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Massless Sine-Gordon and Massive Thirring Models: proof of Coleman's equivalence

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Abstract: We prove Coleman's conjecture on the equivalence between the massless Sine-Gordon model with finite volume interaction and the Thirring model with a finite volume mass term.

1. Introduction

1.1. Coleman's Equivalence. One of the most fascinating aspects of QFT in $d = 1+1$ is the phenomenon of *bosonization*; fermionic systems can be mapped in bosonic ones and viceversa. The simplest example is provided by the equivalence between free massless Dirac fermions and free massless bosons with the identifications (see for instance [ID]):

$$\bar\psi_{\mathbf{x}}(1+\sigma\gamma_5)\psi_{\mathbf{x}} \;\sim\; b_0{:}e^{i\sigma\sqrt{4\pi}\phi_{\mathbf{x}}}{:} \quad , \quad \bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}} \;\sim\; -\frac{1}{\sqrt{\pi}}\,\varepsilon^{\mu\nu}\partial_\nu\phi_{\mathbf{x}} \tag{1.1}$$

where $\sigma = \pm 1$ and b_0 is a suitable constant, depending on the precise definition of the Wick product. Such equivalence can be extended to interacting theories; Coleman [C] showed the equivalence, in the zero charge sector, between the *massive Thirring model*, with Lagrangian (with our conventions)

$$\mathcal{L} = iZ\bar\psi_{\mathbf{x}}\,\partial\!\!\!/\,\psi_{\mathbf{x}} - Z_1\mu\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} - \frac{\lambda}{4}Z^2 j_{\mu,\mathbf{x}} j^\mu_{\mathbf{x}} \tag{1.2}$$

where Z and Z_1 are (formal) renormalization constants, $j_{\mu,\mathbf{x}} = \bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}}$ and the massless Sine-Gordon model, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_{\mathbf{x}}\partial^\mu\varphi_{\mathbf{x}} + \zeta{:}\cos(\alpha\phi_{\mathbf{x}}){:} \tag{1.3}$$

with the identifications

$$Z_1\bar\psi_{\mathbf{x}}(1+\sigma\gamma_5)\psi_{\mathbf{x}} \;\sim\; b_0{:}e^{i\alpha\sigma\phi_{\mathbf{x}}}{:} \quad , \quad Z\bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}} \;\sim\; -b_1\,\varepsilon^{\mu\nu}\partial_\nu\phi_{\mathbf{x}} \tag{1.4}$$

where b_0, b_1 are two suitable constants, depending on λ and the details of the ultraviolet regularization. Moreover, this equivalence is valid if certain relations between the Thirring parameters λ, μ and the Sine-Gordon parameters α, ζ are assumed. The case $\alpha^2 = 4\pi$ is special, as it corresponds to free fermions ($\lambda = 0$); the choice $\zeta = 0$ (free bosons) corresponds to massless fermions ($\mu = 0$).

In order to establish such equivalence, Coleman considered a *fixed* infrared regularizations of the models (1.2) and (1.3), replacing μ in (1.2) with $\mu\chi_\Lambda(\mathbf{x})$ and ζ with $\zeta\chi_\Lambda(\mathbf{x})$, with $\chi_\Lambda(\mathbf{x})$ a compact support function; this means that the mass term in the Thirring model, and the interaction in the Sine-Gordon is concentrated on a finite volume Λ. Such regularization makes possible a perturbative expansion, respectively in μ for the Thirring model and ζ for the Sine-Gordon model; it turned out that the coefficients of such series expansions can be explicitly computed (in the case of the Thirring coefficients this was possible thanks to the explicit formulas for the correlations of the massless Thirring model given first in [Ha, K]) and they are *order by order* identical if the identification (1.4) is done and provided that suitable relations between the parameters are imposed.

The identification of the series expansions coefficients would give a rigorous proof of the equivalence *provided that* the series are convergent. The issue of convergence, which was mentioned but not addressed in [C], is technically quite involved and crucial; there are several physical examples in which order by order arguments without convergence lead to incorrect predictions.

The search for a rigorous proof of Coleman equivalence was the subject of an intense investigation in the framework of constructive QFT, leading to a number of impressive results. The equivalence between the *massive* Sine-Gordon model (with mass M large enough) at $\alpha^2 < 4\pi$ and a Thirring model with a large long-range interaction was rigorously proved in [FS]; similar ideas were also used in [SU]. The properties of the *massive* Sine-Gordon model for $\alpha^2 \geq 4\pi$ were later on deeply investigated. In [BGN] and [NRS] it was proved that the model is stable if one adds a finite number, increasing with α, of vacuum counterterms, while the full construction, through a cluster expansion, of the model was partially realized in [DH]. In [DH] it was also proved that the correlation functions are analytic in ζ, for any $\alpha^2 < 8\pi$. A proof of analyticity, only based on a multiscale analysis of the perturbative expansion, was first given in [B], for $\alpha^2 < 4\pi$, and then extended in [BK] up to $\alpha^2 < 16/3\pi$.

Using the results in [DH] for a fixed finite volume, Dimock [D] was able finally to achieve a proof of Coleman's equivalence, in the Euclidean version of the models, *for the case* $\alpha^2 = 4\pi$; such a value is quite special as it corresponds to $\lambda = 0$, that is the equivalence is with a *free* massive fermionic system, without current-current interaction. Such limitation was mainly due to the fact that the constructive analysis of interacting fermionic systems was much less developed at that time: indeed a rigorous construction of the massive Thirring model in a functional integral approach has been achieved only quite recently [BFM].

A more physically oriented research on Coleman's equivalence was focused in recovering bosonization in the framework of the (formal) path-integral approach, [N,FGS]. The idea is to introduce a vector field A_μ and to use the identity

$$\exp\left\{-\frac{\lambda}{4}\int d\mathbf{x} j_{\mu,\mathbf{x}} j_{\mu,\mathbf{x}}\right\} = \int DA \exp\left\{\int d\mathbf{x}\left[-A^2_{\mu,\mathbf{x}} + \sqrt{\lambda} A_{\mu,\mathbf{x}} j_{\mu,\mathbf{x}})\right]\right\} \quad (1.5)$$

By parameterizing A_μ in terms of scalar fields $\xi_\mathbf{x}, \phi_\mathbf{x}$

$$A_\mu = \partial_\mu \xi_\mathbf{x} + \varepsilon_{\mu,\nu}\partial_\nu \phi_\mathbf{x} \quad (1.6)$$

it turns out that the massive Thirring model can be expressed in terms of the boson fields $\xi_\mathbf{x}$ and $\phi_\mathbf{x}$: the first is a massless free field, while the second one has an exponential interaction when $\mu \neq 0$. In the expectations of the operators $\bar\psi_\mathbf{x}(1+\sigma\gamma_5)\psi_\mathbf{x}$ and $j_{\mu,\mathbf{x}}$, the $\xi_\mathbf{x}$ field has no role and it can be integrated out; the resulting correlations imply the identification (1.4). Such computations are however based on formal manipulations of functional integrals (with no cut-offs, hence formally infinite) and it is well known that such arguments can lead to incorrect result (see for instance the discussion in §1 in [BFM]).

In this paper we will give the first proof of Coleman's equivalence between the Euclidean massive Thirring model with a small interaction and the mass term restricted to a fixed finite volume Λ and the Euclidean massless Sine-Gordon model with the interaction restricted to the same volume Λ and α^2 around 4π. We will follow the Coleman strategy, but an extension of the multiscale techniques developed in [B] for the Sine-Gordon model and in [BM,BFM] for the Thirring model allow us to achieve the convergence of the expansion.

1.2. Main results. We start from a suitable regularization of the Sine-Gordon and Thirring models via the introduction of infrared and ultraviolet cut-offs, which will be removed at the end, by taking fixed the volume Λ in the interaction term of the Sine-Gordon model and in the mass term of the Thirring model.

Let us consider first the (Euclidean) *Sine-Gordon* model. Let $\gamma > 1$, h be a large negative integer (γ^h is the *infrared cutoff*) and N be a large positive integer (γ^N is the *ultraviolet cutoff*). Moreover, let $\varphi_\mathbf{x}$ a 2-dimensional bosonic field and $P_{h,N}(d\varphi)$ be the Gaussian measure with covariance $C_{h,N}(\mathbf{x}) \stackrel{def}{=} \sum_{j=h}^N C_0(\gamma^j\mathbf{x})$, for

$$C_0(\mathbf{x}) \stackrel{def}{=} \frac{1}{(2\pi)^2}\int \frac{d\mathbf{k}}{\mathbf{k}^2}\left[e^{-\mathbf{k}^2} - e^{-(\gamma\mathbf{k})^2}\right] e^{i\mathbf{k}\mathbf{x}} \quad (1.7)$$

Given the two real parameters ζ, the *coupling*, and α (related with the inverse temperature β, in the Coulomb gas interpretation of the model, by the relation $\beta = \alpha^2$), the Sine-Gordon model with finite volume interaction and ultraviolet and infrared cutoffs is defined by the *interacting measure* $P_{h,N}(d\varphi)\exp\{\zeta_N V(\varphi)\}$, with

$$V(\varphi) = \int_\Lambda d\mathbf{x}\, \cos(\alpha\varphi_\mathbf{x}) \quad , \quad \zeta_N = e^{\frac{\alpha^2}{2}C_{0,N}(0)}\zeta \quad (1.8)$$

where Λ is a fixed volume of size 1. Note that $\zeta_N V(\varphi) = \zeta\int_\Lambda d\mathbf{x} :\cos(\alpha\varphi_\mathbf{x}):$, where

$$:e^{i\alpha\varphi_\mathbf{x}}: \stackrel{def}{=} e^{i\alpha\varphi_\mathbf{x}} e^{\frac{\alpha^2}{2}C_{0,N}(0)} \quad (1.9)$$

is the Wick order exponential $e^{ia\varphi_{\mathbf{x}}}$, $a \in \mathbb{R}$, with respect to the measure with covariance $C_{0,N}(\mathbf{x})$ (for any h); hence ζ_N has the role of the *bare strength.*

We consider now the *Thirring model.* The precise regularization of the path integral for fermions was already described in [BFM], §1.2, therefore we only remind the main features. We introduce in $\Lambda_L \equiv [-L/2, L/2] \times [-L/2, L/2]$ a lattice Λ_a whose sites represent the space-time points. We also consider the lattice $\mathcal{D}_a$ of space-time momenta $\mathbf{k} = (k, k_0)$. We introduce a set of Grassmann spinors $\psi_{\mathbf{k}}, \bar\psi_{\mathbf{k}}$, $\mathbf{k} \in \mathcal{D}_a$, such that $\psi_{\mathbf{k}} = (\psi^-_{\mathbf{k},+}, \psi^-_{\mathbf{k},-})$, $\bar\psi = \psi^+\gamma^0$ and $\psi^+_{\mathbf{k}} = (\psi^+_{\mathbf{k},+}, \psi^+_{\mathbf{k},-})$. The γ matrices are explicitly given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^5 = -i\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also define a Grassmann field on the lattice Λ_a by Fourier transform, according to the following convention:

$$\psi^{[h,N]\sigma}_{\mathbf{x},\omega} \stackrel{def}{=} \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}_a} e^{i\sigma\mathbf{k}\mathbf{x}} \widehat\psi^{[h,N]\sigma}_{\mathbf{k},\omega}, \qquad \mathbf{x} \in \Lambda_a . \tag{1.10}$$

Sometimes $\psi^{[h,N]\sigma}_{\mathbf{x},\omega}$ will be shorten into $\psi^\sigma_{\mathbf{x},\omega}$. Moreover, since the limit $a \to 0$ is trivial [BFM], we shall consider in the following $\psi^{[h,N]\sigma}_{\mathbf{x},\omega}$ as defined in the continuous box Λ_L.

In order to introduce an ultraviolet and an infrared cutoff, we could use a gaussian cut-off as in (2.4), but for technical reason, and to use the results of [BFM], we find more convenient to use a compact support cut-off. We define the function $\chi_{h,N}(\mathbf{k})$ in the following way; let $\chi \in C^\infty(\mathbb{R}_+)$ be a *Gevrey function* of class 2, non-negative, non-increasing smooth function such that

$$\chi(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t \ge \gamma_0 , \end{cases} \tag{1.11}$$

for a fixed choice of $\gamma_0 : 1 < \gamma_0 \le \gamma$; then we define, for any $h \le j \le N$,

$$f_j(\mathbf{k}) = \chi\left(\gamma^{-j}|\mathbf{k}|\right) - \chi\left(\gamma^{-(j-1)}|\mathbf{k}|\right) \tag{1.12}$$

and $\chi_{h,N}(\mathbf{k}) = \sum_{j=h}^N f_j(\mathbf{k})$; hence $\chi_{h,N}(\mathbf{k})$ acts as a smooth cutoff for momenta $|\mathbf{k}| \ge \gamma^{N+1}$ and $|\mathbf{k}| \le \gamma^{h-1}$.

Given two real parameters, the *bare coupling* λ and the *bare mass* μ, the Thirring model with finite volume mass term and ultraviolet and infrared cutoffs is defined by the *interacting measure* $P_{h,N}(d\psi)\exp\{\mathcal{V}(\psi)\}$, with

$$\mathcal{V}(\psi) = -\frac{\lambda}{4} Z_N^2 \int_{\Lambda_L} d\mathbf{x} \ \left(\bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}}\right)^2 + Z_N^{(1)}\mu \int_\Lambda d\mathbf{x} \ \bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} + E_{h,N}|\Lambda_L| \tag{1.13}$$

and

$$P_{h,N}(d\psi) \stackrel{def}{=} \mathrm{d}\widehat\psi \prod_{\mathbf{k}\in\mathcal{D}^{[h,N]}} \left[L^{-4}Z_N^2|(-|\mathbf{k}|^2 C^2_{h,N}(\mathbf{k})\right]^{-1} \cdot$$

$$\exp\left\{-Z_N\frac{1}{L^2}\sum_{\omega=\pm}\sum_{\mathbf{k}\in\mathcal{D}^{[h,N]}} \frac{D_\omega(\mathbf{k})}{\chi_{h,N}(\mathbf{k})}\widehat\psi^+_{\mathbf{k},\omega}\widehat\psi^-_{\mathbf{k},\omega'}\right\}, \tag{1.14}$$

where $D_\omega(\mathbf{k}) \stackrel{def}{=} -ik_0 + \omega k_1$ and $E_{h,N}$ is constant chosen so that, if $\mu = 0$, $\int P_{h,N}(d\psi)\exp\{\mathcal{V}(\psi)\} = 1$. We will prove the following theorem.

Theorem 1.1. *Let Λ be a fixed volume of size 1 and assume $|\zeta|, |\lambda|, |\mu|$ small enough, $\alpha^2 < 16\pi/3$; then there exist two constants $\eta_- = a\lambda^2 + O(\lambda^3)$ and $\eta_+ = b\lambda + O(\lambda^2)$, with $a, b > 0$, independent of μ and analytic in λ, such that, if we put*

$$Z_N = \gamma^{-\eta_- N} \quad , \quad Z_N^{(1)} = \gamma^{-\eta_+ N} \tag{1.15}$$

then, if $r = 0$ and $q \geq 2$ or $r \geq 1$, for any choice of the non coinciding points $(\mathbf{x}_1, \ldots, \mathbf{x}_q, \mathbf{y}_1, \ldots, \mathbf{y}_r)$, and of $\sigma_i = \pm 1$, $i = 1, \ldots, q$, $\nu_j = 0, 1$, $j = 1, \ldots, r$,

$$\lim_{-h,N\to\infty} \langle \left[\prod_{i=1}^q :e^{i\sigma_i\alpha\varphi_{\mathbf{x}_i}}:\right] \left[\prod_{j=1}^r (-1)\varepsilon^{\nu_j\mu}\partial^\mu \phi_{\mathbf{y}_j}\right] \rangle_{SG}^T = \tag{1.16}$$
$$= \lim_{-h,N\to\infty} (b_0 Z_N^{(1)})^q (b_1 Z_N)^r \langle \left[\prod_{i=1}^q \bar\psi_{\mathbf{x}_i}\left(\frac{1+\sigma_i\gamma_5}{2}\right)\psi_{\mathbf{x}_i}\right] \left[\prod_{j=1}^r \bar\psi_{\mathbf{y}_j}\gamma^{\nu_j}\psi_{\mathbf{y}_j}\right] \rangle_{Th}^T$$

where $\langle\cdot\rangle_{Th}^T$ and $\langle\cdot\rangle_{SG}^T$ denote the truncated expectations in the Thirring (in the limit $L\to\infty$) and Sine-Gordon models, respectively, b_0 and b_1 are bounded functions of λ and the following relations between the parameters of the two models have to be verified:

$$\frac{\alpha^2}{4\pi} = 1 + \eta_- - \eta_+ \quad , \quad \zeta = b_0\mu \tag{1.17}$$

If $q = 1$ and $r = 0$ both the r.h.s. and the l.h.s. of (1.16) are diverging for $\lambda \leq 0$, while the equality still holds for $\lambda > 0$. A divergence also appears, for $\lambda \leq 0$, in the pressure, but only for the second order term in ζ or μ; however, if we add a suitable vacuum counterterm, also the pressures are equal.

This Theorem proves Coleman's equivalence (1.4). We remark that the relations between the Sine-Gordon parameters and the Thirring parameters in (1.17) are slightly different with respect to those in [C], for $\lambda \neq 0$; this is true in particular for the first equation, involving only quantities which have a physical meaning in the removed cutoff limit, if we express them in terms of λ, as Coleman does. This is not surprising, as the relations between the physical quantities, like the critical indices $\eta_\pm$, and the bare coupling depend on the details of the regularization, and in our Renormalization Group analysis the running coupling constants have a bounded but non trivial flow from the ultraviolet to the infrared scales. Indeed, with a different regularization of the Thirring model (that is starting from a non local current-current interaction and performing the local limit after the limit $N \to \infty$), as in [M1,M2], one would get a simple relation between α and λ. This new relation again is not equal to that of [C], but is in agreement with the regularization procedure of [J], see footnote 7 of [C].

Another important remark concerns the limit $\Lambda \to \infty$. In the case of Sine-Gordon model, one expects that, in this limit, there is exponential decrease of correlations (implying the screening phenomenon in the Coulomb gas interpretation), which is not compatible with convergence of perturbative expansion (in this case the correlations would have a power decay as in the free theory). Up to now, screening has been proved only for $\alpha^2 << 4\pi$ [Y], by extending to dimension two the analogous result obtained in three dimensions by Brydges and Federbush [BF], but screening is expected to be verified in all range of validity of the model ($\alpha^2 < 8\pi$), hence even around $\alpha^2 = 4\pi$. However, if the interaction is restricted to a fixed finite volume, convergence is possible and we could indeed prove it, for $\alpha^2 < 6\pi$; in this paper, for simplicity, we give the proof only for $\alpha^2 < 16\pi/3$, which is sufficient to state the main result.

The situation for the massive Thirring model is slightly different, because it has been shown [BFM] that it is well defined in the limit $\Lambda \to \infty$ and that its correlations decay at least as $\exp(-c\sqrt{|\mu|^{1+O(\lambda)}|\mathbf{x}|})$. Hence, even if the power expansion in the mass can be convergent only if we fix the volume, the proof of Coleman's conjecture strongly supports the related conjecture that even the Sine-Gordon model is well defined around $\alpha^2 = 4\pi$ in the infinite volume limit and has exponential decrease of correlations.

The proof is organized in the following way. In §2 we analyze the massless Sine-Gordon model with finite volume interaction and $\alpha^2 < 16\pi/3$, extending the proof of analyticity in ζ given in [B] for the massive case in the infinite volume limit and $\alpha^2 < 4\pi$. With respect to the technique used in [D], where only the case $\alpha^2 = 4\pi$ was analyzed, our method has the advantage that an explicit expression of the coefficients can be easily achieved; this is probably possible even with the other method, but the proof was given only for $\alpha^2 < 4\pi$ and, as a consequence, the correlations in the model with $\alpha^2 = 4\pi$ were defined as the limit $\alpha^2 \to 4\pi$ of those with $\alpha^2 < 4\pi$.

In §3 we use the methods developed in [BFM, M1, M2] to prove the analyticity in μ of the Thirring model; the explicit expressions of the coefficients are obtained in §4, by using the explicit expression of the field correlation functions given in the Appendix (through the solution of a Schwinger-Dyson equation, based on a rigorous implementation of Ward Identities) and by a rigorous implementation, in a RG context, of the point spitting procedure used in theoretical physics. An important role in the analysis is also played by the proof of the following *exact* relation between critical indices

$$(1+\eta_-)^2 = 1 + \eta_+^2 \tag{1.18}$$

which is used in order to exclude the presence of an extra massless Gaussian field $\xi_{\mathbf{x}}$ in the second of (1.4).

2. The Massless Sine-Gordon Model with a finite volume interaction

We want to study the measure defined in §1.2 in the limit of removed cutoff, $-h, N \to \infty$. To this purpose, we consider the *Generating functional*, $\mathcal{K}_{h,N}(J, A, \zeta)$, defined by the equation

$$\mathcal{K}_{h,N}(J,A,\zeta) = \log \int P_{h,N}(d\varphi)\, e^{\zeta_N V(\varphi)} \cdot \\ \cdot \exp\left\{ \sum_{\sigma=\pm 1} \int d\mathbf{x} J^\sigma_{\mathbf{x}} {:}e^{i\alpha\sigma\varphi_{\mathbf{x}}}{:} + \sum_{\nu=0,1} \int d\mathbf{y} A^\nu_{\mathbf{y}} \left(\partial^\nu \varphi_{\mathbf{y}}\right) \right\} \tag{2.1}$$

where $J^\sigma_{\mathbf{z}}$ and $A^\mu_{\mathbf{y}}$ are two-dimensional, external bosonic fields. Then, given two non negative integers q and r, as well as two sets of labels $\underline{\sigma} = (\sigma_1, \ldots, \sigma_q)$

and $\underline{\nu} = (\nu_1, \ldots, \nu_r)$, together with two sets of two by two distinct points $\underline{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_q)$ and $\underline{\mathbf{y}} = (\mathbf{y}_1, \ldots, \mathbf{y}_q)$, we consider the Schwinger functions, defined by the equation

$$K^{(q,r;\zeta)}_{h,N}(\underline{\mathbf{z}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu}) \stackrel{def}{=} \frac{\partial^{q+r}\mathcal{K}_{h,N}}{\partial J^{\sigma_1}_{\mathbf{z}_1} \cdots \partial J^{\sigma_q}_{\mathbf{z}_q} \partial A^{\nu_1}_{\mathbf{y}_1} \cdots \partial A^{\nu_r}_{\mathbf{y}_r}}(0, 0, \zeta) \tag{2.2}$$

Theorem 2.1. *If $|\zeta|$ is small enough, $\alpha^2 < 16\pi/3$ and $q \geq 2$, if $r = 0$, or $q \geq 0$, if $r \geq 1$, the limit*

$$K^{(q,r;\zeta)}(\underline{\mathbf{z}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu}) \stackrel{def}{=} \lim_{-h,N\to+\infty} K^{(q,r;\zeta)}_{h,N}(\underline{\mathbf{z}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu}) \tag{2.3}$$

exists and is analytic in ζ. In the case $q = r = 0$ (the pressure), the limit does exist and is analytic, up to a divergence in the second order term, present only for $\alpha^2 \geq 4\pi$.

For clarity's sake, we prefer to give the proof of the above theorem in the special cases $(q, r) = (k, 0)$ and $(q, r) = (0, k)$ separately; the proof in the general case is a consequence of the very same ideas that will be discussed for the special ones, but it needs a more involved notation, so we will not report its details.

2.1. The free measure. By the definitions given in §1.2, the regularized free measure is the two–dimensional boson Gaussian measure with covariance

$$C_{h,N}(\mathbf{x}) = \frac{1}{(2\pi)^2}\int \frac{d\mathbf{k}}{\mathbf{k}^2}\left[e^{-(\gamma^{-N}\mathbf{k})^2} - e^{-(\gamma^{-h+1}\mathbf{k})^2}\right] e^{i\mathbf{k}\mathbf{x}} = \sum_{j=h}^{N} C_0(\gamma^j \mathbf{x}) \tag{2.4}$$

The two–dimensional massless boson Gaussian measure is obtained by taking the limits $h \to -\infty$ and $N \to \infty$. It is easy to prove that

$$C_0(0) = \frac{\log\gamma}{2\pi} \quad , \quad \left|\partial^{q_0}_{x_0}\partial^{q_1}_{x_1} C_0(\mathbf{x})\right| \leq A_{q_0,q_1,\kappa} e^{-\kappa|\mathbf{x}|} \tag{2.5}$$

where q_0, q_1 are non negative integers and κ is an arbitrary positive constant. Let us now consider the function

$$C_{h,\infty}(\mathbf{x}) = \lim_{N\to\infty} C_{h,N}(\mathbf{x}) = C_{0,\infty}(\gamma^h \mathbf{x}) \tag{2.6}$$

It is easy to show, by a standard calculation, that there exists a constant c such that

$$|C_{0,\infty}(\mathbf{x}) + \frac{1}{4\pi}\log(c|\mathbf{x}|^2)| \leq C|\mathbf{x}|^2 \tag{2.7}$$

Hence, $C_{h,\infty}(\mathbf{x})$ diverges for $h \to -\infty$ as $-(2\pi)^{-1}\log(\gamma^h|\mathbf{x}|)$. However, if we define

$$\Delta^{-1}_{h,\infty}(\mathbf{x}) = C_{h,\infty}(\mathbf{x}) + \frac{1}{4\pi}\log(c\gamma^{2h}) \tag{2.8}$$

we have, by (2.7):

$$\Delta^{-1}(\mathbf{x}) \stackrel{def}{=} \lim_{h\to-\infty} \Delta^{-1}_{h,\infty}(\mathbf{x}) = -\frac{1}{2\pi}\log|\mathbf{x}| \tag{2.9}$$

Then it is natural to define the *Coulomb potential with ultraviolet cutoff* by

$$\Delta_N^{-1}(\mathbf{x}) \stackrel{def}{=} \lim_{h\to-\infty} \Delta_{h,N}^{-1}(\mathbf{x}) \quad , \quad \Delta_{h,N}^{-1}(\mathbf{x}) \stackrel{def}{=} C_{h,N}(\mathbf{x}) + \frac{1}{4\pi}\log(c\gamma^{2h}) \tag{2.10}$$

Since $C_{h,N}(\mathbf{x}) = C_{h,\infty}(\mathbf{x}) - C_{N,\infty}(\mathbf{x})$, by using (2.5) and (2.9), we see that

$$\left|\Delta_N^{-1}(\mathbf{x}) + \frac{1}{2\pi}\log|\mathbf{x}|\right| \le Ce^{-\kappa\gamma^N|\mathbf{x}|} \quad , \quad \gamma^N|\mathbf{x}| \ge 1 \tag{2.11}$$

and, by using (2.7), we see that

$$|\Delta_N^{-1}(\mathbf{x}) - \frac{1}{4\pi}\log(c\gamma^{2N})| \le C\gamma^{2N}|\mathbf{x}|^2 \quad , \quad \gamma^N|\mathbf{x}| \le 1 \tag{2.12}$$

We define $\mathcal{E}_{h,N}$ and $\mathcal{E}_j$ to be the expectation with respect to the Gaussian measures with covariance $C_{h,N}(\mathbf{x})$ and $C_j(\mathbf{x}) = C_0(\gamma^j\mathbf{x})$, respectively; a superscript T in the expectation will indicate a truncated expectation. Recall that, for a generic probability measure with expectation $\mathcal{E}$, and any family of random variables $(f_1,\ldots,f_s)$, $\mathcal{E}^T$ is defined as

$$\mathcal{E}^T[f_1;\ldots;f_s] = \sum_{\Pi}(-1)^{|\Pi|-1}(|\Pi|-1)! \prod_{X\in\Pi}\mathcal{E}\left[\prod_{i\in X} f_i\right] \tag{2.13}$$

where $\sum_\Pi$ denotes the sum over the partitions of the set $(1,\ldots,s)$. Finally we remind that $:e^{i\sigma\beta\varphi_{\mathbf{x}}}:$ is the Wick normal ordering of $e^{i\sigma\beta\varphi_{\mathbf{x}}}$ *always* taken with respect to the measure with covariance $C_{0,N}(\mathbf{x})$ (see definitions in §1.2).

