Ann. Henri Poincaré 11 (2010), 1409–1452

1424-0637/10/081409-44
published online December 16, 2010
DOI 10.1007/s00023-010-0068-x

Annales Henri Poincaré

Anomalous Behavior in an Effective Model of Graphene with Coulomb Interactions

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Abstract. We analyze by exact Renormalization Group (RG) methods the infrared properties of an effective model of graphene, in which two-dimensional (2D) massless Dirac fermions propagating with a velocity smaller than the speed of light interact with a 3D quantum electromagnetic field. The fermionic correlation functions are written as series in the running coupling constants, with finite coefficients that admit explicit bounds at all orders. The implementation of Ward Identities in the RG scheme implies that the effective charges tend to a line of fixed points. At small momenta, the quasi-particle weight tends to zero and the effective Fermi velocity tends to a finite value. These limits are approached with a power law behavior characterized by non-universal critical exponents.

1. Introduction and Main Result

The charge carriers in graphene at half filling are effectively described by massless Dirac fermions constrained to move on a two-dimensional (2D) manifold embedded in three-dimensional (3D) space [10], with a Fermi velocity v that is approximately 300 times smaller than the speed of light. As a consequence, already without taking into account the interactions, the system displays highly unusual features as compared to standard 2D electron gases, such as an anomalous integer quantum Hall effect and the insensitivity to disordered-induced localization; most of these effects have already been experimentally observed [30,31]. The study of many-body interactions among the charge carriers in graphene is of course very important, particularly in view of recent experiments that suggest their relevant role in several physical properties of graphene [9,25,27,37].

The effect of a weak *short* range interaction in graphene is quite well understood: it turns out that the behavior of the ground state is qualitatively similar to the free one, except that the Fermi velocity and the wave function renormalization are renormalized by a finite amount. This was expected on the basis of a power counting analysis [22–24]; recently, it has been rigorously proven in [18,19], where the *convergence* of the perturbative series was established, using the methods of constructive Quantum Field Theory (QFT) and by taking into full account the lattice effects (i.e., by considering the Hubbard model on the honeycomb lattice).

The situation in the presence of *long* range interactions is much more subtle and still not completely understood. Their effect in graphene is usually studied in terms of a model of Dirac fermions interacting via a *static* Coulomb potential; retardation effects of the electromagnetic (e.m.) field are neglected because the free Fermi velocity v is 300 times smaller than the speed of light c. At weak coupling, a *logarithmic divergence* of the effective Fermi velocity $v(\mathbf{k})$ at the Fermi points $\mathbf{p}_F^{\pm}$ and a finite quasi-particles weight have been predicted, on the basis of one-loop [21] and two-loop [29] computations. An unbounded growth of the effective Fermi velocity was also confirmed by an analysis based on a large-N expansion [26,36], which predicted a *power law* divergence of $v(\mathbf{k})$ at $\mathbf{p}_F^{\pm}$. It is not clear how to reconcile the logarithmic divergence expected from two-loop perturbative computations with the power law behavior found by large-N expansions; moreover, the description in terms of Dirac fermions introduces spurious ultraviolet divergences that can produce ambiguities in the physical predictions [24,29]. These difficulties may be related to a basic inadequacy of the effective model of Dirac fermions with static Coulomb interactions: the fact that the effective Fermi velocity diverges at the Fermi points (as predicted by all the analyses of the model) signals that its physical validity *breaks down* at the infrared scale where $v(\mathbf{k})$ becomes comparable with the speed of light; at lower scales, retardation effects must be taken into account, as first proposed in [20], where a model of massless Dirac fermions propagating with speed $v \ll c$ and interacting with an e.m. field was considered. In [20] it was found that, at small momenta, the wave function renormalization diverges as a power law; this implies that the ground state correlations have an anomalous decay at large distances. Moreover, it was found that the interacting Fermi velocity increases up to the speed of light, again with an anomalous power law behavior. Despite its interest, the model proposed in [20] has not been considered further. The results in [20] were found on the basis of one-loop computations and in the presence of an ultraviolet dimensional regularization scheme. It is interesting to investigate whether the predictions of [20] remain valid even if higher orders corrections are taken into account and in the presence of different regularization schemes closer to the lattice cut-off that is truly present in actual graphene.

