



A Robertson-type uncertainty principle and quantum Fisher information

Paolo Gibilisco ^{a,*}, Daniele Imparato ^b, Tommaso Isola ^c

^a *Dipartimento SEFEMEQ, Facoltà di Economia, Università di Roma “Tor Vergata”, Via Columbia 2, 00133 Rome, Italy*

^b *Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Turin, Italy*

^c *Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, 00133 Rome, Italy*

Received 22 July 2007; accepted 11 October 2007

Submitted by R. Bhatia

Abstract

Let A_1, \dots, A_N be complex self-adjoint matrices and let ρ be a density matrix. The Robertson uncertainty principle

$$\det\{\text{Cov}_\rho(A_h, A_j)\} \geq \det \left\{ -\frac{i}{2} \text{Tr}[\rho[A_h, A_j]] \right\}$$

gives a bound for the quantum generalized variance in terms of the commutators $[A_h, A_j]$. The right side matrix is antisymmetric and therefore the bound is trivial (equal to zero) in the odd case $N = 2m + 1$.

Let f be an arbitrary normalized symmetric operator monotone function and let $\langle \cdot, \cdot \rangle_{\rho, f}$ be the associated quantum Fisher information. We have conjectured the inequality

$$\det\{\text{Cov}_\rho(A_h, A_j)\} \geq \det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\}$$

* Corresponding author.

E-mail addresses: gibilisco@volterra.uniroma2.it (P. Gibilisco), daniele.imparato@polito.it (D. Imparato), isola@mat.uniroma2.it (T. Isola)

URLs: <http://www.economia.uniroma2.it/sefemeq/professori/gibilisco> (P. Gibilisco), <http://www.mat.uniroma2.it/~isola> (T. Isola).

that gives a non-trivial bound for any $N \in \mathbb{N}$ using the commutators $[\rho, A_h]$. In the present paper the conjecture is proved by mean of the Kubo–Ando mean inequality.

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AMS classification: Primary 62B10, 94A17; Secondary 46L30, 46L60

Keywords: Generalized variance; Uncertainty principle; Operator monotone functions; Matrix means; Quantum Fisher information

1. Introduction

Let $M_{n,sa} := M_{n,sa}(\mathbb{C})$ be the space of all $n \times n$ self-adjoint matrices (observables) and let \mathcal{D}_n^1 be the set of strictly positive density matrices (faithful states). Given $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define the (symmetrized) covariance as $\text{Cov}_\rho(A, B) := 1/2[\text{Tr}(\rho AB) + \text{Tr}(\rho BA)] - \text{Tr}(\rho A) \cdot \text{Tr}(\rho B)$ and the variance as $\text{Var}_\rho(A) := \text{Cov}_\rho(A, A)$. In this context the Heisenberg uncertainty principle is the inequality

$$\text{Var}_\rho(A)\text{Var}_\rho(B) \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2. \tag{1.1}$$

Schrödinger and Robertson improved this result to

$$\text{Var}_\rho(A)\text{Var}_\rho(B) - \text{Cov}_\rho(A, B)^2 \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2. \tag{1.2}$$

Robertson himself realized that for N observables A_1, \dots, A_N one can prove the general result

$$\det\{\text{Cov}_\rho(A_h, A_j)\} \geq \det\left\{-\frac{i}{2}\text{Tr}(\rho[A_h, A_j])\right\} \tag{1.3}$$

(see [12,14,25–27]). The left hand side is known as the (quantum) generalized variance of the random vector (A_1, \dots, A_N) . Let us refer to the inequality (1.3) as the “standard” uncertainty principle to distinguish it from other inequalities like the “entropic” uncertainty principle and similar inequalities. It is difficult to overestimate the importance of the uncertainty principle in quantum physics. Examples of recent references where inequality (1.3) plays a role are given by [2,3,13,28–30].

It is worth to write the inequality (1.3) in a different way. Let $(V, g(\cdot, \cdot))$ be a real inner-product vector space and suppose that $v_1, \dots, v_N \in V$. The real $N \times N$ matrix $G := \{g(v_h, v_j)\}$ is positive semidefinite and one can define $\text{Vol}^g(v_1, \dots, v_N) := \sqrt{\det\{g(v_h, v_j)\}}$. If the inner product depends on a further parameter in such a way that $g(\cdot, \cdot) = g_\rho(\cdot, \cdot)$, we write $\text{Vol}^g(v_1, \dots, v_N) = \text{Vol}_\rho^g(v_1, \dots, v_N)$. With these definitions the inequality (1.3) takes the form

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \begin{cases} 0, & N = 2m + 1, \\ \left\{\det\{-\frac{i}{2}\text{Tr}(\rho[A_h, A_j])\}\right\}^{\frac{1}{2}}, & N = 2m, \end{cases} \tag{1.4}$$

since the matrix $\{-\frac{i}{2}\text{Tr}(\rho[A_h, A_j])\}$ is antisymmetric. The above formulation clarifies that for an odd number of observables the standard uncertainty principle states nothing more than the classical property of non-negativeness of the generalized variance.

In order to search for an uncertainty principle which is not trivial for an odd number of observables, one is naturally lead to consider the commutators $[\rho, A_h]$ (to see this consider the case $N = 1$).

Let \mathcal{F}_{op} be the family of symmetric normalized operator monotone functions. To each element $f \in \mathcal{F}_{\text{op}}$ one may associate a ρ -depending scalar product $\langle \cdot, \cdot \rangle_{\rho, f}$ on the self-adjoint (traceless)

matrices, which is a quantum version of the Fisher information (see [22]). Let us denote by Vol_ρ^f the associated volume. In the paper [5] we conjectured that, for any $N \in \mathbb{N}$ (this is one of the main differences from (1.4)) and for arbitrary self-adjoint matrices A_1, \dots, A_N , one has

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]). \tag{1.5}$$

We conjectured inequality (1.5) inspired by the cases $N = 1, 2$ which have been proved by the joint efforts of a number of authors in several papers: Luo [17,18], Luo and Zhang [19–21]; Kosaki [15]; Yanagi et al. [31]; Hansen [11]; Gibilisco et al. [4,5,10]. We refer to [5] for more detailed credit.

The main result of the present paper is the proof of inequality (1.5).

It is well known that the standard uncertainty principle is a consequence of the Cauchy–Schwartz inequality. It is worth to note that the same role in (1.5) is played by the Kubo–Ando inequality

$$2(A^{-1} + B^{-1})^{-1} \leq m_f(A, B) \leq \frac{1}{2}(A + B)$$

that says that any operator mean is larger than the harmonic mean and smaller than the arithmetic mean.

The scheme of the paper is as follows. In Section 2 we describe the preliminary notions of operator monotone functions, matrix means and quantum Fisher information. In Section 3 we discuss a correspondence between regular and non-regular operator monotone functions that is needed in the sequel. In Section 4 we state our main result, namely the inequality (1.5); we also state other two results concerning how the right side depends on $f \in \mathcal{F}_{\text{op}}$ and the conditions to have equality in (1.5). In Sections 5, 6 7 we prove some auxiliary results. In Section 8 we prove the main results. In Section 9 we compare the standard uncertainty principle with the inequality (1.5).

