Commun. Math. Phys. 293, 301–346 (2010)
Digital Object Identifier (DOI) 10.1007/s00220-009-0910-5

Communications in
Mathematical Physics

The Two-Dimensional Hubbard Model on the Honeycomb Lattice

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Received: 19 November 2008 / Accepted: 21 July 2009
Published online: 22 September 2009 –

Abstract: We consider the two-dimensional (2D) Hubbard model on the honeycomb lattice, as a model for a single layer graphene sheet in the presence of screened Coulomb interactions. At half filling and weak enough coupling, we compute the free energy, the ground state energy and we construct the correlation functions up to zero temperature in terms of convergent series; analyticity is proved by making use of constructive fermionic renormalization group methods. We show that the interaction produces a modification of the Fermi velocity and of the wave function renormalization without changing the asymptotic infrared properties of the model with respect to the unperturbed non-interacting case; this rules out the possibility of superconducting or magnetic instabilities in the thermal ground state.

1. Introduction

The recent experimental realization of a monocrystalline graphitic film, known as *graphene* [19], revived the interest in the low temperature physics of two–dimensional electron systems on the honeycomb lattice, which is the typical underlying structure displayed by single–layer graphene sheets. Graphene is quite different from most conventional quasi–two dimensional electron gases, because of the peculiar quasi–particles dispersion relation, which closely resembles the one of massless Dirac fermions in 2 + 1 dimensions. This was already pointed out in [24] and further exploited in [23], where the analogy between graphene and 2 + 1-dimensional quantum electrodynamics (QED) was made explicit, and used to predict a condensed-matter analogue of the axial anomaly in QED. From this point of view, graphene can be considered as a sort of testing bench to investigate the properties of infrared QED in 2 + 1 dimensions. Recently, the experimental observation of graphene greatly enhanced the study of the anomalous effects induced by the pseudo-relativistic dispersion relation of its quasi particles, see [6] for an up-to-date description of the state of the art. Among the most unusual and exciting phenomena displayed by graphene, and already experimentally observed, let us mention the anomalous integer quantum Hall effect and the insensitivity to localization effects generated by disorder. It is reasonable to guess that the unique properties of graphene will have in the next few years several important applications in condensed matter and in nano-technologies.

The main reason behind these anomalous effects lies in the geometry of the Fermi surface, which at half filling is not given by a curve, as in usual 2D Fermi systems, but is completely degenerate: it consists of *two isolated points*, as in one dimensional Fermi systems. From a theoretical point of view, this fact completely changes the infrared scaling properties of the propagator. It has been pointed out, see for instance [12] and references therein, that, in the case of short-range electron-electron interactions, all the operators with four or more fermionic fields are irrelevant in a Renormalization Group (RG) sense; this suggests that the interaction should not affect too much the asymptotic behavior of the model, at least at small coupling. It should be remarked however that such RG analyses were performed only at a perturbative level, without any control on the convergence of the expansion, and directly in the relativistic approximation, consisting in replacing the actual dispersion relation by its linear approximation around the singularity; such approximation implies in particular the validity of a continuous Lorentz $U(1)$ symmetry that is not present in the original model.

The aim of this paper is to present the first rigorous construction of the low temperature and ground state properties of the 2D Hubbard model on the honeycomb lattice with weak local interactions; this is achieved by rewriting the correlation functions in terms of the resummed series, convergent uniformly in the temperature up to zero temperature, as we prove by making use of the constructive fermionic renormalization group. We show that indeed the interaction does not change the asymptotic infrared properties of the model with respect to the unperturbed non-interacting case, but it produces a renormalization of the Fermi velocity and of the wave function. Our result rules out the presence of superconducting or magnetic instabilities at weak coupling; this is in striking contrast with the behavior of the 2D Hubbard model on the square lattice, where quantum instabilities (corresponding to the magnetic or superconducting long range order that are presumably present in the ground state) prevent the convergence of the perturbative expansion in U for low enough temperatures.

2. The Model and the Main Results

2.1. The model. The grandcanonical Hamiltonian of the 2D Hubbard model on the honeycomb lattice at half filling in the second quantized form is given by:

$$
\begin{aligned}
H_\Lambda = & -\sum_{\substack{\vec{x}\in\Lambda\\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \left(a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma} \right) \\
& + \frac{U}{3} \sum_{\substack{\vec{x}\in\Lambda\\ i=1,2,3}} \left[\left(a^+_{\vec{x},\uparrow} a^-_{\vec{x},\uparrow} - \frac{1}{2} \right) \left(a^+_{\vec{x},\downarrow} a^-_{\vec{x},\downarrow} - \frac{1}{2} \right) \right. \\
& \left. + \left(b^+_{\vec{x}+\vec{\delta}_i,\uparrow} b^-_{\vec{x}+\vec{\delta}_i,\uparrow} - \frac{1}{2} \right) \left(b^+_{\vec{x}+\vec{\delta}_i,\downarrow} b^-_{\vec{x}+\vec{\delta}_i,\downarrow} - \frac{1}{2} \right) \right],
\end{aligned} \tag{2.1}
$$

where

1. Λ is a periodic triangular lattice, defined as $\Lambda = \mathbb{B}/L\mathbb{B}$, where $L \in \mathbb{N}$ and $\mathbb{B}$ is the triangular lattice with basis $\vec{a}_1 = \frac{1}{2}(3, \sqrt{3})$, $\vec{a}_2 = \frac{1}{2}(3, -\sqrt{3})$.

2. The vectors $\vec{\delta}_i$ are defined as

$$\vec{\delta}_1 = (1, 0), \quad \vec{\delta}_2 = \frac{1}{2}(-1, \sqrt{3}), \quad \vec{\delta}_3 = \frac{1}{2}(-1, -\sqrt{3}). \tag{2.2}$$

3. $a^{\pm}_{\vec{x},\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec{x} \in \Lambda$, satisfying periodic boundary conditions in $\vec{x}$.
4. $b^{\pm}_{\vec{x}+\vec{\delta}_i,\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec{x} + \vec{\delta}_i \in \Lambda + \vec{\delta}_1$, satisfying periodic boundary conditions in $\vec{x}$.
5. U is the strength of the on–site density–density interaction; it can be either positive or negative.

Note that the Hamiltonian (2.1) is hole-particle symmetric, i.e., it is invariant under the exchange $a^{\pm}_{\vec{x},\sigma} \longleftrightarrow a^{\mp}_{\vec{x},\sigma}$, $b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma} \longleftrightarrow -b^{\mp}_{\vec{x}+\vec{\delta}_1,\sigma}$. This invariance implies in particular that, if we define the average density of the system to be $\rho = (2|\Lambda|)^{-1}\langle N\rangle_{\beta,\Lambda}$, with $N = \sum_{\vec{x},\sigma}(a^{+}_{\vec{x},\sigma}a^{-}_{\vec{x},\sigma} + b^{+}_{\vec{x}+\vec{\delta}_1,\sigma}b^{-}_{\vec{x}+\vec{\delta}_1,\sigma})$ the total particle number operator and $\langle\cdot\rangle_{\beta,\Lambda} = \mathrm{Tr}\{e^{-\beta H_\Lambda}\cdot\}/\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$ the average with respect to the (grandcanonical) Gibbs measure at inverse temperature β, one has $\rho \equiv 1$, for any $|\Lambda|$ and any β.

Our goal is to characterize the low and zero temperature properties of the system described by (2.1), by computing thermodynamic functions (e.g., specific free energy and specific ground state energy) and a complete set of correlations at low or zero temperatures. To this purpose it is convenient to introduce the notions of specific free energy

$$f_\beta(U) = -\frac{1}{\beta}\lim_{|\Lambda|\to\infty}|\Lambda|^{-1}\log\mathrm{Tr}\{e^{-\beta H_\Lambda}\}, \tag{2.3}$$

of specific ground state energy $e(U) = \lim_{\beta\to\infty} f_\beta(U)$, and of Schwinger functions, defined as follows.

Let us introduce the two component fermionic operators $\Psi^{\pm}_{\vec{x},\sigma} = \left(a^{\pm}_{\vec{x},\sigma}, b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}\right)$ and let us write $\Psi^{\pm}_{\vec{x},\sigma,1} = a^{\pm}_{\vec{x},\sigma}$ and $\Psi^{\pm}_{\vec{x},\sigma,2} = b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}$. We shall also consider the operators $\Psi^{\pm}_{\mathbf{x},\sigma} = e^{Hx_0}\Psi^{\pm}_{\vec{x},\sigma}e^{-Hx_0}$ with $\mathbf{x} = (x_0, \vec{x})$ and $x_0 \in [0, \beta]$, for some $\beta > 0$; we shall call x_0 the time variable. We shall write $\Psi^{\pm}_{\mathbf{x},\sigma,1} = a^{\pm}_{\mathbf{x},\sigma}$ and $\Psi^{\pm}_{\mathbf{x},\sigma,2} = b^{\pm}_{\mathbf{x}+,\sigma}$, with $= (0, \vec{\delta}_1)$. We define

$$S^{\beta,\Lambda}_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n) = \langle\mathbf{T}\{\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1}\cdots\Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}\}\rangle_{\beta,\Lambda}, \tag{2.4}$$

where $\mathbf{x}_i \in [0, \beta] \times \Lambda$, $\sigma_i =\uparrow\downarrow$, $\varepsilon_i = \pm$, $\rho_i = 1, 2$ and $\mathbf{T}$ is the operator of fermionic time ordering, acting on a product of fermionic fields as:

$$\mathbf{T}(\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1}\cdots\Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}) = (-1)^\pi\Psi^{\varepsilon_{\pi(1)}}_{\mathbf{x}_{\pi(1)},\sigma_{\pi(1)},\rho_{\pi(1)}}\cdots\Psi^{\varepsilon_{\pi(n)}}_{\mathbf{x}_{\pi(n)},\sigma_{\pi(n)},\rho_{\pi(n)}}, \tag{2.5}$$

where π is a permutation of $\{1, \ldots, n\}$, chosen in such a way that $x_{\pi(1)0} \geq \cdots \geq x_{\pi(n)0}$, and $(-1)^\pi$ is its sign. [If some of the time coordinates are equal to each other, the arbitrariness of the definition is solved by ordering each set of operators with the same time coordinate so that creation operators precede the annihilation operators.]

Taking the limit $\Lambda \to \infty$ in (2.4) we get the finite temperature n-point Schwinger functions, denoted by $S^\beta_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n)$, which describe the properties of the infinite volume system at finite temperature. Taking the $\beta \to \infty$ limit of the finite temperature Schwinger functions, we get the zero temperature Schwinger functions, simply denoted by $S_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \dots ; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n)$, which by definition characterize the properties of the *thermal ground state* of (2.1) in the thermodynamic limit.

2.2. The non interacting case. In the non–interacting case $U = 0$ the Schwinger functions of any order n can be exactly computed as linear combinations of products of two–point Schwinger functions, via the well–known *Wick rule*. The two–point Schwinger function itself, also called the *free propagator*, for $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} - \mathbf{y} \neq (\pm\beta, \vec{0})$, is equal to (see Appendix A for details):

$$S_0^{\beta,\Lambda}(\mathbf{x}-\mathbf{y})_{\rho,\rho'} \equiv S_2^{\beta,\Lambda}(\mathbf{x}, \sigma, -, \rho; \mathbf{y}, \sigma, +, \rho')\Big|_{U=0}$$
$$= \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k_0^2 + |v(\vec{k})|^2} \begin{pmatrix} ik_0 & -v^*(\vec{k}) \\ -v(\vec{k}) & ik_0 \end{pmatrix}_{\rho,\rho'}, \quad (2.6)$$

where

1. $\mathbf{k} = (k_0, \vec{k})$ and $\mathcal{D}_{\beta,L} = \mathcal{D}_\beta \times \mathcal{D}_L$;
2. $\mathcal{D}_\beta = \{k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2}) \ : \ n_0 \in \mathbb{Z}\}$ and $\mathcal{D}_L = \{\vec{k} = \frac{n_1}{L}\vec{b}_1 + \frac{n_2}{L}\vec{b}_2 \ : \ 0 \le n_1, n_2 \le L-1\}$, where $\vec{b}_1 = \frac{2\pi}{3}(1, \sqrt{3}), \vec{b}_2 = \frac{2\pi}{3}(1, -\sqrt{3})$ are a basis of the dual lattice Λ^*;
3. $v(\vec{k}) = \sum_{i=1}^3 e^{i\vec{k}(\vec{\delta}_i - \vec{\delta}_1)} = 1 + 2e^{-i3/2k_1} \cos \frac{\sqrt{3}}{2}k_2$; its modulus $|v(\vec{k})|$ is the *dispersion relation*, given by

$$|v_{\vec{k}}| = \sqrt{\Big(1 + 2\cos(3k_1/2)\cos(\sqrt{3}k_2/2)\Big)^2 + 4\sin^2(3k_1/2)\cos^2(\sqrt{3}k_2/2)}. \quad (2.7)$$

At $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} - \mathbf{y} = (\pm\beta, \vec{0})$, the free propagator has a jump discontinuity, see Appendix A. Note that $S_0^{\beta,\Lambda}(\mathbf{x})$ is antiperiodic in x_0, i.e. $S_0^{\beta,\Lambda}(x_0 + \beta, \vec{x}) = -S_0^{\beta,\Lambda}(x_0, \vec{x})$, and that its Fourier transform $\hat{S}_0^{\beta,\Lambda}(\mathbf{k})$ is well–defined for any $\mathbf{k} \in \mathcal{D}_{\beta,L}$, even in the thermodynamic limit $L \to \infty$, since $|k_0| \ge \frac{\pi}{\beta}$. We shall refer to this last property by saying that the inverse temperature β acts as an infrared cutoff for our theory.

If we take $\beta, L \to \infty$, the limiting propagator $\hat{S}_0(\mathbf{k})$ becomes singular at $\{k_0 = 0\} \times \{\vec{k} = \vec{p}_F^{\,\pm}\}$, where

$$\vec{p}_F^{\,\pm} = \left(\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}}\right) \quad (2.8)$$

are the *Fermi points* (also called *Dirac points*, for an analogy with massive QED_{2+1} that will become clearer below). Note that the asymptotic behavior of $v(\vec{k})$ close to the Fermi points is given by $v(\vec{p}_F^{\,\pm} + \vec{k}') \simeq \frac{3}{2}(ik_1' \pm k_2')$. In particular, if $\omega = \pm$, the Fourier transform of the 2-point Schwinger function close to the Fermi point $\vec{p}_F^{\,\omega}$ can be rewritten in the form:

$$\hat{S}_0(k_0, \vec{p}_F^{\,\omega} + \vec{k}') = \frac{1}{Z_0} \begin{pmatrix} -ik_0 & -v_F^{(0)}(-ik_1' + \omega k_2') + r_\omega(\vec{k}') \\ -v_F^{(0)}(ik_1' + \omega k_2') + r_\omega^*(\vec{k}') & -ik_0 \end{pmatrix}^{-1}, \quad (2.9)$$

where $Z_0 = 1$ is the *free wave function renormalization* and $v_F^{(0)} = 3/2$ is the *free Fermi velocity*. Moreover, $|r_\omega(\vec{k}')| \le C|\vec{k}'|^2$, for small values of $\vec{k}'$ and for some positive constant C.

2.3. The interacting case. We are now interested in what happens by adding a local interaction. In the case $U \neq 0$, the Schwinger functions are not exactly computable anymore. It is well–known that they can be written as formal power series in U, constructed in terms of *Feynmann diagrams*, using as the free propagator the function $S_0(\mathbf{x})$ in (2.6). Our main result consists in a proof of convergence of this formal expansion for U small enough, after the implementation of suitable resummations of the original power series. Our main result can be described as follows.

Theorem 1. *Let us consider the 2D Hubbard model on the honeycomb lattice at half filling, defined by* (2.1)*. There exist a constant $U_0 > 0$ such that, if $|U| \le U_0$, the specific free energy $f_\beta(U)$ and the finite temperature Schwinger functions are analytic functions of U, uniformly in β as $\beta \to \infty$, and so are the specific ground state energy $e(U)$ and the zero temperature Schwinger functions. The Fourier transform of the zero temperature two point Schwinger function $S(\mathbf{x})_{\rho,\rho'} \stackrel{def}{=} S_2(\mathbf{x}, \sigma, -, \rho; \mathbf{0}, \sigma, +, \rho')$, denoted by $\hat{S}(\mathbf{k})$, is singular only at the Fermi points $\mathbf{k} = \mathbf{p}_F^{\pm} = (0, \vec{p}_F^{\,\pm})$, see* (2.8)*, and, close to the singularities, if $\omega = \pm$, it can be written as*

$$\hat{S}(k_0, \vec{p}_F^{\,\omega} + \vec{k}') = \frac{1}{Z}\begin{pmatrix} -ik_0 & -v_F(-ik_1' + \omega k_2') \\ -v_F(ik_1' + \omega k_2') & -ik_0 \end{pmatrix}^{-1} \big(\mathbb{1} + R(\mathbf{k}')\big)\,, \tag{2.10}$$

with $\mathbf{k}' = (k_0, \vec{k}')$, and with Z and v_F two real constants such that

$$Z = 1 + O(U^2), \qquad v_F = \frac{3}{2} + O(U^2). \tag{2.11}$$

Moreover the matrix $R(\mathbf{k}')$ satisfies $||R(\mathbf{k}')|| \le C|\mathbf{k}'|^\vartheta$ for some constants $C, \vartheta > 0$ and for $|\mathbf{k}'|$ small enough.

Remarks. 1) Theorem 1 says that the location of the singularity does not change in the presence of interaction; on the contrary, the wave function renormalization and Fermi velocity are modified by the interaction. Note also that, in the presence of the interaction, the Fermi velocity remains the same in the two coordinate direction even though the model does not display 90^o discrete rotational symmetry, but rather a 120^o rotational symmetry.

2) The resulting theory is not quasi-free: the Wick rule is not valid anymore in the presence of interactions. However, the long distance asymptotics of the higher order Schwinger functions can be estimated by the same methods used to prove Theorem 1, and it is the same suggested by the Wick rule.

3) The fact that the interacting correlations decay as in the non-interacting case implies in particular the absence of magnetic long range order in the thermal ground state of the system at weak coupling (we recall that the thermal ground state is by definition the weak limit as $\beta, |\Lambda| \to \infty$ of the grandcanonical Gibbs state $e^{-\beta H_\Lambda}$, see the end of Sec.2.1). In fact, as a corollary of our construction, we find:

$$\left| \lim_{\beta, |\Lambda| \to \infty} \langle \vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{y}} \rangle_{\beta,\Lambda} \right| \le C \frac{1}{|\vec{x} - \vec{y}|^4}, \tag{2.12}$$

where, if $\vec{x} \in \Lambda$, the spin operator $\vec{S}_{\vec{x}}$ is defined as: $\vec{S}_{\vec{x}} = a^{+}_{\vec{x},\cdot}\vec{\sigma}\, a^{-}_{\vec{x},\cdot}$, with σ_i, $i = 1, 2, 3$, the Pauli matrices; similarly, if $\vec{x} \in \Lambda + \vec{\delta}_1$, $\vec{S}_{\vec{x}} = \sum_\sigma b^{+}_{\vec{x},\cdot}\vec{\sigma} b^{-}_{\vec{x},\cdot}$. Note that it is known, at least in the microcanonical ensemble [15], that the ground state is unique and it has zero total spin; however, so far, existence of Néel order in the ground state was neither proven nor ruled out.

4) Similarly to what was remarked in the previous item, one can exclude the existence of superconducting long range order in the thermal ground state: the Cooper pairs correlations decay to zero at infinity at least as fast as the spin-spin correlations in (2.12).
5) Our analysis can be extended in a straightforward way to the case of exponentially decaying interactions (instead of local interactions). However, if the decay is slower, the result may change. In particular, in the presence of 3D Coulomb interactions, the electron-electron interaction becomes marginal (instead of irrelevant), in a renormalization group sense [13].
6) Previous analyses of the Hubbard model on the honeycomb lattice were performed only at a perturbative level, without any control on the convergence of the weak coupling expansion, and directly in the Quantum Field Theory approximation, consisting in the replacement of $\hat{S}_0(\mathbf{k})$ by its linear approximation around the Fermi points, see for instance [12] and references therein.

The proof of the theorem is based on constructive fermionic Renormalization Group (RG) methods, see [2,17,21] for extensive reviews. It is worth remarking that the result summarized in Theorem 1 is one of the few rigorous constructions of the ground state properties (including correlations) of a weak coupling 2D Hubbard model. The only other example we are aware of is the Fermi liquid construction in [8], applicable to cases of weakly interacting 2D Fermi systems with a highly asymmetric interacting Fermi surface. Related results include the construction of the state at temperatures larger than a BCS-like critical temperature [4,7], or the computation of the first contribution to the ground state energy in a weak coupling limit [11,16,22].

The rest of the paper will be devoted to the proof of Theorem 1. In Sec. 3.1 we review the Grassmann integral representation for the free energy and the Schwinger functions. In Sec.3.2 we start to describe the integration procedure leading to the computation of the free energy, and in particular we describe how to integrate out the ultraviolet degrees of freedom. In Sec.3.3 we complete the proof of convergence of the series for the free energy and the ground state energy. In Sec.3.4 we describe the proof of convergence for the series for the Schwinger functions, with particular emphasis on the case of the two-point Schwinger function. In the Appendices we provide further details concerning the non-interacting theory, the ultraviolet integration, the thermodynamic and zero temperature limits.

3. Renormalization Group Analysis

3.1. Grassmann integration. In this subsection, for any β and L finite, we rewrite the partition function and the Schwinger functions of model (2.1) in terms of Grassmann functional integrals, defined as follows.