The model we consider describes massless Dirac fermions in $2+1$ dimensions propagating with velocity $v < c$, and interacting with a $3+1$ dimensional photon field in the Feynman gauge. We will not be concerned with the instantaneous case ($c \to \infty$); therefore, from now on, for notational simplicity, we shall fix units such that $\hbar = c = 1$. The model is very similar to the one in [20], the main difference being the choice of the ultraviolet cut-off: rather than considering dimensional regularization, in order to mimic the presence of an underlying lattice, we explicitly introduce a (fixed) ultraviolet momentum cut-off both in the electronic and photonic propagators. The correlations can be computed in terms of derivatives of the following Euclidean functional integral:

$$e^{\mathcal{W}(J,\phi)} = \int P(\mathrm{d}\psi)P(\mathrm{d}A)e^{V(A,\psi)+B(J,\phi)} \tag{1.1}$$

with, setting $\mathbf{x} = (x_0, \vec{x})$ and $\vec{x} = (x_1, x_2)$ (repeated indexes are summed; Greek and Latin labels run, respectively, from 0 to 2, 1 to 2),

$$\begin{aligned} V(A,\psi) &:= \int_\Lambda \mathrm{d}\mathbf{x}\ [e\, j_{\mu,\mathbf{x}} A_{\mu,\mathbf{x}} - \nu_\mu A_{\mu,\mathbf{x}} A_{\mu,\mathbf{x}}] \\ B(J,\phi) &:= \int_\Lambda \mathrm{d}\mathbf{x}\ \left[j_{\mu,\mathbf{x}} J_{\mu,\mathbf{x}} + \phi_\mathbf{x} \overline{\psi}_\mathbf{x} + \overline{\phi}_\mathbf{x} \psi_\mathbf{x}\right], \end{aligned} \tag{1.2}$$

where Λ is a 3D box of volume $|\Lambda| = L^3$ with periodic boundary conditions (playing the role of an infrared cutoff, to be eventually removed), the couplings e, ν_μ are real and $\nu_1 = \nu_2$; the couplings ν_μ are *counterterms* to be fixed so that the photon mass is vanishing in the deep infrared. Moreover, $\overline{\psi}_\mathbf{x}$, $\psi_\mathbf{x}$ are four-component *Grassmann spinors*, and the μ-th component $j_{\mu,\mathbf{x}}$ of the current is defined as:

$$j_{0,\mathbf{x}} = i\overline{\psi}_\mathbf{x}\gamma_0\psi_\mathbf{x}, \quad \vec{j}_\mathbf{x} = iv\,\overline{\psi}_\mathbf{x}\vec{\gamma}\psi_\mathbf{x}, \tag{1.3}$$

where γ_μ are euclidean gamma matrices, satisfying the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu,\nu}$. The symbol $P(\mathrm{d}\psi)$ denotes a Grassmann integration with propagator

$$g^{(\le 0)}(\mathbf{x}) := \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\mathbf{x}} \frac{ik_0\gamma_0 + iv\vec{k}\cdot\vec{\gamma}}{k_0^2 + v^2|\vec{k}|^2}\chi_0(\mathbf{k}). \tag{1.4}$$

where $(2\pi)^{-3}\int \mathrm{d}\mathbf{k}$ is a shorthand for $|\Lambda|^{-1}\sum_{\mathbf{k}=2\pi\mathbf{n}/L}$ with $\mathbf{n} \in \mathbb{Z}^3$, and $\chi_0(\mathbf{k}) = \chi(|\mathbf{k}|)$ plays the role of a prefixed ultraviolet cutoff (here $\chi(t)$ is a non-increasing C^∞ function from $\mathbb{R}^+$ to $[0,1]$ such that $\chi(t) = 1$ if $t \le 1$ and $\chi(t) = 0$ if $t \ge M > 1$). Finally, $A_{\mu,\mathbf{x}}$ are gaussian variables and $P(\mathrm{d}A)$ is a gaussian integration with propagator

$$w^{(\le 0)}(\mathbf{x}) := \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\, e^{i\mathbf{p}\mathbf{x}} \frac{\chi_0(\mathbf{p})}{2|\mathbf{p}|} = \int \frac{\mathrm{d}\mathbf{p}\,\mathrm{d}p_3}{(2\pi)^4}\, e^{i\mathbf{p}\mathbf{x}} \frac{\chi_0(\mathbf{p})}{\mathbf{p}^2 + p_3^2}. \tag{1.5}$$

We perform an analysis based on the methods of *constructive Renormalization Group* (RG) for non-relativistic fermions, introduced in [3,14] (see [4,28,33,34] for updated introductions), which have already been proved effective in the study of several low-dimensional critical systems, such as one-dimensional (1D) interacting fermions [3,5,6], 2D critical Ising and vertex models [2,17], the 2D Hubbard model on the square lattice at positive temperatures [7,11,12], interacting fermions with asymmetric Fermi surfaces [13] and the 2D Hubbard model on the honeycomb lattice [18,19], just to mention a few. Compared to other RG approaches, such as those in [32,35], the advantage of the constructive methods we adopt is that they allow us to get a rigorous and complete treatment of the effects of the cut-offs and a full control on the perturbative expansion via explicit bounds at all orders; quite remarkably, in certain cases, such as the ones treated in [6,11,12,7,18,19], these methods even provide a way to prove the *convergence* of the resummed perturbation theory.

