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AN INEQUALITY RELATED TO UNCERTAINTY PRINCIPLE IN VON NEUMANN ALGEBRAS

PAOLO GIBILISCO

Dipartimento SEFEMEQ, Facoltà di Economia
Università di Roma "Tor Vergata"
Via Columbia 2, 00133 Rome, Italy
gibilisco@volterra.uniroma2.it

TOMMASO ISOLA

Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 00133 Rome, Italy
isola@mat.uniroma2.it

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Recently Kosaki proved an inequality for matrices that can be seen as a kind of new uncertainty principle. Independently, the same result was proved by Yanagi *et al.* The new bound is given in terms of Wigner–Yanase–Dyson informations. Kosaki himself asked if this inequality can be proved in the setting of von Neumann algebras. In this paper we provide a positive answer to that question and moreover we show how the inequality can be generalized to an arbitrary operator monotone function.

Keywords: Uncertainty principle; Wigner–Yanase–Dyson information; operator monotone functions.

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1. Introduction

If A, B are selfadjoint matrices and ρ is a density matrix, define

$$\mathrm{Cov}_\rho(A,B) := \mathrm{Re}\{\mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B)\},$$
$$\mathrm{Var}_\rho(A) := \mathrm{Cov}_\rho(A,A).$$

The uncertainty principle reads as

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

This inequality can be refined as

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - \mathrm{Cov}_\rho(A,B)^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2,$$

(see [5, 12]). Recently, a different uncertainty principle has been found [11, 9, 10, 8, 13, 1]. For $\beta \in (0,1)$, define β-correlation and β-information as

$$\mathrm{Corr}_{\rho,\beta}(A,B) := \mathrm{Re}\{\mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho^{\beta} A \rho^{1-\beta} B)\},$$
$$I_{\rho,\beta}(A) := \mathrm{Corr}_{\rho,\beta}(A,A) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^{\beta} A \rho^{1-\beta} A),$$

where the latter coincides with the Wigner–Yanase–Dyson information. It has been proved that

$$\mathrm{Var}_{\rho}(A)\,\mathrm{Var}_{\rho}(B) - \mathrm{Cov}_{\rho}(A,B)^2 \geq I_{\rho,\beta}(A) I_{\rho,\beta}(B) - \mathrm{Corr}_{\rho,\beta}(A,B)^2. \qquad (1.1)$$

The quantities involved in the previous inequality make a perfect sense in a von Neumann algebra setting (see, for example, [7]). In [8], Kosaki asked if the inequality (1.1) is true in this more general setting.

In this paper we provide a positive answer to Kosaki question and moreover we show that, once the inequality is formulated in the context of operator monotone functions, the result can be greatly generalized.

2. Preliminaries

Denote by $M_{n,sa}$ the space of complex self-adjoint $n \times n$ matrices, and recall that a function $f : (0,\infty) \to \mathbb{R}$ is said *operator monotone* if, for any $n \in \mathbb{N}$, any $A, B \in M_{n,sa}$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. Then, $f : (0,\infty) \to \mathbb{R}$ is operator monotone if and only if for any $A, B \in \mathcal{B}(\mathcal{H})$ such that $0 \leq A \leq B$, $f(A) \leq f(B)$ holds. An operator monotone function is *symmetric* if $f(x) := xf(x^{-1})$ and *normalized* if $f(1) = 1$. We denote by $\mathfrak{F}$ the class of positive, symmetric, normalized, operator monotone functions.

Examples of operator monotone functions are the so-called Wigner–Yanase–Dyson functions

$$f_{\beta}(x) := \beta(1-\beta)\frac{(x-1)^2}{(x^{\beta}-1)(x^{1-\beta}-1)}, \quad \beta \in (0,1).$$

Returning to a general $f \in \mathfrak{F}$, we associate to it a function $\tilde{f} \in \mathfrak{F}$ [2] defined by

$$\tilde{f}(x) := \frac{1}{2}\left((x+1) - (x-1)^2\frac{f(0)}{f(x)}\right), \quad x > 0.$$

For example,

$$\tilde{f}_{\beta}(x) = \frac{1}{2}(x^{\beta} + x^{1-\beta}).$$

The left and right multiplication operators are defined as $L_{\rho}(A) := \rho A$ and $R_{\rho}(A) := A\rho$.

