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On the monotonicity of scalar curvature in classical and quantum information geometry

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We study the monotonicity under mixing of the scalar curvature for the α-geometries on the simplex of probability vectors. From the results obtained and from numerical data, we are led to some conjectures about quantum α-geometries and Wigner–Yanase–Dyson information. Finally, we show that this last conjecture implies the truth of the Petz conjecture about the monotonicity of the scalar curvature of the Bogoliubov–Kubo–Mori monotone metric. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1834693]

I. INTRODUCTION

The Bogoliubov–Kubo–Mori (BKM) metric is a distinguished element among the monotone metrics which are the quantum analog of Fisher information on the quantum state space.[35,36] In a definite sense BKM metric is the geometry on the state space that is related to von Neumann entropy (say, Umegaki relative entropy). Other well-known elements of this family are the right logarithmic derivative (RLD) metric, the symmetric logarithmic derivative (SLD or Bures) metric, and the Wigner–Yanase–Dyson (WYD) metrics. In Ref. 34 Petz made a conjecture on the scalar curvature of the BKM metric. Many arguments and numerical calculations suggest that the conjecture is true; nevertheless, a complete proof is still missing (see Refs. 3, 4, 12, 24, and 32).

One can state this conjecture in the following way: the BKM scalar curvature is a quantitative measure of symmetry (like entropy), namely it is increasing under mixing. Let us emphasize that it is also possible to relate the conjecture to quantities with direct physical meaning. An equivalent formulation, still due to Petz,[34] is that "…the scalar curvature is an increasing function of the temperature…." Moreover, the asymptotic relation between volume and curvature in Riemannian geometry and Jeffrey's approach to priors in statistics induced Petz to interpret the scalar curvature as the average statistical uncertainty (that should increase under coarse graining; see Ref. 36).

The original motivations given by Petz for the conjecture rely on the truth of the 2×2 case and on some numerical results for the general case. Petz and Sudar observed in Ref. 38 that "…monotonicity of Kubo metric is not surprising because this result is a kind of reformulation of Lieb convexity theorem.[30] However the monotonicity of the scalar curvature seems to be an inequality of new type (provided the conjecture is really true)…." A recent clear reference for Lieb's result and related inequalities can be found in the paper by Ruskai.[40]

The goals of the present paper are the following.

(1) We want to look at "higher mathematics from an elementary point of view." This means that we want to furnish an elementary motivation for the Petz conjecture. We do this by studying

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the monotonicity of the curvature for α-geometries in the plane. The results obtained in this case are very intuitive if one looks at the unit sphere of the L^p spaces. We conjecture that a similar behavior occurs for α-geometries in higher dimensions and in the noncommutative case, too.

(2) On the basis of the results of point (1) we make a conjecture about the monotonicity of scalar curvature for the WYD metrics. Further, we show that, using a continuity argument, this WYD conjecture would imply the Petz conjecture as a limit case (Theorem 8.1).

(3) We review what is known about monotonicity of scalar curvature for quantum Fisher information. In particular, we emphasize a result on the Bures metric, attributed to Dittmann, according to which the scalar curvature, in this case, is neither Schur-increasing nor Schur-decreasing (see Sec. II for precise definitions). This implies that an example of a monotone metric for which the scalar curvature (or its opposite) is strictly increasing under mixing does not exist yet. Note that Andai (using an integral decomposition of Ref. 15) proved that also in the 2×2 case there exist monotone metrics whose scalar curvature is not monotone.[3]

Finally, let us note that, related to this area, there exist other interesting papers. Some authors have suggested that, when statistical mechanics is geometrized, then the scalar curvature should have important physical meaning (for example, it should be proportional to the inverse of the free energy; see Refs. 8, 9, 25–27, and 39).

II. MAJORIZATION AND SCHUR-INCREASING FUNCTIONS

For the content of this section we refer to Refs. 1, 5–7, and 31.

A. Commutative case

We shall denote by $\mathcal{P}_n$ the manifold of positive vectors of $\mathbb{R}^n$, and by $\mathcal{P}_n^1 \subset \mathcal{P}_n$ the submanifold of density vectors, namely

Definition 2.1:

$$\mathcal{P}_n^1 := \left\{ \rho \in \mathbb{R}^n \middle| \sum_i \rho_i = 1,\ \rho_i > 0 \right\}.$$

We set $e := (1, \ldots, 1)$. The trace of a vector is $\mathrm{Tr}(v) = \Sigma_{i=1}^n v_i$. For an $n \times n$ real matrix, consider the following properties:

(I) $t_{ij} \geqslant 0 \quad i,j = 1, \ldots, n,$
(II) $\Sigma_{i=1}^n t_{ij} = 1 \quad j = 1, \ldots, n,$
(III) $\Sigma_{j=1}^n t_{ij} = 1 \quad i = 1, \ldots, n.$

Definition 2.2:

(a) *T is said to be stochastic if (I),(II) hold*;
(b) *T is said to be doubly stochastic if (I),(II),(III) hold.*

When T is seen as an operator $T: \mathbb{R}^n \to \mathbb{R}^n$ (by $(Tv)_j = \Sigma_{i=1}^n t_{ji} v_i$), then the properties (I),(II),(III) can be written as

(I)′ (positivity preserving) $Tv \geqslant 0$ if $v \geqslant 0$;
(II)′ (trace-preserving) $\mathrm{Tr}(Tv) = \mathrm{Tr}(v) \quad \forall v \in \mathbb{R}^n$;
(III)′ (unital) $Te = e$.

