eta(\lambda) - \eta_2(\lambda) = 0$ (in Theorem 2.1 it is only claimed that $\eta(\lambda) - \eta_2(\lambda) = O(\lambda^3)$).

2.2 Ward identities for the reference model

We have so far reduced the proof that $\eta_b = 0$ in the model (1) to the verification of (20) in the reference model. This result will be achieved by using an identity relating $\hat{G}^2_\omega$ to $\hat{G}^{2,1}$, obtained by performing a *local gauge transformation*, together with equations (16), (17).

In order to derive such identity, we find convenient to introduce a cut-off function $[C^\varepsilon_{h,0}(\mathbf{k})]^{-1}$, where ε is a small positive parameter and $\lim_{\varepsilon\to 0^+}[C^\varepsilon_{h,0}(\mathbf{k})]^{-1} = C_{h,0}(\mathbf{k})]^{-1}$. The functions $[C^\varepsilon_{h,0}(\mathbf{k})]^{-1}$ and $[C_{h,0}(\mathbf{k})]^{-1}$ are equivalent as far as the scaling properties of the theory are concerned but the support of $[C^\varepsilon_{h,0}(\mathbf{k})]^{-1}$ is the set $\mathcal{D}$ instead of $\tilde{\mathcal{D}}$. The definition (10) of the reference model is easily extended to the case in which the cut-off is $[C^\varepsilon_{h,0}(\mathbf{k})]^{-1}$ instead of $[C_{h,0}(\mathbf{k})]^{-1}$, by substituting in the r.h.s. of (10), as well as in the definition of the integration $\int D\psi^{[h,0]}$, the set $\tilde{\mathcal{D}}$ with $\mathcal{D}$. A reason why we find this convenient is that a technically important role in the following is played by the gauge invariance of the integration $\int D\psi^{[h,0]}$, a property which is lost if the Grassmann algebra is restricted to the variables $\hat{\psi}_{\mathbf{k},\omega}$ with $\mathbf{k} \in \tilde{\mathcal{D}}$.

The exact definition of $[C^\varepsilon_{h,0}(\mathbf{k})]^{-1}$ is the following one. Given a positive $\varepsilon << 1$, we define

$$\chi^\varepsilon_{h,0}(\mathbf{k}) = [C^\varepsilon_{h,0}(\mathbf{k})]^{-1} = \sum_{j=h}^{0} f^\varepsilon_j(\mathbf{k}) \;, \tag{21}$$

where $f^\varepsilon_j(\mathbf{k}) = f_j(\mathbf{k})$, if $h+1 \le j \le -1$, while $f^\varepsilon_0(\mathbf{k})$ and $f^\varepsilon_h(\mathbf{k})$ are obtained by slightly modifying $f_0(\mathbf{k})$ and $f_h(\mathbf{k})$ in the following way. $f^\varepsilon_0(\mathbf{k})$ is a C^∞ function of $|\mathbf{k}|$, such that $\lim_{\varepsilon\to 0} f^\varepsilon_0 = f_0$, $f^\varepsilon_0(\mathbf{k}) = f_0(\mathbf{k})$ for $\gamma^{-1} \le |\mathbf{k}| \le 1$, $f^\varepsilon_0(\mathbf{k}) > 0$ for $|\mathbf{k}| \ge 1$ and, if $|\mathbf{k}| \ge \gamma$, $0 < f^\varepsilon_0(\mathbf{k}) \le \varepsilon e^{-|\mathbf{k}|}$. Analogously, $f^\varepsilon_h(\mathbf{k})$ is a C^∞ function of $|\mathbf{k}|$, such that $\lim_{\varepsilon\to 0} f^\varepsilon_h = f_h$, $f^\varepsilon_h(\mathbf{k}) = f_h(\mathbf{k})$ for $\gamma^h \le |\mathbf{k}| \le \gamma^{h+1}$, $f^\varepsilon_h(\mathbf{k}) > 0$, if $0 < |\mathbf{k}| \le \gamma^h$, and if $0 < |\mathbf{k}| \le \gamma^{h-1}$, $0 < f^\varepsilon_h(\mathbf{k}) \le \varepsilon \exp(-|\mathbf{k}|^{-1})$.

Hence, we first study the case $\varepsilon > 0$, for which a Ward identity can be easily obtained, relating the Schwinger functions of interest for us, for which the limit $\varepsilon \to 0$ is trivial, and a "correction term", which is apparently singular as $\varepsilon \to 0$. However we prove that this term can be written as a

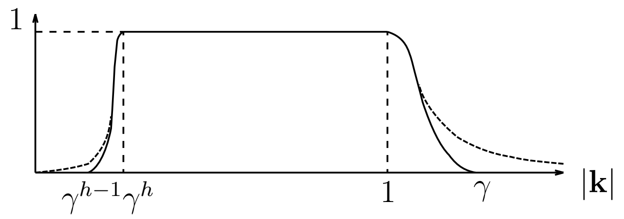

Figure 2: The cutoff functions $[C^{\varepsilon}_{h,0}(\mathbf{k})]^{-1}$ (dashed line) and $[C_{h,0}(\mathbf{k})]^{-1}$ (solid line).

suitable expansion, whose contributions admit "good" bounds uniformly in ε, as well in N, L, β, and have a well defined limit as $\varepsilon \to 0$.

By writing $\psi^{\sigma}_{\mathbf{x},\omega}$ in place of $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$ for simplicity, we can write

$$P(d\psi) = \mathcal{N}^{-1} D\psi \exp\Big[- \int d\mathbf{x}\, \psi^{+}_{\mathbf{x},\omega} D^{[h,0]}_{\omega} \psi^{-}_{\mathbf{x},\omega}\Big]\,, \tag{22}$$

where

$$D^{[h,0]}_{\omega} \psi^{\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k}} e^{i\sigma\mathbf{k}\mathbf{x}} C^{\varepsilon}_{h,0}(\mathbf{k})(i\sigma k_0 - \omega\sigma k)\hat{\psi}^{\sigma}_{\mathbf{k},\omega}\,. \tag{23}$$

By performing the gauge transformation

$$\psi^{\sigma}_{\mathbf{x},\bar\omega} \to e^{i\sigma\alpha_{\mathbf{x},\bar\omega}} \psi^{\sigma}_{\mathbf{x},\bar\omega}\,, \quad \psi^{\sigma}_{\mathbf{x},-\bar\omega} \to \psi^{\sigma}_{\mathbf{x},-\bar\omega} \tag{24}$$

and by using the invariance of $\int D\psi$ after such transformation, we can rewrite the r.h.s. of (11) as

$$\begin{aligned} \mathcal{W}(\phi, J) = \log \int P(d\psi) \exp\Big\{ - \int d\mathbf{x}\, \psi^{+}_{\mathbf{x},\bar\omega}\Big(e^{i\alpha_{\mathbf{x},\bar\omega}} D^{[h,0]}_{\bar\omega} e^{-i\alpha_{\mathbf{x},\bar\omega}} - D^{[h,0]}_{\bar\omega}\Big)\psi^{-}_{\mathbf{x},\bar\omega}\Big\} \cdot \\ \cdot \exp\Big\{ - V(\psi) + \int d\mathbf{x}[\sum_{\omega} J_{\mathbf{x},\omega}\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},\omega} + \\ + e^{-i\alpha_{\mathbf{x},\bar\omega}}\phi^{+}_{\mathbf{x},\bar\omega}\psi^{-}_{\mathbf{x},\bar\omega} + e^{i\alpha_{\mathbf{x},\bar\omega}}\psi^{+}_{\mathbf{x},\bar\omega}\phi^{-}_{\mathbf{x},\bar\omega} + \phi^{+}_{\mathbf{x},-\bar\omega}\psi^{-}_{\mathbf{x},-\bar\omega} + \psi^{+}_{\mathbf{x},-\bar\omega}\phi^{-}_{\mathbf{x},-\bar\omega}]\Big\}\,, \end{aligned} \tag{25}$$

Since $\sum_{\mathbf{x}\in\Lambda_N} \psi^{+}_{\mathbf{x},\bar\omega}[D^{[h,0]}_{\bar\omega}\alpha_{\mathbf{x},\bar\omega}\psi^{-}_{\mathbf{x},\bar\omega}] = -\sum_{\mathbf{x}\in\Lambda_N}[D^{[h,0]}_{\bar\omega}\psi^{+}_{\mathbf{x},\bar\omega}]\alpha_{\mathbf{x},\bar\omega}\psi^{-}_{\mathbf{x},\bar\omega}$ and $\mathcal{W}(\phi, J)$ is independent of $\alpha_{\mathbf{x},\bar\omega}$, differentiating both sides of (25) with respect to $\alpha_{\mathbf{x},\bar\omega}$ and by putting $\alpha_{\mathbf{x},\bar\omega} = 0$, we get

$$\begin{aligned} 0 = \frac{1}{Z(\phi,J)} \int P(d\psi)[D_{\bar\omega}(\psi^{+}_{\mathbf{x},\bar\omega}\psi^{-}_{\mathbf{x},\bar\omega}) + \delta T_{\mathbf{x},\bar\omega} - \phi^{+}_{\mathbf{x},\bar\omega}\psi^{-}_{\mathbf{x},\bar\omega} + \psi^{+}_{\mathbf{x},\bar\omega}\phi^{-}_{\mathbf{x},\bar\omega}] \;\cdot \\ \cdot\; \exp\Big\{ - V(\psi) + \sum_{\omega}\int d\mathbf{x}[J_{\mathbf{x},\omega}\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},\omega} + \phi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},\omega} + \psi^{+}_{\mathbf{x},\omega}\phi^{-}_{\mathbf{x},\omega}]\Big\}, \end{aligned} \tag{26}$$

where $Z(\phi,J)=\exp\{\mathcal{W}(\phi,J)\}$, D_ω is defined as $D_\omega^{[h,0]}$, see (23), with 1 in place of $C^\varepsilon_{h,0}(\mathbf{k})$, so that, if $D_\omega(\mathbf{p})=-ip_0+\omega p$

$$D_\omega(\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega})=\frac{1}{(L\beta)^2}\sum_{\mathbf{p},\mathbf{k}}D_\omega(\mathbf{p})e^{-i\mathbf{p}\mathbf{x}}\hat\psi^+_{\mathbf{k},\omega}\hat\psi^-_{\mathbf{k}-\mathbf{p},\omega}\ . \tag{27}$$

where $\mathbf{p}=(p,p_0)$ is summed over momenta of the form $(2\pi n/L, 2\pi m/\beta)$, with n,m integers Moreover

$$\delta T_{\mathbf{x},\omega}=\frac{1}{(L\beta)^2}\sum_{\mathbf{k}^+\neq\mathbf{k}^-}e^{i(\mathbf{k}^+-\mathbf{k}^-)\mathbf{x}}C^\varepsilon(\mathbf{k}^+,\mathbf{k}^-)\hat\psi^+_{\mathbf{k}^+,\omega}\hat\psi^-_{\mathbf{k}^-,\omega}\ , \tag{28}$$

$$C^\varepsilon(\mathbf{k}^+,\mathbf{k}^-)=[C^\varepsilon_{h,0}(\mathbf{k}^-)-1]D_\omega(\mathbf{k}^-)-[C^\varepsilon_{h,0}(\mathbf{k}^+)-1]D_\omega(\mathbf{k}^+)\ , \tag{29}$$

By differentiating the r.h.s. of (26) with respect to $\phi^+_{\mathbf{y},\bar\omega}$ and $\phi^-_{\mathbf{z},\bar\omega}$ and then setting the external fields equal to 0, we obtain, in terms of the Fourier transform

$$-D_\omega G^{2,1}_\omega(\mathbf{x};\mathbf{y},\mathbf{z})=\delta(\mathbf{x}-\mathbf{y})G^2_\omega(\mathbf{x},\mathbf{z})-\delta(\mathbf{x}-\mathbf{z})G^2_\omega(\mathbf{y},\mathbf{x})+\Delta^{2,1}_\omega(\mathbf{x};\mathbf{y},\mathbf{z})\ , \tag{30}$$

where

$$\Delta^{2,1}_\omega(\mathbf{x};\mathbf{y},\mathbf{z})=<\psi^-_{\mathbf{y},\omega};\psi^+_{\mathbf{z},\omega};\delta T_{\mathbf{x},\omega}>^T\ . \tag{31}$$

If $A_1,\ldots,A_n$ are functions of the field, we are using the symbol

$$<A_1;\ldots;A_n>^T=\frac{\partial^n}{\partial\lambda_1\ldots\partial\lambda_n}\log\int P(d\psi)e^{-V(\psi)+\sum_{i=1}^n\lambda_iA_i}\Big|_{\underline{\lambda}=0}\ . \tag{32}$$

It is convenient to express the *Ward identity* (30) in terms of the Fourier transforms of the connected Schwinger functions; $\hat\Delta^{2,1}_\omega(\mathbf{p},\mathbf{k})$ is defined in a similar way to $\hat G^{2,1}_\omega(\mathbf{p},\mathbf{k})$. In terms of the Fourier transform (30) can be written (see Fig. 3) as

$$D_\omega(\mathbf{p})\hat G^{2,1}_\omega(\mathbf{p},\mathbf{k})=\hat G^2_\omega(\mathbf{k}-\mathbf{p})-\hat G^2_\omega(\mathbf{k})+\hat\Delta^{2,1}_\omega(\mathbf{p},\mathbf{k})\ , \tag{33}$$

If $\mathbf{p}\neq 0$, (33) can be written in the form

$$G^{2,1}_\omega(\mathbf{p},\mathbf{k})=\frac{G^2_\omega(\mathbf{k}-\mathbf{p})-G^2_\omega(\mathbf{k})}{D_\omega(\mathbf{p})}+\hat H^{2,1}_\omega(\mathbf{k},\mathbf{p})\ , \tag{34}$$

where $\hat H^{2,1}_\omega(\mathbf{k},\mathbf{p})$ is the Fourier transform, defined in agreement with (15), of

$$H^{2,1}_\omega(\mathbf{x};\mathbf{y},\mathbf{z})=\frac{\partial}{\partial J_{\mathbf{x},\omega}}\frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}}\mathcal{W}_\Delta(\phi,J)|_{\phi=J=0}\ , \tag{35}$$

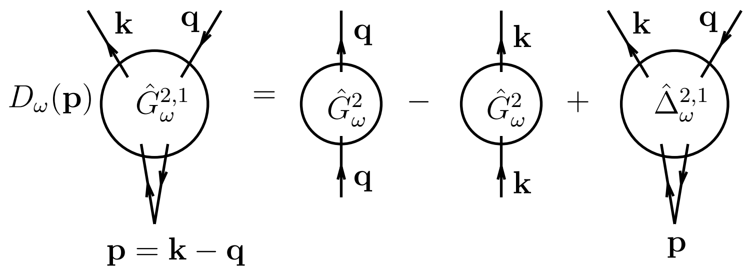

Figure 3: Graphical representation of the identity (33).

with

$$W_\Delta(\phi, J) = \log \int P(d\psi) e^{-V(\psi)+\sum_\omega \int d\mathbf{x}[J_{\mathbf{x},\omega}T_{\mathbf{x},\omega}+\phi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}+\psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]}\;, \tag{36}$$

$$T_{\mathbf{x},\omega} = \frac{1}{(L\beta)^2} \sum_{\mathbf{k}^+\neq\mathbf{k}^-} e^{i(\mathbf{k}^+-\mathbf{k}^-)\mathbf{x}} \frac{C^\varepsilon(\mathbf{k}^+,\mathbf{k}^-)}{D_\omega(\mathbf{k}^+-\mathbf{k}^-)} \hat\psi^+_{\mathbf{k}^+,\omega}\hat\psi^-_{\mathbf{k}^-,\omega}\;. \tag{37}$$

Equation (34) is our Ward identity; it involves not only correlation functions but also the term $\hat H^{2,1}_\omega(\mathbf{k},\mathbf{p})$, which we can call the *correction term* as it would be formally zero in absence of cut-offs. Note that the definition (35) of the correction term $H^{2,1}_\omega$ is similar to the definition (12) of $G^{2,1}_\omega$, but the two quantities have very different properties. In fact $H^{2,1}_\omega$ can be obtained by substituting $\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}$ in (11) with $T_{x,\omega}$, given by (37), which looks as a very singular term as $\varepsilon \to 0$. We are nevertheless able to express also $\hat H^{2,1}_\omega(\mathbf{k},\mathbf{p})$ by a convergent expansion, and we can prove in §4 the following bound.