Let $M \in \mathbb{N}$ and $\chi_0(t)$ be a smooth compact support function that is 1 for $t \le a_0$ and 0 for $t \ge a_0\gamma$, with $\gamma > 1$ and a_0 a constant to be chosen below, see the lines preceding (3.29). Let $\mathcal{D}^*_{\beta,L} = \mathcal{D}_{\beta,L} \cap \{k_0 \,:\, \chi_0(\gamma^{-M}|k_0|) > 0\}$, with $\mathcal{D}_{\beta,L}$ defined after (2.6). We consider the finite Grassmann algebra generated by the Grassmannian variables

$\{\hat{\Psi}^{\pm}_{\mathbf{k},\sigma,\rho}\}^{\sigma=\uparrow\downarrow,\ \rho=1,2}_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}$ and a Grassmann integration $\int\left[\prod_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}\prod_{\sigma=\uparrow\downarrow}^{\rho=1,2} d\hat{\Psi}^{+}_{\mathbf{k},\sigma,\rho}d\hat{\Psi}^{-}_{\mathbf{k},\sigma,\rho}\right]$ defined as the linear operator on the Grassmann algebra such that, given a monomial $Q(\hat{\Psi}^-,\hat{\Psi}^+)$ in the variables $\hat{\Psi}^{\pm}_{\mathbf{k},\sigma,\rho}$, its action on $Q(\hat{\Psi}^-,\hat{\Psi}^+)$ is 0 except in the case $Q(\hat{\Psi}^-,\hat{\Psi}^+)=\prod_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}\prod_{\sigma=\uparrow\downarrow}^{\rho=1,2}\hat{\Psi}^{-}_{\mathbf{k},\sigma,\rho}\hat{\Psi}^{+}_{\mathbf{k},\sigma,\rho}$, up to a permutation of the variables. In this case the value of the integral is determined, by using the anticommuting properties of the variables, by the condition

$$\int\left[\prod_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}\prod_{\sigma=\uparrow\downarrow}^{\rho=1,2} d\hat{\Psi}^{+}_{\mathbf{k},\sigma,\rho}d\hat{\Psi}^{-}_{\mathbf{k},\sigma,\rho}\right]\prod_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}\prod_{\sigma=\uparrow\downarrow}^{\rho=1,2}\hat{\Psi}^{-}_{\mathbf{k},\sigma,\rho}\hat{\Psi}^{+}_{\mathbf{k},\sigma,\rho}=1. \tag{3.13}$$

Defining the free propagator matrix $\hat{g}_{\mathbf{k}}$ as

$$\hat{g}_{\mathbf{k}}=\chi_0(\gamma^{-M}|k_0|)\begin{pmatrix}-ik_0 & -v^*(\vec{k})\\ -v(\vec{k}) & -ik_0\end{pmatrix}^{-1} \tag{3.14}$$

and the "Gaussian integration" $P(d\Psi)$ as

$$P(d\Psi)=\left[\prod_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}^{\sigma=\uparrow\downarrow}\frac{-\beta^2|\Lambda|^2\,[\chi_0(\gamma^{-M}|k_0|)]^2}{k_0^2+|v(\vec{k})|^2}d\hat{\Psi}^{+}_{\mathbf{k},\sigma,1}d\hat{\Psi}^{-}_{\mathbf{k},\sigma,1}d\hat{\Psi}^{+}_{\mathbf{k},\sigma,2}d\hat{\Psi}^{-}_{\mathbf{k},\sigma,2}\right]$$
$$\cdot\ \exp\left\{-(\beta|\Lambda|)^{-1}\sum_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}^{\sigma=\uparrow\downarrow}\hat{\Psi}^{+}_{\mathbf{k},\sigma,\cdot}\,\hat{g}_{\mathbf{k}}^{-1}\,\hat{\Psi}^{-}_{\mathbf{k},\sigma,\cdot}\right\}, \tag{3.15}$$

it turns out that

$$\int P(d\Psi)\hat{\Psi}^{-}_{\mathbf{k}_1,\sigma_1,\rho_1}\hat{\Psi}^{+}_{\mathbf{k}_2,\sigma_2,\rho_2}=\beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\left[\hat{g}_{\mathbf{k}_1}\right]_{\rho_1,\rho_2}, \tag{3.16}$$

so that, if $\mathbf{x}-\mathbf{y}\notin\beta\mathbb{Z}\times\{\vec{0}\}$,

$$\lim_{M\to\infty}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\hat{g}_{\mathbf{k}}=\lim_{M\to\infty}\int P(d\Psi)\Psi^{-}_{\mathbf{x},\sigma}\Psi^{+}_{\mathbf{y},\sigma}=S_0(\mathbf{x}-\mathbf{y}), \tag{3.17}$$

where $S_0(\mathbf{x}-\mathbf{y})$ was defined in (2.5) and the Grassmann fields $\Psi^{\pm}_{\mathbf{x},\sigma}$ are defined by

$$\Psi^{\pm}_{\mathbf{x},\sigma,\rho}=\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}e^{\pm i\mathbf{k}\mathbf{x}}\hat{\Psi}^{\pm}_{\mathbf{k},\sigma,\rho},\qquad \mathbf{x}\in(-\beta/2,\beta/2]\times\Lambda. \tag{3.18}$$

Let us now consider the function on the Grassmann algebra,

$$V(\Psi)=U\sum_{\rho=1,2}\int d\mathbf{x}\,\Psi^{+}_{\mathbf{x},\uparrow,\rho}\Psi^{-}_{\mathbf{x},\uparrow,\rho}\Psi^{+}_{\mathbf{x},\downarrow,\rho}\Psi^{-}_{\mathbf{x},\downarrow,\rho}$$
$$=\frac{U}{(\beta|\Lambda|)^3}\sum_{\rho=1,2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{p}}\hat{\Psi}^{+}_{\mathbf{k}-\mathbf{p},\uparrow,\rho}\hat{\Psi}^{-}_{\mathbf{k},\uparrow,\rho}\hat{\Psi}^{+}_{\mathbf{k}'+\mathbf{p},\downarrow,\rho}\hat{\Psi}^{-}_{\mathbf{k}',\downarrow,\rho}, \tag{3.19}$$

where, in the first line, the symbol $\int d\mathbf{x}$ must be interpreted as

$$\int d\mathbf{x} = \int_{-\beta/2}^{\beta/2} dx_0 \sum_{\vec{x}\in\Lambda}, \tag{3.20}$$

and, in the second line, the sums over $\mathbf{k}$, $\mathbf{k}'$ run over the set $\mathcal{D}^*_{\beta,L}$, while the sums over $\mathbf{p}$ run over the set $2\pi\beta^{-1}\mathbb{Z}\times\mathcal{D}_L$ (with the constraint that $\mathbf{k}-\mathbf{p}, \mathbf{k}'+\mathbf{p}\in\mathcal{D}^*_{\beta,L}$: $\mathbf{p}$ is the *transferred momentum*).

We introduce the following Grassman integrals:

$$e^{-\beta|\Lambda|F_{M,\beta,L}} = \int P(d\Psi)e^{-V(\Psi)}, \tag{3.21}$$

$$\mathcal{S}_n^{M,\beta,\Lambda}(\mathbf{x}_1,\sigma_1,\varepsilon_1,\rho_1;\dots;\mathbf{x}_n,\sigma_n,\varepsilon_n,\rho_n) = \frac{\int P(d\Psi)e^{-V(\Psi)}\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1}\cdots\Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}}{\int P(d\Psi)e^{-V(\Psi)}}. \tag{3.22}$$

Note that these Grassmann integrals are well defined for any U; they are indeed polynomials in U, of degree depending on M and L.

It is well known that the Grassmann integrals in (3.21) and (3.22) can be used to compute the thermodynamic properties of the model with Hamiltonian (2.1), as ensured by the following proposition:

Proposition 1. *For any β, $L<+\infty$, assume that there exists U_0 independent of β and L such that $F_{M,\beta,L}$ and $\mathcal{S}_n^{M,\beta,\Lambda}$ are analytic in the complex domain $|U|\le U_0$, uniformly convergent as $M\to\infty$. Then, if $|U|\le U_0$,*

$$-\frac{1}{\beta|\Lambda|}\log\mathrm{Tr}\{e^{-\beta H_\Lambda}\} = -\frac{2}{\beta|\Lambda|}\sum_{\vec{k}\in\mathcal{D}_L}\log\Big(2+2\cosh(\beta|v(\vec{k})|)\Big) + \lim_{M\to\infty}F_{M,\beta,L}, \tag{3.23}$$

and the Schwinger functions at distinct space-time points, defined in (2.4), can be computed as

$$\begin{aligned} &S_n^{\beta,\Lambda}(\mathbf{x}_1,\sigma_1,\varepsilon_1,\rho_1;\dots;\mathbf{x}_n,\sigma_n,\varepsilon_n,\rho_n)\\ &\quad=\lim_{M\to\infty}\mathcal{S}_n^{M,\beta,\Lambda}(\mathbf{x}_1,\sigma_1,\varepsilon_1,\rho_1;\dots;\mathbf{x}_n,\sigma_n,\varepsilon_n,\rho_n). \end{aligned} \tag{3.24}$$

For completeness, the proof of Proposition 1 is reported in Appendix B; its result guarantees that the thermodynamic properties of the model with Hamiltonian (2.1) can be inferred from the analysis of the Grassmann integrals (3.21) and (3.22), provided that the latter satisfy the uniform analyticity properties assumed in Proposition 1. The rest of the paper is devoted to the study of the Grassmann integrals (3.21) and (3.22); our analysis implies, in particular, the uniform analyticity properties assumed in Proposition 1, see Corollary 1 and Sect. 3.4 below.

It is important to note that both the Gaussian integration $P(d\Psi)$ and the interaction $V(\Psi)$ are invariant under the action of a number of remarkable symmetry transformations, which will be preserved by the subsequent iterative integration procedure and will guarantee the vanishing of some running coupling constants (see below for details). Let us collect in the following lemma all the symmetry properties we will need in the following:

Lemma 1. *For any choice of M, β, Λ, both the quadratic Grassmann measure $P(d\Psi)$ defined in (3.15) and the quartic Grassmann interaction $V(\Psi)$ defined in (3.19) are invariant under the following transformations:*

(1) *spin exchange:* $\hat\Psi^{\varepsilon}_{\mathbf{k},\sigma,\rho} \longleftrightarrow \hat\Psi^{\varepsilon}_{\mathbf{k},-\sigma,\rho}$;

(2) *global $U(1)$:* $\hat\Psi^{\varepsilon}_{\mathbf{k},\sigma,\rho} \to e^{i\varepsilon\alpha_\sigma}\hat\Psi^{\varepsilon}_{\mathbf{k},\sigma,\rho}$, *with $\alpha_\sigma \in \mathbb{R}$ independent of $\mathbf{k}$;*

(3) *spin $SO(2)$:* $\begin{pmatrix}\hat\Psi^{\varepsilon}_{\mathbf{k},\uparrow,\rho}\\ \hat\Psi^{\varepsilon}_{\mathbf{k},\downarrow,\rho}\end{pmatrix} \to R_\theta \begin{pmatrix}\hat\Psi^{\varepsilon}_{\mathbf{k},\uparrow,\rho}\\ \hat\Psi^{\varepsilon}_{\mathbf{k},\downarrow,\rho}\end{pmatrix}$, *with* $R_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ *and $\theta \in \mathbb{T}$ independent of $\mathbf{k}$;*

(4) *discrete spatial rotations:* $\hat\Psi^{\pm}_{(k_0,\vec k),\sigma,\rho} \to e^{\mp i\vec k(\vec\delta_3-\vec\delta_1)(\rho-1)}\hat\Psi^{\pm}_{(k_0,T_1\vec k),\sigma,\rho}$, *with $T_1\vec x \overset{def}{=} R_{2\pi/3}\vec x$; note that in real space this transformation simply reads $a^{\pm}_{(x_0,\vec x),\sigma} \to a^{\pm}_{(x_0,T_1\vec x),\sigma}$ and $b^{\pm}_{(x_0,\vec x),\sigma} \to b^{\pm}_{(x_0,T_1\vec x),\sigma}$;*

(5) *complex conjugation:* $\hat\Psi^{\pm}_{\mathbf{k},\sigma,\rho} \to \hat\Psi^{\pm}_{-\mathbf{k},\sigma,\rho}$, *$c \to c^*$, where c is a generic constant appearing in $P(d\Psi)$ and/or in $V(\Psi)$;*

(6.a) *horizontal reflections:* $\hat\Psi^{\pm}_{(k_0,k_1,k_2),\sigma,1} \longleftrightarrow \hat\Psi^{\pm}_{(k_0,-k_1,k_2),\sigma,2}$;

(6.b) *vertical reflections:* $\hat\Psi^{\pm}_{(k_0,k_1,k_2),\sigma,\rho} \to \hat\Psi^{\pm}_{(k_0,k_1,-k_2),\sigma,\rho}$;

(7) *particle-hole:* $\hat\Psi^{\pm}_{(k_0,\vec k),\sigma,\rho} \to i\hat\Psi^{\mp}_{(k_0,-\vec k),\sigma,\rho}$.

(8) *inversion:* $\hat\Psi^{\pm}_{(k_0,\vec k),\sigma,\rho} \to i(-1)^\rho\hat\Psi^{\pm}_{(-k_0,\vec k),\sigma,\rho}$.

Proof. A moment's thought shows that the invariance of $V(\Psi)$ under the above symmetries is obvious, and so is the invariance of $P(d\Psi)$ under (1)-(2)-(3). Let us then prove the invariance of $P(d\Psi)$ under (4)-(5)-(6.a)-(6.b)-(7)-(8). More precisely, let us consider the term

$$\begin{aligned}\sum_{\mathbf{k}}\hat\Psi^{+}_{\mathbf{k},\sigma,\cdot}\,\hat g_{\mathbf{k}}^{-1}\hat\Psi^{-}_{\mathbf{k},\sigma,\cdot} = \\ -i\sum_{\mathbf{k}}\hat\Psi^{+}_{\mathbf{k},\sigma,1}k_0\hat\Psi^{-}_{\mathbf{k},\sigma,1} - \sum_{\mathbf{k}}\hat\Psi^{+}_{\mathbf{k},\sigma,1}v^*(\vec k)\hat\Psi^{-}_{\mathbf{k},\sigma,2} - \sum_{\mathbf{k}}\hat\Psi^{+}_{\mathbf{k},\sigma,2}v(\vec k)\hat\Psi^{-}_{\mathbf{k},\sigma,1}\\ -i\sum_{\mathbf{k}}\hat\Psi^{+}_{\mathbf{k},\sigma,2}k_0\hat\Psi^{-}_{\mathbf{k},\sigma,2}\end{aligned} \tag{3.25}$$

in (3.15), and let us prove its invariance under the transformations (4)-(5)-(6.a)-(6.b)-(7)-(8).

Under the transformation (4), the first and fourth term in the second line of (3.25) are obviously invariant, while the sum of the second and third is changed into

$$\begin{aligned}&-\sum_{\mathbf{k}}\Big[\hat\Psi^{+}_{(k_0,T_1\vec k),\sigma,1}v^*(\vec k)e^{+i\vec k(\vec\delta_3-\vec\delta_1)}\hat\Psi^{-}_{(k_0,T_1\vec k),\sigma,2} + \hat\Psi^{+}_{(k_0,T_1\vec k),\sigma,2}e^{-i\vec k(\vec\delta_3-\vec\delta_1)}v(\vec k)\hat\Psi^{-}_{(k_0,T_1\vec k),\sigma,1}\Big]\\ &= -\sum_{\mathbf{k}}\Big[\hat\Psi^{+}_{\mathbf{k},\sigma,1}v^*(T_1^{-1}\vec k)e^{+i\vec k(\vec\delta_1-\vec\delta_2)}\hat\Psi^{-}_{\mathbf{k},\sigma,2} + \hat\Psi^{+}_{\mathbf{k},\sigma,2}e^{-i\vec k(\vec\delta_1-\vec\delta_2)}v(T_1^{-1}\vec k)\hat\Psi^{-}_{\mathbf{k},\sigma,1}\Big].\end{aligned} \tag{3.26}$$

Using that $v(T_1^{-1}\vec k) = e^{i\vec k(\vec\delta_1-\vec\delta_2)}v(\vec k)$, as it follows by the definition $v(\vec k) = \sum_{i=1,2,3}e^{i\vec k(\vec\delta_i-\vec\delta_1)}$, we find that the last line of (3.26) is equal to the sum of the second and third term in (3.25), as desired.

The invariance of (3.25) under the transformation (5) is very simple, if one notes that $v(-\vec{k}) = v^*(\vec{k})$, it follows by the definition of $v(\vec{k})$.

Under the transformation (6.a), the sum of the first and fourth term in the second line of (3.25) is obviously invariant, while the sum of the second and third is changed into

$$\begin{aligned}&-\sum_{\mathbf{k}} \hat{\Psi}^+_{(k_0,-k_1,k_2),\sigma,2} v^*(\vec{k}) \hat{\Psi}^-_{(k_0,-k_1,k_2),\sigma,1} - \sum_{\mathbf{k}} \hat{\Psi}^+_{(k_0,-k_1,k_2),\sigma,1} v(\vec{k}) \hat{\Psi}^-_{(k_0,-k_1,k_2),\sigma,2}\\ &= -\sum_{\mathbf{k}} \hat{\Psi}^+_{\mathbf{k},\sigma,2} v^*((-k_1,k_2)) \hat{\Psi}^-_{\mathbf{k},\sigma,1} - \sum_{\mathbf{k}} \hat{\Psi}^+_{\mathbf{k},\sigma,1} v((-k_1,k_2)) \hat{\Psi}^-_{\mathbf{k},\sigma,2}. \end{aligned} \tag{3.27}$$

Noting that $v((-k_1, k_2)) = v^*(\mathbf{k})$, one sees that this is the same as the sum of the second and third term in (3.25), as desired.

Similarly, noting that $v((k_1, -k_2)) = v(\mathbf{k})$, one finds that (3.25) is invariant under the transformation (6.b).

Under the transformation (7), the sum of the first and fourth term in (3.25) is obviously invariant, while the sum of the second and third term is changed into

$$\begin{aligned}&+\sum_{\mathbf{k}} \hat{\Psi}^-_{(k_0,-\vec{k}),\sigma,1} v^*(\vec{k}) \hat{\Psi}^+_{(k_0,-\vec{k}),\sigma,2} + \sum_{\mathbf{k}} \hat{\Psi}^-_{(k_0,-\vec{k}),\sigma,2} v(\vec{k}) \hat{\Psi}^+_{(k_0,-\vec{k}),\sigma,1}\\ &= -\sum_{\mathbf{k}} \hat{\Psi}^+_{\mathbf{k},\sigma,2} v^*(-\vec{k}) \hat{\Psi}^-_{\mathbf{k},\sigma,1} - \sum_{\mathbf{k}} \hat{\Psi}^+_{\mathbf{k},\sigma,1} v(-\vec{k}) \hat{\Psi}^-_{\mathbf{k},\sigma,2}. \end{aligned} \tag{3.28}$$

Using, again, that $v(-\vec{k}) = v^*(\vec{k})$, we see that the latter sum is the same as the sum of the second and third term in (3.25), as desired.

Finally, under the transformation (8), all the terms in the right hand side of (3.25) are separately invariant, and the proof of Lemma 1 is concluded. □

3.2. Free energy: The ultraviolet integration. We start by studying the partition function $\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F_{M,\beta,L}}$ with $F_{M,\beta,L}$ defined in (3.21). Note that our lattice model has an intrinsic ultraviolet cut-off in the $\vec{k}$ variables, while the k_0 variable is not bounded uniformly in M. A preliminary step to our infrared analysis is the integration of the ultraviolet degrees of freedom corresponding to the large values of k_0. We proceed in the following way. We decompose the free propagator $\hat{g}_{\mathbf{k}}$ into a sum of two propagators supported in the regions of k_0 "large" and "small", respectively. The regions of k_0 large and small are defined in terms of the smooth support function $\chi_0(t)$ introduced at the beginning of Sec. 3.1; the constant a_0 entering its definition is chosen so that the support of $\chi_0\left(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,+}|^2}\right)$ and $\chi_0\left(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,-}|^2}\right)$ are disjoint (here $|\cdot|$ is the euclidean norm over $\mathbb{R}^2/\Lambda^*$). In order for this condition to be satisfied, it is enough that $2a_0\gamma < 4\pi/(3\sqrt{3})$; in the following, for reasons that will become clearer later, we shall assume the slightly more restrictive condition $2a_0\gamma < 4\pi/3 - 4\pi/(3\sqrt{3})$. We define

$$f_{u.v.}(\mathbf{k}) = 1 - \chi_0\left(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,+}|^2}\right) - \chi_0\left(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,-}|^2}\right) \tag{3.29}$$

and $f_{i.r.}(\mathbf{k}) = 1 - f_{u.v.}(\mathbf{k})$, so that we can rewrite $\hat{g}_{\mathbf{k}}$ as:

$$\hat{g}_{\mathbf{k}} = f_{u.v.}(\mathbf{k})\hat{g}_{\mathbf{k}} + f_{i.r.}(\mathbf{k})\hat{g}_{\mathbf{k}} \stackrel{def}{=} \hat{g}^{(u.v.)}(\mathbf{k}) + \hat{g}^{(i.r.)}(\mathbf{k}). \tag{3.30}$$

We now introduce two independent sets of Grassmann fields $\{\Psi^{(u.v.)\pm}_{\mathbf{k},\sigma,\rho}\}$ and $\{\Psi^{(i.r.)\pm}_{\mathbf{k},\sigma,\rho}\}$, with $\mathbf{k} \in \mathcal{D}^*_{\beta,L}$, $\sigma =\uparrow\downarrow$, $\rho = 1, 2$, and the Gaussian integrations $P(d\Psi^{(u.v.)})$ and $P(d\Psi^{(i.r.)})$ defined by

$$\int P(d\Psi^{(u.v.)})\hat{\Psi}^{(u.v.)-}_{\mathbf{k}_1,\sigma_1,\rho_1}\hat{\Psi}^{(u.v.)+}_{\mathbf{k}_2,\sigma_2,\rho_2} = \beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat{g}^{(u.v.)}(\mathbf{k}_1)_{\rho_1,\rho_2},$$
$$\int P(d\Psi^{(i.r.)})\hat{\Psi}^{(u.v.)-}_{\mathbf{k}_1,\sigma_1,\rho_1}\hat{\Psi}^{(i.r.)+}_{\mathbf{k}_2,\sigma_2,\rho_2} = \beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat{g}^{(i.r.)}(\mathbf{k}_1)_{\rho_1,\rho_2}. \tag{3.31}$$

Similarly to $P(d\Psi)$, the Gaussian integrations $P(d\Psi^{(u.v.)})$, $P(d\Psi^{(i.r.)})$ also admit an explicit representation analogous to (3.14), with $\hat{g}_{\mathbf{k}}$ replaced by $\hat{g}^{(u.v.)}(\mathbf{k})$ or $\hat{g}^{(i.r.)}(\mathbf{k})$ and the sum over $\mathbf{k}$ restricted to the values in the support of $f_{u.v.}(\mathbf{k})$ or $f_{i.r.}(\mathbf{k})$, respectively. It easy to verify that the ultraviolet propagator $g^{(u.v.)}(\mathbf{x}-\mathbf{y}) = (\beta|\Lambda|)^{-1}\sum_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\hat{g}^{(u.v.)}(\mathbf{k})$ satisfies

$$|g^{(u.v.)}(\mathbf{x}-\mathbf{y})| \le \frac{C_N}{1+||\mathbf{x}-\mathbf{y}||^N}, \tag{3.32}$$

uniformly in M; here $||\mathbf{x}|| = \sqrt{|x_0|^2_\beta + |\vec{x}|^2_\Lambda}$, with $|\cdot|_\beta$ the distance over the one-dimensional torus of length β and $|\cdot|_\Lambda$ the distance over the periodic lattice Λ. The definition of Grassmann integration implies the following identity ("addition principle"):

$$\int P(d\Psi)e^{-V(\Psi)} = \int P(d\Psi^{(i.r.)})\int P(d\Psi^{(u.v.)})e^{-V(\Psi^{(i.r.)}+\Psi^{(u.v.)})}, \tag{3.33}$$

so that we can rewrite the partition function as

$$\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F_{M,L,\beta}} = \int P(d\Psi^{(i.r.)})\exp\left\{\sum_{n\ge1}\frac{1}{n!}\mathcal{E}^T_{u.v.}(-V(\Psi^{(i.r.)}+\cdot);n)\right\}$$
$$\equiv e^{-\beta|\Lambda|F_{0,M}}\int P(d\Psi^{(i.r.)})e^{-\mathcal{V}_M(\Psi^{(i.r.)})}, \tag{3.34}$$

where the *truncated expectation* $\mathcal{E}^T_{u.v.}$ is defined, given any polynomial $V_1(\Psi^{(u.v.)})$ with coefficients depending on $\Psi^{(i.r.)}$, as

$$\mathcal{E}^T_{u.v.}(V_1(\cdot);n) = \frac{\partial^n}{\partial\lambda^n}\log\int P(d\Psi^{(u.v.)})e^{\lambda V_1(\Psi^{(u.v.)})}\Big|_{\lambda=0}, \tag{3.35}$$

and $\mathcal{V}_M$ is fixed by the condition $\mathcal{V}_M(0) = 0$. It can be shown (see Appendix C) that $\mathcal{V}_M$ can be written as

$$\mathcal{V}_M(\Psi) = \sum_{n=1}^{\infty}(\beta|\Lambda|)^{-2n}\sum_{\sigma_1,\dots,\sigma_n=\uparrow\downarrow}\sum_{\rho_1,\dots,\rho_{2n}=1,2}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_{2n}}\left[\prod_{j=1}^{n}\hat{\Psi}^{(i.r.)+}_{\mathbf{k}_{2j-1},\sigma_j,\rho_{2j-1}}\hat{\Psi}^{(i.r.)-}_{\mathbf{k}_{2j},\sigma_j,\rho_{2j}}\right]$$
$$\cdot\hat{W}_{M,2n,\underline{\rho}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n-1})\,\delta\Big(\sum_{j=1}^{n}(\mathbf{k}_{2j-1}-\mathbf{k}_{2j})\Big), \tag{3.36}$$

where $\underline{\rho} = (\rho_1, \ldots, \rho_{2n})$ and we used the notation

$$\delta(\mathbf{k}) = \delta(\vec{k})\delta(k_0), \qquad \delta(\vec{k}) = |\Lambda| \sum_{n_1,n_2\in\mathbb{Z}} \delta_{\vec{k},n_1\vec{b}_1+n_2\vec{b}_2}, \qquad \delta(k_0) = \beta\delta_{k_0,0}, \quad (3.37)$$

with $\vec{b}_1, \vec{b}_2$ a basis of Λ^*. The possibility of representing $\mathcal{V}_M$ in the form (3.36), with the *kernels* $\hat{W}_{M,2n,\underline{\rho}}$ independent of the spin indices σ_i, follows from the symmetries listed in Lemma 1 and from the remark that $P(d\Psi^{(u.v.)})$ and $P(d\Psi^{(i.r.)})$ are separately invariant under the same symmetries. The regularity properties of the kernels are summarized in the following lemma, see Appendix C for a proof.

