PROBABILITY MEASURES IN TERMS OF CREATION, ANNIHILATION, AND NEUTRAL OPERATORS

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Let μ be a probability measure on $\mathbb{R}^d$ with finite moments of all orders. Then we can define the creation operator $a^+(j)$, the annihilation operator $a^-(j)$, and the neutral operator $a^0(j)$ for each coordinate $1 \leq j \leq d$. We use the neutral operators $a^0(i)$ and the commutators $[a^-(j), a^+(k)]$ to characterize polynomially symmetric, polynomially factorizable, and moment-equal probability measures. We also present some results for probability measures on the real line with finite support, infinite support, and compact support.

1. Creation, annihilation, and neutral operators

Let μ be a probability measure on $\mathbb{R}^d$ with finite moments of all orders, namely, for any nonnegative integers $i_1, i_2, \ldots, i_d$,

$$\int_{\mathbb{R}^d} |x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}| \, d\mu(x) < \infty,$$

where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. Let $F_0 = \mathbb{R}$ and for $n \geq 1$ let F_n be the vector space of all polynomials in $x_1, x_2, \ldots, x_d$ of degree $\leq n$. Then we have the inclusion chain

$$F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots \subset L^2(\mu).$$

Next, define $G_0 = \mathbb{R}$ and for $n \geq 1$ define G_n to be the orthogonal complement of F_{n-1} in F_n. Then the spaces G_n, $n \geq 0$, are orthogonal. Define a real Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} G_n \quad \text{(orthogonal direct sum)}.$$

For each $n \geq 0$, let P_n denote the orthogonal projection of $\mathcal{H}$ onto G_n. Let X_j, $1 \leq j \leq d$, be the multiplication operator by x_j. Accardi and Nahni[5] have recently observed that for any $1 \leq j \leq d$ and $n \geq 0$

$$X_j G_n \perp G_k, \quad \forall\, k \neq n-1, n, n+1,$$

where $G_{-1} = \{0\}$ by convention. Then they used this fact to obtain the following *fundamental recursion equality*

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n, \quad n \geq 0, \tag{1}$$

where $P_{-1} = 0$ by convention. When $d = 1$, this equality reduces to the well-known recursion formula

$$x P_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \omega_n P_{n-1}(x), \tag{2}$$

where $P_n(x)$'s are orthogonal polynomials with respect to μ, $P_n(x)$ is a polynomial of degree n with leading coefficient 1, and $\{\alpha_n, \omega_n\}$'s are the Jacobi-Szegö parameters of μ.

Now, for each $n \geq 0$ and $1 \leq j \leq d$, define three operators by

$$D_n^+(j) = P_{n+1} X_j P_n \,:\, G_n \longrightarrow G_{n+1},$$

$$D_n^-(j) = P_{n-1} X_j P_n \,:\, G_n \longrightarrow G_{n-1},$$

$$D_n^0(j) = P_n X_j P_n \,:\, G_n \longrightarrow G_n.$$

Using these operators, we can define for each $1 \leq j \leq d$ three densely defined linear operators $a^+(j)$, $a^-(j)$, and $a^0(j)$ from $\mathcal{H}$ into itself by

$$a^+(j)|_{G_n} = D_n^+(j), \quad n \geq 0,$$

$$a^-(j)|_{G_n} = D_n^-(j), \quad n \geq 0,$$

$$a^0(j)|_{G_n} = D_n^0(j), \quad n \geq 0.$$

The operators $a^+(j)$, $a^-(j)$, and $a^0(j)$ are called *creation, annihilation*, and *neutral operators*, respectively. The collection

$$\{\mathcal{H},\, a^+(j),\, a^-(j),\, a^0(j) \,|\, 1 \leq j \leq d\}$$

is called the *interacting Fock space* of the probability measure μ.

For convenience, we will use the term "CAN operators" to call the creation, annihilation, and neutral operators. By using the multiplication and CAN operators, we can rewrite the fundamental recursion equality in Equation (1) as the equality in the next theorem.

Theorem 1.1. *For each $1 \leq j \leq d$, the following equality holds*

$$X_j = a^+(j) + a^-(j) + a^0(j). \tag{3}$$

We can use the equality in Equation (3) to extend the Accardi-Bożejko unitarity theorem[1] to the multi-dimensional case. In this paper we will present some results to answer the following question.