Theorem 2.3 *There exists a positive constant λ_0, independent of h, such that, if $|\lambda| \le \lambda_0$, then, uniformly in ε small enough and N, L, β large enough*

$$C\gamma^{-2h}\lambda^2 \frac{Z^{(2)}_h}{(Z_h)^2} \le |\hat H^{2,1}_\omega(2\bar{\mathbf{k}}, -\bar{\mathbf{k}})| \le 2C\gamma^{-2h}\lambda^2 \frac{Z^{(2)}_h}{(Z_h)^2}\;. \tag{38}$$

Moreover, $\lim_{\varepsilon\to 0} \hat H^{2,1}_\omega$ *does exist.*

The above result (it was already claimed in [BM] referring for the proof to the present paper) says that $\hat H^{2,1}_\omega$ behaves, as $h \to -\infty$, exactly as $\hat G^{2,1}_\omega(2\bar{\mathbf{k}}, -\bar{\mathbf{k}})$, but its bound has an *extra* λ^2 *factor*. This is just what we need; if we insert (16), (17) and (38) in (33), we obtain (20) and hence, by Theorem 2.1 and 2.2, $\eta_b = 0$.

2.3 Remarks

In the physical literature Ward identities for interacting $d = 1$ fermions with cut-offs are usually derived by various formal arguments, see for example [DL], [ES], [MD], [S]. All arguments are essentially equivalent to expanding $\hat{G}^2_\omega$ and $\hat{G}^{2,1}_\omega$ in Feynman graphs and then "forget" the cut-off function. In fact, if we neglect the cut-offs, the propagator is simply $D_\omega(\mathbf{k})^{-1}$ and the "identity" $G^{2,1}_\omega(\mathbf{p},\mathbf{k}) = [G^2_\omega(\mathbf{k}-\mathbf{p}) - G^2_\omega(\mathbf{k})]/D(\mathbf{p})$, from which $\eta_b = 0$ follows, is derived by the following obvious identity

$$D_\omega(\mathbf{p})^{-1}[D_\omega(\mathbf{k})^{-1} - D_\omega(\mathbf{k}+\mathbf{p})^{-1}] = D_\omega(\mathbf{k}+\mathbf{p})^{-1}D_\omega(\mathbf{k})^{-1}\ , \tag{39}$$

By taking consistently into account the cut-off function one gets, instead of (39), the identity

$$\frac{\hat{g}_\omega(\mathbf{k}) - \hat{g}_\omega(\mathbf{k}+\mathbf{p})}{D_\omega(\mathbf{p})} = \hat{g}_\omega(\mathbf{k})\hat{g}_\omega(\mathbf{k}+\mathbf{p}) + \hat{g}_\omega(\mathbf{k})\hat{g}_\omega(\mathbf{k}+\mathbf{p})\frac{C^\varepsilon(\mathbf{k},\mathbf{k}+\mathbf{p})}{D(\mathbf{p})}\ , \tag{40}$$

which allows in principle to check directly equation (34) at any order (very easily at order 0, which coincides with (34)). Our analysis shows then that one can still derive from the Ward identities the vanishing of η_b in a rigorous way, by taking into account the presence of cut-offs. This however seems *not true* for other consequences of Ward identities for the model (1) claimed in the literature, see [BM1].

Note also that, as $\varepsilon \to 0$, $[C^\varepsilon_{h,0}(\mathbf{k})]^{-1}$ becomes a compact support function, so $C^\varepsilon(\mathbf{k},\mathbf{k}+\mathbf{p})$ becomes singular. However the singularity at $\varepsilon = 0$ of the function $C^\varepsilon(\mathbf{k},\mathbf{k}+\mathbf{p})$ in the second addend of the r.h.s. in (40) is of course compensated by the cut-off functions appearing in the propagators. Hence one could "in principle" derive (34) directly using a compact support cut-off (*i.e.* using $[C_{h,0}]^{-1}$ instead of $[C^\varepsilon_{h,0}]^{-1}$), for instance by a Feynman graph analysis using (40) at $\varepsilon = 0$, but such derivation would be surely much more lengthy.

3 Renormalization Group analysis

3.1 The effective potentials and the beta function

The results in Theorem 2.1 and 2.3 can be derived by expressing $\hat{G}^2_\omega$, $\hat{G}^{2,1}_\omega$ and $\hat{H}^{2,1}_\omega$ by a suitable multiscale expansion based on Renormalization group ideas. In the following sections we will prove (16),(17),(38), referring to [BM] for the proof of many technical lemmas we will need.

We begin our analysis, for clarity reason, by studying the "*free energy*" of the model, which is the simplest quantity which can be studied by our method; it is defined by

$$E_{L,\beta} = -\frac{1}{L\beta}\log\int P(d\psi^{[h,0]})e^{-V(\psi^{[h,0]})} \;. \tag{41}$$

The functional integration in (41) can be performed iteratively by a slight modification of the procedure described (for instance) in sec.(2.5)-(2.8) of [BM]. We prove by induction that, for any negative integer j, there are a constant E_j, a positive function $\tilde{Z}_j(\mathbf{k})$ and a functional $\mathcal{V}^{(j)}$ such that

$$\int P(d\psi^{[h,0]})e^{-V(\psi^{[h,0]})} = \int P_{\tilde{Z}_j,C^{\varepsilon}_{h,j}}(d\psi^{[h,j]})\, e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})-L\beta E_j} \;, \tag{42}$$

with $\mathcal{V}^{(j)}(0)=0$, $Z_j = \max_{\mathbf{k}}\tilde{Z}_j(\mathbf{k})$,

$$P_{\tilde{Z}_j,C^{\varepsilon}_{h,j}}(d\psi^{[h,j]}) = \prod_{\mathbf{k}:C^{\varepsilon}_{h,j}(\mathbf{k})>0}\prod_{\omega=\pm1}\frac{d\hat{\psi}^{[h,j])+}_{\mathbf{k},\omega}d\hat{\psi}^{[h,j]-}_{\mathbf{k},\omega}}{\mathcal{N}_j(\mathbf{k})}\,\cdot$$

$$\cdot\;\exp\left\{-\frac{1}{L\beta}\sum_{\mathbf{k}}C^{\varepsilon}_{h,j}(\mathbf{k})\tilde{Z}_j(\mathbf{k})\sum_{\omega\pm1}\hat{\psi}^{[h,j]+}_{\omega}D_\omega(\mathbf{k})\hat{\psi}^{[h,j]-}_{\mathbf{k},\omega}\right\} \;, \tag{43}$$

$$C^{\varepsilon}_{h,j}(\mathbf{k})^{-1} = \sum_{r=h}^{j} f^{\varepsilon}_r(\mathbf{k}) \equiv \chi_{h,j}(\mathbf{k}) \tag{44}$$

and $\mathcal{N}_j(\mathbf{k}) = (L\beta)^{-1}C^{\varepsilon}_{h,j}(\mathbf{k})\tilde{Z}_j(\mathbf{k})[-k_0^2-k^2]^{1/2}$. Finally, $\mathcal{V}^{(j)}$ which can be written as

$$\mathcal{V}^{(j)}(\psi) = \sum_{n=1}^{\infty}\frac{1}{(L\beta)^{2n}}\sum_{\substack{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}\\ \omega_1,\ldots,\omega_{2n}}}\prod_{i=1}^{2n}\hat{\psi}^{\sigma_i}_{\mathbf{k}_i,\omega_i}\hat{W}^{(j)}_{2n,\underline{\omega}}(\mathbf{k}_1,...,\mathbf{k}_{2n-1})\delta\left(\sum_{i=1}^{2n}\sigma_i\mathbf{k}_i\right) \;, \tag{45}$$

where $\sigma_i = +$ for $i=1,\ldots,n$, $\sigma_i = -$ for $i = n+1,\ldots,2n$ and $\underline{\omega} = (\omega_1,\ldots,\omega_{2n})$.

Equation (42) is in fact true for $j=0$, with

$$\tilde{Z}_0(\mathbf{k}) = 1, \quad E_0 = 0, \quad \mathcal{V}^{(0)}(\psi) = V(\psi) \;. \tag{46}$$

Assume then that it is true for j and we show that it holds also for $j-1$.

First of all, we split $\mathcal{V}^{(j)}$ as $\mathcal{L}\mathcal{V}^{(j)} + \mathcal{R}\mathcal{V}^{(j)}$, where $\mathcal{R} = 1-\mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator on functions of the form (45), defined in the following way by its action on the kernels $\hat{W}^{(j)}_{2n,\underline{\omega}}$.

1. If $2n=4$, then

$$\mathcal{L}\hat{W}^{(j)}_{4,\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\hat{W}^{(j)}_{4,\underline{\omega}}(\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++})\;, \tag{47}$$

where we used the definition

$$\bar{\mathbf{k}}_{\eta\eta'}=\left(\eta\frac{\pi}{L},\eta'\frac{\pi}{\beta}\right)\;,\qquad \eta,\eta'=\pm\;. \tag{48}$$

Note that $\mathcal{L}\hat{W}^{(j)}_{4,\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=0$, if $\sum_{i=1}^4\omega_i\neq 0$, by simple symmetry considerations.

2. If $2n=2$ (in this case there is a non zero contribution only if $\omega_1=\omega_2$)

$$\mathcal{L}\hat{W}^{(j)}_{2,\underline{\omega}}(\mathbf{k})=\frac{1}{4}\sum_{\eta,\eta'=\pm 1}\hat{W}^{(j)}_{2,\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})\left\{1+\eta\frac{L}{\pi}+\eta'\frac{\beta}{\pi}k_0\right\}\;. \tag{49}$$

In order to better understand this definition, note that, if $L=\beta=\infty$,

$$\mathcal{L}\hat{W}^{(j)}_{2,\underline{\omega}}(\mathbf{k})=\hat{W}^{(j)}_{2,\underline{\omega}}(\mathbf{0})+k\frac{\partial\hat{W}^{(j)}_{2,\underline{\omega}}}{\partial k}(\mathbf{0})+k_0\frac{\partial\hat{W}^{(j)}_{2,\underline{\omega}}}{\partial k_0}(0)\;. \tag{50}$$

3. In all other cases

$$\mathcal{L}\hat{W}^{(j)}_{2n,\underline{\omega}}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1})=0\;. \tag{51}$$

The above definitions are such that $\mathcal{L}^2=\mathcal{L}$, a property which plays an important role in the analysis of [BM]. Moreover

$$\mathcal{L}\mathcal{V}^{(j)}(\psi^{[h,j]})=z_jF^{[h,j]}_{\zeta}+a_jF^{[h,j]}_{\alpha}+l_jF^{[h,j]}_{\lambda}\;, \tag{52}$$

where z_j, a_j and l_j are real numbers and

$$\begin{aligned}
F^{[h,j]}_{\alpha} &= \sum_{\omega}\frac{\omega}{(L\beta)}\sum_{\mathbf{k}:C^{\varepsilon}_{h,j}(\mathbf{k})>0}k\hat{\psi}^{[h,j]+}_{\mathbf{k},\omega}\hat{\psi}^{[h,j]-}_{\mathbf{k},\omega}=\\
&= \sum_{\omega}i\omega\int_{\Lambda}d\mathbf{x}\psi^{[h,j]+}_{\mathbf{x},\omega}\partial_x\psi^{[h,j]-}_{\mathbf{x},\omega}\;, \qquad (53)\\
F^{[h,j]}_{\zeta} &= \sum_{\omega}\frac{1}{(L\beta)}\sum_{\mathbf{k}:C^{\varepsilon}_{h,j}(\mathbf{k})>0}(-ik_0)\hat{\psi}^{[h,j]+}_{\mathbf{k},\omega}\hat{\psi}^{[h,j]-}_{\mathbf{k}',\omega}=\\
&= -\sum_{\omega}\int_{\Lambda}d\mathbf{x}\psi^{[h,j]+}_{\mathbf{x},\omega}\partial_0\psi^{[h,j]-}_{\mathbf{x},\omega}\;, \qquad (54)\\
F^{[h,j]}_{\lambda} &= \frac{1}{(L\beta)^4}\sum_{\substack{\mathbf{k}_1,\ldots,\mathbf{k}_4:\\ C^{\varepsilon}_{h,j}(\mathbf{k}_i)>0}}\hat{\psi}^{[h,j]+}_{\mathbf{k}_1,+}\hat{\psi}^{[h,j]-}_{\mathbf{k}_2,+}\hat{\psi}^{[h,j]+}_{\mathbf{k}_3,-}\hat{\psi}^{[h,j]-}_{\mathbf{k}_4,-}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4)=\\
&= \int_{\Lambda}d\mathbf{x}\psi^{[h,j]+}_{\mathbf{x},+}\psi^{[h,j]-}_{\mathbf{x},+}\psi^{[h,j]+}_{\mathbf{x},-}\psi^{[h,j]-}_{\mathbf{x},-}\;. \qquad (55)
\end{aligned}$$

∂_x and ∂_0 are discrete derivatives defined so that the second equality in (53) and (54) is satisfied; if $N = \infty$ they are simply the partial derivative with respect to x and x_0. Note that $\mathcal{L}\mathcal{V}^{(0)} = \mathcal{V}^{(0)}$, hence $l_0 = \lambda$, $a_0 = z_0 = 0$. There is no local term proportional to $\sum_{\mathbf{k}} \hat\psi^{[h,j]+}_{\mathbf{k},\omega} \hat\psi^{[h,j]-}_{\mathbf{k},\omega}$, because of the parity properties of the propagator.

We now *renormalize* $P_{\tilde Z_j, C^\varepsilon_{h,j}}(d\psi^{[h,j]})$, by adding to it part of the quadratic part of the r.h.s. of (52). We get

$$\int P_{\tilde Z_j, C^\varepsilon_{h,j}}(d\psi^{[h,j]})\, e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})} = \\ = e^{-L\beta t_j} \int P_{\tilde Z_{j-1}, C^\varepsilon_{h,j}}(d\psi^{[h,j]})\, e^{-\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})} \;, \tag{56}$$

where

$$\tilde Z_{j-1}(\mathbf{k}) = Z_j(\mathbf{k})[1 + \chi^\varepsilon_{h,j}(\mathbf{k}) z_j] \;, \tag{57}$$

$$\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) = \mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) - z_j Z_j [F^{[h,j]}_\zeta + F^{[h,j]}_\alpha] \,, \tag{58}$$

and the factor $\exp(-L\beta t_j)$ in (56) takes into account the different normalization of the two functional integrals.

If $j > h$, the r.h.s of (56) can be written as

$$e^{-L\beta t_j} \int P_{\tilde Z_{j-1}, C^\varepsilon_{h,j-1}}(d\psi^{[h,j-1]}) \int P_{Z_{j-1}, \tilde f_j^{-1}}(d\psi^{(j)})\, e^{-\tilde{\mathcal{V}}^{(j)}\left(\sqrt{Z_j}[\psi^{[h,j-1]} + \psi^{(j)}]\right)} \;, \tag{59}$$

where $P_{Z_{j-1}, \tilde f_j^{-1}}(d\psi^{(j)})$ is the integration with propagator

$$\hat g^{(j)}_\omega(\mathbf{k}) = \frac{1}{Z_{j-1}} \frac{\tilde f_j(\mathbf{k})}{D_\omega(\mathbf{k})} \;, \tag{60}$$

with $\tilde f_j(\mathbf{k}) = f^\varepsilon_j(\mathbf{k}) Z_{j-1} [\tilde Z_{j-1}(\mathbf{k})]^{-1}$. It is $\tilde Z_{j-1}(\mathbf{k}) = Z_0 + \sum_{i=j}^0 Z_i z_i \chi^\varepsilon_{h,i}(\mathbf{k})$ and, if $j > h$ and $f_j(\mathbf{k}) \neq 0$, then $\tilde Z_{j-1}(\mathbf{k}) = Z_j + Z_j z_j [f^\varepsilon_{j-1}(\mathbf{k}) + f^\varepsilon_i(\mathbf{k})]$, so that the propagators for $j > h$ do not depend of the infrared cut-off and we have

$$\tilde f_j(\mathbf{k}) = f^\varepsilon_j(\mathbf{k}) \frac{Z_j(1 + z_j)}{Z_j + Z_j z_j [f^\varepsilon_{i-1}(\mathbf{k}) + f^\varepsilon_i(\mathbf{k})]} \le f^\varepsilon_j(\mathbf{k})(1 + z_j) \,. \tag{61}$$

This equation also implies that $\hat g^{(j)}_\omega(\mathbf{k})$ is of size $Z^{-1}_{j-1}\gamma^{-j}$.