Lemma 2. *The constant $F_{0,M}$ in (3.34) and the kernels $\hat{W}_{M,2n,\underline{\rho}}$ in (3.36) are given by power series in U, convergent in the complex disc $|U| \le U_0$, for U_0 small enough and independent of β, L, M; after Fourier transform, the $\mathbf{x}$-space counterparts of the kernels $\hat{W}_{M,2n,\underline{\rho}}$ satisfy the following bounds:*

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} \left[\prod_{1\le i<j\le 2n} ||\mathbf{x}_i - \mathbf{x}_j||^{m_{i,j}} \right] \left| W_{M,2n,\underline{\rho}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}) \right|$$
$$\le \beta|\Lambda| C_m^n |U|^{\max\{1,n-1\}}, \qquad (3.38)$$

for some constant $C_m > 0$, where $m = \sum_{1\le i<j\le 2n} m_{i,j}$. Moreover, the limits $F_0 = \lim_{M\to\infty} F_{0,M}$ and $W_{2n,\underline{\rho}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}) = \lim_{M\to\infty} W_{M,2n,\underline{\rho}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n})$ exist and are reached uniformly in M, so that, in particular, the limiting functions are analytic in the same domain $|U| \le U_0$.

Remarks. 1) It is well known that the ultraviolet problem for lattice fermions can be solved in any dimension by a multiscale expansion, see [3,5,4]; for completeness, it will be presented in a self-contained form in Appendix C. Recently, a different proof based on a single scale integration step and using improved bounds on determinants associated to "chronological products" was proposed [20].

2) Once the ultraviolet degrees of freedom have been integrated out, the remaining infrared problem (i.e., the computation of the Grassmann integral in the second line of (3.34)) is essentially independent of M, given the fact that the limit $W_{2n,\underline{\rho}}$ of the kernels $W_{M,2n,\underline{\rho}}$ is reached uniformly and that the limiting kernels are analytic and satisfy the same bounds as (3.38). For this reason, in the infrared integration described in the next two sections, M will not play any essential role and, for this reason, from now on we shall not stress anymore the dependence on M, for notational simplicity.

It is important for the incoming discussion to note that the symmetries listed in Lemma 1 also imply some non-trivial invariance properties of the kernels. We will be particularly interested in the invariance properties of the quadratic part $\hat{W}_{M,2,(\rho_1,\rho_2)}(\mathbf{k})$, which will be used below to show that the structure of the quadratic part of the new effective interaction has the same symmetries as the free integration. The crucial properties that we will need are the following:

Lemma 3. *Let $\hat{W}_{aa}(\mathbf{k}) \equiv \hat{W}_{M,2,(1,1)}(\mathbf{k})$, $\hat{W}_{bb}(\mathbf{k}) = \hat{W}_{M,2,(2,2)}(\mathbf{k})$, $\hat{W}_{ab}(\mathbf{k}) = \hat{W}_{M,2,(1,2)}(\mathbf{k})$ and $\hat{W}_{ba}(\mathbf{k}) = \hat{W}_{M,2,(2,1)}(\mathbf{k})$. Then the following properties are valid:*

(i) $W_{aa}(\mathbf{k}) = W_{bb}(\mathbf{k})$ *and* $W_{ab}(\mathbf{k}) = W^*_{ba}(\mathbf{k})$*;*
(ii) *as* $\beta \to \infty$*, for* $\omega = \pm$*,* $W_{aa}(0, \vec{p}_F^{\,\omega}) = W_{ab}(0, \vec{p}_F^{\,\omega}) = 0$*;*
(iii) *as* $\beta, |\Lambda| \to \infty$*, for* $\omega = \pm$*,*

$$\partial_{\vec{k}} \hat{W}_{aa}(0, \vec{p}_F^{\,\omega}) = \vec{0}, \qquad \mathrm{Re}\left\{\partial_{k_0} \hat{W}_{aa}(0, \vec{p}_F^{\,\omega})\right\} = 0, \qquad \partial_{k_0} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega}) = 0, \quad (3.39)$$
$$\mathrm{Re}\left\{\partial_{k_1} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega})\right\} = \mathrm{Im}\left\{\partial_{k_2} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega})\right\} = 0,$$
$$i\partial_{k_1} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega}) = \omega \partial_{k_2} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega}).$$

Remarks. 1) For simplicity, the properties (ii) and (iii) are spelled out only in the zero temperature limit and in the thermodynamic limit; however, as it will be clear from the proof, those properties all have a finite temperature/volume counterpart.

2) Lemma 3 implies that in the vicinity of the Fermi points the kernel $W_{M,2,(\rho,\rho')}(\mathbf{k})$ can be rewritten in the form

$$W_{M,2,(\rho,\rho')}(k_0, \vec{p}_F^{\,\omega} + \vec{k}') \simeq \begin{pmatrix} -iz_0k_0 & \delta_0(ik_1' - \omega k_2') \\ \delta_0(-ik_1' - \omega k_2') & -iz_0k_0 \end{pmatrix}_{\rho,\rho'}, \quad (3.40)$$

for some real constants z_0, δ_0, modulo higher order terms in $(k_0, \vec{k}')$. Therefore, it is apparent that its structure is the same as the one of $\hat{S}_0(\mathbf{k})$, modulo higher order terms in $(k_0, \vec{k}')$.

Proof. As remarked after (3.37), $P(d\Psi^{(u.v.)})$ and $P(d\Psi^{(i.r.)})$ are separately invariant under the symmetry properties listed in Lemma 1. Therefore $\mathcal{V}(\Psi)$ is also invariant under the same symmetries, and so is the quadratic part of $\mathcal{V}(\Psi)$, that is

$$(\beta|\Lambda|)^{-2} \sum_{\sigma} \sum_{\mathbf{k},\mathbf{p}} \delta(\mathbf{p}) \Big[\hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,1} \hat{\Psi}^{(i.r.)-}_{\mathbf{k+p},\sigma,1} W_{aa}(\mathbf{k}) + \hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,1} \hat{\Psi}^{(i.r.)-}_{\mathbf{k+p},\sigma,2} W_{ab}(\mathbf{k})$$
$$+\hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,2} \hat{\Psi}^{(i.r.)-}_{\mathbf{k+p},\sigma,1} W_{ba}(\mathbf{k}) + \hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,2} \hat{\Psi}^{(i.r.)-}_{\mathbf{k+p},\sigma,2} W_{bb}(\mathbf{k})\Big]. \quad (3.41)$$

Recall that, as assumed in the lines preceding (3.29), the support of $\hat{\Psi}^{(i.r.)}$ consists of two disjoint regions around $\vec{p}_F^{\,+}$ and $\vec{p}_F^{\,-}$, respectively; in particular, we assumed that $2a_0\gamma < 4\pi/3 - 4\pi/(3\sqrt{3})$. Under this condition, it is easy to realize that if both $\mathbf{k}$ and $\mathbf{p}+\mathbf{k}$ belong to the support of $\hat{\Psi}^{(i.r.)}$, then $|\mathbf{p}| < 4\pi/3$. As a consequence, in (3.38), the only non-zero contributions correspond to the terms with $\mathbf{p} = \mathbf{0}$ (in fact, if $\mathbf{p}$ is $\neq \mathbf{0}$ and belongs to the support of $\delta(\mathbf{p})$, then $|\mathbf{p}| \geq 4\pi/3$, which means that either $\mathbf{k}$ or $\mathbf{k}+\mathbf{p}$ is outside the support of $\hat{\Psi}^{(i.r.)}$, and the corresponding term in the sum is identically zero). This means that the sum

$$\sum_{\sigma,\mathbf{k}} \Big[\hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,1} \hat{\Psi}^{(i.r.)-}_{\mathbf{k},\sigma,1} W_{aa}(\mathbf{k}) + \hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,1} \hat{\Psi}^{(i.r.)-}_{\mathbf{k},\sigma,2} W_{ab}(\mathbf{k})$$
$$+\hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,2} \hat{\Psi}^{(i.r.)-}_{\mathbf{k},\sigma,1} W_{ba}(\mathbf{k}) + \hat{\Psi}^{(i.r.)+}_{\mathbf{k},\sigma,2} \hat{\Psi}^{(i.r.)-}_{\mathbf{k},\sigma,2} W_{bb}(\mathbf{k})\Big] \quad (3.42)$$

is invariant under the symmetries (1)–(7) listed in Lemma 1.

Invariance under symmetry (4) implies that:

$$W_{aa}(k_0, \vec{k}) = W_{aa}(k_0, T_1^{-1}\vec{k}), \qquad W_{bb}(k_0, \vec{k}) = W_{bb}(k_0, T_1^{-1}\vec{k}), \quad (3.43)$$
$$W_{ab}(k_0, \vec{k}) = e^{i\vec{k}(\vec{\delta}_1 - \vec{\delta}_2)} W_{ab}(k_0, T_1^{-1}\vec{k}), \qquad W_{ba}(k_0, \vec{k}) = e^{-i\vec{k}(\vec{\delta}_1 - \vec{\delta}_2)} W_{ab}(k_0, T_1^{-1}\vec{k});$$

invariance under (5) implies that:

$$W_{aa}(\mathbf{k}) = W_{aa}(-\mathbf{k})^*, \qquad W_{bb}(\mathbf{k}) = W_{bb}(-\mathbf{k})^*, \tag{3.44}$$
$$W_{ab}(\mathbf{k}) = W_{ab}(-\mathbf{k})^*, \qquad W_{ba}(\mathbf{k}) = W_{ba}(-\mathbf{k})^*;$$

invariance under (6.a) implies that:

$$W_{aa}(k_0, k_1, k_2) = W_{bb}(k_0, -k_1, k_2), \qquad W_{ab}(k_0, k_1, k_2) = W_{ba}(k_0, -k_1, k_2)\ ; \tag{3.45}$$

invariance under (6.b) implies that:

$$W_{aa}(k_0, k_1, k_2) = W_{aa}(k_0, k_1, -k_2), \qquad W_{bb}(k_0, k_1, k_2) = W_{bb}(k_0, k_1, -k_2), \tag{3.46}$$
$$W_{ab}(k_0, k_1, k_2) = W_{ab}(k_0, k_1, -k_2), \qquad W_{ba}(k_0, k_1, k_2) = W_{ba}(k_0, k_1, -k_2)\ ;$$

invariance under (7) implies that:

$$W_{aa}(k_0, \vec{k}) = W_{aa}(k_0, -\vec{k}), \qquad W_{bb}(k_0, \vec{k}) = W_{bb}(k_0, -\vec{k}), \tag{3.47}$$
$$W_{ab}(k_0, \vec{k}) = W_{ba}(k_0, -\vec{k}).$$

Finally, invariance under (8) implies that:

$$W_{aa}(k_0, \vec{k}) = -W_{aa}(-k_0, \vec{k}), \qquad W_{bb}(k_0, \vec{k}) = -W_{bb}(-k_0, \vec{k}), \tag{3.48}$$
$$W_{ab}(k_0, \vec{k}) = W_{ab}(-k_0, \vec{k}), \qquad W_{ba}(k_0, \vec{k}) = W_{ba}(-k_0, \vec{k}).$$

Now, combining the first of (3.45), the second of (3.46) and the second of (3.47), we find that $W_{aa}(\mathbf{k}) = W_{bb}(\mathbf{k})$. Combining the third of (3.44), the third of (3.47) and the last of (3.48), we find that $W_{ab}(\mathbf{k}) = W_{ba}(\mathbf{k})^*$. This concludes the proof of item (i).

The first of (3.48) implies that, as $\beta \to \infty$, $W_{aa}(0, \vec{k}) = 0$, and this proves, in particular, that $W_{aa}(0, \vec{p}_F^{\,\omega}) = 0$ and that, in the limit $|\Lambda| \to \infty$, $\partial_{\vec{k}} W_{aa}(0, \vec{p}_F^{\,\omega}) = \vec{0}$.

Using that $\vec{p}_F^{\,\omega}$ is invariant under the action of T_1, we see that the third of (3.43) implies that $(1 - e^{i\vec{p}_F^{\,\omega}(\vec{\delta}_1 - \vec{\delta}_2)}) W_{ab}(k_0, \vec{p}_F^{\,\omega}) = 0$. Since $e^{i\vec{p}_F^{\,\omega}(\vec{\delta}_1 - \vec{\delta}_2)} = -e^{i\omega\pi/3} \neq 1$, this identity proves, in particular, that $W_{ab}(0, \vec{p}_F^{\,\omega}) = 0$, and $\partial_{k_0} W_{ab}(0, \vec{p}_F^{\,\omega}) = 0$. This concludes the proof of item (ii).

Now, combining the first of (3.44) with the first of (3.47), we find that $W_{aa}(k_0, \vec{k}) = W_{aa}(-k_0, \vec{k})^*$, which implies, in particular, that $\mathrm{Re}\left\{\partial_{k_0} \hat{W}_{aa}(0, \vec{p}_F^{\,\omega})\right\} = 0$.

Finally, let $W_{ab}(0, \vec{p}_F^{\,\omega} + \vec{k}') \simeq \alpha_1^\omega k_1' + \alpha_2^\omega k_2'$, modulo higher order terms in $\vec{k}'$. Using that $T_1^{-1} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$ in the third of (3.43), we find that

$$\alpha_1^\omega k_1' + \alpha_2^\omega k_2' = e^{-i\omega\pi/3}\left[\alpha_1^\omega (k_1'/2 - \sqrt{3}k_2'/2) + \alpha_2^\omega(\sqrt{3}k_1'/2 + k_2'/2)\right], \tag{3.49}$$

which implies $\alpha_1^\omega = -i\omega\alpha_2^\omega$. Moreover, using the third of (3.44) we find that $\alpha_i^\omega = -(\alpha_i^{-\omega})^*$, and using the third of (3.46) we find that $\alpha_2^\omega = -\alpha_2^{-\omega}$. Therefore, $\alpha_2^\omega = -\alpha_2^{-\omega} = -(\alpha_2^{-\omega})^*$, and we see that α_2^ω is real and odd in ω, that is $\alpha_2^\omega = \omega a$, for some real constant a. Therefore, $\alpha_1^\omega = -i\omega\alpha_2^\omega = -ia$, and this concludes the proof of item (iii). $\square$

3.3. Free energy: The infrared integration. Multiscale analysis. In order to compute (3.34) we proceed in an iterative fashion, using standard functional Renormalization Group methods [2,10,17]. As a starting point, it is convenient to decompose the infrared propagator as:

$$g^{(i.r.)}(\mathbf{x},\mathbf{y}) = \sum_{\omega=\pm} e^{-i\vec{p}_F^{\,\omega}(\vec{x}-\vec{y})} g_\omega^{(\le 0)}(\mathbf{x},\mathbf{y}), \tag{3.50}$$

where, if $\mathbf{k}' = (k_0, \vec{k}')$,

$$g_\omega^{(\le 0)}(\mathbf{x},\mathbf{y}) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}'\in\mathcal{D}^\omega_{\beta,L}} \chi_0(|\mathbf{k}'|) e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})} \begin{pmatrix} -ik_0 & -v^*(\vec{k}'+\vec{p}_F^{\,\omega}) \\ -v(\vec{k}'+\vec{p}_F^{\,\omega}) & -ik_0 \end{pmatrix}^{-1}, \tag{3.51}$$

and $\mathcal{D}^\omega_{\beta,L} = \mathcal{D}^*_\beta \times \mathcal{D}^\omega_L$, with $\mathcal{D}^*_\beta = \mathcal{D}_\beta \cap \{k_0 \,:\, \chi_0(\gamma^{-M}|k_0|) > 0\}$ and $\mathcal{D}^\omega_L = \{\frac{n_1}{L}\vec{b}_1 + \frac{n_2}{L}\vec{b}_2 - \vec{p}_F^{\,\omega},\ 0 \le n_1, n_2 \le L-1\}$.

Correspondingly, we rewrite $\Psi^{(i.r.)}$ as a sum of two independent Grassmann fields:

$$\Psi^{(i.r.)\pm}_{\mathbf{x},\sigma,\rho} = \sum_{\omega=\pm} e^{i\vec{p}_F^{\,\omega}\vec{x}} \Psi^{(\le 0)\pm}_{\mathbf{x},\sigma,\rho,\omega}, \tag{3.52}$$

and we rewrite (3.34) in the form:

$$\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F_{0,M}} \int P_{\chi_0,A_0}(d\Psi^{(\le 0)}) e^{-\mathcal{V}^{(0)}(\Psi^{(\le 0)})}, \tag{3.53}$$

where $\mathcal{V}^{(0)}(\Psi^{(\le 0)})$ is equal to $\mathcal{V}_M(\Psi^{(i.r.)})$, once $\Psi^{(i.r.)}$ is rewritten as in (3.52), i.e.,

$$\begin{aligned} &\mathcal{V}^{(0)}(\Psi^{(\le 0)}) \qquad\qquad (3.54)\\ &= \sum_{n=1}^\infty (\beta|\Lambda|)^{-2n} \sum_{\sigma_1,\ldots,\sigma_n=\uparrow\downarrow} \sum_{\rho_1,\ldots,\rho_{2n}=1,2}^{\omega_1,\ldots,\omega_{2n}=\pm} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} \left[\prod_{j=1}^n \hat{\Psi}^{(\le 0)+}_{\mathbf{k}'_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \hat{\Psi}^{(\le 0)-}_{\mathbf{k}'_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\right] \\ &\quad \cdot \hat{W}^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\, \delta\left(\sum_{j=1}^{2n} (-1)^j (\mathbf{p}_F^{\omega_j} + \mathbf{k}'_j)\right) \\ &= \sum_{n=1}^\infty \sum_{\underline{\sigma},\underline{\rho},\underline{\omega}} \int d\mathbf{x}_1\cdots d\mathbf{x}_{2n} \left[\prod_{j=1}^n \Psi^{(\le 0)+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \Psi^{(\le 0)-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\right] W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n}), \end{aligned}$$

with:

1) $\underline{\omega} = (\omega_1,\ldots,\omega_{2n})$, $\underline{\sigma} = (\sigma_1,\ldots,\sigma_n)$ and $\mathbf{p}_F^\omega = (0, \vec{p}_F^{\,\omega})$;
2) $\hat{W}^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1}) = \hat{W}_{M,2n,\underline{\rho}}(\mathbf{k}'_1+\mathbf{p}_F^{\omega_j},\ldots,\mathbf{k}'_{2n-1}+\mathbf{p}_F^{\omega_{2n-1}})$, see (3.36);
3) the kernels $W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})$ are defined as:

$$\begin{aligned} &W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n}) \\ &= (\beta|\Lambda|)^{-2n} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} e^{i\sum_{j=1}^{2n}(-1)^j \mathbf{k}_j\mathbf{x}_j} \hat{W}^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\, \delta\left(\sum_{j=1}^{2n}(-1)^j(\mathbf{p}_F^{\omega_j} + \mathbf{k}'_j)\right). \end{aligned} \tag{3.55}$$

Moreover, $P_{\chi_0,A_0}(d\Psi^{(\le 0)})$ is defined as

$$P_{\chi_0,A_0}(d\Psi^{(\le 0)})$$
$$= \mathcal{N}_0^{-1}\left[\prod_{\mathbf{k}'\in\mathcal{D}^{\omega}_{\beta,L}}^{\chi_0(|\mathbf{k}'|)>0}\prod_{\sigma,\omega,\rho} d\hat\Psi^{(\le 0)+}_{\mathbf{k}',\sigma,\rho,\omega}d\hat\Psi^{(\le 0)-}_{\mathbf{k}',\sigma,\rho,\omega}\right]$$
$$\cdot\exp\left\{-(\beta|\Lambda|)^{-1}\sum_{\omega=\pm,\sigma=\uparrow\downarrow}\sum_{\mathbf{k}'\in\mathcal{D}^{\omega}_{\beta,L}}^{\chi_0(|\mathbf{k}'|)>0}\chi_0^{-1}(|\mathbf{k}'|)\hat\Psi^{(\le 0)+}_{\mathbf{k}',\sigma,\cdot,\omega}A_{0,\omega}(\mathbf{k}')\hat\Psi^{(\le 0)-}_{\mathbf{k}',\sigma,\cdot,\omega}\right\}. \tag{3.56}$$

where:

$$A_{0,\omega}(\mathbf{k}') = \begin{pmatrix} -ik_0 & -v^*(\vec{k}'+\vec{p}_F^{\,\omega}) \\ -v(\vec{k}'+\vec{p}_F^{\,\omega}) & -ik_0\end{pmatrix}$$
$$= \begin{pmatrix} -i\zeta_0 k_0 + s_0(\mathbf{k}') & c_0(ik_1'-\omega k_2') + t_{0,\omega}(\mathbf{k}') \\ c_0(-ik_1'-\omega k_2') + t^*_{0,\omega}(\mathbf{k}') & -i\zeta_0 k_0 + s_0(\mathbf{k}')\end{pmatrix},$$

$\mathcal{N}_0$ is chosen in such a way that $\int P_{\chi_0,A_0}(d\Psi^{(\le 0)}) = 1$, $\zeta_0 = 1$, $c_0 = 3/2$, $s_0 \equiv 0$ and $|t_{0,\omega}(\mathbf{k}')| \le C|\mathbf{k}'|^2$.

It is apparent that the $\Psi^{(\le 0)}$ field has zero mass (i.e., its propagator decays polynomially at large distances in $\mathbf{x}$-space). Therefore, its integration requires an infrared multiscale analysis. We consider the scaling parameter $\gamma > 1$ introduced above, see the lines preceding (3.29), and we define a sequence of geometrically decreasing momentum scales $\gamma^h, h = 0, -1, -2, \ldots$. Correspondingly we introduce compact support functions $f_h(\mathbf{k}') = \chi_0(\gamma^{-h}|\mathbf{k}'|) - \chi_0(\gamma^{-h+1}|\mathbf{k}'|)$ and we rewrite

$$\chi_0(|\mathbf{k}'|) = \sum_{h=-\infty}^{0} f_h(\mathbf{k}'). \tag{3.57}$$

The purpose is to perform the integration of (3.53) in an iterative way. We step by step decompose the propagator into a sum of two propagators, the first supported on momenta $\sim \gamma^h$, $h \le 0$, the second supported on momenta smaller than γ^h. Correspondingly we rewrite the Grassmann field as a sum of two independent fields: $\Psi^{(\le h)} = \Psi^{(h)} + \Psi^{(\le h-1)}$ and we integrate the field $\Psi^{(h)}$. In this way we inductively prove that, for any $h \le 0$, (3.53) can be rewritten as

$$\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F_h}\int P_{\chi_h,A_h}(d\Psi^{(\le h)})e^{-\mathcal{V}^{(h)}(\Psi^{(\le h)})}, \tag{3.58}$$

where $F_h, A_h, \mathcal{V}^{(h)}$ will be defined recursively, $\chi_h(|\mathbf{k}'|) = \sum_{k=-\infty}^{h} f_k(\mathbf{k}')$ and $P_{\chi_h,A_h}(d\Psi^{(\le h)})$ is defined in the same way as $P_{\chi_0,A_0}(d\Psi^{(\le 0)})$ with $\Psi^{(\le 0)}$, χ_0, $A_{0,\omega}$, ζ_0, $c_0, s_0, t_{0,\omega}$ replaced by $\Psi^{(\le h)}$, χ_h, $A_{h,\omega}$, ζ_h, c_h, s_h, $t_{h,\omega}$, respectively. Moreover

$\mathcal{V}^{(h)}(0) = 0$ and

$$\mathcal{V}^{(h)}(\Psi) = \sum_{n=1}^{\infty}(\beta|\Lambda|)^{-2n} \sum_{\underline{\sigma},\underline{\rho},\underline{\omega}} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} \left[\prod_{j=1}^{n} \hat{\Psi}^{(\leq h)+}_{\mathbf{k}'_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \hat{\Psi}^{(\leq h)-}_{\mathbf{k}'_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\right]$$
$$\cdot \hat{W}^{(h)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\ \delta(\sum_{j=1}^{2n}(-1)^j(\mathbf{p}_F^{\omega_j}+\mathbf{k}'_j))$$
$$= \sum_{n=1}^{\infty}\sum_{\underline{\sigma},\underline{\rho},\underline{\omega}} \int d\mathbf{x}_1\cdots d\mathbf{x}_{2n}\left[\prod_{j=1}^{n}\Psi^{(\leq h)+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}}\Psi^{(\leq h)-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\right]$$
$$\times W^{(h)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n}). \tag{3.59}$$

Note that the field $\Psi^{(\leq h)}_{\mathbf{k}',\sigma,\cdot,\omega}$, whose propagator is given by $\chi_h(|\mathbf{k}'|)[A^{(h)}_\omega(\mathbf{k}')]^{-1}$, has the same support as χ_h, that is on a neighborood of size γ^h around the singularity $\mathbf{k}'=\mathbf{0}$ (that, in the original variables, corresponds to the Dirac point $\mathbf{k}=\mathbf{p}_F^\omega$). It is important for the following to think $\hat{W}^{(h)}_{2n,\underline{\rho},\underline{\omega}}$, $h\leq 0$, as functions of the variables $\{\zeta_k, c_k\}_{h<k\leq 0}$. The iterative construction below will inductively imply that the dependence on these variables is well defined.