Using these methods, we construct a renormalized expansion, allowing us to express the Schwinger functions, from which the physical observables can be computed, as series in the effective couplings (the effective charges and the effective photon masses, also called in the following the *running coupling constants*), with *finite* coefficients at all orders, admitting explicit $N!$ bounds (see Theorem 2.1 in Sect. 2 below). If the effective couplings remain small in the infrared, informations obtained from our expansion by lowest order truncations are reliable at weak coupling. The importance of having an expansion with finite coefficients should not be underestimated; the naive perturbative expansion in the fine structure constant is plagued by *logarithmic infrared divergences* and higher orders are more and more divergent.

Of course the renormalized expansion is useful only as long as the running coupling constants are small. In fact, we do prove that they remain small for all infrared scales, by implementing Ward Identities (WIs) in the RG flow, using a technique developed in [6] for the rigorous analysis of Luttinger liquids in situations where bosonization cannot be applied (e.g., in the presence of an underlying lattice and/or of non-linear bands). The WIs that we use are based on an approximate local gauge invariance, the exact gauge symmetry being broken by the ultraviolet cut-off; its presence produces corrections to the "naive" (formal) WIs, which can be resummed and, again, explicitly bounded at all orders. The resulting modified WIs imply that the effective charges tend to a line of fixed points, exactly as in 1D Luttinger liquids. We note that this is one of the very few examples in which Luttinger liquid behavior is found in dimensions higher than 1.

Let us denote by $\langle \ldots \rangle = \lim_{|\Lambda|\to\infty} \langle \ldots \rangle_\Lambda$ the expectation value with respect to the interaction (1.2) in the infinite volume limit; our main result can be informally stated as follows (more rigorous statements will be found below).

Main result. *There exists a choice of ν_μ such that, for $\mathbf{k}$ small,*

$$\langle \psi_{\mathbf{k}} \overline{\psi}_{\mathbf{k}} \rangle = \frac{1}{Z(\mathbf{k})} \frac{ik_0\gamma_0 + iv(\mathbf{k})\vec{k}\cdot\vec{\gamma}}{k_0^2 + v(\mathbf{k})^2|\vec{k}|^2}(1 + B(\mathbf{k})), \tag{1.6}$$

where

$$Z(\mathbf{k}) \sim |\mathbf{k}|^{-\eta}, \quad v_{\text{eff}} - v(\mathbf{k}) \sim (v_{\text{eff}} - v)|\mathbf{k}|^{\tilde{\eta}}, \tag{1.7}$$

$B(\mathbf{k}), \nu_\mu, \eta, \tilde{\eta}, v_{\text{eff}}$ are expressed by series in the effective couplings with finite coefficients that admit $N!$-bounds at all orders. Moreover: (i) *the first non-trivial contribution to $B(\mathbf{k})$ is of second order in e*; (ii) *the first non-trivial contribution to ν_μ is of second order in e and positive*; (iii) *the first non-trivial contributions to $\eta, \tilde{\eta}, v_{\text{eff}}$ are, respectively*:

$$\eta^{(2)} = \frac{e^2}{12\pi^2}, \quad \tilde{\eta}^{(2)} = \frac{2e^2}{5\pi^2}, \quad v_{\text{eff}}^{(2)} = 1 - F(v)\frac{e^2}{6\pi^2}, \tag{1.8}$$

with

$$F(v) = \frac{5}{8}\left[\left(\frac{1}{2v^2} - 2\right)\frac{\xi_0 - \arctan \xi_0}{\xi_0^3} + \frac{1}{2v^2}\frac{\arctan \xi_0}{\xi_0}\right], \quad \xi_0 := \frac{\sqrt{1-v^2}}{v}. \tag{1.9}$$

Note that the theory is not Lorentz invariant, because $v \neq c$; moreover, gauge symmetry is broken by the presence of the ultraviolet momentum cut-off. These two facts produce unusual features as compared to standard QFT models. In particular, the momentum cut-off produces correction terms in the WIs, which can be rigorously bounded at all orders; despite these corrections, one can still use the WIs to prove that the beta function for the effective charges is asymptotically vanishing at all orders, so that the model admits a line of (non-trivial) fixed points.

