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INEQUALITIES FOR QUANTUM FISHER INFORMATION

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ABSTRACT. An inequality relating the Wigner-Yanase information and the SLD-quantum Fisher information was established by Luo (*Proc. Amer. Math. Soc.*, 132, pp. 885–890, 2004). In this paper, we generalize Luo's inequality to any regular quantum Fisher information. Moreover, we show that this general inequality can be derived from the Kubo-Ando inequality, which states that any matrix mean is greater than the harmonic mean and smaller than the arithmetic mean.

1. INTRODUCTION

Fisher information appeared for the first time in [3]. From that seminal work the use of Fisher information spread out, not only in statistics, but also in other mathematical fields, and in a number of applied sciences [4]. Several quantum versions of Fisher information have been studied. Among the first examples one has the Wigner-Yanase information (see [24] or [6], [7], [8], [9] for a recent treatment) and the SLD-information (see [1], [23], [13]) that are defined as follows. As usual $[\cdot,\cdot]$ denotes the commutator. Let ρ be a density matrix and let A be a self-adjoint matrix. Let L be the solution of the operator equation $(L\rho+\rho L)=2i[\rho,A]$. Define the Wigner-Yanase and the SLD-information as

$$I_\rho^{WY}(A) := -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{2}},A]^2), \qquad I_\rho^{SLD}(A) := \frac{1}{4}\mathrm{Tr}(\rho L^2). \tag{1.1}$$

In the paper [16] Luo proved the following three results.

i) If $\rho(t) := e^{-itA}\rho e^{itA}$, the functions of t given by $I_{\rho(t)}^{WY}(A), I_{\rho(t)}^{SLD}(A)$ are constant (this is Theorem 1 in [16]).

ii) The following inequality is true (this is Theorem 2 in [16]):

$$I_\rho^{WY}(A) \le I_\rho^{SLD}(A) \le 2I_\rho^{WY}(A). \tag{1.2}$$

iii) The constant 2 is optimal in the inequality (1.2). Namely, if $1 \le k < 2$, the inequality

$$I_\rho^{SLD}(A) \le kI_\rho^{WY}(A)$$

is false and a counterexample can be found in the elementary 2×2 case (this is the final Example in [16]).

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A full quantum theory for Fisher information was established only a few years ago by Petz in his classification theorem [19]. It is worth noting that the Petz theorem rests on two fundamental breakthroughs due to Rao and Chentsov. Rao observed that Fisher information should be seen as a Riemannian metric on statistical models [22]. Chentsov characterized Fisher information as the unique (in the appropriate setting) Riemannian metric contracting under coarse graining [2].

Starting from this idea, Petz defined the quantum Fisher information (QFI) as Riemannian metrics (on the state manifold) contracting under coarse graining. He was able to prove that QFI are parametrized by functions $f \in \mathcal{F}_{op}$, where $\mathcal{F}_{op}$ is the set of symmetric normalized operator monotone functions. The regular elements of $\mathcal{F}_{op}$ are those for which $f(0) > 0$. The corresponding QFI is said to be regular also. For regular QFI one can define the metric adjusted skew information (or f-information) as

$$I^f_\rho(A) := \frac{f(0)}{2}||i[\rho, A]||^2_{\rho,f}$$

(see [11], [5]). The WY- and SLD-information, defined in (1.1), are particular cases of the above definition.

In this paper we show that the three results proved by Luo are particular cases of the following general results.

i′) Set $\rho_H(t) := e^{-itH}\rho e^{itH}$. If $[A, H] = 0$, then the function $I^f_{\rho_H(t)}(A)$ is constant. Since quantum Fisher information contracts under coarse graining, it is unitary covariant, and this is the crucial ingredient of the proof. This result was stated by Hansen in [11], and we provide here a detailed proof.