The iteration will continue up to the scale h_β, where h_β is the largest scale such that

$$a_0\gamma^{h_\beta-1} < \frac{\pi}{\beta}\zeta_{h_\beta}, \tag{3.60}$$

where a_0 is the constant appearing in the definition of $\chi_0(|\mathbf{k}'|)$. By the properties of ζ_h that will be described and proved below, it will turn out that h_β is finite and larger than $\log_\gamma \frac{\pi}{2a_0\beta}$. The result of the last iteration will be $\Xi_{M,\beta,L}=e^{-\beta|\Lambda|F_{M,\beta,L}}$.

Localization and renormalization. In order to inductively prove (3.58) we write

$$\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}, \tag{3.61}$$

where

$$\mathcal{L}\mathcal{V}^{(h)} = \frac{1}{\beta|\Lambda|}\sum_{\sigma=\uparrow\downarrow}\sum_{\substack{\rho_1,\rho_2=1,2\\ \omega=\pm}}\sum_{\mathbf{k}'}^{\chi_h(|\mathbf{k}'|)>0} \hat{\Psi}^{(\leq h)+}_{\mathbf{k}',\sigma,\rho_1,\omega}\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\sigma,\rho_2,\omega}\hat{W}^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}'), \tag{3.62}$$

and $\mathcal{R}\mathcal{V}^{(h)}$ is given by (3.59) with $\sum_{n=1}^\infty$ replaced by $\sum_{n=2}^\infty$, that is it contains only the monomials with more than four fields.

Note that in (3.62) the ω-index of the Ψ fields is the same; this follows from the fact that in terms with different ω's the momenta verify $\mathbf{k}'_1-\mathbf{k}'_2+\mathbf{p}_F^\omega-\mathbf{p}_F^{-\omega}=n_1\vec{b}_1+n_2\vec{b}_2$, for some choice of n_1,n_2, and such a condition cannot be verified if $\mathbf{k}'_1$, $\mathbf{k}'_2$ are in the support of the $\Psi^{(\leq h)}$ fields, because $\mathbf{p}_F^\omega-\mathbf{p}_F^{-\omega}\notin\Lambda^*$ and $2a_0\gamma$ is smaller than $4\pi/3-4\pi/(3\sqrt{3})$, see the lines preceding (3.29) and the discussion after (3.41).

Remark. The fact that the quadratic terms with different ω's, i.e., the one particle *umklapp processes*, does not contribute to the infrared effective potential is a crucial fact, which reduces the number of *relevant running coupling constants* and, in particular, tells us that the interaction does not generate *mass terms*. Note, in fact, that the presence of one particle umklapp terms with a non zero contribution at the Fermi points could produce an exponential decay of the interacting correlations.

The symmetries of the action, listed in Lemma 1, which are preserved by the iterative integration procedure, imply that, in the zero temperature and thermodynamic limit, $\hat{W}^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{0}) = 0$ and

$$\mathbf{k}'\partial_{\mathbf{k}'}\hat{W}^{(h)}_{2,(\rho_1,\rho_2),(\omega,\omega)}(\mathbf{0}) = \begin{pmatrix} -iz_h k_0 & \delta_h(ik'_1 - \omega k'_2) \\ \delta_h(-ik'_1 - \omega k'_2) & -iz_h k_0 \end{pmatrix}_{\rho_1,\rho_2}, \tag{3.63}$$

for suitable real constants z_h, δ_h. The proof of (3.63) is completely analogous to the proof of Lemma 2 and will not be repeated here.

Once the above definitions are given, we can describe our iterative integration procedure for $h \leq 0$. We start from (3.58) and we rewrite it as

$$\int P_{\chi_h, A_h}(d\Psi^{(\leq h)})\, e^{-\mathcal{L}\mathcal{V}^{(h)}(\Psi^{(\leq h)}) - \mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\leq h)}) - \beta|\Lambda|F_h}, \tag{3.64}$$

with

$$\begin{aligned}\mathcal{L}\mathcal{V}^{(h)}(\Psi^{(\leq h)}) = (\beta|\Lambda|)^{-1}\sum_{\omega,\sigma}\ \sum_{\mathbf{k}'}^{\chi_h(|\mathbf{k}'|)>0} \cdot\hat{\Psi}^{(\leq h)+}_{\mathbf{k}',\sigma,\cdot,\omega} \\ \times \begin{pmatrix} -iz_h k_0 + \sigma_h(\mathbf{k}') & \delta_h(ik'_1 - \omega k'_2) + \tau_{h,\omega}(\mathbf{k}') \\ \delta_h(-ik'_1 - \omega k_2) + \tau^*_{h,\omega}(\mathbf{k}') & -iz_h k_0 + \sigma_h(\mathbf{k}') \end{pmatrix} \hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\sigma,\cdot,\omega}.\end{aligned} \tag{3.65}$$

Then we include $\mathcal{L}\mathcal{V}^{(h)}$ in the fermionic integration, so obtaining

$$\int P_{\chi_h, \overline{A}_{h-1}}(d\Psi^{(\leq h)})\, e^{-\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\leq h)}) - \beta|\Lambda|(F_h + e_h)}, \tag{3.66}$$

where

$$e_h = \frac{1}{\beta|\Lambda|}\sum_{\omega,\sigma}\sum_{\mathbf{k}'}\sum_{n\geq 1}\frac{(-1)^n}{n}\mathrm{Tr}\left\{\left[\chi_h(\mathbf{k}')A^{-1}_{h,\omega}(\mathbf{k}')W^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}')\right]^n\right\} \tag{3.67}$$

is a constant taking into account the change in the normalization factor of the measure and

$$\overline{A}_{h-1,\omega}(\mathbf{k}') = \begin{pmatrix} -i\overline{\zeta}_{h-1}k_0 + \overline{s}_{h-1}(\mathbf{k}') & \overline{c}_{h-1}(ik'_1 - \omega k'_2) + \overline{t}_{h-1,\omega}(\mathbf{k}') \\ \overline{c}_{h-1}(-ik'_1 - \omega k'_2) + \overline{t}^*_{h-1,\omega}(\mathbf{k}') & -i\overline{\zeta}_{h-1}k_0 + \overline{s}_{h-1}(\mathbf{k}') \end{pmatrix} \tag{3.68}$$

with:

$$\begin{aligned} \overline{\zeta}_{h-1}(\mathbf{k}') &= \zeta_h + z_h\chi_h(\mathbf{k}'), & \overline{c}_{h-1}(\mathbf{k}') &= c_h + \delta_h\chi_h(\mathbf{k}'), \\ \overline{s}_{h-1}(\mathbf{k}') &= s_h(\mathbf{k}') + \sigma_h(\mathbf{k}')\chi_h(\mathbf{k}'), & \overline{t}_{h-1,\omega}(\mathbf{k}') &= t_{h,\omega}(\mathbf{k}') + \tau_{h,\omega}(\mathbf{k}')\chi_h(\mathbf{k}'). \end{aligned} \tag{3.69}$$

Now we can perform the integration of the $\Psi^{(h)}$ field. We rewrite the Grassmann field $\Psi^{(\le h)}$ as a sum of two independent Grassmann fields $\Psi^{(\le h-1)} + \Psi^{(h)}$ and correspondingly we rewrite (3.66) as

$$e^{-\beta|\Lambda|(F_h+e_h)} \int P_{\chi_{h-1},A_{h-1}}(d\Psi^{(\le h-1)}) \int P_{f_h,\overline{A}_{h-1}}(d\Psi^{(h)})\, e^{-\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h-1)}+\Psi^{(h)})}, \tag{3.70}$$

where

$$A_{h-1,\omega}(\mathbf{k}') = \begin{pmatrix} -i\zeta_{h-1}k_0 + s_{h-1}(\mathbf{k}') & c_{h-1}(ik_1' - \omega k_2') + t_{h-1,\omega}(\mathbf{k}') \\ c_{h-1}(-ik_1' - \omega k_2') + t^*_{h-1,\omega}(\mathbf{k}') & -i\zeta_{h-1}k_0 + s_{h-1}(\mathbf{k}') \end{pmatrix} \tag{3.71}$$

with:

$$\begin{aligned} \zeta_{h-1} &= \zeta_h + z_h, & c_{h-1} &= c_h + \delta_h, \\ s_{h-1}(\mathbf{k}') &= s_h(\mathbf{k}') + \sigma_h(\mathbf{k}'), & t_{h-1,\omega}(\mathbf{k}') &= t_{h,\omega}(\mathbf{k}') + \tau_{h,\omega}(\mathbf{k}'). \end{aligned} \tag{3.72}$$

The single scale propagator is

$$\int P_{f_h,\overline{A}_{h-1}}(d\Psi^{(h)}) \Psi^{(h)-}_{\mathbf{x}_1,\sigma_1,\rho_1,\omega_1} \Psi^{(h)+}_{\mathbf{x}_2,\sigma_2,\rho_2,\omega_2} = \delta_{\sigma_1,\sigma_2}\delta_{\omega_1,\omega_2} \left[g^{(h)}_\omega(\mathbf{x}_1,\mathbf{x}_2) \right]_{\rho_1,\rho_2}, \tag{3.73}$$

where

$$g^{(h)}_\omega(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}'\in\mathcal{D}^\omega_{\beta,L}} e^{-i\mathbf{k}'(\mathbf{x}_1-\mathbf{x}_2)} f_h(\mathbf{k}') \left[\overline{A}_{h-1,\omega}(\mathbf{k}')\right]^{-1}. \tag{3.74}$$

After the integration of the field on scale h we are left with an integral involving the fields $\Psi^{(\le h-1)}$ and the new effective interaction $\mathcal{V}^{(h-1)}$, defined as

$$e^{-\mathcal{V}^{(h-1)}(\Psi^{(\le h-1)}) - \overline{e}_h\beta|\Lambda|} = \int P_{f_h,\overline{A}_{h-1}}(d\Psi^{(h)})\, e^{-\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h-1)}+\Psi^{(h)})}, \tag{3.75}$$

with $\mathcal{V}^{(h-1)}(0) = 0$. It is easy to see that $\mathcal{V}^{(h-1)}$ is of the form (3.59) and that $F_{h-1} = F_h + e_h + \overline{e}_h$. It is sufficient to use the well known identity

$$\overline{e}_h + \mathcal{V}^{(h-1)}(\Psi^{(\le h-1)}) = \sum_{n\ge 1} \frac{1}{n!}(-1)^{n+1} \mathcal{E}^T_h\left(\mathcal{R}\mathcal{V}^{(h)}\left(\Psi^{(\le h-1)} + \Psi^{(h)}\right); n\right), \tag{3.76}$$

where $\mathcal{E}^T_h(X(\Psi^{(h)}); n)$ is the truncated expectation of order n w.r.t. the propagator $g^{(h)}_\omega$, which is the analogue of (3.35) with $\Psi^{(u.v.)}$ replaced by $\Psi^{(h)}$ and with $P(d\Psi^{(u.v.)})$ replaced by $P_{f_h,\overline{A}_{h-1}}(d\Psi^{(h)})$.

Note that the above procedure allows us to write the *effective renormalizations* $\vec{v}_h = (\zeta_h, c_h)$, $h \le 0$, in terms of $\vec{v}_k$, $h < k \le 0$, namely $\vec{v}_{h-1} = \beta_h(\vec{v}_h, \dots, \vec{v}_0)$, where β_h is the so–called *Beta function*.

Tree expansion for the effective potentials. An iterative implementation of (3.76) leads to a representation of $\mathcal{V}^{(h)}(\Psi^{(\le h)})$ in terms of a tree expansion, defined as follows:

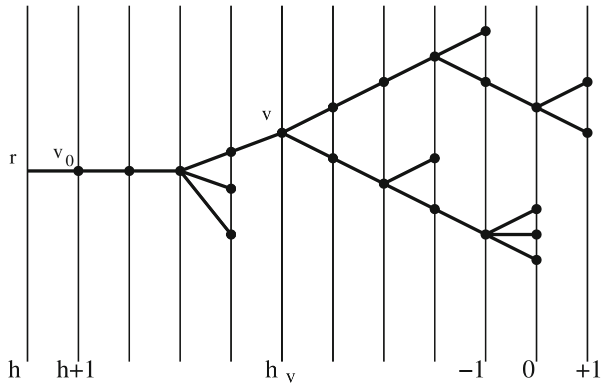

Fig. 1. A tree $\tau \in \mathcal{T}_{h,n}$ with its scale labels

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non-trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n. We shall also consider the *labelled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabelled trees, as explained in the following items.
2) We associate a label $h \leq -1$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, 1]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root r is on the line with index h. In general, the tree will intersect the vertical lines in a set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of the endpoints, of the trivial vertices and of the non-trivial vertices; note that the root is not a vertex. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line to whom v belongs. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.
3) There is only one vertex immediately following the root, which will be denoted v_0 and cannot be an endpoint; its scale is $h + 1$.
4) Given a vertex v of $\tau \in \mathcal{T}_{h,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \geq v$. If a subtree with root v contains only v and an endpoint on scale $h_v + 1$, it will be called a *trivial subtree*.
5) With each endpoint v we associate one of the monomials with four or more Grassmann fields contributing to $\mathcal{RV}^{(0)}(\Psi^{(\leq h_v - 1)})$, corresponding to the terms with $n \geq 2$ in the r.h.s. of (3.54) (with $\Psi^{(\leq 0)}$ replaced by $\Psi^{(\leq h_v - 1)}$) and a set $\mathbf{x}_v$ of space-time points (the corresponding integration variables in the $\mathbf{x}$-space representation).

6) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints described in item 5); the set of field labels associated with the endpoint v will be called I_v; note that $|I_v|$ is the order of the monomial contributing to $\mathcal{V}^{(0)}(\Psi^{(\le h_v-1)})$ and associated to v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\varepsilon(f)$, $\sigma(f)$, $\rho(f)$ and $\omega(f)$ will denote the space-time point, the ε index, the σ index, the ρ index and the ω index, respectively, of the Grassmann field variable with label f.

In terms of these trees, the effective potential $\mathcal{V}^{(h)}$, $h \le -1$, can be written as

$$\mathcal{V}^{(h)}(\Psi^{(\le h)}) + \beta|\Lambda|\bar{e}_{k+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \Psi^{(\le h)}), \tag{3.77}$$

where, if v_0 is the first vertex of τ and $\tau_1, \ldots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)})$ is defined inductively as follows:

i) if $s > 1$, then

$$\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1} \left[\bar{\mathcal{V}}^{(h+1)}(\tau_1, \Psi^{(\le h+1)}); \ldots ; \bar{\mathcal{V}}^{(h+1)}(\tau_s, \Psi^{(\le h+1)}) \right], \tag{3.78}$$

where $\bar{\mathcal{V}}^{(h+1)}(\tau_i, \Psi^{(\le h+1)})$ is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \Psi^{(\le h+1)})$ if the subtree τ_i contains more than one end-point, or if it contains one end-point but it is not a trivial subtree; it is equal to $\mathcal{R}\mathcal{V}^{(0)}(\tau_i, \Psi^{(\le h+1)})$ if τ_i is a trivial subtree;

ii) if $s = 1$, then $\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)})$ is equal to $\mathcal{E}^T_{h+1}\left[\mathcal{R}\mathcal{V}^{(h+1)}(\tau_1, \Psi^{(\le h+1)})\right]$ if τ_1 is not a trivial subtree; it is equal to $\mathcal{E}^T_{h+1}\left[\mathcal{R}\mathcal{V}^{(0)}(\Psi^{(\le h+1)}) - \mathcal{R}\mathcal{V}^{(0)}(\Psi^{(\le h)})\right]$ if τ_1 is a trivial subtree.

Using its inductive definition, the right hand side of (3.77) can be further expanded, and in order to describe the resulting expansion we need some more definitions.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the $s_v \ge 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathcal{I}_v$ of the subsets $P_{v_i} \backslash Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non empty if $s_v > 1$. Given $\tau \in \mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$.

With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)})$ in the r.h.s. of (3.77) as:

$$\mathcal{V}^{(h)}(\tau, \Psi^{(\le h)}) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}),$$
$$\mathcal{V}^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \widetilde{\Psi}^{(\le h)}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}), \tag{3.79}$$

where

$$\widetilde{\Psi}^{(\le h)}(P_v) = \prod_{f \in P_v} \Psi^{(\le h)\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f),\omega(f)}, \tag{3.80}$$

and $K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ is defined inductively by the equation, valid for any $v\in\tau$ which is not an endpoint,

$$K^{(h_v)}_{\tau,\mathbf{P}}(\mathbf{x}_v)=\frac{1}{s_v!}\prod_{i=1}^{s_v}[K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})]\ \ \mathcal{E}^T_{h_v}[\widetilde{\Psi}^{(h_v)}(P_{v_1}\backslash Q_{v_1}),\ldots,\widetilde{\Psi}^{(h_v)}(P_{v_{s_v}}\backslash Q_{v_{s_v}})],\tag{3.81}$$

where $\widetilde{\Psi}^{(h_v)}(P_{v_i}\backslash Q_{v_i})$ has a definition similar to (3.80). Moreover, if v_i is an endpoint $K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})$ is equal to one of the kernels of the monomials contributing to $\mathcal{R}\mathcal{V}^{(0)}(\Psi^{(\le h_v)})$, corresponding to the terms with $n\ge 2$ in the r.h.s. of (3.54) (with $\Psi^{(\le 0)}$ replaced by $\Psi^{(\le h_v)}$); if v_i is not an endpoint, $K^{(h_v+1)}_{v_i}=K^{(h_v+1)}_{\tau_i,\mathbf{P}_i}$, where $\mathbf{P}_i=\{P_w, w\in\tau_i\}$.

Equations (3.77)–(3.81) are not the final form of our expansion; we further decompose $\mathcal{V}^{(h)}(\tau,\mathbf{P})$, by using the following representation of the truncated expectation in the r.h.s. of (3.81). Let us put $s=s_v$, $P_i\equiv P_{v_i}\backslash Q_{v_i}$; moreover we order in an arbitrary way the sets $P_i^{\pm}\equiv\{f\in P_i,\varepsilon(f)=\pm\}$, we call $f^{\pm}_{ij}$ their elements and we define $\mathbf{x}^{(i)}=\cup_{f\in P_i^-}\mathbf{x}(f)$, $\mathbf{y}^{(i)}=\cup_{f\in P_i^+}\mathbf{x}(f)$, $\mathbf{x}_{ij}=\mathbf{x}(f^-_{ij})$, $\mathbf{y}_{ij}=\mathbf{x}(f^+_{ij})$. Note that $\sum_{i=1}^s|P_i^-|=\sum_{i=1}^s|P_i^+|\equiv n$, otherwise the truncated expectation vanishes. A couple $l\equiv(f^-_{ij},f^+_{i'j'})\equiv(f^-_l,f^+_l)$ will be called a line joining the fields with labels f^-_{ij}, $f^+_{i'j'}$, sector indices $\omega^-_l=\omega(f^-_l)$, $\omega^+_l=\omega(f^+_l)$, ρ-indices $\rho^-_l=\rho(f^-_l)$, $\rho^+_l=\rho(f^+_l)$, and spin indices $\sigma^-_l=\sigma(f^-_l)$, $\sigma^+_l=\sigma(f^+_l)$, connecting the points $\mathbf{x}_l\equiv\mathbf{x}_{ij}$ and $\mathbf{y}_l\equiv\mathbf{y}_{i'j'}$, the *endpoints* of l. Moreover, if $\omega^-_l=\omega^+_l$, we shall put $\omega_l\equiv\omega^-_l=\omega^+_l$. Then, we use the *Brydges-Battle-Federbush* formula (e.g., see [10,17]) saying that, up to a sign, if $s>1$,

$$\begin{aligned}&\mathcal{E}^T_h(\widetilde{\Psi}^{(h)}(P_1),\ldots,\widetilde{\Psi}^{(h)}(P_s))\\&\quad=\sum_T\prod_{l\in T}\delta_{\omega^-_l,\omega^+_l}\delta_{\sigma^-_l,\sigma^+_l}\left[g^{(h)}_{\omega_l}(\mathbf{x}_l-\mathbf{y}_l)\right]_{\rho^-_l,\rho^+_l}\int dP_T(\mathbf{t})\,\det G^{h,T}(\mathbf{t}),\end{aligned}\tag{3.82}$$

where T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)}\cup\mathbf{y}^{(i)}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t}=\{t_{ii'}\in[0,1], 1\le i,i'\le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{ii'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i\in\mathbb{R}^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by

$$G^{h,T}_{ij,i'j'}=t_{ii'}\delta_{\omega^-_l,\omega^+_l}\delta_{\sigma^-_l,\sigma^+_l}\left[g^{(h)}_{\omega_l}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'})\right]_{\rho^-_l,\rho^+_l},\tag{3.83}$$

with $(f^-_{ij},f^+_{i'j'})$ not belonging to T. In the following we shall use (3.80) even for $s=1$, when T is empty, by interpreting the r.h.s. as equal to 1, if $|P_1|=0$, otherwise as equal to $\det G^h=\mathcal{E}^T_h(\widetilde{\Psi}^{(h)}(P_1))$.

Remark. It is crucial to note that $G^{h,T}$ is a Gram matrix, i.e., defining $\mathbf{e}_+=\mathbf{e}_\uparrow=(1,0)$ and $\mathbf{e}_-=\mathbf{e}_\downarrow=(0,1)$, the matrix elements in (3.83) can be written in terms of scalar products:

$$\begin{aligned}&t_{ii'}\delta_{\omega^-_l,\omega^+_l}\delta_{\sigma^-_l,\sigma^+_l}\left[g^{(h)}_{\omega_l}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'})\right]_{\rho^-_l,\rho^+_l}\\&\quad=\Big(\mathbf{u}_i\otimes\mathbf{e}_{\omega^-_l}\otimes\mathbf{e}_{\sigma^-_l}\otimes A(\mathbf{x}_{ij}-\cdot),\ \mathbf{u}_{i'}\otimes\mathbf{e}_{\omega^+_l}\otimes\mathbf{e}_{\sigma^+_l}\otimes B(\mathbf{x}_{i'j'}-\cdot)\Big)\equiv(\mathbf{f}_\alpha,\mathbf{g}_\beta),\end{aligned}\tag{3.84}$$

where

$$A(\mathbf{x}) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}' \in \mathcal{D}^{\omega}_{\beta,L}} e^{-i\mathbf{k}'\mathbf{x}} \sqrt{f_h(\mathbf{k}')}\, \mathbb{1},$$
$$B(\mathbf{x}) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}' \in \mathcal{D}^{\omega}_{\beta,L}} e^{-i\mathbf{k}'\mathbf{x}} \sqrt{f_h(\mathbf{k}')} \left[\overline{A}_{h-1,\omega}(\mathbf{k}')\right]^{-1}. \tag{3.85}$$

The symbol $(\cdot,\cdot)$ denotes the inner product, i.e.,

$$\begin{aligned}&\big(\mathbf{u}_i \otimes \mathbf{e}_\omega \otimes \mathbf{e}_\sigma \otimes A(\mathbf{x}-\cdot), \mathbf{u}_{i'} \otimes \mathbf{e}_{\omega'} \otimes \mathbf{e}_{\sigma'} \otimes B(\mathbf{x}'-\cdot)\big)\\ &= (\mathbf{u}_i \cdot \mathbf{u}_{i'})\,(\mathbf{e}_\omega \cdot \mathbf{e}_{\omega'})\,(\mathbf{e}_\sigma \cdot \mathbf{e}_{\sigma'}) \cdot \int d\mathbf{z} A^*(\mathbf{x}-\mathbf{z}) B(\mathbf{x}'-\mathbf{z}),\end{aligned} \tag{3.86}$$

and the vectors $\mathbf{f}_\alpha, \mathbf{g}_\beta$ with $\alpha, \beta = 1, \ldots, n-s+1$ are implicitly defined by (3.84). The usefulness of the representation (3.84) is that, by the Gram-Hadamard inequality (see, e.g., [10]), $|\det(\mathbf{f}_\alpha, \mathbf{g}_\beta)| \le \prod_\alpha ||f_\alpha||\,||g_\alpha||$. In our case, $||\mathbf{f}_\alpha|| \le C\gamma^{3h/2}$ and $||\mathbf{g}_\alpha|| \le C\gamma^{h/2}$. Therefore, $||f_\alpha||\,||g_\alpha|| \le C\gamma^{2h}$, uniformly in α, so that the Gram determinant can be bounded by $C^{n-s+1}\gamma^{2h(n-s+1)}$.