Let $x \in \mathbb{R}^n$ be a vector. We define $x^{\downarrow}$ as a vector with the same components in a decreasing order so that

$$x_1^{\downarrow} \geqslant x_2^{\downarrow} \geqslant \cdots \geqslant x_n^{\downarrow}.$$

Definition 2.3: x is more mixed (more chaotic,…) than y (denoted by $x \succ y$) if and only if

$$x_1^{\downarrow} \leqslant y_1^{\downarrow},$$

$$x_1^{\downarrow} + x_2^{\downarrow} \leqslant y_1^{\downarrow} + y_2^{\downarrow},$$

$$\cdots,$$

$$x_1^{\downarrow} + \cdots + x_{n-1}^{\downarrow} \leqslant y_1^{\downarrow} + \cdots + y_{n-1}^{\downarrow},$$

$$x_1^{\downarrow} + \cdots + x_n^{\downarrow} = y_1^{\downarrow} + \cdots + y_n^{\downarrow}.$$

For example, if $(\rho_1, \ldots, \rho_n)$ is a density vector then

$$\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) \succ (\rho_1, \ldots, \rho_n) \succ (1,0, \ldots, 0,0).$$

The relation $\succ$ is a preordering but not a partial ordering. If $x \succ y$ and $y \succ x$ then $x = Ty$ for some permutation matrix T.

Theorem 2.1:

$$x \succ y \Leftrightarrow x = Ty \text{ where } T \text{ is doubly stochastic}.$$

Definition 2.4 (see Ref. 31 pp. 14 and 54): A real-valued function f defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-increasing on $\mathcal{A}$ if

$$x \succ y \text{ on } \mathcal{A} \Rightarrow \quad f(x) \geqslant f(y).$$

If, in addition, $f(x) > f(y)$ whenever $x \succ y$ but x is not a permutation of y, then f is said to be strictly Schur-increasing. Similarly f is said to be Schur-decreasing on $\mathcal{A}$ if

$$x \succ y \text{ on } \mathcal{A} \Rightarrow f(x) \leqslant f(y),$$

and f is strictly Schur-decreasing if strict inequality $f(x) < f(y)$ holds when x is not a permutation of y.

Of course, f is Schur-increasing if and only if $-f$ is Schur-decreasing.

Remark 2.1 (see Ref. 31, p. 54): $\mathcal{A} \subset \mathbb{R}^n$ is symmetric if $x \in \mathcal{A} \Rightarrow \Pi x \in \mathcal{A}$ for all permutations Π. A function f is symmetric on $\mathcal{A}$ if $f(x) = f(\Pi x)$ for all permutations Π. Let $\mathcal{D} := \{x | x_1 \geqslant \cdots \geqslant x_n\}$. If f is symmetric on a symmetric set $\mathcal{A}$ and Schur-increasing on $\mathcal{D} \cap \mathcal{A}$, then f is Schur-increasing on $\mathcal{A}$.

Remark 2.2: Let us consider the following identification $I: (0, \pi/2) \to \mathcal{P}_2^1$ defined by $I(\theta) := (\cos^2\theta, \sin^2\theta)$. Evidently, if $\theta_1, \theta_2 \leqslant \pi/4$, then

$$\theta_1 \leqslant \theta_2 \Leftrightarrow I(\theta_1) \prec I(\theta_2).$$

Any function on $\mathcal{P}_2^1$ can be seen as a function on $(0, \pi/2)$. By abuse of language we shall use the same symbols to denote the two functions. Since $\mathcal{P}_2^1$ is symmetric we have, because of Remark 2.1

Proposition 2.2: A symmetric function f on $\mathcal{P}_2^1$ is Schur-increasing if and only if f is increasing as a function on $(0, \pi/4)$.

B. Noncommutative case

Let M_n be the space of complex $n \times n$ matrices. We shall denote by H_n the real subspace of Hermitian matrices, by $\mathcal{D}_n$ the manifold of strictly positive elements of M_n, and by $\mathcal{D}_n^1 \subset \mathcal{D}_n$ the submanifold of density matrices, namely

Definition 2.5:

$$\mathcal{D}_n^1 := \{\rho \in M^n | \mathrm{Tr}\ \rho = 1,\ \rho > 0\}.$$

If $A \in M_n$, let $\lambda(A)$ be the n-vector of its eigenvalues, arranged in any order with multiplicities counted. If A is Hermitian then $\lambda(A)$ is a real n-vector. Let A, B be Hermitian.

Definition 2.6:

$$A \succ B \Leftrightarrow \lambda(A) > \lambda(B).$$

Definition 2.7: A linear map Φ on M_n is doubly stochastic if it is positive-preserving, trace-preserving, and unital.

Theorem 2.3:

$$A \succ B \Leftrightarrow A = \Phi(B), \textit{where } \Phi \textit{ is doubly stochastic}.$$

Definition 2.8: A real-valued function f defined on a set $\mathcal{A} \subset H_n$ is said to be Schur-increasing on $\mathcal{A}$ if

$$A \succ B \Rightarrow f(A) \geqslant f(B).$$

Similarly, f is said to be Schur-decreasing on $\mathcal{A}$ if

$$A \succ B \Rightarrow f(A) \leqslant f(B).$$

Of course, f is Schur-increasing if and only if $-f$ is Schur-decreasing.

III. PULL-BACK OF DUALITY PAIRINGS

To make the paper self-contained, we recall some constructions from Ref. 18.

Let V, W be vector spaces over $\mathbb{R}$ (or $\mathbb{C}$). One can say that there is a duality pairing if there exists a separating bilinear form

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{R}.$$

Let $\mathcal{M}, \mathcal{N}, \tilde{\mathcal{N}}$ be differentiable manifolds.