Lemma 2.1. *Let $\sigma_i \in \{-1,+1\}$, $i=1,\ldots,n$, and $\alpha\in\mathbb{R}$. If $Q \stackrel{def}{=} \sum_r \sigma_r$, then*

$$\lim_{h\to-\infty} \mathcal{E}_{h,N}\left[\prod_{r=1}^n :e^{i\alpha\sigma_r\varphi_{\mathbf{x}_r}}:\right] = \delta_{Q,0}c^{-\frac{\alpha^2}{8\pi}n}e^{-\alpha^2\sum_{r<s}\sigma_r\sigma_s\Delta_N^{-1}(\mathbf{x}_r-\mathbf{x}_s)} \tag{2.14}$$

Proof. We first notice that if the Wick product had been defined with respect to the covariance $C_{h,N}$, then $\log\mathcal{E}_{h,N}\left[\prod_{r=1}^n :e^{i\alpha\sigma_r\varphi_{\mathbf{x}_r}}:\right]$ would have been equal to $-\alpha^2\sum_{r<s}\sigma_r\sigma_s C_{h,N}(\mathbf{x}_r-\mathbf{x}_s)$. Hence, by definition (2.10), we get

$$\log\mathcal{E}_{h,N}\left[\prod_{r=1}^n :e^{i\alpha\sigma_r\varphi_{\mathbf{x}_r}}:\right] = \frac{\alpha^2}{4\pi}hn\log\gamma - \alpha^2\sum_{r<s}\sigma_r\sigma_s C_{h,N}(\mathbf{x}_r-\mathbf{x}_s) =$$
$$= \frac{\alpha^2}{4\pi}hQ^2\log\gamma + \frac{\alpha^2}{8\pi}(Q^2-n)\log c - \alpha^2\sum_{r<s}\sigma_r\sigma_s\Delta_{h,N}^{-1}(\mathbf{x}_r-\mathbf{x}_s)$$

which immediately implies the lemma. ∎

If $\mathcal{E}$ is the expectation $\mathcal{E}_{h,N}$ in the limit $-h,N\to\infty$, by taking the limit $N\to\infty$ in the r.h.s. of (2.14), we get, in the case $Q=0$,

$$\mathcal{E}\left[\prod_{r=1}^n :e^{i\alpha\sigma_r\varphi_{\mathbf{x}_r}}:\right] = \delta_{Q,0}c^{-\frac{\alpha^2}{8\pi}n}\prod_{r<s}|\mathbf{x}_r-\mathbf{x}_s|^{\sigma_r\sigma_s\frac{\alpha^2}{2\pi}} \tag{2.15}$$

We are now ready to consider the interacting measure.

2.2. The case $q = r = 0$ (the pressure). To begin with we analyze the pressure:

$$p(\zeta) \stackrel{def}{=} \lim_{-h,N\to\infty} \log Z_{h,N}(\zeta) \quad , \quad Z_{h,N}(\zeta) \stackrel{def}{=} \int P_{h,N}(d\varphi)\, e^{\zeta_N V(\varphi)} \tag{2.16}$$

We proceed as in [B], by studying the multiscale expansion associated with the following decomposition of the covariance:

$$C_{h,N}(\mathbf{x}) = \sum_{j=0}^{N} C_j(\mathbf{x}) + C_{h,-1}(\mathbf{x}) \quad , \quad C_j(\mathbf{x}) \stackrel{def}{=} C_0(\gamma^j \mathbf{x}) \tag{2.17}$$

In comparison with [B], where the case $h = 0$ - the "Yukawa gas" - was considered, here we are collecting in a single integration step all scales below $h = 0$: as we shall see, this is effective since the volume size is fixed to be 1. To simplify the notation, from now on $\mathcal{E}_{-1}$ will denote the expectation w.r.t. $C_{h,-1}(\mathbf{x})$, while $\mathcal{E}_j$ will have the previous meaning for $j \geq 0$.

Let $\mathcal{T}_n^{(N)}$, $n \geq 2$, be the family of labelled trees with the following properties:

1) there is a root r and n ordered *endpoints* e_i, $i = 1, \ldots, n$, which are connected by the tree; the tree is ordered from the root to the endpoints;
2) each vertex v carries a *frequency label* h_v, which is an integer taking values between -1 and $N + 1$, with the condition that $h_u < h_v$, if u precedes v in the order of the tree; moreover, the root has frequency -1 and the endpoint e_i has frequency $h_i + 1$, if h_i is the frequency of the higher vertex preceding it.
3) The endpoint e_i carries two other labels, the *charge* σ_i and the *position* $\mathbf{x}_i$.

These trees differ from those used in [BFM] for the Thirring model, because there are no "trivial vertices" on the lines of the tree.

Since $\mathcal{T}_n^{(N)} \subset \mathcal{T}_n^{(N+1)}$, then $\mathcal{T}_n^{(\infty)} = \lim_{N\to\infty} \mathcal{T}_n^{(N)}$ is obtained from $\mathcal{T}_n^{(N)}$ by letting the frequency indices free to vary between -1 and ∞.

We shall also use the following definitions:

a) Given a tree τ, we shall call *non trivial* (n.t. in the following) the tree vertices different from the root and from the endpoints. If $v \in \tau$ is a n.t. vertex, $s_v \geq 2$ will denote the number of lines branching from v in the positive direction, $v' \in \tau$ is the higher non trivial vertex preceding v, if it does exist, or the root, otherwise. Moreover, X_v will be the set of endpoints following v along the tree; X_v will be called the *cluster* of v and n_v will denote the number of its elements. If v is an endpoint, X_v will denote the endpoint itself. Finally we define $\Phi(X_v) = \sum_{i\,:\,e_i\in X_v} \sigma_i \varphi_{\mathbf{x}_i}$.
b) Given a n.t. vertex v and an integer $j \in [-1, N]$, we shall denote

$$U_j(v) \stackrel{def}{=} \sum_{r,m\,:\,e_r,e_m\in X_v} \sigma_r\sigma_m C_j(\mathbf{x}_r - \mathbf{x}_m) = \mathcal{E}_j\left[\Phi^2(X_v)\right] \geq 0 \tag{2.18}$$

the (double of the) *total energy on scale j* associated with its cluster. If $k'+1 \leq k-1$, we shall also define

$$U_{k',k}(v) \stackrel{def}{=} \sum_{j=k'+1}^{k-1} U_j(v) \tag{2.19}$$

c) If X and Y are two disjoint clusters and j is an integer contained in $[-1, N]$, we denote

$$W_j(X,Y) \stackrel{def}{=} \sum_{r,m\,:\,e_r\in X,\ e_m\in Y} \sigma_r\sigma_m C_j(\mathbf{x}_r-\mathbf{x}_m) = \mathcal{E}_j\left[\Phi(X)\Phi(Y)\right] \quad (2.20)$$

the *interaction energy on scale* j between X and Y.

d) Given a n.t. vertex v, $(v_1,\ldots,v_{s_v})$, will denote the set of vertices following it along the tree; moreover we define

$$G_j(v_1,\ldots,v_{s_v}) = \mathcal{E}_j^T\left[e^{i\alpha\Phi(X_{v_1})};\ldots;e^{i\alpha\Phi(X_{v_{s_v}})}\right] \quad (2.21)$$

By proceeding as in [B], it is easy to see that

$$Z_{h,N}(\zeta) = \int P_{h,-1}(\varphi)\; e^{\zeta V(\varphi)+\sum_{n=2}^\infty \zeta^n V_n^{(N)}(\varphi)} \quad (2.22)$$

where

$$V_n^{(N)}(\varphi) = \sum_{\tau\in\mathcal{T}_n^{(N)}} \frac{1}{2^n}\sum_{\sigma_1,\ldots,\sigma_n}\int_{\Lambda^n} d\mathbf{x}_1\cdots d\mathbf{x}_n\; e^{i\alpha\sum_{r=1}^n \sigma_r\varphi_{x_r}} V_\tau(\underline{\sigma},\underline{\mathbf{x}}) \quad (2.23)$$

$$V_\tau(\underline{\sigma},\underline{\mathbf{x}}) = \left(\prod_{i=1}^n \gamma^{\frac{\alpha^2}{4\pi}h_{e_i}}\right)\prod_{n.t.\ v\in\tau}\frac{G_{h_v}(v_1,\ldots,v_{s_v})}{s_v!}e^{-\frac{\alpha^2}{2}U_{h_{v'},h_v}(v)} \quad (2.24)$$

We note that $V_\tau(\underline{\sigma},\underline{\mathbf{x}})$ is independent of N and h.

In order to prove that the pressure, see (2.16), is well defined, the main step is to verify that, uniformly in h and N, $Z_{h,N}(\zeta) = 1+O(\zeta)$. As we will discuss later in this section, since the only dependence on h in (2.22) is through the measure $P_{h,-1}(d\varphi)$, which has support on smooth functions for any h, the wanted bound for $Z_{h,N}(\zeta)$ is an easy consequence of a uniform C^n bound of $V_n^{(N)}(\varphi)$. Since

$$|V_n^{(N)}(\varphi)| \le \sum_{\tau\in\mathcal{T}_n^{(N)}} b_\tau \quad , \quad b_\tau \stackrel{def}{=} \frac{1}{2^n}\sum_{\sigma_1,\ldots,\sigma_n}\int_{\Lambda^n} d\mathbf{x}_1\cdots d\mathbf{x}_n\; |V_\tau(\underline{\sigma},\underline{\mathbf{x}})| \quad (2.25)$$

and the number of trees is of order C^n, we shall look for a "good" bound of b_τ. The main ingredients in this task are the positivity of $U_j(v)$, see (2.18), and the Battle-Federbush formula for the truncated expectations (see [Br]):

$$G_j(v_1,\ldots,v_s) = \sum_{T\in\bar{\mathcal{T}}_s}\prod_{\langle r,m\rangle\in T}\left[-\alpha^2 W_j(X_{v_r},X_{v_m})\right]\int dp_T(\underline{t})\; e^{-\frac{\alpha^2}{2}U_j(v,\underline{t})} \quad (2.26)$$

where $s=s_v$, $\bar{\mathcal{T}}_s$ is the family of connected tree graphs on the set of integers $\{1,\ldots,s\}$, $\frac{1}{2}U_j(v,\underline{t})$ is obtained by taking a sequence of convex linear combination, with parameters $\underline{t}$, of the energies of suitable subsets of $X_v = \cup_i X_{v_i}$ (hence $U_j(v,\underline{t})\ge 0$) and $dp_T(\underline{t})$ is a probability measure.

By using (2.24), (2.26) and (2.20), we get

$$|V_\tau(\underline{\sigma},\underline{\mathbf{x}})| \le \left(\prod_{i=1}^{n}\gamma^{\frac{\alpha^2}{4\pi}h_{e_i}}\right)\cdot$$
$$\cdot\prod_{n.t.\ v\in\tau}\frac{\alpha^{2(s_v-1)}}{s_v!}\sum_{T\in\bar{\mathcal{T}}_{s_v}}\prod_{\langle r,m\rangle\in T}\sum_{\substack{e\in X_{v_r}\\ e'\in X_{v_m}}}|C_{h_v}(\mathbf{x}_e-\mathbf{x}_{e'})| \tag{2.27}$$

On the other hand, for any given $\varepsilon>0$, we can use the bound

$$\sum_{T\in\bar{\mathcal{T}}_s}\prod_{\langle r,m\rangle\in T} n_{v_r}n_{v_m}\le s!\,\varepsilon^{-2(s-1)}\prod_{r=1}^{s}e^{2\varepsilon n_{v_i}} \tag{2.28}$$

Moreover, by (2.17) and (2.5), for $h_v\ge 0$,

$$\int_\Lambda d\mathbf{x}\,|C_{h_v}(\mathbf{x})|\le C\gamma^{-2h_v}\ . \tag{2.29}$$

The trees, as defined after (2.17), satisfy the following identity: $\sum_{w\ge v}(s_w-1)=n_v-1$; as a consequence, if v_0 is the first non trivial vertex of τ,

$$\sum_{n.t.\ v\in\tau}h_v(s_v-1)=h_r(n_{v_0}-1)+\sum_{n.t.\ v\in\tau}(h_v-h_{v'})(n_v-1)$$

where v' is the n.t. vertex immediately preceding v or the root, if $v=v_0$. This allows us to write:

$$b_\tau\le C_\varepsilon^n\left(\prod_{i=1}^{n}\gamma^{\frac{\alpha^2}{4\pi}h_{e_i}}\right)\prod_{n.t.\ v\in\tau}\gamma^{-2h_v(s_v-1)}e^{2\varepsilon n_v}\le \tag{2.30}$$
$$\le C_\varepsilon^n\prod_{n.t.\ v\in\tau}\gamma^{-(h_v-h_{v'})[D(n_v)-2\varepsilon n_v]}$$

where the *dimension* $D(n)$ is given by

$$D(n)=2(n-1)-\frac{\alpha^2}{4\pi}n\ . \tag{2.31}$$

Let us consider first the case $\alpha^2<4\pi$. This condition implies that $D(n)>0$ for any $n\ge 2$; hence, the bound (2.30) implies in the usual way that $|V_\tau(\underline{\sigma},\underline{\mathbf{x}})|\le C_\alpha^n$, for a constant C_α, which diverges as $\alpha^2\to(4\pi)^-$; since $|\Lambda|=1$, this bound is valid also for b_τ. By a little further effort, see below, one can then prove that the pressure $p(\zeta)$ is an analytic function of ζ, for ζ small enough.

If $4\pi\le\alpha^2<16\pi/3$, $D(n)>0$ only for $n\ge 3$, so that the previous bound is divergent for all trees containing at least one vertex with $n_v=2$. In particular, $V_2^{(N)}(\varphi)$ diverges as $N\to\infty$; this divergence is related with the fact that the term of order ζ^2 and $\sigma_1=-\sigma_2$ in the perturbative expansion of the pressure is really divergent, as one can easily check. The only way to cure this specific divergence is to renormalize the model by subtracting a suitable constant of order ζ^2 from the potential, as we shall see below.

However, all other terms, even those associated with a tree containing at least one vertex with $n_v=2$, are indeed bounded uniformly in N; in order to prove this claim, we need to improve the bound (2.30) by the two following lemmas.

Lemma 2.2. *If $n_v = 2$ and v_1, v_2 are the two endpoints following v with positions $\mathbf{x}_1$, $\mathbf{x}_2$ respectively and equal charges $\sigma_1 = \sigma_2$, then*

$$|G_{h_v}(v_1, v_2)e^{-\frac{\alpha^2}{2}U_{h_{v'},h_v}(v)}| \leq C\gamma^{-\frac{\alpha^2}{\pi}(h_v - h_{v'})}e^{-\gamma^{h_v}|\mathbf{x}_1-\mathbf{x}_2|} \tag{2.32}$$

Proof. Since $h_{v'} + 1 \geq 0$, it is easy to check that

$$G_{h_v}(v_1, v_2)e^{-\frac{\alpha^2}{2}U_{h_{v'},h_v}(v)} = \left[e^{-\alpha^2 C_{h_v}(\mathbf{x}_1-\mathbf{x}_2)} - 1\right] e^{-\alpha^2 C_0(0)} \cdot$$
$$\cdot\, e^{-\alpha^2 \sum_{k=h_{v'}+1}^{h_v-1}\left[C_0(0)+C_k(\mathbf{x}_1-\mathbf{x}_2)\right]} \stackrel{def}{=} F(\mathbf{x}_1 - \mathbf{x}_2)$$

Hence, by using (2.5) with $\kappa > 1$ and since $C_0(\mathbf{x}) \leq C_0(0)$, we get

$$|F(\mathbf{z})| \leq Ce^{-\kappa\gamma^{h_v}|\mathbf{z}|}e^{-2\alpha^2 C_0(0)(h_v-h_{v'})}e^{-\alpha^2\sum_{k=h_{v'}+1}^{h_v-1}\left[C_0(\gamma^k\mathbf{z})-C_0(0)\right]} \leq$$
$$\leq C\gamma^{-\frac{\alpha^2}{\pi}(h_v-h_{v'})}e^{2\alpha^2 RC_0(0)}e^{-(\kappa-\kappa_R)\gamma^{h_v}|\mathbf{z}|}$$

where $\kappa_R \stackrel{def}{=} \alpha^2 B\sum_{r=R}^{\infty}\gamma^{-r}$, the constant B is such that $|C_0(\mathbf{x}) - C_0(0)| \leq B|\mathbf{x}|$ (it exists by (2.5) for $(q_0, q_1) = (1, 0), (0, 1)$) and R is an arbitrary positive integer, that we can choose so that $\kappa - \kappa_R \geq 1$. ∎

Lemma 2.3. *For $j \geq 0$, if the cluster X is made of two endpoints with positions $\mathbf{x}_1$, $\mathbf{x}_2$ and opposite charges, and Y is another arbitrary cluster, then*

$$|W_j(X, Y)| \leq C\gamma^j|\mathbf{x}_1 - \mathbf{x}_2|\sum_{\mathbf{y}\in Y}\int_0^1 dt\; e^{-\gamma^j|\mathbf{x}_2+t(\mathbf{x}_1-\mathbf{x}_2)-\mathbf{y}|} \tag{2.33}$$

Moreover, if also the cluster Y is made of two endpoints with opposite charge and positions $\mathbf{y}_1$, $\mathbf{y}_2$, then

$$|W_j(X, Y)| \leq C\gamma^{2j}|\mathbf{x}_1 - \mathbf{x}_2|\;|\mathbf{y}_1 - \mathbf{y}_2|\int_0^1 dtds\; e^{-\gamma^j\left|\mathbf{x}_2+t(\mathbf{x}_1-\mathbf{x}_2)-\mathbf{y}_2-s(\mathbf{y}_1-\mathbf{y}_2)\right|} \tag{2.34}$$

Proof. By using the identity

$$C_0(\mathbf{x}_1-\mathbf{y})-C_0(\mathbf{x}_2-\mathbf{y}) = \sum_{a=0,1}(\mathbf{x}_1-\mathbf{x}_2)_a\int_0^1 dt\;\big(\partial_a C_0\big)[\mathbf{x}_2+t(\mathbf{x}_1-\mathbf{x}_2)-\mathbf{y}] \tag{2.35}$$

together with (2.5) (with $\kappa \geq 1$), we get the bound

$$|C_j(\mathbf{x}_1 - \mathbf{y}) - C_j(\mathbf{x}_2 - \mathbf{y})| \leq C\gamma^j|\mathbf{x}_1 - \mathbf{x}_2|\int_0^1 dt\; e^{-\gamma^j|\mathbf{x}_2+t(\mathbf{x}_1-\mathbf{x}_2)-\mathbf{y}|} \tag{2.36}$$

which immediately implies (2.33). The bound (2.34) is proved in a similar way, by using the identity

$$C_0(\mathbf{x}_1 - \mathbf{y}_1) - C_0(\mathbf{x}_2 - \mathbf{y}_1) - C_0(\mathbf{x}_1 - \mathbf{y}_2) + C_0(\mathbf{x}_2 - \mathbf{y}_2) = \sum_{a,b=0,1}(\mathbf{x}_1 - \mathbf{x}_2)_a\,\cdot$$
$$\cdot(\mathbf{y}_1 - \mathbf{y}_2)_b\int_0^1 dtds\;\big(\partial_a\partial_b C_0\big)[\mathbf{x}_2 + t(\mathbf{x}_1 - \mathbf{x}_2) - \mathbf{y}_2 - s(\mathbf{y}_1 - \mathbf{y}_2)] \tag{2.37}$$

∎

Let us now consider a tree with $n \geq 3$ endpoints. By using Lemma 2.2, we can improve the bound (2.30) by replacing $D(n_v)$ with $D(n_v) + \alpha^2/\pi$ in all vertices with $n_v = 2$ and $\sigma_1 = \sigma_2$. Since $D(2) + \alpha^2/\pi = 2 + \alpha^2/(2\pi) > 0$, this is sufficient to make the corresponding sum over $h_v - h_{v'}$ convergent. It follows that the sum over all trees with $n \geq 3$ and no vertex with $n_v = 2$ and $Q = 0$ is finite, uniformly in h, if $\alpha^2 < 16\pi/3$.

A similar argument can be used to control the vertices with $n_v = 2$ and $Q = 0$. In fact, if v is a vertex with $n_v = 2$, then v' is certainly a n.t. vertex, otherwise n would be equal to 2 and we are supposing $n \geq 3$. Hence, we can use Lemma 2.3 in (2.26) for the vertex v', which allows us to improve the bound of (2.26) for the vertex v: since $\gamma^{h_{v'}}|\mathbf{x}_1 - \mathbf{x}_2|\,|G_{h_v}(v_1, v_2)| \leq C\gamma^{-(h_v - h_{v'})}e^{-\gamma^{h_v}|\mathbf{x}_1 - \mathbf{x}_2|/2}$, if v_1 and v_2 are the two endpoints following v, we can modify the bound (2.30) by adding 1 to the dimension $D(n_v)$ of v; this is sufficient, since $D(2) + 1 = 3 - \alpha^2/(2\pi)$ is positive for $\alpha^2 < 6\pi$. It follows that $|V_\tau(\underline{\sigma}, \underline{\mathbf{x}})| \leq C_\alpha^n$ holds for all $n \geq 3$, with $C(\alpha) \to \infty$ as $\alpha^2 \to (16\pi/3)^-$. By a further effort, one could prove that $C(\alpha)$

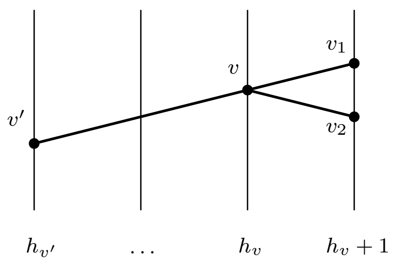

Fig. 2.1. A subtree of τ with $n_v = 2$ and $Q = 0$. While v_1 and v_2 are endpoints, therefore their scale has to be $h_v + 1$, v' is the higher non trivial vertex of τ preceding v, hence the only constraint is that $h_v - h_{v'} \leq N$.

can be substituted with a new constant, which is indeed finite up to 6π, but we do not need here this stronger property.

Let us now come back to the terms of order two. It is easy to see that

$$V_2^{(N)}(\varphi) = \frac{\zeta^2}{2} \sum_{\sigma = \pm 1} V_{2,\sigma}^{(N)}(\varphi) \tag{2.38}$$

$$V_{2,\sigma}^{(N)}(\varphi) = \int_{\Lambda^2} d\mathbf{x} d\mathbf{y}\ \cos[\alpha\varphi_{\mathbf{x}} + \sigma\alpha\varphi_{\mathbf{y}}] W_\sigma^{(N)}(\mathbf{x} - \mathbf{y}) \tag{2.39}$$

$$W_\sigma^{(N)}(\mathbf{x} - \mathbf{y}) = \frac{1}{2} \sum_{j=0}^{N} \gamma^{\frac{\alpha^2}{2\pi}(j-1)} \left[e^{-\sigma\alpha^2 C_j(\mathbf{x}-\mathbf{y})} - 1\right] e^{-\alpha^2 \sum_{r=0}^{j-1}[C_r(0) + \sigma C_r(\mathbf{x}-\mathbf{y})]} \tag{2.40}$$

By proceeding as in Lemma 2.2, it is easy to see that $V_{2,+}^{(N)}(\varphi)$ is bounded uniformly in N for any α. This is not true for $V_{2,-}^{(N)}(\varphi)$; in fact, if we define

$$c_N = \mathcal{E}_{h,-1}(V_{2,-}^{(N)}(\varphi)) \tag{2.41}$$

one can easily check that c_N diverges for $N \to \infty$ and that $\zeta^2 c_N/2$ is equal to the term of order ζ^2 and $\sigma_1 = -\sigma_2$ in the perturbative expansion of the pressure. However, if we define $\widetilde{Z}_{h,N}(\zeta) = Z_{h,N}(\zeta) e^{-\frac{\zeta^2}{2} c_N}$, we can show that the *renormalized pressure* (in presence of the cutoffs) $\widetilde{p}_{h,N}(\zeta) = \log \widetilde{Z}_{h,N}(\zeta)$ has a power expansion uniformly convergent as $-h, N \to \infty$, for $\alpha^2 < 16\pi/3$ (the result is indeed true for $\alpha^2 < 6\pi$).

It is easy to see that

$$\widetilde{p}_{h,N}(\zeta) = \zeta \gamma^{\frac{\alpha^2 h}{4\pi}} + \sum_{n=2}^{\infty} \zeta^n \sum_{\tau \in \widetilde{\mathcal{T}}_n^{(N)}} p_\tau^{(h)} \tag{2.42}$$

where $\widetilde{\mathcal{T}}_n^{(N)}$ is a family of trees defined as $\mathcal{T}_n^{(N)}$, with the following differences:

1) the root has scale -2;
2) there is no tree which has only two endpoint with opposite charge.

Moreover

$$p_\tau^{(h)} = \frac{1}{2^n} \sum_{\sigma_1,\ldots,\sigma_n} \int_{\Lambda^n} d\mathbf{x}_1 \cdots d\mathbf{x}_n \, \widetilde{V}_\tau^{(h)}(\underline{\sigma}, \underline{\mathbf{x}}) \tag{2.43}$$

$$\widetilde{V}_\tau^{(h,N)}(\underline{\sigma}, \underline{\mathbf{x}}) = \left(\prod_{i=1}^n \gamma^{\frac{\alpha^2}{4\pi}(h_i+1)} \right) \prod_{n.t.\ v \in \tau} \frac{\widetilde{G}_{h_v}(v_1, \ldots, v_{s_v})}{s_v!} e^{-\frac{\alpha^2}{2} U_{h_{v'}, h_v}(v)} \tag{2.44}$$

where $\widetilde{G}_{h_v}(v_1, \ldots, v_{s_v}) = G_{h_v}(v_1, \ldots, v_{s_v})$, if $h_v \geq 0$, while, if $h_v = -1$ and $s_v = s$,

$$\widetilde{G}_{-1}(v_1, \ldots, v_s) = \mathcal{E}_{h,-1}^T \left[F(\varphi, X_{v_1}); \ldots; F(\varphi, X_{v_s}) \right] \tag{2.45}$$

with, given a cluster X,

$$F(\varphi, X) = \begin{cases} \cos[\alpha \Phi(X)] - 1 & \text{, if } |X| = 2 \text{ and } \sigma_1 = -\sigma_2 \\ \cos[\alpha \Phi(X)] & \text{, otherwise} \end{cases} \tag{2.46}$$

where we subtracted a -1 in the terms with $|X| = 2$ and $\sigma_1 = -\sigma_2$ (without changing the value of the truncated expectation, since $s \geq 2$), in order to improve the bound in the corresponding vertex, with an argument similar to that used before. In fact, in order to bound (2.45), we shall use the definition (2.13) and the bound

$$\left| \mathcal{E}_{h,-1} \left[\prod_{i=1}^m F(\varphi, X_i) \right] \right| \leq \left[\prod_{i:|X_i|=2} |\mathbf{x}_1^{(i)} - \mathbf{x}_2^{(i)}|^2 \right] \cdot$$
$$\cdot \left(\frac{\alpha^2}{2} \right)^{m_2} \sup_{\mathbf{y}_1, \ldots, \mathbf{y}_{m_2} \in \Lambda} \mathcal{E}_{h,-1} \left[|\partial \varphi_{\mathbf{y}_1}|^2 \cdots |\partial \varphi_{\mathbf{y}_{m_2}}|^2 \right] \tag{2.47}$$

where $m_2 \le m$ is the number of clusters with 2 endpoints and, for each cluster of this type, $\mathbf{x}_1^{(i)}$ and $\mathbf{x}_2^{(i)}$ are the two endpoint positions; $|\partial\varphi_{\mathbf{y}}|^2 \stackrel{def}{=} (\partial_0\varphi_{\mathbf{y}})^2 + (\partial_1\varphi_{\mathbf{y}})^2$. On the other hand, it is easy to see that there is a constant c_0, independent of h, such that $|\mathcal{E}_{h,-1}[\partial_{a_1}\varphi_{\mathbf{x}_1}\partial_{a_2}\varphi_{\mathbf{x}_2}]| \le c_0$. It follows, by using the Wick Theorem, that $\mathcal{E}_{h,-1}\left[|\partial\varphi_{\mathbf{y}_1}|^2\cdots|\partial\varphi_{\mathbf{y}_q}|^2\right] \le 2^q c_0^q (2q-1)!! \le C^q q!$, so that, if we choose $C \ge 1$ (which allows us to substitute m_2 with m) and use (2.13), we obtain the following bound

$$|\widetilde{G}_{-1}(v_1,\ldots,v_s)| \le \left[\prod_{i:|X_{v_i}|=2} |\mathbf{x}_1^{(i)} - \mathbf{x}_2^{(i)}|^2\right] \cdot \tag{2.48}$$

$$\cdot \sum_{k=1}^{s}(k-1)!\frac{1}{k!}\sum_{\substack{m_1,\ldots,m_k\ge 1\\ \sum_{r=1}^k m_r = s}} \frac{s!}{m_1!\cdots m_k!}\prod_{r=1}^{k}(C^{m_r} m_r!)$$

The sum in the second line is equal to $C^s s! \sum_{k=1}^s \frac{1}{k}\begin{pmatrix} s-1 \\ k-1\end{pmatrix} \le 2^{s-1}C^s s!$, so that

$$|\widetilde{G}_{-1}(v_1,\ldots,v_s)| \le C^s s! \left[\prod_{i:|X_{v_i}|=2} |\mathbf{x}_1^{(i)} - \mathbf{x}_2^{(i)}|^2\right] \tag{2.49}$$

The factors $|\mathbf{x}_1^{(i)} - \mathbf{x}_2^{(i)}|^2$ can be used control the sum over the scale labels of the vertices with $|X_{v_i}| = 2$, by the same argument used in the discussion following (2.37). Hence, if we define the function $W_{n,h,N}(\underline{\mathbf{x}})$ so that $\sum_{\tau\in\widetilde{\mathcal{T}}_n^{(N)}} p_\tau^{(h)} = \int_{\Lambda^n} d\mathbf{x}_1\cdots d\mathbf{x}_n\, W_{n,h,N}(\underline{\mathbf{x}})$, the previous arguments imply that there exist positive functions $f_\tau(\underline{\mathbf{x}})$, independent of h and N, and a constant C, such that

$$|W_{n,h,N}(\underline{\mathbf{x}})| \le \sum_{\tau\in\widetilde{\mathcal{T}}_n^{(N)}} f_\tau(\underline{\mathbf{x}}) \stackrel{def}{=} H_{n,N}(\underline{\mathbf{x}}) \quad , \quad \int_{\Lambda^n} d\underline{\mathbf{x}}\, H_{n,N}(\underline{\mathbf{x}}) \le C^n \tag{2.50}$$

Since $\widetilde{\mathcal{T}}_n^{(N)} \subset \widetilde{\mathcal{T}}_n^{(N+1)}$, $H_{n,N}(\underline{\mathbf{x}})$ is monotone in N. Hence, by the Monotone Convergence Theorem, $H_{n,N}(\underline{\mathbf{x}})$ has a L^1 limit $H_n(\underline{\mathbf{x}})$, as $N\to\infty$; by (2.50), $|W_{n,h,N}(\underline{\mathbf{x}})| \le H_n(\underline{\mathbf{x}})$. On the other hand, by definition we have

$$W_{n,h,N}(\underline{\mathbf{x}}) = \frac{1}{n!}\frac{1}{2^n}\sum_{\sigma_1,\ldots,\sigma_n} \mathcal{E}_{h,N}^T\left[:e^{i\alpha\sigma_1\varphi_{\mathbf{x}_1}}:;\ldots;:e^{i\alpha\sigma_n\varphi_{\mathbf{x}_n}}:\right] \tag{2.51}$$

and Lemma 2.1, (2.11) and (2.13) imply that $W_{n,h,N}(\underline{\mathbf{x}})$ is almost everywhere convergent as $-h, N \to \infty$. Then, by the Dominated Convergence Theorem, (2.42) and (2.50), $\widetilde{p}(\zeta) = \lim_{-h,N\to\infty}\widetilde{p}_{h,N}(\zeta)$ does exist and is an analytic function of ζ, for ζ small enough; moreover, $\widetilde{p}(\zeta) = \sum_{n=2}^\infty p_n\zeta^n$ and, if $n\ge 3$, by (2.15)

$$p_n = \frac{c^{-\frac{\alpha^2}{8\pi}n}}{n!}\frac{1}{2^n}\sum_{\sigma_1,\ldots,\sigma_n}^{Q=0}\int_{\Lambda^n} d\mathbf{x}_1\cdots d\mathbf{x}_n \cdot \tag{2.52}$$

$$\left\{ \sum_{\Pi} (-1)^{|\Pi|-1} (|\Pi|-1)! \prod_{Y \in \Pi} \prod_{\substack{r,s \in Y \\ r<s}} |\mathbf{x}_r - \mathbf{x}_s|^{\sigma_r \sigma_s \frac{\alpha^2}{2\pi}} \right\}$$

where $\sum_{\Pi}$ denotes the sum over the partitions of the set $\{1, \ldots, n\}$. If $\alpha^2 < 4\pi$, the previous expression is well defined also for $n = 2$, and gives the coefficient of order 2 of $p(\zeta)$. We stress that the integral and the sum over the partitions can not be exchanged.