2. Operator monotone functions, matrix means and quantum Fisher information

Let $M_n := M_n(\mathbb{C})$ (resp. $M_{n,sa} := M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). We shall denote general matrices by X, Y, \dots while letters A, B, \dots will be used for self-adjoint matrices, endowed with the Hilbert–Schmidt scalar product $\langle A, B \rangle = \text{Tr}(A^*B)$. The adjoint of a matrix X is denoted by X^\dagger while the adjoint of a superoperator $T : (M_n, \langle \cdot, \cdot \rangle) \rightarrow (M_n, \langle \cdot, \cdot \rangle)$ is denoted by T^* . Let \mathcal{D}_n be the set of strictly positive elements of M_n and $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of strictly positive density matrices, namely $\mathcal{D}_n^1 = \{\rho \in M_n | \text{Tr}\rho = 1, \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, namely $\rho > 0$.

A function $f : (0, +\infty) \rightarrow \mathbb{R}$ is said *operator monotone (increasing)* if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said *symmetric* if $f(x) = xf(x^{-1})$ and *normalized* if $f(1) = 1$.

Definition 2.1. \mathcal{F}_{op} is the class of functions $f : (0, +\infty) \rightarrow (0, +\infty)$ such that

- (i) $f(1) = 1$,
- (ii) $tf(t^{-1}) = f(t)$,
- (iii) f is operator monotone.

Example 2.1. Examples of elements of \mathcal{F}_{op} are given by the following list:

$$f_{\text{RLD}}(x) := \frac{2x}{x+1}, \quad f_{\text{WY}}(x) := \left(\frac{1+\sqrt{x}}{2}\right)^2,$$

$$f_{\text{SLD}}(x) := \frac{1+x}{2}, \quad f_{\text{WYD}(\beta)}(x) := \beta(1-\beta) \frac{(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}, \quad \beta \in \left(0, \frac{1}{2}\right).$$

We now report Kubo–Ando theory of matrix means (see [16]) as exposed in [24].

Definition 2.2. A mean for pairs of positive matrices is a function $m : \mathcal{D}_n \times \mathcal{D}_n \rightarrow \mathcal{D}_n$ such that

- (i) $m(A, A) = A$,
- (ii) $m(A, B) = m(B, A)$,
- (iii) $A < B \implies A < m(A, B) < B$,
- (iv) $A < A', B < B' \implies m(A, B) < m(A', B')$,
- (v) m is continuous,
- (vi) $Cm(A, B)C^* \leq m(CAC^*, CBC^*)$, for every $C \in M_n$.

Property (vi) is known as the transformer inequality. We denote by \mathcal{M}_{op} the set of matrix means. The fundamental result, due to Kubo and Ando, is the following.

Theorem 2.1. *There exists a bijection between \mathcal{M}_{op} and \mathcal{F}_{op} given by the formula*

$$m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Example 2.2. The arithmetic, geometric and harmonic (matrix) means are given respectively by

$$A \nabla B := \frac{1}{2}(A + B),$$

$$A \# B := A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}},$$

$$A ! B := 2(A^{-1} + B^{-1})^{-1}.$$

They correspond respectively to the operator monotone functions $\frac{x+1}{2}, \sqrt{x}, \frac{2x}{x+1}$.

Kubo and Ando [16] proved that, among matrix means, arithmetic is the largest while harmonic is the smallest.

Proposition 2.2. *For any $f \in \mathcal{F}_{\text{op}}$ one has*

$$2(A^{-1} + B^{-1})^{-1} \leq m_f(A, B) \leq \frac{1}{2}(A + B),$$

which is equivalent to

$$\frac{2x}{1+x} \leq f(x) \leq \frac{1+x}{2} \quad \forall x > 0.$$

In what follows, if \mathcal{N} is a differentiable manifold we denote by $T_\rho \mathcal{N}$ the tangent space to \mathcal{N} at the point $\rho \in \mathcal{N}$. Recall that there exists a natural identification of $T_\rho \mathcal{D}_n^1$ with the space of self-adjoint traceless matrices; namely, for any $\rho \in \mathcal{D}_n^1$

$$T_\rho \mathcal{D}_n^1 = \{A \in M_n \mid A = A^*, \text{Tr}(A) = 0\}.$$

A Markov morphism is a completely positive and trace preserving operator $T : M_n \rightarrow M_m$. A *monotone metric* is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g_{T(\rho)}^m(TX, TX) \leq g_\rho^n(X, X)$$

holds for every Markov morphism $T : M_n \rightarrow M_m$, for every $\rho \in \mathcal{D}_n^1$ and for every $X \in T_\rho \mathcal{D}_n^1$. Usually monotone metrics are normalized in such a way that $[A, \rho] = 0$ implies $g_\rho(A, A) = \text{Tr}(\rho^{-1}A^2)$. A monotone metric is also said a *quantum Fisher information* (QFI) because of Chentsov uniqueness theorem for commutative monotone metrics (see [1]).

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$, and observe that they are commuting positive super-operators on $M_{n,sa}$. For any $f \in \mathcal{F}_{op}$ one can define the positive superoperator $m_f(L_\rho, R_\rho)$. Now we can state the fundamental theorem about monotone metrics.

Theorem 2.3 (see [22]). *There exists a bijective correspondence between monotone metrics (quantum Fisher informations) on \mathcal{D}_n^1 and normalized symmetric operator monotone functions $f \in \mathcal{F}_{op}$. This correspondence is given by the formula*

$$\langle A, B \rangle_{\rho, f} := \text{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

The metrics associated with the functions f_β are very important in information geometry and are related to Wigner–Yanase–Dyson information (see for example [6–9] and references therein).

3. The function \tilde{f} and its properties

For $f \in \mathcal{F}_{op}$ define $f(0) := \lim_{x \rightarrow 0} f(x)$. The condition $f(0) \neq 0$ is relevant because it is a necessary and sufficient condition for the existence of the so-called radial extension of a monotone metric to pure states (see [23]). Following [11] we say that a function $f \in \mathcal{F}_{op}$ is regular iff $f(0) \neq 0$. The corresponding operator mean, associated QFI, etc. are said regular too.

Definition 3.1. We introduce the sets

$$\mathcal{F}_{op}^r := \{f \in \mathcal{F}_{op} \mid f(0) \neq 0\}, \quad \mathcal{F}_{op}^n := \{f \in \mathcal{F}_{op} \mid f(0) = 0\}.$$

Trivially one has $\mathcal{F}_{op} = \mathcal{F}_{op}^r \dot{\cup} \mathcal{F}_{op}^n$.

Proposition 3.1 [4]. *For $f \in \mathcal{F}_{op}^r$ and $x > 0$ set*

$$\tilde{f}(x) := \frac{1}{2} \left[(x + 1) - (x - 1)^2 \frac{f(0)}{f(x)} \right].$$

Then $\tilde{f} \in \mathcal{F}_{op}^n$.

By the very definition one has the following result (see Proposition 5.3 in [4]).