Definition 3.1: Suppose we have a pair of immersions (φ, χ), where $\varphi : \mathcal{M} \to \mathcal{N}$ and $\chi : \mathcal{M} \to \tilde{\mathcal{N}}$, such that a duality pairing exists between $T_{\varphi(\rho)}\mathcal{N}$ and $T_{\chi(\rho)}\tilde{\mathcal{N}}$ for any $\rho \in \mathcal{M}$. Then, we may pull-back this pairing on $\mathcal{M}$ by defining

$$\langle u, v \rangle_\rho^{\varphi,\chi} := \langle D_\rho \varphi(u), D_\rho \chi(u) \rangle, \quad u, v \in T_\rho \mathcal{M}.$$

The most elementary example is given by the case where $\mathcal{N} = \tilde{\mathcal{N}}$ is a Riemannian manifold, $\varphi = \chi$, and the duality pairing is just given by the Riemannian scalar product on $T_{\varphi(\rho)}\mathcal{N}$ (this is the pull-back metric induced by the map φ).

A nontrivial example is the following. Let X be a uniformly convex Banach space such that the dual $\tilde{X}$ is uniformly convex. We denote by $\langle \cdot, \cdot \rangle$ the standard duality pairing between X and $\tilde{X}$. Let $J : X \to \tilde{X}$ be the duality mapping, that is J is the differential of the map $v \to \frac{1}{2}\|v\|^2$. $J(v)$ is the unique element of the dual such that $\langle v, J(v) \rangle = \|v\|^2 = \|J(v)\|^2$.

Definition 3.2: Let $\mathcal{M}$ be a manifold. If we have a map $\varphi : \mathcal{M} \to X$ we can consider a dualized pull-back that is a bilinear form defined on the tangent space of $\mathcal{M}$ by

$$\langle A, B \rangle_\rho^\varphi := \langle A, B \rangle_\rho^{\varphi, J \circ \varphi} = \langle D_\rho \varphi(A), D_\rho (J \circ \varphi)(B) \rangle.$$

Example 3.1: For X a Hilbert space, J is the identity, and this is again the definition of pull-back metric induced by the map φ.

In what follows, if $p \in \mathbb{R} \setminus \{0\}$ then $\tilde{p}$ is defined by $(1/p) + (1/\tilde{p}) = 1$. If $p = 1$ then $\tilde{p} = +\infty$.

Example 3.2: Let $(X, \mathcal{F}, \mu)$ be a measure space. If f is a measurable function and $p \in (1, +\infty)$ then $\|f\|_p := (\int |f|^p d\mu)^{1/p}$. Set

$$L^p = L^p(X,\mathcal{F},\mu) = \{f \text{ is measurable and } \|f\|_p < \infty\}.$$

Define N^p as L^p with the norm

$$\|f\|_{N^p} := \frac{\|f\|_p}{p}.$$

Obviously $\tilde{N^p}$ (the dual of N^p) can be identified with $N^{\tilde{p}}$.

Now, suppose that $\rho > 0$ is measurable and $\int \rho = 1$, namely ρ is a strictly positive density. Then, $v = p\rho^{1/p}$ is an element of the unit sphere of N^p and it is easy to see that $J(v) = \tilde{p}\rho^{1/p}$. The family of maps $\rho \to p\rho^{1/p}$ is known as Amari embeddings.

Let $X = \{1, \ldots, n\}$ and let μ be the counting measure. In this case N^p is just $\mathbb{R}^n$ with the norm $\|\cdot\|_p / p$.

Proposition 3.1: Consider the Amari embedding $\varphi: \rho \in \mathcal{P}_n^1 \to p\rho^{1/p} \in N^p$ *for an arbitrary* $p \in (1, +\infty)$. *Then, the bilinear form*

$$\langle A,B\rangle_\rho^\varphi := \langle A,B\rangle_\rho^{\varphi, J\circ\varphi} = \langle D_\rho\varphi(A), D_\rho(J\circ\varphi)(B)\rangle, \quad A,B \in T_\rho\mathcal{P}_n^1,$$

is just the Fisher information.

Proof:

$$\langle D_\rho\varphi(A), D_\rho(J\circ\varphi)(B)\rangle = \int (\rho^{(1/p)-1}A)(\rho^{(1/p)-1}B) = \int \frac{AB}{\rho}.$$

□

The above result can be stated in much greater generality using the machinery of Refs. 19 and 14.

IV. SCALAR CURVATURE OF α-GEOMETRIES

The α-geometries are one of the fundamental objects of information geometry (see Refs. 2 and 20). The study of the monotonicity of their curvatures does not appear in the literature as far as we know. In this section we start such an investigation.

A. The plane case

Definition 4.1: The α-*geometry on* $\mathcal{P}_2^1$ *is the pull-back geometry induced by the map* $A_p(\rho): \mathcal{P}_2^1 \to \mathbb{R}^2$ *defined by*

$$A_p(\rho) := \begin{cases} p\rho^{1/p}, & p \in \mathbb{R}\setminus\{0\}, \\ \log(\rho), & p = \infty, \end{cases}$$

where $p = 2/(1-\alpha)$.

Definition 4.2: We denote by $c_p(\rho)$ *the curvature of the* α-*geometry* [*with* $p = 2/(1-\alpha)$] *at the point* $\rho \in \mathcal{P}_2^1$.

Remark 4.1: For the curvature $c_p(\cdot)$ there are two easy cases:

—if $p = 1$ then $c_p(\cdot) = \text{const} = 0$;

—if $p = 2$ then $c_p(\cdot) = \text{const} = \frac{1}{2}$.

Taking a look at the unit sphere of $\mathbb{R}^2$ with respect to the L^p-norm, one can easily understand the following general result.