ii′) The inequality (1.2) is a particular case of the following inequality:

$$I^f_\rho(A) \leq I^{SLD}_\rho(A) \leq \frac{1}{2f(0)} I^f_\rho(A), \tag{1.3}$$

which is true for any (regular) quantum Fisher information. Inequality (1.3) is a consequence of the Kubo-Ando inequality

$$2(A^{-1} + B^{-1})^{-1} \leq m(A, B) \leq \frac{A + B}{2},$$

which states that any matrix mean is greater than the harmonic mean and smaller than the arithmetic mean.

iii′) The constant $\frac{1}{2f(0)}$ is optimal in inequality (1.3). Namely, if $1 \leq k < \frac{1}{2f(0)}$, then the inequality

$$I^{SLD}_\rho(A) \leq k I^f_\rho(A)$$

is false and a counterexample can be found in the elementary 2×2 case.

Let us observe that in the papers [15], [17] Luo also proved another inequality for the WY and SLD information, namely

$$I^{WY}_\rho(A) \leq \mathrm{Var}_\rho(A), \qquad I^{SLD}_\rho(A) \leq \mathrm{Var}_\rho(A). \tag{1.4}$$

From inequalities (1.3) and (1.4) one immediately obtains that this result is also completely general, namely

$$I^f_\rho(A) \leq \mathrm{Var}_\rho(A),$$

a result recently proved by Hansen in [11] and with a different approach by the authors in [5].

2. Operator monotone functions, matrix means and quantum Fisher information

Let $M_n := M_n(\mathbb{C})$ (resp. $M_{n,sa} := M_n(\mathbb{C})_{sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). We shall denote general matrices by $X, Y, ...$ while the letters $A, B, ...$ will be used for self-adjoint matrices (the Hilbert-Schmidt scalar product is denoted by $\langle A, B \rangle = \mathrm{Tr}(A^* B)$). The adjoint of a matrix X is denoted by $X^\dagger$ while the adjoint of a superoperator $T : (M_n, \langle \cdot, \cdot \rangle) \to (M_n, \langle \cdot, \cdot \rangle)$ is denoted by T^*. Let $\mathcal{D}_n$ be the set of strictly positive elements of M_n while $\mathcal{D}_n^1 \subset \mathcal{D}_n$ is the set of strictly positive density matrices, namely $\mathcal{D}_n^1 = \{\rho \in M_n | \mathrm{Tr}\rho = 1,\, \rho > 0\}$. If it is not specified from now on, we treat the case of faithful states, namely $\rho > 0$.

Definition 2.1. Suppose that $\rho \in \mathcal{D}_n^1$ is fixed. Define $X_0 := X - \mathrm{Tr}(\rho X) I$.

Definition 2.2. For $A, B \in M_{n,sa}$ and $\rho \in \mathcal{D}_n^1$ define covariance and variance as

$$\mathrm{Cov}_\rho(A, B) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A) \cdot \mathrm{Tr}(\rho B) = \mathrm{Tr}(\rho A_0 B_0),$$
$$\mathrm{Var}_\rho(A) := \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho A)^2 = \mathrm{Tr}(\rho A_0^2).$$

Let $\mathbb{R}^+ := (0, \infty)$. A function $f : \mathbb{R}^+ \to \mathbb{R}$ is said to be *operator monotone (increasing)* if, for any $n \in \mathbb{N}$, any A, $B \in M_n$ such that $0 \le A \le B$, the inequalities $0 \le f(A) \le f(B)$ hold. An operator monotone function is said to be *symmetric* if $f(x) = x f(x^{-1})$ and *normalized* if $f(1) = 1$.

Definition 2.3. $\mathcal{F}_{op}$ is the class of functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
(i') $f(1) = 1$,
(ii') $t f(t^{-1}) = f(t)$,
(iii') f is operator monotone.

Example 2.4. Two important elements of $\mathcal{F}_{op}$ are

$$f_{WY}(x) := \left(\frac{1 + \sqrt{x}}{2} \right)^2, \qquad f_{SLD}(x) = \frac{1 + x}{2}.$$

We now review briefly the Kubo-Ando theory of matrix means (see [14]) as exposed in [21].