Proposition 3.2. Let $f \in \mathcal{F}_{\text{op}}^r$. The following three conditions are equivalent:

- (1) $\tilde{f} \leq \tilde{g}$;
- (2) $m_{\tilde{f}} \leq m_{\tilde{g}}$;
- (3) $\frac{f(0)}{f(t)} \geq \frac{g(0)}{g(t)} \forall t > 0$.

Let us give some more definitions.

Definition 3.2. Suppose that $\rho \in \mathcal{D}_n^1$ is fixed. Define $X_0 := X - \text{Tr}(\rho X)I$.

Definition 3.3. For $A_1, A_2 \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define covariance and variance as

$$\begin{aligned} \text{Cov}_\rho(A_1, A_2) &:= \frac{1}{2}[\text{Tr}(\rho A_1 A_2) + \text{Tr}(\rho A_2 A_1)] - \text{Tr}(\rho A_1) \cdot \text{Tr}(\rho A_2) \\ &= \frac{1}{2}[\text{Tr}(\rho(A_1)_0(A_2)_0) + \text{Tr}(\rho(A_2)_0(A_1)_0)] = \text{Re}\{\text{Tr}(\rho(A_1)_0(A_2)_0)\}, \\ \text{Var}_\rho(A) &:= \text{Cov}_\rho(A, A) = \text{Tr}(\rho A^2) - \text{Tr}(\rho A)^2 = \text{Tr}(\rho A_0^2). \end{aligned} \tag{3.1}$$

Suppose, now, that $A_1, A_2 \in M_{n,sa}$, $\rho \in \mathcal{D}_n^1$ and $f \in \mathcal{F}_{\text{op}}^r$. The fundamental result for our present purpose is given by Proposition 6.3 in [4], which is stated as follows.

Theorem 3.3

$$\frac{f(0)}{2} \langle i[\rho, A_1], i[\rho, A_2] \rangle_{\rho, f} = \text{Cov}_\rho(A_1, A_2) - \text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_1)_0)(A_2)_0).$$

As a consequence of both the spectral theorem and Theorem 3.3 one has the following relations.

Proposition 3.4 [4]. Let $\{\varphi_i\}$ be a complete orthonormal base composed of eigenvectors of ρ , and $\{\lambda_i\}$ the corresponding eigenvalues. To self-adjoint matrices A_1, A_2 we associate matrices $\mathcal{A}^j = \mathcal{A}^j(\rho)$, $j = 1, 2$, whose entries are given by $\mathcal{A}_{kl}^j := \langle (A_j)_0 \varphi_k | \varphi_l \rangle$.

We have the following identities.

$$\begin{aligned} \text{Cov}_\rho(A_1, A_2) &= \text{Re}\{\text{Tr}(\rho(A_1)_0(A_2)_0)\} = \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^1 \mathcal{A}_{lk}^2\} \\ \frac{f(0)}{2} \langle i[\rho, A_1], i[\rho, A_2] \rangle_{\rho, f} &= \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^1 \mathcal{A}_{lk}^2\} \\ &\quad - \sum_{k,l} m_{\tilde{f}}(\lambda_i, \lambda_j) \text{Re}\{\mathcal{A}_{kl}^1 \mathcal{A}_{lk}^2\}. \end{aligned}$$

We also need the following result (Corollary 11.2 in [4]).

Proposition 3.5. On pure states

$$\text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_1)_0)(A_2)_0) = 0.$$

4. Volume theorems for quantum Fisher informations

Given a matrix $A = \{A_{kl}\}$, we denote its determinant by $\det(A) = \det\{A_{kl}\}$. Let $(V, g(\cdot, \cdot))$ be a real inner-product vector space. By $\langle u, v \rangle$ we denote the standard scalar product for vectors $u, v \in \mathbb{R}^N$. One has

Proposition 4.1. *Let $v_1, \dots, v_N \in V$. The real $N \times N$ matrix $G := \{g(v_h, v_j)\}$ is positive semi-definite and therefore $\det\{g(v_h, v_j)\} \geq 0$.*

Motivated by the case $(V, g(\cdot, \cdot)) = (\mathbb{R}^N, \langle \cdot, \cdot \rangle)$ one can give the following definition.

Definition 4.1

$$\text{Vol}^g(v_1, \dots, v_N) := \sqrt{\det\{g(v_h, v_j)\}}.$$

Remark 4.1

(i) Obviously,

$$\text{Vol}^g(v_1, \dots, v_N) \geq 0,$$

where the equality holds if and only if $v_1, \dots, v_N \in V$ are linearly dependent.

(ii) If the inner product depends on a further parameter so that $g(\cdot, \cdot) = g_\rho(\cdot, \cdot)$, we write

$$\text{Vol}_\rho^g(v_1, \dots, v_N) = \text{Vol}^g(v_1, \dots, v_N).$$

(iii) In the case of a probability space $(V, g_\rho(\cdot, \cdot)) = (\mathcal{L}_{\mathbb{R}}^2(\Omega, \mathcal{G}, \rho), \text{Cov}_\rho(\cdot, \cdot))$ the number $\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N)^2$ is known as the *generalized variance* of the random vector (A_1, \dots, A_N) .

In what follows we move to the noncommutative case. Here A_1, \dots, A_N are self-adjoint matrices, ρ is a (faithful) density matrix and $g(\cdot, \cdot) = \text{Cov}_\rho(\cdot, \cdot)$ has been defined in (3.1). By Vol_ρ^f we denote the volume associated to the quantum Fisher information $\langle \cdot, \cdot \rangle_{\rho, f}$ given by the (regular) normalized symmetric operator monotone function f .

Definition 4.2. The function

$$I_\rho^f(A) := \frac{f(0)}{2} \text{Vol}_\rho^f(i[\rho, A]) = \frac{f(0)}{2} (i[\rho, A], i[\rho, A])_{\rho, f}$$

is known as the *metric adjusted skew information* or *f-information* (see [4,11]).

Let $N \in \mathbb{N}$, $f \in \mathcal{F}_{op}^r$, $\rho \in \mathcal{D}_n^1$ and $A_1, \dots, A_N \in M_{n,sa}$ be arbitrary. We shall prove in Section 8 the following results.

Theorem 4.2

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]). \tag{4.1}$$

Theorem 4.3. *The above inequality is an equality if and only if A_{10}, \dots, A_{N0} are linearly dependent.*

Theorem 4.4. Fix $N \in \mathbb{N}$, $\rho \in \mathcal{D}_n^1$ and $A_1, \dots, A_N \in M_{n,sa}$. Define for $f \in \mathcal{F}_{op}^r$

$$V(f) := \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]).$$

Then, for any $f, g \in \mathcal{F}_{op}^r$

$$\tilde{f} \leq \tilde{g} \implies V(f) \geq V(g).$$

Remark 4.2. The inequality

$$\det\{\text{Cov}_\rho(A_h, A_j)\} \geq \det\left\{\text{Cov}_\rho(A_h, A_j) - \text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_h)_0)(A_j)_0)\right\}$$

also makes sense for not-faithful states, which is true by continuity as a consequence of Theorem 8.1.