Question: What properties of μ are determined by the associated CAN operators?

2. Polynomially symmetric measures

Definition 2.1. A probability measure μ on $\mathbb{R}^d$ is said to be *polynomially symmetric* if

$$\int_{\mathbb{R}^d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}\, d\mu(x) = 0$$

for all nonnegative integers $i_1, i_2, \ldots i_d$ with $i_1 + i_2 + \cdots + i_d$ being an odd integer.

Note that if μ is a symmetric measure with finite moments of all orders, then it is polynomially symmetric. But the converse is not true. Consider the function

$$\theta(x) = e^{-(\ln x)^2} \sin(2\pi \ln x), \quad x > 0. \tag{4}$$

It is well-known that

$$\int_0^\infty x^n \theta(x)\, dx = 0, \quad \forall\, n = 0, 1, 2, \ldots \tag{5}$$

Define a function

$$f(x) = \begin{cases} c\,\theta^+(x), & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ c\,\theta^-(-x), & \text{if } x < 0, \end{cases}$$

where θ^+ and θ^- are the positive and negative parts of θ, respectively, and the constant c is chosen such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. By using Equation (5) one can easily check that the probability measure

$$d\mu(x) = f(x)\,dx$$

is polynomially symmetric. Obviously, μ is not symmetric.

The next theorem has been proved in our paper[2].

Theorem 2.1. *A probability measure μ on $\mathbb{R}^d$ with finite moments of all orders is polynomially symmetric if and only if $a^0(j) = 0$ for all $j = 1, 2, \ldots, d$.*

3. Polynomially factorizable measures

Definition 3.1. A probability measure μ on $\mathbb{R}^d$ is said to be *polynomially factorizable* if

$$\int_{\mathbb{R}^d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}\,d\mu(x) = \int_{\mathbb{R}^d} x_1^{i_1}\,d\mu(x) \int_{\mathbb{R}^d} x_2^{i_2}\,d\mu(x) \cdots \int_{\mathbb{R}^d} x_d^{i_d}\,d\mu(x)$$

for all nonnegative integers $i_1, i_2, \ldots i_d$.

Obviously, if μ is a product measure with finite moments of all orders, then it is polynomially factorizable. However, the converse is not true. Consider two modified functions of the function θ in Equation (4):

$$\theta_1(x) = e^{-(\ln x)^2}\big[1 + \sin(2\pi \ln x)\big], \quad x > 0,$$

$$\theta_2(x) = e^{-(\ln x)^2}\big[1 - \sin(2\pi \ln x)\big], \quad x > 0.$$

Define a function $g(x, y)$ on $\mathbb{R}^2$ by

$$g(x,y) = \begin{cases} k\big[\theta_1(x)\sin^2 y + \theta_2(x)\cos^2 y\big]e^{-y}, & \text{if } x > 0, y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where the constant k is chosen so that $\int_{\mathbb{R}^2} g(x,y)\,dxdy = 1$. Then the probability measure

$$d\mu(x,y) = g(x,y)\,dxdy$$

can be shown to be polynomially factorizable, but not a product measure.

The next theorem follows from Theorem 4.10 in our paper[2].

Theorem 3.1. *A probability measure* μ *on* $\mathbb{R}^d$ *with finite moments of all orders is polynomially factorizable if and only if for any* $i \neq j$, *the operators in* $\{a^+(i), a^-(i), a^0(i)\}$ *commute with the operators in* $\{a^+(j), a^-(j), a^0(j)\}$.

4. Probability measures by means of the CAN operators

Let μ be a probability measure on $\mathbb{R}^d$ with finite moments of all orders. We have the associated CAN operators $a^+(j)$, $a^-(j)$, and $a^0(j)$. Define a $d \times 1$ matrix A_μ^0 and a $d \times d$ matrix $A_\mu^{-,+}$ by

$$A_\mu^0 = \begin{pmatrix} a^0(1) \\ a^0(2) \\ \vdots \\ a^0(d) \end{pmatrix},$$

$$\begin{aligned} A_\mu^{-,+} &= \Big(\big[a^-(j), a^+(k) \big] \Big)_{j,k=1}^{d} \\ &= \begin{pmatrix} \big[a^-(1), a^+(1)\big] & \cdots & \big[a^-(1), a^+(d)\big] \\ \vdots & \ddots & \vdots \\ \big[a^-(d), a^+(1)\big] & \cdots & \big[a^-(d), a^+(d)\big] \end{pmatrix}, \end{aligned}$$

where $[a, b] = ab - ba$, the commutator of a and b.