2.3. The case $r = 0$ (the charge correlation functions). Let $\xi_i = (\mathbf{z}_i, \sigma_i)$, $i = 1, \ldots, k$, a family of fixed positions and charges, such that $\mathbf{z}_i \neq \mathbf{z}_j$ for $i \neq j$ and μ_i, $i = 1, \ldots, k$, a set of real numbers. If

$$Z_{h,N}(\zeta, \underline{\xi}, \underline{\mu}) = \int P_{[h,N]}(d\varphi) e^{\zeta_N V(\varphi) + \sum_{r=1}^{k} \mu_r :e^{i\alpha\sigma_r \varphi(\mathbf{z}_r)}:} \tag{2.53}$$

the *charge correlation function of order k*, $k \geq 1$, defined by (2.2), is given by

$$K_{h,N}^{(k,\zeta)}(\underline{\mathbf{z}}, \underline{\sigma}) = \frac{\partial^k}{\partial\mu_1 \cdots \partial\mu_k} \log Z_{h,N}(\zeta, \underline{\xi}, \underline{\mu}) \Big|_{\underline{\mu}=0} \tag{2.54}$$

By proceeding as in Sect. 2.2, one can show that

$$Z_{h,N}(\zeta, \underline{\xi}, \underline{\mu}) = \int P_{[h,-1]}(d\varphi) e^{V_{eff}^{(N)}(\zeta,\varphi) + B^{(N)}(\zeta,\varphi,\underline{\xi},\underline{\mu}) + R^{(N)}(\zeta,\varphi,\underline{\xi},\underline{\mu})} \tag{2.55}$$

where $V_{eff}^{(N)}(\zeta, \varphi) = \zeta V(\varphi) + \sum_{n=2}^{\infty} \zeta^n V_n^{(N)}(\varphi)$, $B^{(N)}(\zeta, \varphi, \underline{\xi}, \underline{\mu})$ is the sum over the terms of order at most 1 in each of the μ_r, and $R^{(N)}(\zeta, \varphi, \underline{\xi}, \underline{\mu})$ is the rest. (2.54) implies that

$$K_{h,N}^{(k,\zeta)}(\underline{\mathbf{z}}, \underline{\sigma}) = \frac{\partial^k}{\partial\mu_1 \cdots \partial\mu_k} \log \widetilde{Z}_{h,N}(\zeta, \underline{\xi}, \underline{\mu}) \Big|_{\underline{\mu}=0} \tag{2.56}$$

where

$$\widetilde{Z}_{h,N}(\zeta, \underline{\xi}, \underline{\mu}) = \int P_{h,-1}(d\varphi)\, e^{V_{eff}^{(N)}(\zeta,\varphi) - \zeta^2 c_N/2 + B^{(N)}(\zeta,\varphi,\underline{\xi},\underline{\mu})} \tag{2.57}$$

In order to describe the functional $B^{(N)}(\zeta, \varphi, \underline{\xi}, \underline{\mu})$, we need to introduce a new definition. We shall call $\mathcal{T}_{n,k}^{(N)}$ the family of labelled trees whose properties are very similar to those of $\mathcal{T}_n^{(N)}$, with the only difference that there are $n + m$ endpoints, $n \geq 0$, $1 \leq m \leq k$; n endpoints, to be called *normal*, are associated as before to the interaction, while the others, to be called *special*, are associated with m different variables ξ_i, whose set of indices we shall denote I_τ, while ξ_τ will denote the set of variables itself. It is easy to see that

$$B^{(N)}(\zeta, \varphi, \underline{\xi}, \underline{\mu}) = \sum_{n=0}^{\infty} \zeta^n B_n^{(N)}(\varphi, \underline{\xi}, \underline{\mu}) \tag{2.58}$$

$$B_n^{(N)}(\varphi, \underline{\xi}, \underline{\mu}) = \delta_{n,0} \sum_{r=1}^{k} \mu_r {:}e^{i\alpha\sigma\varphi_{\mathbf{z}_r}}{:} + \sum_{\substack{\tau\in\mathcal{T}^{(N)}_{n,k}\\ n+m\geq 2}} \frac{1}{2^n} \sum_{\sigma'_1,\ldots,\sigma'_n} \int_{\Lambda^n} d\mathbf{x}_1 \cdots d\mathbf{x}_n \cdot$$

$$\cdot \prod_{s\in I_\tau} \mu_s e^{i\alpha\left[\sum_{r=1}^n \sigma'_r\varphi_{\mathbf{x}_r} + \sum_{s\in I_\tau} \sigma_s\varphi_{\mathbf{z}_s}\right]} V_\tau(\underline{\sigma}', \underline{\mathbf{x}}, \xi_\tau) \tag{2.59}$$

where $V_\tau(\underline{\sigma}', \underline{\mathbf{x}}, \xi_\tau)$ is defined exactly as in (2.24), with $(\underline{\sigma}', \underline{\sigma}_\tau, \underline{\mathbf{x}}, \underline{\mathbf{z}}_\tau)$ in place of $(\underline{\sigma}, \underline{\mathbf{x}})$.

One can easily check that, if $\alpha^2 \geq 4\pi$, the terms with $n = k = 1$ and $\sigma_1 = -\bar{\sigma}_1$ in the r.h.s. of (2.59) have a divergent bound as $N \to \infty$. This is related to the fact that the function $K_{h,N}^{(1,\zeta)}(\mathbf{z};\sigma)$ is indeed divergent at the first order in ζ. However, if we regularize these terms by subtracting their value at $\varphi = 0$, the counterterms give no contribution to $K_{h,N}^{(k,\zeta)}(\underline{\mathbf{z}};\underline{\sigma})$, for $k \geq 2$. Hence, we can proceed as in the bound of the pressure and we get similar results. There are however a few differences to discuss.

Given a tree τ (with root of scale -2) contributing to $K_{h,N}^{(k,\zeta)}(\underline{\mathbf{z}};\underline{\sigma})$, we call τ^* the tree which is obtained from τ by erasing all the vertices which are not needed to connect the $m \leq k$ special endpoints. The endpoints of τ^* are the m special endpoints of τ, which we denote e_i^*, $i = 1, \ldots, m$. Given a vertex $v \in \tau^*$, we shall call $\underline{\mathbf{z}}_v$ the subset of the positions associated with the endpoints following v in τ^*; moreover, we shall call s_v^* the number of branches following v in τ^*. The positions in $\underline{\mathbf{z}}_v$ are connected in our bound by a *spanning tree* of propagators of scales $j \geq h_v$; hence, if we use the bound

$$e^{-2\gamma^h|\mathbf{x}|} \leq e^{-\gamma^h|\mathbf{x}|} \cdot e^{-c\sum_{j=0}^h \gamma^j|\mathbf{x}|} \quad , \quad c = \sum_{j=0}^{\infty} \gamma^{-j/2} \tag{2.60}$$

and define $\delta = \min_{1\leq i<j\leq k} |\mathbf{z}_i - \mathbf{z}_j|$, it is easy to see that we can extract, for any $v \in \tau^*$, a factor $e^{-c\gamma^{h_v}\delta}$ from the propagators bound, by leaving a decaying factor $e^{-\gamma^j|\mathbf{x}|}$ for each propagator (of scale j) of the spanning tree. On the other hand, the fact that the points in $\underline{\mathbf{z}}_v$ are not integrated implies that there are $s_v^* - 1$ less integrations to do by using propagators of scale h_v, for each vertex $v \in \tau^*$. In conclusion, with respect to the pressure bound, we have to add, for each tree τ, a factor

$$\prod_{v\in\tau^*} \gamma^{2h_v(s_v^*-1)} e^{-c\gamma^{h_v}\delta} \leq (c\delta)^{-2(m-1)}(2m-2)! \tag{2.61}$$

where we used the identity $\sum_{v\in\tau^*}(s_v^* - 1) = m - 1$. Since $m \leq k$, the sum over the scale labels can be done exactly as in the pressure case, up to a $C^k(2k)!$ overall factor.

There is another difference to analyze, related with the fact that, in the analogue of (2.45), the function $F(\varphi, X_v)$ corresponding to a cluster with two endpoints of opposite charge, one normal and one special, is bounded by $|\mathbf{x} - \mathbf{z}| \sup_{\mathbf{y}} |\partial\varphi_{\mathbf{y}}|$, rather than bounded by $|\mathbf{x} - \mathbf{z}|^2 \sup_{\mathbf{y}} |\partial\varphi_{\mathbf{y}}|^2$. The fact that the zero in the positions is of order one has no consequence, since such a zero is sufficient to regularize the bound over a cluster with two endpoints of opposite charges. The fact that $|\partial\varphi_{\mathbf{y}}|$ appears, instead of its square, is also irrelevant, since the only consequence is that, in the bound analogue to (2.47), one has to substitute $\mathcal{E}_{h,-1}\left[|\partial\varphi_{\mathbf{y}_1}|^2\cdots|\partial\varphi_{\mathbf{y}_{m_2}}|^2\right]$ with $\mathcal{E}_{h,-1}\left[|\partial\varphi_{\mathbf{y}_1}|\cdots|\partial\varphi_{\mathbf{y}_{m_2'}}|\right]$, with $m_2'\le 2m_2$. However, by Schwartz inequality, $\mathcal{E}_{h,-1}\left[|\partial\varphi_{\mathbf{y}_1}|\cdots|\partial\varphi_{\mathbf{y}_{m_2'}}|\right]\le\sqrt{\mathcal{E}_{h,-1}\left[|\partial\varphi_{\mathbf{y}_1}|^2\cdots|\partial\varphi_{\mathbf{y}_{m_2'}}|^2\right]}$ and we can still use the Wick Theorem to get an even better bound.

The previous arguments allow us to prove that $K^{(k,\zeta)}(\underline{\mathbf{z}};\underline{\sigma})=\lim_{-h,N\to+\infty}K^{(k,z)}_{h,N}(\underline{\mathbf{z}},\underline{\sigma})$ does exist and is an analytic function of ζ around $\zeta=0$, with a radius of convergence independent of δ (the minimum distance between two points in $\underline{\mathbf{z}}$). On the other hand, it is easy to check the well known identity

$$K^{(k,\zeta)}_{h,N}(\underline{\mathbf{z}};\underline{\sigma})=\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\frac{1}{2^n}\sum_{\underline{\sigma}'}\int d\mathbf{x}_1\cdots d\mathbf{x}_n$$
$$\mathcal{E}^T_{h,N}\left[:e^{i\alpha\sigma_1\varphi_{\mathbf{z}_1}}:;\ldots;:e^{i\alpha\sigma_k\varphi_{\mathbf{z}_k}}:;:e^{i\alpha\sigma'_1\varphi_{\mathbf{x}_1}}:;\ldots;:e^{i\alpha\sigma'_n\varphi_{\mathbf{x}_n}}:\right] \tag{2.62}$$

An argument similar to that used at the end of §2.2 allows us to prove that the power expansion of $K^{(k,\zeta)}(\underline{\mathbf{z}};\underline{\sigma})$ is obtained by the previous equation, by substituting in the r.h.s $\mathcal{E}^T_{h,N}$ with $\mathcal{E}^T$. Hence, by using (2.15), we get that $K^{(k,\zeta)}(\underline{\mathbf{z}};\underline{\sigma})=\sum_{n=0}^{\infty}\zeta^n g_{k,n}(\underline{\mathbf{z}};\underline{\sigma})$, with

$$g_{k,n}(\underline{\mathbf{z}};\underline{\sigma})=\frac{c^{-\frac{\alpha^2}{8\pi}(n+k)}}{n!}\frac{1}{2^n}\sum_{\substack{\sigma'_1,\ldots,\sigma'_n\\ \sum_{i=1}^n\sigma'_i+\sum_{r=1}^k\sigma_r=0}}\int_{\Lambda^n}d\mathbf{x}_1\cdots d\mathbf{x}_n\cdot \tag{2.63}$$
$$\left\{\sum_{\Pi}(-1)^{|\Pi|-1}(|\Pi|-1)!\prod_{Y\in\Pi}\prod_{\substack{r,s\in Y\\ r<s}}|\mathbf{y}_r-\mathbf{y}_s|^{\bar\sigma_r\bar\sigma_s\frac{\alpha^2}{2\pi}}\right\}$$

where $\underline{\mathbf{y}}=(\underline{\mathbf{x}},\underline{\mathbf{z}})$, $\underline{\bar\sigma}=(\underline{\sigma}',\underline{\sigma})$ and $\sum_\Pi$ denotes the sum over the partitions of the set $(1,\ldots,n+k)$.

2.4. The case $r>0$ (the $\partial\varphi$ correlation functions). Let $\underline{\mathbf{y}}=(\mathbf{y}_1,\ldots,\mathbf{y}_k)$ a set of $k\ge1$ distinct fixed positions, $\underline{\nu}=(\nu_1,\ldots,\nu_k)$ a set of derivative indices and $\underline{\mu}=(\mu_1,\ldots,\mu_k)$ a set of real numbers. If

$$Z_{h,N}(\zeta,\underline{\mathbf{y}},\underline{\nu},\underline{\mu})=\int P_{[h,N]}(d\varphi)e^{\zeta_N V(\varphi)+\sum_{r=1}^k\mu_r\partial^{\nu_r}\varphi(\mathbf{y}_r)} \tag{2.64}$$

the $\partial\varphi$ *correlation function of order* k, $k\ge1$, is given by

$$K^{(k,\zeta)}_{h,N}(\underline{\mathbf{y}};\underline{\nu})=\left.\frac{\partial^k}{\partial\mu_1\cdots\partial\mu_k}\log Z_{h,N}(\zeta,\underline{\mathbf{y}},\underline{\nu},\underline{\mu})\right|_{\underline{\mu}=0} \tag{2.65}$$

We can proceed as in §2.3 and we can represent $K_{h,N}^{(k,\zeta)}(\underline{\mathbf{y}};\underline{\nu})$ as in (2.56), that is we can substitute in (2.65) $Z_{h,N}(\zeta,\underline{\mathbf{y}},\underline{\nu},\underline{\mu})$ with

$$\widetilde{Z}_{h,N}(\zeta,\underline{\mathbf{y}},\underline{\nu},\underline{\mu}) = \int P_{[h,-1]}(d\varphi) e^{V_{eff}^{(N)}(\zeta,\varphi)-2\zeta^2 c_N + B^{(N)}(\zeta,\varphi,\underline{\mathbf{y}},\underline{\nu},\underline{\mu})} \tag{2.66}$$

It is not hard to see that $B^{(N)}(\zeta,\varphi,\underline{\mathbf{y}},\underline{\nu},\underline{\mu}) = \sum_{n=0}^{\infty}\zeta^n\ B_n^{(N)}(\varphi,\underline{\mathbf{y}},\underline{\nu},\underline{\mu})$, with

$$B_n^{(N)}(\varphi,\underline{\mathbf{y}},\underline{\nu},\underline{\mu}) = \delta_{n,0}\sum_{r=1}^{k}\mu_r\partial^{\nu_r}\varphi(\mathbf{y}_r) + \sum_{\substack{\tau\in\mathcal{T}_{n,k}^{(N)}\\ n+m\ge 2}}\frac{1}{2^n}\sum_{\sigma_1,\ldots,\sigma_n}\int_{\Lambda^n} d\mathbf{x}_1\cdots d\mathbf{x}_n\ \cdot$$
$$\cdot e^{i\alpha\sum_{r=1}^n\sigma_r\varphi(\mathbf{x}_r)}\widetilde{V}_\tau(\underline{\sigma},\underline{\nu},\underline{\mathbf{x}},\underline{\mathbf{y}}_\tau)\prod_{r\in I_\tau}\mu_r \tag{2.67}$$

where $\mathcal{T}_{n,k}^{(N)}$ is defined exactly as in (2.59), except for the fact that the m special endpoints ($1\le m\le k$) are associated with the $\partial\varphi$ terms; moreover $\widetilde{V}_\tau(\underline{\sigma},\underline{\mathbf{x}},\underline{\mathbf{y}}_\tau)$ is defined in a way similar to $V_\tau(\underline{\sigma},\underline{\mathbf{x}},\underline{\xi}_\tau)$, but, before giving its expression, we need a few new definitions.

If v is a non trivial vertex, we shall call $I_v\subset I_\tau$ the set of special endpoints immediately following v (that is the set of $\partial\varphi$ endpoints which are contracted in v), $\bar{s}_v$ the number of vertices immediately following v, which are not special endpoints, and $s_v^*=|I_v|$ (hence $s_v=\bar{s}_v+s_v^*$). Moreover, we shall use X_v to denote the set of normal endpoints (instead of all endpoints) following v. Then we can write

$$\widetilde{V}_\tau(\underline{\sigma},\underline{\nu},\underline{\mathbf{x}},\underline{\mathbf{y}}_\tau) = \left(\prod_{i=1}^n\gamma^{\frac{\alpha^2}{4\pi}h_{e_i}}\right)\prod_{n.t.\ v\in\tau}\frac{\widetilde{G}_{h_v}(v_1,\ldots,v_{s_v})}{s_v!}e^{-\frac{\alpha^2}{2}U_{h_{v'},h_v}(v)} \tag{2.68}$$

with $\widetilde{G}_j(v_1,\ldots,v_s)=\mathcal{E}_j^T\left[F_{v_1}(\varphi);\ldots;F_{v_s}(\varphi)\right]$, where $F_v(\varphi)=\partial^\nu\varphi_{\mathbf{y}}$, if the vertex v is a special endpoint with position $\mathbf{y}$ and label ν, otherwise $F_v(\varphi)=\exp(i\alpha\Phi(X_v))$. We can always rearrange the order of the arguments so that the first possibility happens for $i=1,\ldots,m$. If $m=0$, we can use the identity (2.26), otherwise we can write

$$\widetilde{G}_j(v_1,\ldots,v_s) = \frac{\partial^m}{\partial\lambda_1\cdots\partial_{\lambda_m}}H_j(\lambda_1,\ldots,\lambda_m)\bigg|_{\underline{\lambda}=0} \tag{2.69}$$

where

$$H_j(\underline{\lambda}) = \mathcal{E}_j^T(e^{\lambda_1\partial^{\nu_1}\varphi_{\mathbf{y}_1}};\ldots;e^{\lambda_m\partial^{\nu_m}\varphi_{\mathbf{y}_m}};e^{i\alpha\Phi(X_{v_{m+1}})};\ldots;e^{i\alpha\Phi(X_{v_s})}) \tag{2.70}$$

is a quantity which satisfies an identity similar to (2.26), that is

$$H_j(\underline{\lambda}) = \sum_{T\in\mathcal{T}_s}\prod_{\langle a,b\rangle\in T}c_{a,b}\int dp_T(\underline{t})\ e^{-\frac{1}{2}\widetilde{U}_j(v,\underline{t},\underline{\lambda})} \tag{2.71}$$

where

$$c_{a,b} \stackrel{def}{=} \begin{cases} \widetilde{c}_{a,b} \stackrel{def}{=} -\alpha^2 W_j(X_{v_a}, X_{v_b}) & \text{if } a,b > m \\ \lambda_a \widetilde{c}_{a,b} \stackrel{def}{=} i\alpha\lambda_a \sum_{r\,:\,e_r\in X_{v_b}} \sigma_r \left(\partial^{\nu_a} C_j\right)(\mathbf{y}_a - \mathbf{x}_r) & \text{if } a \le m < b \\ \lambda_a\lambda_b \widetilde{c}_{a,b} \stackrel{def}{=} -\lambda_a\lambda_b \left(\partial^{\nu_a}\partial^{\nu_b} C_j\right)(\mathbf{y}_a - \mathbf{y}_b) & \text{if } a,b \le m\,. \end{cases} \tag{2.72}$$

It follows that

$$\widetilde{G}_j(v_1,\dots,v_s) = \sum_{T\in\bar{\mathcal{T}}'_s} \prod_{\langle a,b\rangle\in T} \widetilde{c}_{a,b} \int dp_T(\underline{t}) e^{-\frac{1}{2}\widetilde{U}_j(v,\underline{t},0)} \tag{2.73}$$

where $\bar{\mathcal{T}}'_s$ is the set of $T \in \bar{\mathcal{T}}_s$, such that all special endpoints are leaves of T. Note that $\widetilde{U}_j(v,\underline{t},0)$ is a positive quantity, since it is a convex combination of "interaction energies" which do not involve the special vertices; hence we can safely bound the r.h.s. of (2.73), as in the previous sections. Let us define

$$\widetilde{b}_\tau(\underline{\mathbf{y}}) = \frac{1}{2^n} \sum_{\underline{\nu},\underline{\sigma}} \int_{\Lambda^n} d\underline{\mathbf{x}}\, |\widetilde{V}_\tau(\underline{\sigma},\underline{\nu},\underline{\mathbf{x}},\underline{\mathbf{y}}_\tau)| \tag{2.74}$$

The bound of $\widetilde{b}_\tau(\underline{\mathbf{y}})$ differs from the r.h.s. of (2.30) for the following reasons:

1) there is a γ^{k_i} factor more, coming from the field derivative, for the i-th special endpoint, if k_i is the scale label of the higher n.t. vertex preceding it (the vertex where it is contracted);
2) there is a factor $\gamma^{h_v(s_v^*-1)}$ more, which takes into account the fact that the special endpoints positions are not integrate, for each n.t. vertex v such that $s_v^* > 0$;
3) if $\delta = \min_{1\le i<j\le k} |\mathbf{y}_i - \mathbf{y}_j|$, there is a factor $\exp(-c\gamma_{h_v}\delta)$ for each n.t. vertex v such that $s_v^* > 0$, coming from the same argument used in the case of the charge correlation functions.

Hence, if $m_\tau \ge 1$ is the number of special endpoints in τ, we get

$$\widetilde{b}_\tau(\underline{\mathbf{y}}) \le C_\varepsilon^{n+m_\tau} \left(\prod_{i=1}^n \gamma^{\frac{\alpha^2}{4\pi}h_{e_i}}\right) \left(\prod_{n.t.\ v\in\tau} \gamma^{-2h_v(s_v-1)} e^{2\varepsilon n_v} \prod_{i=1}^{m_\tau} \gamma^{k_i}\right) \cdot \\ \cdot \prod_{n.t. v: s_v^*>0} \gamma^{2h_v(s_v^*-1)} e^{-c\gamma^{h_v}\delta} \tag{2.75}$$

The last product can be bounded as in (2.61), so that, by "distributing along the tree" the other factors, we get

$$\widetilde{b}_\tau(\underline{\mathbf{y}}) \le C_\varepsilon^{n+k} (\delta)^{-2(m_\tau-1)} (2m_\tau - 2)! \prod_{n.t.\ v\in\tau} \gamma^{-(h_v-h_{v'})(\widetilde{D}(n_v,m_v)-2\varepsilon n_v)} \tag{2.76}$$

where n_v and m_v denote the number of normal and special endpoints following v, respectively, and

$$\widetilde{D}(n,m) = 2(n-1) - \frac{\alpha^2}{4\pi} n + m \tag{2.77}$$

Let us consider first the case $\alpha^2 < 4\pi$. Since $n_v + m_v \geq 2$, $\widetilde{D}(n_v, m_v)$ is always positive, except if $n_v = 0$ and $m_v = 2$. However, no tree may have a non trivial vertex of this type, except the trees with only two special endpoints and no normal endpoint, that is the trees belonging to $\mathcal{T}_{0,2}^{(N)}$, and it is very easy to see that

$$\sum_{\tau\in\mathcal{T}_{0,2}^{(N)}} \widetilde{V}_\tau(\nu_1,\nu_2,\mathbf{y}_1,\mathbf{y}_2) = -\sum_{j=0}^{N}\gamma^{2j}\left(\partial^{\nu_1}\partial^{\nu_2}C_0\right)\left(\gamma^j(\mathbf{y}_1-\mathbf{y}_2)\right) \tag{2.78}$$

By (2.5), this quantity has a finite limit as $N\to\infty$, if $\mathbf{y}_1 \neq \mathbf{y}_2$, as we are supposing. Hence there is no ultraviolet divergence in the expansion (2.67) of $B_n^{(N)}(\varphi, \underline{\mathbf{y}}, \underline{\nu}, \underline{\mu})$ and we have only to check that there is no infrared problem related with the integration over the φ field in (2.66). This follows as in §2.2, by using the identity (2.13); it is sufficient to observe that

$$\left|\mathcal{E}_{h,-1}\left[\left(\prod_{i=1}^{m}(\partial^{\nu_i}\varphi_{\mathbf{y}_i})\right)\prod_{j=1}^{s}e^{i\alpha\Phi(X_{v_j})}\right]\right| \leq \sqrt{\mathcal{E}_{h,-1}\left[\prod_{i=1}^{m}|\partial\varphi_{\mathbf{y}_i}|^2\right]} \tag{2.79}$$

and then apply the arguments used in §2.2 to bound the sum over the partitions.

Let us now suppose that $4\pi \leq \alpha^2 < 16\pi/3$. In this case $\widetilde{D}(n_v, m_v)$ can be non positive only if either $m_v = 0$ and $n_v = 2$ or $m_v = n_v = 1$. The vertices satisfying the first condition can be regularized as before, for the others we can use the factor $e^{-\frac{\alpha^2}{2}U_{h_{v'},h_v}(v)} = \gamma^{-\frac{\alpha^2}{4\pi}(h_v - h_{v'}-1)}$ to make their dimension positive; in fact $\widetilde{D}(1,1) + \alpha^2/(4\pi) = 1$. The integration over the φ field in (2.66) can now be done by an obvious modification of the argument used for the charge correlation functions.