All the dependence on the infrared cut-off is restricted to the integration of the field of scale h, whose propagator (see (56) with $j = h$) is

$$\hat g^{(h)}(\mathbf{k}) = \frac{f^\varepsilon_h(\mathbf{k})}{\tilde Z_{h-1}(\mathbf{k}) D_\omega(\mathbf{k})} = \frac{f^\varepsilon_h(\mathbf{k})}{D_\omega(\mathbf{k})} \frac{1}{Z_0 + \sum_{i=h}^0 Z_i z_i \chi^\varepsilon_{h,i}(\mathbf{k})} \;. \tag{62}$$

The latter propagator $\hat{g}^{(h)}(\mathbf{k})$ depends strongly on $\mathbf{k}$ near the cut-off; in fact, if $f_h(\mathbf{k}) \neq 0$ but $f_{h+1}(\mathbf{k}) = 0$, then

$$\hat{g}^{(h)}(\mathbf{k}) = \frac{f_h^\varepsilon(\mathbf{k})}{D_\omega(\mathbf{k})} \frac{1}{Z_0 + (Z_{h-1} - Z_0) f_h^\varepsilon(\mathbf{k})} . \tag{63}$$

However, $\hat{g}^{(j)}(\mathbf{k})$ is of size $Z_{j-1}^{-1}\gamma^{-j}$ even for $j = h$, because

$$\frac{Z_{h-1} f_h^\varepsilon(\mathbf{k})}{Z_0 + (Z_{h-1} - Z_0) f_h^\varepsilon(\mathbf{k})} \leq 2 . \tag{64}$$

We now *rescale* the field so that

$$\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) = \hat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}\psi^{[h,j]}) ; \tag{65}$$

it follows that

$$\mathcal{L}\hat{\mathcal{V}}^{(j)}(\psi^{[h,j]}) = \delta_j F_\alpha^{[h,j]} + \lambda_j F_\lambda^{[h,j]} , \tag{66}$$

where

$$\delta_j = \frac{Z_j}{Z_{j-1}}(a_j - z_j) , \quad \lambda_j = \left(\frac{Z_j}{Z_{j-1}}\right)^2 l_j . \tag{67}$$

We call the pairs $\vec{v}_j = (\delta_j, \lambda_j)$ the *running coupling constants* on scale j. A simple perturbative calculation shows that $\lambda_{-1} = \lambda + O(\lambda^2)$, $a_{-1} = O(\lambda^2)$, $z_{-1} = O(\lambda^2)$.

Finally

$$e^{-\mathcal{V}^{(j-1)}(\sqrt{Z_{j-1}}\psi^{[h,j-1]}) - L\beta\tilde{E}_j} = \int P_{Z_{j-1},\tilde{f}_j^{-1}}(d\psi^{(j)})\, e^{-\tilde{\mathcal{V}}^{(j)}\left(\sqrt{Z_{j-1}}[\psi^{[h,j-1]}+\psi^{(j)}]\right)} , \tag{68}$$

and $\mathcal{V}^{(j-1)}(\sqrt{Z_{j-1}}\psi^{[h,j-1]})$ is of the form (45); moreover it satisfies the identity (42), with $E_{j-1} = E_j + t_j + \tilde{E}_j$. This completes the iterative step.

We finally define

$$e^{-L\beta\tilde{E}_h} = \int P_{Z_h,\tilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}\left(\sqrt{Z_h}\psi^{(h)}\right)} , \tag{69}$$

so that

$$E_{L,\beta} = E_h = \sum_{j=h}^{-1} \tilde{E}_j + \sum_{j=h+1}^{-1} t_j . \tag{70}$$

Note that the above procedure allows us to write, in particular, the running coupling constants $\vec{v}_j$, $0 < j \leq h$, in terms of $\vec{v}_{j'}$, $0 \geq j' \geq j+1$:

$$\vec{v}_j = \vec{\beta}(\vec{v}_{j+1}, \dots, \vec{v}_0) , \qquad \vec{v}_0 = (\lambda, 0) . \tag{71}$$

The function $\vec{\beta}(\vec{v}_{j+1}, ..., \vec{v}_0)$ is called the *Beta function*. The fact that it is well defined, for small values of λ, in the limit $L, \beta \to \infty$, is a highly non trivial result, see [BG, BGPS, BoM1, BM].

Finally note that Z_h represents the *wave function renormalization* of the fermionic field, δ_j the renormalization of its velocity and λ_j is the effective coupling of the theory at scale j.

3.2 The tree expansion

One can write the effective potential on scale j, if $h \le j < 0$, as a sum of terms, which is in fact a finite sum for finite values of N, L, β. Each term of this expansion is associated with a *tree* in the following way.

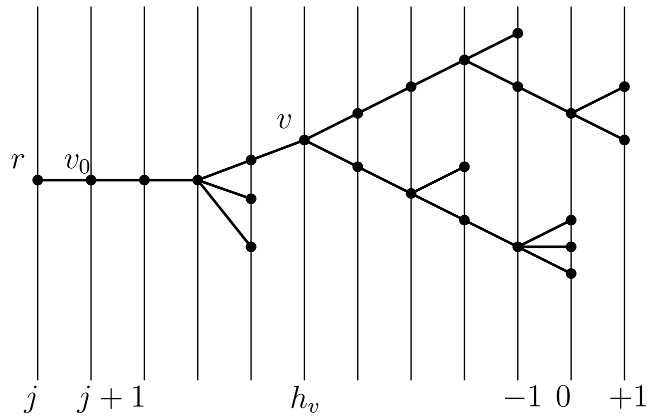

Figure 4: Example of a tree.

1. Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \ge 1$ points, the *endpoints* of the *unlabeled tree* (see Fig. 4), so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

   Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n.

We shall consider also *labeled trees* (which we shall call simply *trees* in the following); they are defined by associating certain labels with the unlabeled trees, as explained in the following items.

2. We associate a label $j \leq 0$ with the root and we denote $\mathcal{T}_{j,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[j, 1]$, and we represent any tree $\tau \in \mathcal{T}_{j,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index j. There is the constraint that, if v is an endpoint, $h_v > j + 1$.

   The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

   Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $j + 1$.

3. With each endpoint v of scale h_v we associate one of the two local terms contributing to $\mathcal{L}\hat{\mathcal{V}}^{(h_v)}(\psi^{[h,h_v-1]})$ in the r.h.s. of (66) and one space-time point $\mathbf{x}_v$. We shall say that the endpoint is of type δ or λ, with an obvious correspondence with the two terms. Note that there is no endpoint of type δ, if $h_v = +1$.

   Given a vertex v, which is not an endpoint, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v.

   Moreover, we impose the constraint that, if v is an endpoint, $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v.

4. If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

5. We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\sigma(f)$ and $\omega(f)$ will

denote the space-time point, the σ index and the ω index, respectively, of the field variable with label f.

6. If the endpoint v is of type δ, one of the field variables belonging to I_v carries also a derivative. In (53) this derivative acts on the field ψ^-, but we could also choose a representation of $F_\zeta^{[h,j]}$ such that the derivative acts on the field ψ^+. Which representation is used depends on detailed properties of the different terms associated with the tree, which are discussed in [BM], see Remark after eq. 3.40 there. Once this choice is done, we can associate an integer $m(f) \in \{0,1\}$ to each field label f, denoting the order of the derivative acting on the corresponding field variable.

7. We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the s_v vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$, that is if v is a non trivial vertex.

   Given $\tau \in \mathcal{T}_{j,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with the previous constraints; let us call $\mathbf{P}$ one of this choices. Given $\mathbf{P}$, we consider the family $\mathcal{G}_\mathbf{P}$ of all connected Feynman graphs, such that, for any $v \in \tau$, the internal fields of v are paired by propagators of scale h_v, so that the following condition is satisfied: for any $v \in \tau$, the subgraph built by the propagators associated with all vertices $v' \geq v$ is connected. The sets P_v have, in this picture, the role of the external legs of the subgraph associated with v. The graphs belonging to $\mathcal{G}_\mathbf{P}$ will be called *compatible with* $\mathbf{P}$ and we shall denote $\mathcal{P}_\tau$ the family of all choices of $\mathbf{P}$ such that $\mathcal{G}_\mathbf{P}$ is not empty.

As explained in detail in §3.2 of [BM], we can write, if $h \leq j \leq -1$,

$$\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + L\beta\tilde{E}_{j+1} = \tag{72}$$
$$= \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{j,n}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sqrt{Z_j}^{|P_{v_0}|} \int d\mathbf{x}_{v_0} \tilde{\psi}^{[h,j]}(P_{v_0}) K_{\tau,\mathbf{P}}^{(j+1)}(\mathbf{x}_{v_0}) \; ,$$

where

$$\tilde{\psi}^{[h,j]}(P_v) = \prod_{f \in P_v} \psi^{[h,j]\sigma(f)}_{\mathbf{x}(f),\omega(f)} \tag{73}$$

and $K_{\tau,\mathbf{P}}^{(j+1)}(\mathbf{x}_{v_0})$ is a suitable function, which is obtained by summing the values of all the Feynman graphs compatible with $\mathbf{P}$, see item 7) above, and applying iteratively in the vertices of the tree, different from the endpoints and v_0, the $\mathcal{R}$-operation, starting from the vertices with higher scale. Note that there is no derivative acting on the fields with label $f \in P_{v_0}$, even if the field is associated with the endpoint of type δ; this result is achieved by using the freedom discussed in item 6) about the choice of the field with $m(f) = 1$.

In a similar way we get

$$\tilde{E}_h = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h-1,n}} \sum_{\mathbf{P} \in \mathcal{P}_\tau : P_{v_0} = \emptyset} K_{\tau,\mathbf{P}}^{(h)}(\mathbf{x}_{v_0}) \; . \tag{74}$$

3.3 The main bound

In order to control, uniformly in L and β, the various sums in (72), one has to exploit in a careful way the $\mathcal{R}$ operation acting on the vertices of the tree, as explained in full detail in [BM], §3. The result of this analysis, which applies essentially unchanged to the model studied in this paper, is a general bound which has a simple dimensional interpretation.

Let us see what happens if we erase the $\mathcal{R}$ operation in all the vertices of the tree. In this case one gets the dimensional bound

$$\int d\mathbf{x}_{v_0} |K_{\tau,\mathbf{P}}^{(j+1)}(\mathbf{x}_{v_0})| \le L\beta \; (C\bar{\varepsilon})^n \gamma^{-j(-2+|P_{v_0}|/2)} \; \cdot$$
$$\cdot \prod_{\text{v not e.p}} (\frac{Z_{h_v}}{Z_{h_v-1}})^{\frac{|P_v|}{2}} \gamma^{-(-2+\frac{|P_v|}{2})} \; , \tag{75}$$

where C is a suitable constant and $\bar{\varepsilon} = \max_{j+1 \le j' \le 0} |\vec{v}_{j'}|$. Note that the good dependence on n derives from the anticommuting properties of the field variables.

The bound (75) allows us to associate a factor $\gamma^{2-|P_v|/2}$ with any trivial or non trivial vertex of the tree. This would allow us to control the sums over the scale labels and $\mathcal{P}_\tau$, provided $|P_v|$ were larger than 4 in all vertices, which is however not true. The effect of the $\mathcal{R}$ operation is to improve the bound, so that there is a factor less than 1 associated even with the vertices where $|P_v|$ is equal to 2 or 4. In order to explain how this works, we need a more detailed discussion of the $\mathcal{R}$ operation. We shall do that below by using the simpler expressions that one obtains in the (formal) limit $L = \beta = \infty$; this is sufficient to explain the essential points and makes clearer the notation.

1) If $2n = 4$, by (47) (with $L = \beta = \infty$),

$$\mathcal{L} \int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \prod_{i=1}^{4} \psi_{\mathbf{x}_i,\omega_i}^{[h,j]\sigma_i} = \int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \prod_{i=1}^{4} \psi_{\mathbf{x}_4,\omega_i}^{[h,j]\sigma_i}] \; , \tag{76}$$

where $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_4)$ and $W(\underline{\mathbf{x}})$ is the Fourier transform of $\hat{W}^{(j)}_{4,\underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$. Note that $W(\underline{\mathbf{x}})$ is translation invariant; hence $\psi^{[h,j]\sigma_i}_{\mathbf{x}_4,\omega_i}$ in the r.h.s. of (76) can be substituted with $\psi^{[h,j]\sigma_i}_{\mathbf{x}_k,\omega_i}$, $k = 1, 2, 3$ and we have four equivalent representations of the localization operation, which differ by the choice of the *localization point*.

If the localization point is chosen as in (76), we have

$$\begin{aligned}
&\mathcal{R} \int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \prod_{i=1}^4 \psi^{[h,j]\sigma_i}_{\mathbf{x}_i,\omega_i} = \int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \left[\prod_{i=1}^4 \psi^{[h,j]\sigma_i}_{\mathbf{x}_i,\omega_i} - \prod_{i=1}^4 \psi^{[h,j]\sigma_i}_{\mathbf{x}_4,\omega_i} \right] = \\
&= \int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \Big[\psi^{[h,j]\sigma_1}_{\mathbf{x}_1,\omega_1} \psi^{[h,j]\sigma_2}_{\mathbf{x}_2,\omega_2} D^{1[h,j]\sigma_3}_{\mathbf{x}_3,\mathbf{x}_4,\omega_3} \psi^{[h,j]\sigma_4}_{\mathbf{x}_4,\omega_4} + \qquad (77)\\
&+ \psi^{[h,j]\sigma_1}_{\mathbf{x}_1,\omega_1} D^{1[h,j]\sigma_2}_{\mathbf{x}_2,\mathbf{x}_4,\omega_2} \psi^{[h,j]\sigma_3}_{\mathbf{x}_4,\omega_3} \psi^{[h,j]\sigma_4}_{\mathbf{x}_4,\omega_4} + D^{1[h,j]\sigma_1}_{\mathbf{x}_1,\mathbf{x}_4,\omega_1} \psi^{[h,j]\sigma_2}_{\mathbf{x}_4,\omega_2} \psi^{[h,j]\sigma_3}_{\mathbf{x}_4,\omega_3} \psi^{[h,j]\sigma_4}_{\mathbf{x}_4,\omega_4} \Big] ,
\end{aligned}$$

where (again if $L = \beta = \infty$)

$$D^{1[h,j]\sigma}_{\mathbf{y},\mathbf{x},\omega} = \psi^{[h,j]\sigma}_{\mathbf{y},\omega} - \psi^{[h,j]\sigma}_{\mathbf{x},\omega} . \qquad (78)$$

The field $D^{1[h,j]\sigma}_{\mathbf{y},\mathbf{x},\omega}$ is dimensionally equivalent to the product of $|\mathbf{y} - \mathbf{x}|$ and the derivative of the field, so that the bound of its contraction with another field variable on a scale $j' < j$ will produce a "gain" $\gamma^{-(j-j')}$ with respect to the contraction of $\psi^{[h,j]\sigma}_{\mathbf{y},\omega}$. On the other hand, each term in the r.h.s. of (77) differs from the term which $\mathcal{R}$ acts on mainly because one $\psi^{[h,j]}$ field is substituted with a $D^{1[h,j]}$ field and some of the other $\psi^{[h,j]}$ fields are "translated" in the localization point. All three terms share the property that the field whose $\mathbf{x}$ coordinate is equal to the localization point is not affected by the action of $\mathcal{R}$.

2) If $2n = 2$, by (50),

$$\begin{aligned}
&\mathcal{R} \int d\mathbf{x}_1 d\mathbf{x}_2 W(\mathbf{x}_1 - \mathbf{x}_2) \psi^{[h,j]+}_{\mathbf{x}_1,\omega} \psi^{[h,j]-}_{\mathbf{x}_2,\omega} = \qquad (79)\\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W(\mathbf{x}_1 - \mathbf{x}_2) \psi^{[h,j]+}_{\mathbf{x}_1,\omega} D^{2[h,j]-}_{\mathbf{x}_2,\mathbf{x}_1,\omega} = \int d\mathbf{x}_1 d\mathbf{x}_2 W(\mathbf{x}_1 - \mathbf{x}_2) D^{2[h,j]+}_{\mathbf{x}_1,\mathbf{x}_2,\omega} \psi^{[h,j]-}_{\mathbf{x}_2,\omega} ,
\end{aligned}$$

where $W(\mathbf{x})$ is the Fourier transform of $\hat{W}^{(j)}_{2,\omega,\omega}(\mathbf{k})$ and

$$D^{2[h,j]\sigma}_{\mathbf{y},\mathbf{x},\omega} = \psi^{[h,j]\sigma}_{\mathbf{y},\omega} - \psi^{[h,j]\sigma}_{\mathbf{x},\omega} - (\mathbf{y} - \mathbf{x}) \cdot \nabla \psi^{[h,j]\sigma}_{\mathbf{x},\omega} . \qquad (80)$$

As in item 1) above, we define the localization point as the $\mathbf{x}$ coordinate of the field which is left unchanged by $\mathcal{L}$ or $\mathcal{R}$. We are free to choose it equal to $\mathbf{x}_1$ or $\mathbf{x}_2$. Hence the effect of $\mathcal{R}$ can be described as the replacement of a $\psi^{[h,j]\sigma}$ field with a $D^{2[h,j]\sigma}$ field, with a gain in the bounds of a factor $\gamma^{-2(j-j')}$.