Definition 2.1. For $A, B \in M_{n,sa}$, $f \in \mathfrak{F}$, and ρ a faithful density matrix, define f-correlation and f-information as

$$\begin{aligned}\mathrm{Corr}_\rho^f(A,B) &:= \mathrm{Re}\{\mathrm{Tr}(\rho AB) - \mathrm{Tr}(R_\rho \tilde{f}(L_\rho R_\rho^{-1})(A)\cdot B)\},\\ I_\rho^f(A) &:= \mathrm{Corr}_\rho^f(A,A).\end{aligned}$$

Recall that f-information is also known as metric adjusted skew information (see [4]). The following generalization of inequality (1.1) is proved in [2].

Theorem 2.2. $\mathrm{Var}_\rho(A)\,\mathrm{Var}_\rho(B) - \mathrm{Cov}_\rho(A,B)^2 \geq I_\rho^f(A) I_\rho^f(B) - \mathrm{Corr}_\rho^f(A,B)^2$.

In the next section, we prove that the above inequality holds true in a general von Neumann algebra, thus answering, in particular, the question raised by Kosaki in [8], and recalled above. A different generalization of Theorem 2.2 has been proved in [3].

3. The Main Result

Let $\mathcal{M}$ be a von Neumann algebra, and ω a normal faithful state on $\mathcal{M}$, and denote by $\mathcal{H}_\omega$ and ξ_ω the GNS Hilbert space and vector, and by S_ω, J_ω and Δ_ω the modular operators associated to ω.

The proof of the main result is divided in a series of lemmas. In order to deal with unbounded operators, we introduce some sesquilinear forms on $\mathcal{H}_\omega$, and take [6] as our standard reference.

Definition 3.1. Let $f \in \mathfrak{F}$ and define the following sesquilinear forms

$$\begin{aligned}\mathcal{E}(\xi,\eta) &:= \langle \Delta_\omega^{1/2}\xi, \Delta_\omega^{1/2}\eta\rangle,\\ \mathcal{E}_1(\xi,\eta) &:= \mathcal{E}(\xi,\eta) + \langle \xi,\eta\rangle,\\ \mathcal{F}^f(\xi,\eta) &:= \langle \tilde{f}(\Delta_\omega)^{1/2}\xi, \tilde{f}(\Delta_\omega)^{1/2}\eta\rangle,\\ \mathcal{G}^f(\xi,\eta) &:= \frac{1}{2}\mathcal{E}_1(\xi,\eta) - \mathcal{F}^f(\xi,\eta).\end{aligned}$$

It follows from [6, Example VI.1.13] that $\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{F}^f$ are closed, positive and symmetric sesquilinear forms.

Lemma 3.2. *Let $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}$, $\{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$ be such that $\xi_n \to \xi$, $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$ as $n \to \infty$, and analogously for η_n and η. Then*

$$\begin{aligned}\mathcal{E}(\xi,\eta) &= \lim_{n\to\infty} \mathcal{E}(\xi_n,\eta_n) = \lim_{n\to\infty}\langle \xi_n, \Delta_\omega \eta_n\rangle,\\ \mathcal{F}^f(\xi,\eta) &= \lim_{n\to\infty} \mathcal{F}^f(\xi_n,\eta_n) = \lim_{n\to\infty}\langle \xi_n, \tilde{f}(\Delta_\omega)\eta_n\rangle.\end{aligned}$$

Proof. It follows from [6, Theorem VI.2.1] that $\mathcal{D}(\Delta_\omega)$ is a core for $\mathcal{D}(\mathcal{E}) \equiv \mathcal{D}(\Delta_\omega^{1/2})$, so that, from [6, Theorem VI.1.21], for any $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$ there is $\{\xi_n\} \subset \mathcal{D}(\Delta_\omega)$ such that $\xi_n \to \xi$ and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$ as $n \to \infty$. Then

$\mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0$ as $m, n \to \infty$. Now observe that $0 \leq \tilde{f}(x) \leq \frac{1}{2}(x+1)$, for $x > 0$ [2], so that

$$\begin{aligned}\mathcal{F}^f(\xi_n - \xi_m, \xi_n - \xi_m) &= \langle \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m), \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m)\rangle \\ &= \langle \xi_n - \xi_m, \tilde{f}(\Delta_\omega)(\xi_n - \xi_m)\rangle \\ &\leq \frac{1}{2}\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle + \frac{1}{2}\langle \xi_n - \xi_m, \Delta_\omega(\xi_n - \xi_m)\rangle \\ &= \frac{1}{2}\|\xi_n - \xi_m\| + \frac{1}{2}\mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0 \text{ as } m, n \to \infty.\end{aligned}$$

This implies $\xi \in \mathcal{D}(\mathcal{F}^f)$ and $\mathcal{F}^f(\xi_n - \xi, \xi_n - \xi) \to 0$ as $n \to \infty$.