Definition 2.5. A *mean* for pairs of positive matrices is a function $m : \mathcal{D}_n \times \mathcal{D}_n \to \mathcal{D}_n$ such that
(i) $m(A, A) = A$,
(ii) $m(A, B) = m(B, A)$,
(iii) $A < B \implies A < m(A, B) < B$,
(vi) $A < A', \quad B < B' \implies m(A, B) < m(A', B')$,
(v) m is continuous,
(vi) $C m(A, B) C^* \le m(CAC^*, CBC^*)$, for every $C \in M_n$.

Property (vi) is known as the transformer inequality. We denote by $\mathcal{M}_{op}$ the set of matrix means. The fundamental result, due to Kubo and Ando, is the following.

Theorem 2.6. *There exists a bijection between* $\mathcal{M}_{op}$ *and* $\mathcal{F}_{op}$ *given by the formula*

$$m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

When A and B commute (for example if $A = x, B = y$ are positive numbers) we have that

$$m_f(A,B) := A \cdot f(BA^{-1}).$$

Example 2.7. The arithmetic, geometric and harmonic (matrix) means are given respectively by

$$\begin{aligned} m_{\mathcal{A}}(A,B) &:= A \nabla B := \frac{1}{2}(A+B), \\ m_{\mathcal{G}}(A,B) &:= A \# B := A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}, \\ m_{\mathcal{H}}(A,B) &:= A!B := 2(A^{-1}+B^{-1})^{-1}. \end{aligned}$$

Any convex combination of two means is still a mean (see [14]). Kubo and Ando [14] proved that, among matrix means, the arithmetic is the largest while the harmonic is the smallest.

Corollary 2.8. *For any $f \in \mathcal{F}_{op}$ and for any $x, y > 0$ one has*

$$\begin{aligned} f_{RLD}(x) := \frac{2x}{1+x} \leq f(x) \leq \frac{1+x}{2}, \\ \frac{2xy}{x+y} \leq m_f(x,y) \leq \frac{x+y}{2}. \end{aligned}$$

In what follows if $\mathcal{N}$ is a differential manifold we denote by $T_\rho \mathcal{N}$ the tangent space to $\mathcal{N}$ at the point $\rho \in \mathcal{N}$. Recall that there exists a natural identification of $T_\rho \mathcal{D}_n^1$ with the space of self-adjoint traceless matrices; namely, for any $\rho \in \mathcal{D}_n^1$,

$$T_\rho \mathcal{D}_n^1 = \{A \in M_n | A = A^* \, , \, \mathrm{Tr}(A) = 0\}.$$

A Markov morphism is a completely positive and trace-preserving operator $T : M_n \to M_m$. A monotone metric (also called a quantum Fisher information) is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

holds for every Markov morphism $T : M_n \to M_m$, for every $\rho \in \mathcal{D}_n^1$ and for every $X \in T_\rho \mathcal{D}_n^1$. Usually monotone metrics are normalized in such a way that $[A, \rho] = 0$ implies $g_{f,\rho}(A, A) = \mathrm{Tr}(\rho^{-1} A^2)$.

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$, and observe that they are commuting self-adjoint superoperators on $M_{n,sa}$. Now we can state the fundamental theorems about monotone metrics.

Theorem 2.9 (see [19])**.** *There exists a bijective correspondence between monotone metrics (quantum Fisher information) on $\mathcal{D}_n^1$ and normalized symmetric operator monotone functions $f \in \mathcal{F}_{op}$. This correspondence is given by the formula*

$$\langle A, B \rangle_{\rho,f} := \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

We set $||A||^2_{\rho,f} := \langle A, A \rangle_{\rho,f}$.