Definition 4.1. Two probability measures μ and ν with finite moments of all orders are said to be *moment-equal* if

$$\int_{\mathbb{R}^d} m(x)\, d\mu(x) = \int_{\mathbb{R}^d} m(x)\, d\nu(x)$$

for all monomial functions $m(x)$.

The next two theorems have been proved in our paper[3].

Theorem 4.1. *Two probability measures* μ *and* ν *on* $\mathbb{R}^d$ *with finite moments of all orders are moment-equal if and only if* $A_\mu^0 = A_\nu^0$ *and* $A_\mu^{-,+} = A_\nu^{-,+}$.

Theorem 4.2. *A probability measure* μ *on* $\mathbb{R}^d$ *with finite moments of all orders is the standard Gaussian measure on* $\mathbb{R}^d$ *if and only if* $A_\mu^0 = 0_d$ *and* $A_\mu^{-,+} = I_d$*, namely,* $a^0(i) = 0$ *for all* $1 \le i \le d$ *and* $\big[a^-(j), a^+(k)\big] = \delta_{j,k} I$ *for all* $1 \le j, k \le d$.

The above discussion leads to the following problem for specifying a probability measure μ in terms of the matrices A_μ^0 and $A_\mu^{-,+}$.

Problem: Let V be the vector space of all polynomials on $\mathbb{R}^d$. Let a_i and $a_{j,k}$ be linear operators on V for $1 \le i, j, k \le d$. Find conditions on $\{a_i\}_{i=1}^d$ and $\{a_{j,k}\}_{j,k=1}^d$ so that there exists a probability measure μ on $\mathbb{R}^d$ satisfying $a_i = a^0(i)$ and $a_{j,k} = \big[a^-(j), a^+(k)\big]$ for all $1 \le i, j, k \le d$.

In the next section we will give some results on the solution to the above problem for the case when $d = 1$.

5. Probability measures on the real line

Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. Let V be the vector space of all polynomials in x and let V_n be its subspace consisting of all polynomials of degree $\le n$.

Let $F_n = V_n/\sim$. Here the equivalence relation $\sim$ is given by μ-almost everywhere, namely, $f \sim g$ if $f = g$ holds μ-a.e.

Assumption. In this section all linear operators $T : V \to V$ are assumed to satisfy the condition that $T(V_n) \subset V_n$ for all $n \ge 0$, namely, all subspaces $V_n, n \ge 0$, are invariant under T.

5.1. *Probability measures on* $\mathbb{R}$ *with finite support*

Observe that if a probability measure μ on $\mathbb{R}$ is supported by m distinct points, then

$$F_j = V_j, \quad j = 0, 1, 2, \ldots, m-1,$$

$$F_j = V_{m-1}, \quad j = m, m+1, \ldots.$$

The following theorem can be easily verified.

Theorem 5.1. *Suppose* μ *is a probability measure on* $\mathbb{R}$ *supported by* m *distinct points. Then the following equalities hold:*

(1) $\mathrm{Tr}\big(a_\mu^0\,\big|_{V_k}\big) = \mathrm{Tr}\big(a_\mu^0\,\big|_{V_{m-1}}\big)$ *for all* $k \ge m-1$.

(2) $\mathrm{Tr}\big(\big[a_\mu^-, a_\mu^+\big]\,\big|_{V_k}\big) = 0$ *for all* $k \ge m-1$.

Definition 5.1. Two linear operators S and T from V into itself are called *trace equivalent* on V, denoted by $S \stackrel{t}{\sim} T$ on V, if

$$\mathrm{Tr}\big(S\,|_{V_k} \big) = \mathrm{Tr}\big(T\,|_{V_k} \big), \quad \forall\, k \geq 0.$$

They are called *trace equivalent* on V_n, denoted by $S \stackrel{t}{\sim} T$ on V_n, if

$$\mathrm{Tr}\big(S\,|_{V_k} \big) = \mathrm{Tr}\big(T\,|_{V_k} \big), \quad \forall\, 0 \leq k \leq n.$$

The next theorem is from our paper[3]. It characterizes those measures supported by finitely many points in $\mathbb{R}$ in terms of the CAN operators.