Therefore, if $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}$, $\{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$, approximating ξ, η in the above sense, we obtain, from [6, Theorem VI.1.12], that $\mathcal{F}^f(\xi, \eta) = \lim_{n\to\infty} \mathcal{F}^f(\xi_n, \eta_n)$, and analogously for $\mathcal{E}$. □

Lemma 3.3.

(i) $\mathcal{D}(\mathcal{F}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$,

(ii) $\mathcal{G}^f$ *is a symmetric sesquilinear form on* $\mathcal{D}(\mathcal{G}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$, *which is positive on* $\mathcal{D}(\Delta_\omega^{1/2})$.

Proof. (i) It follows from the proof of the previous lemma.

(ii) We only need to prove positivity. To begin, let $\xi \in \mathcal{D}(\Delta_\omega)$. Then, setting $g(x) := \frac{1}{2}(x+1) - \tilde{f}(x) \geq 0$, for all $x > 0$, we have $\mathcal{G}^f(\xi, \xi) = \frac{1}{2}\mathcal{E}_1(\xi, \xi) - \mathcal{F}^f(\xi, \xi) = \frac{1}{2}\langle \xi, \xi\rangle + \frac{1}{2}\langle \xi, \Delta_\omega \xi\rangle - \langle \xi, \tilde{f}(\Delta_\omega)\xi\rangle = \langle \xi, g(\Delta_\omega)\xi\rangle \geq 0$.

Moreover, if $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\xi_n \in \mathcal{D}(\Delta_\omega)$ is such that $\xi_n \to \xi$, and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0$, then, from Lemma 3.2, it follows $\mathcal{G}^f(\xi, \xi) = \lim_{n\to\infty} \mathcal{G}^f(\xi_n, \xi_n) \geq 0$. □

We can now introduce the main objects of study. In the sequel, we denote by $T \widehat{\in} \mathcal{M}$ the fact that T is a closed, densely defined, linear operator on $\mathcal{H}_\omega$, and is affiliated with $\mathcal{M}$.

Definition 3.4. For any $A, B \widehat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we set $A_0 := A - \langle \xi_\omega, A\xi_\omega\rangle$, $B_0 := B - \langle \xi_\omega, B\xi_\omega\rangle$, and define the bilinear forms

$$\begin{aligned}\mathrm{Cov}_\omega(A, B) &:= \mathrm{Re}\langle A_0\xi_\omega, B_0\xi_\omega\rangle, \\ \mathrm{Var}_\omega(A) &:= \mathrm{Cov}_\omega(A, A), \\ \mathrm{Corr}^f_\omega(A, B) &:= \mathrm{Re}\langle A_0\xi_\omega, B_0\xi_\omega\rangle - \mathrm{Re}\langle \tilde{f}(\Delta_\omega)^{1/2}A_0\xi_\omega,\ \tilde{f}(\Delta_\omega)^{1/2}B_0\xi_\omega\rangle, \\ I^f_\omega(A) &:= \mathrm{Corr}^f_\omega(A, A).\end{aligned}$$

Remark 3.5. Observe that in the matrix case $\omega = \mathrm{Tr}(\rho\cdot)$, for some density matrix ρ, and $\Delta_\omega = L_\rho R_\rho^{-1}$, so that the previous definition is a true generalization of covariance and f-correlation in the matrix case.

For the reader's convenience, we prove the following folklore result.

Lemma 3.6. $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

Proof. (1) Let us first prove that $\mathcal{D}(\Delta_\omega^{1/2}) \subset \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$. Indeed, let $\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and define the linear operator $T_0 : x'\xi_\omega \in \mathcal{M}'\xi_\omega \mapsto x'\eta \in \mathcal{H}_\omega$, which is densely defined, and affiliated with $\mathcal{M}$. Let us show that if is preclosed: indeed, if $x'_n\xi_\omega \to 0$, and $x'_n\eta \to \zeta$, then, for any $y' \in \mathcal{M}'$, we get