Because of Proposition 3.5 one has (by an obvious extension of the definition) the following result.

Proposition 4.5. If ρ is a pure state, then for any $N \in \mathbb{N}$, $f \in \mathcal{F}_{op}^r$, $A_1, \dots, A_N \in M_{n,sa}$ one has

$$\text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) = \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]).$$

5. Some combinatorics

The following simple combinatorial results are needed in order to prove the main results. For the sequel, set $\underline{n} := \{1, \dots, n\}$ and denote by S^N the symmetric group of order N . Furthermore, given $z \in \mathbb{C}$, we shall introduce the operator

$$C^k(z) := \begin{cases} \text{Re}(z) & \text{if } k = 0, \\ \text{Im}(z) & \text{if } k = 1. \end{cases}$$

Given a finite set $X \subset \mathbb{N}$ and $N \in \mathbb{N}^+$, for any tensor $\{Q_j^k\}$ one has

$$\prod_{j=1}^N \sum_{k \in X} Q_j^k = \sum_{u \in X^{\underline{N}}} \prod_{j=1}^N Q_j^{u(j)}. \tag{5.1}$$

Therefore, taking $X = \{0, 1\}$ and $Q_j^k = C^k(z_j)C^k(w_j)$, one gets the following result.

Lemma 5.1. If $z_j, w_j \in \mathbb{C}$ then

$$\prod_{j=1}^N \left(\sum_{k \in \{0,1\}} C^k(z_j)C^k(w_j) \right) = \sum_{u \in \{0,1\}^{\underline{N}}} \left(\prod_{j=1}^N C^{u(j)}(z_j)C^{u(j)}(w_j) \right).$$

Indeed, with similar arguments (5.1) can be generalized to a tensor $\{Q_{kl}^j\}$, so that one obtains the following lemma.

Lemma 5.2. For a finite set $X \subset \mathbb{N}$ and $N \in \mathbb{N}^+$ one has

$$\prod_{j=1}^N \left(\sum_{k,l \in X} Q_{kl}^j \right) = \sum_{\alpha, \beta \in X^{\mathbb{N}}} \left(\prod_{j=1}^N Q_{\alpha(j)\beta(j)}^j \right).$$

For finite X , consider a bijection $g : X \rightarrow X$. For any function $r : X \rightarrow \mathbb{R}$ one has

$$\sum_{x \in X} r(x) = \sum_{x \in X} r(g(x)). \tag{5.2}$$

From this result, one obtains the following proposition.

Proposition 5.3. Let X be a finite set and let G be a group of bijections $g : X \rightarrow X$. For any function $r : X \rightarrow \mathbb{R}$ one has

$$\sum_{x \in X} r(x) = \frac{1}{\#(G)} \sum_{x \in X} \sum_{g \in G} r(g(x)).$$

Now consider $X := \{0, 1\}^{\mathbb{N}}$ which can be identified with the power set of \mathbb{N} . If $u \in \{0, 1\}^{\mathbb{N}}$, each permutation $\sigma \in S_N$ can be seen as a bijection $\sigma : X \rightarrow X$ defining $\sigma(u) := u \circ \sigma$. Therefore, from (5.2) we get the following lemma.

Lemma 5.4. For any function $r : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ and for any $\sigma \in S_N$ one has

$$\sum_{u \in \{0,1\}^{\mathbb{N}}} r(u) = \sum_{u \in \{0,1\}^{\mathbb{N}}} r(\sigma(u)).$$

6. The function H

Let $\mathbb{R}_+ := (0, +\infty)$ and $\mathbf{x} = (x_1, \dots, x_N), \mathbf{y} = (y_1, \dots, y_N) \in \mathbb{R}_+^N$. In the sequel, we need to study the following function.

Definition 6.1. For any $f \in \mathcal{F}_{\text{op, set}}^r$

$$H^f(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^N \frac{x_j + y_j}{2} - \prod_{j=1}^N \left(\frac{x_j + y_j}{2} - m_{\tilde{f}}(x_j, y_j) \right).$$

Proposition 6.1. For any $\mathcal{F} \in \mathcal{F}_{\text{op}}^r, \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$,

$$H^f(\mathbf{x}, \mathbf{y}) > 0.$$

Proof. Since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$,

$$0 < m_{\tilde{f}}(x_j, y_j) \leq \frac{x_j + y_j}{2}, \quad j = 1, \dots, N,$$

we have

$$\prod_{j=1}^N \left(\frac{x_j + y_j}{2} - m_{\tilde{f}}(x_j, y_j) \right) < \prod_{j=1}^N \frac{x_j + y_j}{2},$$

so that we obtain the result. \square

Proposition 6.2

$$\begin{aligned} \tilde{f} &\leq \tilde{g} \\ &\Downarrow \\ H^f(\mathbf{x}, \mathbf{y}) &\leq H^g(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N. \end{aligned}$$

Proof. Since for any $x, y > 0$

$$\frac{x + y}{2} - m_{\tilde{f}}(x, y) = \frac{(x - y)^2}{2y} \cdot \frac{f(0)}{f\left(\frac{x}{y}\right)}, \tag{6.1}$$

we have

$$H^f(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^N \frac{x_j + y_j}{2} - \prod_{j=1}^N \left(\frac{(x_j - y_j)^2}{2y_j} \cdot \frac{f(0)}{f\left(\frac{x_j}{y_j}\right)} \right).$$

Because of Proposition 3.2 we have

$$\tilde{f} \leq \tilde{g} \Rightarrow \frac{f(0)}{f(t)} \geq \frac{g(0)}{g(t)} > 0 \quad \forall t > 0;$$

hence, we obtain

$$H^f(\mathbf{x}, \mathbf{y}) \leq H^g(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$$

by elementary computations. \square

Corollary 6.3. For any $f \in \mathcal{F}_{\text{op}}$,

$$0 < H^{SLD}(\mathbf{x}, \mathbf{y}) \leq H^f(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2^N} \prod_{j=1}^N (x_j + y_j) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N.$$

Define

$$\mathcal{C} := \underline{n}^N = \{(x_1, \dots, x_N) : x_i \in \{1, \dots, n\}, i = 1, \dots, N\}.$$

Definition 6.2. Fix $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$. Given $\alpha, \beta \in \mathcal{C} = \underline{n}^N$, let $H_{\alpha, \beta}^f := H^f(\lambda_\alpha, \lambda_\beta)$, where $\lambda_\alpha := (\lambda_{\alpha_1}, \dots, \lambda_{\alpha_N}), \lambda_\beta := (\lambda_{\beta_1}, \dots, \lambda_{\beta_N})$.