Proposition 2.10.

$$||A||_{\rho, f_{SLD}} \leq ||A||_{\rho,f} \leq ||A||_{\rho, f_{RLD}}.$$

Proof. This is an immediate consequence of Corollary 2.8. ∎

Proposition 2.11 (See [19], p. 83)**.** *Monotone metrics are unitarily covariant; namely, if U is unitary, then*

$$||U^*AU||^2_{U^*\rho U,f} = ||A||^2_{\rho,f}.$$

3. The function $\tilde{f}$ and the f-information

For $f \in \mathcal{F}_{op}$ define $f(0) := \lim_{x\to 0} f(x)$. The condition $f(0) \neq 0$ is relevant because it is a necessary and sufficient condition for the existence of the so-called radial extension of a monotone metric to pure states (see [20]). Following [11] we say that a function $f \in \mathcal{F}_{op}$ is *regular* iff $f(0) \neq 0$. The corresponding operator mean, associated QFI, etc., are said to be regular too.

Definition 3.1.

$$\mathcal{F}^r_{op} := \{f \in \mathcal{F}_{op} \mid f(0) \neq 0\}, \quad \mathcal{F}^n_{op} := \{f \in \mathcal{F}_{op} \mid f(0) = 0\}.$$

Trivially one has $\mathcal{F}_{op} = \mathcal{F}^r_{op} \dot{\cup} \mathcal{F}^n_{op}$.

Definition 3.2. For $f \in \mathcal{F}^r_{op}$ and $x > 0$ set

$$\tilde{f}(x) := \frac{1}{2}\left[(x+1) - (x-1)^2\frac{f(0)}{f(x)}\right].$$

Example 3.3.

$$\tilde{f}_{WY}(x) = \sqrt{x}, \qquad \tilde{f}_{SLD}(x) = \frac{2x}{1+x}.$$

Observe [5] that $f \in \mathcal{F}^r_{op}$ implies $\tilde{f} \in \mathcal{F}^n_{op}$.

A self-adjoint operator A determines the evolution of the state ρ by the formula $\rho_A(t) := e^{-iAt}\rho e^{iAt}$. The evolution satisfies the equation $\dot{\rho}_A(t) = i[\rho_A(t), A]$. We set

$$\dot{\rho}_A := \dot{\rho}_A(0) = i[\rho, A].$$

Observe that $L := 2(L_\rho + R_\rho)^{-1}(i[\rho, A])$ can be seen as a quantum analogue of the symmetric logarithmic derivative (see [16]).

Definition 3.4.

$$I^{WY}_\rho(A) := -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{2}}, A]^2), \qquad I^{SLD}_\rho(A) := \frac{1}{4}\mathrm{Tr}\big(\rho L^2\big).$$

Proposition 3.5.

$$I^{WY}_\rho(A) = \frac{f_{WY}(0)}{2}||\dot{\rho}_A||^2_{\rho,f_{WY}}, \qquad I^{SLD}_\rho(A) = \frac{f_{SLD}(0)}{2}||\dot{\rho}_A||^2_{\rho,f_{SLD}}.$$

Proof. For the first equality see [12] or [6], [11]. For the second equality remember that $f_{SLD}(x) := \frac{1+x}{2}$.

Therefore one has

$$\begin{aligned}
I^{SLD}_\rho(A) &= \mathrm{Tr}\big(\rho(L_\rho + R_\rho)^{-1}(i[\rho, A])(L_\rho + R_\rho)^{-1}(i[\rho, A])\big) \\
&= \frac{1}{2}\mathrm{Tr}\big((L_\rho + R_\rho)(L_\rho + R_\rho)^{-1}(\dot{\rho}_A)(L_\rho + R_\rho)^{-1}(\dot{\rho}_A)\big) \\
&= \frac{1}{4}\mathrm{Tr}(2(L_\rho + R_\rho)^{-1}(\dot{\rho}_A)(\dot{\rho}_A)) \\
&= \frac{f_{SLD}(0)}{2}\mathrm{Tr}(m_{SLD}(L_\rho, R_\rho)^{-1}(\dot{\rho}_A)(\dot{\rho}_A)) \\
&= \frac{f_{SLD}(0)}{2}||\dot{\rho}_A||^2_{\rho,f_{SLD}}.
\end{aligned}$$

□

Definition 3.6. For $f \in \mathcal{F}_{op}^r$ the metric adjusted skew information (or f-information) is defined as

$$I_\rho^f(A) := \frac{f(0)}{2} ||\dot{\rho}_A||^2_{\rho,f}.$$

Of course, if ρ and A commute, then $I_\rho^f(A) = 0$. In what follows the following definition is very important.