It is now easy to prove, as in the previous sections, that $K_{h,N}^{(k,\zeta)}(\underline{\mathbf{y}};\underline{\nu})$ has a finite limit, as $-h, N\to\infty$, if $\delta > 0$, and that this limit is an analytic function of ζ around $\zeta = 0$, with a radius of convergence independent of δ. On the other hand,

$$K_{h,N}^{(k,\zeta)}(\underline{\mathbf{y}};\underline{\nu}) = \sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\frac{1}{2^n}\sum_{\underline{\sigma}}\int d\mathbf{x}_1\cdots d\mathbf{x}_n$$
$$\mathcal{E}_{h,N}^T\left[\partial^{\nu_1}\varphi_{\mathbf{y}_1};\ldots;\partial^{\nu_k}\varphi_{\mathbf{y}_k};{:}e^{i\alpha\sigma_1\varphi_{\mathbf{x}_1}}{:};\ldots;{:}e^{i\alpha\sigma_n\varphi_{\mathbf{x}_n}}{:}\right] \tag{2.80}$$

An argument similar to that used at the end of §2.2 allows us to prove that the power expansion of $K^{(k,\zeta)}(\underline{\mathbf{y}};\underline{\nu})$ is obtained by the previous equation, by substituting in the r.h.s $\mathcal{E}_{h,N}^T$ with $\mathcal{E}^T$. Moreover, it is not hard to check that, if $n > 0$, $Q \equiv \sum_{i=1}^n \sigma_i$ and $h_{k,n}(\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu})$ is the limiting value of the truncated expectation in (2.80), we have

$$h_{k,n}(\underline{\mathbf{x}},\underline{\mathbf{y}};\underline{\sigma},\underline{\nu}) = \delta_{Q,0}\, c^{-\frac{\alpha^2}{8\pi}n}\left(\prod_{r=1}^{k}W(\mathbf{y}_r,\underline{\mathbf{x}};\nu_r,\underline{\sigma})\right)\cdot \tag{2.81}$$
$$\cdot\left\{\sum_{\Pi}(-1)^{|\Pi|-1}(|\Pi|-1)!\prod_{Y\in\Pi}\prod_{\substack{r,s\in Y\\ r<s}}|\mathbf{x}_r-\mathbf{x}_s|^{\sigma_r\sigma_s\frac{\alpha^2}{2\pi}}\right\}$$

where $\sum_{\Pi}$ denotes the sum over the partitions of the set $(1,\ldots,n)$ and

$$W(\mathbf{y},\underline{\mathbf{x}};\underline{\sigma},\nu) = i\alpha\sum_{i=1}^{n}\sigma_i\left(\partial^\nu\Delta^{-1}\right)(\mathbf{y}-\mathbf{x}_i) = \frac{i\alpha}{2\pi}\sum_{i=1}^{n}\sigma_i\frac{(\mathbf{x}_i-\mathbf{y})^\nu}{|\mathbf{y}-\mathbf{x}_i|^2} \tag{2.82}$$

while, if $n=0$,

$$h_{k,0}(\underline{\mathbf{y}},\underline{\nu}) = \delta_{k,2}\mathcal{E}^T\Big[(\partial^{\nu_1}\varphi_{\mathbf{y}_1});(\partial^{\nu_2}\varphi_{\mathbf{y}_2})\Big] = \delta_{k,2}h^{\nu_1,\nu_2}(\mathbf{y}_1-\mathbf{y}_2) \tag{2.83}$$

with

$$h^{\nu_1,\nu_2}(\mathbf{y}) = \frac{1}{2\pi|\mathbf{y}|^2}\left[\delta^{\nu_1,\nu_2}-2\frac{\mathbf{y}^{\nu_1}\mathbf{y}^{\nu_2}}{|\mathbf{y}|^2}\right] \tag{2.84}$$

Hence, we get that $K^{(k,\zeta)}(\underline{\mathbf{y}};\underline{\nu}) = \sum_{n=0}^\infty \zeta^n \bar h_{k,n}(\underline{\mathbf{y}};\underline{\nu})$, with

$$\bar h_{k,n}(\underline{\mathbf{y}};\underline{\nu}) = \frac{1}{n!}\frac{1}{2^n}\sum_{\substack{\sigma_1,\ldots,\sigma_n\\ \sum_{i=1}^n\sigma_i=0}}\int_{\Lambda^n} d\mathbf{x}_1\cdots d\mathbf{x}_n\; h_{k,n}(\underline{\mathbf{x}},\underline{\mathbf{y}};\underline{\sigma},\underline{\nu}) \tag{2.85}$$

Note that $\bar h_{1,n}(\mathbf{y},\nu)=0$ for any n, since $W(y,\underline{\mathbf{x}},\underline{\sigma},\nu)$ is odd in $\underline{\sigma}$ and the sum in (2.85) is restricted to the $\underline{\sigma}$ such that $Q=0$; hence $K^{(1,\zeta)}(\mathbf{y},\nu)=0$.

3. The Thirring model with a finite volume mass term

The *Generating Functional*, $\mathcal{W}_{h,N}(J,A,\mu)$, of the Thirring model with cutoff and with a mass term in finite volume is defined by the equation

$$\begin{aligned}\mathcal{W}_{h,N}(J,A,\mu) &\stackrel{def}{=} \log\int P_{h,N}(d\psi)\;\exp\Big\{-\lambda Z_N^2 V_L(\psi)+\mu Z_N^{(1)}\int_\Lambda d\mathbf{x}\;\bar\psi_\mathbf{x}\psi_\mathbf{x}+\\ &+Z_N^{(1)}\sum_{\sigma=\pm1}\int d\mathbf{x}\;J^\sigma_\mathbf{x}\left(\bar\psi_\mathbf{x}\Gamma^\sigma\psi_\mathbf{x}\right)+Z_N\sum_{\nu=0,1}\int d\mathbf{x}\;A^\nu_\mathbf{x}\left(\bar\psi_\mathbf{x}\gamma^\nu\psi_\mathbf{x}\right)\Big\}\end{aligned} \tag{3.1}$$

where the free measure $P_{h,N}(d\psi)$ is defined by (1.14), Z_N and $Z_N^{(1)}$ are defined in (1.15), $J^\sigma_\mathbf{z}$ and $A^\mu_\mathbf{y}$ are two-dimensional, external bosonic fields and

$$Z_N^2 V_L(\psi) \stackrel{def}{=} \frac{1}{4}\int_{\Lambda_L} d\mathbf{x}\;\left(Z_N\bar\psi_\mathbf{x}\gamma^\mu\psi_\mathbf{x}\right)^2 + E_{h,N}|\Lambda_L| \quad , \quad \Gamma^\sigma \stackrel{def}{=} \frac{I+\sigma\gamma^5}{2} \tag{3.2}$$

$E_{h,N}$ being the *vacuum counterterm* introduced in (1.13); it is chosen so that $\mathcal{W}_{h,N}(0,0,0)=0$.

Given the set of non coinciding points $\underline{\mathbf{x}}=(\mathbf{x}_1,\ldots,\mathbf{x}_q)$ and the set $\underline{\sigma}=(\sigma_1,\ldots,\sigma_q)$, $\sigma_i=\pm1$, we want to study the Schwinger functions

$$G^{(q,r;\mu)}_{h,N}(\underline{\mathbf{x}},\underline{\mathbf{y}};\underline{\sigma},\underline{\nu}) \stackrel{def}{=} \lim_{a^{-1},L\to\infty}\frac{\partial^{q+r}\mathcal{W}_{h,N}}{\partial J^{\sigma_1}_{\mathbf{x}_1}\cdots\partial J^{\sigma_q}_{\mathbf{x}_q}\partial A^{\nu_1}_{\mathbf{y}_1}\cdots\partial A^{\nu_r}_{\mathbf{y}_r}}(0,0,\mu)\;. \tag{3.3}$$

Theorem 3.1. *If μ and λ are small enough and $q \geq 2$, if $r = 0$, or $q \geq 0$, if $r \geq 1$, the limit*

$$G^{(q,r;\zeta)}(\underline{\mathbf{z}}, \underline{\mathbf{w}}; \underline{\sigma}, \underline{\nu}) \stackrel{def}{=} \lim_{-h,N\to+\infty} G^{(q,r;\zeta)}_{h,N}(\underline{\mathbf{z}}, \underline{\mathbf{w}}; \underline{\sigma}, \underline{\nu}) \tag{3.4}$$

exists and is analytic in μ. In the case $q = r = 0$ (the pressure), the limit does exist and is analytic, up to a divergence in the second order term, present only for $\lambda \leq 0$.

As in §2, we shall give the proof of the above theorem only in the special cases $(q,r) = (k,0)$ and $(q,r) = (0,k)$ separately.

In order to prove Theorem 3.1, we note first that definition (3.3) and the identity $\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} = \sum_\sigma \bar\psi_{\mathbf{x}} \Gamma^\sigma \psi_{\mathbf{x}}$ imply that

$$G^{(q,r;\mu)}_{h,N}(\underline{\mathbf{z}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu}) = \sum_{\substack{p=2n-q\\ n\geq 0}} \sum_{\underline{\sigma}'} \frac{\mu^p}{p!} \int d\underline{\mathbf{x}}\, \overline{\chi}_\Lambda(\underline{\mathbf{x}}) S^{(2n,r)}_{h,N}(\underline{\mathbf{z}}\,\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\sigma}\,\underline{\sigma}', \underline{\nu}) \tag{3.5}$$

where $\underline{\mathbf{z}}\,\underline{\mathbf{x}} = (\mathbf{z}_1, \ldots, \mathbf{z}_q, \mathbf{x}_1, \ldots, \mathbf{x}_p)$, $\underline{\sigma}\,\underline{\sigma}' = (\sigma_1, \ldots, \sigma_q, \sigma'_1, \ldots, \sigma'_p)$, we defined

$$\overline{\chi}_\Lambda(\underline{\mathbf{x}}) \stackrel{def}{=} \chi_\Lambda(\mathbf{x}_1)\cdots\chi_\Lambda(\mathbf{x}_p)\;, \qquad S^{(m,r)}_{h,N}(\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu}) \stackrel{def}{=} G^{(m,r;0)}_{h,N}(\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu})\;. \tag{3.6}$$

and we used the fact that $G^{(m,r;0)}_{h,N}(\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu})$ can be different from 0 only if $\sum_{i=1}^m \sigma_i = 0$, implying in particular that m is even.

In the following we shall give a bound for the functions $S^{(m,r)}_{h,N}(\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu})$, uniform in the cutoffs and implying (by an argument similar to that used for the Sine-Gordon model, that we shall skip here) that the limit exists, is integrable and is exchangeable with the integral in (3.5). It follows that

$$G^{(q,r;\mu)}(\underline{\mathbf{z}}, \underline{\mathbf{y}}; \underline{\sigma}, \underline{\nu}) = \sum_{\substack{p=2n-q\\ n\geq 0}} \sum_{\underline{\sigma}'} \frac{\mu^p}{p!} \int d\underline{\mathbf{x}}\, \overline{\chi}_\Lambda(\underline{\mathbf{x}}) S^{(2n,r)}(\underline{\mathbf{z}}\,\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\sigma}\,\underline{\sigma}', \underline{\nu}) \tag{3.7}$$

We remark that $\overline{\chi}_\Lambda(\underline{\mathbf{x}})$ *is not a regular test function* since it is not vanishing for coinciding points, and hence we could encounter divergences caused by the ultraviolet problem. Indeed, as we shall see, the integration of $G^{(2,0;0)}$ will be finite only for $\lambda > 0$ (and small in absolute value), so that the pressure $G^{(0,0;\mu)}$ and the, if $\mathbf{x} \in \Lambda$, "density" $G^{(1,0;\mu)}(\mathbf{x}, \sigma)$ are really divergent for $\lambda \leq 0$, since this is true for the terms with $2n = 2$ and $r = 0$ in the r.h.s. of (3.7).

As announced in the introduction, we first consider the case $q, r = 0$ then we discuss the case $q > 0$ and $r = 0$; and finally the case $q = 0$ and $r > 0$.

3.1. Case $q = r = 0$ (the pressure). Our definitions imply that $S^{(0,0)} = 0$. If $m \geq 2$ and even (otherwise it is 0 by symmetry), the m-points Schwinger function $S^{(m,0)}(\underline{\mathbf{x}}, \underline{\sigma})$ is obtained as the m-th order functional derivative of the generating function $\mathcal{W}_{h,N}(J, 0, 0)$ with respect to $J^{\sigma_1}_{\mathbf{x}_1}, \ldots, J^{\sigma_m}_{\mathbf{x}_m}$ at $J = 0$. We can proceed as in [BFM] and we get an expansion similar to eq. (2.28) of that paper, which we refer to for the notation. The only difference is that the special endpoints of type

J are associated with the terms $Z_j^{(1)}\sum_\sigma \bar\psi_{\mathbf{x}}\Gamma^\sigma\psi_{\mathbf{x}} = Z_j^{(1)}\sum_\sigma \psi^+_{\mathbf{x},-\sigma}\psi^-_{\mathbf{x},\sigma}$ instead of $Z_j\sum_\sigma \psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}$, but this does not change the structure of the expansion; we only have to add, for each special endpoint of scale h_i, a factor $Z^{(1)}_{h_i}/Z_{h_i}$, which can be controlled by studying the flow of $Z_j^{(1)}$. It turns out that there are two constants $\eta_+(\lambda) = b\lambda + O(\lambda^2)$, $b > 0$, and $c_+(\lambda) = 1 + O(\lambda)$, such that, in the limit $N \to \infty$, $Z_j^{(1)} = c_+(\lambda)\gamma^{-\eta_+ j}$; this result is obtained by an argument similar to that used in [BFM] to prove that there are two constants $\eta_-(\lambda) = a\lambda^2+O(\lambda^3)$, $a > 0$, and $c_-(\lambda) = 1 + O(\lambda)$, such that, in the limit $N \to \infty$, $Z_j = c_-(\lambda)\gamma^{-\eta_- j}$ (in [BFM] $c_-(\lambda) = 1$, since the definition of Z_N differs by a constant chosen so to get this result). In analogy to eq. (2.40) of [BFM], we can write

$$S^{(m,0)}(\underline{\mathbf{x}},\underline{\sigma}) = m! \lim_{|h|,N\to\infty} \sum_{n=0}^{\infty}\sum_{j_0=-\infty}^{N-1}\sum_{\tau\in\mathcal{T}^{0,m}_{j_0,n}}\sum_{\mathbf{P}\in\mathcal{P}} S_{0,m,\tau,\underline{\sigma}}(\underline{\mathbf{x}}) \;, \tag{3.8}$$

Given a tree τ contributing to the r.h.s. of (3.8), we call τ^* the tree which is obtained from τ by erasing all the vertices which are not needed to connect the m special endpoints (all of type J). The endpoints of τ^* are the m special endpoints of τ, which we denote v_i^*, $i = 1,\ldots,m$; with each of them a space-time point $\mathbf{x}_i$ and a label σ_i are associated. Given a vertex $v \in \tau^*$, we shall call $\underline{\mathbf{u}}_v$ the set of the space-time points associated with the normal endpoints of τ that follow v in τ (in [BFM] they were called *internal points*); $\underline{\mathbf{x}}_v$ will denote the subset of $\underline{\mathbf{x}}$ made of all points associated with the endpoints of τ^* following v.

Furthermore, we shall call s_v^* the number of branches of τ^* following $v \in \tau^*$, $s_v^{*,1}$ the number of branches containing only one endpoint and $s_v^{*,2} = s_v^* - s_v^{*,1}$. For each n.t. vertex or endpoint $v \in \tau^*$, shortening the notation of s_v^* into s, we choose one point in $\underline{\mathbf{x}}_v$, let it be called $\mathbf{w}_v$, with the only constraint that, if $v_1,\ldots,v_s$ are the n.t. vertices or endpoints following v, then $\mathbf{w}_v$ is one among $\underline{\mathbf{w}}_v \equiv \{\mathbf{w}_{v_1},\ldots,\mathbf{w}_{v_s}\}$.

The bound of $S_{0,m,\tau,\underline{\sigma}}(\underline{\mathbf{x}})$ will be done as in [BFM], by comparing it with the bound of its integral over $\underline{\mathbf{x}}$, given by eq. (2.36) of that paper. However, we shall slightly modify the procedure, to get an estimate more convenient for our actual needs.

Given the space-time points $\underline{\mathbf{v}} = (\mathbf{v}_1,\ldots,\mathbf{v}_p)$ connected by the tree T, we shall define, if $\mathbf{v}_{l,i}$ and $\mathbf{v}_{l,f}$ denote the endpoints of the line $l \in T$,

$$D_T(\underline{\mathbf{v}}) \stackrel{def}{=} \|T\| \stackrel{def}{=} \sum_{l\in T}\sqrt{|\mathbf{v}_{l,i}-\mathbf{v}_{l,f}|} \tag{3.9}$$

Now we want to show that, from the bounds of the propagators associated with the lines l of the spanning tree $T_\tau = \bigcup_v T_v$, we can extract a factor $e^{-c'\sqrt{\gamma^{h_v}}D_{C_v}(\underline{\mathbf{w}}_v)}$ for each n.t. vertex $v \in \tau^*$, where C_v is a chain of segments that only depends on τ and T_τ, and connects the space-time points $\underline{\mathbf{w}}_v$.

Indeed, given a n.t. $v \in \tau^*$, there is a subtree T_v^* of T_τ connecting the points $\underline{\mathbf{w}}_v$ together with a subset of $\underline{\mathbf{x}}_v \cup \underline{\mathbf{u}}_v$. Since T_v^* is made of lines of scale $j \geq h_v$, the decaying factors in the bounds of the propagator in T_v^* can be written as

$$e^{-c\sqrt{\gamma^h|\mathbf{x}|}} = e^{-\frac{c}{2}\sqrt{\gamma^h|\mathbf{x}|}} \cdot e^{-2c'\sum_{j=-\infty}^{h}\sqrt{\gamma^j|\mathbf{x}|}} \quad , \tag{3.10}$$

for $c' = c/\left[4\sum_{j=0}^{\infty}\gamma^{-j/2}\right]$. Hence, collecting the latter factor for each of the lines T_v^* we obtain $e^{-2c'\sqrt{\gamma^{h_v}}\|T_v^*\|}$.

We finally would like to replace, in the previous bound, $\|T_v^*\|$ with $\|C_v\|$, up to a constant, *for a* C_v *which does not depend on the position of the internal points of* T_v^*. This is possible as a consequence of the following lemma.

Lemma 3.1. *Let* T *be a tree graph connecting the points* $\{\mathbf{w}_j\}_{j=1}^l$ *together with other "internal points",* $\{\mathbf{u}_j\}_{j=1}^q$*. Then there exists a chain* C *connecting all and only the points* $\{\mathbf{w}_j\}_{j=1}^l$ *such that* $2\|T\| \geq \|C\|$ *and* C *only depends on* T*.*

Proof. Suppose that the points $\{\mathbf{u}_j\}_{j=1}^q$ are fixed in an arbitrary way and let us consider the oriented closed path $\bar{\mathcal{C}}$ obtained by "circumnavigating" T, for example in the clockwise direction; this path contains twice each branch of T, with both possible orientations. We shall call $\mathcal{C}$ the oriented closed path obtained by continuous deformation of $\bar{\mathcal{C}}$, as the points $\{\mathbf{u}_j\}_{j=1}^q$ vary in $\mathbb{R}^2$. The path $\mathcal{C}$ allows us to reorder the points $\mathbf{w}_1, \ldots, \mathbf{w}_l$ into $\mathbf{w}_{t(1)}, \ldots, \mathbf{w}_{t(l)}$, by putting $t(1) = 1$ and by choosing $t(i+1)$, $1 \leq i \leq l-1$, so that $\mathbf{w}_{t(i+1)}$ is the point following $\mathbf{w}_{t(i)}$ on $\mathcal{C}$. The chain C is obtained by joining with a segment $\mathbf{w}_{t(i)}$ and $\mathbf{w}_{t(i+1)}$, for $i = 1, \ldots, l-1$; the condition $2\|T\| \geq \|C\|$ then easily follows from the triangle inequality for the function $x \to |x|^{1/2}$. ∎

As a consequence of the above lemma and (3.10), we can extract from the propagator bounds, for each choice of T_v, a factor $\prod_{v\in\tau^*} e^{-c'\sqrt{\gamma^{h_v}}D_{C_v}(\underline{\mathbf{w}}_v)}$, which does not depend on the internal points positions, by leaving a factor $e^{-(c/2)\sqrt{\gamma^j|\mathbf{x}-\mathbf{y}|}}$ for each propagator of T_τ, to be used for bounding the integral over the internal points.

The final bound of $S_{0,m,\tau,\underline{\sigma}}(\underline{\mathbf{x}})$ will be obtained by "undoing", in the r.h.s of eq. (2.36) of [BFM], the sum over T_v for any $v \in \tau^*$ (note that C_v depends on T_v^* and hence on T_v), then adding the factors coming from the previous considerations, together with a factor taking into account that there are $1 + \sum_{v\in\tau^*}(s_v^* - 1) = m$ integrations less to do. By suitably choosing them, the lacking integrations produce in the bound an extra factor $\prod_{v\in\tau^*}\gamma^{2h_v(s_v^*-1)}L^{-2}$, so that we get

$$|S_{0,m,\tau,\underline{\sigma}}(\underline{\mathbf{x}})| \leq C^m(C\bar{\lambda}_{j_0})^n\gamma^{-j_0(-2+m)}\left[\prod_{\text{v not e.p.}}\left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{|P_v|/2}\gamma^{-d_v}\right]\cdot$$
$$\left(\prod_{i=1}^m\frac{Z_{h_i}^{(1)}}{Z_{h_i}}\right)\left(\prod_{n.t.v\in\tau^*}\frac{1}{s_v!}\sum_{T_v}\gamma^{2h_v(s_v^*-1)}e^{-c'\sqrt{\gamma^{h_v}}D_{C_v}(\underline{\mathbf{w}}_v)}\right) \tag{3.11}$$

where h_i is the scale of the i-th endpoint of type J and

$$d_v \stackrel{def}{=} -2 + m_v + |P_v|/2 + z_v\ , \tag{3.12}$$

with $m_v = |\mathbf{X}_v|$ and z_v equal to the parameter $\widetilde{z}(P_v)$ defined by eq. (2.38) of [BFM].

We have now to bound the integral of $S_{0,m,\tau,\underline{\sigma}}(\underline{\mathbf{x}})\overline{\chi}_\Lambda(\underline{\mathbf{x}})$, let us call it $I_{m,\tau,\underline{\sigma}}$. In order to exploit the improvement related with the restriction of the integration variables to a fixed volume of size 1, we shall proceed in a way different with respect to that followed in [BFM], that is we bound the integral *before* the sums over the trees T_v. We use the bound:

$$\int d\mathbf{x}\ \chi_\Lambda(\mathbf{x}) e^{-c'\sqrt{\gamma^h|\mathbf{x}-\mathbf{y}|}} \le C \begin{cases} \gamma^{-2h} & \text{if } h>0 \\ 1 & \text{if } h\le 0 \end{cases} \tag{3.13}$$

The sum over the tree graphs is done in the usual way and we get

$$\begin{aligned} I_{m,\tau,\underline{\sigma}} \le & C^m (C\bar\lambda_{j_0})^n \gamma^{-j_0(-2+m)} \left(\prod_{n.t. v\in\tau^*} \gamma^{2h_v(s_v^*-1)} \right) \left(\prod_{i=1}^m \frac{Z^{(1)}_{h_i}}{Z_{h_i}} \right) \cdot \\ & \cdot \left[\prod_{\text{v not e.p.}} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{|P_v|/2} \gamma^{-d_v} \right] \left(\prod_{n.t. v\in\tau^*}^{h_v>0} \gamma^{-2h_v(s_v^*-1)} \right) \end{aligned} \tag{3.14}$$

Let us now call E_i the family of trivial vertices belonging to the branch of τ^* which connects v_i^* with the higher non trivial vertex of τ^* preceding it and note that, by the remark preceding (3.8), $Z^{(1)}_{h_i}/Z_{h_i} \le C\gamma^{-h_i\bar\eta}$, with $\bar\eta = c_0\lambda + O(\bar\lambda^2_{j_0})$, $c_0>0$. Hence, the definition of $s_v^{*,1}$ implies that, if $E=\cup_i E_i$,

$$\prod_{i=1}^m \frac{Z^{(1)}_{h_i}}{Z_{h_i}} \le C^m \left(\prod_{v\in E} \gamma^{-\bar\eta} \right) \prod_{n.t. v\in\tau^*} \gamma^{-h_v\bar\eta s_v^{*,1}} \ . \tag{3.15}$$

Let v_0^* be the first vertex with $s_v^*\ge 2$ following v_0 (recall that v_0 is the vertex immediately following the root of τ, of scale j_0+1); since $m\ge 2$, this vertex is certainly present. Then, since $m_v=m$ for $v_0\le v\le v_0^*$, we have the identity

$$\gamma^{-j_0(-2+m)} \prod_{v_0\le v<v_0^*} \gamma^{-d_v} = \gamma^{-h_{v_0^*}(-2+m_{v_0^*})} \prod_{v_0\le v<v_0^*} \gamma^{-\widetilde d_v} \ , \tag{3.16}$$

where we used the definition $\widetilde d_v = d_v-(-2+m_v) = \frac{|P_v|}{2}+z_v$; note that $\widetilde d_v \ge 1/2$, for any $v\in\tau^*$, $v>v_0$.

By inserting (3.15) and (3.16) in the r.h.s. of (3.14), we get

$$\begin{aligned} I_{m,\tau,\underline{\sigma}} \le & C^m (C\bar\lambda_{j_0})^n \left[\prod_{\substack{v\notin\tau^* \\ v\ not\ e.p.}} \gamma^{-d_v} \right] \left[\prod_{v\in E} \gamma^{-d_v-\bar\eta} \right] \left[\prod_{v_0\le v<v_0^*} \gamma^{-\widetilde d_v} \right] \cdot \\ & \cdot \left[\prod_{v\ not\ e.p.} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{|P_v|/2} \right] \left(\prod_{n.t. v\in\tau^*}^{h_v>0} \gamma^{-2h_v(s_v^*-1)} \right) F_\tau \ , \end{aligned} \tag{3.17}$$

where

$$F_\tau = \gamma^{-h_{v_0^*}(-2+m_{v_0^*})} \left[\prod_{n.t. v\in\tau^*} \gamma^{h_v\left[2(s_v^*-1)-\bar\eta s_v^{*,1}\right]} \right] \left(\prod_{v\in(\tau^*\setminus E)}^{v\ge v_0^*} \gamma^{-d_v} \right) \ . \tag{3.18}$$

Given a n.t. vertex $v \in \tau^*$, let $s = s_v^*$, $s_1 = s_v^{*,1}$, $s_2 = s - s_1$ and $v_1, \dots, v_{s_2}$ the n.t. vertices immediately following v in τ^*. Since $m_v = s_1 + \sum_{i=1}^{s_2} m_{v_i}$, we can write

$$-(-2+m_v) + [2(s-1) - \bar{\eta}s_1] = s_1(1-\bar{\eta}) - \sum_{i=1}^{s_2}(-2+m_{v_i}) \; . \tag{3.19}$$

This identity, applied to the vertex v_0^*, implies that, if $v_1, \dots, v_{s_2}$, $s_2 = s_{v_0^*}^* - s_{v_0^*}^{*,1}$, are the n.t. vertices immediately following v_0^* in τ^*, then

$$\begin{aligned} &\gamma^{-h_{v_0^*}(-2+m_{v_0^*})}\gamma^{h_{v_0^*}\left[2(s_{v_0^*}^*-1)-\bar{\eta}s_{v_0^*}^{*,1}\right]} = \\ &= \gamma^{\alpha'_{v_0^*}h_{v_0^*}}\left[\prod_{i=1}^{s_2}\gamma^{-h_{v_i}(-2+m_{v_i})}\cdot\prod_{v\in\mathcal{C}_i}\gamma^{-2+m_v}\right] \; , \end{aligned} \tag{3.20}$$

where $\mathcal{C}_i$ is the path connecting v_0^* with v_i in τ^* (not including v_i) and we used the definition

$$\alpha'_v = s_v^{*,1}(1-\bar{\eta}) \; . \tag{3.21}$$

The presence of the factor $\gamma^{-h_{v_i}(-2+m_{v_i})}$ for each vertex v_i in the r.h.s. of (3.20) implies that an identity similar to (3.20) can be used for each n.t. vertex $v \in \tau^*$. It is then easy to show that

$$F_\tau = \left[\prod_{n.t. v\in\tau^*}\gamma^{\alpha'_v h_v}\right]\left[\prod_{v\in(\tau^*\setminus E)}^{v\ge v_0^*}\gamma^{-\widetilde{d}_v}\right] \; . \tag{3.22}$$

By inserting this equation in (3.17), we get

$$\begin{aligned} I_{m,\tau,\underline{\sigma}} \le\; & C^m(C\bar{\lambda}_{j_0})^n\left(\prod_{n.t. v\in\tau^*}\gamma^{\alpha'_v h_v}\right)\left(\prod_{n.t. v\in\tau^*}^{h_v>0}\gamma^{-2h_v(s_v^*-1)}\right)\cdot \\ &\cdot\left[\prod_{v\ not\ e.p.}\left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{|P_v|/2}\gamma^{-d'_v}\right] \; , \end{aligned} \tag{3.23}$$

where

$$d'_v = \begin{cases} \widetilde{d}_v & \text{if } v\in\tau^*,\ v_0\le v\notin E \\ d_v+\bar{\eta} & \text{if } v\in E \\ d_v & \text{otherwise} \end{cases} \tag{3.24}$$

and it is always strictly greater than zero, if $\bar{\eta}$ (hence λ) is small enough. This allows us to control the sum over the τ scale labels in the usual way, by keeping fixed $h_{v_0^*}$. However, before doing that, it is necessary to extract from the r.h.s. of (3.23) some factors needed to control the sum over $h_{v_0^*}$ too. First of all, in order to control the sum over the negative values of $h_{v_0^*}$, we try to replace the non-negative quantity α'_v with another non-negative one, α_v, s.t. $\alpha_{v_0^*}$ is *strictly positive*, paying a price in the dimension of the vertices; this can be easily achieved by fixing $\varepsilon > 0$ and using the inequality

$$1 \le \gamma^{\varepsilon h_{v_0^*}}\left(\prod_{n.t. v\in\tau^*}\gamma^{\varepsilon(s_v^{*,2}-1)h_v}\right)\prod_{v\in\tau^*, v\notin E}\gamma^{\varepsilon} \tag{3.25}$$

which allows us to replace (3.23) with

$$I_{m,\tau,\underline{\sigma}} \le C^m (C\bar\lambda_{j_0})^n \gamma^{\varepsilon h_{v_0^*}} \left(\prod_{n.t.v\in\tau^*} \gamma^{\alpha_v h_v} \right) \left(\prod_{n.t.v\in\tau^*}^{h_v>0} \gamma^{-2h_v(s_v^*-1)} \right) \cdot$$
$$\cdot \left[\prod_{v\ not\ e.p.} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{|P_v|/2} \gamma^{-\bar d_v} \right] , \tag{3.26}$$

with $\alpha_v = \alpha'_v + \varepsilon(s_v^{*,2} - 1)$ and $\bar d_v = d'_v - \varepsilon$, if $v_0^* \le v \notin E$; and $\bar d_v = d'_v$ otherwise.