Theorem 4.1: *For the function* $c_p(\cdot): \mathcal{P}_2^1 \to R$ *one has the following properties*:

—*if* $p \in (1,2)$ *then* $c_p(\cdot)$ *is a strictly Schur-decreasing function*;

—*if* $p \in (2, +\infty]$ *then* $c_p(\cdot)$ *is a strictly Schur-increasing function.*

Proof: Let us first consider $p \in (1,\infty)$. Then, the α-geometry, $\alpha = (p-2)/p$, on $\mathcal{P}_2^1$ is the geometry of the set

$$\mathcal{B}:=\left\{(x,y)\in\mathbb{R}^2:\left(\frac{x}{p}\right)^p+\left(\frac{y}{p}\right)^p=1,\quad x>0,\quad y>0\right\}.$$

Let us introduce the parametrization

$$x=p(\cos\ \vartheta)^{2/p},\quad y=p(\sin\ \vartheta)^{2/p},\quad 0<\vartheta<\frac{\pi}{2}.$$

Then

$$x'=2(\cos\ \vartheta)^{(2/p)-1}(-\sin\ \vartheta),\quad y'=2(\sin\ \vartheta)^{(2/p)-1}\cos\ \vartheta,$$

$$x''=2(\cos\ \vartheta)^{(2/p)-2}\left(\frac{2}{p}\sin^2\ \vartheta-1\right),\quad y''=2(\sin\ \vartheta)^{(2/p)-2}\left(\frac{2}{p}\cos^2\ \vartheta-1\right).$$

Let us parametrize density vectors as $(\cos^2\ \theta,\sin^2\ \vartheta)$. In this way the curvature of α-geometry at the point ρ, namely $c_p(\rho)$, is

$$\begin{aligned}c_p(\vartheta):&=\frac{|x'y''-x''y'|}{[(x')^2+(y')^2]^{3/2}}=\frac{p-1}{p}\frac{(\sin\ \vartheta\cos\ \vartheta)^{(2/p)+2}}{[(\sin\ \vartheta(\cos\ \vartheta)^{1/p})^4+((\sin\ \vartheta)^{1/p}\cos\ \vartheta)^4]^{3/2}}\\&=\frac{p-1}{p}\left(\frac{1}{2}\right)^{2[1-(2/p)]}\cdot\frac{(\sin 2\vartheta)^{2[1-(2/p)]}}{[(\cos\ \vartheta)^{4/\tilde{p}}+(\sin\ \vartheta)^{4/\tilde{p}}]^{3/2}}=A_p\cdot\frac{g_p(\vartheta)}{f_p(\vartheta)^{\frac{3}{2}}},\end{aligned}$$

where we set

$$A_p:=\frac{p-1}{p}\left(\frac{1}{2}\right)^{2[1-(2/p)]},$$

$$g_p(\vartheta):=(\sin 2\vartheta)^{2-(4/p)},$$

$$f_p(\vartheta):=(\cos\ \vartheta)^{4/\tilde{p}}+(\sin\ \vartheta)^{4/\tilde{p}}.$$

We want to compute the monotonicity properties of c_p with respect to the preordering $>$. We have

$$g_p'(\vartheta):=4(\sin 2\vartheta)^{1-(4/p)}\cdot(\cos\ \vartheta+\sin\ \vartheta)\left(1-\frac{2}{p}\right)(\cos\ \vartheta-\sin\ \vartheta);$$

since $0<\vartheta<\pi/2$, then

$$4(\sin 2\vartheta)^{1-(4/p)}\cdot(\cos\ \vartheta+\sin\ \vartheta)>0,$$

and therefore

$$g_p'(\vartheta)>0\Leftrightarrow\left(1-\frac{2}{p}\right)(\cos\ \vartheta-\sin\ \vartheta)>0.$$

Moreover

$$f_p'(\vartheta)=\frac{4}{\tilde{p}}\sin\ \vartheta\cos\ \vartheta((\sin\ \vartheta)^{(2/\tilde{p})-1}+(\cos\ \vartheta)^{(2/\tilde{p})-1})((\sin\ \vartheta)^{(2/\tilde{p})-1}-(\cos\ \vartheta)^{(2/\tilde{p})-1});$$

again, since $0<\vartheta<\pi/2$, then

$$4\sin\ \vartheta\cos\ \vartheta((\sin\ \vartheta)^{(2/\tilde{p})-1}+(\cos\ \vartheta)^{(2/\tilde{p})-1})>0,$$

and therefore

$$f_p'(\vartheta) > 0 \Leftrightarrow \frac{1}{\tilde{p}}((\sin\ \vartheta)^{(2/\tilde{p})-1} - (\cos\ \vartheta)^{(2/\tilde{p})-1}) > 0.$$

$c_p(\cdot)$ is evidently symmetric on $\mathcal{P}_2^1$ and therefore (because of Proposition 2.2) the fact that the curvature is strictly Schur-increasing (decreasing) is equivalent to the fact that $c_p(\vartheta)$ is strictly increasing (decreasing) for $0<\vartheta<\pi/4$.

We have the following cases:

Case: $1<p<2$.

This implies $1-(2/p)<0, (2/\tilde{p})-1<0$, and therefore

$$g_p'(\vartheta) > 0 \Leftrightarrow \cos\ \vartheta < \sin\ \vartheta \Leftrightarrow \frac{\pi}{4} < \vartheta < \frac{\pi}{2},$$

$$f_p'(\vartheta) > 0 \Leftrightarrow (\sin\ \vartheta)^{(2/\tilde{p})-1} > (\cos\ \vartheta)^{(2/\tilde{p})-1} \Leftrightarrow \sin\ \vartheta < \cos\ \vartheta \Leftrightarrow 0 < \vartheta < \frac{\pi}{4}.$$

Therefore, for $0<\vartheta<\pi/4$, g is decreasing, f is increasing, and $1/f^{3/2}$ is decreasing. This implies that

$$c_p = A_p \frac{g_p}{f_p^{3/2}}$$

is strictly decreasing for $0<\vartheta<\pi/4$.