Definition 3.7.

$$\mathcal{C}_\rho^f(A_0) := \mathrm{Tr}(m_f(L_\rho, R_\rho)(A_0) \cdot A_0).$$

Observe [5] that $I_\rho^f(A) = \mathrm{Var}_\rho(A) - \mathcal{C}_\rho^{\tilde{f}}(A_0)$. Note that this formula allows us to consider the f-information also for those states which are not faithful.

Definition 3.8. For any state (faithful or not faithful) and for f regular define:

$$I_\rho^f(A) := \mathrm{Var}_\rho(A) - \mathcal{C}_\rho^{\tilde{f}}(A_0).$$

Proposition 3.9 (See [5])**.** *If $g \leq f$, one has*

$$0 \leq \mathcal{C}_\rho^g(A_0) \leq \mathcal{C}_\rho^f(A_0).$$

Moreover if ρ is pure,

$$\mathcal{C}_\rho^g(A_0) = 0.$$

We have immediately the following result.

Proposition 3.10.

$$I_\rho^f(A) \leq \mathrm{Var}_\rho(A)$$

with equality on pure states.

Luo (see [18]) suggested that if one considers the variance as a measure of "uncertainty" of an observable A in the state ρ, then the equality

$$\mathrm{Var}_\rho(A) = I_\rho^f(A) + \mathcal{C}_\rho^{\tilde{f}}(A_0)$$

splits the variance into a "quantum" part ($I_\rho^f(A)$) and a "classical" part ($\mathcal{C}_\rho^{\tilde{f}}(A_0)$).

4. The main results

Theorem 1 in [16] is a particular case of the following result (which was stated by Hansen in [11]).

Theorem 4.1. *If $[A, H] = 0$, then $I^f_{\rho_H(t)}(A) = I_\rho^f(A)$, for all $t \in \mathbb{R}$.*

Proof. Set $U_t := e^{itH}$. Then

$$\rho_H(t) := e^{-itH} \rho e^{itH} = U_t^* \rho U_t.$$

Since $[A, U_t] = 0$ we have (using Proposition 2.11)

$$\begin{aligned} I^f_{\rho_H(t)}(A) &= \frac{f(0)}{2} ||i[\rho_H(t), A]||^2_{\rho_H(t),f} = \frac{f(0)}{2} ||i[U_t^* \rho U_t, A]||^2_{U_t^* \rho U_t, f} \\ &= \frac{f(0)}{2} ||U_t^*(i[\rho, A]) U_t||^2_{U_t^* \rho U_t, f} = \frac{f(0)}{2} ||i[\rho, A]||^2_{\rho,f} = I_\rho^f(A). \end{aligned}$$ □

Proposition 4.2. *If $\tilde{g} \leq \tilde{f}$, one has*

$$I_\rho^f(A) \leq I_\rho^g(A).$$

Proof. This is an immediate consequence of Proposition 3.9. □

Theorem 2 in [16] is a particular case of the following result.