The lack of gauge invariance due to the ultraviolet momentum cut-off makes it necessary (as in [8]) to introduce positive counterterms to keep the photon mass equal to zero. Similarly, it implies that the effective couplings with the temporal and spatial components of the gauge field are different and that the effective Fermi velocity v_{eff} is not equal to the speed of light. However, it is possible to introduce in the bare interaction two different charges, e_0 and e_1, describing the couplings of the photon field with the temporal and spatial components of the current, which can be tuned so that the dressed charges are equal and $v_{\text{eff}} = 1$.

A more realistic model for single layer graphene could be obtained by considering tight binding electrons hopping on the honeycomb lattice, whose lattice currents are coupled to a 3D photon field. A Renormalization Group analysis similar to the one in the present paper could be repeated for the lattice model, by extending the formalism in [18,19]. If the lattice model is chosen in such a way that lattice gauge invariance is preserved, we expect that its photon mass counterterms are exactly zero and that its effective Fermi velocity is equal to the speed of light. In any case (i.e., both in the presence or in the absence of lattice gauge invariance), we expect the lattice model to have the same infrared asymptotic behavior of the continuum model considered here, provided that the bare parameters e_μ, ν_μ of the continuum model are properly tuned.

Finally, let us comment about the possibility of providing a full *non-perturbative* construction of the ground state of the present model or, possibly, of a more realistic model of tight binding electrons hopping on the honeycomb lattice and interacting with e.m. forces. In this paper, we express the physical observables in terms of series in the running coupling constants (with bounded coefficients at all orders) and we show that the running coupling constants remain close to their initial value, thanks to Ward Identities and cancellations in the beta function. Thus, the usual problem that so far prevented the non-perturbative construction of the ground state of systems of interacting fermions in $d > 1$ with convex symmetric Fermi surface (namely, the presence of a beta function driving the infrared flow for the effective couplings out of the weak coupling regime) is absent in the present case. Therefore, a full non-perturbative construction of the ground state of the present model appears to be feasible, using determinant bounds for the fermionic sector and cluster expansion techniques for the bosonic sector. Of course, the construction is expected to be much more difficult than the one in [6] or [18,19], due to the simultaneous presence of bosons and fermions; if one succeeded in providing it, it would represent the first rigorous example of anomalous Luttinger-liquid behavior in more than one dimension.

The paper is organized as follows: in Sect. 2 we describe how to evaluate the functional integrals defining the partition function and the correlations of our model in terms of an exact RG scheme (details are discussed in Appendices A and B); in Sect. 3 we describe the infrared flow of the effective couplings and prove the emergence of an effective Fermi velocity different from the speed of light (the explicit lowest order computations of the beta function are presented in Appendix C); in Sect. 4 we derive the Ward Identity allowing us to control the flow of the effective charges (details are discussed in Appendix D) and proving that the beta function for the charges is asymptotically vanishing; finally, in Sect. 5 we draw the conclusions.

2. Renormalization Group Analysis

2.1. The Effective Potential

In this section we show how to evaluate the functional integral (1.1); the integration will be performed in an iterative way, starting from the momenta "close" to the ultraviolet cutoff moving towards smaller momentum scales. At the n-th step of the iteration the functional integral (1.1) is rewritten as an integral involving only the momenta smaller than a certain value, proportional to M^{-n}, where $M > 1$ is the same constant (to be chosen sufficiently close to 1) appearing in the definition of the cut-off function (see lines after (1.4)), and both the propagators and the interaction will be replaced by "effective" ones; they differ from their "bare" counterparts because the physical parameters appearing in their definitions (the Fermi velocity v, the charge e, and the "photon mass" ν_μ) are *renormalized* by the integration of the momenta on higher scales. In the following, it will be convenient to introduce the *scale label* $h \leq 0$ as $h := -n$.