If we apply the expansion (3.82) in each vertex of τ different from the endpoints, we get an expression of the form

$$\mathcal{V}^{(h)}(\tau, \mathbf{P}) = \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} \widetilde{\Psi}^{(\le h)}(P_{v_0}) W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) \equiv \sum_{T \in \mathbf{T}} \mathcal{V}^{(h)}(\tau, \mathbf{P}, T), \tag{3.87}$$

where $\mathbf{T}$ is a special family of graphs on the set of points $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non-trivial vertex v. Note that any graph $T \in \mathbf{T}$ becomes a tree graph on $\mathbf{x}_{v_0}$, if one identifies all the points in the sets $\mathbf{x}_v$, with v an endpoint. Given $\tau \in \mathcal{T}_{h,n}$ and the labels $\mathbf{P}, T$, calling $v_i^*, \ldots, v_n^*$ the endpoints of τ and putting $h_i = h_{v_i^*}$, the explicit representation of $W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0})$ in (3.87) is

$$\begin{aligned}W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) &= \left[\prod_{i=1}^{n} K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})\right]\\ &\cdot \left\{\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{h_v,T_v}(\mathbf{t}_v) \left[\prod_{l \in T_v} \delta_{\omega_l^-,\omega_l^+} \delta_{\sigma_l^-,\sigma_l^+} \left[g^{(h_v)}_{\omega_l}(\mathbf{x}_l - \mathbf{y}_l)\right]_{\rho_l^-,\rho_l^+}\right]\right\},\end{aligned} \tag{3.88}$$

Analyticity of the effective potentials. The tree expansion described above allows us to express the effective potential $\mathcal{V}^{(h)}$ in terms of the *running coupling constants* ζ_h, c_h and of the *renormalization functions* $\sigma_k(\mathbf{k}), t_{k,\omega}(\mathbf{k})$.

The next goal is the proof of the following result.

Theorem 2. *There exists a constant* $U_0 > 0$*, independent of* M*,* β *and* L*, such that the kernels* $W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})$ *in* (3.59)*,* $h \le -1$*, are analytic functions of* U *in the complex domain* $|U| \le U_0$*, satisfying, for any* $0 \le \theta < 1$ *and a suitable constant* $C_\theta > 0$

(independent of M, β, L), the following estimates:

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le \gamma^{h(3-2l+\theta)}\left(C_\theta\,|U|\right)^{max(1,l-1)}. \tag{3.89}$$

Moreover, the constants e_h and $\bar{e}_h$ defined by (3.67) *and* (3.76) *are analytic functions of U in the same domain $|U|\le U_0$, and there they satisfy the estimate $|e_h|+|\bar{e}_h|\le C_\theta|U|\gamma^{h(3+\theta)}$.*

Proof of Theorem 2. Let us preliminarily assume that, for $h'\le h\le -1$, and for suitable constants c, c_n, the corrections $z_h, \delta_h, \sigma_h(\mathbf{k}')$ and $\tau_h(\mathbf{k}')$ defined in (3.63) and (3.65), satisfy the following estimates:

$$\begin{aligned}&\max\{|z_h|,|\delta_h|\}\le c|U|\gamma^{\theta h},\\ &\sup_{\gamma^{h'-1}\le|\mathbf{k}'|\le\gamma^{h'+1}}\{||\partial^n_{\mathbf{k}'}\sigma_h(\mathbf{k}')||,||\partial^n_{\mathbf{k}'}\tau_{h,\omega}(\mathbf{k}')||\}\le c_n|U|\gamma^{2(h'-h)}\gamma^{(1+\theta-n)h}.\end{aligned} \tag{3.90}$$

Using (3.90) we inductively see that the running coupling functions $\zeta_h, c_h, s_h(\mathbf{k}')$ and $t_h(\mathbf{k}')$ satisfy similar estimates:

$$\begin{aligned}&\max\{|\zeta_h-1|,|c_h-3/2|\}\le c|U|,\\ &\sup_{\gamma^{h'-1}\le|\mathbf{k}'|\le\gamma^{h'+1}}\{||\partial^n_{\mathbf{k}'}s_h(\mathbf{k}')||,||\partial^n_{\mathbf{k}'}(t_{h,\omega}(\mathbf{k}')-t_{0,\omega}(\mathbf{k}'))||\}\le c_n|U|\gamma^{2(h'-h)}\gamma^{(1+\theta-n)h}.\end{aligned} \tag{3.91}$$

Now, using the definition of $g^{(h)}_\omega$, see (3.74) and (3.68), we get, after integration by parts, for any $N\ge 0$,

$$\left\|\left[g^{(h)}_\omega(\mathbf{x}_1,\mathbf{x}_2)\right]_{\rho,\rho'}\right\|\le C_N\frac{\gamma^{2h}}{1+(\gamma^h||\mathbf{x}_1-\mathbf{x}_2||)^N}, \tag{3.92}$$

where C_N is a suitable constant and $||\mathbf{x}_1-\mathbf{x}_2||$ is the distance on the torus, defined after (3.32)

Using the tree expansion described above and, in particular, Eqs.(3.77), (3.79), (3.87) and (3.88), we find that the l.h.s. of (3.89) can be bounded from above by

$$\begin{aligned}&\sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}\int\prod_{l\in T^*}d(\mathbf{x}_l-\mathbf{y}_l)\left[\prod_{i=1}^n|K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})|\right]\\ &\cdot\left[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\max_{\mathbf{t}_v}\left|\det G^{h_v,T_v}(\mathbf{t}_v)\right|\prod_{l\in T_v}||\,g^{(h_v)}_{\omega_l}(\mathbf{x}_l-\mathbf{y}_l)\,||\right],\end{aligned} \tag{3.93}$$

where $||\cdot||$ is the spectral norm and where T^* is a tree graph obtained from $T=\cup_v T_v$, by adding in a suitable (obvious) way, for each endpoint v_i^*, $i=1,\ldots,n$, one or more lines connecting the space-time points belonging to $\mathbf{x}_{v_i^*}$.

A standard application of the Gram–Hadamard inequality, combined with the dimensional bound on $g_\omega^{(h)}(\mathbf{x})$ given by (3.92), see the remark after (3.83), implies that

$$|\det G^{h_v,T_v}(\mathbf{t}_v)| \le c^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)} \cdot \gamma^{h_v\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)}. \quad (3.94)$$

By the decay properties of $g_\omega^{(h)}(\mathbf{x})$ given by (3.92), it also follows that

$$\prod_{v \text{ not e.p.}} \frac{1}{s_v!}\int \prod_{l\in T_v} d(\mathbf{x}_l-\mathbf{y}_l)\,||g_{\omega_l}^{(h_v)}(\mathbf{x}_l-\mathbf{y}_l)|| \le c^n \prod_{v \text{ not e.p.}} \frac{1}{s_v!}\gamma^{-h_v(s_v-1)}. \quad (3.95)$$

The bound (3.38) on the kernels produced by the ultraviolet integration implies that

$$\int \prod_{l\in T^*\setminus\cup_v T_v} d(\mathbf{x}_l-\mathbf{y}_l)\prod_{i=1}^n |K_{v_i^*}^{(h_i)}(\mathbf{x}_{v_i^*})| \le \prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}, \quad (3.96)$$

where $p_i = |P_{v_i^*}|$. Combining the previous bounds, we find that (3.93) can be bounded above by

$$\sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}} C^n\left[\prod_{v \text{ not e.p.}}\frac{1}{s_v!}\gamma^{h_v\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-3(s_v-1)\right)}\right]\left[\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\right]. \quad (3.97)$$

Let us define $n(v)=\sum_{i:v_i^*>v} 1$ as the number of endpoints following v on τ and v' as the vertex immediately preceding v on τ. Recalling that $|I_v|$ is the number of field labels associated to the endpoints following v on τ (note that $|I_v|\ge 4n(v)$) and using that

$$\begin{aligned}
&\sum_{v \text{ not e.p.}}\left[\left(\sum_{i=1}^{s_v}|P_{v_i}|\right)-|P_v|\right] = |I_{v_0}|-|P_{v_0}|,\\
&\sum_{v \text{ not e.p.}}(s_v-1) = n-1,\\
&\sum_{v \text{ not e.p.}}(h_v-h)\left[\left(\sum_{i=1}^{s_v}|P_{v_i}|\right)-|P_v|\right] = \sum_{v \text{ not e.p.}}(h_v-h_{v'})(|I_v|-|P_v|),\\
&\sum_{v \text{ not e.p.}}(h_v-h)(s_v-1) = \sum_{v \text{ not e.p.}}(h_v-h_{v'})(n(v)-1),
\end{aligned} \quad (3.98)$$

we find that (3.97) can be bounded above by

$$\begin{aligned}
&\sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}} C^n\gamma^{h(3-|P_{v_0}|+|I_{v_0}|-3n)}\\
&\cdot\left[\prod_{v \text{ not e.p.}}\frac{1}{s_v!}\gamma^{(h_v-h_{v'})(3-|P_v|+|I_v|-3n(v))}\right]\left[\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\right].
\end{aligned} \quad (3.99)$$

Using the identities

$$\gamma^{hn} \prod_{v \text{ not e.p.}} \gamma^{(h_v - h_{v'})n(v)} = \prod_{v \text{ e.p.}} \gamma^{h_{v'}},$$
$$\gamma^{h|I_{v_0}|} \prod_{v \text{ not e.p.}} \gamma^{(h_v - h_{v'})|I_v|} = \prod_{v \text{ e.p.}} \gamma^{h_{v'}|I_v|}, \tag{3.100}$$

we obtain

$$\frac{1}{\beta|\Lambda|} \int d\mathbf{x}_1 \cdots d\mathbf{x}_{2l} |W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})| \le \sum_{n \ge 1} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau \\ |P_{v_0}|=2l}} \sum_{T \in \mathbf{T}} C^n \gamma^{h(3-|P_{v_0}|)}$$
$$\cdot \left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-(h_v - h_{v'})(|P_v|-3)} \right] \left[\prod_{v \text{ e.p.}} \gamma^{h_{v'}(|I_v|-3)} \right] \left[\prod_{i=1}^{n} C^{p_i} |U|^{\frac{p_i}{2}-1} \right]. \tag{3.101}$$

Note that, if v is not an endpoint, $|P_v| - 3 \ge 1$ by the definition of $\mathcal{R}$. Moreover, if v is an endpoint, $|I_v| - 3 \ge 1$; in particular, we get

$$\prod_{v \text{ e.p.}} \gamma^{h_{v'}(|I_v|-3)} \le \gamma^{h_* - 1}, \tag{3.102}$$

with h_* the highest scale label of the tree. Now, note that the number of terms in $\sum_{T \in \mathbf{T}}$ can be bounded by $C^n \prod_{v \text{ not e.p.}} s_v!$. Using also that $|P_v| - 3 \ge 1$ and $|P_v| - 3 \ge |P_v|/4$, we find that the l.h.s. of (3.101) can be bounded as

$$\frac{1}{\beta|\Lambda|} \int d\mathbf{x}_1 \cdots d\mathbf{x}_{2l} |W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})| \le \gamma^{h(3-|P_{v_0}|)} \sum_{n \ge 1} C^n \sum_{\tau \in \mathcal{T}_{h,n}} \gamma^{h_* - 1}$$
$$\cdot \left(\prod_{v \text{ not e.p.}} \gamma^{-\theta(h_v - h_{v'})} \gamma^{-(1-\theta)(h_v - h_{v'})/2} \right)$$
$$\times \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau \\ |P_{v_0}|=2l}} \left(\prod_{v \text{ not e.p.}} \gamma^{-(1-\theta)|P_v|/8} \right) \prod_{i=1}^{n} C^{p_i} |U|^{\frac{p_i}{2}-1}. \tag{3.103}$$

Now, the sum over $\mathbf{P}$ can be bounded using the following combinatorial inequality (see for instance Sect. A6.1 of [10]): let $\{p_v, v \in \tau\}$, with $\tau \in \mathcal{T}_{h,n}$, a set of integers such that $p_v \le \sum_{i=1}^{s_v} p_{v_i}$ for all $v \in \tau$ which are not endpoints; then, if $\alpha > 0$,

$$\prod_{\text{v not e.p.}} \sum_{p_v} \gamma^{-\alpha p_v} \le C^n_\alpha.$$

This implies that

$$\sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau \\ |P_{v_0}|=2l}} \left(\prod_{v \text{ not e.p.}} \gamma^{-(1-\theta)|P_v|/8} \right) \prod_{i=1}^{n} C^{p_i} |U|^{\frac{p_i}{2}-1} \le C^n_\theta |U|^n.$$

Finally, using that $\gamma^{h_*} \prod_{v \text{ not e.p.}} \gamma^{-\theta(h_v - h_{v'})} \le \gamma^{\theta h}$, and that, for $0 < \theta < 1$,

$$\sum_{\tau \in \mathcal{T}_{h,n}} \prod_{v \text{ not e.p.}} \gamma^{-(1-\theta)(h_v - h_{v'})/2} \le C^n,$$

as it follows by the fact that the number of non-trivial vertices in τ is smaller than $n-1$ and that the number of trees in $\mathcal{T}_{h,n}$ is bounded by const^n, and collecting all the previous bounds, we obtain

$$\frac{1}{\beta|\Lambda|} \int d\mathbf{x}_1 \cdots d\mathbf{x}_{2l} |W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})| \le \gamma^{h(3-|P_{v_0}|+\theta)} \sum_{n \ge 1} C^n |U|^n, \tag{3.104}$$

which is the desired result.

We now need to prove the assumption (3.90). We proceed by induction. The assumption is valid for $h = 0$, as it follows by (3.38) and by the discussion in Appendix C. Now, assume that (3.90) is valid for all $h \ge k+1$, and let us prove it for $k-1$. The functions $-iz_k k_0 + \sigma_k(\mathbf{k}')$ and $\delta_k(ik_1' - \omega k_2') + \tau_{k,\omega}(\mathbf{k}')$ admit a representation in terms of $W^{(k)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{x}, \mathbf{y})$. In particular,

$$\max\{|z_k|, |\delta_k|\} \le \frac{1}{\beta|\Lambda|} \int d\mathbf{x}_1 d\mathbf{x}_2 ||\mathbf{x} - \mathbf{y}|| \, |W^{(k)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{x}, \mathbf{y})|, \tag{3.105}$$

and

$$\sup_{\gamma^{h'-1} \le |\mathbf{k}'| \le \gamma^{h'+1}} \{||\partial^n_{\mathbf{k}'} \sigma_k(\mathbf{k}')||, ||\partial^n_{\mathbf{k}'} \tau_{k,\omega}(\mathbf{k}')||\} \le \frac{C\gamma^{2h'}}{\beta|\Lambda|} \int d\mathbf{x}_1 d\mathbf{x}_2 ||\mathbf{x} - \mathbf{y}||^{n+2} |W^{(k)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{x}, \mathbf{y})|. \tag{3.106}$$

The same proof leading to (3.104) shows that the r.h.s. of (3.105) can be bounded by the r.h.s. of (3.104) times γ^{-k} (that is the dimensional estimate for $||\mathbf{x} - \mathbf{y}||$), and that the r.h.s. of (3.105) can be bounded by the r.h.s. of (3.104) times $\gamma^{2h'}\gamma^{-(n+2)k}$ (where $\gamma^{-k(n+2)}$ is the dimensional estimate for $||\mathbf{x} - \mathbf{y}||^{n+2}$). This concludes the proof of Theorem 2.

It remains to prove the estimates on e_h, $\overline{e}_h$. The bound on $\overline{e}_h$ is an immediate corollary of the discussion above, simply because $\overline{e}_h$ can be bounded by (3.93) with $l = 0$. Finally, remember that e_h is given by (3.67): an explicit computation of $A^{-1}_{h,\omega}(\mathbf{k}') W^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}')$ and the use of (3.90)–(3.91) imply that $||A^{-1}_{h,\omega}(\mathbf{k}') W^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}')|| \le C|U|\gamma^{\theta h}$, from which: $|e_h| \le C'\gamma^{3h} \sum_{n \ge 1} (C|U|\gamma^{\theta h})^n$, as desired. □

The existence and analyticity of the specific free energy is a corollary of Theorem 2, see Appendix D for the proof.

Corollary 1. *The limit $f_\beta(U) = \lim_{L\to\infty} \lim_{M\to\infty} F_{M,\beta,L}$, with $F_{M,\beta,L}$ defined in* (3.21)*, exists and is reached uniformly in U; in particular, $f_\beta(U)$ is analytic in $|U| \le U_0$, with U_0 the same constant of Theorem* 2*. Moreover, the limit $e(U) = \lim_{\beta\to\infty} f_\beta(U)$ exists and is reached uniformly in U; in particular, $e(U)$ is analytic in $|U| \le U_0$.*

Corollary 1 implies the part of the statement of Theorem 1 concerning the free energy and the ground state energy. For the proof of analyticity of the Schwinger functions, see the next section.

3.4. The two point Schwinger function. In this section we describe how to modify the expansion for the free energy described in previous sections in order to compute the Schwinger functions at distinct space-time points. For simplicity, we shall restrict our attention to the case of the two point Schwinger function. The general case can be worked out along the same lines.

The Schwinger functions can be derived from the *generating function* defined as

$$\mathcal{W}(\phi) = \log \int P(d\Psi) e^{-\mathcal{V}(\psi) + \int d\mathbf{x} \left[\phi^+_{\mathbf{x},\sigma,\rho} \Psi^-_{\mathbf{x},\sigma,\rho} + \Psi^+_{\mathbf{x},\sigma,\rho} \phi^-_{\mathbf{x},\sigma,\rho}\right]}, \tag{3.107}$$

where summation over repeated indices is understood and the variables $\phi^\varepsilon_{\mathbf{x},\sigma,\rho}$ are Grassmann variables, anticommuting among themselves and with the variables $\Psi^\varepsilon_{\mathbf{x},\sigma,\rho}$. The two–point Schwinger function $\mathcal{S}(\mathbf{x}-\mathbf{y})_{\rho,\rho'} \stackrel{def}{=} \mathcal{S}_2^{M,\beta,\Lambda}(\mathbf{x},\sigma,-,\rho;\mathbf{y},\sigma,+,\rho')$ is given by

$$\mathcal{S}(\mathbf{x}-\mathbf{y})_{\rho,\rho'} = \frac{\partial^2}{\partial\phi^+_{\mathbf{x},\sigma,\rho}\partial\phi^-_{\mathbf{y},\sigma,\rho'}} \mathcal{W}(\phi)\Big|_{\phi=0}. \tag{3.108}$$

We start by studying the generating function and, in analogy with the procedure described before, we begin by decomposing the field Ψ in an ultraviolet and an infrared component: $\Psi = \Psi^{(u.v.)} + \Psi^{(i.r.)}$, with $\Psi^{(i.r.)\pm}_{\mathbf{x},\sigma,\rho} = \sum_{\omega=\pm} e^{i\vec{p}^{\,\omega}_F \vec{x}} \Psi^{(\le 0)\pm}_{\mathbf{x},\sigma,\rho,\omega}$. After the integration of the $\Psi^{(u.v.)}$ variables, and after rewriting $\phi^\pm_{\mathbf{x},\sigma,\rho} = \sum_{\omega=\pm} e^{i\vec{p}^{\,\omega}_F \vec{x}} \phi^\pm_{\mathbf{x},\sigma,\rho}$, we get:

$$\begin{aligned} e^{\mathcal{W}(\phi)} &= e^{-\beta|\Lambda|F_0 + S^{(\ge 0)}(\phi)} \int P_{\chi_0,A_0}(d\Psi^{(\le 0)}) \cdot \\ &\cdot e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)}) - B^{(0)}(\Psi^{(\le 0)},\phi) + \int d\mathbf{x}\left[\phi^+_{\mathbf{x},\sigma,\rho,\omega}\Psi^{(\le 0)-}_{\mathbf{x},\sigma,\rho,\omega} + \Psi^{(\le 0)+}_{\mathbf{x},\sigma,\rho,\omega}\phi^-_{\mathbf{x},\sigma,\rho,\omega}\right]}, \end{aligned} \tag{3.109}$$

where $S^{(\ge 0)}(\phi)$ (chosen in such a way that $S^{(\ge 0)}(0) = 0$) collects the terms depending on ϕ but not on $\Psi^{(\le 0)}$ and $B^{(0)}(\Psi^{(\le 0)},\phi)$ the terms depending both on ϕ and $\Psi^{(\le 0)}$ generated by the ultraviolet integration.

Proceeding as in Sec. 3.3, we inductively show (see below for details) that, if $h \le 0$, $e^{\mathcal{W}(\phi)}$ can be rewritten as:

$$\begin{aligned} e^{\mathcal{W}(\phi)} &= e^{-\beta|\Lambda|F_h + S^{(\ge h)}(\phi)} \int P_{\chi_h,A_h}(d\Psi^{(\le h)}) \\ &\cdot e^{-\mathcal{V}^{(h)}(\Psi^{(\le h)}) - B^{(h)}(\Psi^{(\le h)},\phi) + \int d\mathbf{k}'\left[\hat\phi^+_{\mathbf{k}',\sigma,\rho,\omega}\hat Q^{(h+1)}_{\mathbf{k}',\omega,\rho,\rho'}\hat\Psi^{(\le h)-}_{\mathbf{k}',\sigma,\rho',\omega} + \hat\Psi^{(\le h)+}_{\mathbf{k}',\sigma,\rho,\omega}\hat Q^{(h+1)}_{\mathbf{k}',\omega,\rho,\rho'}\hat\phi^-_{\mathbf{k}',\sigma,\rho',\omega}\right]}, \end{aligned} \tag{3.110}$$

where $\int d\mathbf{k}'$ must be interpreted as equal to $(\beta|\Lambda|)^{-1}\sum_{\mathbf{k}\in\mathcal{D}^\omega_{\beta,L}}$; $B^{(h)}(\Psi^{(\le h)},\phi)$ can be written as $B^{(h)}_\phi(\Psi^{(\le h)}) + W^{(h)}_R$, with $W^{(h)}_R$ containing the terms of third or higher order in ϕ and $B^{(h)}_\phi(\Psi^{(\le h)})$ of the form

$$\begin{aligned} &\int d\mathbf{x}\Bigg[\phi^+_{\cdot,\sigma,\rho_1,\omega} * G^{(h+1)}_{\omega,\rho_1,\rho_2} * \frac{\partial\mathcal{V}^{(h)}(\Psi^{(\le h)})}{\partial\Psi^{(\le h)+}_{\cdot,\sigma,\rho_2,\omega}} + \frac{\partial\mathcal{V}^{(h)}(\Psi^{(\le h)})}{\partial\Psi^{(\le h)-}_{\cdot,\sigma,\rho_1,\omega}} * G^{(h+1)}_{\omega,\rho_1,\rho_2} * \phi^-_{\cdot,\sigma,\rho_2,\omega} \\ &+\phi^+_{\cdot,\sigma_1,\rho_1,\omega_1} * G^{(h+1)}_{\omega_1,\rho_1,\rho_2} * \frac{\partial^2}{\partial\Psi^{(\le h)+}_{\cdot,\sigma_1,\rho_2,\omega_1}\partial\Psi^{(\le h)-}_{\cdot,\sigma_2,\rho_3,\omega_2}}\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h)}) * G^{(h+1)}_{\omega_2,\rho_3,\rho_4} * \phi^-_{\cdot,\sigma_2,\rho_4,\omega_2}\Bigg], \end{aligned} \tag{3.111}$$

with

$$\hat{G}^{(h+1)}_{\omega,\rho,\rho'}(\mathbf{k}') = \sum_{k=h+1}^{1} \hat{g}^{(k)}_{\omega,\rho,\rho''}(\mathbf{k}')\hat{Q}^{(k)}_{\mathbf{k}',\omega,\rho'',\rho'} \tag{3.112}$$

and $\hat{Q}^{(h)}_{\mathbf{k}',\omega,\rho,\rho'}$ defined inductively by the relations

$$\hat{Q}^{(h)}_{\mathbf{k}',\omega,\rho,\rho'} = \hat{Q}^{(h+1)}_{\mathbf{k}',\omega,\rho,\rho'} - W^{(h)}_{2,\rho,\rho'',(\omega,\omega)}(\mathbf{k}')\hat{G}^{(h+1)}_{\omega,\rho'',\rho'}(\mathbf{k}'), \qquad Q^{(1)}_{\mathbf{k}',\omega,\rho,\rho'} \equiv \delta_{\rho,\rho'}, \tag{3.113}$$

where $W^{(h)}_{2,\underline{\rho},\underline{\omega}}$ is the kernel of $\mathcal{LV}^{(h)}$, as defined in (3.62). In (3.112), $\hat{g}^{(1)}_{\omega}$ is defined as

$$\hat{g}^{(1)}_{\omega}(\mathbf{k}') = \hat{g}^{(u.v.)}(\mathbf{k}' + \mathbf{p}^{\omega}_F)\Big[\mathbb{1}\left(||\mathbf{k}'|| < ||\mathbf{k}' + \mathbf{p}^{\omega}_F - \mathbf{p}^{-\omega}_F||\right) + \frac{1}{2}\mathbb{1}\left(||\mathbf{k}'|| = ||\mathbf{k}' + \mathbf{p}^{\omega}_F - \mathbf{p}^{-\omega}_F||\right)\Big],$$

where $\mathbf{p}^{\omega}_F \overset{def}{=} (0, \vec{p}^{\,\omega}_F)$. Note that, by the compact support properties of $\hat{g}^{(h)}_{\omega}(\mathbf{k}')$, if $\hat{g}^{(h)}_{\omega}(\mathbf{k}') \neq 0$, $h < 0$, then $\hat{g}^{(j)}(\mathbf{k}) = 0$ for $|j-h| > 1$, so that

$$\hat{Q}^{(h)}_{\mathbf{k}',\omega,\rho,\rho'} = 1 - \hat{W}^{(h)}_{2,\rho,\rho_1,(\omega,\omega)}(\mathbf{k}')\hat{g}^{(h+1)}_{\omega,\rho_1,\rho_2}(\mathbf{k}')\hat{Q}^{(h+1)}_{\mathbf{k}',\omega,\rho_2,\rho'},$$

and, therefore, proceeding by induction, we see that on the support of $\hat{g}^{(h)}_{\omega}(\mathbf{k}')$ we have

$$||\hat{Q}^{(h)}_{\mathbf{k}',\omega} - 1|| \leq C|U|\gamma^{\theta h}, \qquad ||\partial^n_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}',\omega}|| \leq C_n|U|\gamma^{(\theta-n)h}. \tag{3.114}$$

In order to derive (3.114), we used Theorem 2 and the decay bounds (3.92).