Case: $2<p<\infty$.

This implies $1-(2/p)>0, (2/\tilde{p})-1>0$, and therefore

$$g_p'(\vartheta) > 0 \Leftrightarrow \cos\ \vartheta > \sin\ \vartheta \Leftrightarrow 0 < \vartheta < \frac{\pi}{4},$$

$$f_p'(\vartheta) > 0 \Leftrightarrow (\sin\ \vartheta)^{(2/\tilde{p})-1} > (\cos\ \vartheta)^{(2/\tilde{p})-1} \Leftrightarrow \sin\ \vartheta > \cos\ \vartheta \Leftrightarrow \frac{\pi}{4} < \vartheta < \frac{\pi}{2}.$$

Therefore, for $0<\vartheta<\pi/4$, g is increasing, f is decreasing, and $1/f^{3/2}$ is increasing. This implies that

$$c_p = A_p \frac{g_p}{f_p^{3/2}}$$

is strictly increasing for $0<\vartheta<\pi/4$.

Case: $p=\infty$.

Use now the following parametrization:

$$x = 2\ \log(\cos\ \theta), \quad y = 2\ \log(\sin\ \theta),$$

for the curve $e^x+e^y=1$. Then

$$x' = -2\frac{\sin\ \theta}{\cos\ \theta}, \quad y' = 2\frac{\cos\ \theta}{\sin\ \theta},$$

$$x'' = \frac{-2}{\cos^2\ \theta}, \quad y'' = \frac{-2}{\sin^2\ \theta},$$

$$c_\infty(\vartheta) := \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{3/2}} = \frac{(\sin\ \theta \cos\ \theta)^2}{[(\cos\ \theta)^4 + (\sin\ \theta)^4]^{3/2}}$$
$$= \lim_{p\to+\infty} \frac{p-1}{p}\left(\frac{1}{2}\right)^{2[1-(2/p)]} \cdot \frac{(\sin 2\vartheta)^{2[1-(2/p)]}}{[(\cos\ \vartheta)^{4/\tilde{p}} + (\sin\ \vartheta)^{4/\tilde{p}}]^{3/2}}.$$

Note that

$$c_\infty(\vartheta) = \lim_{p\to+\infty} c_p(\theta).$$

If we set

$$g_\infty(\vartheta) := (\sin\ \theta \cos\ \theta)^2, \quad f_\infty(\vartheta) := (\cos\ \theta)^4 + (\sin\ \theta)^4,$$

then

$$g'_\infty(\vartheta) = 2 \sin\ \vartheta \cos\ \vartheta(\cos\ \vartheta + \sin\ \vartheta)(\cos\ \vartheta - \sin\ \vartheta),$$

$$f'_\infty(\vartheta) = 4 \sin\ \vartheta \cos\ \vartheta(\cos\ \vartheta + \sin\ \vartheta)(\sin\ \vartheta - \cos\ \vartheta).$$

This implies

$$g'_\infty(\vartheta) > 0 \Leftrightarrow \cos\ \vartheta > \sin\ \vartheta,$$

$$f'_\infty(\vartheta) > 0 \Leftrightarrow \sin\ \vartheta > \cos\ \vartheta.$$

We have the same situation of the case $2<p<\infty$ and therefore the same conclusion.

This ends the proof. □

Note that we have also

Proposition 4.2: For the function $c_p(\cdot):\mathcal{P}_2^1 \to \mathbb{R}$ *one has the following properties: if* $p \in (-\infty, 0)$ *then* $c_p(\cdot)$ *is strictly Schur-increasing.*

Proof: Since

$$1 - \frac{2}{p} > 0, \quad \frac{2}{\tilde{p}} - 1 > 1 > 0, \quad 0 < \tilde{p} < 1,$$

we have the same situation of the case $2<p<\infty$ in the preceding Theorem 4.1 and therefore the same conclusion. □

If $p \in (0,1)$ then $c_p(\cdot)$ can have an arbitrary behavior (Schur-increasing, Schur-decreasing, neither of the two).

B. The general case

Definition 4.3: The α*-geometry on* $\mathcal{P}_n^1$ *is the pull-back geometry induced by the map* $A_p(\rho):\mathcal{P}_n^1 \to \mathbb{R}^n$ *defined by*

$$A_p(\rho) := \begin{cases} p\rho^{1/p}, & p \in \mathbb{R}\setminus\{0\}, \\ \log(\rho), & p = \infty, \end{cases}$$

where $p = 2/(1-\alpha)$.

Definition 4.4: We denote by $\mathrm{Scal}_p(\rho)$ *the scalar curvature of the* α*-geometry* [*with* $p = (2/(1-\alpha))$] *at the point* $\rho \in \mathcal{P}_n^1$.

Of course the cases $p=1$ (flat geometry) and $p=2$ (geometry of a $(n-1)$-dimensional sphere with radius 2) are easy to study. One has

—if $p=1$ then $\mathrm{Scal}_p(\cdot) = \mathrm{const} = 0$;

—if $p=2$ then $\mathrm{Scal}_p(\cdot) = \mathrm{const} = \frac{1}{4}(n-1)(n-2)$.

Again taking a look at the unit sphere of $\mathbb{R}^n$ equipped with L^p-norm, one can easily understand the following conjecture.