Setting $\chi_h(\mathbf{k}) := \chi(M^{-h}|\mathbf{k}|)$, we start from the following identity:

$$\chi_0(\mathbf{k}) = \sum_{h=-\infty}^{0} f_h(\mathbf{k}), \quad f_h(\mathbf{k}) := \chi_h(\mathbf{k}) - \chi_{h-1}(\mathbf{k}); \tag{2.1}$$

let $\psi = \sum_{h=-\infty}^{0} \psi^{(h)}$ and $A = \sum_{h=-\infty}^{0} A^{(h)}$, where $\{\psi^{(h)}\}_{h\leq 0}$, $\{A^{(h)}\}_{h\leq 0}$ are independent free fields with the same support of the functions f_h introduced above.

We evaluate the functional integral (1.1) by integrating the fields in an iterative way starting from $\psi^{(0)}$, $A^{(0)}$; for simplicity, we start by treating the case $J = \phi = 0$. We define $\mathcal{V}^{(0)}(A,\psi) := V(A,\psi)$ and we want to inductively prove that after the integration of $\psi^{(0)}, A^{(0)}, \ldots, \psi^{(h+1)}, A^{(h+1)}$ we can rewrite:

$$e^{\mathcal{W}(0,0)} = e^{|\Lambda|E_h} \int P(\mathrm{d}\psi^{(\le h)})P(\mathrm{d}A^{(\le h)})e^{\mathcal{V}^{(h)}(A^{(\le h)},\sqrt{Z_h}\psi^{(\le h)})}, \tag{2.2}$$

where $P(d\psi^{(\le h)})$ and $P(dA^{(\le h)})$ have propagators

$$g^{(\le h)}(\mathbf{k}) = \frac{\chi_h(\mathbf{k})}{\tilde{Z}_h(\mathbf{k})}\frac{i\gamma_0 k_0 + i\tilde{v}_h(\mathbf{k})\vec{k}\cdot\vec{\gamma}}{k_0^2 + \tilde{v}_h(\mathbf{k})^2|\vec{k}|^2}, \quad w^{(\le h)}(\mathbf{p}) = \frac{\chi_h(\mathbf{p})}{2|\mathbf{p}|}, \tag{2.3}$$

$\mathcal{V}^{(h)}$ has the form

$$\begin{aligned}\mathcal{V}^{(h)}(A,\psi) = &\sum_{\substack{n,m\ge 0\\ n+m\ge 1}} \sum_{\underline{\rho},\underline{\mu}} \int \left[\prod_{i=1}^{2n}\frac{\mathrm{d}\mathbf{k}_i}{(2\pi)^3}\right]\left[\prod_{j=1}^{m}\frac{\mathrm{d}\mathbf{p}_j}{(2\pi)^3}\right]\prod_{i=1}^{n}\overline{\psi}_{\mathbf{k}_{2i-1},\rho_{2i-1}}\psi_{\mathbf{k}_{2i},\rho_{2i}},\\ &\times\prod_{i=1}^{m}A_{\mu_i,\mathbf{p}_i}W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}(\{\mathbf{k}_i\},\{\mathbf{p}_j\})\delta\left(\sum_{j=1}^{m}\mathbf{p}_j + \sum_{i=1}^{2n}(-1)^i\mathbf{k}_i\right),\end{aligned} \tag{2.4}$$

and E_h, $\tilde{Z}_h(\mathbf{k})$, $\tilde{v}_h(\mathbf{k})$ and the kernels $W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}$ will be defined recursively.

In order to inductively prove (2.2), we split $\mathcal{V}^{(h)}$ as $\mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator on functions of the form (2.4), defined by its action on the kernels $W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}$ in the following way:

$$\begin{aligned}\mathcal{L}W^{(h)}_{0,1,\underline{\rho}}(\mathbf{k}) &:= W^{(h)}_{0,1,\underline{\rho}}(\mathbf{0}) + \mathbf{k}\partial_{\mathbf{k}}W^{(h)}_{0,1,\underline{\rho}}(\mathbf{0}),\\ \mathcal{L}W^{(h)}_{1,1,\underline{\rho},\mu}(\mathbf{p},\mathbf{k}) &:= W^{(h)}_{1,1,\underline{\rho},\mu}(\mathbf{0},\mathbf{0}),\\ \mathcal{L}W^{(h)}_{2,0,\underline{\mu}}(\mathbf{p}) &:= W^{(h)}_{2,0,\underline{\mu}}(\mathbf{0}) + \mathbf{p}\partial_{\mathbf{p}}W^{(h)}_{2,0,\underline{\mu}}(\mathbf{0}),\\ \mathcal{L}W^{(h)}_{3,0,\underline{\mu}}(\mathbf{p}_1,\mathbf{p}_2) &:= W^{(h)}_{3,0,\underline{\mu}}(\mathbf{0},\mathbf{0}),\end{aligned} \tag{2.5}$$