Let us now define $\chi_v = 1$ if $h_v > 0$ and $\chi_v = 0$ for $h_v \le 0$. If we put $w = v_0^*$, we can write

$$\gamma^{\varepsilon h_{v_0^*}} \left[\prod_{n.t.v\in\tau^*} \gamma^{\alpha_v h_v} \right] \left(\prod_{n.t.v\in\tau^*}^{h_v>0} \gamma^{-2h_v(s_v^*-1)} \right) = \tag{3.27}$$
$$= \gamma^{\left[\alpha_w + \varepsilon - 2\chi_w(s_w^*-1)\right] h_w} \prod_{\substack{n.t.v\in\tau^* \\ v\neq w}} \gamma^{\left[\alpha_v - 2\chi_v(s_v^*-1)\right] h_v}$$

and, if $|\lambda| << |\varepsilon| < 1/2$, we use the two straightforward inequalities

$$\begin{aligned} \alpha_v &= \varepsilon(s_v^* - 1) + s_v^{*,1}(1 - \bar\eta - \varepsilon) \ge \varepsilon \\ \alpha_v - 2(s_v^* - 1) &= (2-\varepsilon) - (1+\bar\eta)s_v^* - s_v^{*,2}(1 - \varepsilon - \bar\eta) \\ &\le (2-\varepsilon) - (1+\bar\eta)s_v^* < 0 \end{aligned} \tag{3.28}$$

Hence, the two terms in square brackets in the r.h.s. of (3.27) can be bounded by c^m, *irrespective of the sign of* $\bar\eta$, that is the sign of λ.

Thanks to these arguments, we can replace (3.26) with

$$I_{m,\tau,\underline{\sigma}} \le C^m (C\bar\lambda_{j_0})^n \gamma^{\left[\alpha_w + \varepsilon - 2\chi_w(s_w^*-1)\right] h_w} \left[\prod_{v\ not\ e.p.} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{|P_v|/2} \gamma^{-\bar d_v} \right] \tag{3.29}$$

Now the sum over the tree scale labels, as well as the sum over the trees with a fixed value of h_w, can be performed in the usual way, using the fact that $\bar d_v$ is always strictly positive and bounded below by a quantity proportional to $\bar d_v$ itself; this gives a C^m bound. We finally have to bound the sum over h_w; note that

$$\sum_{h_w=-\infty}^{+\infty} \gamma^{\left[\alpha_w + \varepsilon - 2\chi_w(s_w^*-1)\right] h_w} = \sum_{h>0} \gamma^{\left[\alpha_w + \varepsilon - 2(s_w^*-1)\right] h} + \sum_{h\le 0} \gamma^{\left[\alpha_w + \varepsilon\right] h} \tag{3.30}$$

The second sum is always finite since $\alpha_w + \varepsilon \ge \varepsilon s_w^* \ge 2\varepsilon$. Regarding the first sum, we note that

$$\begin{aligned} \alpha_w + \varepsilon - 2(s_w^* - 1) &= 2 - (1+\bar\eta)s_w^* - s_w^{*,2}(1 - \varepsilon - \bar\eta) \\ &\le 2 - (1+\bar\eta)s_w^* \end{aligned} \tag{3.31}$$

Hence, the sum is always bounded, except in the case $\bar\eta \le 0$ (that is $\lambda \le 0$) with $s_w^* = s_w^{*,1} = 2$. It follows, by (3.8), that

$$\int d\underline{\mathbf{x}}\, \overline{\chi}_\Lambda(\underline{\mathbf{z}}) |S^{(m)}(\underline{\mathbf{z}}, \underline{\sigma})| \le m! C^m \quad , \quad m \ge 3 \tag{3.32}$$

so that, by (3.7), the pressure can be defined only by subtracting from $G^{(0,\mu)}$ the term with $m = 2$. The *renormalized pressure* is analytic in μ, for μ small enough.

3.2. Case $q \geq 2$, $r = 0$. We have for $S^{(m)}(\underline{\mathbf{z}}\,\underline{\mathbf{x}};\underline{\sigma},\underline{\sigma}')$ an expansion analogous to (3.8), but now the special endpoints are associated with two different types of space-time points, those which have to be integrated as before ($\underline{\mathbf{x}}$) and those which are fixed ($\underline{\mathbf{z}}$). We denote by $\underline{\mathbf{x}}_v$ and $\underline{\mathbf{z}}_v$ the points following v of the two types and we slightly modify the definition of the point $\mathbf{w}_v$ to be one point in $\underline{\mathbf{z}}_v$, if $\underline{\mathbf{z}}_v \neq \emptyset$, or one point in $\underline{\mathbf{x}}_v$, otherwise; we still require that $\mathbf{w}_v \in \underline{\mathbf{w}}_v$.

We want to mimic the strategy used for the Sine-Gordon correlations functions. Therefore we introduce a new tree τ^o, that is obtained from τ^* by erasing all the vertices which are not needed to connect the q special endpoints carrying a space-time point of type $\mathbf{z}$. (We remark that the roles of the trees τ and τ^* of the bosonic theory here are played by τ^* and τ^o respectively). Correspondingly, we define s_v^o the number of the branches of τ^o following $v \in \tau^o$; mote that the space-time points associated with the endpoints of τ^o following $v \in \tau^o$ are those in $\underline{\mathbf{z}}_v$, hence $\sum_{w\geq v}(s_w^o - 1) = |\underline{\mathbf{z}}_v| - 1$.

A bound similar to (3.11) holds. In this case, anyway, we prefer to have a separate decaying factor in the distance of the points $\underline{\mathbf{z}}$: for each nontrivial vertex v of τ^o

$$D_{C_v}(\underline{\mathbf{w}}_v) \geq \frac{1}{2} D_{\widetilde{C}_v}(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v) + \frac{1}{2} D_{C_v}(\underline{\mathbf{w}}_v)\ . \tag{3.33}$$

where $\widetilde{C}_v$ denotes the ordered path connecting the points in $(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v)$, made of lines which connect a point with that following it in the ordered path C_v, see Lemma 3.1.

Therefore, in place of (3.11), we have:

$$\begin{aligned}|S_{0,m,\tau,\underline{\sigma}}(\underline{\mathbf{z}},\underline{\mathbf{x}})| &\leq C^m (C\bar{\lambda}_{j_0})^n \gamma^{-j_0(-2+m)} \left[\prod_{\text{v not e.p.}} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{|P_v|/2} \gamma^{-d_v} \right] \cdot \\ &\cdot \left(\prod_{i=1}^m \frac{Z_{h_i}^{(1)}}{Z_{h_i}} \right) \prod_{n.t.v\in\tau^*} \frac{1}{s_v!} \sum_{T_v} \gamma^{2h_v(s_v^*-1)} \exp\left\{-\frac{c'}{2}\gamma^{\frac{h_v}{2}} D_{C_v}(\underline{\mathbf{w}}_v)\right\} \cdot \\ &\cdot \prod_{n.t.v\in\tau^o} \exp\left\{-\frac{c'}{2}\gamma^{\frac{h_v}{2}} D_{\widetilde{C}_v}(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v)\right\}\end{aligned} \tag{3.34}$$

We can repeat, with no essential modification, the steps that from (3.11) have led to (3.14). Hence, if we call $I_{m,\tau,\underline{\sigma}}(\underline{\mathbf{z}})$ the integral over $\underline{\mathbf{x}}$ of $S_{0,m,\tau,\underline{\sigma}}(\underline{\mathbf{z}\mathbf{x}})\overline{\chi}_\Lambda(\underline{\mathbf{x}})$, we get the bound:

$$\begin{aligned}I_{m,\tau,\underline{\sigma}}(\underline{\mathbf{z}}) &\leq C^m (C\bar{\lambda}_{j_0})^n \gamma^{-j_0(-2+m)} \left(\prod_{n.t.v\in\tau^*} \gamma^{2h_v(s_v^*-1)} \right) \cdot \\ &\cdot \left(\prod_{i=1}^m \frac{Z_{h_i}^{(1)}}{Z_{h_i}} \right) \left[\prod_{\text{v not e.p.}} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{|P_v|/2} \gamma^{-d_v} \right] \left(\prod_{n.t.v\in\tau^*}^{h_v>0} \gamma^{-2h_v(s_v^*-1)} \right) .\end{aligned}$$

$$\cdot \left(\prod_{n.t.v \in \tau^o}^{h_v > 0} \gamma^{2h_v(s_v^o - 1)} \right) \prod_{n.t.v \in \tau^o} \exp\left\{ -\frac{c'}{2} \gamma^{\frac{h_v}{2}} D_{\widetilde{C}_v}(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v) \right\} \tag{3.35}$$

Indeed, we observe that the chain C_v is a spanning tree of propagators with root in one of the $\mathbf{z}_v$ points (if any, see the definition of $\mathbf{w}_v$). Hence, integrating down the position of the vertices $\underline{\mathbf{x}}_v$ from the endpoints of such a tree to the root, in the case at hand there are, with respect to the procedure for $q = 0$, $s_v^o - 1$ missing integration for each nontrivial vertex v of the tree τ^o. By (3.13), this means a factor $\gamma^{-2h_v(s_v^o - 1)}$ less, if $h_v > 0$, and a constant factor less, if $h_v \leq 0$; this explains the last line of (3.35). Going on in parallel with §3.1, we obtain the analogous of (3.29); recalling that w is the lowest n.t. vertex of the tree τ^*,

$$I_{m,\tau,\underline{\sigma}}(\underline{\mathbf{z}}) \leq C^m (C\bar{\lambda}_{j_0})^n \gamma^{\left[\alpha_w + \varepsilon - 2\chi_w(s_w^* - 1)\right] h_w} \left[\prod_{v \ not \ e.p.} \left(\frac{Z_{h_v}}{Z_{h_v - 1}} \right)^{|P_v|/2} \gamma^{-\bar{d}_v} \right] \cdot$$

$$\cdot \left(\prod_{n.t.v \in \tau^o}^{h_v > 0} \gamma^{2h_v(s_v^o - 1)} \right) \prod_{n.t.v \in \tau^o} \exp\left\{ -\frac{c'}{2} \gamma^{\frac{h_v}{2}} D_{\widetilde{C}_v}(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v) \right\} \tag{3.36}$$

At this point, in contrast with the pressure bound, we want to take advantage of the exponential fall off in the diameter of $\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v$ to prove the convergence of the correlations (with $q \geq 2$) *for any sign of* $\bar{\eta}$.

Note that our definitions imply that $\cup_{n.t.v \in \tau^0} \underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v = \underline{\mathbf{z}}$ and that $\cup_{n.t.v \in \tau^0} \widetilde{C}_v$ is a tree connecting all the points in $\underline{\mathbf{z}}$. This remark, together with the trivial bound $h_v \geq h_{v_0^*}$, implies that

$$\prod_{n.t.v \in \tau^o} \exp\left\{ -\frac{c'}{4} \gamma^{\frac{h_v}{2}} D_{\widetilde{C}_v}(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v) \right\} \leq \exp\left\{ -\frac{c'}{4} \sqrt{\gamma^{h_{v_0^*}} \mathrm{diam}(\underline{\mathbf{z}})} \right\} \tag{3.37}$$

On the other hand, since $|\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v| \geq 2$ for any *n.t.* $v \in \tau^0$, if we define $\delta \stackrel{def}{=} \min_{i,j} |\mathbf{z}_i - \mathbf{z}_j|$, we have

$$\gamma^{2h_v(s_v^o - 1)} \exp\left\{ -\frac{c'}{4} \gamma^{\frac{h_v}{2}} D_{\widetilde{C}_v}(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v) \right\} \leq \left(\frac{C}{\delta} \right)^{2(s_v^0 - 1)} (s_v^0 - 1)^{4(s_v^0 - 1)} \tag{3.38}$$

so that, by using also the identity $\sum_{v \in \tau^0} (s_v^0 - 1) = q - 1$,

$$\left(\prod_{n.t.v \in \tau^o}^{h_v > 0} \gamma^{2h_v(s_v^o - 1)} \right) \prod_{n.t.v \in \tau^o} \exp\left\{ -\frac{c'}{2} \sqrt{\gamma^{h_v} \mathrm{diam}(\underline{\mathbf{z}}_v \cap \underline{\mathbf{w}}_v)} \right\}$$

$$\leq [(q-1)!]^4 \left(\frac{C}{\delta} \right)^{2(q-1)} \exp\left\{ -\frac{c'}{4} \sqrt{\gamma^{h_{v_0^*}} \mathrm{diam}(\underline{\mathbf{z}})} \right\} \tag{3.39}$$

Let us now remark that the quantity

$$\left[1 + \mathrm{diam}(\underline{\mathbf{z}})^{(\alpha_w + \varepsilon)} \right] \sum_{h=-\infty}^{+\infty} \gamma^{[\alpha_w + \varepsilon - 2\chi_w(s_w^* - 1)]h} \exp\left\{ -c_0 \sqrt{\gamma^h \mathrm{diam}(\underline{\mathbf{z}})} \right\} \tag{3.40}$$

is bounded by a constant. In fact, the series is convergent also without the exponential, as shown before, and this is sufficient, if $\mathrm{diam}(\underline{\mathbf{z}}) \le 1$; if $\mathrm{diam}(\underline{\mathbf{z}}) = \gamma^{-h_0}$, $h_0 \le 0$, we can bound the series by $2\sum_{h=-\infty}^{+\infty} \gamma^{(\alpha_w+\varepsilon)h} \exp[-c_0\sqrt{\gamma^h}]$, which is convergent, sice $\alpha_w + \varepsilon \ge 2\varepsilon$. Hence we get, by using $2\varepsilon \le a_w + \varepsilon \le q(1+\varepsilon-\bar\eta)$, that there is a constant C_q, such that

$$\int d\underline{\mathbf{x}}\; \overline{\chi}_\Lambda(\underline{\mathbf{x}}) |S^{(m)}(\underline{\mathbf{z}}, \underline{\mathbf{x}}, \underline{\sigma})| \le m! \left(1+\delta^{-2(q-1)}\right) \frac{C_q}{1+\mathrm{diam}(\underline{\mathbf{z}})^{2\varepsilon}} \tag{3.41}$$

3.3. Case $r \ge 1$, $q = 0$. This case is very similar to the previous one; therefore we limit ourself to the discussion of the differences.

Formula (3.34) still holds, with $\underline{\mathbf{y}}_v$ in place of $\underline{\mathbf{z}}_v$ (to be consistent with notation in (3.7)) and with the replacement $\prod_{i=1}^m (Z^{(1)}_{h_i}/Z_{h_i}) \longrightarrow \prod_{i=1}^p (Z^{(1)}_{h_{\underline{i}}}/Z_{h_i})$, following from the fact that the strength renormalization of the field $\bar\psi\gamma^\mu\psi$ is equal to Z_h. It is easy to go along the developments of §3.2 again, up to a couple of differences. The minor one is that in formulas (3.15) and (3.24) the set E has to be replaced with the set $E\backslash Y$, where Y is the family of trivial vertices of τ^* belonging to the branches ending up with an endpoint of type $\mathbf{y}$; but this is not a problem, since the dimensions of all the vertices remain strictly positive. The major difference is that in (3.15), in the case at hand, there is $h_v\bar\eta(s_v^{*,1} - t_v^{*,1})$ in place of $h_v\bar\eta s_v^{*,1}$, if $t_v^{*,1}$ is the number of branches departing from v and ending up with one endpoint of type $\mathbf{y}$ (hence $0 \le t_v^{*,1} \le s_v^{*,1}$). At the end of the developments, the latter fact generates a new α_v, that we have to prove to be positive in order to control the bound in the vertices $v \ne w$ such that $h_v \ge 0$ (as done in (3.28) for the old one). With simple computations we find:

$$\alpha_v = \varepsilon(s_v^* - 1) + (s_v^{*,1} - t_v^{*,1})(1-\bar\eta-\varepsilon) + t_v^{*,1}(1-\varepsilon) \ge \varepsilon \tag{3.42}$$

Also, we need to prove that $\alpha_v - 2(s_v^* - 1)$ is negative, in order to to control the bound in the vertices $v \ne w$ such that $h_v > 0$; and indeed:

$$\begin{aligned}\alpha_v - 2(s_v^*-1) &= (2-\varepsilon) - s_v^* - (s_v^{*,1} - t_v^{*,1})\bar\eta - s_v^{*,2}(1-\varepsilon)\\ &\le (2-\varepsilon) - (1-|\bar\eta|)s_v^* < 0\end{aligned} \tag{3.43}$$

Finally, the summation on the scale of w is controlled by the the exponential fall off in the diameter of $\underline{\mathbf{y}}$, as in (3.40).

4. Explicit expression of the coefficients in the mass expansion and proof of Theorem 1.1

4.1. The case $r = 0$. As explained in the remark preceding (3.7), in order to get an explicit expression for the coefficients of the expansion (3.7), it is sufficient to calculate the correlations $S^{(m,0)}(\underline{\mathbf{x}}, \sigma)$. We now show how to get this result by computing the correlations of the ψ field at non coinciding points. We consider the following generating function

$$\mathcal{W}_{N,\varepsilon}(J) = \lim_{h\to-\infty} \log \int P_{h,N}(d\psi) e^{-\lambda Z_N^2 V(\psi) + \bar Z_N^{(1)} \sum_\sigma \int d\mathbf{x} d\mathbf{y}\, J^\sigma_{\mathbf{x}} \delta_\varepsilon(\mathbf{x}-\mathbf{y}) \bar\psi_{\mathbf{x}} \Gamma^\sigma \psi_{\mathbf{y}}} \tag{4.1}$$

where $\delta_\varepsilon(\mathbf{x})$ is a smooth approximation of the delta function, rotational invariant, whose support does not contain the point $\mathbf{x}=0$; for definiteness we will choose $\delta_\varepsilon(\mathbf{x}) = \varepsilon^{-2} v(\varepsilon^{-1}|\mathbf{x}|)$, $v(\rho)$ being a function on $\mathbb{R}^1$ with support in $[1,2]$, such that $\int d\rho\rho v(\rho) = (2\pi)^{-1}$ (so that $\int d\mathbf{x}\delta_\varepsilon(\mathbf{x}) = 1$). We define

$$\bar S^{(m)}_{N,\varepsilon}(\underline{\mathbf{x}},\underline{\sigma}) = \frac{\partial^m}{\partial J^{\sigma_1}_{\mathbf{x}_1}\ldots\partial J^{\sigma_m}_{\mathbf{x}_m}} \mathcal{W}_{N,\varepsilon}(J)|_{J=0} \tag{4.2}$$

while $S^{(m)}_{N,0}(\underline{\mathbf{x}},\underline{\sigma})$ will denote the analogous quantity with $\delta_\varepsilon(\mathbf{x}-\mathbf{y}) \to \delta(\mathbf{x}-\mathbf{y})$. Note that $S^{(m,0)}(\underline{\mathbf{x}},\underline{\sigma}) = \lim_{N\to\infty} S^{(m)}_{N,0}(\underline{\mathbf{x}},\underline{\sigma})$.

Lemma 4.1. *If λ is small enough, there exists a constant $c_1 = 1 + O(\lambda)$, such that, if we put $\bar Z^{(1)}_N = c_1\varepsilon^{\eta_+}$, then, for any set $\underline{\mathbf{x}}$ of m distinct points,*

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} \bar S^{(m)}_{N,\varepsilon}(\underline{\mathbf{x}},\underline{\sigma}) = \lim_{N\to\infty} S^{(m)}_{N,0}(\underline{\mathbf{x}},\underline{\sigma}) \tag{4.3}$$

Dim. - The proof of the Lemma is based on a multiscale analysis of the functional $\mathcal{W}_{N,\varepsilon}(J)$, performed by using the techniques explained in sect. 2 of [BFM]. We shall not give here the detailed proof, but we shall stress only the relevant differences with respect to the case studied there.

First of all, the external field φ is zero and the free measure has mass zero. Moreover the terms linear in J and quadratic in ψ contains the monomial $\psi^+_{\mathbf{x},-\sigma}\psi^-_{\mathbf{y},\sigma} = \bar\psi_{\mathbf{x}}\Gamma^\sigma\psi_{\mathbf{y}}$, instead of $\psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{y},\sigma}$. This difference is unimportant from the point of view of the dimensional analysis, so that, in the case $\varepsilon = 0$, we can essentially repeat the analysis of [BFM] with obvious minor changes. The situation is different for $\varepsilon > 0$, since in this case these terms (which are marginal) are not local on the scale N, so that they need a more accurate discussion.

Let us call $B^{(j)}_J(\psi)$ the contribution to the effective potential on scale j, which is linear in J and has as external fields $\psi^{[h,j]+}_{\mathbf{x},\omega}$ and $\psi^{[h,j]-}_{\mathbf{y},-\omega}$ and let h_ε be the largest integer such that $\gamma^{-h_\varepsilon} \ge \varepsilon$ and let $N > h_\varepsilon$. We want to show that, if $N \ge j \ge h_\varepsilon$, this term, which is dimensionally marginal, is indeed irrelevant, so there is no need to localize it. This follows from the observation that $B^{(j)}_J(\psi)$ is of the form

$$\begin{aligned} B^{(j)}_J(\psi) = \bar Z^{(1)}_N \sum_\omega \int d\mathbf{x}d\mathbf{y} J^\omega_{\mathbf{z}} \delta_\varepsilon(\mathbf{x}-\mathbf{y}) \psi^{[h,j]+}_{\mathbf{x},-\omega}\psi^{[h,j]-}_{\mathbf{y},\omega} + \\ + \sum_\omega \int d\mathbf{z} J^\omega_{\mathbf{z}} \int d\bar{\mathbf{z}} d\mathbf{x} d\mathbf{y} \delta_\varepsilon(\mathbf{z}-\bar{\mathbf{z}}) W_j(\mathbf{z},\bar{\mathbf{z}},\mathbf{x},\mathbf{y}) \psi^{[h,j]+}_{\mathbf{x},-\omega}\psi^{[h,j]-}_{\mathbf{y},\omega} \end{aligned} \tag{4.4}$$

where $W_j(\mathbf{z},\bar{\mathbf{z}},\mathbf{x},\mathbf{y})$ is the kernel of the sum over all graphs containing at least one λ vertex. It is easy to see that it is of the form

$$W_j(\mathbf{z},\bar{\mathbf{z}},\mathbf{x},\mathbf{y}) = \widetilde W_j(\mathbf{z},\mathbf{x})\widetilde W_j(\bar{\mathbf{z}},\mathbf{y}) + \bar W_j(\mathbf{z},\bar{\mathbf{z}},\mathbf{x},\mathbf{y}) \tag{4.5}$$

where the second term is given by the sum over the graphs which stay connected after cutting the line δ_ε, while the first term is associated with the other graphs. The first term do not need a localization, even for $j < h_\varepsilon$, because $\widetilde W_j(\mathbf{z},\mathbf{x})$ and $\widetilde W_j(\bar{\mathbf{z}},\mathbf{y})$ are sum over graphs with two external lines, one (the one contracted with the J vertex) of scale $h_1 > j$, the other one of scale $h_2 \le j$. The momentum conservation and the compact support properties of the single scale propagators imply that $h_1 = j+1$, so that there is no diverging sum associated with h_1, as one could expect since the first term has a bound $C|\lambda|$. On the other hand, it easy to see that the second term satisfies the bound

$$\int d\bar{\mathbf{z}} d\mathbf{x} d\mathbf{y} \delta_\varepsilon(\mathbf{z}-\bar{\mathbf{z}}) |\bar{W}_j(\mathbf{z},\bar{\mathbf{z}},\mathbf{x},\mathbf{y})| \le C|\lambda|\gamma^{-2(j-h_\varepsilon)} \tag{4.6}$$

This immediately follows by comparing this bound with the analogous one for $\varepsilon = 0$, which is $C|\lambda|$ for dimensional reasons. With respect to the case $\varepsilon = 0$, we have a new vertex $\bar{\mathbf{z}}$, which is linked to the graph by the line δ_ε and a propagator of scale $j' > j$. The bound (4.6) is obtained by using the decaying properties of this propagator to integrate over $\bar{\mathbf{z}}$ and by bounding δ_ε by $C\varepsilon^{-2}$.