By suitably using the definition of the $\mathcal{R}$, it is shown in §3 of [BM] that

$$\mathcal{R}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) = \tag{81}$$
$$= \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{j,n}} \quad \sum_{\mathbf{P}\in\mathcal{P}_\tau}\sum_{\alpha\in A_{\tau,\mathbf{P}}} \sqrt{Z_j}^{|P_{v_0}|}\int d\mathbf{x}_{v_0} D_\alpha \tilde{\psi}^{[h,j]}(P_{v_0}) K^{(j+1)}_{\tau,\mathbf{P},\alpha}(\mathbf{x}_{v_0}) \ ,$$

where $A_{\tau,\mathbf{P}}$ labels a finite set of different terms, of counting power C^n, and, for any $\alpha \in A_{\tau,\mathbf{P}}$, D_α denotes an operator dimensionally equivalent to a derivative of order m_α. The important property of (81) is that, see eq. 3.110 of [BM],

$$\int d\mathbf{x}_{v_0}|K^{(j+1)}_{\tau,\mathbf{P},\alpha}(\mathbf{x}_{v_0})| \leq L\beta\ (C\bar{\varepsilon})^n \gamma^{-j(-2+|P_{v_0}|/2+m_\alpha)} \ \cdot$$
$$\cdot \prod_{\mathrm{v\ not\ e.p}} (\frac{Z_{h_v}}{Z_{h_v-1}})^{|P_v|/2}\gamma^{-[-2+|P_v|/2+z(P_v)]} \ , \tag{82}$$

where $m_\alpha \geq z(P_{v_0})$ and

$$z(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4 \ , \\ 2 & \text{if } |P_v| = 2 \ , \\ 0 & \text{otherwise.} \end{cases} \tag{83}$$

We now consider the action of $\mathcal{L}$ on $\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})$. We get an expansion similar to (81), that we can write in the form

$$\mathcal{L}V^{(j)}(\tau,\sqrt{Z_j}\psi^{[h,j]}) = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{j,n}} [z_j(\tau)Z_jF_\zeta^{[h,j]} + a_j(\tau)Z_jF_\alpha^{[h,j]} + l_j(\tau)Z_j^2F_\lambda^{[h,j]}] \ , \tag{84}$$

where (in the limit $L = \beta = \infty$)

$$\begin{aligned} z_j(\tau) &= \frac{1}{L\beta}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\alpha\in A_{\tau,\mathbf{P}} \\ P_{v_0}=(f_1,f_2),\omega(f_1)=\omega(f_2)=+1}} \int d\mathbf{x}_{v_0}[x(f_2)-x(f_1)]K^{(j+1)}_{\tau,\mathbf{P},\alpha}(\mathbf{x}_{v_0}) \ , \\ a_j(\tau) &= \frac{1}{L\beta}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\alpha\in A_{\tau,\mathbf{P}} \\ P_{v_0}=(f_1,f_2),\omega(f_1)=\omega(f_2)=+1}} \int d\mathbf{x}_{v_0}[x(f_2)-x(f_1)]K^{(j+1)}_{\tau,\mathbf{P},\alpha}(\mathbf{x}_{v_0}) \ , \\ l_j(\tau) &= \frac{1}{L\beta}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\alpha\in A_{\tau,\mathbf{P}} \\ |P_{v_0}|=4,\underline{\sigma}=(+,-,+,-),\underline{\omega}=(+1,-1,-1,+1)}} \int d\mathbf{x}_{v_0}K^{(j+1)}_{\tau,\mathbf{P},\alpha}(\mathbf{x}_{v_0}) \ . \end{aligned} \tag{85}$$

The constants z_j, a_j and l_j, which characterize the local part of the effective potential, can be obtained from (85) by summing over $n \geq 1$ and

$\tau \in \mathcal{T}_{j,n}$. Finally, the constant $\tilde{E}_{j+1}$ appearing in the l.h.s. of (72) can be written in the form $\tilde{E}_{j+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{j,n}} \tilde{E}_{j+1}(\tau)$, with

$$\tilde{E}_{j+1}(\tau) = \frac{1}{L\beta} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau, \alpha \\ P_{v_0} = \emptyset}} \int d\mathbf{x}_{v_0} K^{(j)}_{\tau, \mathbf{P}, \alpha}(\mathbf{x}_{v_0}) \; . \tag{86}$$

All the kernels appearing in (85) and (86) satisfy the bound (82), with $m_\alpha = 0$

Note that, by the remark preceding (62), the effective potential is independent of the infrared cut-off for $j > h$. This means in particular that, if we add a superscript (h) to keep track of the infrared cut-off, $\vec{v}_j^{(h)} = \vec{v}_j^{-\infty}$ for $j > h$. On the other hand, in previous papers (see [BGPS], [GS], [BoM1], [BM]) it was shown, by using several properties of the exact solution of the Luttinger model (see [ML], [BGM]), that $\lambda_j^{-\infty} = \lambda + O(\lambda^2)$ and $\delta_j^{-\infty} = O(\lambda^2)$. Moreover, since $\lambda_h - \lambda_h^{-\infty} = (\lambda_h - \lambda_{h+1}^{(h)}) - (\lambda_h^{-\infty} - \lambda_{h+1}^{-\infty})$, the previous result implies that $\lambda_h = \lambda_h^{-\infty} + O(\lambda^2)$, since, by (82) and (85), both $\lambda_h - \lambda_{h+1}^{(h)}$ and $\lambda_h^{-\infty} - \lambda_{h+1}^{-\infty}$ are of order λ^2. We can resume this results in the following Theorem.

Theorem 3.1 *There is a constant ε_0, such that, if $|\lambda| \le \varepsilon_0$, then, uniformly in the infrared cut-off,*

$$\lambda_j = \lambda + O(\lambda^2) \; , \qquad \delta_j = O(\lambda^2) \; , \qquad h \le j \le -1 \; . \tag{87}$$

3.4 The expansion for the Schwinger functions

The procedure described in sections (3.1)-(3.3) can be generalized to get an expansion for the connected Schwinger functions of the model, in particular those defined by (12) and (13). The main difference with respect to the "free energy" case is that the external fields $J_{\mathbf{x},\omega}$ and $\phi_{\mathbf{x},\omega}$ have to be taken into account.

We start from the generating function (11) and we perform iteratively the integration of the ψ variables, to be defined iteratively in the following way. After the fields $\psi^{(0)}, ...\psi^{(j+1)}$ have been integrated, we can write

$$e^{\mathcal{W}(\phi,J)} = e^{-L\beta E_j} \int P_{\tilde{Z}_j, C^\varepsilon_{h,j}}(d\psi^{[h,j]}) e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + \mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{[h,j]},\phi,J)} \; , \tag{88}$$

where $\mathcal{B}^{(j)}(\sqrt{Z_j}\psi, \phi, J)$ denotes the sum over the terms containing at least one ϕ or J field; we shall write it in the form

$$\mathcal{B}^{(j)}(\sqrt{Z_j}\psi, \phi, J) = \mathcal{B}^{(j)}_\phi(\sqrt{Z_j}\psi) + \mathcal{B}^{(j)}_J(\sqrt{Z_j}\psi) + W^{(j)}_R(\sqrt{Z_j}\psi, \phi, J) \; , \tag{89}$$

where $\mathcal{B}_\phi^{(j)}(\psi)$ and $\mathcal{B}_J^{(j)}(\psi)$ denote the sums over all the terms containing only one ϕ or J field, respectively. For $j=0$, the comparison with (11) shows that $W_R^{(0)}=0$, $\mathcal{B}_\phi^{(0)}(\psi)=\sum_\omega\int d\mathbf{x}[\phi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}+\psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]$ and $\mathcal{B}_J^{(0)}(\psi)=\sum_\omega\int d\mathbf{x}J_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}$.

In order to control the expansion of the connected Schwinger functions, we have to extend the definition of the localization operation $\mathcal{L}$ to $\mathcal{B}^{(j)}(\sqrt{Z_j}\psi,\phi,J)$. First of all, we put $\mathcal{L}W_R^{(j)}=W_R^{(j)}$. Let us now consider $\mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\psi)$; we want to show that, by a suitable choice of the localization procedure, it can be written in the form

$$\begin{aligned}\mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\psi)&=\sum_\omega\sum_{i=j+1}^{0}\int d\mathbf{x}d\mathbf{y}\Big[\phi^+_{\mathbf{x},\omega}g_\omega^{Q,(i)}(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial\psi^+_{\mathbf{y}\omega}}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi)+\\&+\frac{\partial}{\partial\psi^-_{\mathbf{y},\omega}}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi)g_\omega^{Q,(i)}(\mathbf{y}-\mathbf{x})\phi^-_{\mathbf{x},\omega}\Big]+\\&+\frac{1}{\sqrt{Z_j}}\frac{1}{L\beta}\sum_{\omega,\mathbf{k}}\Big[(\sqrt{Z_j}\hat\psi^+_{\mathbf{k},\omega})\hat Q_\omega^{(j+1)}(\mathbf{k})\hat\phi^-_{\mathbf{k},\omega}+\hat\phi^+_{\mathbf{k},\omega}\hat Q_\omega^{(j+1)}(\mathbf{k})(\sqrt{Z_j}\hat\psi^-_{\mathbf{k},\omega})\Big]\;,\end{aligned}\qquad(90)$$

where

$$\hat g_\omega^{Q,(i)}(\mathbf{k})=\hat g_\omega^{(i)}(\mathbf{k})\hat Q_\omega^{(i)}(\mathbf{k})\qquad(91)$$

and $Q_\omega^{(j)}(\mathbf{k})$ is defined inductively by the relations

$$\hat Q_\omega^{(j)}(\mathbf{k})=\hat Q_\omega^{(j+1)}(\mathbf{k})-z_jZ_jD_\omega(\mathbf{k})\sum_{i=j+1}^{0}\hat g_\omega^{Q,(i)}(\mathbf{k})\;,\quad \hat Q_\omega^{(0)}(\mathbf{k})=1\;.\qquad(92)$$

In fact, the terms in the first two lines of (90) have a simple interpretation in terms of Feynman graphs; they are obtained by taking all the graphs contributing to $\mathcal{V}^{(j)}(\sqrt{Z_h}\psi)$ and, given a single graph, by adding a new space-time-point $\mathbf{x}$ associated with a term $\phi_\mathbf{x}\psi_\mathbf{x}$ and contracting the correspondent ψ field with one of the external fields of the graph through a propagator $\sum_{i=j+1}^0 g_\omega^{Q,(i)}(\mathbf{x}-\mathbf{y})$. Hence, it is very easy to see that (90) is satisfied for $j=-1$. The fact that it is valid for any j follows from our choice to localize $\mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\psi)$ by the following procedure: first of all we substitute in the r.h.s. of (90) $\mathcal{V}^{(j)}$ with $\mathcal{L}\mathcal{V}^{(j)}+\mathcal{R}\mathcal{V}^{(j)}$, $\mathcal{L}\mathcal{V}^{(j)}$ being defined by (52); then we extract from $\mathcal{L}\mathcal{V}^{(j)}$ the terms proportional to z_j, as in (58), which are absorbed in the terms in the third line of (90). Finally we rescale the field ψ by (65) and perform the integration of the scale j field. It is then easy to check that (90) is satisfied for $j=\bar j+1$, if it is satisfied for $j=\bar j$, together with (92).

Note that $f_j(\mathbf{k})=0$ for $|\mathbf{k}|<\gamma^{j-1}$ or $|\mathbf{k}|>\gamma^{j+1}$, so that

$$f_{h_1}(\mathbf{k})f_{h_2}(\mathbf{k})=0\quad\text{if } |h_1-h_2|>1\;.\qquad(93)$$

It follows that, if $\hat{g}^{(j)}_\omega(\mathbf{k}) \neq 0$, by using also (62) and (92),

$$\hat{Q}^{(j)}_\omega(\mathbf{k}) = 1 - z_j f^\varepsilon_{j+1}(\mathbf{k}) \frac{Z_j}{\tilde{Z}_j(\mathbf{k})} \ . \tag{94}$$

Hence, the propagator $\hat{g}^{Q,(i)}_\omega(\mathbf{k})$ is equivalent to $\hat{g}^{(i)}_\omega(\mathbf{k})$, as concerns the dimensional bounds.

Finally let us consider $\mathcal{B}^{(j)}_J(\sqrt{Z_j}\psi)$. It is easy to see that the field J is equivalent, from the point of view of dimensional considerations, to two ψ fields. Hence, the only terms which need a regularization are those of second order in ψ, which are indeed marginal. We shall use for them the definition

$$\begin{aligned}\mathcal{B}^{(j,2)}_J(\sqrt{Z_j}\psi) &= \sum_{\omega,\tilde{\omega}} \int d\mathbf{x} d\mathbf{y} d\mathbf{z} B_{\omega,\tilde{\omega}}(\mathbf{x},\mathbf{y},\mathbf{z}) J_{\mathbf{x},\omega}(\sqrt{Z_j}\psi^+_{\mathbf{y},\tilde{\omega}})(\sqrt{Z_j}\psi^-_{\mathbf{z},\tilde{\omega}}) = \\ &= \frac{1}{(L\beta)^2} \sum_{\omega,\tilde{\omega},\mathbf{k},\mathbf{p}} \hat{B}_{\omega,\tilde{\omega}}(\mathbf{p},\mathbf{k}) \hat{J}(\mathbf{p})(\sqrt{Z_j}\hat{\psi}^+_{\mathbf{p}+\mathbf{k},\tilde{\omega}})(\sqrt{Z_j}\hat{\psi}^-_{\mathbf{k},\tilde{\omega}}) \ . \end{aligned} \tag{95}$$

We write

$$\mathcal{B}^{(j,2)}_J(\sqrt{Z_j}\psi) = \mathcal{L}\mathcal{B}^{(j,2)}_J(\sqrt{Z_j}\psi) + \mathcal{R}\mathcal{B}^{(j,2)}_J(\sqrt{Z_j}\psi) \ , \tag{96}$$

where $\mathcal{L}$ is defined through its action on $\hat{B}_\omega(\mathbf{p},\mathbf{k})$ in the following way:

$$\mathcal{L}\hat{B}_{\omega,\tilde{\omega}}(\mathbf{p},\mathbf{k}) = \frac{1}{4} \sum_{\eta,\eta'=\pm 1} \hat{B}_{\omega,\tilde{\omega}}(\bar{\mathbf{p}}_{\eta'}, \bar{\mathbf{k}}_{\eta,\eta'}) \ , \tag{97}$$

where $\bar{\mathbf{k}}_{\eta,\eta'}$ is defined as in (48) and $\bar{\mathbf{p}}_{\eta'} = (0, 2\pi\eta'/\beta)$. In the $L = \beta = \infty$ it reduces simply to $\mathcal{L}\hat{B}_{\omega,\tilde{\omega}}(\mathbf{p},\mathbf{k}) = \hat{B}_{\omega,\tilde{\omega}}(0,0)$.

This definition apparently implies that we have to introduce two new renormalization constants. However, this is not the case. One can show that, in the $L = \beta = \infty$ limit

$$\hat{B}_{\omega,-\omega}(0,0) = 0 \ , \tag{98}$$

by using the symmetry property of the propagators

$$\hat{g}^{(j)}_\omega(\mathbf{k}) = -i\omega \hat{g}^{(j)}_\omega(\mathbf{k}^*) \ , \qquad \mathbf{k} = (k, k_0), \quad \mathbf{k}^* = (-k_0, k) \ . \tag{99}$$

In fact, the contribution of order n in $\bar{\varepsilon}$ to $B_{\omega,-\omega}(\mathbf{p},\mathbf{k})$ can be written as a sum of connected Feynman graphs obtained by contracting $2n+2$ fields of type ω and $2n-2$ fields of type $-\omega$, so that, by (99), $B_{\omega,-\omega}(\mathbf{p},\mathbf{k}) = (-i\omega)^{n+1}(i\omega)^{n-1} B_{\omega,-\omega}(\mathbf{p}^*,\mathbf{k}^*) = -B_{\omega,-\omega}(\mathbf{p}^*,\mathbf{k}^*)$, which implies (98).

If L and β are finite, the identity (98) is not true anymore, but the corrections do not give rise to any divergence, as $j \to \infty$, and go to zero, for any fixed j, as $L, \beta \to \infty$. In fact it is not hard to see, by comparing $\mathcal{L}\hat{B}_{\omega,\tilde{\omega}}(\mathbf{p}, \mathbf{k})$ with its limit as $L, \beta \to \infty$ and using the properties of the multiscale expansion described above, that the corrections are of order $\gamma^{-j} \max\{L^{-1}, \beta^{-1}\}$, as one can guess by dimensional arguments. In other words, one can say that $\mathcal{L}\hat{B}_{\omega,-\omega}$ behaves as an irrelevant term (hence no renormalization constant is associated to it).

The previous considerations imply that we can write

$$\mathcal{L}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\psi) = \sum_\omega \frac{Z_j^{(2)}}{Z_j} \int d\mathbf{x} J_{\mathbf{x},\omega} (\sqrt{Z_j}\psi^+_{\mathbf{x},\omega})(\sqrt{Z_j}\psi^-_{\mathbf{x},\omega}) \; , \tag{100}$$

which defines a new renormalization constant $Z_j^{(2)}$, the *density renormalization*. It is easy to see, by proceeding as in §3.3, that $\mathcal{R}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\psi)$ can be written as a sum of terms of the form (95), with one of the fields ψ replaced by a field D^1 (see (78)). This allows us to improve the bounds in the usual way, see §3.3. The definition of $\mathcal{R}$ is extended to all the other contributions to $\mathcal{B}_J^{(j)}(\sqrt{Z_j}\psi)$ as the identity.