$$\begin{aligned}\langle \zeta, y'\xi_\omega\rangle &= \lim_{n\to\infty} \langle x'_n\eta, y'\xi_\omega\rangle = \lim_{n\to\infty} \langle \eta, {x'_n}^* y'\xi_\omega\rangle = \lim_{n\to\infty} \langle \eta, S_\omega^*({y'}^* x'_n\xi_\omega)\rangle \\ &= \lim_{n\to\infty} \langle {y'}^* x'_n\xi_\omega, S_\omega\eta\rangle = \lim_{n\to\infty} \langle x'_n\xi_\omega, y'S_\omega\eta\rangle = 0,\end{aligned}$$

which shows that T_0 is preclosed. Let $T_\eta := \overline{T_0}$. Then, $T_\eta\widehat{\in}\mathcal{M}$, and $T_\eta\xi_\omega = \eta$. It remains to be proved that $\xi_\omega \in \mathcal{D}(T_\eta^*)$. Since $S_\omega\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, we can also consider $T_{S_\omega\eta}$. Let us show that $T_{S_\omega\eta} \subset T_\eta^*$. Indeed, for any $x', y' \in \mathcal{M}'$, we have

$$\begin{aligned}\langle T_{S_\omega\eta}x'\xi_\omega, y'\xi_\omega\rangle &= \langle x'S_\omega\eta, y'\xi_\omega\rangle = \langle S_\omega\eta, {x'}^* y'\xi_\omega\rangle \\ &= \langle {y'}^* x'\xi_\omega, \eta\rangle = \langle x'\xi_\omega, y'\eta\rangle = \langle x'\xi_\omega, T_\eta y'\xi_\omega\rangle.\end{aligned}$$

Then, $\xi_\omega \in \mathcal{D}(T_{S_\omega\eta}) \subset \mathcal{D}(T_\eta^*)$, which shows that $\mathcal{D}(\Delta_\omega^{1/2}) \subset \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

(2) Let us now prove that $\mathcal{D}(\Delta_\omega^{1/2}) \supset \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T)\cap\mathcal{D}(T^*)\}$. Indeed, if $T\widehat{\in}\mathcal{M}$ is such that $\xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)$, we can consider its polar decomposition $T = v|T|$, and let $e_n := \chi_{[0,n]}(|T|)$, $T_n := v|T|e_n$, for any $n \in \mathbb{N}$. Since $\xi_\omega \in \mathcal{D}(T)$, we have $T_n\xi_\omega = ve_n|T|\xi_\omega \to T\xi_\omega$. Moreover, since $\xi_\omega \in \mathcal{D}(T^*)$, we have $T_n^*\xi_\omega = |T|e_nv^*\xi_\omega = e_nT^*\xi_\omega \to T^*\xi_\omega$. Since S_ω is a closed operator, it follows that $T\xi_\omega \in \mathcal{D}(S_\omega) = \mathcal{D}(\Delta_\omega^{1/2})$ [and $S_\omega T\xi_\omega = T^*\xi_\omega$], which is what we wanted to prove. □

Lemma 3.7. *For any $A, B\widehat{\in}\mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A)\cap\mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we have*

(i) $\mathrm{Cov}_\omega(A, B) = \frac{1}{2}\,\mathrm{Re}\,\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega)$ *is a positive bilinear form.*
(ii) $\mathrm{Corr}_\omega^f(A, B) = \mathrm{Re}\,\mathcal{G}^f(A_0\xi_\omega, B_0\xi_\omega)$ *is a positive bilinear form.*

Proof. (i) Observe that

$$\begin{aligned}\langle B_0\xi_\omega, A_0\xi_\omega\rangle &= \langle B_0^*\xi_\omega, A_0^*\xi_\omega\rangle = \langle J_\omega\Delta_\omega^{1/2}B_0\xi_\omega, J_\omega\Delta_\omega^{1/2}A_0\xi_\omega\rangle \\ &= \langle \Delta_\omega^{1/2}A_0\xi_\omega, \Delta_\omega^{1/2}B_0\xi_\omega\rangle = \mathcal{E}(A_0\xi_\omega, B_0\xi_\omega).\end{aligned}$$

The result follows from this and the fact that $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

(ii) It follows from (i) and Lemma 3.3(ii). □

Lemma 3.8. *Let $\xi, \eta \in \mathcal{H}_\omega$, $\Delta_\omega = \int_0^\infty t\, de(t)$ and define, for Ω a Borel subset of $[0, \infty)$, $\mu_{\xi\eta}(\Omega) := \operatorname{Re}\langle \xi, e(\Omega)\eta \rangle$ and*

$$\mu := \mu_{\xi\xi} \otimes \mu_{\eta\eta} + \mu_{\eta\eta} \otimes \mu_{\xi\xi} - 2\mu_{\xi\eta} \otimes \mu_{\xi\eta}.$$

Then μ is a bounded positive Borel measure on $[0, \infty)^2$.