Proposition 6.4. For all $\sigma \in S^N$ one has

$$H_{\alpha(\sigma), \beta(\sigma)}^f = H_{\alpha, \beta}^f.$$

Proof. It is left to the reader. \square

7. The function K

In order to prove the main result of this paper, we introduce some notations. Let $\{\varphi_i\}$ be a complete orthonormal basis composed of eigenvectors of ρ , and $\{\lambda_i\}$ the corresponding eigenvalues. As in Proposition 3.4, set

$$\mathcal{A}_{kl}^j := \langle (A_j)_0 \varphi_k | \varphi_l \rangle, \quad j = 1, \dots, N; \quad k, l = 1, \dots, n.$$

Note that $\mathcal{A}_{kl}^j = \overline{\mathcal{A}_{lk}^j}$, since the A_j 's are self-adjoint; namely

$$\operatorname{Re}\{\mathcal{A}_{kl}^j\} = \operatorname{Re}\{\mathcal{A}_{lk}^j\}, \quad \operatorname{Im}\{\mathcal{A}_{kl}^j\} = -\operatorname{Im}\{\mathcal{A}_{lk}^j\}.$$

Since

$$\operatorname{Re}\{zw\} = \operatorname{Re}\{z\}\operatorname{Re}\{w\} - \operatorname{Im}\{z\}\operatorname{Im}\{w\},$$

we obtain the following lemma.

Lemma 7.1

$$\operatorname{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^m\} = \operatorname{Re}\{\mathcal{A}_{kl}^j\}\operatorname{Re}\{\mathcal{A}_{lk}^m\} + \operatorname{Im}\{\mathcal{A}_{kl}^j\}\operatorname{Im}\{\mathcal{A}_{lk}^m\}.$$

If $\alpha, \beta \in \mathcal{C} = \underline{n}^N$ and $\sigma \in S^N$ we define a $N \times N$ matrix $\mathcal{B}^{\alpha\sigma, \beta\sigma}$ setting

$$(\mathcal{B}^{\alpha\sigma, \beta\sigma})_{hj} := \operatorname{Re}\{\mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^h \mathcal{A}_{\beta_{\sigma(h)}, \alpha_{\sigma(h)}}^j\}, \quad h, j = 1, \dots, N; \quad \alpha_{\sigma(h)}, \beta_{\sigma(h)} = 1, \dots, n.$$

When $\sigma := I$ is the identity in S_N , we shall simply denote by $\mathcal{A}_{\alpha, \beta}^h$, $h = 1, \dots, N$, and $\mathcal{B}^{\alpha, \beta}$ the corresponding matrices.

Definition 7.1

$$K_{\alpha, \beta} := K_{\alpha, \beta}(\rho; A_1, \dots, A_N) := \sum_{\sigma \in S_N} \det(\mathcal{B}^{\alpha\sigma, \beta\sigma}).$$

Definition 7.2. If $u \in \{0, 1\}^N$ and $\alpha, \beta \in \mathcal{C} = \underline{n}^N$ we define an $N \times N$ matrix $D(u; \alpha, \beta)$ setting

$$\{D(u; \alpha, \beta)_{hj}\} := \{\mathbf{C}^{u(j)} \mathcal{A}_{\alpha_j \beta_j}^h\}, \quad h, j = 1, \dots, N.$$

Proposition 7.2. We have

$$K_{\alpha, \beta} = \sum_{u \in \{0, 1\}^N} \det(D(u; \alpha, \beta))^2 \geq 0.$$

Proof. Applying: Lemmas 7.1, 5.1 and 5.4 to the function

$$r(u) = r_{\sigma, \tau}(u) := \prod_{j=1}^N \mathbf{C}^{u(j)} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \mathbf{C}^{u(j)} \mathcal{A}_{\alpha_{\tau(j)}, \beta_{\tau(j)}}^{\tau(j)},$$

we get

$$K_{\alpha, \beta} = \sum_{\sigma \in S_N} \det \mathcal{B}^{\alpha\sigma, \beta\sigma}$$

$$\begin{aligned}
 &= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N (\mathcal{B}^{\alpha_\sigma, \beta_\sigma})_{j, \tau(j)} \\
 &= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N \operatorname{Re} \left\{ \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \mathcal{A}_{\beta_{\sigma(j)}, \alpha_{\sigma(j)}}^{\tau(j)} \right\} \\
 &= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N \left(\operatorname{Re} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \operatorname{Re} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} + \operatorname{Im} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \operatorname{Im} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} \right) \\
 &= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N \left(\sum_{u \in \{0,1\}} C^u \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \cdot C^u \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} \right) \\
 &= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \sum_{u \in \{0,1\}^N} \prod_{j=1}^N C^{u(j)} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j C^{u(j)} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)} \\
 &= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \sum_{u \in \{0,1\}^N} \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^{\tau(j)}.
 \end{aligned}$$

Hence, since for any $E = \{E_{jk}\}$ and $\sigma \in S^N$, $\operatorname{sgn} \sigma \cdot \det\{E_{\sigma(j)k}\} = \det(E)$, one has

$$\begin{aligned}
 K_{\alpha, \beta} &= \sum_{\sigma \in S_N} \det \mathcal{B}^{\alpha_\sigma, \beta_\sigma} \\
 &= \sum_{u \in \{0,1\}^N} \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \prod_{h=1}^N C^{u(\sigma(h))} \mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^{\tau(h)} \\
 &= \sum_{u \in \{0,1\}^N} \sum_{\sigma \in S_N} \operatorname{sgn} \sigma \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \operatorname{sgn} \sigma \sum_{\tau \in S_N} \operatorname{sgn} \tau \prod_{h=1}^N C^{u(\sigma(h))} \mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^{\tau(h)} \\
 &= \sum_{u \in \{0,1\}^N} \left(\sum_{\sigma \in S_N} \operatorname{sgn} \sigma \prod_{j=1}^N C^{u(\sigma(j))} \mathcal{A}_{\alpha_{\sigma(j)}, \beta_{\sigma(j)}}^j \right) \left(\operatorname{sgn} \sigma \det \left\{ C^{u(\sigma(h))} \mathcal{A}_{\alpha_{\sigma(h)}, \beta_{\sigma(h)}}^{\tau(h)} \right\} \right) \\
 &= \sum_{u \in \{0,1\}^N} \det \left\{ C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h \right\} \det \left\{ C^{u(h)} \mathcal{A}_{\alpha_h, \beta_h}^j \right\} \\
 &= \sum_{u \in \{0,1\}^N} \left(\det \left\{ C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h \right\} \right)^2 \\
 &= \sum_{u \in \{0,1\}^N} \det(D(u; \alpha, \beta))^2. \quad \square
 \end{aligned}$$

Lemma 7.3. *If $\mathcal{A}^1, \dots, \mathcal{A}^N \in M_{n,sa}$ are linearly independent then there exist $\alpha, \beta \in \mathcal{C}$ and $u \in \{0, 1\}^N$ such that*

$$\det(D(u; \alpha, \beta)) = \det\{C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h\} \neq 0.$$