At the end of the iterative integration procedure, we get

$$\mathcal{W}(\varphi, J) = -L\beta E_{L,\beta} + \sum_{m^\phi + n^J \geq 1} S^{(h)}_{2m^\phi, n^J}(\phi, J) \; . \tag{101}$$

We can expand the functional $S^{(h)}_{2m^\phi,n^J}(\phi, J)$ and the various terms in the r.h.s. of (89) in terms of trees, as we did for the effective potential, by suitably modifying the definitions given in §3.2.

1. First of all, we have to add two new types of endpoints, to be called of type ϕ and J; the first one is associated with the terms in the third line of (90), the second one with the terms in the r.h.s. of (100). They will be sometimes called *special endpoints*; as for the other endpoints, the scale of a special endpoint $\bar{v}$ is $h_v + 1$, if h_v is the scale of the non trivial vertex immediately preceding $\bar{v}$. Given $v \in \tau$, we shall call n_v^ϕ and n_v^J the number of endpoints of type ϕ and J belonging to the cluster L_v, defined as in item 4) of §3.3, while n_v will denote the number of endpoints of type λ or δ, to be called *normal*. Analogously, given τ, we shall call n_τ^ϕ and n_τ^J the number of endpoint of type ϕ and J, while n_τ will denote the number of normal endpoints. Finally, $\mathcal{T}_{j,n,n^\phi,n^J}$ will denote the set of trees with n normal endpoints, n^ϕ endpoints of type ϕ and n^J endpoints of type J.

2. The definition of the sets P_v (of the external fields in the vertex v) is modified, in the sense that the set P_v includes both the field variables of type ψ which are not yet contracted in the vertex v, to be called *normal external fields*, and those which belong to an endpoint normal or of type J and are contracted with a field variable belonging to an endpoint of type ϕ through a propagator $g^{Q,(h_v)}$, to be called *special external fields* of v.

3. As explained above, we regularize the terms linear in ϕ by extracting from the effective potential, in the r.h.s. of (90), its local part, defined by (52). This implies that one of the ψ variables contracted in the propagator linked to the ϕ variable is treated as an external field variable, see item 2) above. However, in order to exploit the regularizing effect of the $\mathcal{R}$ operation on the terms with 2 or 4 external fields (see remark after (78)), we have to be sure that the field variable which "acquires a derivative" is not yet contracted on the vertex scale. This can be realized by choosing the localization point as the space-time point of the special external field, that is the field which is contracted with the ψ field of the type ϕ endpoint.

It is easy to see that

$$S^{(h)}_{2m^\phi,n^J}(\phi,J) = \sum_{n=0}^{\infty}\sum_{j_0=h-1}^{-1}\sum_{\underline{\omega}}\sum_{\tau\in\mathcal{T}_{j_0,n,2m^\phi,n^J}}\sum_{\mathbf{P}\in\mathcal{P}_\tau:|P_{v_0}|=2m^\phi}$$
$$\int d\underline{\mathbf{x}}\prod_{i=1}^{2m^\phi}\phi^{\sigma_i}_{\mathbf{x}_i,\omega_i}\prod_{r=1}^{n^J}J_{\mathbf{x}_{2m^\phi+r},\omega_{2m^\phi+r}}S_{2m^\phi,n^J,\tau,\underline{\omega}}(\underline{\mathbf{x}})\;, \qquad (102)$$

where $\underline{\omega} = \underline{\omega} = \underline{\omega} = \underline{\omega} = \{\omega_1,\ldots,\omega_{2m^\phi+n^J}\}$, $\underline{\mathbf{x}} = \{\mathbf{x}_1,\ldots,\mathbf{x}_{2m^\phi+n^J}\}$ and $\sigma_i = +$ if i is odd, $\sigma_i = -$ if i is even.

The Schwinger functions are simply related to the kernels of the functionals $S^{(h)}_{2m^\phi,n^J}(\phi,J)$ and (102) allows us to get an expansion for them. For example, $G^2_\omega(\mathbf{x}_1,\mathbf{x}_2)$ is equal to the sum over the terms in the r.h.s. of (102) with $m^\phi = 1$, $n^J = 0$ and $\underline{\omega} = (\omega,\omega)$, while $G^{2,1}_\omega(\mathbf{x};\mathbf{y},\mathbf{z})$ is obtained by selecting the terms with $m^\phi = 1$, $n^J = 1$ and $\underline{\omega} = (\omega,\omega,\omega)$. Hence, a bound for the Fourier transform of the Schwinger functions can be obtained by using the dimensional bound

$$\int d\underline{\mathbf{x}}\qquad |S_{2m^\phi,n^J,\tau,\underline{\omega}}(\underline{\mathbf{x}})| \le L\beta\;(C\bar\varepsilon)^n\gamma^{-j_0(-2+m^\phi+n^J)}\prod_{i=1}^{2m^\phi}\frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}}\;\cdot$$
$$\cdot\prod_{r=1}^{n^J}\frac{Z^{(2)}_{\bar h_r}}{Z_{\bar h_r}}\prod_{\text{v not e.p}}(\frac{Z_{h_v}}{Z_{h_v-1}})^{|P_v|/2}\gamma^{-d_v}\;, \qquad (103)$$

where h_i is the scale of the propagator linking the i-th endpoint of type ϕ to the tree, $\bar{h}_r$ is the scale of the r-th endpoint of type J and

$$d_v = -2 + |P_v|/2 + n_v^J + \tilde{z}(P_v)\;, \tag{104}$$

with

$$\tilde{z}(P_v) = \begin{cases} z(P_v) & \text{if } n_v^\phi \le 1\;, n_v^J = 0\;, \\ 1 & \text{if } n_v^\phi = 0\;, n_v^J = 1\;, |P_v| = 2\;, \\ 0 & \text{otherwise} \end{cases} \tag{105}$$

The bound (103) can be easily obtained by the same arguments leading to the bound (82), by taking into account the remarks in items 1)-3) above. Essentially one has to modify the bound (82) in the following way.

a) Insert a factor $\gamma^{-h_i}(Z_{h_i})^{-1/2}$ for each endpoint of type ϕ; this factor bounds the product of the propagator linking the i-th endpoint of type ϕ to the tree and the $(Z_{h_i})^{1/2}$ renormalization constant of the corresponding special external field variable. We use here (60), (61) and (64).

b) Insert a factor $Z^{(2)}_{\bar{h}_r}/Z_{\bar{h}_r}$ for each endpoint of type J.

c) Substitute the "regularization index" z_v with $\tilde{z}_v$, to take into account that the $\mathcal{R}$ operation is trivial in all the vertices with $n_v^\phi + n_v^J > 1$.

d) Insert a factor $\gamma^{n_v^J}$ for each non trivial vertex v, to take into account that any J variable is dimensionally equivalent to two ψ external fields, so that the dimension of any vertex v increases by one unit for any endpoint of type J belonging to the cluster L_v.

The bound (103) is sufficient to get a bound for the Schwinger functions Fourier transforms, because, by translation invariance, the Fourier transform of $S_{2m^\phi,n^J,\tau,\underline{\omega}}(\underline{\mathbf{x}})$ is bounded by $(L\beta)^{-1}\int d\underline{\mathbf{x}}|S_{2m^\phi,n^J,\tau,\underline{\omega}}(\underline{\mathbf{x}})|$. We only have to sum over τ the r.h.s. of (103) (without the $L\beta$ factor), by using the techniques described in detail in §5 of [BM]. The main point is to control the sums over the sets P_v and the scale indices h_v, for fixed values of the external propagators scale indices h_i, which are determined up to one unit by the external momenta. Hence, if all the "vertex dimensions" d_v were greater than 0, one would get a dimensional bound of the type

$$(C\bar{\varepsilon})^{\bar{n}} \sum_{j_0=h}^{\bar{h}} \gamma^{-j_0(-2+m^\phi+n^J)} \prod_{i=1}^{2m^\phi} \frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}} \prod_{r=1}^{n^J} \frac{Z^{(2)}_{\bar{h}_r}}{Z_{\bar{h}_r}}\;, \tag{106}$$

where $\bar{n}$ is the minimal order in λ of the graphs contributing to the Schwinger function and $\bar{h}$ is an upper bound on the scale of the tree lower vertex v_0, which depends on the external momenta.

However, it is not true that, given τ, $d_v > 0$ for all non trivial $v \in \tau$; in fact $d_v = 0$, if $|P_v| = 2$ and $n_v^\phi = n_v^J = 1$ or $n_v^\phi = 2, n_v^J = 0$. This implies that the sum over the scale indices of some special paths on the tree can produce a result different from the "trivial one", leading to (106). Hence, in order to get the right bound, one has to analyze case by case the constraints on the endpoint scale indices, related with the support properties of the single scale propagators and the fact that the ϕ and J momenta are fixed.

The result of this analysis, rather difficult to describe in general, will be given only for the bound of the connected Schwinger functions appearing in Theorem 2.1. Moreover, we shall use the expansion (102) also to extract some "dominant terms" and get an improved bound on the rest, as we shall see below.

3.5 Proof of Theorem 2.1

The bounds (18) and the equations (19) are proved in [BM], Theorem 4.9; so it remains to prove (16) and (17).

By using (102), we can write, for any $\mathbf{k}$

$$\hat{G}^{(2)}_\omega(\mathbf{k}) = \sum_{j=h}^{0} \hat{g}^{Q,(j)}_\omega(\mathbf{k}) + \sum_{n=1}^{\infty} \sum_{j_0=h-1}^{-1} \sum_{\substack{\tau\in\mathcal{T}_{j_0,n,2,0} \\ |P_{v_0}|=2}} \hat{G}_{2,\tau}(\mathbf{k}) \; , \tag{107}$$

where $G_{2,\tau} = S_{2,0,\tau,,\tau,\{\omega,\omega\}}$. The choice of $\bar{k}$ implies that, given τ, the scale of the external propagators has to be equal to h or $h+1$, hence $\hat{G}_{2,\tau}$ can be different from 0 only if the index j_0 in the r.h.s. of (107) (which is also the scale index of v_0, the lower tree vertex) takes the value h or $h+1$. In this case, by using the bound (103) and translation invariance, we get, by using also the fact that $Z_j/Z_{j-1} < 1$ (see [BM], Theorem 4.9) and Theorem 3.1,

$$|\hat{G}_{2,\tau}(\bar{\mathbf{k}})| \le \frac{1}{L\beta}\int d\mathbf{x}_1 d\mathbf{x}_2 |G_{2,\tau}(\mathbf{x}_1-\mathbf{x}_2)| \le (C\bar{\varepsilon})^n \gamma^{-h} Z_h^{-1} \prod_v \gamma^{-d_v} \; . \tag{108}$$

The previous considerations also imply that the only vertices of τ with $d_v = 0$ have scale h or $h+1$, so that there is no problem in performing the sum over the scale indices in the r.h.s. of (107). Moreover, by symmetry reasons, $G_{2,\tau} = 0$ if $n_\tau = 1$; hence the sum over all the trees with $n \ge 1$ can be bounded by $(C\bar{\varepsilon})^2\gamma^{-h}Z_h^{-1}$. Finally, the terms of order 0 in the r.h.s. of (107) sum up to

$$\frac{1}{Z_h}\frac{\tilde f_{h+1}(\bar{\mathbf{k}})+\tilde f_h(\bar{\mathbf{k}})}{D_\omega(\bar{\mathbf{k}})}[1+O(\bar\varepsilon^2]=\frac{1}{Z_hD_\omega(\bar{\mathbf{k}})}[1+O(\bar\varepsilon^2]\ , \tag{109}$$

which easily implies (16).

We finally prove (17). By using (102), we can write, if $\mathbf{p}=\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2$ and $\mathbf{k}=\bar{\mathbf{k}}_2$,

$$\hat G^{(2,1)}_\omega(\mathbf{p},\mathbf{k})=\sum_{n=0}^\infty\sum_{j_0=h-1}^{-1}\sum_{\substack{\tau\in\mathcal{T}_{j_0,n,2,1}\\ |P_{v_0}|=2}}\hat G_{2,1,\tau}(\mathbf{p},\mathbf{k})\ , \tag{110}$$

where $G_{2,1,\tau}=S_{2,1,\tau,,\underline{\tau},\{\omega,\omega,\omega\}}$.

The condition on $\bar{\mathbf{k}}_1$ and $\bar{\mathbf{k}}_2$ implies that, for any τ, the only vertices with $d_v=0$ have scale h or $h+1$. Hence, the sum over the trees with n normal endpoints of $|\hat G_{2,1,\tau}(\mathbf{p},\mathbf{k})|$ satisfies, by (103), the dimensional bound

$$\sum_{\tau:n_\tau=n}|\hat G_{2,1,\tau}(\mathbf{p},\mathbf{k})|\le (C\bar\varepsilon)^n\frac{\gamma^{-2h}}{Z_h}\frac{Z_h^{(2)}}{Z_h}\ . \tag{111}$$

Moreover, by symmetry reasons, $\hat G_{2,1,\tau}(\mathbf{p},\mathbf{k})=0$, if $n_\tau=1$, and

$$\sum_{\tau:n_\tau=0}\hat G_{2,1,\tau}(\mathbf{p},\mathbf{k})=\frac{Z_h^{(2)}}{Z_h}Z_h\frac{1}{Z_hD_\omega(\bar{\mathbf{k}}_1)}\frac{1}{Z_hD_\omega(\bar{\mathbf{k}}_2)}[1+O(\bar\varepsilon^2]\ , \tag{112}$$

since $\tilde f_{h+1}(\bar{\mathbf{k}}_i)+\tilde f_h(\bar{\mathbf{k}}_i)=1$. Hence, we get (17).

4 The expansion for $H^{2,1}_\omega$

4.1 Preliminary remark

In this section we have to find an expansion for *correction term* $H^{2,1}_\omega$. The definition of $H^{2,1}_\omega$ as a derivative of a functional integral, see (35-37), is apparently very similar to the expression for $G^{2,1}_\omega$ given by (11-12). In fact, the definition (11) of the generating function $\mathcal{W}(\phi,J)$ differs from the definition (36) of $\mathcal{W}_\Delta(\phi,J)$ only because $\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}$ is replaced by $T_{\mathbf{x},\omega}$. However, such difference is not trivial at all, because of the singularity of $C(\mathbf{k},\mathbf{k}+\mathbf{p})$, as $\varepsilon\to0$, and of $D^{-1}(\mathbf{p})$ at $\mathbf{p}=0$. Nevertheless, we are still able to prove the bound (29), which differs from the analogous bound for $\hat G^{2,1}$ by an extra λ^2 factor.

In order to get this result, we will define a multiscale expansion similar to the previous ones, so getting a few terms of a new kind, for which the $\mathcal{L}$

operation must be defined in a proper way. Correspondingly new renormalization constants will appear, which we can prove are strictly related to $Z_h^{(2)}$, what is crucial to get (29).

4.2 Properties of $T_{\mathbf{x},\omega}$

We begin our analysis by studying the quantity

$$\begin{aligned}
&\Delta_\omega^{(i,j)}(\mathbf{k}^+,\mathbf{k}^-) = \frac{C(\mathbf{k}^+,\mathbf{k}^-)}{D_\omega(\mathbf{p})}\tilde g_\omega^{(i)}(\mathbf{k}^+)\tilde g_\omega^{(j)}(\mathbf{k}^-) = \\
&= \frac{1}{Z_{i-1}Z_{j-1}}\frac{1}{D_\omega(\mathbf{p})}\left\{\frac{\tilde f_i^\varepsilon(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)}\Big[\frac{\tilde f_j^\varepsilon(\mathbf{k}^-)}{\chi_{h,0}^\varepsilon(\mathbf{k}^-)} - \tilde f_j^\varepsilon(\mathbf{k}^-)\Big] - \right. \\
&\left. -\frac{\tilde f_j^\varepsilon(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)}\Big[\frac{\tilde f_i^\varepsilon(\mathbf{k}^+)}{\chi_{h,0}^\varepsilon(\mathbf{k}^+)} - \tilde f_i^\varepsilon(\mathbf{k}^+)\Big]\right\} ,
\end{aligned} \tag{113}$$

where $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$. The above quantity appears in the expansion for $\hat H_{2,1}$ when both the fields of $T_{\mathbf{x},\omega}$ are contracted. Note first that

$$\Delta_\omega^{(i,j)}(\mathbf{k}^+,\mathbf{k}^-) = 0 \ , \qquad \text{if } 0 > i,j > h \ , \tag{114}$$

since $\chi_{h,0}^\varepsilon(\mathbf{k}^\pm) = 1$, if $h < i,j < 0$. We will see that this property plays a crucial role; it says that, contrary to what happens for $G^{2,1}$, at least one of the two fermionic lines connected to J must have scale 0 or h.