Note that this procedure is convenient only because $j \ge h_\varepsilon$, otherwise it would be convenient to integrate over $\bar{\mathbf{z}}$ by using δ_ε and we should get the dimensional bound $C|\lambda|$ of the case $\varepsilon = 0$. It follows that, starting from $j = h_\varepsilon$, we have to apply to $B_J^{(j)}(\psi)$ the localization procedure; then we define, if $j \ge h_\varepsilon$,

$$\mathcal{L}B_J^{(j)}(\psi) = \sum_\omega \bar{Z}_j^{(1)} \int d\mathbf{z} J_{\mathbf{z}}^\omega \psi_{\mathbf{z},-\omega}^{[h,j]+} \psi_{\mathbf{z},\omega}^{[h,j]-} \tag{4.7}$$

and we perform the limit $N \to \infty$. In this limit, $\bar{Z}_j^{(1)}$ can be represented as an expansion in terms of trees, which have one special vertex (the J vertex) and an arbitrary number of normal vertices, the normal vertices being associated with the limiting value $\lambda_{-\infty}$ of the running coupling (whose flow is independent of the $\bar{Z}_j^{(1)}$ flow). It follows that $\bar{Z}_{h_e}^{(1)} = c_1\gamma^{-h_\varepsilon\eta_+}[1+O(\lambda)]$ and that, if $j < h_\varepsilon$,

$$\bar{Z}_{j-1}^{(1)} = \bar{Z}_j^{(1)}\gamma^{\eta_+} + O(|\lambda|\gamma^{-h_\varepsilon\eta_+}\gamma^{-(h_\varepsilon-j)/2}) \tag{4.8}$$

where the first term comes from the trees with the special vertex of scale $\le h_\varepsilon$; it is exactly equal to the term one would get in the theory with $\varepsilon = 0$, in the limit $N \to \infty$. The second term is the contribution of the trees with the special vertex of scale $> h_\varepsilon$ (these trees must have at least one normal vertex); it is of course proportional to ε^{η_+} and takes into account the "short memory property" (exponential decrease of the irrelevant terms influence). The flow (4.8) immediately implies that, for any fixed j and $|\eta_+| < 1/2$, $\lim_{\varepsilon\to 0}(\bar{Z}_{j-1}^{(1)}/\bar{Z}_j^{(1)}) = \gamma^{\eta_+} = (Z_{j-1}^{(1)}/Z_j^{(1)})$ and that $\bar{Z}_j^{(1)} = c_1[1+O(\lambda)]Z_j^{(1)}$. Hence, by suitably choosing c_1, we can get $\lim_{\varepsilon\to 0}\bar{Z}_j^{(1)} = Z_j^{(1)}$. ∎

Note that $S^{(m)}(\underline{\mathbf{x}},\underline{\omega})$ is different from 0 only if m is even and $\sum_i \omega_i = 0$; moreover the truncated correlations can be written as sums over the non truncated ones. Hence, in order to get an explicit formula for $S^{(m)}(\underline{\mathbf{x}},\underline{\omega})$, it is sufficient to calculate the correlation

$$K^{(n)}(\underline{\mathbf{x}},\underline{\mathbf{u}}) = \lim_{-h,N\to\infty}(Z_N^{(1)})^{2n}\langle \prod_{j=1}^n \left(\bar{\psi}_{\mathbf{x}_j}\Gamma^+\psi_{\mathbf{x}_j}\right)\left(\bar{\psi}_{\mathbf{u}_j}\Gamma^-\psi_{\mathbf{u}_j}\right)\rangle \tag{4.9}$$

where $\langle \cdot \rangle$ denotes the expectation with respect to the zero mass Thirring measure. By using Lemma 4.1, we have

$$K^{(n)}(\underline{\mathbf{x}},\underline{\mathbf{u}}) = c_1^{2n} \lim_{\varepsilon\to 0} \varepsilon^{2n\eta_+} \cdot \tag{4.10}$$
$$\cdot \int d\underline{\mathbf{y}} d\underline{\mathbf{v}} [\prod_{i=1}^n \delta_\varepsilon(\mathbf{x}_i - \mathbf{y}_i)\delta_\varepsilon(\mathbf{u}_i - \mathbf{v}_i)] \widetilde{K}^{(2n)}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}})$$

where

$$\widetilde{K}^{(2n)}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}}) = \langle \prod_{j=1}^n \left(\bar\psi_{\mathbf{y}_j} \Gamma^+ \psi_{\mathbf{x}_j}\right) \left(\bar\psi_{\mathbf{v}_j} \Gamma^- \psi_{\mathbf{u}_j}\right) \rangle \tag{4.11}$$

On the other hand, by using the results of [BFM], see Theorem A.1 below, one can prove that, if $\langle\cdot\rangle_0$ is the mean value for $\lambda = 0$ and $\psi_i^- \stackrel{def}{=} \psi^-_{\mathbf{x}_i,\omega_i}$, $\psi_i^+ \stackrel{def}{=} \psi^+_{\mathbf{y}_i,\omega'_i}$,

$$\langle \psi_n^- \cdots \psi_1^- \psi_1^+ \cdots \psi_n^+ \rangle = c_0^{\lambda A(a-\bar a)n} \langle \psi_n^- \cdots \psi_1^- \psi_1^+ \cdots \psi_n^+ \rangle_0 \cdot$$
$$\cdot \frac{\prod_{s,t\in X}^{s<t} |\mathbf{x}_s - \mathbf{x}_t|^{\frac{\lambda A}{4\pi}(a-\bar a\omega_s\omega_t)} \cdot \prod_{s,t\in X}^{s<t} |\mathbf{y}_s - \mathbf{y}_t|^{\frac{\lambda A}{4\pi}(a-\bar a\omega'_s\omega'_t)}}{\prod_{s,t\in X} |\mathbf{x}_s - \mathbf{y}_t|^{\frac{\lambda A}{4\pi}(a-\bar a\omega_s\omega'_t)}} \tag{4.12}$$

where c_0 is an arbitrary constant, to be determined by fixing, for example, the value of the 2-points function at some value of $\mathbf{x}_1 - \mathbf{y}_1$, while a and $\bar a$ are the parameters (function of λ) defined in eq. (1.6) of [BFM] and A is equal to the expression $[1-\lambda\sum_{\varepsilon=\pm1} A_\varepsilon(\alpha_\varepsilon+\rho_\varepsilon)]^{-1}$, appearing in eq. (1.36) of [BFM]. Hence, since $\bar\psi_{\mathbf{x}} \Gamma^\omega \psi_{\mathbf{y}} = \psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{y},\omega}$, we get

$$\widetilde{K}^{(2n)}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}}) = c_0^{2\lambda A(a-\bar a)n} \langle \prod_{j=1}^n \left(\bar\psi_{\mathbf{y}_j} \Gamma^+ \psi_{\mathbf{x}_j}\right) \left(\bar\psi_{\mathbf{v}_j} \Gamma^- \psi_{\mathbf{u}_j}\right) \rangle_0 \cdot$$
$$\cdot \frac{\prod_{s,t\in X}^{s<t} |\mathbf{x}_s - \mathbf{x}_t|^{\frac{\lambda A}{4\pi}(a-\bar a)} |\mathbf{u}_s - \mathbf{u}_t|^{\frac{\lambda A}{4\pi}(a-\bar a)} \cdot \prod_{s,t\in X} |\mathbf{x}_s - \mathbf{u}_t|^{\frac{\lambda A}{4\pi}(a+\bar a)}}{\prod_{s,t\in X} |\mathbf{x}_s - \mathbf{y}_t|^{\frac{\lambda A}{4\pi}(a+\bar a)} |\mathbf{u}_s - \mathbf{y}_t|^{\frac{\lambda A}{4\pi}(a-\bar a)}} \cdot \tag{4.13}$$
$$\cdot \frac{\prod_{s,t\in X}^{s<t} |\mathbf{y}_s - \mathbf{y}_t|^{\frac{\lambda A}{4\pi}(a-\bar a)} |\mathbf{v}_s - \mathbf{v}_t|^{\frac{\lambda A}{4\pi}(a-\bar a)} \cdot \prod_{s,t\in X} |\mathbf{y}_s - \mathbf{v}_t|^{\frac{\lambda A}{4\pi}(a+\bar a)}}{\prod_{s,t\in X} |\mathbf{u}_s - \mathbf{v}_t|^{\frac{\lambda A}{4\pi}(a+\bar a)} |\mathbf{x}_s - \mathbf{v}_t|^{\frac{\lambda A}{4\pi}(a-\bar a)}}$$

A well known identity for the free fermions correlations, equivalent to the so called *Cauchy Lemma* [H] (since $g_\omega^{-1}(\mathbf{x}) = 2\pi(x_0 + i\omega x_1)$), is

$$\langle \prod_{j=1}^n \psi^-_{\mathbf{x}_j,\omega} \psi^+_{\mathbf{v}_j,\omega} \rangle_0 = \sum_{\pi\in P(1,\dots,n)} (-1)^\pi \prod_{j=1}^n g_\omega(\mathbf{x}_j - \mathbf{v}_{\pi(j)}) =$$
$$= (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{i,j\in X}^{i<j} g_\omega^{-1}(\mathbf{x}_i - \mathbf{x}_j) g_\omega^{-1}(\mathbf{v}_i - \mathbf{v}_j)}{\prod_{i,j\in X} g_\omega^{-1}(\mathbf{x}_i - \mathbf{v}_j)} \tag{4.14}$$

Then

$$\langle \prod_{j=1}^n \left(\bar\psi_{\mathbf{y}_j} \Gamma^+ \psi_{\mathbf{x}_j}\right) \left(\bar\psi_{\mathbf{v}_j} \Gamma^- \psi_{\mathbf{u}_j}\right) \rangle_0 = (-1)^n \langle \prod_{j=1}^n \psi^-_{\mathbf{x}_j,+} \psi^+_{\mathbf{v}_j,+} \rangle_0 \langle \prod_{j=1}^n \psi^-_{\mathbf{u}_j,-} \psi^+_{\mathbf{y}_j,-} \rangle_0$$
$$= \frac{\prod_{i,j\in X}^{i<j} g_+^{-1}(\mathbf{x}_i - \mathbf{x}_j) g_+^{-1}(\mathbf{v}_i - \mathbf{v}_j)}{\prod_{i,j\in X} g_+^{-1}(\mathbf{v}_i - \mathbf{x}_j)} \cdot \frac{\prod_{i,j\in X}^{i<j} g_-^{-1}(\mathbf{u}_i - \mathbf{u}_j) g_-^{-1}(\mathbf{y}_i - \mathbf{y}_j)}{\prod_{i,j\in X} g_-^{-1}(\mathbf{u}_i - \mathbf{y}_j)} \tag{4.15}$$

and since $g_+^{-1}(\mathbf{x}-\mathbf{v})g_-^{-1}(\mathbf{x}-\mathbf{v}) = (2\pi)^2|\mathbf{x}-\mathbf{v}|^2$, we get, by some straightforward calculations,

$$\widetilde{K}^{(2n)}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}}) = \left(\frac{c_0^{\lambda A(a-\bar{a})}}{2\pi}\right)^{2n} \frac{F(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}})}{\prod_s (|\mathbf{x}_s-\mathbf{y}_s||\mathbf{u}_s-\mathbf{v}_s|)^{\frac{\lambda A}{4\pi}(a+\bar{a})}} \tag{4.16}$$

where $F(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}})$ is a continuous function such that

$$\lim_{\substack{\mathbf{y}_j\to\mathbf{x}_j\\ \mathbf{v}_j\to\mathbf{u}_j}} F(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}}) = \frac{\prod_{s<t}|\mathbf{x}_s-\mathbf{x}_t|^{2(1-\frac{\lambda A}{2\pi}\bar{a})}|\mathbf{u}_s-\mathbf{u}_t|^{2(1-\frac{\lambda A}{2\pi}\bar{a})}}{\prod_{s,t}|\mathbf{u}_s-\mathbf{x}_t|^{2(1-\frac{\lambda A}{2\pi}\bar{a})}} \tag{4.17}$$

By using the previous identities, together with (4.9), we get

$$K^{(n)}(\underline{\mathbf{x}},\underline{\mathbf{u}}) = \left(\frac{c_1 c_0^{\lambda A(a-\bar{a})}}{2\pi}\right)^{2n} \lim_{\varepsilon\to 0}\int d\underline{\mathbf{y}}d\underline{\mathbf{v}}[\prod_{i=1}^n \delta_\varepsilon(\mathbf{x}_i-\mathbf{y}_i)\delta_\varepsilon(\mathbf{u}_i-\mathbf{v}_i)]\cdot$$
$$\cdot \frac{\varepsilon^{2n\eta_+}F(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{u}},\underline{\mathbf{v}})}{\prod_s(|\mathbf{x}_s-\mathbf{y}_s||\mathbf{u}_s-\mathbf{v}_s|)^{\frac{\lambda A}{4\pi}(a+\bar{a})}} \tag{4.18}$$

By using the tree expansion, one can see that the limit $\varepsilon\to 0$ is bounded and different from zero, at least for $n=1$. It follows, by taking into account the support properties of $\delta_\varepsilon(\mathbf{x})$, that $\eta_+ = \frac{\lambda A}{4\pi}(a+\bar{a})$; note that in [BFM] it is stated that $\eta_- = \frac{\lambda A}{4\pi}(a-\bar{a})$, so that we have

$$\eta_\sigma = \frac{\lambda A}{4\pi}(a+\sigma\bar{a}) \tag{4.19}$$

Hence it is easy to see that, if we put $c_3(\eta) = \int d\mathbf{x}\delta_0(\mathbf{x})|\mathbf{x}|^{-\eta}$,

$$K^{(n)}(\underline{\mathbf{x}},\underline{\mathbf{u}}) = \left(\frac{c_3(\eta_+)c_1 c_0^{\lambda A(a-\bar{a})}}{2\pi}\right)^{2n} \frac{\prod_{s<t}|\mathbf{x}_s-\mathbf{x}_t|^{2(1-\frac{\lambda A}{2\pi}\bar{a})}|\mathbf{u}_s-\mathbf{u}_t|^{2(1-\frac{\lambda A}{2\pi}\bar{a})}}{\prod_{s,t}|\mathbf{u}_s-\mathbf{x}_t|^{2(1-\frac{\lambda A}{2\pi}\bar{a})}} \tag{4.20}$$

If we compare (4.20) with (2.63) and use the remark at the beginning of this section, we get the formal equivalence

$$\lim_{-h,N\to\infty} Z_N^{(1)}\bar{\psi}_\mathbf{x}\Gamma^\sigma\psi_\mathbf{x} \sim b_0{:}e^{i\alpha\sigma\varphi_\mathbf{x}}{:} \tag{4.21}$$

with

$$b_0 = c^{\frac{\alpha^2}{8\pi}}\frac{c_3(\eta_+)c_1 c_0^{\lambda A(a-\bar{a})}}{2\pi} \tag{4.22}$$

if the following relation between α and λ is satisfied:

$$\frac{\alpha^2}{4\pi} = 1-\frac{\lambda A}{2\pi}\bar{a} = 1+\eta_- - \eta_+ \tag{4.23}$$

where we also used (4.19).

This completes the proof of Theorem 1.1 for $r=0$.

4.2. The case $r>0$. Let us define

$$j^\mu_{\mathbf{x}} = \bar\psi_{\mathbf{x}}\gamma^\mu\psi_{\mathbf{x}} \quad , \quad \rho_{\mathbf{x},\omega} = \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} \quad , \quad \mathit{\Pi}_{\mathbf{x},\omega} = \psi^+_{\mathbf{x},-\omega}\psi^-_{\mathbf{x},\omega} \tag{4.24}$$

We have

$$j^0_{\mathbf{x}} = \sum_\omega \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} \quad , \quad j^1_{\mathbf{x}} = i\sum_\omega \omega\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} \tag{4.25}$$

Hence, in order to calculate $S^{(2n,r)}(\underline{\mathbf{z}}\,\underline{\mathbf{x}},\underline{\mathbf{y}};\underline{\omega}\,\underline{\omega}',\underline{\nu})$, it is sufficient to calculate the correlation function

$$\begin{aligned} D_{k_+,k_-,n}(\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{y}},\underline{\mathbf{v}}) \equiv & \lim_{-h,N\to\infty}(Z_N)^{k_++k_-}(Z_N^{(1)})^{2n} \cdot \\ & \langle(\prod_{i=1}^{k_+}\rho_{\mathbf{a}_i,+})(\prod_{i=1}^{k_-}\rho_{\mathbf{c}_i,-})(\prod_{i=1}^{n}\mathit{\Pi}_{\mathbf{y}_i,+})(\prod_{i=1}^{n}\mathit{\Pi}_{\mathbf{v}_i,-})\rangle^{(h,N)} \end{aligned} \tag{4.26}$$

where $<\cdot>^{(h,N)}$ denotes the expectation w.r.t. the massless Thirring measure.

By an obvious extension of Lemma 4.1, we know that there are two constants c_1 and c_2 (smooth functions of λ, equal to 1 for $\lambda=0$), such that

$$\begin{aligned} D_{k_+,k_-,n}(\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{y}},\underline{\mathbf{v}}) = & \lim_{\varepsilon_1,\varepsilon_2\to 0}(c_2\varepsilon_2^{\eta_-})^{(k_++k_-)}(c_1\varepsilon_1^{\eta_+})^{2n}\int d\underline{\mathbf{b}}\,d\underline{\mathbf{d}}\,d\underline{\mathbf{x}}\,d\mathbf{u}\ \cdot \\ \cdot\delta_{\varepsilon_2}(\underline{\mathbf{b}}-\underline{\mathbf{a}})\delta_{\varepsilon_2}(\underline{\mathbf{d}}-\underline{\mathbf{b}})&\delta_{\varepsilon_1}(\underline{\mathbf{x}}-\underline{\mathbf{y}})\delta_{\varepsilon_1}(\mathbf{u}-\underline{\mathbf{v}})\mathit{\Omega}(\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},\underline{\mathbf{d}},\underline{\mathbf{y}},\underline{\mathbf{x}},\underline{\mathbf{v}},\mathbf{u}) \end{aligned} \tag{4.27}$$

where, if $\underline{\mathbf{x}}=(\mathbf{x}_1,\ldots,\mathbf{x}_k)$, $\delta_\varepsilon(\underline{\mathbf{x}})=\prod_{i=1}^k\delta_\varepsilon(\mathbf{x}_i)$, with $\delta_\varepsilon(\mathbf{x})$ defined as in §4.1; moreover,

$$\begin{aligned} \mathit{\Omega}(\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},\underline{\mathbf{d}},\underline{\mathbf{y}},\underline{\mathbf{x}},\underline{\mathbf{v}},\mathbf{u}) = & \langle(\prod_{i=1}^{k_+}\psi^+_{\mathbf{a}_i,+}\psi^-_{\mathbf{b}_i,+})(\prod_{i=1}^{k_-}\psi^+_{\mathbf{c}_i,-}\psi^-_{\mathbf{d}_i,-})\ \cdot \\ & \cdot\ (\prod_{i=1}^{n}\psi^+_{\mathbf{y}_i,-}\psi^-_{\mathbf{x}_i,+})(\prod_{i=1}^{n}\psi^+_{\mathbf{v}_i,+}\psi^-_{\mathbf{u}_i,-})\rangle \end{aligned} \tag{4.28}$$

By using the identities (4.12) and (4.14) and by doing some simple algebra, one can see that, if $\underline{\mathbf{z}}=(\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},\underline{\mathbf{d}})$ and $\underline{\mathbf{w}}=(\underline{\mathbf{y}},\underline{\mathbf{x}},\underline{\mathbf{v}},\mathbf{u})$,

$$\begin{aligned} \mathit{\Omega}(\underline{\mathbf{z}},\underline{\mathbf{w}}) = c_0^{\lambda A(a-\bar a)(k_++k_-)} & \frac{F_1(\underline{\mathbf{z}},\underline{\mathbf{w}})}{\prod_{i=1}^{k_+}g_+^{-1}(\mathbf{a}_i-\mathbf{b}_i)\prod_{i=1}^{k_-}g_-^{-1}(\mathbf{c}_i-\mathbf{d}_i)}\ \cdot \\ & \cdot\frac{F_2(\underline{\mathbf{z}},\underline{\mathbf{w}})}{\prod_{i=1}^{k_+}|\mathbf{b}_i-\mathbf{a}_i|^{\eta_-}\prod_{i=1}^{k_-}|\mathbf{d}_i-\mathbf{c}_i|^{\eta_-}}\ \widetilde{K}^{2n}(\underline{\mathbf{w}}) \end{aligned} \tag{4.29}$$

where $\widetilde{K}^{2n}(\underline{\mathbf{w}})$ is defined as in (4.16), and

$$\begin{aligned} F_1(\underline{\mathbf{z}},\underline{\mathbf{w}}) = & \prod_{s<t}\widetilde{h}^{(+)}_{s,t}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{a}_t,\mathbf{b}_t)\prod_{s<t}\widetilde{h}^{(-)}_{s,t}(\mathbf{c}_s,\mathbf{d}_s,\mathbf{c}_t,\mathbf{d}_t)\ \cdot \\ & \cdot\ \prod_{s,t}\widetilde{h}^{(+)}_{s,t}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{v}_t,\mathbf{x}_t)\prod_{s,t}\widetilde{h}^{(-)}_{s,t}(\mathbf{c}_s,\mathbf{d}_s,\mathbf{y}_t,\mathbf{u}_t) \end{aligned} \tag{4.30}$$

$$
\begin{aligned}
F_2(\underline{\mathbf{z}},\underline{\mathbf{w}}) = & \prod_{s<t} h_{s,t}^{(-)}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{a}_t,\mathbf{b}_t) \prod_{s<t} h_{s,t}^{(-)}(\mathbf{c}_s,\mathbf{d}_s,\mathbf{c}_t,\mathbf{d}_t)\cdot \\
& \cdot \prod_{s,t} h_{s,t}^{(-)}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{v}_t,\mathbf{x}_t) \prod_{s,t} h_{s,t}^{(-)}(\mathbf{c}_s,\mathbf{d}_s,\mathbf{y}_t,\mathbf{u}_t)\cdot \\
& \cdot \prod_{s,t} h_{s,t}^{(+)}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{c}_t,\mathbf{d}_t) \prod_{s,t} h_{s,t}^{(+)}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{y}_t,\mathbf{u}_t) \prod_{s,t} h_{s,t}^{(+)}(\mathbf{c}_s,\mathbf{d}_s,\mathbf{v}_t,\mathbf{x}_t)
\end{aligned}
\tag{4.31}
$$

We defined

$$
\widetilde{h}_{s,t}^{(\omega)}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{v}_t,\mathbf{x}_t) = \frac{g_\omega^{-1}(\mathbf{a}_s-\mathbf{v}_t)g_\omega^{-1}(\mathbf{b}_s-\mathbf{x}_t)}{g_\omega^{-1}(\mathbf{a}_s-\mathbf{x}_t)g_\omega^{-1}(\mathbf{b}_s-\mathbf{v}_t)} \tag{4.32}
$$

$$
h_{s,t}^{(\sigma)}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{v}_t,\mathbf{x}_t) = \left(\frac{|\mathbf{a}_s-\mathbf{v}_t||\mathbf{b}_s-\mathbf{x}_t|}{|\mathbf{a}_s-\mathbf{x}_t||\mathbf{b}_s-\mathbf{v}_t|}\right)^{\eta_\sigma} \tag{4.33}
$$

Let us first evaluate, in the r.h.s. of (4.27), the limit $\varepsilon_1 \to 0$. This can be done exactly as in §4.1 and we get

$$
\begin{aligned}
& D_{k_+,k_-,n}(\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{y}},\underline{\mathbf{v}}) = \lim_{\varepsilon_2\to 0}(c_2\varepsilon_2^{\eta_-})^{(k_++k_-)}\int d\underline{\mathbf{b}}\, d\underline{\mathbf{d}}\, d\underline{\mathbf{x}}\, d\mathbf{u}\ \cdot \\
& \cdot\delta_{\varepsilon_2}(\underline{\mathbf{b}}-\underline{\mathbf{a}})\delta_{\varepsilon_2}(\underline{\mathbf{d}}-\underline{\mathbf{b}})\Omega_0(\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},\underline{\mathbf{d}},\underline{\mathbf{y}},\underline{\mathbf{v}})
\end{aligned}
\tag{4.34}
$$

where, if we put $\underline{\mathbf{w}}_0 = (\underline{\mathbf{y}},\underline{\mathbf{v}})$,

$$
\begin{aligned}
\Omega_0(\underline{\mathbf{z}},\underline{\mathbf{w}}_0) = c_0^{\lambda A(a-\bar{a})(k_++k_-)} & \frac{F_1(\underline{\mathbf{z}},\underline{\mathbf{y}},\underline{\mathbf{y}},\underline{\mathbf{v}},\underline{\mathbf{v}})}{\prod_{i=1}^{k_+} g_+^{-1}(\mathbf{a}_i-\mathbf{b}_i)\prod_{i=1}^{k_-} g_-^{-1}(\mathbf{c}_i-\mathbf{d}_i)}\cdot \\
& \cdot\frac{F_2(\underline{\mathbf{z}},\underline{\mathbf{y}},\underline{\mathbf{y}},\underline{\mathbf{v}},\underline{\mathbf{v}})}{\prod_{i=1}^{k_+}|\mathbf{b}_i-\mathbf{a}_i|^{\eta_-}\prod_{i=1}^{k_-}|\mathbf{d}_i-\mathbf{c}_i|^{\eta_-}}\ K^{(n)}(\underline{\mathbf{w}}_0)
\end{aligned}
\tag{4.35}
$$

$K^{(n)}(\underline{\mathbf{w}}_0)$ being defined as in (4.9); its explicit expression is given by (4.20).

Let us now perform the limit $\varepsilon_2 \to 0$. We use the identity, following from obvious symmetry arguments,

$$
\begin{aligned}
& \lim_{\varepsilon_2\to 0}\int d\underline{\mathbf{b}}\, d\underline{\mathbf{d}}\, \delta_{\varepsilon_2}(\underline{\mathbf{b}}-\underline{\mathbf{a}})\delta_{\varepsilon_2}(\underline{\mathbf{d}}-\underline{\mathbf{b}})\frac{\varepsilon^{\eta_-(k_++k_-)}}{\prod_{i=1}^{k_+}|\mathbf{b}_i-\mathbf{a}_i|^{\eta_-}\prod_{i=1}^{k_-}|\mathbf{d}_i-\mathbf{c}_i|^{\eta_-}}\cdot \\
& \quad\cdot\frac{F(\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}},\underline{\mathbf{d}})}{\prod_{i=1}^{k_+} g_+^{-1}(\mathbf{b}_i-\mathbf{a}_i)\prod_{i=1}^{k_-} g_-^{-1}(\mathbf{d}_i-\mathbf{c}_i)} = \\
& = \lim_{\varepsilon_2\to 0}\int d\underline{\mathbf{b}}\, d\underline{\mathbf{d}}\, \delta_{\varepsilon_2}(\underline{\mathbf{b}}-\underline{\mathbf{a}})\delta_{\varepsilon_2}(\underline{\mathbf{d}}-\underline{\mathbf{b}})\prod_{i=1}^{k_+}\left(\frac{\varepsilon_2}{|\mathbf{b}_i-\mathbf{a}_i|}\right)^{\eta_-}\prod_{i=1}^{k_-}\left(\frac{\varepsilon_2}{|\mathbf{d}_i-\mathbf{c}_i|}\right)^{\eta_-}\cdot \\
& \quad\cdot\frac{\prod_{i=1}^{k_+}[(\mathbf{b}_i-\mathbf{a}_i)\cdot\partial_{\mathbf{b}_i}]\prod_{i=1}^{k_-}[(\mathbf{d}_i-\mathbf{c}_i)\cdot\partial_{\mathbf{d}_i}]F(\underline{\mathbf{a}},\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{c}})}{\prod_{i=1}^{k_+} g_+^{-1}(\mathbf{b}_i-\mathbf{a}_i)\prod_{i=1}^{k_-} g_-^{-1}(\mathbf{d}_i-\mathbf{c}_i)} = \cdot \\
& = \left(\frac{c_3(\eta_-)}{4\pi}\right)^{k_++k_-}\prod_{i=1}^{k_+} D_{\mathbf{b}_i}^{-}\prod_{i=1}^{k_+} D_{\mathbf{d}_i}^{+} F(\underline{\mathbf{a}},\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{c}})
\end{aligned}
\tag{4.36}
$$

where $c_3(\eta)$ is defined as in §4.1 and $D_{\mathbf{x}}^{\omega} \equiv \frac{\partial}{\partial x_0} + i\omega\frac{\partial}{\partial x_1}$.

To complete the calculation, note that, up to terms which are of the second order in at least one of the differences $\mathbf{b}_s - \mathbf{a}_s$ or $\mathbf{d}_s - \mathbf{c}_s$ (these terms give no contribution in the limit $\varepsilon_2 \to 0$),

$$h^{(\sigma)}_{s,t}(\mathbf{a}_s, \mathbf{b}_s, \mathbf{v}_t, \mathbf{y}_t) \simeq 1 + \eta_\sigma (\mathbf{b}_s - \mathbf{a}_s) \cdot \left[\frac{\mathbf{a}_s - \mathbf{y}_t}{|\mathbf{a}_s - \mathbf{y}_t|^2} - \frac{\mathbf{a}_s - \mathbf{v}_t}{|\mathbf{a}_s - \mathbf{v}_t|^2} \right] \tag{4.37}$$

$$h^{(\sigma)}_{s,t}(\mathbf{a}_s, \mathbf{b}_s, \mathbf{a}_t, \mathbf{b}_t) \simeq 1 + \eta_\sigma \left[- \frac{(\mathbf{b}_s - \mathbf{a}_s) \cdot (\mathbf{b}_t - \mathbf{a}_t)}{|\mathbf{a}_s - \mathbf{a}_t|^2} + \right. \\ \left. + 2 \frac{[(\mathbf{b}_s - \mathbf{a}_s) \cdot (\mathbf{a}_s - \mathbf{a}_t)][(\mathbf{b}_t - \mathbf{a}_t) \cdot (\mathbf{a}_s - \mathbf{a}_t)]}{|\mathbf{a}_s - \mathbf{a}_t|^4} \right] \tag{4.38}$$

$$h^{(\sigma)}_{s,t}(\mathbf{a}_s, \mathbf{b}_s, \mathbf{c}_t, \mathbf{d}_t) \simeq 1 + \eta_\sigma \left[- \frac{(\mathbf{b}_s - \mathbf{a}_s) \cdot (\mathbf{d}_t - \mathbf{c}_t)}{|\mathbf{a}_s - \mathbf{c}_t|^2} + \right. \\ \left. + 2 \frac{[(\mathbf{b}_s - \mathbf{a}_s) \cdot (\mathbf{a}_s - \mathbf{c}_t)][(\mathbf{d}_t - \mathbf{c}_t) \cdot (\mathbf{a}_s - \mathbf{c}_t)]}{|\mathbf{a}_s - \mathbf{c}_t|^4} \right] \tag{4.39}$$

$$\widetilde{h}^{(\sigma)}_{s,t}(\mathbf{a}_s, \mathbf{b}_s, \mathbf{v}_t, \mathbf{y}_t) \simeq 1 + g_\sigma^{-1}(\mathbf{b}_s - \mathbf{a}_s) \left[\frac{1}{g_\sigma^{-1}(\mathbf{a}_s - \mathbf{y}_t)} - \frac{1}{g_\sigma^{-1}(\mathbf{a}_s - \mathbf{v}_t)} \right] \tag{4.40}$$

$$\widetilde{h}^{(\sigma)}_{s,t}(\mathbf{a}_s, \mathbf{b}_s, \mathbf{a}_t, \mathbf{b}_t) \simeq 1 + g_\sigma^{-1}(\mathbf{b}_s - \mathbf{a}_s) g_\sigma^{-1}(\mathbf{b}_t - \mathbf{a}_t) \frac{1}{[g_\sigma^{-1}(\mathbf{a}_s - \mathbf{a}_t)]^2} \tag{4.41}$$

In order to get the final result, we have to substitute these expression in the r.h.s of (4.30) and (4.31), expand their product, keep the terms which are of the first order in all the differences $\mathbf{b}_i - \mathbf{a}_i$ and $\mathbf{d}_i - \mathbf{c}_i$ and, finally, apply to them the differential operator $\prod_{i=1}^{k_+} D^-_{\mathbf{b}_i} \prod_{i=1}^{k_+} D^+_{\mathbf{d}_i}$, whose effect can be easily obtained by using the trivial identities

$$D^{-\omega}_{\mathbf{b}_s} g_\omega^{-1}(\mathbf{b}_s - \mathbf{a}_s) = 4\pi \tag{4.42}$$

$$D^{\omega}_{\mathbf{b}_s} \frac{(\mathbf{b}_s - \mathbf{a}_s) \cdot \mathbf{z}}{|\mathbf{z}|^2} = 2\pi g_{-\omega}(\mathbf{z}) \tag{4.43}$$

$$D^{\omega}_{\mathbf{b}_s} D^{\omega}_{\mathbf{b}_t} (\mathbf{b}_s - \mathbf{a}_s) \cdot (\mathbf{b}_t - \mathbf{a}_t) = 0 \tag{4.44}$$

$$D^{\omega}_{\mathbf{b}_s} D^{-\omega}_{\mathbf{b}_t} (\mathbf{b}_s - \mathbf{a}_s) \cdot (\mathbf{b}_t - \mathbf{a}_t) = 2 \tag{4.45}$$