Conjecture 4.1: Suppose $n>2$. *For the function* $\mathrm{Scal}_p(\cdot):\mathcal{P}_n^1\to R$ *one has the following properties:*

—*if* $p\in(1,2)$ *then* $\mathrm{Scal}_p(\cdot)$ *is a strictly Schur-decreasing function*;

—*if* $p\in(2,+\infty]$ *then* $\mathrm{Scal}_p(\cdot)$ *is a strictly Schur-increasing function*.

C. Non-commutative case

Definition 4.5: The α*-geometry on* $\mathcal{D}_n^1$ *is the geometry induced by the pull-back of the map* $A_p(\rho):\mathcal{D}_n^1\to M_n$ *defined by*

$$A_p(\rho):=\begin{cases}p\rho^{1/p}, & p\in\mathbb{R}\setminus\{0\},\\ \log(\rho), & p=\infty,\end{cases}$$

where $p=2/(1-\alpha)$.

Since the commutativity or noncommutativity of the context will always be clear, we perform a little abuse of language in the following definition.

Definition 4.6: We denote by $\mathrm{Scal}_p(\rho)$ *the scalar curvature of the* α*-geometry* [*with* $p=(2/(1-\alpha))$] *at the point* $\rho\in\mathcal{D}_n^1$.

Again, the case $p=1$ (flat geometry) is obvious. The case $p=2$ is known (see Refs. 16 and 17 or Theorem 7.2 below), and we have

—if $p=1$ then $\mathrm{Scal}_p(\cdot)=\mathrm{const}=0$;

—if $p=2$ then $\mathrm{Scal}_p(\cdot)=\mathrm{const}=\frac{1}{4}(n^2-1)(n^2-2)$.

Motivated by the commutative plane case we formulate the following conjecture.

Conjecture 4.2: Suppose $n\geqslant 2$. *For the function* $\mathrm{Scal}_p(\cdot):\mathcal{D}_n^1\to\mathbb{R}$ *one has the following properties:*

—*if* $p\in(1,2)$ *then* $\mathrm{Scal}_p(\cdot)$ *is a strictly Schur-decreasing function*;

—*if* $p\in(2,+\infty]$ *then* $\mathrm{Scal}_p(\cdot)$ *is a strictly Schur-increasing function*.

V. MONOTONE METRICS AND THEIR SCALAR CURVATURES

A commutative Markov morphism $T:\mathbb{R}^n\to\mathbb{R}^m$ is a stochastic map. A noncommutative Markov morphism is a linear map $T:M_n\to M_m$ that is completely positive and trace-preserving (note that in the commutative case complete positivity is equivalent to positivity; see, for example, Ref. 41).

In the commutative case a monotone metric is a family of Riemannian metrics $g=\{g^n\}$ on $\{\mathcal{P}_n^1\}$, $n\in\mathbb{N}$ such that

$$g^m_{T(\rho)}(TX,TX)\leqslant g^n_\rho(X,X)$$

holds for every Markov morphism $T:\mathbb{R}^n\to\mathbb{R}^m$ and all $\rho\in\mathcal{P}_n^1$ and $X\in T_\rho\mathcal{P}_n$.

In perfect analogy, a monotone metric in the noncommutative case is a family of Riemannian metrics $g=\{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n\in\mathbb{N}$ such that

$$g^m_{T(\rho)}(TX,TX)\leqslant g^n_\rho(X,X)$$

holds for every Markov morphism $T:M_n\to M_m$ and all $\rho\in\mathcal{D}_n^1$ and $X\in T_\rho\mathcal{D}_n^1$.

Let us recall that a function $f:(0,\infty)\to\mathbb{R}$ is called an operator monotone if for any $n\in\mathbb{N}$, any A, $B\in M_n$ such that $0\leqslant A\leqslant B$, the inequalities $0\leqslant f(A)\leqslant f(B)$ hold. An operator monotone function is said to be symmetric if $f(x)=xf(x^{-1})$ and normalized if $f(1)=1$. In what follows, by operator monotone we mean normalized symmetric operator monotone. With each operator monotone function f one associates also the so-called Chentsov–Morotzova function

$$c_f(x,y) := \frac{1}{yf\left(\frac{x}{y}\right)} \quad \text{for } x,y>0.$$

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A\rho$. Since L_ρ, R_ρ commute we may define $c(L_\rho, R_\rho)$. Now, we can state the fundamental theorems about monotone metrics (uniqueness and classification are up to scalars).

Theorem 5.1 (Ref. 10): *There exists a unique monotone metric on $\mathcal{P}_n^1$ given by the Fisher information.*

Theorem 5.2 (Ref. 35): *There exists a bijective correspondence between monotone metrics on $\mathcal{D}_n^1$ and operator monotone functions given by the formula*

$$\langle A,B\rangle_{\rho,f} := \mathrm{Tr}(A\cdot c_f(L_\rho,R_\rho)(B)).$$

To state the general formula for the scalar curvature of a monotone metric we need some auxiliary functions. In what follows $c', (\log c)'$ denote derivatives with respect to the first variable, and $c=c_f$.

$$h_1(x,y,z) := \frac{c(x,y)-zc(x,z)c(y,z)}{(x-z)(y-z)c(x,z)c(y,z)},$$

$$h_2(x,y,z) := \frac{(c(x,z)-c(y,z))^2}{(x-y)^2c(x,y)c(x,z)c(y,z)},$$

$$h_3(x,y,z) := z\frac{(\ln c)'(z,x)-(\ln c)'(z,y)}{x-y},$$

$$h_4(x,y,z) := z(\ln c)'(z,x)(\ln c)'(z,y),$$

$$h := h_1 - \tfrac{1}{2}h_2 + 2h_3 - h_4. \tag{5.1}$$

The functions h_i have no essential singularities if arguments coincide.