Theorem 4.3. *For any* $f \in \mathcal{F}_{op}^r$*, for any* $\rho \in \mathcal{D}_n^1$ *and for any* $A \in M_{n,sa}$*, we have that*

$$I_\rho^f(A) \le I_\rho^{SLD}(A) \le \frac{1}{2f(0)} I_\rho^f(A).$$

Proof. The first inequality is an immediate consequence of Proposition 4.2, Example 3.3 and Corollary 2.8. The second inequality is a consequence of Proposition 2.10, because we have

$$||\dot\rho_A||_{\rho,f_{SLD}} \le ||\dot\rho_A||_{\rho,f}$$

and therefore

$$\frac{f_{SLD}(0)}{2}||\dot\rho_A||^2_{\rho,f_{SLD}} \le \frac{1}{4}||\dot\rho_A||^2_{\rho,f}$$

so that

$$I_\rho^{f_{SLD}}(A) = \frac{f_{SLD}(0)}{2}||\dot\rho_A||^2_{\rho,f_{SLD}} \le \frac{1}{2f(0)} \cdot \frac{f(0)}{2} \cdot ||\dot\rho_A||^2_{\rho,f} = \frac{1}{2f(0)} \cdot I_\rho^f(A).$$

□

A different proof can be given for the second inequality. It is more complicated but can shed light on Luo's proof and on the optimality of the constant $\frac{1}{2f(0)}$.

Proposition 4.4. *Let* $k \ge 1$*. The following inequalities are equivalent:*

$$(i) \qquad I_\rho^{SLD}(A) \le k \cdot I_\rho^f(A) \qquad \forall A \in M_{n,sa}, \forall \rho \in \mathcal{D}_n^1,$$

$$(ii) \qquad m_{\tilde f} \le \left(1 - \tfrac{1}{k}\right) m_{\mathcal{A}} + \tfrac{1}{k} m_{\mathcal{H}},$$

$$(iii) \qquad f(x) \le 2kf(0) \cdot \tfrac{1+x}{2}, \qquad \forall x > 0.$$

Proof. Let $\{\varphi_i\}$ be a complete orthonormal basis composed of eigenvectors of ρ, and $\{\lambda_i\}$ the corresponding eigenvalues. Set $a_{ij} \equiv \langle A_0\varphi_i|\varphi_j\rangle$. Note that $a_{ij} \neq A_{ij} :=$ the i,j entry of A.

As a consequence of the spectral theorem for commuting self-adjoint operators, one gets the following formulas (see [5]):

$$\mathrm{Var}_\rho(A) = \mathrm{Tr}(\rho A_0^2) = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)a_{ij}a_{ji},$$

$$\mathcal{C}_\rho^{\tilde f}(A_0) = \sum_{i,j} m_{\tilde f}(\lambda_i, \lambda_j)a_{ij}a_{ji}.$$

$(i) \Longleftrightarrow (ii)$.

$$\begin{aligned} k \cdot I^f_\rho(A) - I^{SLD}_\rho(A) &= [k \cdot \mathrm{Var}_\rho(A) - k \cdot \mathcal{C}^{\tilde{f}}_\rho(A_0)] - [\mathrm{Var}_\rho(A) - \mathcal{C}^{\tilde{f}_{SLD}}_\rho(A_0)] \\ &= (k-1)\,\mathrm{Var}_\rho(A) + \mathcal{C}^{\tilde{f}_{SLD}}_\rho(A_0) - k\mathcal{C}^{\tilde{f}}_\rho(A_0) \\ &= (k-1)\sum_{i,j} \frac{1}{2}\cdot(\lambda_i+\lambda_j)a_{ij}a_{ji} + \sum_{i,j} m_{\mathcal{H}}(\lambda_i,\lambda_j)a_{ij}a_{ji} \\ &\quad - k\cdot\sum_{i,j} m_{\tilde{f}}(\lambda_i,\lambda_j)a_{ij}a_{ji} \\ &= k\sum_{i,j}\left[\Big(1-\frac{1}{k}\Big)m_{\mathcal{A}}(\lambda_i,\lambda_j) + \frac{1}{k}m_{\mathcal{H}}(\lambda_i,\lambda_j) - m_{\tilde{f}}(\lambda_i,\lambda_j)\right]|a_{ij}|^2. \end{aligned}$$

Therefore, because of the arbitrariness of both ρ and A, one has that

$$kI^f_\rho(A) - I^{SLD}_\rho(A) \geq 0$$

is equivalent to

$$m_{\tilde{f}} \leq \Big(1-\frac{1}{k}\Big)m_{\mathcal{A}} + \frac{1}{k}m_{\mathcal{H}}.$$