Proof. Let Ω_1, Ω_2 be Borel subsets of $[0, \infty)$, and set $e_j := e(\Omega_j)$, $j = 1, 2$. Observe that $|\operatorname{Re}\langle \xi, e_1\eta\rangle \cdot \operatorname{Re}\langle \xi, e_2\eta\rangle| \le \|e_1\xi\| \cdot \|e_1\eta\| \cdot \|e_2\xi\| \cdot \|e_2\eta\|$, so that

$$\begin{aligned}\mu(\Omega_1 \times \Omega_2) \ge \|e_1\xi\|^2 \cdot \|e_2\eta\|^2 + \|e_2\xi\|^2 \cdot \|e_1\eta\|^2 \\ - 2\|e_1\xi\| \cdot \|e_1\eta\| \cdot \|e_2\xi\| \cdot \|e_2\eta\| \ge 0.\end{aligned}$$

The result follows by standard measure theoretic arguments. □

Theorem 3.9. *For any $A, B \widehat{\in} \mathcal{M}_{sa}$ such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and any $f \in \mathfrak{F}$, we have*

$$\operatorname{Var}_\omega(A) \operatorname{Var}_\omega(B) - \operatorname{Cov}_\omega(A, B)^2 \ge I_\omega^f(A) I_\omega^f(B) - \operatorname{Corr}_\omega^f(A, B)^2.$$

Proof. Set

$$\begin{aligned}
G(A, B) &:= \operatorname{Var}_\omega(A) \operatorname{Var}_\omega(B) - \operatorname{Cov}_\omega(A, B)^2 - I_\omega^f(A) I_\omega^f(B) + \operatorname{Corr}_\omega^f(A, B)^2 \\
&\overset{(a)}{=} \frac{1}{2}\mathcal{E}_1(A_0\xi_\omega, A_0\xi_\omega) \cdot \frac{1}{2}\mathcal{E}_1(B_0\xi_\omega, B_0\xi_\omega) - \left(\frac{1}{2}\operatorname{Re}\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega)\right)^2 \\
&\quad - \left(\frac{1}{2}\mathcal{E}_1(A_0\xi_\omega, A_0\xi_\omega) - \mathcal{F}^f(A_0\xi_\omega, A_0\xi_\omega)\right) \\
&\quad \times \left(\frac{1}{2}\mathcal{E}_1(B_0\xi_\omega, B_0\xi_\omega) - \mathcal{F}^f(B_0\xi_\omega, B_0\xi_\omega)\right) \\
&\quad + \left(\frac{1}{2}\operatorname{Re}\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega) - \operatorname{Re}\mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega)\right)^2 \\
&= \frac{1}{2}\mathcal{E}_1(A_0\xi_\omega, A_0\xi_\omega) \cdot \mathcal{F}^f(B_0\xi_\omega, B_0\xi_\omega) \\
&\quad + \frac{1}{2}\mathcal{F}^f(A_0\xi_\omega, A_0\xi_\omega) \cdot \mathcal{E}_1(B_0\xi_\omega, B_0\xi_\omega) \\
&\quad - \mathcal{F}^f(A_0\xi_\omega, A_0\xi_\omega) \cdot \mathcal{F}^f(B_0\xi_\omega, B_0\xi_\omega) \\
&\quad - \operatorname{Re}\mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega) \cdot \operatorname{Re}\mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega) \\
&\quad + (\operatorname{Re}\mathcal{F}^f(A_0\xi_\omega, B_0\xi_\omega))^2,
\end{aligned}$$

where in (a), we have used Lemma 3.7. Let us now introduce the function, for $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$,

$$\begin{aligned}H(\xi, \eta) := \frac{1}{2}\mathcal{E}_1(\xi, \xi) \cdot \mathcal{F}^f(\eta, \eta) + \frac{1}{2}\mathcal{F}^f(\xi, \xi) \cdot \mathcal{E}_1(\eta, \eta) - \mathcal{F}^f(\xi, \xi) \cdot \mathcal{F}^f(\eta, \eta) \\ - \operatorname{Re}\mathcal{E}_1(\xi, \eta) \cdot \operatorname{Re}\mathcal{F}^f(\xi, \eta) + \left(\operatorname{Re}\mathcal{F}^f(\xi, \eta)\right)^2,\end{aligned}$$