Note that $\langle A,B\rangle_\rho^f := \mathrm{Tr}(A\cdot c_f(L_\rho,R_\rho)(B))$ defines a Riemannian metric also over $\mathcal{D}_n$ ($\mathcal{D}_n^1$ is a submanifold of codimension 1). Let $\mathrm{Scal}_f(\rho)$ be the scalar curvature of $(\mathcal{D}_n,\langle\cdot,\cdot\rangle_\rho^f)$ at ρ and $\mathrm{Scal}_f^1(\rho)$ be the scalar curvature of $(\mathcal{D}_n^1,\langle\cdot,\cdot\rangle_\rho^f)$.

Theorem 5.3 (Ref. 12): *Let $\sigma(\rho)$ be the spectrum of ρ. Then*

$$\mathrm{Scal}_f(\rho) = \sum_{x,y,z\in\sigma(\rho)} h(x,y,z) - \sum_{x\in\sigma(\rho)} h(x,x,x),$$

$$\mathrm{Scal}_f^1(\rho) = \mathrm{Scal}_f(\rho) + \tfrac{1}{4}(n^2-1)(n^2-2).$$

These results have the following form in the simplest case (2×2 matrices). From Theorem 5.3 it follows that (see Ref. 3)

Corollary 5.4: If $\rho\in D_2$ has eigenvalues λ_1,λ_2, one has

$$\mathrm{Scal}(\rho) = h(\lambda_1,\lambda_1,\lambda_2) + h(\lambda_1,\lambda_2,\lambda_1) + h(\lambda_2,\lambda_1,\lambda_1) + h(\lambda_2,\lambda_2,\lambda_1) + h(\lambda_2,\lambda_1,\lambda_2) + h(\lambda_1,\lambda_2,\lambda_2) + \tfrac{3}{2}.$$

Theorem 5.5 (Ref. 3): *If $\rho\in D_2$ has eigenvalues λ_1,λ_2, and $a=2\lambda_1-1$, then*

$$r_f(a) := \mathrm{Scal}_f(\rho) = \frac{14(a-1)\left[f'\left(\frac{1-a}{1+a}\right)\right]^2}{(1+a)^3\left[f\left(\frac{1-a}{1+a}\right)\right]^2} + \frac{2(a^2+7a-6)f'\left(\frac{1-a}{1+a}\right)}{(1+a)^2 a f\left(\frac{1-a}{1+a}\right)} + \frac{8(1-a)f''\left(\frac{1-a}{1+a}\right)}{(1+a)^3 f\left(\frac{1-a}{1+a}\right)} + \frac{2(1+a)f\left(\frac{1-a}{1+a}\right)}{a^2} + \frac{3a^3+5a^2+8a-4}{2(1+a)a^2}.$$

VI. THE WYD METRICS

We are going to study a particular class of monotone metrics.

Definition 6.1:

$$f_p(x) := \frac{1}{p\tilde{p}} \cdot \frac{(x-1)^2}{(x^{1/p}-1)(x^{1/\tilde{p}}-1)}, \quad p \in \mathbb{R} \setminus \{0,1\},$$

$$f_1(x) = f_\infty(x) := \frac{x-1}{\log(x)}, \quad p = 1,\infty.$$

Obviously $f_p = f_{\tilde{p}}$ and

$$f_1 = \lim_{p\to 1} f_p = \lim_{p\to\infty} f_p = f_\infty.$$

Theorem 6.1 (Refs. 22 and 23): *The function f_p is operator monotone if and only if $p \in A := (-\infty,-1] \cup [\frac{1}{2},+\infty]$.*

Note that $p \in A$ if and only if $\alpha \in [-3,3]$.

Definition 6.2: The WYD(p) metric of parameter p is the monotone metric associated with f_p (where $p \in A$).

We have that f_{-1} is the function of the RLD-metric, $f_1 = f_\infty$ is the function of the BKM-metric, and f_2 is the function of the Wigner–Yanase metric.

In what follows $p \in (1,+\infty)$ and we use again the symbol N_p to denote M_n with the norm

$$\|A\|_{N^p} = p^{-1}(\mathrm{Tr}(|A|^p))^{1/p}.$$

All the commutative construction of Example 3.2 goes through. The following Proposition is the noncommutative analogous of Proposition 3.1 (see also Refs. 16, 21, 23, 28, and 37).

Proposition 6.2 (Ref. 18): Let $\varphi: \rho \in \mathcal{D}_n^1 \to p\rho^{1/p} \in N_p$ be the Amari embedding. The dualized pull-back

$$\langle A,B\rangle_\rho^\varphi := \langle A,B\rangle_\rho^{\varphi,J\circ\varphi} = \langle D_\rho\varphi(A), D_\rho(J\circ\varphi)(B)\rangle$$

coincides with the Wigner–Yanase–Dyson information.

VII. KNOWN RESULTS ON MONOTONICITY

In this short section we review what is known about monotonicity of scalar curvature for monotone metrics. This is useful to emphasize that, up to now, no examples exist of a monotone metrics with Schur-increasing (or Schur-decreasing) scalar curvature.

The Bures or SLD metric is the monotone metric associated with the function $f=(1+x)/2$.

Theorem 7.1 (Refs. 11 and 13): *The scalar curvature of SLD metric is neither Schur-increasing nor Schur-decreasing.*

Proof: By Ref. 11 the SLD-metric has a global minimum at the most mixed state for any n. On

the other hand (this is due to Ref. 13), if $\sigma=\mathrm{diag}(\frac{2}{9},\frac{1}{9},\frac{2}{3})$ and $\rho=\mathrm{diag}(\frac{1}{6},\frac{1}{6},\frac{2}{3})$ then $\rho\succ\sigma$. Using Theorem 5.3 one can calculate $\mathrm{Scal}(\sigma)=3078/25>3447/28=\mathrm{Scal}(\rho)$ and so the conclusion follows. □

Theorem 7.2 (Ref. 17): *The scalar curvature of WY metric is a constant equal to* $\frac{1}{4}(n^2-1)\times(n^2-2)$.