Using (3.114), the definition (3.112) and the decay bounds (3.92), we find that

$$\int d\mathbf{x}\, |\mathbf{x}|^j\, ||G^{(h)}_{\omega}(\mathbf{x})|| \leq C_j\gamma^{-(1+j)h}. \tag{3.115}$$

Let us now prove (3.110). We proceed by induction. For $h = 0$ (3.110) is clearly true (it coincides with (3.109)). Assuming inductively that the representation (3.110) is valid up to a certain value of $h < 0$, we can show that the same representation is valid for $h-1$. In fact, we can rewrite the term $\mathcal{V}^{(h)}$ in the exponent of (3.110) as $\mathcal{V}^{(h)} = \mathcal{LV}^{(h)} + \mathcal{RV}^{(h)}$, as in (3.61), and we can "absorb" $\mathcal{LV}^{(h)}$ in the fermionic integration, as explained in Sec. 3.3, see (3.64)–(3.66). Similarly we rewrite

$$\frac{\partial}{\partial\Psi^{(\leq h)\pm}_{\mathbf{x},\sigma,\rho,\omega}}\mathcal{V}^{(h)}(\Psi^{(\leq h)}) = \int d\mathbf{y}\, W^{(h)}_{2,(\rho,\rho'),(\omega,\omega)}(\mathbf{x},\mathbf{y})\Psi^{(\leq h)\mp}_{\mathbf{y},\sigma,\rho',\omega} + \frac{\partial}{\partial\Psi^{(\leq h)\pm}_{\mathbf{x},\sigma,\rho,\omega}}\mathcal{RV}^{(h)}(\Psi^{(\leq h)}). \tag{3.116}$$

This rewriting induces a decomposition of the first line of (3.111) into two pieces, the first proportional to $W^{(h)}_2$, the second identical to the first line of (3.111) itself, with $\mathcal{V}^{(h)}$ replaced by $\mathcal{RV}^{(h)}$, that we will call $\mathcal{R}B^{(h)}_{\phi}(\Psi^{(\leq h)})$. We choose to "absorb" the term proportional to $W_2^{(h)}$ into the definition of $Q^{(h)}$, and this gives the recursion relation (3.113). Moreover, note that combining $\mathcal{R}B_\phi^{(h)}(\Psi^{(\le h)})$ with $\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h)})$ we find:

$$\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h)}) + \mathcal{R}B_\phi^{(h)}(\Psi^{(\le h)}) = \mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h)} + G^{(h+1)} * \phi) + W_{R,1}^{(h)}, \quad (3.117)$$

with $W_{R,1}^{(h)}$ containing terms of third or higher order in ϕ. We define $\overline{W}_R^{(h)} = W_R^{(h)} + W_{R,1}^{(h)}$.

After these splittings and redefinitions, we can rewrite (3.110) as

$$e^{\mathcal{W}(\phi)} = e^{-\beta|\Lambda|(F_h+e_h)+S^{(\ge h)}(\phi)} \int P_{\chi_{h-1},A_{h-1}}(d\Psi^{(\le h-1)}) \int P_{f_h,\overline{A}_{h-1}}(d\Psi^{(h)})$$
$$\cdot e^{-\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h)}+G^{(h+1)}*\phi)-\overline{W}_R^{(h)}+\int d\mathbf{k}'\left[\hat{\phi}^+_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}'}\hat{\Psi}^{(\le h)-}_{\mathbf{k}'}+\hat{\Psi}^{(\le h)+}_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}'}\hat{\phi}^-_{\mathbf{k}'}\right]}. \quad (3.118)$$

Integrating the field $\Psi^{(h)}$, we get the analogue of (3.75):

$$\int P_{f_h,\overline{A}_{h-1}}(d\Psi^{(h)})\, e^{-\mathcal{R}\mathcal{V}^{(h)}(\Psi^{(\le h)}+G^{(h+1)}*\phi)-\overline{W}_R^{(h)}+\int d\mathbf{k}'\left[\hat{\phi}^+_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}'}\hat{\Psi}^{(\le h)-}_{\mathbf{k}'}+\hat{\Psi}^{(\le h)+}_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}'}\hat{\phi}^-_{\mathbf{k}'}\right]}$$
$$= e^{-\overline{e}_h\beta|\Lambda|-\mathcal{V}^{(h-1)}(\Psi^{(\le h-1)}+G^{(h)}*\phi)+\int d\mathbf{k}'\hat{\phi}^+_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}'}\hat{g}^{(h)}(\mathbf{k}')\hat{Q}^{(h)}_{\mathbf{k}'}\hat{\phi}^-_{\mathbf{k}'}-W_{R,2}^{(h-1)}}\;.$$
$$\cdot e^{\int d\mathbf{k}'\left[\hat{\phi}^+_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}'}\hat{\Psi}^{(\le h-1)-}_{\mathbf{k}'}+\hat{\Psi}^{(\le h-1)+}_{\mathbf{k}'}\hat{Q}^{(h)}_{\mathbf{k}'}\hat{\phi}^-_{\mathbf{k}'}\right]}, \quad (3.119)$$

with $G^{(h)}$ defined by the recursion relation (3.112) and $W_{R,2}^{(h-1)}$ a term of third or higher order in ϕ. Equation (3.119) can be proved by making use of a formal change of Grassmann variables $\hat{\Psi}'_{\mathbf{k}'} = \hat{\Psi}_{\mathbf{k}'} - \hat{g}^{(h)}(\mathbf{k}')Q^{(h)}_{\mathbf{k}'}\hat{\phi}_{\mathbf{k}'}$, as described in Ch.4 of [2]. At this point it is straightforward to check that the final expression for $e^{\mathcal{W}(\phi)}$ that we end up with is given by the r.h.s. of (3.110), with h replaced by $h-1$, and the inductive assumption is proved.

From the definitions and the construction above, we get

$$\mathcal{S}_{\rho,\rho'}(\mathbf{x}-\mathbf{y}) = \sum_{\omega=\pm} e^{-i\vec{p}_F^\omega(\vec{x}-\vec{y})}\mathcal{S}_{\omega,\rho,\rho'}(\mathbf{x}-\mathbf{y}) \equiv \sum_{\omega=\pm} e^{-i\vec{p}_F^\omega(\vec{x}-\vec{y})}\;.$$
$$\cdot\sum_{h=-\infty}^{1}\Big[\Big(Q^{(h)}_{\omega,\rho,\rho_1} * g^{(h)}_{\omega,\rho_1,\rho_2} * Q^{(h)}_{\omega,\rho_2,\rho'}\Big)(\mathbf{x}-\mathbf{y})$$
$$-\Big(G^{(h)}_{\omega,\rho,\rho_1} * W^{(h-1)}_{2,(\rho_1,\rho_2),(\omega,\omega)} * G^{(h)}_{\omega,\rho_2,\rho'}\Big)(\mathbf{x}-\mathbf{y})\Big]. \quad (3.120)$$

Analyticity of $\mathcal{S}_{\rho,\rho'}(\mathbf{x}-\mathbf{y})$ follows from this representation and the results of Theorem 2. Concerning the representation (2.10), let us take the Fourier transform of $\mathcal{S}_{\omega,\rho,\rho'}(\mathbf{x}-\mathbf{y})$. If we define $h_\mathbf{k} = \min\{h : \hat{g}^{(h)}_\omega(\mathbf{k}') \not\equiv 0\}$, we get, for $\mathbf{k}'$ inside the support of $\Psi^{(\le 0)}_{\mathbf{k}',\sigma,\rho,\omega}$,

$$\hat{\mathcal{S}}_{\omega,\rho,\rho'}(\mathbf{k}') = \sum_{j=h_\mathbf{k}}^{h_\mathbf{k}+1} Q^{(j)}_{\mathbf{k}',\omega,\rho,\rho_1} g^{(j)}_{\omega,\rho_1,\rho_2}(\mathbf{k}')^{(j)} Q^{(j)}_{\mathbf{k}',\omega,\rho_2,\rho'}$$
$$-\sum_{j=h_\mathbf{k}}^{h_\mathbf{k}+1} G^{(j)}_{\omega,\rho,\rho_1}(\mathbf{k}')W^{(j-1)}_{2,(\rho_1,\rho_2),(\omega,\omega)}(\mathbf{k}')G^{(j)}_{\omega,\rho_2,\rho'}(\mathbf{k}'), \quad (3.121)$$

which readily implies (2.10): in fact, using the explicit expression of $g_\omega^{(h)}$ and the inductive bounds on $Q^{(h)}$, see (3.114), it is easy to see that the term in the first line of (3.121) can be written as in (2.10) and that their only singularity is located at $\mathbf{k}' = \mathbf{0}$.

The contributions from the second line can be bounded using the bounds on $W_2^{(h)}$ proved in Theorem 2, and we find that they can be bounded by $C|U|\gamma^{h_{\mathbf{k}'}(-1+\theta)}$, which means that they only contribute to the error term appearing in (2.10). This also implies that no other singularity, besides the one at the Fermi points, can be produced by such terms.

Finally, if $\mathbf{k}$ does not belong to the support of $\Psi^{(\leq 0)}$, we can write

$$\hat{\mathcal{S}}_{\rho,\rho'}(\mathbf{k}) = \hat{\mathcal{S}}^{(u.v.)}_{\rho,\rho'}(\mathbf{k}) = g^{(u.v.)}(\mathbf{x}-\mathbf{y}) - \left(g^{(u.v.)}_{\rho,\rho_1} * W_{2,(\rho_1,\rho_2)} * g^{(u.v.)}_{\rho_2,\rho'}\right)(\mathbf{x}-\mathbf{y}), \tag{3.122}$$

with $W_{2,\underline{\rho}}$ defined by (3.36). The bounds discussed in Sec. 3.2 and Appendix C imply that $\mathcal{S}^{(u.v.)}_{\rho,\rho'}(\mathbf{x}-\mathbf{y})$ decays faster than any power, so that no singularity can appear in its Fourier transform.

Note that all the bounds discussed in this section are uniform in M, β, L and this fact, in analogy with the results and proofs of Lemma 2 and Corollary 1, implies the existence of the two-point Schwinger function $\mathcal{S}_2^\beta$ and of its zero temperature limit $\mathcal{S}_2$, see Appendix D for details. A similar expansion can be obtained for higher order Schwinger functions, but we will not belabor the details here. This concludes the proof of the uniform analyticity properties of 3.22 assumed in Proposition 1 and of Theorem 1. □

Appendix A. The Non-interacting Theory

In this Appendix we give some details about the computation of the Schwinger functions of the non-interacting theory, i.e., of model (2.1) with $U = 0$. In this case the Hamiltonian of interest reduces to

$$H_{0,\Lambda} = -\sum_{\substack{\vec{x}\in\Lambda\\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \left(a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma}\right), \tag{A.1}$$

with Λ, $a^\pm_{\vec{x},\sigma}$, $b^\pm_{\vec{x}+\vec{\delta}_i,\sigma}$ defined as in items (1)–(4) after (2.1).

First of all, let us recall that, being $H_{0,\Lambda}$ quadratic, the $2n$-point Schwinger functions satisfy the Wick rule, i.e.,

$$\begin{aligned} \langle \mathbf{T}\{\Psi^-_{\mathbf{x}_1,\sigma_1,\rho_1}\cdots\Psi^-_{\mathbf{x}_n,\sigma_n,\rho_n}\Psi^+_{\mathbf{y}_1,\sigma'_1,\rho'_1}\cdots\Psi^+_{\mathbf{y}_n,\sigma'_n,\rho'_n}\}\rangle_{\beta,\Lambda} &= -\det G,\\ G_{ij} = \delta_{\sigma_i\sigma'_j}\langle \mathbf{T}\{\Psi^-_{\mathbf{x}_i,\sigma_i,\rho_i}\Psi^+_{\mathbf{y}_j,\sigma'_j,\rho'_j}\}\rangle_{\beta,\Lambda}&. \end{aligned} \tag{A.2}$$

Moreover, every n–point Schwinger function $S_n^{\beta,\Lambda}(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n)$ with $\sum_{i=1}^n \varepsilon_i \neq 0$ is identically zero. Therefore, in order to construct the whole set of Schwinger functions of $H_{0,\Lambda}$, it is enough to compute the 2–point function $S_0^{\beta,\Lambda}(\mathbf{x}-\mathbf{y}) = \langle \mathbf{T}\{\Psi^-_{\mathbf{x},\sigma,\rho}\Psi^+_{\mathbf{y},\sigma,\rho'}\}\rangle_{\beta,\Lambda}$, and in order to do this, it is convenient to first diagonalize $H_{0,\Lambda}$. Let us proceed as follows:

We identify Λ with the set of vectors in a fundamental cell, and we write

$$\Lambda = \{n_1\vec{a}_1 + n_2\vec{a}_2 \ :\ 0 \leq n_1, n_2 \leq L-1\}, \tag{A.3}$$

with $\vec{a}_1 = \frac{1}{2}(3, \sqrt{3})$ and $\vec{a}_2 = \frac{1}{2}(3, -\sqrt{3})$. The reciprocal lattice Λ^* is the set of vectors such that $e^{i\vec{K}\vec{x}} = 1$, if $\vec{x} \in \Lambda$. A basis $\vec{b}_1, \vec{b}_2$ for Λ^* can be obtained by the inversion formula:

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = 2\pi \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}^{-1}, \tag{A.4}$$

which gives

$$\vec{b}_1 = \frac{2\pi}{3}(1, \sqrt{3}), \qquad \vec{b}_2 = \frac{2\pi}{3}(1, -\sqrt{3}). \tag{A.5}$$

We call $\mathcal{D}_L$ the set of quasi-momenta $\vec{k}$ of the form

$$\vec{k} = \frac{m_1}{L}\vec{b}_1 + \frac{m_2}{L}\vec{b}_2, \qquad m_1, m_2 \in \mathbb{Z}, \tag{A.6}$$

identified modulo Λ^*; this means that $\mathcal{D}_L$ can be identified with the vectors $\vec{k}$ of the form (2.2) and restricted to the *first Brillouin zone*:

$$\mathcal{D}_L = \left\{ \vec{k} = \frac{m_1}{L}\vec{b}_1 + \frac{m_2}{L}\vec{b}_2 \ : \ 0 \le m_1, m_2 \le L-1 \right\}. \tag{A.7}$$

Given a periodic function $f : \Lambda \to \mathbb{R}$, its Fourier transform is defined as

$$f(\vec{x}) = \frac{1}{|\Lambda|} \sum_{\vec{k} \in \mathcal{D}_L} e^{i\vec{k}\vec{x}} \hat{f}(\vec{k}), \tag{A.8}$$

which can be inverted into

$$\hat{f}(\vec{k}) = \sum_{\vec{x} \in \Lambda} e^{-i\vec{k}\vec{x}} f(\vec{x}), \qquad \vec{k} \in \mathcal{D}_L, \tag{A.9}$$

where we used the identity

$$\sum_{\vec{x} \in \Lambda} e^{i\vec{k}\vec{x}} = |\Lambda| \delta_{\vec{k},\vec{0}}, \tag{A.10}$$

and δ is the periodic Kronecker delta function over Λ^*.

We now associate to the set of creation/annihilation operators $a^{\pm}_{\vec{x},\sigma}$, $b^{\pm}_{\vec{x}+\vec{\delta}_i,\sigma}$ the corresponding set of operators in momentum space:

$$a^{\pm}_{\vec{x},\sigma} = \frac{1}{|\Lambda|} \sum_{\vec{k} \in \mathcal{D}_L} e^{\pm i\vec{k}\vec{x}} \hat{a}^{\pm}_{\vec{k},\sigma}, \qquad b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma} = \frac{1}{|\Lambda|} \sum_{\vec{k} \in \mathcal{D}_L} e^{\pm i\vec{k}\vec{x}} \hat{b}^{\pm}_{\vec{k},\sigma}. \tag{A.11}$$

Note that, using (A.8)–(A.10), we find that

$$\hat{a}^{\pm}_{\vec{k},\sigma} = \sum_{\vec{x} \in \Lambda} e^{\mp i\vec{k}\vec{x}} a^{\pm}_{\vec{x},\sigma}, \qquad \hat{b}^{\pm}_{\vec{k},\sigma} = \sum_{\vec{x} \in \Lambda} e^{\mp i\vec{k}\vec{x}} b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma} \tag{A.12}$$

are fermionic creation/annihilation operators satisfying

$$\{a^{\varepsilon}_{\vec{k},\sigma}, a^{\varepsilon'}_{\vec{k}',\sigma'}\} = |\Lambda| \delta_{\vec{k},\vec{k}'} \delta_{\varepsilon,-\varepsilon'} \delta_{\sigma,\sigma'}, \qquad \{b^{\varepsilon}_{\vec{k},\sigma}, b^{\varepsilon'}_{\vec{k}',\sigma'}\} = |\Lambda| \delta_{\vec{k},\vec{k}'} \delta_{\varepsilon,-\varepsilon'} \delta_{\sigma,\sigma'}, \tag{A.13}$$

and $\{a^{\varepsilon}_{\vec{k},\sigma}, b^{\varepsilon'}_{\vec{k}',\sigma'}\} = 0$. With these definitions, we can rewrite

$$H_{0,\Lambda} = -\sum_{\substack{\vec{x}\in\Lambda \\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} (a^{+}_{\vec{x},\sigma} b^{-}_{\vec{x}+\vec{\delta}_i,\sigma} + b^{+}_{\vec{x}+\vec{\delta}_i,\sigma} a^{-}_{\vec{x},\sigma}) = -\frac{1}{|\Lambda|^2} \sum_{\substack{\vec{x}\in\Lambda \\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow}$$
$$\times \sum_{\vec{k},\vec{k}'\in\mathcal{D}_L} \Big(e^{+i\vec{k}\vec{x}} e^{-i\vec{k}'(\vec{x}+\vec{\delta}_i-\vec{\delta}_1)} \hat{a}^{+}_{\vec{k},\sigma} \hat{b}^{-}_{\vec{k}',\sigma} + e^{-i\vec{k}\vec{x}} e^{+i\vec{k}'(\vec{x}+\vec{\delta}_i-\vec{\delta}_1)} \hat{b}^{+}_{\vec{k}',\sigma} \hat{a}^{-}_{\vec{k},\sigma} \Big)$$
$$= -\frac{1}{|\Lambda|} \sum_{\vec{k}\in\mathcal{D}_L} \sum_{\sigma=\uparrow\downarrow} \Big(v^{*}_{\vec{k}} \hat{a}^{+}_{\vec{k},\sigma} \hat{b}^{-}_{\vec{k},\sigma} + v_{\vec{k}} \hat{b}^{+}_{\vec{k},\sigma} \hat{a}^{-}_{\vec{k},\sigma} \Big), \tag{A.14}$$

with

$$v_{\vec{k}} = \sum_{i=1}^{3} e^{i(\vec{\delta}_i - \vec{\delta}_1)\vec{k}} = 1 + 2e^{-i\frac{3}{2}k_1} \cos \frac{\sqrt{3}}{2} k_2. \tag{A.15}$$

The Hamiltonian $H_{0,\Lambda}$ can be diagonalized by introducing the fermionic operators

$$\hat{\alpha}_{\vec{k},\sigma} = \frac{\hat{a}_{\vec{k},\sigma}}{\sqrt{2}} + \frac{v^{*}_{\vec{k}}}{\sqrt{2}|v_{\vec{k}}|} \hat{b}_{\vec{k},\sigma}, \qquad \hat{\beta}_{\vec{k},\sigma} = \frac{\hat{a}_{\vec{k},\sigma}}{\sqrt{2}} - \frac{v^{*}_{\vec{k}}}{\sqrt{2}|v_{\vec{k}}|} \hat{b}_{\vec{k},\sigma}, \tag{A.16}$$

in terms of which we can rewrite

$$H_{0,\Lambda} = \frac{1}{|\Lambda|} \sum_{\vec{k}\in\mathcal{D}_L} \sum_{\sigma=\uparrow\downarrow} \Big(-|v_{\vec{k}}| \hat{\alpha}^{+}_{\vec{k},\sigma} \hat{\alpha}_{\vec{k},\sigma} + |v_{\vec{k}}| \hat{\beta}^{+}_{\vec{k},\sigma} \hat{\beta}_{\vec{k},\sigma} \Big), \tag{A.17}$$

with

$$|v_{\vec{k}}| = \sqrt{\Big(1 + 2\cos(3k_1/2)\cos(\sqrt{3}k_2/2)\Big)^2 + 4\sin^2(3k_1/2)\cos^2(\sqrt{3}k_2/2)}, \tag{A.18}$$

which is vanishing iff $\vec{k} = \vec{p}^{\,\omega}_F$, $\omega = \pm$, with

$$\vec{p}^{\,\omega}_F = \Big(\frac{2\pi}{3}, \omega\frac{2\pi}{3\sqrt{3}}\Big). \tag{A.19}$$

Now, for $\vec{x} \in \Lambda$, we define $\alpha^{\pm}_{\vec{x},\sigma} = |\Lambda|^{-1} \sum_{\vec{k}\in\mathcal{D}_L} e^{\pm i\vec{k}\vec{x}} \hat{\alpha}_{\vec{k},\sigma}$ and $\beta^{\pm}_{\vec{x},\sigma} = |\Lambda|^{-1} \sum_{\vec{k}\in\mathcal{D}_L} e^{\pm i\vec{k}\vec{x}} \hat{\alpha}_{\vec{k},\sigma}$; moreover, if $\mathbf{x} = (x_0, \vec{x})$ we define $\alpha^{\pm}_{\mathbf{x},\sigma} = e^{H_{0,\Lambda}x_0} \alpha^{\pm}_{\vec{x},\sigma} e^{-H_{0,\Lambda}x_0}$ and $\beta^{\pm}_{\mathbf{x},\sigma} = e^{H_{0,\Lambda}x_0} \beta^{\pm}_{\vec{x},\sigma} e^{-H_{0,\Lambda}x_0}$. A straightforward computation, see, e.g., Appendix 1 of [2], shows that, if $-\beta < x_0 - y_0 \leq \beta$,

$$\langle \mathbf{T}\{\alpha^{-}_{\mathbf{x},\sigma} \alpha^{+}_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda}$$
$$= \frac{\delta_{\sigma,\sigma'}}{|\Lambda|} \sum_{\vec{k}\in\mathcal{D}_L} e^{-i\vec{k}(\vec{x}-\vec{y})} \Bigg[\mathbb{1}\,(x_0 - y_0 > 0)\, \frac{e^{(x_0-y_0)|v_{\vec{k}}|}}{1 + e^{\beta|v_{\vec{k}}|}} - \mathbb{1}\,(x_0 - y_0 \leq 0)\, \frac{e^{(x_0-y_0+\beta)|v_{\vec{k}}|}}{1 + e^{\beta|v_{\vec{k}}|}} \Bigg], \tag{A.20}$$

$$\langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda}$$
$$= \frac{\delta_{\sigma,\sigma'}}{|\Lambda|}\sum_{\vec{k}\in\mathcal{D}_L} e^{-i\vec{k}(\vec{x}-\vec{y})}\left[\mathbb{1}\,(x_0-y_0>0)\,\frac{e^{-(x_0-y_0)|v_{\vec{k}}|}}{1+e^{-\beta|v_{\vec{k}}|}} - \mathbb{1}\,(x_0-y_0\le 0)\,\frac{e^{-(x_0-y_0+\beta)|v_{\vec{k}}|}}{1+e^{-\beta|v_{\vec{k}}|}}\right], \tag{A.21}$$

and $\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = 0$. A priori Eq. (A.21) and (A.22) are defined only for $-\beta < x_0-y_0 \le \beta$, but we can extend them periodically over the whole real axis; the periodic extension of the propagator is continuous in the time variable for $x_0-y_0 \notin \beta\mathbb{Z}$, and it has jump discontinuities at the points $x_0-y_0\in\beta\mathbb{Z}$. Note that at $x_0-y_0=\beta n$, the difference between the right and left limits is equal to $(-1)^n\delta_{\vec{x},\vec{y}}$, so that the propagator is discontinuous only at $\mathbf{x}-\mathbf{y}=\beta\mathbb{Z}\times\vec{0}$. For $\mathbf{x}-\mathbf{y}\notin\beta\mathbb{Z}\times\vec{0}$, we can write

$$\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{-ik_0-|v_{\vec{k}}|}, \tag{A.22}$$

$$\langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{-ik_0+|v_{\vec{k}}|}. \tag{A.23}$$