In the the cases in which $\Delta_\omega^{(i,j)}(\mathbf{k}^+,\mathbf{k}^-)$ is not identically equal to 0, since $\Delta_\omega^{(i,j)}(\mathbf{k}^+,\mathbf{k}^-) = \Delta_\omega^{(j,i)}(\mathbf{k}^-,\mathbf{k}^+)$, we can restrict the analysis to the case $i \ge j$.

1) If $i = j = 0$, by using (21), it is easy to see that the r.h.s. of (113) has a well defined limit as $\varepsilon \to 0$, given by

$$\Delta_\omega^{(0,0)}(\mathbf{k}^+,\mathbf{k}^-) = \frac{1}{D_\omega(\mathbf{p})}\left[\frac{f_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)}u_0(\mathbf{k}^-) - \frac{f_0(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)}u_0(\mathbf{k}^+)\right] , \tag{115}$$

where $u_0(\mathbf{k})$ is a C^∞ function such that

$$u_0(\mathbf{k}) = \begin{cases} 0 & \text{if } |\mathbf{k}| \le 1 \\ 1 - f_0(\mathbf{k}) & \text{if } 1 \le |\mathbf{k}| \end{cases} . \tag{116}$$

We want to show that

$$\Delta_\omega^{(0,0)}(\mathbf{k}^+,\mathbf{k}^-) = \frac{\mathbf{p}}{D_\omega(\mathbf{p})}\mathbf{S}_\omega^{(0)}(\mathbf{k}^+,\mathbf{k}^-) = \frac{p_0 S_{\omega,0}^{(0)}(\mathbf{k}^+,\mathbf{k}^-) + p S_{\omega,1}^{(0)}(\mathbf{k}^+,\mathbf{k}^-)}{D_\omega(\mathbf{p})} , \tag{117}$$

where $S^{(0)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)$ are smooth functions such that

$$|\partial^{m_+}_{\mathbf{k}^+}\partial^{m_-}_{\mathbf{k}^-} S^{(0)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)| \le C_{m_++m_-} \;, \tag{118}$$

if $\partial^m_{\mathbf{k}}$ denotes a generic derivative of order m with respect to the variables $\mathbf{k}$ and C_m is a suitable constant, depending on m.

The proof of (117) is trivial if $\mathbf{p}$ is bounded away from 0, for example $|\mathbf{p}| \ge 1/2$. It is sufficient to remark that $\Delta^{(0,0)}_\omega(\mathbf{k}^+,\mathbf{k}^-)$, by the compact support properties of $f_0(\mathbf{k})$, is a smooth function and put $S^{(0)}_{\omega,0} = -i\Delta^{(0,0)}_\omega$, $S^{(0)}_{\omega,1} = \omega\Delta^{(0,0)}_\omega$. If $|\mathbf{p}| \le 1/2$, we can use the identity

$$\Delta^{(0,0)}_\omega(\mathbf{k}^+,\mathbf{k}^-) = -\frac{f_0(\mathbf{k}^+)u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)D_\omega(\mathbf{k}^-)} + \tag{119}$$
$$+\frac{\mathbf{p}}{D_\omega(\mathbf{p})}\int_0^1 dt \frac{\mathbf{k}^+-t\mathbf{p}}{|\mathbf{k}^+-t\mathbf{p}|}\left[f'_0(\mathbf{k}^+-t\mathbf{p})\frac{u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^-)} - u'_0(\mathbf{k}^+-t\mathbf{p})\frac{f_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)}\right] \;,$$

from which (118) follows.

2) If $i=0$ and $h \le j < 0$, we get

$$\Delta^{(0,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-) = -\frac{1}{Z_{j-1}}\frac{\tilde f_j(\mathbf{k}^-)u_0(\mathbf{k}^+)}{D_\omega(\mathbf{p})D_\omega(\mathbf{k}^-)} + \delta_{j,h}\frac{1}{\tilde Z_{h-1}(\mathbf{k}^-)}\frac{f_0(\mathbf{k}^+)u_h(\mathbf{k}^-)}{D_\omega(\mathbf{p})D_\omega(\mathbf{k}^+)} \;, \tag{120}$$

where

$$u_h(\mathbf{k}) = \begin{cases} 0 & \text{if } |\mathbf{k}| \ge \gamma^h \\ 1-f_h(\mathbf{k}) & \text{if } |\mathbf{k}| \le \gamma^h \end{cases} \;. \tag{121}$$

If $j < -1$, the first term in the r.h.s. of (120) vanishes for $|\mathbf{p}| \le 1-\gamma^{-1}$, since $u_0(\mathbf{k}^+) \ne 0$ implies that $|\mathbf{k}^+| \ge 1$, so that $|\mathbf{k}^-| = |\mathbf{k}^+-\mathbf{p}| \ge 1-(1-\gamma^{-1}) = \gamma^{-1}$ and, as a consequence, $\tilde f_j(\mathbf{k}^-) = 0$. Analogously, the second term in the r.h.s. of (120) vanishes for $|\mathbf{p}| \le 1-\gamma^{-1}-\gamma^h$, since $f_0(\mathbf{k}^+) \ne 0$ implies that $|\mathbf{k}^+| \ge 1-\gamma^{-1}$, so that $|\mathbf{k}^-| \ge \gamma^h$ and, as a consequence, $u_h(\mathbf{k}^-) = 0$. On the other hand, if $j=-1$, because $\tilde f_{-1}(\mathbf{k})u_0(\mathbf{k}) = 0$, we can write

$$u_0(\mathbf{k}^+)\tilde f_{-1}(\mathbf{k}^-) = -u_0(\mathbf{k}^+)\;\mathbf{p}\int_0^1 dt\frac{\mathbf{k}^+-t\mathbf{p}}{|\mathbf{k}^+-t\mathbf{p}|}\tilde f'_{-1}(\mathbf{k}^+-t\mathbf{p}) \;. \tag{122}$$

It follows that

$$\Delta^{(0,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-) = \frac{\mathbf{p}}{D_\omega(\mathbf{p})}\mathbf{S}^{(j)}_\omega(\mathbf{k}^+,\mathbf{k}^-) \;, \tag{123}$$

where $S^{(j)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)$ are smooth functions such that

$$|\partial^{m_0}_{\mathbf{k}^+}\partial^{m_j}_{\mathbf{k}^-} S^{(j)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)| \le C_{m_0+m_j}\frac{\gamma^{-j(1+m_j)}}{\tilde Z_{j-1}(\mathbf{k}^-)} \;, \quad h \le j < 0 \;. \tag{124}$$

3) If $i = j = h$ we get

$$\Delta_\omega^{(h,h)}(\mathbf{k}^+,\mathbf{k}^-) = \frac{1}{D_\omega(\mathbf{p})}\frac{1}{\tilde Z_{h-1}(\mathbf{k}^+)\tilde Z_{h-1}(\mathbf{k}^-)} \cdot \cdot \left[\frac{f_h(\mathbf{k}^+)u_h(\mathbf{k}^-)}{D_\omega(\mathbf{k}^+)} - \frac{u_h(\mathbf{k}^+)f_h(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)}\right] . \quad (125)$$

Since this expression can appear only at the last integration step, it is not involved in any regularization procedure. Hence we only need its size for values of $\mathbf{p}$ of order γ^h or larger. It is easy to see that

$$|\Delta_\omega^{(h,h)}(\mathbf{k}^+,\mathbf{k}^-)| \le \frac{C}{M}\frac{\gamma^{-2h}}{\tilde Z_{h-1}(\mathbf{k}^+)\tilde Z_{h-1}(\mathbf{k}^-)} , \quad \text{if } |\mathbf{p}| \ge M\gamma^h . \quad (126)$$

4) If $j = h < i < -1$, we get

$$\Delta_\omega^{(i,h)}(\mathbf{k}^+,\mathbf{k}^-) = \frac{1}{\tilde Z_{h-1}(\mathbf{k}^-)Z_{i-1}}\frac{\tilde f_i(\mathbf{k}^+)u_h(\mathbf{k}^-)}{D_\omega(\mathbf{p})D_\omega(\mathbf{k}^+)} , \quad (127)$$

which satisfies the bound

$$|\Delta_\omega^{(i,h)}(\mathbf{k}^+,\mathbf{k}^-)| \le \frac{C}{M}\frac{\gamma^{-h-i}}{\tilde Z_{h-1}(\mathbf{k}^-)Z_{i-1}} , \quad \text{if } |\mathbf{p}| \ge M\gamma^h . \quad (128)$$

4.3 The multiscale expansion of the correction term

We are now ready to begin the description of the iterative integration procedure. As in §3.4, we can write

$$e^{\mathcal{W}_\Delta(\phi,J)} = e^{-L\beta E_j}\int P_{\tilde Z_j, C^\varepsilon_{h,j}}(d\psi^{[h,j]})e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})+K^{(j)}(\sqrt{Z_j}\psi^{[h,j]},\phi,J)} , \quad (129)$$

where $K^{(j)}(\sqrt{Z_j}\psi,\phi,J)$ denotes the sum over the terms containing at least one ϕ or J field; we shall write it in the form

$$K^{(j)}(\sqrt{Z_j}\psi,\phi,J) = \mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\psi) + K_J^{(j)}(\sqrt{Z_j}\psi) + \tilde W_R^{(j)}(\sqrt{Z_j}\psi,\phi,J) , \quad (130)$$

where $\mathcal{B}_\phi^{(j)}(\psi)$ and $K_J^{(j)}(\psi)$ denote the sums over the terms containing only one ϕ or J field, respectively. Note that $\mathcal{B}_\phi^{(j)}(\psi)$ it is the same function appearing in (89) and the action of $\mathcal{L}$ on it is defined exactly as before.

As in §3.4, the only terms contributing to $K_J^{(j)}(\sqrt{Z_j}\psi)$, for which the localization has to be defined different from the identity, are those of second order in ψ, which behave as marginal terms; we shall denote their sum

$K_J^{(j,2)}(\sqrt{Z_j}\psi)$. For $j=0$, $K_J^{(0,2)}(\sqrt{Z_0}\psi) = K_J^{(0)}(\sqrt{Z_0}\psi) = \sum_\omega \int d\mathbf{x} J_{\mathbf{x},\omega} T^{[h,0]}_{\mathbf{x},\omega}$ and we define the $\mathcal{L}$ operation on it as the identity, that is

$$\mathcal{L}K_J^{(0,2)}(\sqrt{Z_0}\psi^{[h,0]}) = K_J^{(0,2)}(\sqrt{Z_0}\psi^{[h,0]}) \ . \qquad (131)$$

Let us now analyze the structure of $K_J^{(-1,2)}(\sqrt{Z_{-1}}\psi^{[h,-1]})$, as it appears after integrating the $\psi^{(0)}$ field and rescaling $\psi^{[h,-1]}$. We have

$$K_J^{(-1,2)}(\psi) = \frac{1}{Z_{-1}} \sum_\omega \int d\mathbf{x} J_{\mathbf{x},\omega} \Big\{ T_{\mathbf{x},\omega} +$$
$$\sum_{\tilde\omega} \int d\mathbf{y} d\mathbf{z} [F^{(-1)}_{2,\omega,\tilde\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) + \delta_{\omega,\tilde\omega} F^{(-1)}_{1,\omega}(\mathbf{x},\mathbf{y},\mathbf{z})] \psi^+_{\mathbf{y},\tilde\omega} \psi^-_{\mathbf{z},\tilde\omega} \Big\} \ . \qquad (132)$$

$F^{(-1)}_{2,\omega,\tilde\omega}$ denotes the sum over all Feynman graphs built by contracting both ψ fields of $T_{\mathbf{x},\omega}$ (on scale 0) and by choosing equal to $\tilde\omega = \pm 1$ the ω-index of the two external ψ fields. $F^{(-1)}_{1,\omega}$ represents the sum over the graphs build by leaving external one of these ψ fields of $T_{\mathbf{x},\omega}$. See Fig. (5), where the J field and the external ψ fields are represented as dashed lines and the small circle represents the non local kernel of $T_{\mathbf{x},\omega}$.

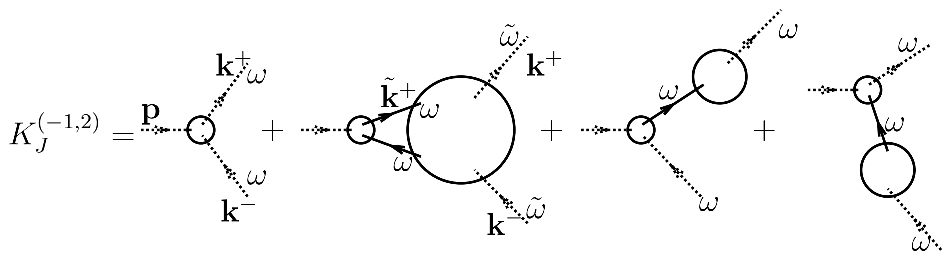

Figure 5: Graphical representation of equation (132).

It is easy to see that the Fourier transform of $F^{(-1)}_{2,\omega,\tilde\omega}$ can be written, if we choose the momenta $\mathbf{k}^+$ and $\mathbf{k}^-$ of the ψ external fields as independent variables, as

$$\hat F^{(-1)}_{2,\omega,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) = \frac{\mathbf{p}}{D_\omega(\mathbf{p})} \int d\tilde{\mathbf{k}}^+ \mathbf{S}_\omega(\tilde{\mathbf{k}}^+, \tilde{\mathbf{k}}^+ - \mathbf{p}) G^{(-1)}_{\omega,\tilde\omega}(\tilde{\mathbf{k}}^+, \mathbf{k}^+, \mathbf{k}^-) \ , \qquad (133)$$

where $\mathbf{S}(\mathbf{k}^+,\mathbf{k}^-)$ is given by (117), $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$ and $G^{(-1)}_{\omega,\tilde\omega}(\tilde{\mathbf{k}}^+,\mathbf{k}^+,\mathbf{k}^-)$ is of the form $G^{(-1)}_{\omega,\tilde\omega}(\tilde{\mathbf{k}}^+,\mathbf{k}^+,\mathbf{k}^-) = G_0(\tilde{\mathbf{k}}^+,\mathbf{k}^+,\mathbf{k}^-) + G_1(\mathbf{k}^+)G_2(\mathbf{k}^-)\delta(\tilde{\mathbf{k}}^+ - \mathbf{k}^+)$, where G_0 represents a suitable sum over connected graphs with four external lines, while G_1 and G_2 represent suitable sums over connected graphs with two external lines. Note that all these three functions can be written, at order n of perturbation theory, as sums of C^n terms, each term being represented as a truncated expectation of ψ monomials, which can then be expanded as a sum over tree graphs of suitable determinants, thanks to the anticommuting properties of the Grassmanian variables [Le], hence the argument in [BM] to avoid bad factorials in the bounds can be used.

$G^{(-1)}_{\omega,\tilde\omega}$ has special symmetry properties, which it is very important to exploit. Consider first the case $\omega=\tilde\omega$; then each term contributing to $G^{(-1)}_{\omega,\omega}$ is obtained by taking n interaction terms (each having two ψ fields of type ω and two of type $-\omega$) and by building a graph with four external lines, two of type ω and two of type $\tilde\omega$. It follows that $n\ge 2$ and that in the graph there are $(2n-4)/2$ propagators of type ω and $(2n)/2$ propagators of type $-\omega$. By using the symmetry property of the propagators (99), one gets

$$G^{(-1)}_{\omega,\omega}(\tilde{\mathbf{k}}^+,\mathbf{k}^+,\mathbf{k}^-) = -G^{(-1)}_{\omega,\omega}(\tilde{\mathbf{k}}^{+*},\mathbf{k}^{+*},\mathbf{k}^{-*})\;. \tag{134}$$

In a similar way, one can check that

$$G^{(-1)}_{\omega,-\omega}(\tilde{\mathbf{k}}^+,\mathbf{k}^+,\mathbf{k}^-) = G^{(-1)}_{\omega,-\omega}(\tilde{\mathbf{k}}^{+*},\mathbf{k}^{+*},\mathbf{k}^{-*})\;, \tag{135}$$

$$\mathbf{p}\cdot\mathbf{S}_\omega(\mathbf{k}^+,\mathbf{k}^-) = -i\omega\mathbf{p}^*\cdot\mathbf{S}_\omega(\mathbf{k}^{+*},\mathbf{k}^{-*})\;. \tag{136}$$

(133), (134), (135) and (136) imply that

$$\hat F^{(-1)}_{2,\omega,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) = \frac{1}{D_\omega(\mathbf{p})}[p_0\hat A^{(-1)}_{\omega,\tilde\omega,0}(\mathbf{k}^+,\mathbf{k}^-) + p\hat A^{(-1)}_{\omega,\tilde\omega,1}(\mathbf{k}^+,\mathbf{k}^-)]\;, \tag{137}$$

where $\hat A^{(-1)}_{\omega,\tilde\omega,i}(\mathbf{k}^+,\mathbf{k}^-)$ are smooth functions verifying the condition

$$\hat A^{(-1)}_{\omega,\tilde\omega,1}(\mathbf{k}^+,\mathbf{k}^-) = i\tilde\omega\hat A^{(-1)}_{\omega,\tilde\omega,0}(\mathbf{k}^{+*},\mathbf{k}^{-*})\;. \tag{138}$$

It follows that, if we define (in the $L=\beta=\infty$ limit, see the discussion after (97))

$$\mathcal{L}\hat F^{(-1)}_{2,\omega,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) = \frac{1}{D_\omega(\mathbf{p})}[p_0\hat A^{(-1)}_{\omega,\tilde\omega,0}(0,0) + p\hat A^{(-1)}_{\omega,\tilde\omega,1}(0,0)]\;, \tag{139}$$

then

$$\mathcal{L}\hat F^{(-1)}_{2,\omega,\omega}(\mathbf{k}^+,\mathbf{k}^-) = Z^{(3,+)}_{-1}\;, \tag{140}$$

$$\mathcal{L}\hat F^{(-1)}_{2,\omega,-\omega}(\mathbf{k}^+,\mathbf{k}^-) = Z^{(3,-)}_{-1}\frac{D_{-\omega}(\mathbf{p})}{D_\omega(\mathbf{p})}\;, \tag{141}$$

where $Z_{-1}^{(3,+)} = i\hat{A}^{(-1)}_{\omega,\omega,0}(0,0)$ and $Z_{-1}^{(3,-)} = i\hat{A}^{(-1)}_{\omega,-\omega,0}(0,0)$ are constants, which one can easily show to be real.