Proof. Note that the independence hypothesis implies $N \leq \dim_{\mathbb{R}}(M_{n,sa}) = n^2$. Therefore the $N \times n^2$ matrix

$$\begin{pmatrix} \mathcal{A}_{11}^1 & \cdots & \mathcal{A}_{1n}^1 & \mathcal{A}_{21}^1 & \cdots & \mathcal{A}_{2n}^1 & \cdots & \mathcal{A}_{n1}^1 & \cdots & \mathcal{A}_{nn}^1 \\ \mathcal{A}_{11}^2 & \cdots & \mathcal{A}_{1n}^2 & \mathcal{A}_{21}^2 & \cdots & \mathcal{A}_{2n}^2 & \cdots & \mathcal{A}_{n1}^2 & \cdots & \mathcal{A}_{nn}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathcal{A}_{11}^N & \cdots & \mathcal{A}_{1n}^N & \mathcal{A}_{21}^N & \cdots & \mathcal{A}_{2n}^N & \cdots & \mathcal{A}_{n1}^N & \cdots & \mathcal{A}_{nn}^N \end{pmatrix}$$

has rank N because it has N independent rows. This means that there exist N columns that are linearly independent, which is, in turn, equivalent to say that there exists $\alpha, \beta \in \mathcal{C}$ such that the matrix

$$\begin{pmatrix} \mathcal{A}_{\alpha_1 \beta_1}^1 & \mathcal{A}_{\alpha_2 \beta_2}^1 & \cdots & \mathcal{A}_{\alpha_N \beta_N}^1 \\ \mathcal{A}_{\alpha_1 \beta_1}^2 & \mathcal{A}_{\alpha_2 \beta_2}^2 & \cdots & \mathcal{A}_{\alpha_N \beta_N}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{A}_{\alpha_1 \beta_1}^N & \mathcal{A}_{\alpha_2 \beta_2}^N & \cdots & \mathcal{A}_{\alpha_N \beta_N}^N \end{pmatrix}$$

has rank N . This implies that the $N \times 2N$ matrix

$$\begin{pmatrix} \operatorname{Re} \mathcal{A}_{\alpha_1 \beta_1}^1 & \operatorname{Im} \mathcal{A}_{\alpha_1 \beta_1}^1 & \operatorname{Re} \mathcal{A}_{\alpha_2 \beta_2}^1 & \operatorname{Im} \mathcal{A}_{\alpha_2 \beta_2}^1 & \cdots & \operatorname{Re} \mathcal{A}_{\alpha_N \beta_N}^1 & \operatorname{Im} \mathcal{A}_{\alpha_N \beta_N}^1 \\ \operatorname{Re} \mathcal{A}_{\alpha_1 \beta_1}^2 & \operatorname{Im} \mathcal{A}_{\alpha_1 \beta_1}^2 & \operatorname{Re} \mathcal{A}_{\alpha_2 \beta_2}^2 & \operatorname{Im} \mathcal{A}_{\alpha_2 \beta_2}^2 & \cdots & \operatorname{Re} \mathcal{A}_{\alpha_N \beta_N}^2 & \operatorname{Im} \mathcal{A}_{\alpha_N \beta_N}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \operatorname{Re} \mathcal{A}_{\alpha_1 \beta_1}^N & \operatorname{Im} \mathcal{A}_{\alpha_1 \beta_1}^N & \operatorname{Re} \mathcal{A}_{\alpha_2 \beta_2}^N & \operatorname{Im} \mathcal{A}_{\alpha_2 \beta_2}^N & \cdots & \operatorname{Re} \mathcal{A}_{\alpha_N \beta_N}^N & \operatorname{Im} \mathcal{A}_{\alpha_N \beta_N}^N \end{pmatrix}$$

has rank N , so that this matrix must have also N independent columns. This last assertion is equivalent to the desired conclusion. \square

Corollary 7.4. *If $(A_1)_0, \dots, (A_N)_0 \in M_{n,sa}$ are linear independent then there exist $\alpha, \beta \in \mathcal{C}$ and $u \in \{0, 1\}^N$ such that*

$$\det(D(u; \alpha, \beta)) = \det\{C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h\} \neq 0.$$

Proof. By definition of $\mathcal{A}^j, j = 1, \dots, N$, the hypothesis of linear independence of

$$(A_1)_0, \dots, (A_N)_0$$

implies the linear independence of

$$\mathcal{A}^1, \dots, \mathcal{A}^N.$$

Hence, by Lemma 7.3 there exist $\alpha, \beta \in \mathcal{C}$ and $u \in \{0, 1\}^N$ such that

$$\det(D(u; \alpha, \beta)) = \det\{C^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^h\} \neq 0. \quad \square$$

8. Proof of the main results

Theorem 8.1. *Let $N \in \mathbb{N}, f \in \mathcal{F}_{op}^r, \rho \in \mathcal{D}_n^1$ and $A_1, \dots, A_N \in M_{n,sa}$ be arbitrary. Then*

$$\operatorname{Vol}_{\rho}^{\operatorname{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \operatorname{Vol}_{\rho}^f(i[\rho, A_1], \dots, i[\rho, A_N]).$$

Proof. Theorem 8.1 is equivalent to the following inequality:

$$\det\{\text{Cov}_\rho(A_h, A_j)\} \geq \det\left\{\frac{f(0)}{2}\langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho, f}\right\}.$$

If ρ and A_1, \dots, A_N are fixed we set

$$F(f) := \det\{\text{Cov}_\rho(A_h, A_j)\} - \det\left\{\frac{f(0)}{2}\langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho, f}\right\}.$$

Because of Theorem 3.3 one has

$$F(f) = \det\{\text{Cov}_\rho(A_h, A_j)\} - \det\left\{\text{Cov}_\rho(A_h, A_j) - \text{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)((A_h)_0)(A_j)_0)\right\},$$

so that Theorem 8.1 is equivalent to

$$F(f) \geq 0.$$

From Proposition 3.4, we have

$$\begin{aligned} \text{Cov}_\rho(A_h, A_j) &= \text{Re}\{\text{Tr}(\rho(A_h)_0(A_j)_0)\} = \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^h A_{lk}^j\} \\ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho, f} &= \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) \text{Re}\{\mathcal{A}_{kl}^h \mathcal{A}_{lk}^j\} - \sum_{k,l} m_{\tilde{f}}(\lambda_k, \lambda_l) \text{Re}\{\mathcal{A}_{kl}^h A_{lk}^j\}. \end{aligned}$$

and therefore one has

$$\begin{aligned} F(f) &= \sum_{\sigma \in S_N} \text{sgn}\sigma \left[\prod_{j=1}^N \text{Cov}_\rho(A_j, A_{\sigma(j)}) - \prod_{j=1}^N \frac{f(0)}{2} \langle i[\rho, A_j], i[\rho, A_{\sigma(j)}]\rangle_{\rho, f} \right] \\ &= \sum_{\sigma \in S_N} \text{sgn}\sigma \xi_\sigma, \end{aligned}$$

where

$$\xi_\sigma = \prod_{j=1}^N \sum_{k,l=1}^n \frac{\lambda_k + \lambda_l}{2} \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\} - \prod_{j=1}^N \sum_{k,l=1}^n \left[\frac{\lambda_k + \lambda_l}{2} - m_{\tilde{f}}(\lambda_k, \lambda_l) \right] \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\}.$$