VIII. A CONJECTURE ON THE WYD SCALAR CURVATURE AND ITS RELATION TO PETZ CONJECTURE

In this section we want to suggest that a whole family of monotone metrics with Schur-increasing scalar curvature may exist.

Conjecture 8.1: There exist $\varepsilon>0$ *such that for* p *in the interval* $I:=(1,1+\varepsilon)$ *the scalar curvature of the* WYD(p) *metrics is a Schur-increasing function.*

Conjecture 8.2 (Petz conjecture): The scalar curvature of BKM metric is a Schur-increasing function. This can be rephrased as

$$\rho\succ\sigma\Rightarrow \mathrm{Scal}_{f_1}(\rho)\geqslant \mathrm{Scal}_{f_1}(\sigma).$$

The motivations for Conjecture 8.1 are the following. The WYD(p) metrics come from the dualized pull-back of Proposition 6.2. This means that the WYD(p) metrics depend, indeed, on the pair $(p,\tilde{p})$. Note that when p is in the Schur-decreasing region $(1,2)$ we have that $\tilde{p}$ is in the Schur-increasing region $(2,+\infty)$ (Theorem 4.1, Conjectures 4.1, 4.2). When p approaches 1 then $\tilde{p}$ goes to infinity. Near the boundary values $\{1,+\infty\}$ the increasing–decreasing "symmetry" should be broken: in this case WYD(p) geometry comes from a geometry converging to a flat limit ($p\to1$) and a geometry converging to a (conjectured) Schur-increasing scalar curvature ($\tilde{p}\to\infty$).

Theorem 8.1: *If Conjecture 8.1 is true then Conjecture 8.2 (Petz conjecture) is true.*

Proof: For an arbitrary manifold M let us denote by $\mathcal{M}(M)$ the manifold of Riemannian metrics of M. If $\rho\in M$ is fixed and $g\in\mathcal{M}(M)$ then the function $F_\rho(\cdot):\mathcal{M}(M)\to\mathbb{R}$ defined by $F_\rho(g):=\mathrm{Scal}_g(\rho)$ is a smooth function (see Refs. 29 and 33). Identifying f_p with the metric

$$\langle A,B\rangle_{\rho,f_p}:=\mathrm{Tr}(Ac_{f_p}(L_\rho,R_\rho)(B)),$$

we may consider the function $p\to f_p$ as a continuous curve in $\mathcal{M}(\mathcal{D}_n^1)$. This implies that, by composition, the function $p\to\mathrm{Scal}_{f_p}(\rho)$ is a real, continuous function for each $\rho\in\mathcal{D}_n^1$. Suppose now that Conjecture 8.1 is true.

We have for arbitrary $\rho,\sigma\in\mathcal{D}_n^1$, such that $\rho\succ\sigma$

$$\mathrm{Scal}_{f_1}(\rho)=\lim_{p\to1}\mathrm{Scal}_{f_p}(\rho)\geqslant\lim_{p\to1}\mathrm{Scal}_{f_p}(\sigma)=\mathrm{Scal}_{f_1}(\sigma).$$

But, this is precisely the Petz conjecture. □

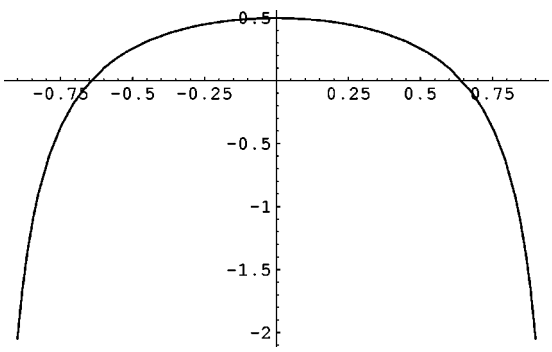

FIG. 1. Case $p=1+10^{-1}$.

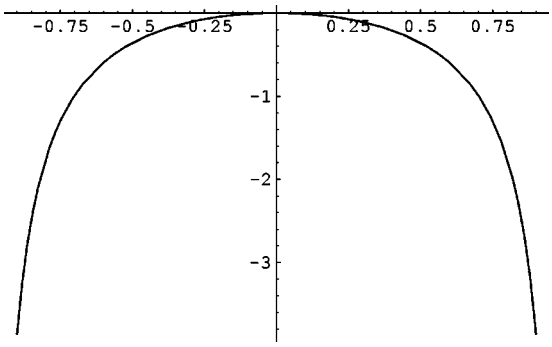

FIG. 2. Case $p=1+10^{-6}$.

A. Numerical results

Conjecture 8.1 would have many consequences. An example is the following theorem.

Theorem 8.2: *Conjecture 8.1 implies that there exists* $\varepsilon>0$ *such that for* $p\in(1,1+\varepsilon)$ *the functions* $r_p:=r_{f_p}$ *of Theorem 5.5 are concave and have their maximum at zero.*

Proof: It follows immediately by Theorem 5.5. □

Using MATHEMATICA, one has the following graphs for the function r_p:

Case $p=1+10^{-1}$; see Fig. 1;

Case $p=1+10^{-6}$; see Fig. 2.

Let us emphasize what we said in the Introduction: a recent result of Andai[3] shows the nontriviality of the above behavior. Indeed, also in the 2×2 case there exist many monotone metrics with nonincreasing scalar curvature.

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