The action of $\mathcal{L}$ on $F^{-1}_{2,\omega,\tilde\omega}$ was given above in momentum space; it is however very easy to write it in coordinate space. The support properties of the external propagator imply that $|\mathbf{p}| \le \gamma + \gamma^h$, hence $|\mathbf{p}| \le \gamma^2$, if γ^h is small enough, as we shall suppose (to simplify the notation). Then we can freely multiply $\Delta^{(i,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)$ by $\chi_0(\gamma^{-2}|\mathbf{k}^+ - \mathbf{k}^-|)$. Hence, in space-time coordinates, the contribution to $K_J^{(-1,2)}(\psi)$ containing $F^{(-1)}_{2,\omega,\tilde\omega}$ can be written, by using the representation (137) as

$$\frac{1}{Z_{-1}} \sum_{\omega,\tilde\omega} \int d\mathbf{x} J_{\mathbf{x},\omega} \int d\mathbf{x}' \mathbf{V}_\omega(\mathbf{x}-\mathbf{x}') \int d\mathbf{y} d\mathbf{z} \mathbf{A}^{(-1)}_{\omega,\tilde\omega}(\mathbf{x}',\mathbf{y},\mathbf{z}) \psi^+_{\mathbf{y},\tilde\omega} \psi^-_{\mathbf{z},\tilde\omega} \;, \tag{142}$$

where

$$\begin{aligned} \mathbf{V}_\omega(\mathbf{x}) &= \int \frac{d\mathbf{p}}{(2\pi)^2} \sqrt{\chi_0(\gamma^{-2}|\mathbf{p}|)} e^{i\mathbf{p}\mathbf{x}} \frac{\mathbf{p}}{D_\omega(\mathbf{p})} \;, \\ \mathbf{A}^{(-1)}_{\omega,\tilde\omega}(\mathbf{x}',\mathbf{y},\mathbf{z}) &= \int \frac{d\mathbf{k}^+ d\mathbf{k}^-}{(2\pi)^4} \sqrt{\chi_0(\gamma^{-2}|\mathbf{k}^+ - \mathbf{k}^-|)} \cdot \\ &\quad \cdot e^{i\mathbf{k}^+(\mathbf{x}'-\mathbf{y}) - i\mathbf{k}^-(\mathbf{x}'-\mathbf{z})} \hat{\mathbf{A}}^{(-1)}_{\omega,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) \;. \end{aligned} \tag{143}$$

It follows that the operation $\mathcal{L}$ can be described as the *localization* of the ψ fields in the point $\mathbf{x}'$ and that the corresponding $\mathcal{R}$ operation produces a term with $\psi^+_{\mathbf{y},\tilde\omega}\psi^-_{\mathbf{z},\tilde\omega} - \psi^+_{\mathbf{x}',\tilde\omega}\psi^-_{\mathbf{x}',\tilde\omega}$ in place of $\psi^+_{\mathbf{y},\tilde\omega}\psi^-_{\mathbf{z},\tilde\omega}$. We can then apply the argument following (78) to explain the regularization effect of $\mathcal{R}$, since $\hat{A}^{(-1)}_{\omega,\tilde\omega,i}(\mathbf{k}^+,\mathbf{k}^-)$ are smooth functions.

If λ is small enough, the size of $Z_{-1}^{(3,+)}$ and $Z_{-1}^{(3,-)}$ is determined by the contributions of lower order in their expansions in power of λ. It is easy to see that $Z_{-1}^{(3,+)}$ is of order λ^2 and that there is only one graph of that order different from zero in its expansion, that on the left of Fig. 6, while $Z_{-1}^{(3,-)}$ is of order λ and the only corresponding first order graph is represented on the right of the picture.

By an explicit calculation, one can check that the contributions to $Z_{-1}^{(3,+)}$ and $Z_{-1}^{(3,-)}$ of the two graphs are different from zero. Hence

$$Z_{-1}^{(3,+)} = -c_+\lambda^2 < 0 \;, \qquad Z_{-1}^{(3,-)} = \frac{\lambda}{4\pi} \;. \tag{144}$$

We now consider the contribution to $F^{(-1)}_{1,\omega}(\mathbf{x},\mathbf{y},\mathbf{z})$ associated with the third term in Fig. 5. Its Fourier transform can be written as

$$\hat{F}^{(-1,+)}_{1,\omega}(\mathbf{k}^+,\mathbf{k}^-) = \frac{[C^\varepsilon_{h,0}(\mathbf{k}^-) - 1] D_\omega(\mathbf{k}^-) \hat{g}^{(0)}_\omega(\mathbf{k}^+) - u_0(\mathbf{k}^+)}{D_\omega(\mathbf{p})} G^{(2)}_\omega(\mathbf{k}^+) \;, \tag{145}$$

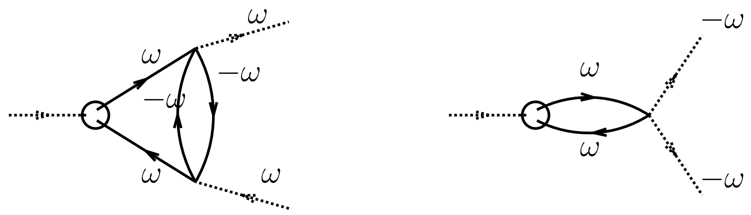

Figure 6: Terms of lower order contributing to $Z_{-1}^{(3,+)}$ and $Z_{-1}^{(3,-)}$.

where $G_\omega^{(2)}(\mathbf{k}^+)$ represents the sum over the connected Feynman graphs with propagator $g^{(0)}$ and two external lines. Since, by symmetry reasons, $G_\omega^{(2)}(0) = 0$, the simplest way to regularize $F_{1,\omega}^{(-1,+)}$ is to define

$$\begin{aligned} \mathcal{R}\hat{F}_{1,\omega}^{(-1,+)}(\mathbf{k}^+, \mathbf{k}^-) &= \frac{[C_{h,0}^\varepsilon(\mathbf{k}^-) - 1] D_\omega(\mathbf{k}^-) \hat{g}_\omega^{(0)}(\mathbf{k}^+) - u_0(\mathbf{k}^+)}{D_\omega(\mathbf{p})} \cdot \\ &\cdot [G_\omega^{(2)}(\mathbf{k}^+) - G_\omega^{(2)}(0)] \;, \end{aligned} \tag{146}$$

whose corresponding local part is vanishing. In other words the dimensional gain is here obtained without the introduction of a renormalization constant. Note that there is a simple description of this operation in terms of a localization operation on the ψ fields, as in the remark following (141). A similar procedure can be defined for the contribution to $F_{1,\omega}^{(-1)}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ associated with the fourth term in Fig. (5).

We can summarize the previous discussion by defining

$$\begin{aligned} \mathcal{L}K_J^{(-1,2)}(\psi) &= \sum_\omega \int d\mathbf{x} \Big\{ J_{\mathbf{x},\omega} \left[\frac{T_{\mathbf{x},\omega}}{Z_{-1}} + \frac{Z_{-1}^{(3,+)}}{Z_{-1}} \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} \right] + \\ &\frac{Z_{-1}^{(3,-)}}{Z_{-1}} J^{(-)}_{\mathbf{x},\omega} \psi^+_{\mathbf{x},-\omega} \psi^-_{\mathbf{x},-\omega} \Big\} \;, \end{aligned} \tag{147}$$

where $J^{(-)}_{\mathbf{x},\omega}$ is the Fourier transform of

$$\hat{J}^{(-)}_{\mathbf{p},\omega} = \hat{J}_{\mathbf{p},\omega} \frac{D_{-\omega}(\mathbf{p})}{D_\omega(\mathbf{p})} \;. \tag{148}$$

Equation (147) implies that the integration of the scale $j = -1$ has to take into account two new local terms, to be called of type Z^+ and of type Z^-, similar to those introduced in §3.4 to analyze the Schwinger functions, see (100). There is however an important difference in the term of type Z^-, related with the fact that, in this term, we have absorbed in the external J field a bounded but not smooth function of $\mathbf{p}$, in order to avoid that it is involved in the regularization operations.

We can now describe the general step, by defining the action of $\mathcal{L}$ on $K_J^{(j,2)}(\psi)$, which can be written, if $j < -1$, after rescaling $\psi^{[h,j]}$, as

$$\begin{aligned} K_J^{(j,2)}(\psi) &= \frac{1}{Z_j}\sum_\omega \int d\mathbf{x}\Big\{ J_{\mathbf{x},\omega}T_{\mathbf{x},\omega} + \sum_{\tilde\omega}\int d\mathbf{y}d\mathbf{z}\Big[J_{\mathbf{x},\omega}F^{(j)}_{Z^+,\omega,\tilde\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) + \\ &+ J^{(-)}_{\mathbf{x},\omega}F^{(j)}_{Z^-,\omega,\tilde\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) + J_{\mathbf{x},\omega}F^{(j)}_{2,\omega,\tilde\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) + \\ &+ \delta_{\omega,\tilde\omega}J_{\mathbf{x},\omega}F^{(j)}_{1,\omega}(\mathbf{x},\mathbf{y},\mathbf{z})\Big]\psi^+_{\mathbf{y},\tilde\omega}\psi^-_{\mathbf{z},\tilde\omega}\Big\}\ , \end{aligned} \tag{149}$$

where $F^{(j)}_{Z^\pm,\omega,\tilde\omega}$ represents the sum over all graphs with one vertex of type $Z^\pm$ and two ψ external fields of type $\tilde\omega$, build by using propagators of scale $i\in[j+1,-1]$, $F^{(j)}_{2,\omega,\tilde\omega}$ is the sum over the same kind of graphs with one vertex $T_{\mathbf{x},\omega}$, whose ψ fields are both contracted and $F^{(j)}_{1,\omega}$ is the sum over the graphs with one vertex $T_{\mathbf{x},\omega}$, such that one of its ψ fields is external.

It is important to stress that, thanks to the identity (114), given a graph contributing to $F^{(j)}_{2,\omega,\tilde\omega}$, at least one among the ψ fields belonging to $T_{\mathbf{x},\omega}$ is contracted on scale 0, so that we can write

$$\hat F^{(j)}_{2,\omega,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) = \frac{\mathbf{p}}{D_\omega(\mathbf{p})}\sum_{i=j}^{0}\int d\tilde{\mathbf{k}}^+\tilde{\mathbf{S}}^{(i)}_\omega(\tilde{\mathbf{k}}^+,\tilde{\mathbf{k}}^+-\mathbf{p})G^{(i)}_{\omega,\tilde\omega}(\tilde{\mathbf{k}}^+,\mathbf{k}^+,\mathbf{k}^-)\ , \tag{150}$$

where $\tilde{\mathbf{S}}^{(i)}(\mathbf{k}^+,\mathbf{k}^-)$ is given by (117) for $i=0$ and (123) for $i<0$, if no derivative acts on $\mathbf{S}^{(i)}(\mathbf{k}^+,\mathbf{k}^-)$ as a consequence of the regularization on a scale r such that $j<r\le i$, otherwise it is given by a suitable derivative of $\mathbf{S}^{(i)}(\mathbf{k}^+,\mathbf{k}^-)$. Moreover, $G^{(i)}_{\omega,\tilde\omega}(\tilde{\mathbf{k}}^+,\mathbf{k}^+,\mathbf{k}^-)$ is a suitable smooth function, which can be expressed as a sum over products of propagators $\tilde g^{(i')}$, $i'\in[j+1,0]$, or their derivatives, integrated over suitable loop variables. Hence we can extend to the case $j<-1$ the definition of $\mathcal{L}$ given for $j=-1$, for what concerns its action on all terms in the r.h.s. of (149) except $F^{(j)}_{Z^+,\omega,\tilde\omega}$ and $F^{(j)}_{Z^-,\omega,\tilde\omega}$, for which we put (for $L=\beta=\infty$, see otherwise the discussion after (99))

$$\mathcal{L}\hat F^{(j)}_{Z^\pm,\omega,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) = \hat F^{(j)}_{Z^\pm,\omega,\tilde\omega}(0,0)\ . \tag{151}$$

Note that $\tilde{\mathbf{S}}^{(i)}(\mathbf{k}^+,\mathbf{k}^-)$ is a smooth function, by our definition of $\mathcal{R}\hat F^{(j)}_{1,\omega}$, which generalizes (146). And, by the same argument leading to (98), we have

$$\hat F^{(j)}_{Z^+,\omega,-\omega}(0,0) = \hat F^{(j)}_{Z^-,\omega,\omega}(0,0) = 0\ . \tag{152}$$

It follows that we can write

$$\mathcal{L}K_J^{(j,2)}(\psi) = \sum_\omega \int d\mathbf{x} \Big\{ J_{\mathbf{x},\omega} \left[\frac{T_{\mathbf{x},\omega}}{Z_j} + \frac{Z_j^{(3,+)}}{Z_j} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} \right] + \frac{Z_j^{(3,-)}}{Z_j} J^{(-)}_{\mathbf{x},\omega} \psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{x},-\omega} \Big\} , \tag{153}$$

which defines the new renormalization constants $Z_j^{(3,+)}$ and $Z_j^{(3,-)}$, for $j \in [h, -1]$.

4.4 The bounds

The previous considerations allow to define a tree expansion for $H^{2,1}_\omega$, similar to that used for $G^{2,1}_\omega$ in §3 and described in §3.4 after (103). The only important difference is that we have now three different special endpoints associated with the field J, corresponding to the three different terms in the r.h.s. of (153); we shall call these endpoints of type J and *subtype* T, Z^+ and Z^-, respectively.

There is of course a tree expansion also for the renormalization constants $Z_j^{(3,+)}$ and $Z_j^{(3,-)}$, involving trees with root at scale $j-1$, one endpoint of type J, $|P_{v_0}| = 2$ and the operation $\mathcal{L}$ acting on v_0. One can show in the usual way that, given a tree τ with n normal endpoints and the special endpoint of subtype $Z^\pm$ and scale $i+1$, its contribution $Z_\tau^{(3,\pm)}$ to $Z_j^{(3,\pm)}$ satisfies the bound

$$|Z_\tau^{(3,\pm)}| \le (C\bar\varepsilon)^n |Z_i^{(3,\pm)}| \prod_{v\in\tau} \gamma^{-d_v} , \quad \text{with } d_v \ge 1, \forall v \in \tau . \tag{154}$$

A similar bound is satisfied if the special endpoint is of subtype T, without the $|Z_i^{(3,\pm)}|$ factor. However, in this case, the scale index of the special endpoint has to be equal to 1, because of the properties of the function $\Delta^{(i,j)}_\omega$ described in §4.2. Therefore there is a path $\mathcal{C}$ in the tree connecting the special endpoint with v_0 and we can extract from $\prod_{v\in\tau}\gamma^{-d_v}$ a small factor $\gamma^{1/2}$ for each $v \in \mathcal{C}$, without loosing the summability properties of the bound; hence we write

$$|Z_\tau^{(3,\pm)}| \le (C\bar\varepsilon)^n \gamma^{j/2} \prod_{v\in\mathcal{C}} \gamma^{-(d_v-\frac{1}{2})} \prod_{v\in\tau\setminus\mathcal{C}} \gamma^{-d_v} , \quad \text{with } d_v \ge 1, \forall v \in \tau . \tag{155}$$

Another important property, following from (151) and (152), is that $Z_\tau^{(3,+)} = 0$, if the special endpoint is of subtype Z^-, and viceversa. Hence we can write, if $j \in [h+1, -1]$,

$$Z_{j-1}^{(3,\pm)} = Z_j^{(3,\pm)} + \sum_{i=j}^{-1} \beta_{j,i} Z_i^{(3,\pm)} + \tilde\beta_j^{(3,\pm)} , \tag{156}$$

where $\beta_{j,i} Z_i^{(3,\pm)}$ is the sum over the contributions associated with trees whose special endpoint is of subtype $Z^\pm$ and scale $i+1$, while $\tilde\beta_j^{(3,\pm)}$ is the sum over the trees whose special endpoint is of subtype T. Note that $\beta_{j,i}$ is equal for $Z_{j-1}^{(3,+)}$ and $Z_{j-1}^{(3,-)}$, by the symmetry of the interaction (9) under the transformation $\omega \to -\omega$.