From Definition 6.2 and applying Proposition 5.2 to the case $X = \underline{n}$ we get

$$\begin{aligned} \xi_\sigma &= \prod_{j=1}^N \sum_{k,l=1}^n \frac{\lambda_k + \lambda_l}{2} \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\} - \prod_{j=1}^N \sum_{k,l=1}^n \left[\frac{\lambda_k + \lambda_l}{2} - m_{\tilde{f}}(\lambda_k, \lambda_l) \right] \text{Re}\{\mathcal{A}_{kl}^j \mathcal{A}_{lk}^{\sigma(j)}\} \\ &= \sum_{\alpha, \beta \in \mathcal{C}} \left\{ \prod_{j=1}^N \frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} - \prod_{j=1}^N \left[\frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - m_{\tilde{f}}(\lambda_{\alpha_j}, \lambda_{\beta_j}) \right] \right. \\ &\quad \left. \times \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} \right\} \\ &= \sum_{\alpha, \beta \in \mathcal{C}} \left\{ \prod_{j=1}^N \frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} \prod_{j=1}^N \text{Re}\{\mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)}\} - \prod_{j=1}^N \left[\frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - m_{\tilde{f}}(\lambda_{\alpha_j}, \lambda_{\beta_j}) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{j=1}^N \operatorname{Re} \left\{ \mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)} \right\} \\
 & = \sum_{\alpha, \beta \in \mathcal{C}} \left[\prod_{j=1}^N \frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - \prod_{j=1}^N \left(\frac{\lambda_{\alpha_j} + \lambda_{\beta_j}}{2} - m_{\bar{f}}(\lambda_{\alpha_j}, \lambda_{\beta_j}) \right) \right] \prod_{j=1}^N \operatorname{Re} \left\{ \mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)} \right\} \\
 & = \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha, \beta} \prod_{j=1}^N \operatorname{Re} \left\{ \mathcal{A}_{\alpha_j \beta_j}^j \mathcal{A}_{\beta_j \alpha_j}^{\sigma(j)} \right\}.
 \end{aligned}$$

Hence, applying Proposition 5.3 to the case $G = S^N$, $X = \mathcal{C} \times \mathcal{C}$ and $r(x) := r(\alpha, \beta) := H_{\alpha, \beta}^f \det \mathcal{B}^{\alpha, \beta}$ and Proposition 6.4 we get

$$\begin{aligned}
 F(f) & = \sum_{\sigma \in S_N} \operatorname{sgn} \sigma \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha, \beta}^f \prod_{i=1}^N \operatorname{Re} \left\{ \mathcal{A}_{\alpha_i \beta_i}^i \mathcal{A}_{\beta_i \alpha_i}^{\sigma(i)} \right\} \\
 & = \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha, \beta}^f \sum_{\sigma \in S_N} \operatorname{sgn} \sigma \prod_{i=1}^N \operatorname{Re} \left\{ \mathcal{A}_{\alpha_i \beta_i}^i \mathcal{A}_{\beta_i \alpha_i}^{\sigma(i)} \right\} \\
 & = \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha, \beta}^f \det \mathcal{B}^{\alpha, \beta} \\
 & = \frac{1}{N!} \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha, \beta}^f \sum_{\sigma \in S_N} \det \mathcal{B}^{\alpha \sigma, \beta \sigma} \\
 & = \frac{1}{N!} \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha, \beta}^f K_{\alpha, \beta}.
 \end{aligned}$$

By Corollary 6.3, $H_{\alpha, \beta}^f$ is strictly positive; on the other hand, Proposition 7.2 ensures the nonnegativity of $K_{\alpha, \beta}$, so that we obtain the result. \square

Theorem 8.2. *Let $N \in \mathbb{N}$, $f \in \mathcal{F}_{op}^r$, $\rho \in \mathcal{D}_n^1$ and $A_1, \dots, A_N \in M_{n,sa}$ be arbitrary. The inequality*

$$\operatorname{Vol}_{\rho}^{\operatorname{Cov}}(A_1, \dots, A_N) \geq \left(\frac{f(0)}{2} \right)^{\frac{N}{2}} \operatorname{Vol}_{\rho}^f(i[\rho, A_1], \dots, i[\rho, A_N])$$

is an equality if and only if $(A_1)_0, \dots, (A_N)_0$ are linearly dependent.

Proof. Since

$$\operatorname{Cov}_{\rho}(A_1, A_2) = \operatorname{Tr}(\rho(A_1)_0(A_2)_0) = \operatorname{Cov}_{\rho}((A_1)_0, (A_2)_0),$$

we have that

$$\operatorname{Cov}_{\rho}(A_1, A_2) = \operatorname{Cov}_{\rho}((A_1)_0, (A_2)_0).$$

From this it follows

$$\operatorname{Vol}_{\rho}^{\operatorname{Cov}}(A_1, \dots, A_N) = \operatorname{Vol}_{\rho}^{\operatorname{Cov}}((A_1)_0, \dots, (A_N)_0).$$

Therefore, if $(A_1)_0, \dots, (A_N)_0$ are linearly dependent then

$$\begin{aligned}
 0 &= \text{Vol}_\rho^{\text{Cov}}((A_1)_0, \dots, (A_N)_0) = \text{Vol}_\rho^{\text{Cov}}(A_1, \dots, A_N) \\
 &\geq \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]) \geq 0
 \end{aligned}$$

and we are done.

Conversely, suppose that $(A_1)_0, \dots, (A_N)_0$ are not linear dependent; then we want to show that $F(f) > 0$. Since for any $\alpha, \beta \in \mathcal{C}$, $H_{\alpha,\beta}$ is strictly positive and $K_{\alpha,\beta}$ is nonnegative, this is equivalent to prove that $K_{\alpha,\beta}$ is not null for some $\alpha, \beta \in \mathcal{C}$. Because of Proposition 7.2, this is, in turn, equivalent to show that $\det(\mathbf{C}^{u(j)} \mathcal{A}_{\alpha_j, \beta_j}^i)$ is not null for some $\alpha, \beta \in \mathcal{C}$ and $u \in \{0, 1\}^N$. This is a consequence of Corollary 7.4. \square

Theorem 8.3. *Define*

$$V(f) := \left(\frac{f(0)}{2}\right)^{\frac{N}{2}} \text{Vol}_\rho^f(i[\rho, A_1], \dots, i[\rho, A_N]).$$

Then

$$\tilde{f} \leq \tilde{g} \implies V(f) \geq V(g).$$

Proof. Because of Proposition 6.1 and Proposition 6.2, one has that

$$\tilde{f} \leq \tilde{g} \implies 0 < H_{\alpha,\beta}^f \leq H_{\alpha,\beta}^g.$$

Since $K_{\alpha,\beta} \geq 0$ does not depend on f and

$$F(f) = \frac{1}{N!} \sum_{\alpha, \beta \in \mathcal{C}} H_{\alpha,\beta}^f K_{\alpha,\beta}$$

we get that

$$0 \leq F(f) \leq F(g).$$

By definition of F , we obtain the thesis. \square

9. Relation with the standard uncertainty principle

In this section we prove that the inequality (1.5) cannot be seen as a refinement of the Robertson uncertainty principle and viceversa.