and $\mathcal{L}W^{(h)}_P := 0$ otherwise. As it will be clear from the dimensional analysis performed in Sect. 2.4 below, these are the only terms that need renormalization; in particular, $\mathcal{L}W^{(h)}_{0,1,\underline{\rho}}(\mathbf{k})$ will contribute to the wave function renormalization and to the effective Fermi velocity, $\mathcal{L}W^{(h)}_{2,0,\underline{\mu}}(\mathbf{p})$ to the effective photon mass, and $\mathcal{L}W^{(h)}_{1,1,\underline{\rho},\mu}(\mathbf{p},\mathbf{k})$ to the effective charge.

As a consequence of the symmetries of our model, see Appendix A, it turns out that

$$\begin{aligned}&W^{(h)}_{1,0,\mu}(\mathbf{0}) = 0, \quad W^{(h)}_{3,0,\underline{\mu}}(\mathbf{0},\mathbf{0}) = 0, \quad W^{(h)}_{0,1,\underline{\rho}}(\mathbf{0}) = 0,\\ &\hat{W}^{(h)}_{2,0,\underline{\mu}}(\mathbf{0}) = -\delta_{\mu_1,\mu_2}M^h\nu_{\mu_1,h}, \quad \partial_{\mathbf{p}}\hat{W}^{(h)}_{2,0,\underline{\mu}}(\mathbf{0}) = 0\end{aligned} \tag{2.6}$$

and, moreover, that

$$\begin{aligned}\overline{\psi}_{\mathbf{k}}\,\mathbf{k}\partial_{\mathbf{k}}W^{(h)}_{0,1}(\mathbf{0})\psi_{\mathbf{k}} &= -iz_{\mu,h}k_\mu\overline{\psi}_{\mathbf{k}}\gamma_\mu\psi_{\mathbf{k}}\\ \overline{\psi}_{\mathbf{k}+\mathbf{p}}W^{(h)}_{1,1,\mu}(\mathbf{0},\mathbf{0})\psi_{\mathbf{k}}A_{\mu,\mathbf{p}} &= i\lambda_{\mu,h}\overline{\psi}_{\mathbf{k}+\mathbf{p}}\gamma_\mu\psi_{\mathbf{k}}A_{\mu,\mathbf{p}},\end{aligned} \tag{2.7}$$

with $z_{\mu,h}$, $\lambda_{\mu,h}$ real, and $z_{1,h} = z_{2,h}$, $\lambda_{1,h} = \lambda_{2,h}$. We can *renormalize* $P(\mathrm{d}\psi^{(\le h)})$ by adding to the exponent of its gaussian weight the local part of the quadratic terms in the fermionic fields; we get that

$$\begin{aligned}&\int P(\mathrm{d}\psi^{(\le h)})P(\mathrm{d}A^{(\le h)})e^{\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)}\\ &\quad= e^{|\Lambda|t_h}\int \widetilde{P}(\mathrm{d}\psi^{(\le h)})P(\mathrm{d}A^{(\le h)})e^{\widetilde{\mathcal{V}}^{(h)}(A,\sqrt{Z_h}\psi)},\end{aligned} \tag{2.8}$$

where t_h takes into account the different normalization of the two functional integrals, $\widetilde{\mathcal{V}}^{(h)}$ is given by

$$\begin{aligned}\widetilde{\mathcal{V}}^{(h)}(A,\psi) &= \mathcal{V}^{(h)}(A,\psi) + \int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\,iz_{\mu,h}k_\mu\overline{\psi}_{\mathbf{k}}\gamma_\mu\psi_{\mathbf{k}}\\ &=: \mathcal{V}^{(h)}(A,\psi) - \mathcal{L}_\psi\mathcal{V}^{(h)}(A,\psi),\end{aligned} \tag{2.9}$$

and $\widetilde{P}(\mathrm{d}\psi^{(\le h)})$ has propagator equal to

$$\tilde{g}^{(\le h)}(\mathbf{k}) = \frac{\chi_h(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})}\frac{i\gamma_0k_0 + i\tilde{v}_{h-1}(\mathbf{k})\vec{k}\cdot\vec{\gamma}}{k_0^2+\tilde{v}_{h-1}(\mathbf{k})^2|\vec{k}|^2}, \tag{2.10}$$

with

$$\begin{aligned}\tilde{Z}_{h-1}(\mathbf{k}) &= \tilde{Z}_h(\mathbf{k}) + Z_hz_{0,h}\chi_h(\mathbf{k}),\\ \tilde{Z}_{h-1}(\mathbf{k})\tilde{v}_{h-1}(\mathbf{k}) &= \tilde{Z}_h(\mathbf{k})\tilde{v}_h(\mathbf{k}) + Z_hz_{1,h}\chi_h(\mathbf{k}).\end{aligned} \tag{2.11}$$