Let us now observe that the the trees contributing to $\tilde\beta_j^{(3,+)}$ have at least two normal endpoints, since it is not possible to build a graph contributing to $Z_j^{(3,+)}$ with only one endpoint of type λ and the local part of the graphs with one endpoint of type δ is equal to zero. This last property is of course true also for $Z_j^{(3,-)}$, but in this case it is possible to build a graph contributing to it with one endpoint of type λ, see Fig. 6. However, the considerations of §4.2, item 2), imply that this graph could give a contribution different from zero to $\tilde\beta_j^{(3,-)}$ only for $j=-1$, but also in this case a simple explicit calculation implies that its value is zero.

By using (155) and the previous remark, one can easily show that

$$|\tilde\beta_j^{(3,\pm)}| \le C\bar\varepsilon^2 \gamma^{j/2} \; . \tag{157}$$

In a similar way one can prove that

$$|\beta_{j,i}| \le C\bar\varepsilon^2 \gamma^{-\frac{i-j}{2}} \; . \tag{158}$$

We want to compare the flow equation (156) with the flow equation of the renormalization constant $Z_j^{(2)}$ introduced in §3.4 to study the Schwinger Functions, see (100). In this case the involved trees have one endpoint of type J, which can have scale $\ge +1$, while in the previous case the scale of the special endpoints of subtype $Z^\pm$ was ≥ 0. However, if the scale of the special endpoint is ≥ 0, the contribution of corresponding trees is equal to $\beta_{j,i} Z_i^{(2)}$, where $\beta_{j,i}$ is the same number appearing in (156). Hence we can write, if $j \in [h+1, -1]$,

$$Z_{j-1}^{(2)} = Z_j^{(2)} + \sum_{i=j}^{-1} \beta_{j,i} Z_i^{(2)} + \tilde\beta_j^{(2)} \; , \tag{159}$$

where $\beta_{j,i} Z_i^{(2)}$ is the sum over the contributions to $Z_j^{(2)}$ associated with trees whose special endpoint has scale $i+1 \ge 0$, while $\tilde\beta_j^{(2)}$ is the sum over the trees whose special endpoint has scale $+1$. By proceeding as in the proof of (157), one can show that

$$|\tilde\beta_j^{(2)}| \le C\bar\varepsilon^2 \gamma^{j/2} \; . \tag{160}$$

As we shall see, the renormalization constants $Z_j^{(3,\pm)}$ and $Z_j^{(2)}$ are divergent as $j \to -\infty$, but $Z_j^{(3,\pm)}/Z_j^{(2)}$ is bounded and of order $\bar\varepsilon^2$, uniformly in j.

Lemma 4.1 *If $\bar\varepsilon \le 2|\lambda|$ and $|\lambda|$ is small enough, there is a constant c_0, independent of j and h, such that*

$$c_0\lambda^2 \le \left|\frac{Z_j^{(3,+)}}{Z_j^{(2)}}\right| \le 2c_0\lambda^2 \;, \quad c_0|\lambda| \le \left|\frac{Z_j^{(3,-)}}{Z_j^{(2)}}\right| \le 2c_0|\lambda| \;, \quad j\in[h,-1] \;. \quad (161)$$

Proof - Let us consider first $Z_j^{(3,+)}$. In order to control its dependence on j, we have to analyze in a different way the regions $j\in[j_0,-1]$ and $j<j_0$, with j_0 chosen so that

$$\gamma^{j_0/2} = c_1|\lambda|^2 \;, \quad (162)$$

for some constant c_1. If $j\ge j_0$, we put $Z_j^{(3,+)} = a_j + b_j$, where a_j is the contribution of order λ^2, while b_j is the sum over the terms of order ≥ 3. The analysis of §4.2 implies that a_j is obtained by applying the $\mathcal{L}$ operation to the graph in the left of Fig. (6), with the two propagators of type ω on scale 0 or -1 (by the remark after (121), the local part is zero, if this condition is not satisfied) and the two propagators of type $-\omega$ on scale $i\in[j+1,0]$. By an explicit calculation, we can show that

$$c_2\lambda^2 \le -a_j \le 2c_2\lambda^2 \;, \quad \text{uniformly in } j \;. \quad (163)$$

On the other hand, if we extract from both sides of (156) the terms of order $\bar\varepsilon^2$, we get

$$b_{j-1} = b_j + \sum_{i=j}^{-1}\beta_{j,i}Z_i^{(3,+)} + \bar\beta_j \;, \quad |\bar\beta_j| \le C\bar\varepsilon^3\gamma^{j/2} \;, \quad (164)$$

which allows very easily to prove the bound $|b_j| \le c_3\bar\varepsilon^3(1+c_2\bar\varepsilon)|j|$, for some constant c_3, as far as $c_3\bar\varepsilon(1+c_2\bar\varepsilon)|j| \le c_2/2$, a condition which is certainly satisfied for $j\ge j_0$, if $\bar\varepsilon$ is small enough, and allows also to prove, under the further hypothesis $\bar\varepsilon \le 2|\lambda|$, that

$$\frac{c_2}{2}\bar\varepsilon^2 \le |Z_j^{(3,+)}| \le \frac{5c_2}{2}\bar\varepsilon^2 \;, \quad j\in[j_0,-1] \;. \quad (165)$$

Moreover, by using (159), the fact that $Z_0^{(2)}=1$ and an explicit second order calculation, one can show very easily by induction that there exists a positive constant c_4 such that

$$\gamma^{c_4\bar\varepsilon^2} \le \frac{Z_{j-1}^{(2)}}{Z_j^{(2)}} \le \gamma^{2c_4\bar\varepsilon^2} \quad \Rightarrow \quad \gamma^{c_4\bar\varepsilon^2|j|} \le Z_j^{(2)} \le \gamma^{2c_4\bar\varepsilon^2|j|} \;, \quad j\in[h+1,-1] \;. \quad (166)$$

(165) and (166) immediately imply the first of the bounds (161), if $j \geq j_0$ and $|\lambda| \geq \bar{\varepsilon}/2$ is so small that $\gamma^{C\bar{\varepsilon}j_0} \geq 1/2$.

Let us now suppose that $j < j_0$. Equation (156) can be rewritten in the form

$$Z_{j-1}^{(3,+)} = Z_j^{(3,+)} + \sum_{i=j}^{j_0} \beta_{j,i} Z_i^{(3,+)} + \bar{\beta}_j \ , \quad \bar{\beta}_j = \sum_{i=j_0+1}^{-1} \beta_{j,i} Z_i^{(3,+)} + \tilde{\beta}_j^{3,+} \ . \quad (167)$$

By using (157), (158), (162) and (165), we get the bound

$$|\bar{\beta}_j| \leq C\bar{\varepsilon}^4 \gamma^{-(j_0-j)/2} \ , \quad (168)$$

which allows to prove by induction that there is a constant, which can of course be chosen equal to the constant c_4 of (166) if it is large enough, such that, if $j \in [h+1, j_0]$,

$$\gamma^{c_4\bar{\varepsilon}^2} \leq \frac{Z_{j-1}^{(3,+)}}{Z_j^{(3,+)}} \leq \gamma^{2c_4\bar{\varepsilon}^2} \quad \Rightarrow \quad \frac{c_2}{2}\bar{\varepsilon}^2 \gamma^{c_4\bar{\varepsilon}^2(j_0-j)} \leq Z_j^{(3,+)} \leq \frac{5c_2}{2}\bar{\varepsilon}^2 \gamma^{2c_4\bar{\varepsilon}^2(j_0-j)} \ . \quad (169)$$

We want now to show that, if $r < j_0$, there exists a constant c_5 such that

$$\left| \frac{Z_{r-1}^{(3,+)}}{Z_r^{(3,+)}} - \frac{Z_{r-1}^{(2)}}{Z_r^{(2)}} \right| \leq c_5 \bar{\varepsilon}^2 \gamma^{-(j_0-r)/4} \ . \quad (170)$$

Note that, by (159), (166), (167) and (169), if $j < j_0$,

$$\frac{Z_{j-1}^{(3,+)}}{Z_j^{(3,+)}} - \frac{Z_{j-1}^{(2)}}{Z_j^{(2)}} = \sum_{i=j}^{j_0} \beta_{j,i} \left[\frac{Z_i^{(3,+)}}{Z_j^{(3,+)}} - \frac{Z_i^{(2)}}{Z_j^{(2)}} \right] + \eta_j \ , \quad (171)$$

with

$$|\eta_j| \leq C\bar{\varepsilon}^2 \gamma^{-(j_0-j)/2} \leq \frac{c_5}{2} \bar{\varepsilon}^2 \gamma^{-(j_0-j)/2} \ , \quad (172)$$

if c_5 is chosen large enough. Hence, the bound (170) follows immediately from (171), if $r = j_0$; let us suppose that it is satisfied for $r \in [j+1, j_0]$, $j \leq j_0 - 1$ and note that, if $\bar{\varepsilon}$ is small enough, by the first of (166) and (169), $|Z_r^{(3,+)}/Z_{r-1}^{(3,+)} - Z_r^{(2)}/Z_{r-1}^{(2)}| \leq |Z_{r-1}^{(3,+)}/Z_r^{(3,+)} - Z_{r-1}^{(2)}/Z_r^{(2)}|$, since $\gamma^{c_4\bar{\varepsilon}^2} > 1$. Hence, if $i \in [j+1, j_0]$,

$$\left| \frac{Z_i^{(3,+)}}{Z_j^{(3,+)}} - \frac{Z_i^{(2)}}{Z_j^{(2)}} \right| = \left| \prod_{r=j+1}^{i} \frac{Z_r^{(3,\pm)}}{Z_{r-1}^{(3,\pm)}} - \prod_{r=j+1}^{i} \frac{Z_r^{(2)}}{Z_{r-1}^{(2)}} \right| \leq \quad (173)$$

$$\leq \sum_{r=j+1}^{i} c_5 \bar{\varepsilon}^2 \gamma^{-(j_0-r)/4} \gamma^{-c_4\bar{\varepsilon}^2(i-j-1)} \leq c_5 c_6 \bar{\varepsilon}^2 (i-j) \gamma^{-(j_0-i)/4} \ ,$$

for some constant c_6. This bound, together with (158), (171) and (172), imply that there exists a constant c_7 such that

$$\left|\frac{Z^{(3,+)}_{j-1}}{Z^{(3,+)}_{j}} - \frac{Z^{(2)}_{j-1}}{Z^{(2)}_{j}}\right| \le c_5\bar\varepsilon^2\gamma^{-(j_0-j)/4}[\frac{1}{2}+c_7\bar\varepsilon^2] \;, \tag{174}$$

which implies (170) for $r=j$, if $\bar\varepsilon$ is small enough.

By using (166), (169) and (170), we get, if $i<j_0$

$$\begin{aligned}&\left|\frac{Z^{(3,+)}_{i-1}}{Z^{(2)}_{i-1}} - \frac{Z^{(3,+)}_{i}}{Z^{(2)}_{i}}\right| \le \left|\frac{Z^{(3,+)}_{i-1}}{Z^{(3,+)}_{i}} - \frac{Z^{(2)}_{i-1}}{Z^{(2)}_{i}}\right|\left|\frac{Z^{(3,+)}_{i}}{Z^{(2)}_{i-1}}\right| \le \\ &C\bar\varepsilon^4\gamma^{-(j_0-i)/4+c_4\bar\varepsilon^2(j_0-i)} \le C\bar\varepsilon^4\gamma^{-(j_0-j)/8}\;,\end{aligned} \tag{175}$$

so that

$$\left|\frac{Z^{(3,+)}_{j}}{Z^{(2)}_{j}} - \frac{Z^{(3,+)}_{j_0}}{Z^{(2)}_{j_0}}\right| \le C\bar\varepsilon^4\sum_{i=j+1}^{j_0}\gamma^{-(j_0-j)/8} \le C\bar\varepsilon^4\;, \tag{176}$$

which implies (161) for $Z^{(3,+)}_j/Z^{(2)}_j$, if $\bar\varepsilon\le 2|\lambda|$.

The proof for $Z^{(3,-)}_j/Z^{(2)}_j$ is very similar. However, one can avoid the different treatment of the regions $j\ge j_0$ and $j<j_0$, since $\tilde\beta^{(3,-)}_j/Z^{(3,-)}_j$ is of order $\bar\varepsilon$, by the bound on the right of (143). This bound also justifies the presence of $\bar\varepsilon$ in place of $\bar\varepsilon^2$. ∎

Lemma 4.2 *If $\bar{\mathbf{k}}_1=-\bar{\mathbf{k}}_2=\bar{\mathbf{k}}$ and $|\bar{\mathbf{k}}|=\gamma^h$, then there exists a constant C such that*

$$C\gamma^{-2h}\bar\varepsilon\frac{Z^{(2)}_h}{(Z_h)^2} \le |\hat H^{2,1}_\omega(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_2)| \le 2C\gamma^{-2h}\bar\varepsilon\frac{Z^{(2)}_h}{(Z_h)^2}\;. \tag{177}$$

Proof - As explained at the beginning of §4.4, $\hat H^{2,1}_\omega(\mathbf{p},\mathbf{k})$ admits an expansion similar to that of $\hat G^{(2,1)}_\omega(\mathbf{p},\mathbf{k})$, see (110). The main difference is that the special endpoint of type J can be of three different subtypes, so that it is convenient to write

$$\hat H^{2,1}_\omega(\mathbf{p},\mathbf{k}) = \hat H^{2,1}_{\omega,Z^+}(\mathbf{p},\mathbf{k}) + \hat H^{2,1}_{\omega,Z^-}(\mathbf{p},\mathbf{k}) + \hat H^{2,1}_{\omega,T}(\mathbf{p},\mathbf{k})\;, \tag{178}$$

where $\hat H^{2,1}_{\omega,Z^\pm}$ denotes the sum over the trees whose special endpoint is of subtype $Z^\pm$, while $\hat H^{2,1}_{\omega,T}$ is the sum over the trees whose special endpoint is of subtype T.

Let us now suppose that $\mathbf{p}=\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2$ and $\mathbf{k}=\bar{\mathbf{k}}_2$, with $|\bar{\mathbf{k}}|=\gamma^h$. Then it is obvious that the sum over the trees with n normal endpoints contributing to $\hat{H}^{2,1}_{\omega,Z^+}(\mathbf{p},\mathbf{k})$ can be bounded as in the case of $\hat{G}^{(2,1)}_{\omega}(\mathbf{p},\mathbf{k})$, see (111), by substituting $Z^{(2)}_h$ with $Z^{(3,+)}_h$. A similar argument can be used for $\hat{H}^{2,1}_{\omega,Z^-}(\mathbf{p},\mathbf{k})$, but in this case one has to take into account the fact that the trivial trees containing only one endpoint, the special one, does not contribute. Hence we have

$$|\hat{H}^{2,1}_{\omega,Z^+}(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_2)|+|\hat{H}^{2,1}_{\omega,Z^-}(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_2)|\le C\gamma^{-2h}\frac{|Z^{(3,+)}_h|+\bar{\varepsilon}|Z^{(3,-)}_h|}{(Z_h)^2}\;. \tag{179}$$

Let us now consider $\hat{H}^{2,1}_{\omega,T}$. The analysis of the previous sections (in particular the bounds (126) and (128)) implies that a bound like (111) is still valid for the sum over the trees with n normal endpoints, but now $Z^{(2)}_h$ has to be substituted with 1. Moreover, the contributions corresponding to the trivial trees without normal endpoints are given by $\Delta^{(h,h)}_{\omega}(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)+\Delta^{(h,h+1)}_{\omega}(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)+\Delta^{(h+1,h)}_{\omega}(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)$, because of the support properties of the propagators. However, by (125) and (127), this quantity is exactly equal to 0, so that

$$|\hat{H}^{2,1}_{\omega,T}(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_2)|\le C\gamma^{-2h}\frac{\bar{\varepsilon}}{(Z_h)^2}\;. \tag{180}$$

The bound (179) and (180), immediately imply the upper bound of Theorem 2.3. The lower bound follows from the explicit calculation of the leading contributions to $\hat{H}^{2,1}_{\omega,Z^+}(\mathbf{p},\mathbf{k})$ and $\hat{H}^{2,1}_{\omega,Z^-}(\mathbf{p},\mathbf{k})$, which are both of order λ^2; one has essentially to check that they do not cancel out. ■

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