Note indeed that for $x_0-y_0\notin\beta\mathbb{Z}$ the sums over k_0 in (A.22) are convergent, uniformly in M; if $x_0-y_0=\beta n$ and $\vec{x}\neq\vec{y}$, the r.h.s. of (A.22) is equal to

$$\frac{1}{2}\left(\lim_{x_0-y_0\to(\beta n)^+}\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} + \lim_{x_0-y_0\to(\beta n)^-}\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda}\right)$$
$$= \langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda}\Big|_{x_0-y_0=\beta n}. \tag{A.24}$$

A similar remark is valid for $\langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda}$. If we now re-express $\alpha^\pm_{\mathbf{x},\sigma}$ and $\beta^\pm_{\mathbf{x},\sigma}$ in terms of $a^\pm_{\mathbf{x},\sigma}$ and $b^\pm_{\mathbf{x}+\delta_1,\sigma}$, using (A.16), we get (2.6) and (3.17). Note that if, on the contrary, $\mathbf{x}=\mathbf{y}$, then (3.17) is not valid. In fact

$$\lim_{M\to\infty}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}}\hat{g}_{\mathbf{k}} = \sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}}\frac{1}{k_0^2+|v(\vec{k})|^2}\begin{pmatrix}0 & -v^*(\vec{k})\\ -v(\vec{k}) & 0\end{pmatrix}. \tag{A.25}$$

In particular, the diagonal part of (A.25) is vanishing, while, using (A.16) and the fact that $\langle\hat{\alpha}^+_{\vec{k},\sigma}\hat{\beta}_{\vec{k}',\sigma'}\rangle_{\beta,\Lambda} = \langle\hat{\beta}^+_{\vec{k},\sigma}\hat{\alpha}_{\vec{k}',\sigma'}\rangle_{\beta,\Lambda} = 0$, we have that

$$S_0(0^-,\vec{0})_{1,1} = S_0(0^-,\vec{0})_{2,2} = -\frac{1}{2}\left(\langle\alpha^+_{\vec{x},\sigma}\alpha_{\vec{x},\sigma}\rangle_{\beta,\Lambda} + \langle\beta^+_{\vec{x},\sigma}\beta_{\vec{x},\sigma}\rangle_{\beta,\Lambda}\right)$$
$$= -\frac{1}{2|\Lambda|}\sum_{\vec{k}\in\mathcal{D}_L}\left(\frac{e^{\beta|v_{\vec{k}}|}}{1+e^{\beta|v_{\vec{k}}|}} + \frac{e^{-\beta|v_{\vec{k}}|}}{1+e^{-\beta|v_{\vec{k}}|}}\right) = -\frac{1}{2}. \tag{A.26}$$

Appendix B. Proof of Proposition 1

Let us start by proving (3.23), which is equivalent to

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\}} = \exp\left\{-\beta|\Lambda| \lim_{M\to\infty} F_{M,\beta,L}\right\}. \tag{B.1}$$

The first key remark is that, if β, L are finite, the left hand side of (B.1) is a priori well defined and analytic on the whole complex plane. In fact, by the Pauli principle, the Fock space generated by the fermion operators $a^\pm_{\vec{x},\sigma}$, $b^\pm_{\vec{x}+\vec{\delta}_1,\sigma}$, with $\vec{x} \in \Lambda, \sigma =\uparrow\downarrow$, is finite dimensional. Therefore, writing $H_\Lambda = H^0_\Lambda + UV_\Lambda$, with H^0_Λ and V_Λ two bounded operators, we see that $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$ is an entire function of U, simply because $e^{-\beta H_\Lambda}$ converges in norm over the whole complex plane:

$$||e^{-\beta H_\Lambda}|| \le \sum_{n=0}^\infty \frac{\beta^n}{n!}\left(||H^0_\Lambda|| + |U|\,||V_\Lambda||\right)^n = \sum_{k=0}^\infty \frac{\beta^k|U|^k||V_\Lambda||^k}{k!}\sum_{n\ge k}\frac{\beta^{n-k}||H^0_\Lambda||^{n-k}}{(n-k)!}$$
$$= e^{\beta||H^0_\Lambda||+\beta|U|\,||V_\Lambda||}, \tag{B.2}$$

where the norm $||\cdot||$ is, e.g., the Hilbert-Schmidt norm $||A|| = \sqrt{\mathrm{Tr}(A^\dagger A)}$.

On the other hand, by assumption, $F_{M,\beta,L}$ is analytic in $|U| \le U_0$, with U_0 independent of β, L, M, and uniformly convergent as $M \to \infty$. Hence, by the Weierstrass theorem, the limit $F_{\beta,L} = \lim_{M\to\infty} F_{M,\beta,L}$ is analytic in $|U| \le U_0$ and its Taylor coefficients coincide with the limits as $M \to \infty$ of the Taylor coefficients of $F_{M,\beta,L}$. Moreover, $\lim_{M\to\infty} e^{-\beta|\Lambda|F_{M,\beta,L}} = e^{-\beta|\Lambda|F_{\beta,L}}$, again by the Weierstrass theorem.

It is well known that the Taylor coefficients of $e^{-\beta|\Lambda|F_{\beta,L}}$ coincide with the Taylor coefficients of $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}/\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\}$: this can be shown by developing the trace in power series by using Trotter's product formula; the coefficients of the resulting expansion are expressed in terms of Feynman graphs, which are order by order finite for any fixed β and L (in fact at any fixed order they can be written as a finite combination of integrals over imaginary time and spatial momenta of products of propagators, which are bounded and integrable). Note that, in order to guarantee that the two formal power series are the same, the correct choice of the interaction (3.19) expressed in Grassmann variables does not include terms bilinear in the fields, contrary to the interaction in second quantized form, see (2.1): in fact, with this choice, in both perturbative expansions the "tadpoles" are exactly vanishing, as required by the condition that the system is at half filling, even though the Grassmann propagator at the origin does not coincide with $S_0(0^-, \vec{0})$, see (A.25) and (A.26).

In conclusion, $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}/\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\} = e^{-\beta|\Lambda|F_{\beta,L}}$ in the complex region $|U| \le U_0$, simply because the l.h.s. is entire in U, the r.h.s. is analytic in $|U| \le U_0$ and the Taylor coefficients at the origin of the two sides are the same. Taking logarithms at both sides proves (3.23).

Regarding (3.24), we note that, by analyticity, $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}/\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\}$ never vanishes on $|U| \le U_0$; therefore, the same argument used above for the pressure can be now repeated for the Schwinger functions. □

Appendix C. The Ultraviolet Integration

In this Appendix we prove Lemma 2, that is we prove Eq.(3.38) and the existence (and uniformity) of the $M \to \infty$ limit. Note that in order to get (3.38), a simple application of

(3.82) and determinant bounds is not enough, because $g^{(u.v.)}(\mathbf{x})$ does not admit a Gram representation, which is a key property needed for the implementation of standard fermionic cluster expansion methods. As mentioned in Sect. 3.2, a way out of this problem is to decompose the ultraviolet propagator into a sum of propagators, each admitting a Gram representation, and performing a simple multiscale analysis of the ultraviolet problem, in analogy with the standard strategy for ultraviolet problems in fermionic Quantum Field Theories [9,14]. This multiscale analysis is very similar to (but much simpler than) the one described in Sect. 3.3; it has been performed in several previous papers [3,5,4] and it is reported here for completeness.

Appendix C.1. Proof of (3.38). Let M be the integer introduced at the beginning of Sect. 3.1, let β, L be fixed throughout this Appendix and let us rewrite the Fourier transform of $\hat{g}^{(u.v.)}(\mathbf{k})$ as

$$g^{(u.v.)}(\mathbf{x}) = \sum_{h=1}^{M} g^{(h)}(\mathbf{x}), \tag{C.1}$$

where

$$g^{(h)}(\mathbf{x}) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{D}^*_{\beta,L}} f_{u.v.}(\mathbf{k}) H_h(k_0) e^{-i\mathbf{k}\mathbf{x}} \hat{g}_{\mathbf{k}}, \tag{C.2}$$

with $H_1(k_0) = \chi_0(\gamma^{-1}|k_0|)$ and, if $h \geq 2$, $H_h(k_0) = \chi_0(\gamma^{-h}|k_0|) - \chi_0(\gamma^{-h+1}|k_0|)$. Note that $[g^{(h)}(\mathbf{0})]_{\rho\rho} = 0$, $\rho = 1, 2$, and, for any integer $K \geq 0$, $g^{(h)}(\mathbf{x})$ satisfies the bound

$$||g^{(h)}(\mathbf{x})|| \leq \frac{C_K}{1 + (\gamma^h |x_0|_\beta + |\vec{x}|_\Lambda)^K}, \tag{C.3}$$

where $|\cdot|_\beta$ is the distance on the one dimensional torus of size β and $|\cdot|_\Lambda$ is the distance on the periodic lattice Λ. Moreover, $g^{(h)}(\mathbf{x})$ admits a Gram representation: $g^{(h)}(\mathbf{x}-\mathbf{y}) = \int d\mathbf{z}\, A_h^*(\mathbf{x}-\mathbf{z}) \cdot B_h(\mathbf{y}-\mathbf{z})$, with

$$\begin{aligned} A_h(\mathbf{x}) &= \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} \sqrt{f_{u.v.}(\mathbf{k}) H_h(k_0)} \frac{e^{-i\mathbf{k}\mathbf{x}}}{k_0^2 + |v(\vec{k})|^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ B_h(\mathbf{x}) &= \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} \sqrt{f_{u.v.}(\mathbf{k}) H_h(k_0)}\, e^{-i\mathbf{k}\mathbf{x}} \begin{pmatrix} ik_0 & -v^*(\vec{k}) \\ -v(\vec{k}) & ik_0 \end{pmatrix}, \end{aligned} \tag{C.4}$$

and

$$||A_h||^2 = \int d\mathbf{z} |A_h(\mathbf{z})|^2 \leq C\gamma^{-3h}, \qquad ||B_h||^2 \leq C\gamma^{3h}, \tag{C.5}$$

for a suitable constant C.

Our goal is to compute

$$\begin{aligned} e^{-\beta|\Lambda|F_0 - \mathcal{V}(\Psi^{(i.r)})} &= \lim_{M\to\infty} \int P(d\Psi^{[1,M]}) e^{V(\Psi^{(i.r.)}+\Psi^{[1,M]})} \\ &= \exp\left\{ \lim_{M\to\infty} \log \int P(d\Psi^{[1,M]}) e^{V(\Psi^{(i.r.)}+\Psi^{[1,M]})} \right\}, \end{aligned} \tag{C.6}$$

where $P(d\Psi^{[1,M]})$ is the fermionic "Gaussian integration" associated with the propagator $\sum_{h=1}^{M}\hat{g}^{(h)}(\mathbf{k})$ (i.e., it is the same as $P(d\Psi^{(u.v.)})$); the fact that the limit $M\to\infty$ can be exchanged with the logarithm in (C.6) follows from the analysis below. We perform the integration of (C.6) in an iterative fashion, analogous to the procedure described in Sec. 3.3 for the infrared integration. We can inductively prove the analogue of (3.58), i.e.,

$$e^{-\beta|\Lambda|F_{0,M}-\mathcal{V}_M(\Psi^{(i.r)})}=e^{-\beta|\Lambda|F_h}\int P(d\Psi^{[1,h]})e^{\mathcal{V}_M^{(h)}(\Psi^{(i.r.)}+\Psi^{[1,h]})},\tag{C.7}$$

where $P(d\Psi^{[1,h]})$ is the fermionic "Gaussian integration" associated with the propagator $\sum_{k=1}^{h}\hat{g}^{(k)}(\mathbf{k})$ and

$$\mathcal{V}_M^{(h)}(\Psi^{[1,h]})=\sum_{n=1}^{\infty}\sum_{\underline{\rho},\underline{\sigma}}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2n}\left[\prod_{j=1}^{n}\Psi^{[1,h]+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1}}\Psi^{[1,h]-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j}}\right]$$
$$\times W^{(h)}_{M,2n,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n}).\tag{C.8}$$

In order to inductively prove (C.7)–(C.8) we simply use the addition principle to rewrite

$$\int P(d\Psi^{[1,h]})e^{\mathcal{V}_M^{(h)}(\Psi^{(i.r.)}+\Psi^{[1,h]})}=\int P(d\Psi^{[1,h-1]})$$
$$\times\int P(d\Psi^{(h)})e^{\mathcal{V}_M^{(h)}(\Psi^{(i.r.)}+\Psi^{[1,h-1]}+\Psi^{(h)})},\tag{C.9}$$

where $P(d\Psi^{(h)})$ is the fermionic Gaussian integration with propagator $\hat{g}^{(h)}(\mathbf{k})$. After the integration of $\Psi^{(h)})$ we define

$$e^{\mathcal{V}_M^{(h-1)}(\Psi^{(i.r.)}+\Psi^{[1,h-1]})-\beta|\Lambda|\bar{e}_h}=\int P(d\Psi^{(h)})e^{\mathcal{V}_M^{(h)}(\Psi^{(i.r.)}+\Psi^{[1,h-1]}+\Psi^{(h)})},\tag{C.10}$$

which proves (C.7). In analogy with (3.76) we have

$$\bar{e}_h+\mathcal{V}_M^{(h-1)}(\Psi)=\sum_{n\ge 1}\frac{1}{n!}(-1)^{n+1}\mathcal{E}_h^T\big(\mathcal{V}_M^{(h)}\big(\Psi+\Psi^{(h)}\big);n\big).\tag{C.11}$$

As described in Sect. 3.3, the iterative action of $\mathcal{E}_{h_i}^T$ can be conveniently represented in terms of trees $\tau\in\mathcal{T}_{M;h,n}$, where $\mathcal{T}_{M;h,n}$ is a set of labelled trees, completely analogous to the set $\mathcal{T}_{h,n}$ described before Eq.(3.77), unless for the following modifications:

1. a tree $\tau\in\mathcal{T}_{M;h,n}$ has vertices v associated with scale labels $h+1\le h_v\le M+1$, while the root r has scale h;
2. with each end-point v we associate $V(\Psi^{[1,M]})$, with $V(\Psi)$ defined in (3.19).

In terms of these trees, the effective potential $\mathcal{V}_M^{(h)}$, $0\le h\le M$ (with $\mathcal{V}_M^{(0)}(\Psi^{(i.r.)})$ identified with $\mathcal{V}(\Psi^{(i.r.)})$), can be written as

$$\mathcal{V}_M^{(h)}(\Psi^{[1,h]})+\beta|\Lambda|\bar{e}_{h+1}=\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{M;h,n}}\mathcal{V}^{(h)}(\tau,\Psi^{[1,h]}),\tag{C.12}$$

where, if v_0 is the first vertex of τ and $\tau_1,\ldots,\tau_s$ ($s=s_{v_0}$) are the subtrees of τ with root v_0, $\mathcal{V}^{(h)}(\tau,\Psi^{[1,h]})$ is defined inductively as follows:

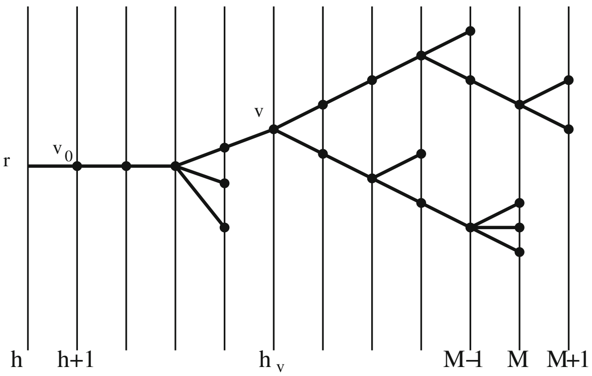

Fig. 2. A tree $\tau \in \mathcal{T}_{M;h,n}$ with its scale labels

i) if $s > 1$, then

$$\mathcal{V}^{(h)}(\tau, \Psi^{[1,h]}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1} \left[\bar{\mathcal{V}}^{(h+1)}(\tau_1, \Psi^{[1,h+1]}); \ldots ; \bar{\mathcal{V}}^{(h+1)}(\tau_s, \Psi^{[1,h+1]}) \right], \tag{C.13}$$

where $\bar{\mathcal{V}}^{(h+1)}(\tau_i, \Psi^{[1,h+1]})$ is equal to $\mathcal{V}^{(h+1)}(\tau_i, \Psi^{[1,h+1]})$ if the subtree τ_i contains more than one end-point, or if it contains one end-point but it is not a trivial subtree; it is equal to $V(\Psi^{[1,h+1]})$ if τ_i is a trivial subtree;

ii) if $s = 1$, then $\mathcal{V}^{(h)}(\tau, \Psi^{(\leq h)})$ is equal to $\mathcal{E}^T_{h+1} \left[\mathcal{V}^{(h+1)}(\tau_1, \Psi^{[1,h+1]}) \right]$ if τ_1 is not a trivial subtree; it is equal to $\mathcal{E}^T_{h+1} \left[V(\Psi^{[1,h+1]}) - V(\Psi^{[1,h]}) \right]$ if τ_1 is a trivial subtree.

Note that, with $V(\Psi)$ defined as in (3.19) and with the present choice of the ultraviolet cutoff (such that $[g^{(h)}(\mathbf{0})]_{\rho\rho} = 0$), we get $\mathcal{E}^T_{h+1} \left[V(\Psi^{[1,h+1]}) - V(\Psi^{[1,h]}) \right] = 0$. This implies that, if v is not an endpoint and $n(v)$ is the number of endpoints following v on τ, and if τ has a vertex v with $n(v) = 1$, then its value vanishes: therefore, in the sum over the trees, we can freely impose the constraint that $n(v) > 1$ for all vertices $v \in \tau$. From now on we shall assume that the trees in $\mathcal{T}_{M;h,n}$ satisfy this constraint.

Repeating step by step the discussion leading to (3.79), (3.87) and (3.88), and using analogous definitions, we find that

$$\mathcal{V}^{(h)}(\tau, \mathbf{P}) = \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} \widetilde{\Psi}^{[1,h]}(P_{v_0}) W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) \equiv \sum_{T \in \mathbf{T}} \mathcal{V}^{(h)}(\tau, \mathbf{P}, T), \tag{C.14}$$

where

$$\widetilde{\Psi}^{[1,h]}(P_v) = \prod_{f \in P_v} \Psi^{[1,h]\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f)} \tag{C.15}$$

and

$$W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = U^n \left\{ \prod_{\substack{v \\ \text{not e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{h_v,T_v}(\mathbf{t}_v) \right.$$
$$\left. \left[\prod_{l \in T_v} \delta_{\sigma_l^-,\sigma_l^+} \left[g^{(h_v)}(\mathbf{x}_l - \mathbf{y}_l) \right]_{\rho_l^-,\rho_l^+} \right] \right\}. \tag{C.16}$$

Moreover, $G^{h_v,T_v}(\mathbf{t}_v)$ is a matrix, analogous to (3.83), with $\delta_{\omega_l^+,\omega_l^-}$ replaced by 1 and $g^{(h)}_{\omega_l}$ replaced by $g^{(h)}$. Note that $W_{\tau,\mathbf{P},T}$ and, therefore, $\mathcal{V}^{(h)}(\tau,\mathbf{P})$ do not depend on M: $\mathcal{V}^{(h)}_M(\Psi)$ depends on M only through the choice of the scale labels (i.e., the dependence on M is all encoded in $\mathcal{T}_{M;h,n}$).

As in the proof of Theorem 2, we get the bound

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le \sum_{n\ge1}|U|^n\sum_{\tau\in\mathcal{T}_{M;h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}\int\prod_{l\in T}d(\mathbf{x}_l-\mathbf{y}_l)$$
$$\cdot\left[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\max_{\mathbf{t}_v}\left|\det G^{h_v,T_v}(\mathbf{t}_v)\right|\prod_{l\in T_v}||g^{(h_v)}(\mathbf{x}_l-\mathbf{y}_l)||\right] \tag{C.17}$$

and, using the analogues of the estimates (3.94), (3.95) and (3.96), taking into account the new scaling of the propagator, we find that (C.17) can be bounded above by

$$\sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{M;h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}C^n|U|^n\left[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\gamma^{-h_v(s_v-1)}\right]. \tag{C.18}$$

Using (3.98) we find that the latter expression can be rewritten as

$$\sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{M;h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}C^n|U|^n\gamma^{-h(n-1)}\left[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\gamma^{-(h_v-h_{v'})(n(v)-1)}\right], \tag{C.19}$$

where we remind the reader that $n(v)>1$ for any $\tau\in\mathcal{T}_{M;h,n}$. Performing the sums over T, $\mathbf{P}$ and τ as in the proof of Theorem 2, we finally find

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})|\le C|U|^{\max\{1,n-1\}}, \tag{C.20}$$

which (3.38) with $m=0$. The proof of the general case, $m\ge0$, is completely analogous. By the uniformity of the constant C with respect to M,β,L, the bounds above imply analyticity of the kernels in $|U|\le U_0$, for a suitable U_0 independent of M,β,L.

Appendix C.2. The $M\to\infty$ limit. In this subsection we prove that, if $M'\ge M$,

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}\left|W^{(0)}_{M',2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})-W^{(0)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})\right|$$
$$\le C_1|U|^{\max\{1,n-1\}}\gamma^{-M/2}, \tag{C.21}$$

which readily implies the last statement of Lemma 2. In fact, (C.21) implies that $\{\hat W^{(0)}_{k,2l,\underline{\rho}}\}_{k\in\mathbb{N}}$ is a Cauchy sequence, uniformly in U for $|U|\le U_0$. Since the kernels are analytic in U in the same domain, by the Weierstrass theorem the kernels $\hat W^{(0)}_{M,2l,\underline{\rho}}$ admit a limit $\hat W^{(0)}_{2l,\underline{\rho}}$ as $M\to\infty$; the limit is analytic in $|U|\le U_0$ and its Taylor coefficients are the limits of the coefficients of $\hat W^{(0)}_{M,2l,\underline{\rho}}$.

Using the same representation leading to (C.17) and following the steps leading to (C.18) and (C.19), we see that the l.h.s. of (C.21) can be bounded as

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}\left|W^{(0)}_{M',2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})-W^{(0)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})\right|$$
$$\leq\sum_{n\geq 1}\sum_{\tau\in\mathcal{T}_{M';0,n}\setminus\mathcal{T}_{M;0,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}C^n|U|^n\left[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\gamma^{-(h_v-h_{v'})(n(v)-1)}\right].\tag{C.22}$$

Note that the set of trees over which we are summing is $\mathcal{T}_{M';h,n}\setminus\mathcal{T}_{M;h,n}$, i.e., the trees contributing to the difference $W^{(0)}_{M',2l,\underline{\rho}}-W^{(0)}_{M,2l,\underline{\rho}}$ must have at least one endpoint on scale $M<h^*\leq M'+1$. By using the fact that $n(v)\geq 2$, we can bound the r.h.s. of (C.22) from above by

$$\gamma^{-\frac{M}{2}}\sum_{n\geq 1}\sum_{\tau\in\mathcal{T}_{M';0,n}\setminus\mathcal{T}_{M;0,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}C^n|U|^n\left[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\gamma^{-(h_v-h_{v'})(n(v)-3/2)}\right]$$
$$\leq C_1|U|^{\max\{1,n-1\}}\gamma^{-M/2},\tag{C.23}$$

which proves (C.21).