After this, defining $Z_{h-1} := \tilde{Z}_{h-1}(\mathbf{0})$, we *rescale* the fermionic field so that

$$\widetilde{\mathcal{V}}^{(h)}(A,\sqrt{Z_h}\psi) = \hat{\mathcal{V}}^{(h)}(A,\sqrt{Z_{h-1}}\psi); \tag{2.12}$$

therefore, setting

$$v_{h-1} := \tilde{v}_{h-1}(\mathbf{0}),\quad e_{0,h} := \frac{Z_h}{Z_{h-1}}\lambda_{0,h},\quad e_{1,h}v_{h-1} = e_{2,h}v_{h-1} := \frac{Z_h}{Z_{h-1}}\lambda_{1,h}, \tag{2.13}$$

we have that:

$$\begin{aligned}&\mathcal{L}\hat{\mathcal{V}}^{(h)}(A^{(\le h)},\sqrt{Z_{h-1}}\,\psi^{(\le h)})\\ &\quad= \int_\Lambda \mathrm{d}\mathbf{x}\,\Big(Z_{h-1}e_{\mu,h}j^{(\le h)}_{\mu,\mathbf{x}}A^{(\le h)}_{\mu,\mathbf{x}} - M^h\nu_{\mu,h}A^{(\le h)}_{\mu,\mathbf{x}}A^{(\le h)}_{\mu,\mathbf{x}}\Big),\end{aligned} \tag{2.14}$$

where

$$j^{(\le h)}_{0,\mathbf{x}} := i\overline{\psi}^{(\le h)}_{\mathbf{x}}\gamma_0\psi^{(\le h)}_{\mathbf{x}},\quad \vec{j}^{(\le h)}_{\mathbf{x}} := iv_{h-1}\,\overline{\psi}^{(\le h)}_{\mathbf{x}}\vec{\gamma}\psi^{(\le h)}_{\mathbf{x}}. \tag{2.15}$$

After this rescaling, we can rewrite (2.8) as

$$\int P(\mathrm{d}\psi^{(\le h)})P(\mathrm{d}A^{(\le h)})e^{\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)} = e^{|\Lambda|t_h}\int P(\mathrm{d}\psi^{(\le h-1)})P(\mathrm{d}A^{(\le h-1)})$$
$$\times\int P(\mathrm{d}\psi^{(h)})P(\mathrm{d}A^{(h)})e^{\hat{\mathcal{V}}^{(h)}(A^{(\le h-1)}+A^{(h)},\sqrt{Z_{h-1}}(\psi^{(\le h-1)}+\psi^{(h)}))}, \qquad (2.16)$$

where $\psi^{(\le h-1)}, A^{(\le h-1)}$ have propagators given by (2.3) (with h replaced by $h-1$) and $\psi^{(h)}, A^{(h)}$ have propagators given by

$$\frac{g^{(h)}(\mathbf{k})}{Z_{h-1}} = \frac{\tilde f_h(\mathbf{k})}{Z_{h-1}}\frac{i\gamma_0 k_0 + i\tilde v_{h-1}(\mathbf{k})\vec k\cdot\vec\gamma}{k_0^2+\tilde v_{h-1}(\mathbf{k})^2|\vec k|^2}, \quad w^{(h)}(\mathbf{p}) = \frac{f_h(\mathbf{p})}{2|\mathbf{p}|},$$
$$f_h(\mathbf{k}) = \chi_h(\mathbf{k})-\chi_{h-1}(\mathbf{k}), \quad \tilde f_h(\mathbf{k}) = \frac{Z_{h-1}}{\tilde Z_{h-1}(\mathbf{k})}f_h(\mathbf{k}). \qquad (2.17)$$