Theorem 9.1 (Hadamard inequality). *If $H \in M_{N,sa}$ is positive semidefinite then*

$$\det(H) \leq \prod_{j=1}^N h_{jj}.$$

Particular cases of the theorem below have been proved by Kosaki ($N = 2, f = f_{\text{WYD}(\beta)}$; see [15]), Yanagi–Furuichi–Kuriyama ($N = 2, f = f_{\text{WY}}$; see [31]) and Gibilisco–Imparato–Isola ($N = 2, f$ arbitrary; see [4]).

Theorem 9.2. Let $f \in \mathcal{F}_{\text{op}}^r$. The inequality

$$\det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\} \geq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}$$

is (in general) false for any $N = 2m$.

Proof. Let $n = N = 2m$. By the Hadamard inequality it is enough to find $A_1, \dots, A_N \in M_{N,sa}$ and a state $\rho \in \mathcal{D}_N^1$ such that

$$\prod_{j=1}^N I_{\rho}^f(A_j) < \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}. \tag{9.1}$$

Let $\rho := \text{diag}(\lambda_1, \dots, \lambda_N)$, where $\lambda_1 < \lambda_2 < \dots < \lambda_N$. The aim is to construct A_1, \dots, A_N that are block-diagonal matrices, each matrix consisting of exactly one non-null block equal to a 2×2 Pauli matrix.

More precisely, given $h = 2q + 1$, where $q = 0, \dots, N - 1$, define the Hermitian matrices A_h and A_{h+1} such that $(A_h)_{hh+1} = i = (A_h^*)_{h+1h}$, $(A_{h+1})_{hh+1} = 1 = (A_{h+1})_{h+1h}$ and $(A_h)_{kl} = (A_{h+1})_{kl} = 0$ elsewhere.

Since the state ρ is diagonal and A_h are null diagonal matrices, $A_h \equiv \mathcal{A}^h$, where $(\mathcal{A}^h)_{kl} = \langle (A_h)_0 \phi_k, \phi_l \rangle$ is defined as in Proposition 3.4. Therefore, say, if h is odd one obtains from Proposition 3.4

$$\begin{aligned} I_{\rho}^f(A_h) &= \frac{1}{2} \sum_{k,l} (\lambda_k + \lambda_l) |\mathcal{A}_{kl}^h|^2 - \sum_{k,l} m_{\bar{f}}(\lambda_k, \lambda_l) |\mathcal{A}_{kl}^h|^2 \\ &= \lambda_h + \lambda_{h+1} - 2m_{\bar{f}}(\lambda_h, \lambda_{h+1}) \\ &= I_{\rho}^f(A_{h+1}). \end{aligned}$$

Suppose now that h is odd and $h < k$. We have

$$\begin{aligned} \text{Tr}(\rho[A_h, A_k]) &= \sum_{j,l,m} \rho_{jl} ((A_h)_{lm} (A_k)_{mj} - (A_k)_{lm} (A_h)_{mj}) \\ &= \sum_{j,m} \lambda_j ((A_h)_{jm} (A_k)_{mj} - (A_k)_{jm} (A_h)_{mj}) \\ &= \sum_{j,m} \lambda_j ((A_h)_{jm} (A_k)_{mj} - (A_k)_{jm} (A_h)_{mj}) \\ &= 2i(\lambda_h - \lambda_{h+1}) \delta_k^{h+1}, \end{aligned}$$

where δ_h^{k+1} denotes the Kronecker delta function. Since $\text{Tr}(\rho[A_h, A_k]) = -\text{Tr}(\rho[A_k, A_h])$,

$$\left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\} = \begin{pmatrix} 0 & \lambda_1 - \lambda_2 & 0 & \dots & 0 \\ \lambda_2 - \lambda_1 & 0 & \lambda_2 - \lambda_3 & \dots & 0 \\ 0 & \lambda_3 - \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \lambda_{N-1} - \lambda_N \\ 0 & 0 & \dots & \lambda_N - \lambda_{N-1} & 0 \end{pmatrix}$$

so that

$$\det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\} = \prod_{h < N, h=2q+1} (\lambda_{h+1} - \lambda_h)^2.$$

Finally, since for any $f \in \mathcal{F}_{\text{op}}^r$ the function $m_{\tilde{f}}(\cdot, \cdot)$ is a mean, one has $\lambda_h < m_{\tilde{f}}(\lambda_h, \lambda_{h+1}) < \lambda_{h+1}$. This implies, for any odd h ,

$$I_{\rho}^f(A_h) = I_{\rho}^f(A_{h+1}) = \lambda_h + \lambda_{h+1} - 2m_{\tilde{f}}(\lambda_h, \lambda_{h+1}) < \lambda_{h+1} - \lambda_h,$$

so that one can get (9.1) by taking the product over all h . \square

Theorem 9.3. *Let $f \in \mathcal{F}_{\text{op}}^r$. The inequality*

$$\det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\} \leq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}$$

is (in general) false for any $N = 2m$.

Proof. It suffices to find selfadjoint matrices A_1, \dots, A_N which are pairwise commuting but not commuting with a given state ρ and such that $[\rho, A_1], \dots, [\rho, A_N]$ are linearly independent.

Consider a state of the form $\rho = \text{diag}(\lambda_1, \dots, \lambda_n)$ where the eigenvalues λ_i are all distinct.

Let $A_1, \dots, A_N \in \mathcal{M}_{n,sa}(\mathbb{R})$ be N linearly independent symmetric real matrices such that $(A_j)_{kk} = 0$ for any $j = 1, \dots, N$ and $k = 1, \dots, n$. Note that the linear independence of A_1, \dots, A_N implies the condition $n(n - 1)/2 \geq N$.

Obviously, $[A_j, A_m] = 0$ for any $j, m = 1, \dots, N$, while a direct computation shows that

$$\begin{aligned} ([\rho, A_j])_{kl} &= \sum_{h=1}^n \rho_{kh} (A_j)_{hl} - \sum_{h=1}^n (A_j)_{kh} \rho_{hl} \\ &= (A_j)_{kl} (\lambda_k - \lambda_l) \end{aligned}$$

Observe that also $[\rho, A_1] \dots, [\rho, A_N]$ are linearly independent. Suppose, in fact, that there exists a vector $\alpha \in \mathbb{R}^N$ such that

$$\sum_{j=1}^N \alpha_j [\rho, A_j] \equiv 0,$$

that is, for any $k, l = 1, \dots, n$

$$0 = \sum_{j=1}^N \alpha_j ([\rho, A_j])_{kl} = (\lambda_k - \lambda_l) \sum_{j=1}^N \alpha_j (A_j)_{kl}.$$

This implies that $\sum_j \alpha_j (A_j)_{kl} = 0$, and hence $\alpha \equiv 0$, because of the linear independence of A_1, \dots, A_N . \square

Acknowledgements

After the completion of the present paper we received by A. Andai the preprint ‘Uncertainty principle with quantum Fisher information, arXiv:0707.1147’ where he gives a completely different (and very interesting) proof of inequality (1.5).

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