Let us consider, for example, the case $n = 0$. Then we see immediately that $D_{k_+,k_-,0}(\underline{\mathbf{a}}, \underline{\mathbf{c}})$ is different from 0 only if $k_+ + k_- = 2m$ and that, if we put $\underline{\mathbf{z}} = (\underline{\mathbf{a}}, \underline{\mathbf{c}})$ and $\omega_i = +1$ if $\mathbf{z}_i \in \underline{\mathbf{a}}$, $\omega_i = -1$ if $\mathbf{z}_i \in \underline{\mathbf{c}}$, $D_{k_+,k_-,0}(\underline{\mathbf{a}}, \underline{\mathbf{c}})$ satisfies the Wick Theorem with covariance $C_{\omega_1,\omega_2}(\mathbf{z}_1 - \mathbf{z}_2) = \lim_{-h,N\to\infty} (Z^{(2)}_N)^2 < \rho_{\mathbf{z}_i,\omega_i} \rho_{\mathbf{z}_j,\omega_j} >$, that is $Z^{(2)}_N \rho_{\mathbf{x},\omega}$, in the removed cutoffs limit, is a Gaussian field in the massless Thirring model. It is easy to check that

$$C_{\omega_1,\omega_2}(\mathbf{z}_1 - \mathbf{z}_2) = \delta_{\omega_1,\omega_2} \frac{\left[\frac{c_0^{\lambda A(a-\bar{a})} c_2 c_3}{2\pi} \right]^2 (1 + \eta_-/2)}{[(z_{1,0} - z_{2,0}) + i\omega(z_{1,1} - z_{2,1})]^2} \tag{4.46}$$

It follows that, if we call $D^T_{k_+,k_-,n}(\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{y}},\underline{\mathbf{v}})$ the truncated expectation corresponding to $D_{k_+,k_-,n}(\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{y}},\underline{\mathbf{v}})$, we have

$$D^T_{k_+,k_-,0}(\underline{\mathbf{a}},\underline{\mathbf{c}}) = \delta_{k,2} C_{\omega_1,\omega_2}(\mathbf{z}_1 - \mathbf{z}_2) \tag{4.47}$$

Hence, by using (4.25) and the definition (2.84), we get

$$S^{(0,k)}(\mathbf{z}_1,\mathbf{z}_2;\nu_1,\nu_2) = -\delta_{k,2} b^2 h^{\nu_1,\nu_2}(\mathbf{z}_1 - \mathbf{z}_2) \tag{4.48}$$

with

$$b^2 = \frac{1}{\pi}[c_0^{\lambda A(a-\bar{a})} c_2 c_3]^2 (1 + \frac{\eta_-}{2}) \tag{4.49}$$

Let us now consider the case $n > 0$; in this case we shall give the explicit expression of $D^T_{k_+,k_-,n}(\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{y}},\underline{\mathbf{v}})$. This quantity can be obtained, by expanding $K^{(n)}(\underline{\mathbf{w}}_0)$ in terms of products of connected expectations in the usual way and then trying to get a connected quantity by using the terms which survive to the limit $\varepsilon_2 \to 0$, see discussion above. It is obvious that a connected contribution can be obtained only by keeping the products of different first order zeros coming from the functions $h^{(\sigma)}_{s,t}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{v}_t,\mathbf{y}_t)$, $\widetilde{h}^{(\sigma)}_{s,t}(\mathbf{a}_s,\mathbf{b}_s,\mathbf{v}_t,\mathbf{y}_t)$ and the analogous with $(\mathbf{c}_s,\mathbf{d}_s)$ in place of $(\mathbf{a}_s,\mathbf{b}_s)$, together with the truncated expectation in the expansion of $K^{(n)}(\underline{\mathbf{w}}_0)$, that we shall call $K^{(n)}_T(\underline{\mathbf{w}}_0)$. It follows that

$$D^T_{k_+,k_-,n}(\underline{\mathbf{a}},\underline{\mathbf{c}},\underline{\mathbf{y}},\underline{\mathbf{v}}) = K^{(n)}_T(\underline{\mathbf{w}}_0) \prod_{r=1}^{k_+ + k_-} W(\mathbf{z}_r,\omega_r,\underline{\mathbf{w}}_0,\underline{\sigma}) \tag{4.50}$$

where we defined $\underline{\sigma}$ so that $\sigma_i = +1$ if $\mathbf{w}_i \in \underline{\mathbf{y}}$, $\sigma_i = -1$ if $\mathbf{w}_i \in \underline{\mathbf{v}}$ and

$$W(\mathbf{z},\omega,\underline{\mathbf{w}}_0,\underline{\sigma}) = -\omega\, c_0^{\lambda A(a-\bar{a})} c_2 c_3 \left(1 + \frac{\eta_- - \eta_+}{2}\right) \sum_{j=i}^{n} \sigma_j g_\omega(\mathbf{z} - \mathbf{w}_j) \tag{4.51}$$

By using (4.25), we easily get the final result

$$S^{(2n,k)}(\underline{\mathbf{w}},\underline{\mathbf{y}};\underline{\sigma},\underline{\nu}) = S^{(2n,0)}(\underline{\mathbf{w}};\underline{\sigma}) \prod_{r=1}^{k} \widetilde{W}^{\nu_r}(\mathbf{y}_r,\underline{\mathbf{w}},\underline{\sigma}) \tag{4.52}$$

with

$$\widetilde{W}^{\nu}(\mathbf{y},\underline{\mathbf{w}}_0,\underline{\sigma}) = \frac{i}{\pi} c_0^{\lambda A(a-\bar{a})} c_2 c_3 \left(1 + \frac{\eta_- - \eta_+}{2}\right) \sum_{j=i}^{n} \sigma_j \frac{\varepsilon_{\nu,\mu}(\mathbf{y} - \mathbf{w}_j)^\mu}{|\mathbf{y} - \mathbf{w}_j|^2} \tag{4.53}$$

It follows that (4.52) has the same structure than (2.81), so that, by comparing (4.53) with (2.82), as well as (4.48) with (2.83), we get, in agreement with the considerations after (1.6), the equivalence

$$\lim_{-h,N\to\infty} Z_N^{(2)} j^\nu_{\mathbf{x}} \sim -\varepsilon_{\nu,\mu}(b_1 \partial^\mu \varphi_{\mathbf{x}} + b_2 \partial^\mu \xi_{\mathbf{x}}) \tag{4.54}$$

where ξ is a free boson field of zero mass, independent of ϕ, and

$$b_1 = \frac{2}{\alpha} c_0^{\lambda A(a-\bar{a})} c_2 c_3 \left(1 + \frac{\eta_- - \eta_+}{2}\right) \tag{4.55}$$

One can check, by using the relations $a^{-1} = 1 - \lambda/(4\pi) + O(\lambda^2)$, $\bar{a}^{-1} = 1 + \lambda/(4\pi) + O(\lambda^2)$ and $A = 1 + O(\lambda^2)$, that, if b^2 is the constant defined in (4.49),

$$b_2^2 = b^2 - b_1^2 = O(|\lambda|^3) \tag{4.56}$$

However, one can prove that $b_2 = 0$. This follows from the remark that

$$\frac{b_1^2}{b^2} = \frac{\left(1 + \frac{\eta_- - \eta_+}{2}\right)^2}{(1 + \eta_- - \eta_+)\left(1 + \frac{\eta_-}{2}\right)} \tag{4.57}$$

where we used (4.55), (4.49) and (4.23). Hence, in order to prove that $b_2 = 0$, it is sufficient to prove that the r.h.s. of (4.57) is equal to 1; by a simple calculation one can check that this condition is equivalent, since $\eta_- > 0$, to the condition (1.18). Our solution of the Thirring model allows us to represent η_- and η_+ as well defined power series in the physical value $\lambda_{-\infty}$ of the running coupling (see eq. (2.35) of [BFM] for η_-). This representation is not convenient to verify the identity (1.18); however, (1.18) is independent of the details of the ultraviolet regularization of the model, hence it can be checked also by using the explicit (rigorous) representations of η_- and η_+ in terms of the bare coupling, which were found in [M1] and [M2] with a different ultraviolet regularization (by the way, they are also in agreement with the heuristic procedure proposed in [J] and [K]). In this approach, if we call $\widetilde{\lambda}$ the bare constant and put $x = \widetilde{\lambda}/(4\pi)$, one gets

$$\eta_- = \frac{2x^2}{1 - x^2} \quad , \quad \eta_+ = \frac{2x}{1 - x^2} \tag{4.58}$$

and one can check that (1.18) is indeed satisfied.

This completes the proof of Theorem 1.1 for $r > 0$.

A. The explicit formula for the field correlation functions

A.1. The Schwinger-Dyson equation. In this appendix we will derive the explicit expression of the n-point Schwinger functions (4.12), by extending the arguments used in [BFM], to which we refer for details, to analyze the 2-point function. Let us define $\mathcal{W}(J, \varphi) = \log \int P_{h,N}(d\psi) \exp\{\mathcal{V}^{(N)}(\sqrt{Z_N}\psi, J, \varphi)\}$, where the free measure $P_{h,N}(d\psi)$ is defined by (1.14), $\mathcal{V}^{(N)} = -\lambda V\left(\sqrt{Z_N}\psi\right) + \sum_\omega \int d\mathbf{x}[J_{\mathbf{x},\omega} Z_N \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \varphi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega}\varphi^-_{\mathbf{x},\omega}]$ and the fields $\varphi^\pm$ are anticommuting between themselves and with $\psi^\pm$. We shall introduce the Fourier transform of various fields. In doing that, we shall consider the fields $\rho = \psi^+\psi^-$, ψ^+ e φ^+ as incoming fields, while α, J, ψ^- e φ^- will be outcoming fields.

First of all, we note that the Schwinger-Dyson equations are generated by the identity

$$D_\omega(\mathbf{k}) \frac{\partial e^{\mathcal{W}}}{\partial \widehat{\varphi}^+_{\mathbf{k},\omega}} = \chi_{h,N}(\mathbf{k}) \left[\frac{\widehat{\varphi}^-_{\mathbf{k},\omega} e^{\mathcal{W}}}{Z_N} - \lambda \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{\partial^2 e^{\mathcal{W}}}{\partial \widehat{J}_{\mathbf{p},-\omega} \partial \widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}} \right] \tag{A.1}$$

Indeed, given any $F(\psi)$ which is a power series in the field, we have, by the Wick Theorem,

$$\langle \widehat{\psi}^-_{\mathbf{k},\omega} F(\psi) \rangle_0 = \frac{\widehat{g}^{[h,N]}_\omega(\mathbf{k})}{Z_N} \langle \frac{\partial F(\psi)}{\partial \widehat{\psi}^+_{\mathbf{k},\omega}} \rangle_0. \tag{A.2}$$

where $\langle\cdot\rangle_0$ is the mean value with respect to $P_{h,N}$ and $\widehat{g}_\omega^{[h,N]}(\mathbf{k}) = \chi_{h,N}(\mathbf{k})/D_\omega(\mathbf{k})$. Then, (A.1) is a consequence of the identity

$$\frac{\partial e^{\mathcal{W}}}{\partial \widehat{\varphi}^+_{\mathbf{k},\omega}} = \langle \widehat{\psi}^-_{\mathbf{k},\omega} e^{\mathcal{V}^{(N)}(\sqrt{Z_N}\psi,\varphi)}\rangle_0 = \frac{\widehat{g}_\omega^{[h,N]}(\mathbf{k})}{Z_N} \langle \frac{\partial}{\partial \widehat{\psi}^+_{\mathbf{k},\omega}} e^{\mathcal{V}^{(N)}(\sqrt{Z_N}\psi,\varphi)}\rangle_0 \quad \text{(A.3)}$$

and the remark that $V(\psi) = \int d\mathbf{x} \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-}$.

A.2. The approximate Ward–Takahashi identities. We consider a new generating functional, $\mathcal{W}_\mathcal{A}(\alpha, J, \varphi) = \log \int P_{h,N}(d\psi) \exp\{\mathcal{V}^{(N)}(\sqrt{Z_N}\psi, J, \varphi) + [Z_N\mathcal{A}_0 + Z_N \sum_{\sigma=\pm} \nu_N^{(\sigma)} \mathcal{A}_\sigma](\alpha,\psi)\}$ where $\nu_N^{(\pm)}$ are two suitable constants, to be chosen below as functions of λ, and

$$\mathcal{A}_0(\alpha,\psi) \stackrel{def}{=} \sum_{\omega=\pm} \int \frac{d\mathbf{q}\, d\mathbf{p}}{(2\pi)^4} C_\omega(\mathbf{q},\mathbf{p}) \widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q},\omega} \widehat{\psi}^-_{\mathbf{p},\omega} \, , \quad \text{(A.4)}$$

$$\mathcal{A}_\sigma(\alpha,\psi) \stackrel{def}{=} \sum_{\omega=\pm} \int \frac{d\mathbf{q}\, d\mathbf{p}}{(2\pi)^4} D_{\sigma\omega}(\mathbf{p}-\mathbf{q}) \widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q},\sigma\omega} \widehat{\psi}^-_{\mathbf{p},\sigma\omega} \quad \text{(A.5)}$$

having defined, as in [BFM], $D_\omega(\mathbf{k}) = -ik_0 + \omega k_1$ and $C_\omega(\mathbf{q},\mathbf{p}) = [\chi^{-1}_{h,N}(\mathbf{p}) - 1]D_\omega(\mathbf{p}) - [\chi^{-1}_{h,N}(\mathbf{q}) - 1]D_\omega(\mathbf{q})$.

By doing the transformation $\psi^\pm_{\mathbf{x},\omega} \to e^{i\alpha_{\mathbf{x},\omega}}\psi^\pm_{\mathbf{x},\omega}$, see [BM] for a rigorous definition, we get

$$D_\mu(\mathbf{p}) \frac{\partial \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\mu}}(J,\varphi) = \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\mu} \frac{\partial \mathcal{W}}{\partial \widehat{\varphi}^+_{\mathbf{k},\mu}} - \frac{\partial \mathcal{W}}{\partial \widehat{\varphi}^-_{\mathbf{k}+\mathbf{p},\mu}} \widehat{\varphi}^-_{\mathbf{k},\mu} \right] - \\ - \frac{\partial \mathcal{W}_{\mathcal{A}_0}}{\partial \widehat{\alpha}_{\mathbf{p},\mu}}(0, J, \varphi); \quad \text{(A.6)}$$

where $\mathcal{W}_{\mathcal{A}_0}$ is the same as $\mathcal{W}_\mathcal{A}$ but neglecting the interactions $\mathcal{A}_\sigma(\alpha,\psi)$. The last term in the r.h.s. of (A.6) is not negligible in the removed cutoffs limit, but we can extract its leading contribution by introducing suitable counterterms [BFM], so that the rest will vanish, by putting

$$(1-\nu_N^{(+)}) D_\mu(\mathbf{p}) \frac{\partial \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\mu}}(J,\varphi) - \nu_N^{(-)} D_{-\mu}(\mathbf{p}) \frac{\partial \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},-\mu}}(J,\varphi) = \quad \text{(A.7)}$$

$$= \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\mu} \frac{\partial \mathcal{W}}{\partial \widehat{\varphi}^+_{\mathbf{k},\mu}}(J,\varphi) - \frac{\partial \mathcal{W}}{\partial \widehat{\varphi}^-_{\mathbf{k}+\mathbf{p},\mu}}(J,\varphi) \widehat{\varphi}^-_{\mathbf{k},\mu} \right] - \frac{\partial \mathcal{W}^{(h)}_\mathcal{A}}{\partial \widehat{\alpha}_{\mathbf{p},\mu}}(0, J, \varphi)$$

If we define

$$a_N = [1 - \nu_N^{(-)} - \nu_N^{(+)}]^{-1} \quad , \quad \bar{a}_N = [1 + \nu_N^{(-)} - \nu_N^{(+)}]^{-1} \quad \text{(A.8)}$$

by some simple algebra we obtain the identity

$$\frac{\partial e^{\mathcal{W}}}{\partial \widehat{J}_{\mathbf{p},\sigma}}(0,\varphi) + \sum_{\sigma'} \frac{A_{N,\sigma\sigma'}}{D_\sigma(\mathbf{p})} \frac{\partial e^{\mathcal{W}_\mathcal{A}}}{\partial \widehat{\alpha}_{\mathbf{p},\sigma'}}(0,0,\varphi) = \quad \text{(A.9)}$$

$$= \sum_{\sigma'} \frac{A_{N,\sigma\sigma'}}{D_\sigma(\mathbf{p})} \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\sigma'} \frac{\partial e^{\mathcal{W}}}{\partial \widehat{\varphi}^+_{\mathbf{k},\sigma'}}(0,\varphi) - \frac{\partial e^{\mathcal{W}}}{\partial \widehat{\varphi}^-_{\mathbf{k}+\mathbf{p},\sigma'}}(0,\varphi) \widehat{\varphi}^-_{\mathbf{k},\sigma'} \right]$$

where $A_{N,\sigma} = (a_N + \sigma \bar{a}_N)/2$. With the argument explained in [BFM], it would be easy to prove that the term in $\mathcal{W}_{\mathcal{A}}$ is vanishing in the limit of removed cutoff; anyway this is not our current objective.

A.3. A Closed Equation for the field correlation functions. By doing an arbitrary number of functional derivatives with respect to the φ external field in (A.1) and then putting $\varphi = 0$, one can obtain an infinite number of relations between the field correlation functions and other correlations involving several fields and one current, integrated over the current momentum. We want now to show that, by using the identity (A.9), it is possible to get a closed equation for the field correlation functions, in the limit of removed cutoffs. Let us define $\partial^{\omega}_{\mathbf{x}} = \partial_0 + i\omega\partial_1$.

Lemma A.1. *For $|\lambda|$ small enough, there exists a constant $A = 1 + O(\lambda)$, such that the equations of motion for the truncated Schwinger functions -except the two point Schwinger function- in the limit of removed cutoffs are generated, at non coinciding points, by the identity:*

$$\begin{aligned}\partial^{\omega_1}_{\mathbf{x}_1} \frac{\partial \mathcal{W}}{\partial \varphi^+_{\mathbf{x}_1,\omega_1}} &= \lambda A \sum_{\mu} A_{-\omega_1 \mu} \int d\mathbf{z}\; g_{-\omega_1}(\mathbf{x}_1 - \mathbf{z}) \left[\varphi^+_{\mathbf{z},\mu} \frac{\partial^2 \mathcal{W}}{\partial \varphi^+_{\mathbf{z},\mu} \partial \varphi^+_{\mathbf{x}_1,\omega_1}} - \frac{\partial^2 \mathcal{W}}{\partial \varphi^+_{\mathbf{x}_1,\omega_1} \partial \varphi^-_{\mathbf{z},\mu}} \varphi^-_{\mathbf{z},\mu} \right] \\ &+ \lambda A \sum_{\mu} A_{-\omega_1 \mu} \int d\mathbf{z}\; g_{-\omega_1}(\mathbf{x}_1 - \mathbf{z}) \left[\varphi^+_{\mathbf{z},\mu} \frac{\partial \mathcal{W}}{\partial \varphi^+_{\mathbf{z},\mu}} \frac{\partial \mathcal{W}}{\partial \varphi^+_{\mathbf{x}_1,\omega_1}} - \frac{\partial \mathcal{W}}{\partial \varphi^+_{\mathbf{x}_1,\omega_1}} \frac{\partial \mathcal{W}}{\partial \varphi^-_{\mathbf{z},\mu}} \varphi^-_{\mathbf{z},\mu} \right] \quad \text{(A.10)}\end{aligned}$$

Proof. If we make a derivative with respect to $\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}$ in both sides of (A.9), with $\omega = \sigma$, and then integrate over $\mathbf{p}$ (which is meaningful, since the correlation functions can not have a singularity at $\mathbf{p} = 0$ and have compact support in $\mathbf{p}$ for h and N finite), we get

$$\begin{aligned}\int \frac{d\mathbf{p}}{(2\pi)^2} \frac{\partial^2 e^{\mathcal{W}}}{\partial \widehat{J}_{\mathbf{p},-\omega} \partial \widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}} &= -\sum_{\mu} \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{A_{N,-\omega\mu}}{D_{-\omega}(\mathbf{p})} \frac{\partial^2 e^{\mathcal{W}_{\mathcal{A}}}}{\partial \widehat{\alpha}_{\mathbf{p},\mu} \partial \widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}} + \quad \text{(A.11)} \\ &+ \sum_{\mu} \int \frac{d\mathbf{p}\, d\mathbf{q}}{(2\pi)^4} \frac{A_{N,-\omega\mu}}{D_{-\omega}(\mathbf{p})} \left[\widehat{\varphi}^+_{\mathbf{q}+\mathbf{p},\mu} \frac{\partial^2 e^{\mathcal{W}}}{\partial \partial \widehat{\varphi}^+_{\mathbf{q},\mu} \widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}} - \frac{\partial^2 e^{\mathcal{W}}}{\partial \widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega} \partial \widehat{\varphi}^-_{\mathbf{q}+\mathbf{p},\mu}} \widehat{\varphi}^-_{\mathbf{q},\mu} \right]\end{aligned}$$

where both sides are calculated at $J = \alpha = 0$ and we used the fact that $D_{\omega}(\mathbf{p})$ is odd in $\mathbf{p}$ to cancel one term in the r.h.s. of (A.11).

We introduce the generating functionals $\mathcal{W}_{\mathcal{T},\mu}(\beta, \varphi)$, for $\mu = \pm$, defined as

$$\begin{aligned} e^{\mathcal{W}_{\mathcal{T},\mu}(\beta,\varphi)} &\stackrel{def}{=} \int P_{h,N}(d\psi)\; e^{-\lambda_N Z_N^2 V(\psi)} \exp\left\{ \int \varphi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} + \int \psi^+_{\mathbf{x},\omega} \varphi^-_{\mathbf{x},\omega} \right\} \cdot \\ &\cdot \exp\left\{ \left[\mathcal{T}^{(\mu)}_0 + \sum_{\sigma=\pm} \nu^{(\sigma)}_N \mathcal{T}^{(\mu)}_{\sigma} \right] \left(\sqrt{Z_N}\psi, \sqrt{Z_N}\beta \right) \right\} \cdot \quad \text{(A.12)} \\ &\cdot \exp\left\{ \sum_{\omega=\pm} \left[-\alpha^{(\mu\omega)} \lambda \mathcal{B}^{(3)}_{\omega} - \rho^{(\mu\omega)} \mathcal{B}^{(1)}_{\omega} \right] \left(\sqrt{Z_N}\psi, \sqrt{Z_N}\beta \right) \right\}\end{aligned}$$

with $\{\alpha^{(\mu)}\}_{\mu=\pm}$, $\{\rho^{(\mu)}\}_{\mu=\pm}$, four real parameters to be fixed later and

$$\begin{aligned}
\mathcal{T}_0^{(\mu)}(\psi,\beta) &\stackrel{def}{=} \sum_{\omega=\pm}\int \frac{d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}}{(2\pi)^6}\, \frac{C_\mu(\mathbf{q},\mathbf{q}-\mathbf{p})}{D_{-\omega}(\mathbf{p})} \widehat{\beta}_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q},\mu}\widehat{\psi}^-_{-\mathbf{p}+\mathbf{q},\mu}\ ,\\
\mathcal{T}_\sigma^{(\mu)}(\psi,\beta) &\stackrel{def}{=} \sum_{\omega=\pm}\int \frac{d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}}{(2\pi)^6}\, \frac{D_{\sigma\mu}(-\mathbf{p})}{D_{-\omega}(\mathbf{p})}\, \widehat{\beta}_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q},\sigma\mu}\widehat{\psi}^-_{-\mathbf{p}+\mathbf{q},\sigma\mu}\ ,\\
\mathcal{B}_\omega^{(3)}(\psi,\beta) &\stackrel{def}{=} \int \frac{d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}}{(2\pi)^6}\, \widehat{\beta}_{\mathbf{k},\omega}\widehat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\widehat{\psi}^+_{\mathbf{q},-\omega}\widehat{\psi}^-_{-\mathbf{p}+\mathbf{k},-\omega}\ ,\\
\mathcal{B}_\omega^{(1)}(\beta,\psi) &\stackrel{def}{=} \int \frac{d\mathbf{k}}{(2\pi)^2}\, \widehat{\beta}_{\mathbf{k},\omega} D_\omega(\mathbf{k})\widehat{\psi}^-_{\mathbf{k},\omega}
\end{aligned} \tag{A.13}$$

We remark that $\mathcal{W}_{\mathcal{T},\mu}(\beta,\varphi)$ differs from the analogous generating functional introduced in [BFM] because of the presence of the *interactions* $\mathcal{B}_\omega^{(1)}(\beta,\psi)$ and $\mathcal{B}_\omega^{(3)}(\beta,\psi)$, that in the cited paper were - in a sense - reconstructed a posteriori; here we describe a faster way to implement the same procedure of [BFM]. We have the following identity:

$$\begin{aligned}
&\int \frac{d\mathbf{p}}{(2\pi)^2}\, \frac{1}{D_{-\omega}(\mathbf{p})}\frac{\partial^2 e^{\mathcal{W}_\mathcal{A}}}{\partial\widehat{\alpha}_{\mathbf{p},\mu}\partial\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,0,\varphi) = \frac{1}{Z_N}\frac{\partial e^{\mathcal{W}_{\mathcal{T},\mu}}}{\partial\widehat{\beta}_{\mathbf{k},\omega}}(0,\varphi) + \qquad (A.14)\\
&+\alpha^{(\mu\omega)}\lambda\int\frac{d\mathbf{p}}{(2\pi)^2}\, \frac{\partial^2 e^{\mathcal{W}}}{\partial\widehat{J}_{\mathbf{p},-\omega}\partial\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}}(0,\varphi) + \rho^{(\mu\omega)}D_\omega(\mathbf{k})\frac{\partial e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{\mathbf{k},\omega}}(0,\varphi)
\end{aligned}$$

which, plugged into (A.11), gives

$$\begin{aligned}
&\left(1+\lambda\sum_\mu A_{N,-\mu}\ \alpha^{(\mu)}\right)\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{\partial^2 e^{\mathcal{W}}}{\partial\widehat{J}_{\mathbf{p},-\omega}\partial\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}} =\\
&= -\sum_\mu \frac{A_{N,-\omega\mu}}{Z_N}\frac{\partial e^{\mathcal{W}_{\mathcal{T},\mu}}}{\partial\widehat{\beta}_{\mathbf{k},\omega}} - \left(\sum_\mu A_{N,-\mu}\rho^{(\mu)}\right) D_\omega(\mathbf{k})\frac{\partial e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{\mathbf{k},\omega}} + \qquad (A.15)\\
&+\sum_\mu\int\frac{d\mathbf{p}\, d\mathbf{q}}{(2\pi)^4}\, \frac{A_{N,-\omega\mu}}{D_{-\omega}(\mathbf{p})}\left[\widehat{\varphi}^+_{\mathbf{q}+\mathbf{p},\mu}\frac{\partial^2 e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{\mathbf{q},\mu}\partial\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}} - \frac{\partial^2 e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{\mathbf{k}+\mathbf{p},\omega}\partial\widehat{\varphi}^-_{\mathbf{q}+\mathbf{p},\mu}}\widehat{\varphi}^-_{\mathbf{q},\mu}\right]\ .
\end{aligned}$$

This equation, together with (A.1), the identity $\delta_{\mu,\omega} = (1+\mu\omega)/2$ and the remark that $\mathcal{W}_{\mathcal{T},\mu}(0,\varphi) = \mathcal{W}(0,\varphi)$, implies that

$$\begin{aligned}
&D_\omega(\mathbf{k})\frac{\partial\mathcal{W}}{\partial\widehat{\varphi}^+_{\mathbf{k},\omega}} = \chi_{h,N}(\mathbf{k})\frac{B_N}{Z_N}\widehat{\varphi}^-_{\mathbf{k},\omega} + \chi_{h,N}(\mathbf{k})\frac{\lambda A_N}{Z_N}\sum_\mu A_{N,-\omega\mu}\frac{\partial\mathcal{W}_{\mathcal{T},\mu}}{\partial\widehat{\beta}_{\mathbf{k},\omega}}(0,\varphi) + \qquad (A.16)\\
&+\lambda A_N\sum_\mu \chi_{h,N}(\mathbf{k})\int\frac{d\mathbf{q}\, d\mathbf{p}}{(2\pi)^4}\, \frac{A_{N,-\omega\mu}}{D_{-\omega}(\mathbf{p})}e^{-\mathcal{W}}\left[\widehat{\varphi}^+_{\mathbf{q},\mu}\frac{\partial^2 e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{\mathbf{q}+\mathbf{p},\mu}\partial\widehat{\varphi}^+_{\mathbf{k}-\mathbf{p},\omega}} - \frac{\partial^2 e^{\mathcal{W}}}{\partial\widehat{\varphi}^+_{\mathbf{k}-\mathbf{p},\omega}\partial\widehat{\varphi}^-_{\mathbf{q},\mu}}\widehat{\varphi}^-_{\mathbf{q}+\mathbf{p},\mu}\right]
\end{aligned}$$

where $A_N = [1+\lambda\sum_\mu A_{N,-\mu}(\alpha^{(\mu)}-\rho^{(\mu)})]^{-1}$ and $B_N = [1+\lambda\sum_\mu A_{N,-\mu}\alpha^{(\mu)}]A_N$.