Theorem 5.2. *Let $m \geq 1$ be a fixed integer. Let a^0 and $a^{-,+}$ be two linear operators from V_{m-1} into itself. Then there exists a probability measure μ on $\mathbb{R}$ supported by m distinct points such that $a^0 \stackrel{t}{\sim} a_\mu^0$ and $a^{-,+} \stackrel{t}{\sim} \big[a_\mu^-, a_\mu^+\big]$ on V_{m-1} if and only if the following conditions hold:*

(1) *The spaces V_k, $0 \leq k \leq m-2$, are invariant under a^0 and $a^{-,+}$.*
(2) $\mathrm{Tr}\big(a^{-,+}\,\big|_{V_k}\big) > 0$ *for all* $0 \leq k \leq m-2$.
(3) $\mathrm{Tr}\big(a^{-,+}\,\big|_{V_{m-1}}\big) = 0$.

5.2. *Probability measures on $\mathbb{R}$ with infinite support*

Let μ be a probability measure on $\mathbb{R}$ with infinite support, namely, the support of μ contains infinitely many points. In this case, we have

$$F_n = V_n, \quad \forall\, n \geq 0.$$

The next theorem has been proved in our paper[3].

Theorem 5.3. *Let a^0 and $a^{-,+}$ be two linear operators from V into itself. Then there exists a probability measure μ on $\mathbb{R}$ with infinite support such that $a^0 \stackrel{t}{\sim} a_\mu^0$ and $a^{-,+} \stackrel{t}{\sim} \big[a_\mu^-, a_\mu^+\big]$ on V if and only if the following conditions hold:*

(1) *The spaces V_n, $n \geq 0$, are invariant under a^0 and $a^{-,+}$.*
(2) $\mathrm{Tr}\big(a^{-,+}\,\big|_{V_n}\big) > 0$ *for all* $n \geq 0$.

Let Ξ denote the set of all trace equivalent classes of ordered pairs $(a^0, a^{-,+})$ of linear operators from V into itself satisfying either one of the following conditions (a) and (b):

(a) $\mathrm{Tr}\big(a^{-,+}\,\big|_{V_n}\big) > 0$, $\forall\, n \geq 0$.

(b) There exists m such that
  (1) $\mathrm{Tr}\big(a^0\,\big|_{V_k}\big) = \mathrm{Tr}\big(a^0\,\big|_{V_{m-1}}\big)$, $\forall\, k \geq m-1$,
  (2) $\mathrm{Tr}\big(a^{-,+}\,\big|_{V_k}\big) > 0$, $\forall\, 0 \leq k \leq m-2$,
  (3) $\mathrm{Tr}\big(a^{-,+}\,\big|_{V_k}\big) = 0$, $\forall\, k \geq m-1$.

Theorem 5.4. *There is a one-to-one correspondence between the set Ξ and the set of all probability measures on $\mathbb{R}$ with finite moments of all orders.*

5.3. *Probability measures on $\mathbb{R}$ with compact support*

The Paley-Wiener type problem is to characterize probability measures with compact support. We have the next theorem from our paper[3].

Theorem 5.5. *A probability measure μ on $\mathbb{R}$ with finite moments of all orders has compact support if and only if the following two sequences of real numbers are bounded:*

(1) $\mathrm{Tr}\big(a^0_\mu\,\big|_{F_n}\big) - \mathrm{Tr}\big(a^0_\mu\,\big|_{F_{n-1}}\big)$, $n \geq 1$.

(2) $\mathrm{Tr}\big(a^{-,+}_\mu\,\big|_{F_n}\big)$, $n \geq 1$.