Appendix D. The Thermodynamic and Zero Temperature Limits

In this Appendix we first prove Corollary 1, discussing the existence (and uniformity) of the thermodynamic limit and of the zero temperature limit of the free energy. Finally, we discuss the existence of the thermodynamic and zero temperature limits for the Schwinger functions.

Let us start by studying the thermodynamic limit of the free energy. The discussion in Appendix C implies that $\lim_{M\to\infty}F_{\beta,L}=F_0+\sum_{h=h_\beta}^0(e_h+\bar{e}_h)$, where F_0 was defined in Lemma 2 and e_h and $\bar{e}_h$ were defined in (3.67) and (3.76), respectively [1]. Note that both F_0 and $e_h,\bar{e}_h$ depend on L and β, through the propagators (which depend on β, L) and through the definition of the integration interval and of the sum over the scale labels. In order to make this dependence apparent, let us rename them as $F_{0,\beta,L}$, $e_{h,\beta,L}$ and $\bar{e}_{h,\beta,L}$. Similarly, when needed, we shall attach extra labels β, L to the kernels of the effective potentials, to the propagators and to the Gram determinants, to make their dependence on β, L apparent. We already know that $F_{0,\beta,L}$, $e_{h,\beta,L}$ and $\bar{e}_{h,\beta,L}$ are analytic in the uniform domain $|U|\leq U_0$, where they satisfy bounds of the form: $|F_{0,\beta,L}|\leq C|U|$, $|e_h|+|\bar{e}_h|\leq C|U|\gamma^{h(3+\theta)}$, $0\leq\theta<1$. Our first goal is to prove that, for any $\beta<+\infty$ and for any $0<K<4$,

$$|F_{0,\beta,L}-F_{0,\beta}|+\sum_{h=h_\beta}^{0}\left(|e_{h,\beta,L}-e_{h,\beta}|+|\bar{e}_{h,\beta,L}-\bar{e}_{h,\beta}|\right)\leq\frac{C_K|U|}{L^K},\tag{D.1}$$

[1] With some abuse of notation, we are denoting by the same symbols both the functions e_h and $\bar{e}_h$ computed at finite M, and their limits as $M\to\infty$ (which exist, by Lemma 2, Theorem 2 and an application of the Weierstrass theorem: note in fact that e_h and $\bar{e}_h$ in (3.67) and (3.76) have a very weak dependence on M, induced by the kernels of $\mathcal{V}^{(0)}_M$, that is essentially irrelevant, as proved in Appendix C).

for suitable functions $F_{0,\beta}, e_{h,\beta}, \bar{e}_{h,\beta}$, analytic in $|U| \leq U_0$, and a suitable constant C_K.

Let us start by considering

$$F_{0,\beta,L} = \frac{1}{\beta|\Lambda|} \sum_{n\geq 1} \sum_{M\geq 1} \sum_{\tau\in\mathcal{T}^*_{0,n}(M)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau \\ |P_{v_0}|=0}} \sum_{T\in\mathbf{T}} \int d\mathbf{x}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta,L}(\mathbf{x}_{v_0}), \tag{D.2}$$

where $\mathcal{T}^*_{0,n}(M)$ is the set of trees with root on scale 0, n endpoints and the highest scale label equal to $M+1$. We observe that, by using translation invariance, we can fix one variable at the origin: remember that $\mathbf{x}_{v_0} = \{\mathbf{x}_v \,:\, v \text{ is an e.p. of } \tau\}$, so that, if v^* is one arbitrarily chosen endpoint of τ,

$$F_{0,\beta,L} = \sum_{n\geq 1} \sum_{M\geq 1} \sum_{\tau\in\mathcal{T}^*_{0,n}(M)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau \\ |P_{v_0}|=0}} \sum_{T\in\mathbf{T}} \int_{(\beta,\Lambda)} d\bar{\mathbf{x}}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta,L}(\bar{\mathbf{x}}_{v_0}), \tag{D.3}$$

where $\bar{\mathbf{x}}_{v_0} = \{\mathbf{x}_v - \mathbf{x}_{v^*} \,:\, v \text{ is an e.p. of } \tau\}$ and

$$\int_{(\beta,\Lambda)} d\bar{\mathbf{x}}_{v_0} = \prod_{\substack{v \text{ e.p.} \\ v\neq v^*}} \left[\int_{-\beta/2}^{\beta/2} dx_{0,v} \sum_{\vec{x}_v\in\Lambda} \right]. \tag{D.4}$$

We want to estimate

$$\begin{aligned} &F_{0,\beta,L} - F_{0,\beta} \\ &= \sum_{n\geq 1} \sum_{M\geq 1} \sum_{\tau\in\mathcal{T}^*_{0,n}(M)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau \\ |P_{v_0}|=0}} \sum_{T\in\mathbf{T}} \left[\int_{(\beta,\Lambda)} d\bar{\mathbf{x}}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta,L}(\bar{\mathbf{x}}_{v_0}) - \int_{(\beta,\mathbb{B})} d\bar{\mathbf{x}}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta}(\bar{\mathbf{x}}_{v_0}) \right], \end{aligned} \tag{D.5}$$

where $\mathbb{B}$ is the infinite triangular lattice and $W^{(0)}_{\tau,\mathbf{P},T,\beta}$ the kernel obtained from $W^{(0)}_{\tau,\mathbf{P},T,\beta,L}$ by replacing all the propagators $g^{(h)}_{\beta,L}(\mathbf{x})$ by their infinite volume limits $g^{(h)}_{\beta}(\mathbf{x}) = \lim_{L\to\infty} g^{(h)}_{\beta,L}(\mathbf{x})$. Let us fix $0 < \delta < 1/4$ and let us define

$$\Lambda_\delta = \{n_1\vec{a}_1 + n_2\vec{a}_2 \,:\, |n_1|, |n_2| \leq \delta L\}. \tag{D.6}$$

We rewrite (D.5) as

$$F_{0,\beta,L} - F_{0,\beta} = \sum_{n\geq 1} \sum_{M\geq 1} \sum_{\tau\in\mathcal{T}^*_{0,n}(M)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau \\ |P_{v_0}|=0}} \sum_{T\in\mathbf{T}} \Big(R^{(1)}_{\tau,\mathbf{P},T,\beta,L} + R^{(2)}_{\tau,\mathbf{P},T,\beta,L} + R^{(3)}_{\tau,\mathbf{P},T,\beta,L} \Big), \tag{D.7}$$

where

$$\begin{aligned} R^{(1)}_{\tau,\mathbf{P},T,\beta,L} &= \int_{(\beta,\Lambda)} d\bar{\mathbf{x}}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta,L}(\bar{\mathbf{x}}_{v_0}) - \int_{(\beta,\Lambda_\delta)} d\bar{\mathbf{x}}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta,L}(\bar{\mathbf{x}}_{v_0}), \\ R^{(2)}_{\tau,\mathbf{P},T,\beta,L} &= \int_{(\beta,\Lambda_\delta)} d\bar{\mathbf{x}}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta}(\bar{\mathbf{x}}_{v_0}) - \int_{(\beta,\mathbb{B})} d\bar{\mathbf{x}}_{v_0} W^{(0)}_{\tau,\mathbf{P},T,\beta}(\bar{\mathbf{x}}_{v_0}), \\ R^{(3)}_{\tau,\mathbf{P},T,\beta,L} &= \int_{(\beta,\Lambda_\delta)} d\bar{\mathbf{x}}_{v_0} \left(W^{(0)}_{\tau,\mathbf{P},T,\beta,L}(\bar{\mathbf{x}}_{v_0}) - W^{(0)}_{\tau,\mathbf{P},T,\beta}(\bar{\mathbf{x}}_{v_0}) \right). \end{aligned} \tag{D.8}$$

The contributions to $F_{0,\beta,L} - F_{0,\beta}$ associated to $R^{(1)}_{\tau,\mathbf{P},T,\beta,L}$ and $R^{(2)}_{\tau,\mathbf{P},T,\beta,L}$, in analogy with (C.17), can be bounded from above by

$$\sum_{n\geq 1}|U|^n \sum_{M\geq 1}\sum_{\tau\in\mathcal{T}^*_{h,n}(M)}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=0}}\sum_{T\in\mathbf{T}}\int^* \prod_{l\in T} d(\mathbf{x}_l-\mathbf{y}_l)$$
$$\cdot\left[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\max_{\mathbf{t}_v}\left|\det G^{h_v,T_v}_{\beta,L_v}(\mathbf{t}_v)\right|\prod_{l\in T_v}||\, g^{(h_v)}_{\beta,L_l}(\mathbf{x}_l-\mathbf{y}_l)\,||\right], \qquad \text{(D.9)}$$

where L_v and L_l can only assume the values L or $+\infty$, and $\int^*$ means that the integration region satisfies the following constraint: there exists a subtree $\tilde{T}\subseteq T$ such that $|\sum_{l\in\tilde{T}}(\mathbf{y}_l-\mathbf{x}_l)|\geq \delta L$. Using this constraint and (C.3), we get the following improved version of the analogue of (3.95):

$$\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\int\prod_{l\in T_v} d(\mathbf{x}_l-\mathbf{y}_l)\,||g^{(h_v)}_{\beta,L_l}(\mathbf{x}_l-\mathbf{y}_l)|| \leq \frac{c_K^n}{1+(\delta L)^K}\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\gamma^{-h_v(s_v-1)}. \qquad \text{(D.10)}$$

This implies that the contributions to $F_{0,\beta,L} - F_{0,\beta}$ associated to $R^{(1)}_{\tau,\mathbf{P},T,\beta,L}$ and $R^{(2)}_{\tau,\mathbf{P},T,\beta,L}$ are bounded by $C_K|U|L^{-K}$, as desired.

Let us now look at the contributions to $F_{0,\beta,L} - F_{0,\beta}$ associated to $R^{(3)}_{\tau,\mathbf{P},T,\beta,L}$. Using (C.16) we can bound it from above by $R^{(4)}_{\beta,L} + R^{(5)}_{\beta,L}$, with

$$R^{(4)}_{\beta,L} = \sum_{n\geq 1}|U|^n \sum_{M\geq 1}\sum_{\tau\in\mathcal{T}^*_{h,n}(M)}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=0}}\sum_{T\in\mathbf{T}}\int_{(\beta,\Lambda_\delta)}\prod_{l\in T} d(\mathbf{x}_l-\mathbf{y}_l)\left\{\left(\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\right.\right.$$
$$\left.\left.\cdot\max_{\mathbf{t}_v}\left|\det G^{h_v,T_v}_{\beta,L_v}(\mathbf{t}_v)\right|\right)\cdot\sum_{l\in T}||g^{(h_l)}_{\beta,L}(\mathbf{x}_l-\mathbf{y}_l)-g^{(h_l)}_{\beta}(\mathbf{x}_l-\mathbf{y}_l)||\prod_{\substack{l'\in T\\ l'\neq l}}||\, g^{(h_l)}_{\beta,L_{l'}}(\mathbf{x}_{l'}-\mathbf{y}_{l'})\,||\right\}$$

and

$$R^{(5)}_{\beta,L} = \sum_{n\geq 1}|U|^n \sum_{M\geq 1}\sum_{\tau\in\mathcal{T}^*_{h,n}(M)}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=0}}\sum_{T\in\mathbf{T}}\int_{(\beta,\Lambda_\delta)}\prod_{l\in T} d(\mathbf{x}_l-\mathbf{y}_l)\left\{\prod_{l\in T}||g^{(h_l)}_{\beta,L_l}(\mathbf{x}_l-\mathbf{y}_l)||\right.$$
$$\left.\cdot\sum_{w\text{ not e.p.}}\frac{1}{s_w!}\max_{\mathbf{t}_w}|\det G^{h_w,T_w}_{\beta,L}(\mathbf{t}_w)-\det G^{h_w,T_w}_{\beta}(\mathbf{t}_w)|\cdot\left(\prod_{\substack{v\text{ not e.p.}\\ v\neq w}}\frac{1}{s_v!}\max_{\mathbf{t}_v}\left|\det G^{h_v,T_v}_{\beta,L_v}(\mathbf{t}_v)\right|\right)\right\}$$

where, again, L_l and L_v can only assume the values L or $+\infty$. By Poisson's summation formula, $g^{(h)}_{\beta,L}(\mathbf{x}) - g^{(h)}_{\beta}(\mathbf{x}) = \sum_{\vec{n}\neq\vec{0}} g^{(h)}_\beta(x_0, \vec{x}+\vec{a}_1 n_1 L+\vec{a}_2 n_2 L)$, with $\vec{a}_{1,2}$ the two basis' vectors of the triangular lattice $\mathbb{B}$; therefore,

$$\int_{(\beta,\Lambda_\delta)} d(\mathbf{x}_l-\mathbf{y}_l)||g^{(h_l)}_{\beta,L}(\mathbf{x}_l-\mathbf{y}_l)-g^{(h_l)}_{\beta}(\mathbf{x}_l-\mathbf{y}_l)|| \leq \frac{C_K\gamma^{-h_l}}{L^K}, \qquad \text{(D.11)}$$

which, combined with the same bounds leading from (C.17) to (C.20), implies that $R^{(4)}_{\beta,L} \le C_K |U| L^{-K}$. Now, in order to get a bound on $R^{(5)}_{\beta,L}$, if G^{h_w,T_w} is an $s \times s$ matrix, we rewrite

$$\begin{aligned} &\det G^{h_w,T_w}_{\beta,L}(\mathbf{t}_w) - \det G^{h_w,T_w}_{\beta}(\mathbf{t}_w) \\ &\quad = \sum_{\mathbf{p}} (-1)^{\mathbf{p}} \left[(g^{(h)}_{\beta,L})_{1,p(1)} \cdots (g^{(h)}_{\beta,L})_{s,p(s)} - (g^{(h)}_{\beta})_{1,p(1)} \cdots (g^{(h)}_{\beta})_{s,p(s)} \right] \\ &\quad = \sum_{\mathbf{p}} (-1)^{\mathbf{p}} \sum_{j=1}^{s} \cdot \cdot \left((g^{(h)}_{\beta,L})_{1,p(1)} \cdots (g^{(h)}_{\beta,L})_{j-1,p(j-1)} \right) \\ &\qquad \left((g^{(h)}_{\beta,L})_{j,p(j)} - (g^{(h)}_{\beta})_{j,p(j)} \right) \left((g^{(h)}_{\beta})_{j+1,p(j+1)} \cdots (g^{(h)}_{\beta})_{s,p(s)} \right), \end{aligned} \tag{D.12}$$

where $\mathbf{p} = (p(1), \dots, p(s))$ is a permutation of the indices in the (unordered) set $J = \{1, \dots, s\}$. We rewrite the two sums over $\mathbf{p}$ and j in the following way:

$$\sum_{\mathbf{p}} \sum_{j=1}^{s} = \sum_{j=1}^{s} \sum_{k=1}^{s} \sum_{J_1,J_2}^{*} \sum_{\mathbf{p}}^{**}, \tag{D.13}$$

where the $*$ on the second sum means that the (unordered) sets J_1 and J_2 are s.t. (J_1, J_2) is a partition of $J \setminus \{k\}$; the $**$ on the third sum means that $p(1), \dots, p(j-1)$ belong to J_1, $p(j) = k$ and $p(j+1), \dots, p(s)$ belong to J_2. Using (D.13), we rewrite (D.12) as

$$\begin{aligned} \det G^{h_w,T_w}_{\beta,L}(\mathbf{t}_w) - \det G^{h_w,T_w}_{\beta}(\mathbf{t}_w) &= \sum_{j=1}^{s} \sum_{k=1}^{s} \left((g^{(h)}_{\beta,L})_{j,k} - (g^{(h)}_{\beta})_{j,k} \right) \\ \cdot \sum_{J_1,J_2}^{*} (-1)^{\pi} \sum_{\mathbf{p}_1,\mathbf{p}_2} (-1)^{\mathbf{p}_1+\mathbf{p}_2} &\left(\prod_{i \in J_1} (g^{(h)}_{\beta,L})_{i,p_1(i)} \right) \left(\prod_{i' \in J_2} (g^{(h)}_{\beta})_{i',p_2(i')} \right), \end{aligned} \tag{D.14}$$

where: $(-1)^{\pi}$ is the sign of the permutation leading from the ordering $(1, \dots, s)$ to the ordering $(f, \bar{J}_1, \bar{J}_2)$, with $\bar{J}_i$ a fixed (arbitrary) reordering of J_i; $\mathbf{p}_i$, $i = 1, 2$ is a permutation of $\bar{J}_i$ and $(-1)^{\mathbf{p}_i}$ is its sign. In conclusion, using the obvious notation,

$$\begin{aligned} &\det G^{h_w,T_w}_{\beta,L}(\mathbf{t}_w) - \det G^{h_w,T_w}_{\beta}(\mathbf{t}_w) \\ &= \sum_{j=1}^{s} \sum_{k=1}^{s} \left((g^{(h)}_{\beta,L})_{j,k} - (g^{(h)}_{\beta})_{j,k} \right) \sum_{J_1,J_2}^{*} (-1)^{\pi} \det G^{h_w}_{\beta,L}(J_1) \cdot \det G^{h_w}_{\beta}(J_2), \end{aligned} \tag{D.15}$$

where $G^{h_w}_{\beta,L}(J_1)$ and $G^{h_w}_{\beta,L}(J_1)$ are two Gram matrices. Note that the number of terms in the sum $\sum^{*}_{J_1.J_2}$ is equal to 2^s. By the Gram-Hadamard inequality and Poisson's summation formula, we get

$$\max_{\mathbf{t}_w} \left| \det G^{h_w,T_w}_{\beta,L}(\mathbf{t}_w) - \det G^{h_w,T_w}_{\beta}(\mathbf{t}_w) \right| \le \frac{c^s_K}{L^K}, \tag{D.16}$$

which, combined with the same bounds leading from (C.17) to (C.20), implies the desired bound, $R^{(5)}_{\beta,L} \le C_K |U| L^{-K}$.

We now need to prove that $\sum_{h=h_\beta}^{0}\big(|\bar e_{h,\beta,L}-\bar e_{h,\beta}|+|e_{h,\beta,L}-e_{h,\beta}|\big)\le C|U|L^{-K}$. The quantity $|\bar e_{h,\beta,L}-\bar e_{h,\beta}|$ can be bounded by following a strategy completely analogous to the one used to bound $|F_{0,\beta,L}-F_{0,\beta}|$, the only difference being that now the trees involved in the expansions are the infrared ones (with root on scale h and highest scale ≤ 1); therefore, the analogue of (D.10) is changed into

$$\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\int\prod_{l\in T_v}d(\mathbf{x}_l-\mathbf{y}_l)\,||g^{(h_v)}_{\omega_l,\beta,L_l}(\mathbf{x}_l-\mathbf{y}_l)|| \\ \le\frac{c_K^n}{1+(\gamma^h\delta L)^K}\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\gamma^{-h_v(s_v-1)}\ ; \qquad \text{(D.17)}$$

the analogue of (D.11) is changed into

$$\int_{(\beta,\Lambda_\delta)}d(\mathbf{x}_l-\mathbf{y}_l)||g^{(h_l)}_{\omega_l,\beta,L}(\mathbf{x}_l-\mathbf{y}_l)-g^{(h_l)}_{\omega_l,\beta}(\mathbf{x}_l-\mathbf{y}_l)||\le\frac{C_K\gamma^{-h_l}}{(\gamma^{h_l}L)^K}\ ; \qquad \text{(D.18)}$$

the analogue of (D.16) is changed into

$$\max_{\mathbf{t}_w}\big|\det G^{h_w,T_w}_{\beta,L}(\mathbf{t}_w)-\det G^{h_w,T_w}_{\beta}(\mathbf{t}_w)|\le\frac{c_K^s\gamma^{2h_w s}}{(\gamma^{h_w}L)^K}. \qquad \text{(D.19)}$$

These estimates imply that, for any $0\le\theta<1$ and any $K>0$, $|\bar e_{h,\beta,L}-\bar e_{h,\beta}|\le C_{K,\theta}|U|\gamma^{(3+\theta)h}(\gamma^hL)^{-K}$; therefore, for any $K<3+\theta$, we get the desired bound, $\sum_{h\ge h_\beta}|\bar e_{h,\beta,L}-\bar e_{h,\beta}|\le C_K|U|L^{-K}$. A similar estimate is valid for $\sum_{h\ge h_\beta}|e_{h,\beta,L}-e_{h,\beta}|$, but we will not belabor the details here. This concludes the proof of the first claim of Corollary 1, concerning the thermodynamic limit of the free energy.

We are now left with discussing the zero temperature limit $\lim_{\beta\to\infty}f_\beta(U)$. More precisely, we need to prove that, for any $\beta'>\beta$ and $0<K<4$,

$$|F_{0,\beta}-F_{0,\beta'}|+\left|\sum_{h=h_\beta}^{0}(e_{h,\beta}+\bar e_{h,\beta})-\sum_{h=h_{\beta'}}^{0}(e_{h,\beta'}+\bar e_{h,\beta'})\right|\le\frac{C_K|U|}{\beta^K}. \qquad \text{(D.20)}$$

If we follow step by step the discussion above, leading to the estimate $|F_{0,\beta,L}-F_{0,\beta}|\le C_K|U|L^{-K}$, we find that, similarly, $|F_{0,\beta}-F_{0,\beta'}|\le C_K|U|\beta^{-K}$, $K>0$; the proof of this bound is based on a decomposition of the difference $F_{0,\beta}-F_{0,\beta'}$ into a sum of terms involving either integrals over constrained regions (such that $\sum_{l\in T}|\mathbf{x}_l-\mathbf{y}_l|\ge\delta\beta$) or differences of propagators $|g^{(h)}_\beta(\mathbf{x})-g^{(h)}(\mathbf{x})|\le\sum_{n\ne0}|g^{(h)}(x_0+n\beta,\vec x)|\le C_K(\gamma^h\beta)^{-K}$; the technical details are similar to those discussed above for the thermodynamic limit and will not be repeated here.

Let us now consider the difference $\sum_{h=h_\beta}^{0}\bar e_{h,\beta}-\sum_{h=h_{\beta'}}^{0}\bar e_{h,\beta'}$: its absolute value can be bounded from above by $\sum_{h_{\beta'}\le h<h_\beta}|\bar e_{h,\beta'}|+\sum_{h_\beta\le h\le0}|\bar e_{h,\beta}-\bar e_{h,\beta'}|$. The terms in the latter sum admit bounds similar to those obtained above for $|\bar e_{h,\beta,L}-\bar e_{h,\beta}|$, leading to $|\bar e_{h,\beta}-\bar e_{h,\beta'}|\le C_{K,\theta}|U|\gamma^{(3+\theta)h}(\gamma^h\beta)^{-K}$; therefore, for any $K<3+\theta$, we get $\sum_{h\ge h_\beta}|\bar e_{h,\beta}-\bar e_{h,\beta'}|\le C_K|U|\beta^{-K}$, as desired. Finally, $\sum_{h_{\beta'}\le h<h_\beta}|\bar e_{h,\beta'}|\le\sum_{h<h_\beta}c|U|\gamma^{(3+\theta)h}\le C|U|\gamma^{(3+\theta)h_\beta}\le C'|U|\beta^{-3-\theta}$, as desired. A similar discussion

implies the same bound for $e_{h,\beta}$ but we will not belabor the details here. This concludes the proof of Corollary 1. □

Finally, let us consider the two-point Schwinger function $\mathcal{S}(\mathbf{x}) \equiv \mathcal{S}_2^{M,\beta,\Lambda}(\mathbf{x}, \sigma, -; \mathbf{0}, \sigma, +)$. We recall that $\mathcal{S}(\mathbf{x})$ can be expressed in terms of the kernels $W_2^{(j)}$ of the effective potential and of the propagators $g_\omega^{(j)}$, see (3.120)–(3.121) and (3.112)–(3.113). Therefore, for any fixed $\mathbf{k}$, convergence of $\hat{\mathcal{S}}(\mathbf{k})$ follows from the uniform convergence of $W_2^{(j)}(\mathbf{x})$ and of $g_\omega^{(j)}(\mathbf{x})$. Note that the uniform convergence of $g_\omega^{(j)}$ was already proven above, in the discussion on the convergence of the free energy. Moreover, the convergence of the kernels of the effective potential can be proven by expressing $W_2^{(j)}$ in a way analogous to the r.h.s. of (D.5), then by decomposing the integral of $W_{\tau,\mathbf{P},T,\beta,L}^{(j)}$ in a way analogous to (D.9), and finally by bounding the analogues of $R^{(1)}$, $R^{(2)}$, $R^{(3)}$ in the same way explained above for the free energy. The n-point Schwinger functions can be treated in a similar way, but we will not belabour the details here.

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Communicated by M. Salmhofer