Before doing the limit $-h, N\to\infty$, we can rewrite the previous identity in the space coordinates, by doing the Fourier transform in both sides. Since we want to get an identity involving only the correlations with at least four points, the first term in the r.h.s. gives no contribution. Hence, it is easy to see that we get the identity (A.10), with $A = \lim_{N\to\infty} A_N$, if the correlations obtained from derivatives of the last term are proved to be vanishing in the limit of removed cutoffs and if, in this limit, we can safely substitute $\chi_{h,N}(\mathbf{k})$ with 1. Let us first consider this problem, without giving the technical details. If we make a certain number of derivatives with respect to the field φ at non coinciding points and put $\varphi = 0$, we are faced with the problem of calculating the limit of expressions of the type

$$\int d\mathbf{z}\ \delta_{h,N}(\mathbf{z}) g_\omega(\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{z}) G_{h,N}(\mathbf{x}_1 - \mathbf{z}, \mathbf{y}_2, \ldots, \mathbf{y}_n) \tag{A.17}$$

where $\delta_{h,N}(\mathbf{z})$ is the Fourier transform of $\chi_{h,N}(\mathbf{k})$ and the points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are all different, except $\mathbf{x}_2$ and $\mathbf{y}_2$, which can be equal. The function $G_{h,N}(\underline{\mathbf{y}})$ is a (truncated) Schwinger function, which was proved in [BFM] (eq. 2.58) to decay as $|\mathbf{y}_i|^{-\varepsilon'}$, $0 < \varepsilon' < 1$, if $|\mathbf{y}_i| \to \infty$, while the other points are fixed. By using the bound 2.52 of [BFM] it is also possible to prove that it diverges as $|\mathbf{y}_i - \mathbf{y}_j|^{-\varepsilon''}$, $0 < \varepsilon'' < 1$, if $|\mathbf{y}_i - \mathbf{y}_j| \to 0$, while the other points stay constant. On the other hand, it is easy to prove that $|\delta_{h,N}(\mathbf{z})| \le C(\gamma^{-2N} + |\mathbf{z}|^2)^{-1}$. These properties and the good convergence properties of $G_{h,N}(\underline{\mathbf{y}})$ as $-h, N \to \infty$ (uniform if the points vary in non intersecting neighborhoods of the arguments) imply that one can make without any problem in (A.17) the limit $h \to -\infty$ and substitute $G_{h,N}(\underline{\mathbf{y}})$ with $G(\underline{\mathbf{y}}) = \lim_{-h,N\to\infty} G_{h,N}(\underline{\mathbf{y}})$. The previous remarks imply also that the function $g_\omega(\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{z})\, G(\mathbf{x}_1 - \mathbf{z}, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ is a L^1 function of $\mathbf{z}$ with a finite number of singularities; moreover, $\delta_{-\infty,N}(\mathbf{z}) \to \delta(\mathbf{z})$ as $N \to \infty$. It follows that the limit of (A.17) does exist and is given by $g_\omega(\mathbf{x}_1 - \mathbf{x}_2)\, G(\mathbf{x}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$; one can also see that the limit is uniform, if the $\mathbf{x}_i$ vary in small non intersecting neighborhoods, so implying that one can exchange the derivative with the removed cutoff limit in the l.h.s. of (A.16), written in the space coordinates.

We still have to discuss the main point, that is the fact that the correlations obtained from derivatives of the last term of (A.16) vanish in the removed cutoff limit. Since we assume some familiarity with [BM], we shall do that very briefly, by studying the flow of the marginal terms proportional to the field β in the effective potential related with the generating functional (A.12).

After the integration of the fields $\psi^{(j')}$, $j' > j$, we obtain an expression of the type:

$$\begin{aligned} e^{\mathcal{W}^{(h)}_{\mathcal{T},\mu}(\beta,\varphi)} &= \int P_{[h,j]}(d\psi)\ \exp\left\{\mathcal{V}^{(j)}\left(\varphi, \sqrt{Z_j}\psi\right) + \mathcal{W}^{(j)}_{\mathcal{T},\mathrm{irr}}\left(\beta,\varphi,\sqrt{Z_j}\psi\right)\right\}\cdot \\ &\cdot \exp\left\{\left[\left(\frac{Z_N}{Z_j}\right)^2 \mathcal{T}^{(\mu)}_0 + \frac{Z_N}{Z_j}\sum_{\sigma=\pm} \nu^{(\sigma)}_j \mathcal{T}^{(\mu)}_\sigma\right]\left(\sqrt{Z_j}\psi, \sqrt{Z_j}\beta\right)\right\}\cdot \\ &\cdot \exp\left\{\left[\tilde\zeta^{(3,\mu\omega)}_j \mathcal{B}^{(3)} + \frac{Z_N}{Z_j}\sum_{k=j}^{N} \tilde\zeta^{(1,\mu\omega)}_k \mathcal{B}^{(1)}\right]\left(\sqrt{Z_j}\psi, \sqrt{Z_j}\beta\right)\right\} \end{aligned} \tag{A.18}$$

where $\mathcal{V}^{(j)}$ is the effective potential for $\beta = 0$, $\mathcal{W}^{(j)}_{\mathcal{T},\mathrm{irr}}$ is the irrelevant part of the terms of order at least 1 in β, while the rest represents the corresponding marginal terms, written in terms of two running coupling constants:

$$\widetilde{\zeta}_j^{(3,\mu)} \stackrel{def}{=} \begin{cases} -\alpha_N^{(\mu)}\lambda & \text{for } j = N \\ \widetilde{\lambda}_j^{(\mu)} - \alpha_N^{(\mu)}\lambda_j & \text{for } j \leq N-1 \; ; \end{cases}$$

$$\widetilde{\zeta}_j^{(1,\mu)} \stackrel{def}{=} \begin{cases} -\rho_N^{(\mu)} & \text{for } j = N \\ \left(\widetilde{z}_j^{(\mu)} - \alpha_N^{(\mu)} z_j\right)\frac{Z_j}{Z_N} & \text{for } j \leq N-1 \; ; \end{cases} \tag{A.19}$$

where $\{\widetilde{\lambda}_j^{\mu}\}$ and $\{\widetilde{z}_j^{\mu}\}$ are *exactly* the coupling studied in [BM]; while $\{\lambda_j\}$ and $\{z_j\}$ are *exactly* the effective coupling and the field renormalization of the original generating functional, $\mathcal{W}$. In [BM] (equation (144)) it was proved that there exist α_N^{μ} such that the following two bounds are both satisfied

$$|\widetilde{\lambda}_j^{(\mu)} - \alpha_N^{(\mu)}\lambda_j| \leq C\gamma^{-(N-j)/2} \quad , \quad |\widetilde{z}_j^{(\mu)} - \alpha_N^{(\mu)} z_i| \leq C\gamma^{-(N-j)/2} \tag{A.20}$$

Thereby, if we put $\rho_N^{(\mu)} = \sum_{j\leq N-1}\widetilde{\zeta}_j^{(1,\mu)}$, the factor in front of $\mathcal{B}^{(1)}$ in (A.18) is

$$\frac{Z_N}{Z_j}\sum_{k=j}^{N}\widetilde{\zeta}_k^{(1,\mu)} = \frac{Z_N}{Z_j}\left(\sum_{k=j}^{N-1}\widetilde{\zeta}_k^{(1,\mu)} - \rho_N^{(\mu)}\right)$$

$$= -\frac{Z_N}{Z_j}\sum_{k\leq j-1}\widetilde{\zeta}_k^{(1,\mu)} = \sum_{k\leq j-1}(\widetilde{z}_k^{(\mu)} - \alpha_N^{(\mu)} z_k)\frac{Z_k}{Z_j} \tag{A.21}$$

and the last term, by the second (A.20), can be bounded by $C\gamma^{-(N-j)/4}$. This remark, together with the first of (A.20), allows us to prove that contribution to the correlation functions of the last term in (A.18) vanishes in the limit of removed cutoffs, as a consequence of the short memory property, see [BM] for details.

A.4. Solution of the closed equations. Let us define $a = \lim_{N\to\infty} a_N$, $\bar{a} = \lim_{N\to\infty}\bar{a}_N$, $\Delta^{-1}(\mathbf{x}|\mathbf{y}) = \frac{1}{2\pi}\ln\left(\frac{|\mathbf{y}|}{|\mathbf{x}|}\right)$ and

$$G_{\underline{\omega},\underline{\omega}'}^{(2n)}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = e^{-\mathcal{W}(\prime,\prime)}\frac{\partial^{2n}e^{\mathcal{W}}}{\partial\varphi^{+}_{\mathbf{x}_n,\omega_n}\cdots\partial\varphi^{+}_{\mathbf{x}_1,\omega_1}\partial\varphi^{-}_{\mathbf{y}_1,\omega_1'}\cdots\partial\varphi^{-}_{\mathbf{y}_n,\omega_n'}}(0,0) \tag{A.22}$$

Theorem A.1. *For $|\lambda|$ small enough and $\mathbf{x}\neq\mathbf{y}$,*

$$S_{\omega,\omega'}^{(2)}(\mathbf{x},\mathbf{y}) \equiv G_{\omega,\omega'}^{(2)}(\mathbf{x},\mathbf{y}) = \delta_{\omega,\omega'}g_\omega(\mathbf{x}-\mathbf{y})e^{\frac{\lambda A(a-\bar{a})}{2}\Delta^{-1}(\mathbf{x}-\mathbf{y}|\mathbf{z})} \tag{A.23}$$

where $\mathbf{z}$ is a fixed, non-zero position, whose arbitrariness reflects the arbitrariness of a factor in front of $S_{\omega,\omega'}^{(2)}(\mathbf{x},\mathbf{y})$. Furthermore, if $n>1$ and $(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{y}_1,\ldots,\mathbf{y}_n)$ is a family of two by two distinct points,

$$G_{\underline{\omega},\underline{\omega}'}^{(2n)}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \sum_{\pi\in P_X}(-1)^{\pi}\mathcal{G}_{\underline{\omega},\pi(\underline{\omega}')}^{(2n)}(\underline{\mathbf{x}},\pi(\underline{\mathbf{y}})) \tag{A.24}$$

where $X \stackrel{def}{=} \{1,\dots,n\}$, P_X *is the set of the permutations of the elements of* X *and*

$$\mathcal{G}^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \left(\prod_{j=1}^n S^{(2)}_{\omega_j,\omega'_j}(\mathbf{x}_j,\mathbf{y}_j)\right)\cdot$$
$$\cdot \prod_{s,t\in X}^{s<t} e^{\lambda A\frac{a-\bar a\omega_s\omega'_t}{2}\left[\Delta^{-1}(\mathbf{x}_s-\mathbf{y}_t|\mathbf{x}_s-\mathbf{x}_t)-\Delta^{-1}(\mathbf{y}_s-\mathbf{y}_t|\mathbf{y}_s-\mathbf{x}_t)\right]} \tag{A.25}$$

Proof. The equation (A.23) has been proved in [BFM]. The proof of (A.24) will be done by checking that the *truncated* correlation functions corresponding to the functions (A.24), assumed to be the right *not truncated* correlation functions, solve the identity (A.10) for $n>1$. The reason for this procedure is that, as we have discussed §A.3, we were able to get a closed equation, in the limit of removed cutoffs, only for the truncated correlation functions with $n>1$. However, it is worthwhile to give first the heuristic argument which allows us to conjecture that (A.24) is the right expression for the not truncated correlation functions.

In the limit $N\to\infty$, if we put $Z=\lim_{N\to\infty}$ and we ignore the fact that $Z=0$, the identity (A.16) can be written in terms of the space coordinates as

$$\partial^{\mathbf{x}_1}_{\omega_1}\frac{\partial e^{\mathcal{W}}}{\partial\varphi^+_{\mathbf{x}_1,\omega_1}} = \frac{B}{Z}e^{\mathcal{W}}\varphi^-_{\mathbf{x}_1,\omega_1} + \lambda A\sum_\varepsilon A_{-\omega_1\varepsilon}\cdot$$
$$\cdot\int d\mathbf{z}\; g_{-\omega_1}(\mathbf{x}_1-\mathbf{z})\left[\varphi^+_{\mathbf{z},\varepsilon}\frac{\partial e^W}{\partial\varphi^+_{\mathbf{z},\varepsilon}\partial\varphi^+_{\mathbf{x}_1,\omega_1}} - \frac{\partial e^W}{\partial\varphi^+_{\mathbf{x}_1,\omega_1}\partial\varphi^-_{\mathbf{z},\varepsilon}}\varphi^-_{\mathbf{z},\varepsilon}\right] \tag{A.26}$$

This implies, if $\eta=\lambda AA_-$ and $Z=\lim_{N\to\infty}Z_N$ (we will ignore then fact that $Z=0$), that $S^{(2)}_{\omega,\omega'}(\mathbf{x},\mathbf{y})=\delta_{\omega,\omega'}S_\omega(\mathbf{x}-\mathbf{y})$, with

$$\partial_\omega S_\omega(\mathbf{x}) = \frac{B}{Z}\delta(\mathbf{x}) - \eta g_{-\omega}(\mathbf{x})S_\omega(\mathbf{x}) \tag{A.27}$$

Hence, since, for any value of $\mathbf{z}$, $g_{-\omega}(\mathbf{x})=-\partial_\omega\Delta^{-1}(\mathbf{x}|\mathbf{z})$, we get

$$S_\omega(\mathbf{x}) = \frac{B}{Z}e^{\eta\left[\Delta^{-1}(\mathbf{x}|\mathbf{z})-\Delta^{-1}(0|\mathbf{z})\right]}g_\omega(\mathbf{x}) \tag{A.28}$$

where $\Delta^{-1}(0|\mathbf{z})=+\infty$, which should balance, in this formal calculation, the fact that $Z=0$. In fact, this equation implies the correct value (A.23) of $S_\omega(\mathbf{x})$, if we choose $\mathbf{z}$ so that

$$\frac{B}{Z} = e^{\eta\Delta^{-1}(0|\mathbf{z})} \tag{A.29}$$

Hence, we are encouraged to pursue this formal procedure. If we take $2n-1$ suitable functional derivatives in both sides of (A.26) and we call $\underline{\mathbf{x}}_j$ the vector $\underline{\mathbf{x}}$ without the element $\mathbf{x}_j$, we find the following equation:

$$\partial^{\mathbf{x}_1}_{\omega_1}G^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \frac{B}{Z}\sum_{k=1}^n(-1)^{k-1}\delta_{\omega_1,\omega'_k}\delta(\mathbf{x}_1-\mathbf{y}_k)G^{(2n-2)}_{\underline{\omega}_1,\underline{\omega}'_k}(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_k) \tag{A.30}$$
$$+\lambda A\left[\sum_{k=2}^n A_{-\omega_1\omega_k}g_{-\omega_1}(\mathbf{x}_1-\mathbf{x}_k) - \sum_{k=1}^n A_{-\omega_1\omega'_k}g_{-\omega_1}(\mathbf{x}_1-\mathbf{y}_k)\right]G^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}})$$

By using (A.29), this equation can be written as

$$\partial^{\mathbf{x}_1}_{\omega_1}\left[\left(\prod_{h=2}^n e^{\lambda AA_{-\omega_1\omega_h}\Delta^{-1}(\mathbf{x}_1-\mathbf{x}_h|\mathbf{z})}\right)\left(\prod_{h=1}^n e^{-\lambda AA_{-\omega_1\omega'_h}\Delta^{-1}(\mathbf{x}_1-\mathbf{y}_h|z)}\right)G^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}})\right]=$$

$$=\sum_{k=1}^n(-1)^{k-1}\delta_{\omega_1,\omega'_k}\delta(\mathbf{x}_1-\mathbf{y}_k)\cdot \tag{A.31}$$

$$\cdot\left(\prod_{h=2}^n e^{\lambda AA_{-\omega_1\omega_h}\Delta^{-1}(\mathbf{y}_k-\mathbf{x}_h|\mathbf{z})}\right)\left(\prod_{\substack{h=1\\h\neq k}}^n e^{-\lambda AA_{-\omega_1\omega'_h}\Delta^{-1}(\mathbf{y}_k-\mathbf{y}_h|\mathbf{z})}\right)G^{(2n-2)}_{\underline{\omega}_1,\underline{\omega}'_k}(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_k)$$

and hence we arrive at a formula for $G^{(2n)}$ in terms of $G^{(2n-2)}$:

$$G^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}})=\sum_{k=1}^n(-1)^{k-1}S^{(2)}_{\omega_1,\omega'_k}(\mathbf{x}_1-\mathbf{y}_k)G^{(2n-2)}_{\underline{\omega}_1,\underline{\omega}'_k}(\underline{\mathbf{x}}_1,\underline{\mathbf{y}}_k)\cdot \tag{A.32}$$

$$\cdot\left(\prod_{h=2}^n e^{\lambda AA_{-\omega_1\omega_h}\Delta^{-1}(\mathbf{y}_k-\mathbf{x}_h|\mathbf{x}_1-\mathbf{x}_h)}\right)\left(\prod_{\substack{h=1\\h\neq k}}^n e^{-\lambda AA_{-\omega_1\omega'_h}\Delta^{-1}(\mathbf{y}_k-\mathbf{y}_h|\mathbf{x}_1-\mathbf{y}_h)}\right)$$

where all Z factors disappeared. Such an iterative relation is clearly solved by (A.24) together with (A.25).

Let us now assume that the expression (A.24) is correct; we want to check if the corresponding truncated correlation functions satisfy the identity (A.10), for $n>1$. First of all, we remind the connection between the two kind of functions. In order to abridge the notation, we put $\varphi^+_j\stackrel{def}{=}\varphi^+_{\mathbf{x}_j,\omega_j}$ and $\varphi^-_j\stackrel{def}{=}\varphi^-_{\mathbf{y}_j,\omega'_j}$; furthermore, if $X_j\subset X=\{1,\dots,n\}$, we define $\varphi^\varepsilon_{X_j}\stackrel{def}{=}\prod_{k\in X_j}\varphi^\varepsilon_k$, with the factors ordered with decreasing k, if $\varepsilon=+$, and with increasing k otherwise. Expanding $e^{\mathcal{W}}(0,\varphi)$ in powers of $\mathcal{W}$, we find:

$$\frac{\partial^{2n}e^{\mathcal{W}}}{\partial\varphi^+_X\partial\varphi^-_X}(0,0)=\sum_{m=0}^n\frac{1}{m!}\sum^{*}_{X_1,\dots,X_m}\sum_{\pi\in\mathcal{P}_X^{X_1,\dots,X_m}}(-1)^\pi\cdot \tag{A.33}$$

$$\frac{\partial^{2|X_1|}\mathcal{W}}{\partial\varphi^+_{X_1}\partial\varphi^-_{\pi(X_1)}}(0,0)\dots\frac{\partial^{2|X_m|}\mathcal{W}}{\partial\varphi^+_{X_m}\partial\varphi^-_{\pi(X_m)}}(0,0)$$

where $\sum^*_{X_1,\dots,X_m}$ denotes the sum over all the possible partitions of X into m, distinguishable and non empty subsets $X_1,\dots,X_m$. Furthermore, $\mathcal{P}_X^{X_1,\dots,X_m}$ is the quotient set of the permutations of the elements of X, $\mathcal{P}_X$, where two elements are identified if they differ only for a permutation in $\mathcal{P}_{X_1}\otimes\dots\otimes\mathcal{P}_{X_m}$.

To find out the explicit expression of the truncated functions, we define

$$\begin{aligned}&V(\mathbf{x}_s|P_t)\stackrel{def}{=}\lambda AA_{-\omega_s\omega'_t}\Delta^{-1}(\mathbf{x}_s-\mathbf{y}_t|\mathbf{x}_s-\mathbf{x}_t)\\&V_{s,t}\stackrel{def}{=}V(\mathbf{x}_s|P_t)-V(\mathbf{y}_s|P_t)\end{aligned} \tag{A.34}$$

and, exploiting the analogy of the expression (A.25) with the partition function of a lattice gas, we perform the Mayer expansion:

$$\mathcal{G}^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \left[\prod_{s,t\in X}^{s<t}\left(e^{V_{s,t}}-1+1\right)\right]\prod_{j=1}^{n} S^{(2)}_{\omega_j,\omega'_j}(\mathbf{x}_j,\mathbf{y}_j) = \tag{A.35}$$
$$= \sum_{m=0}^{n}\frac{1}{m!}\sum_{X_1,\ldots,X_m}^{*}\prod_{i=1}^{m}\sum_{g\in\mathcal{C}(X_i)}\prod_{\langle s,t\rangle\in g}\left(e^{V_{s,t}}-1\right)\prod_{k\in X_i} S^{(2)}_{\omega_k,\omega'_k}(\mathbf{x}_k,\mathbf{y}_k)$$

where the link $\langle s,t\rangle$ is the order pair of the elements $s,t\in X$; $\mathcal{C}(X)$ is the set of the graphs containing a path which connects every element of X; when X_j is made of only one point, there is no possible graph; as usual, the product over empty sets gives 1 by definition. Therefore the comparison with (A.33) gives the following expression for the truncated functions:

$$S^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \sum_{\pi\in P_X}(-1)^{\pi}\widetilde{S}^{(2n)}_{\underline{\omega},\pi(\underline{\omega}')}(\underline{\mathbf{x}},\pi(\underline{\mathbf{y}})) \tag{A.36}$$

with

$$\widetilde{S}^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \sum_{g\in\mathcal{C}(X)}\prod_{\langle s,t\rangle\in g}\left(e^{V_{s,t}}-1\right)\prod_{k\in X} S^{(2)}_{\omega_k,\omega'_k}(\mathbf{x}_k,\mathbf{y}_k) \tag{A.37}$$

We now perform some manipulations. From the previous expression we get

$$\left(\prod_{s\in X}^{s\neq 1} e^{-V(\mathbf{x}_1|P_s)}\right) e^{-\lambda AA_-\Delta^{-1}(\mathbf{x}_1-\mathbf{y}_1|\mathbf{z})}\widetilde{S}^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}}) =$$
$$= \delta_{\omega_1,\omega'_1} g_{\omega_1}(\mathbf{x}_1-\mathbf{y}_1)\left(\prod_{k=2}^{n} S^{(2)}_{\omega_k,\omega'_k}(\mathbf{x}_k-\mathbf{y}_k)\right)\cdot \tag{A.38}$$
$$\cdot\sum_{g\in\mathcal{C}(X)}\prod_{s\in X}^{\langle s,1\rangle\notin g}\left(e^{-V(\mathbf{x}_1|P_s)}\right)\left(\prod_{s\in X}^{\langle s,1\rangle\in g} e^{-V(\mathbf{y}_1|P_s)} - e^{-V(\mathbf{x}_1|P_s)}\right)\prod_{\langle s,t\rangle\in g}^{s,t\neq 1}\left(e^{V_{s,t}}-1\right)$$

and therefore, by taking a derivative w.r.t. $\mathbf{x}_1$, we find

$$\partial^{\mathbf{x}_1}_{\omega_1}\left[\left(\prod_{s\in X}^{s\neq 1} e^{-V(\mathbf{x}_1|P_s)}\right) e^{-\lambda AA_-\Delta^{-1}(\mathbf{x}_1-\mathbf{y}_1|\mathbf{z})}\widetilde{S}^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}})\right] =$$
$$= \delta_{\omega_1,\omega'_1} g_{\omega_1}(\mathbf{x}_1-\mathbf{y}_1)\left(\prod_{k=2}^{n} S^{(2)}_{\omega_k,\omega'_k}(\mathbf{x}_k-\mathbf{y}_k)\right)\cdot \tag{A.39}$$
$$\cdot\sum_{h\in X}^{h\neq 1}\sum_{g\in\mathcal{C}(X)}\left(\prod_{s\in X}^{\substack{s\neq h\\ \langle s,1\rangle\notin g}} e^{-V(\mathbf{x}_1|P_s)}\right)\prod_{s\in X}^{\substack{s\neq h\\ \langle s,1\rangle\in g}}\left(e^{-V(\mathbf{y}_1|P_s)} - e^{-V(\mathbf{x}_1|P_s)}\right)\cdot$$
$$\cdot\prod_{\langle s,1\rangle\in g}^{s,t\neq 1}\left(e^{V_{s,t}}-1\right)(-1)^{\Theta[\langle h,1\rangle\notin g]} e^{-V(\mathbf{x}_1|P_h)}(\partial_1 V)(\mathbf{x}_1|P_h)$$

where $\Theta[\cdot]$ is equal to 1 if the relation $[\cdot]$ is true; $\Theta[\cdot]$ is zero otherwise. If the graph $g \in \mathcal{C}(X)$ does not contain the link $\langle h,1\rangle$, then also the graph $g' \stackrel{def}{=} g \cup \langle h,1\rangle$ is in $\mathcal{C}(X)$; because of the factor $(-1)^{\Theta[\langle h,1\rangle \notin g]}$ their contribution cancel each other. We call $\mathcal{C}_{\langle h,1\rangle}(X)$ the remaining set of graphs: it is made of the graphs in $\mathcal{C}(X)$ which are no longer in $\mathcal{C}(X)$ if the link $\langle h,1\rangle$ is erased. Clearly they can be also constructed by joining with the link $\langle h,1\rangle$ two graphs $g_1 \in \mathcal{C}(X_1)$ and $g_h \in \mathcal{C}(X_h)$, for any choice of disjoint X_1 and X_h s.t. $1 \in X_1$, $h \in X_h$ and $X_1 \cup X_h = X$. Because of these considerations we arrive at the expression

$$\partial^{\mathbf{x}_1}_{\omega_1}\left[\left(\prod_{s\in X}^{s\neq 1} e^{-V(\mathbf{x}_1|P_s)}\right) e^{-\lambda A A_- \Delta^{-1}(\mathbf{x}_1-\mathbf{y}_1|z)} \widetilde{S}^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}})\right] = \tag{A.40}$$
$$= \left(\prod_{s\in X}^{s\neq 1} e^{-V(\mathbf{x}_1|P_s)}\right) e^{-\lambda A A_- \Delta^{-1}(\mathbf{x}_1-\mathbf{y}_1|z)} \left(\prod_{k=1}^{n} S^{(2)}_{\omega_k,\omega'_k}(\mathbf{x}_k-\mathbf{y}_k)\right)\cdot$$
$$\cdot\sum_{h\in X}^{h\neq 1}\sum_{X_1,X_h}^{**}\sum_{\substack{g_1\in\mathcal{C}(X_1)\\ g_h\in\mathcal{C}(X_h)}} (\partial_1 V)(\mathbf{x}_1|P_h)\left(\prod_{(s,t)\in g_1} e^{V_{s,t}}-1\right)\left(\prod_{(s,t)\in g_h} e^{V_{s,t}}-1\right)$$

where $\sum^{**}_{X_1,X_h}$ is the same of $\sum^{*}_{X_1,X_h}$, with the further constraint that $1 \in X_1$ and $h \in X_h$ (such a notation is abusive, but quite clear): therefore X_1, X_h is ordered, and there is no factor $(1/2!)$. As consequence, for $n > 1$, $S^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}})$ satisfy (A.10), that, after suitable derivatives in the fields, reads

$$\partial^{x_1}_{\omega_1} S^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \lambda A \sum_{h=2}^{n}\sum_{X_1,X_h}^{**}\sum_{\pi\in P_X^{X_1,X_h}} (-1)^{\pi} M^{n,h}_{X_1,X_h}(\underline{\mathbf{x}},\underline{\mathbf{y}}_{\circ\pi}) + \tag{A.41}$$
$$+\lambda A\left[\sum_{h=2}^{n} A_{-\omega_1\omega_h} g_{-\omega_1}(\mathbf{x}_1-\mathbf{x}_h) - \sum_{h=1}^{n} A_{-\omega_1\omega'_h} g_{-\omega_1}(\mathbf{x}_1-\mathbf{y}_h)\right] S^{(2n)}_{\underline{\omega},\underline{\omega}'}(\underline{\mathbf{x}},\underline{\mathbf{y}})$$

with

$$M^{n,h}_{X_1,X_h}(\underline{\mathbf{x}},\underline{\mathbf{y}}) \stackrel{def}{=} \left[A_{-\omega_1\omega_h} g_{-\omega_1}(\mathbf{x}_1-\mathbf{x}_h) - A_{-\omega_1\omega'_h} g_{-\omega_1}(\mathbf{x}_1-\mathbf{y}_h)\right]\cdot$$
$$\cdot S^{(2|X_1|)}_{\underline{\omega}_{X_1},\underline{\omega}'_{X_1}}(\underline{\mathbf{x}}_{X_1},\underline{\mathbf{y}}_{X_1}) S^{(2|X_h|)}_{\underline{\omega}_{X_h},\underline{\omega}'_{X_h}}(\underline{\mathbf{x}}_{X_h},\underline{\mathbf{y}}_{X_h}) \tag{A.42}$$

■

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