$(ii) \Longleftrightarrow (iii)$. Suppose $x>0$, $x \neq 1$. Then

$$m_{\tilde{f}} \leq \Big(1-\frac{1}{k}\Big)m_{\mathcal{A}} + \frac{1}{k}m_{\mathcal{H}}$$

is equivalent to

$$\tilde{f}(x) \leq \Big(1-\frac{1}{k}\Big)\left(\frac{1+x}{2}\right) + \frac{1}{k}\left(\frac{2x}{x+1}\right) \qquad \forall x>0,$$

which, using the definition of $\tilde{f}$, can be transformed into

$$2kf(0)\cdot\frac{1+x}{2} \geq f(x) \qquad \forall x>0,$$

and this ends the proof. □

Example 4.5. In the case of the Wigner-Yanase metric, one has $f_{WY}(0) = \frac{1}{4}$ and $\tilde{f}_{WY}(x) = \sqrt{x}$. The inequality of Proposition 4.4(ii) (when $k = 2 = \frac{1}{2f_{WY}(0)}$) states that

$$m_{\mathcal{G}} \leq \frac{1}{2}(m_{\mathcal{A}} + m_{\mathcal{H}});$$

that is, the geometric mean is smaller than the "midpoint" between the arithmetic and harmonic means. The calculations used by Luo in the proof of inequality (1.1) can be seen as an application of the above inequality.

We now prove that $\frac{1}{2f(0)}$ is the best constant we can have in Theorem 4.3.

Proposition 4.6. *Let* $1 \leq k \leq \frac{1}{2f(0)}$. *The inequality*

$$I^{SLD}_\rho(A) \leq k\cdot I^f_\rho(A) \qquad \forall A \in M_{n,sa}, \forall\rho\in\mathcal{D}^1_n$$

is false.

Proof. From the hypothesis we get that the inequality

$$f(x) \le 2kf(0) \cdot \frac{1+x}{2} \qquad \forall x > 0$$

cannot be true; otherwise one would have

$$1 = f(1) \le 2kf(0) < 1,$$

which is absurd. From Proposition 4.4 we get the conclusion. □

5. The inequality on the Bloch sphere

As an example we discuss in detail what happens for 2×2 matrices. We show that also in this case the constant $\frac{1}{2f(0)}$ is optimal. The final Example in [16] is a particular case of this discussion.

Recall that the Pauli matrices are the following:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A generic 2×2 density matrix in the Stokes parameterization is written as

$$\rho = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} = \frac{1}{2}(I + x\sigma_1 + y\sigma_2 + z\sigma_3),$$

where $(x, y, z) \in \mathbb{R}^3$, and $x^2 + y^2 + z^2 \le 1$. Let $r := \sqrt{x^2 + y^2 + z^2} \in [0, 1]$. The eigenvalues of ρ are $\lambda_1 = \frac{1-r}{2}$ and $\lambda_2 = \frac{1+r}{2}$.

Proposition 5.1.

$$I_\rho^f(A) = \Big[1 - m_{\tilde{f}}(1-r, 1+r)\Big] \cdot |a_{12}|^2.$$

Proof. We use the notation as in the proof of Proposition 4.4. Observe that

$$\frac{\lambda_i + \lambda_j}{2} - m_{\tilde{f}}(\lambda_i, \lambda_j) = \begin{cases} 0, & i = j, \\ \frac{1}{2} - m_{\tilde{f}}(\lambda_i, \lambda_j), & i \ne j. \end{cases}$$