6. Classical measures on the real line

Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. We have the associated orthogonal polynomials $\{P_n\}$ and the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ as given in Equation (2). The corresponding CAN operators are given by

$$a^+_\mu P_n = P_{n+1}, \quad a^-_\mu P_n = \omega_n P_{n-1}, \quad a^0_\mu P_n = \alpha_n P_n, \quad n \geq 0,$$

where $P_{-1} = 0$ by convention. Therefore, the commutator $a^{-,+}_\mu = \big[a^-_\mu, a^+_\mu\big]$ is given by

$$a^{-,+}_\mu P_n = \begin{cases} \omega_1 P_0, & \text{if } n = 0, \\ (\omega_{n+1} - \omega_n) P_n, & \text{if } n \geq 1. \end{cases}$$

Consider the following classical probability measures on the real line:

(1) Gaussian: $d\mu(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{x^2}{2\sigma^2}}\, dx$, $x \in \mathbb{R}$ $(\sigma > 0)$.

(2) Poisson: $\mu(\{k\}) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$ $(\lambda > 0)$.

(3) gamma: $\alpha > 0$.

$$d\mu(x) = \frac{1}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-x}\, dx,\ x > 0.$$

(4) Pascal: $r > 0,\ 0 < p < 1$.

$$\mu(\{k\}) = p^r \binom{-r}{k} (-1)^k (1-p)^k,\ k = 0, 1, 2, \ldots$$

(5) uniform: $d\mu(x) = \dfrac{1}{2}\, dx, \ -1 \le x \le 1.$

(6) arcsine: $d\mu(x) = \dfrac{1}{\pi} \dfrac{1}{\sqrt{1-x^2}}\, dx, \ -1 < x < 1.$

(7) semi-circle: $d\mu(x) = \dfrac{2}{\pi} \sqrt{1-x^2}\, dx, \ -1 \le x \le 1.$

(8) beta-type: $\beta > -1/2,\ \beta \ne 0$.

$$d\mu(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})} (1-x^2)^{\beta-1/2}\, dx, \ -1 < x < 1.$$

For the above probability measures, the Jacobi-Szegö parameters are given in the next table. By convention, $\omega_0 = 1$.

measure μ	$\alpha_n,\ n \ge 0$	$\omega_n,\ n \ge 1$
Gaussian	0	$\sigma^2 n$
Poisson	$(\lambda + n)$	λn
gamma	$\alpha + 2n$	$n(\alpha + n - 1)$
Pascal	$\dfrac{(2-p)n + r(1-p)}{p}$	$\dfrac{n(n+r-1)(1-p)}{p^2}$
uniform	0	$\dfrac{n^2}{(2n+1)(2n-1)}$
arcsine	0	$\begin{cases} \frac{1}{2}, & \text{if } n = 1 \\ \frac{1}{4}, & \text{if } n \ge 2 \end{cases}$
semi-circle	0	$\dfrac{1}{4}$
beta-type	0	$\dfrac{n(n-1+2\beta)}{4(n+\beta)(n-1+\beta)}$

Furthermore, the operator a^0_μ and the commutator $a^{-,+}_\mu = \left[a^-_\mu, a^+_\mu\right]$ are given in the following table.

measure μ	$a^0_\mu P_n,\ n \geq 0$	$a^{-,+}_\mu P_n,\ n \geq 0$
Gaussian	0	$\sigma^2 P_n$
Poisson	$(\lambda + n)P_n$	λP_n
gamma	$(\alpha + 2n)P_n$	$(\alpha + 2n)P_n$
Pascal	$\dfrac{(2-p)n + r(1-p)}{p} P_n$	$\dfrac{(2n+r)(1-p)}{p^2} P_n$
uniform	0	$-\dfrac{1}{(2n+3)(2n+1)(2n-1)} P_n$
arcsine	0	$\begin{cases} \frac{1}{2} P_0, & \text{if } n = 0 \\ -\frac{1}{4} P_1, & \text{if } n = 1 \\ 0, & \text{if } n \geq 2 \end{cases}$
semi-circle	0	$\begin{cases} \frac{1}{4} P_0, & \text{if } n = 0 \\ 0, & \text{if } n \geq 1 \end{cases}$
beta-type	0	$\dfrac{\beta^2 - \beta}{2(n+1+\beta)(n+\beta)(n-1+\beta)} P_n$

It is interesting to compare the above table with Theorems 2.1, 5.1, 5.2, 5.3, and 5.5.

Acknowledgments

This research was initiated in May-June 2002 during the visits of H.-H. Kuo and A. Stan to the Centro Vito Volterra (CVV), Università di Roma “Tor Vergata”. They are grateful to the CVV for financial support and would like to thank Professor L. Accardi and his staff members for the warm hospitality during the visits.

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