and recall that $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T)\cap\mathcal{D}(T^*)\}$, so that, if A, B are as in the statement of the theorem, we obtain $G(A,B) = H(A_0\xi_\omega, B_0\xi_\omega)$, and to prove the theorem it suffices to show that $H(\xi,\eta) \geq 0$, for all $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$. Observe that, for $\xi,\eta \in \mathcal{D}(\Delta_\omega)$, we get

$$
\begin{aligned}
H(\xi,\eta) &= \frac{1}{2}\langle \xi,(1+\Delta_\omega)\xi\rangle\cdot\langle\eta,\tilde f(\Delta_\omega)\eta\rangle + \frac{1}{2}\langle\eta,(1+\Delta_\omega)\eta\rangle\cdot\langle\xi,\tilde f(\Delta_\omega)\xi\rangle \\
&\quad - \langle\xi,\tilde f(\Delta_\omega)\xi\rangle\cdot\langle\eta,\tilde f(\Delta_\omega)\eta\rangle \\
&\quad - \operatorname{Re}\langle\xi,(1+\Delta_\omega)\eta\rangle\cdot\operatorname{Re}\langle\xi,\tilde f(\Delta_\omega)\eta\rangle + \big(\operatorname{Re}\langle\xi,\tilde f(\Delta_\omega)\eta\rangle\big)^2 \\
&\overset{\text{(b)}}{=} \frac{1}{2}\int_0^\infty (s+1)\,d\mu_{\xi\xi}(s)\int_0^\infty \tilde f(t)\,d\mu_{\eta\eta}(t) \\
&\quad + \frac{1}{2}\int_0^\infty \tilde f(s)\,d\mu_{\xi\xi}(s)\int_0^\infty (t+1)\,d\mu_{\eta\eta}(t) \\
&\quad - \int_0^\infty \tilde f(s)\,d\mu_{\xi\xi}(s)\int_0^\infty \tilde f(t)\,d\mu_{\eta\eta}(t) \\
&\quad - \frac{1}{2}\int_0^\infty (s+1)\,d\mu_{\xi\eta}(s)\int_0^\infty \tilde f(t)\,d\mu_{\xi\eta}(t) \\
&\quad - \frac{1}{2}\int_0^\infty \tilde f(s)\,d\mu_{\xi\eta}(s)\int_0^\infty (t+1)\,d\mu_{\xi\eta}(t) \\
&\quad - \int_0^\infty \tilde f(s)\,d\mu_{\xi\eta}(s)\int_0^\infty \tilde f(t)\,d\mu_{\xi\eta}(t) \\
&\overset{\text{(c)}}{=} \frac{1}{2}\int_{[0,\infty)^2} ((s+1)\tilde f(t) + (t+1)\tilde f(s) - 2\tilde f(s)\tilde f(t))\,d\mu_{\xi\xi}\otimes\mu_{\eta\eta}(s,t) \\
&\quad - \frac{1}{2}\int_{[0,\infty)^2} ((s+1)\tilde f(t) + (t+1)\tilde f(s) - 2\tilde f(s)\tilde f(t))\,d\mu_{\xi\eta}\otimes\mu_{\xi\eta}(s,t) \\
&\overset{\text{(d)}}{=} \frac{1}{4}\iint_{[0,\infty)^2} ((s+1)\tilde f(t) + (t+1)\tilde f(s) - 2\tilde f(s)\tilde f(t))\,d\mu(s,t),
\end{aligned}
$$

where we used in (b) notation as in Lemma 3.8, in (c) Fubini–Tonelli theorem, and in (d) the symmetries of the first integrand and notation as in Lemma 3.8.

Since μ is a positive measure, and

$$(s+1)\tilde f(t) + (t+1)\tilde f(s) - 2\tilde f(s)\tilde f(t) = (s+1-\tilde f(s))\tilde f(t) + (t+1-\tilde f(t))\tilde f(s) \geq 0,$$

we obtain $H(\xi,\eta) \geq 0$, for any $\xi,\eta \in \mathcal{D}(\Delta_\omega)$.

It follows from Lemma 3.2 that, for any $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, we have $H(\xi,\eta) = \lim_{n\to\infty} H(\xi_n,\eta_n) \geq 0$, which ends the proof. □

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