Therefore

$$\begin{aligned} I_\rho^f(A) &= \sum_{i,j} \left[\frac{\lambda_i + \lambda_j}{2} - m_{\tilde{f}}(\lambda_i, \lambda_j)\right] \cdot |a_{ij}|^2 \\ &= \left[\frac{1}{2} - m_{\tilde{f}}(\frac{1-r}{2}, \frac{1+r}{2})\right] |a_{12}|^2 + \left[\frac{1}{2} - m_{\tilde{f}}(\frac{1+r}{2}, \frac{1-r}{2})\right] |a_{21}|^2 \\ &= \Big[1 - m_{\tilde{f}}(1-r, 1+r)\Big] \cdot |a_{12}|^2. \end{aligned}$$

□

Corollary 5.2. *If $r \ne 0$, then*

$$I_\rho^{SLD}(A) = \left[\frac{r^2}{1 - m_{\tilde{f}}(1-r, 1+r)}\right] \cdot I_\rho^f(A).$$

Proof. If $f_{SLD}(x) = \frac{1+x}{2}$, then $\tilde{f}_{SLD} = \frac{2x}{x+1}$. In this case,

$$m_{\tilde{f}_{SLD}}(1-r, 1+r) = (1+r)\tilde{f}_{SLD}\left(\frac{1-r}{1+r}\right) = 1 - r^2.$$

Therefore, from the above proposition,

$$I_\rho^{SLD}(A) = \Big[1 - m_{\tilde f_{SLD}}(1-r, 1+r)\Big] \cdot |a_{12}|^2 = \big[1-(1-r^2)\big]\cdot |a_{12}|^2 = r^2 \cdot |a_{12}|^2,$$

and this ends the proof. □

Example 5.3. In the case $f_{WY}(x) = \left(\frac{1+\sqrt{x}}{2}\right)^2$ one has $\tilde f_{WY}(x) = \sqrt{x}$. In this case (see [16]),

$$\begin{aligned} I_\rho^{SLD}(A) &= \left[\frac{r^2}{1 - m_{\tilde f_{WY}}(1-r,1+r)}\right] \cdot I_\rho^{WY}(A) \left[\frac{r^2}{1-\sqrt{1-r^2}}\right] \cdot I_\rho^{WY}(A) \\ &= [1+\sqrt{1-r^2}] \cdot I_\rho^{WY}(A). \end{aligned}$$

Remark 5.4. Note that for any regular f the function $\tilde f$ is not regular and therefore

$$\lim_{r\to 1} \frac{r^2}{1 - m_{\tilde f}(1-r,1+r)} = \lim_{r\to 1} \frac{r^2}{1-(1+r)\tilde f\left(\frac{1-r}{1+r}\right)} = \frac{1}{1-\tilde f(0)} = 1.$$

We already know such a result because the case $r = 1$ is that of pure states where any f-information coincides with the variance.

Proposition 5.5. *If f is regular, then*

$$\lim_{r\to 0} \frac{r^2}{1 - m_{\tilde f}(1-r,1+r)} = -\frac{1}{2\tilde f''(1)} = \frac{1}{2f(0)}.$$

Proof. Let $g(r) := 1 - m_{\tilde f}(1-r, 1+r)$. For any $f \in \mathcal{F}_{op}$ one has $f'(1) = \frac{1}{2}$ (because of symmetry), and this implies that $g(0) = g'(0) = 0$. Therefore we have to use twice the l'Hôpital theorem. An easy calculation shows that $\tilde f''(1) = -f(0)$; therefore we get

$$\begin{aligned} \lim_{r\to 0} \frac{r^2}{1 - m_{\tilde f}(1-r,1+r)} &= \lim_{r\to 0} \frac{\frac{\mathrm{d}^2}{\mathrm{d}r^2} r^2}{\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big[1 - m_{\tilde f}(1-r,1+r)\Big]} \\ &= \lim_{r\to 0} \frac{2}{-\frac{4}{(1+r)^3}\tilde f''\left(\frac{1-r}{1+r}\right)} = \frac{2}{-4\tilde f''(1)} = \frac{1}{2f(0)}. \end{aligned}$$

□

From the above proposition we get a different proof of the fact that the constant $\frac{1}{2f(0)}$ is optimal also in the 2×2 matrix case.

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