At this point, we can integrate the scale h and, defining

$$e^{\mathcal{V}^{(h-1)}(A,\sqrt{Z_{h-1}}\psi)+|\Lambda|\tilde E_h}$$
$$:= \int P(\mathrm{d}\psi^{(h)})P(\mathrm{d}A^{(h)})e^{\hat{\mathcal{V}}^{(h)}(A+A^{(h)},\sqrt{Z_{h-1}}(\psi+\psi^{(h)}))}, \qquad (2.18)$$

our inductive assumption (2.2) is reproduced at the scale $h-1$ with $E_{h-1} := E_h + t_h + \tilde E_h$. Notice that (2.18) can be seen as a recursion relation for the effective potential, since from (2.9), (2.12) it follows that

$$\hat{\mathcal{V}}^{(h)}(A,\sqrt{Z_{h-1}}\psi) = \widetilde{\mathcal{V}}^{(h)}(A,\sqrt{Z_h}\psi)$$
$$= \mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi) - \mathcal{L}_\psi\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi). \qquad (2.19)$$

The integration in (2.18) is performed by expanding in series the exponential in the r.h.s. (which involves interactions of any order in ψ and A, as apparent from (2.4)), and integrating term by term with respect to the gaussian integration $P(\mathrm{d}\psi^{(h)})P(\mathrm{d}A^{(h)})$. This procedure gives rise to an expansion for the effective potentials $\mathcal{V}^{(h)}$ (and to an analogous expansion for the correlations) in terms of the renormalized parameters $\{e_{\mu,k},\nu_{\mu,k},Z_{k-1},v_{k-1}\}_{h<k\le 0}$, which can be conveniently represented as a sum over *Feynman graphs* according to rules that will be explained below. We will call $\{e_{\mu,k},\nu_{\mu,k}\}_{h<k\le 0}$ *effective couplings* or *running coupling constants* while $\{e_{\mu,k}\}_{h<k\le 0}$ are the *effective charges*.

Note that such *renormalized* expansion is significantly different from the power series expansion in the bare couplings e,ν_μ; while the latter is plagued by *logarithmic divergences*, the former is *order by order finite*.

By comparing (1.1) and (1.2) with (2.2), (2.4) and (2.14), we see that the integration of the fields living on momentum scales $\ge M^h$ produces an *effective theory* very similar to the original one, modulo the presence of a new propagator, involving a renormalized velocity v_h and a renormalized wave function Z_h, and the presence of a modified interaction $\mathcal{V}^{(h)}$. The lack of Lorentz symmetry in our model (implied by the fact that $v\neq 1$) has two main effects: (1) the Fermi velocity has a non-trivial flow; (2) the marginal terms in the effective potential are defined in terms of *two* charges, namely $e_{0,h}$ and $e_{1,h} = e_{2,h}$, which are *different*, in general.

2.2. Tree Expansion

The iterative integration procedure described above leads to a representation of the effective potentials in terms of a sum over connected Feynman diagrams, as explained in the following. The key formula, which we start from, is (2.18), which can be rewritten as

$$|\Lambda|\tilde{E}_h + \mathcal{V}^{(h-1)}(A^{(\leq h-1)}, \sqrt{Z_{h-1}}\,\psi^{(\leq h-1)}) \\ = \sum_{n\geq 1} \frac{1}{n!}\mathcal{E}_h^T\left(\hat{\mathcal{V}}^{(h)}(A^{(\leq h)}, \sqrt{Z_{h-1}}\psi^{(\leq h)}); n\right), \tag{2.20}$$

with $\mathcal{E}_h^T$ the truncated expectation on scale h, defined as

$$\mathcal{E}_h^T(X(A^{(h)}, \psi^{(h)}); n) := \frac{\partial^n}{\partial\lambda^n}\log\int P(\mathrm{d}\psi^{(h)})P(\mathrm{d}A^{(h)})e^{\lambda X(A^{(h)},\psi^{(h)})}\Big|_{\lambda=0} \tag{2.21}$$

If X is graphically represented as a vertex with external lines $A^{(h)}$ and $\psi^{(h)}$, the truncated expectation (2.21) can be represented as the sum over the Feynman diagrams obtained by contracting in all possible connected ways the lines exiting from n vertices of type X. Every contraction corresponds to a propagator on scale h, as defined in (2.17). Since $\hat{\mathcal{V}}^{(h)}$ is related to $\mathcal{V}^{(h)}$ by a rescaling and a subtraction, see (2.9) and (2.12), Eq. (2.20) can be iterated until scale 0, and $\mathcal{V}^{(h-1)}$ can be written as a sum over connected Feynman diagrams with lines on all possible scales between h and 0. The iteration of (2.20) induces a natural hierarchical organization of the scale labels of every Feynman diagram, which will be conveniently represented in terms of tree diagrams. In fact, let us rewrite $\hat{\mathcal{V}}^{(h)}$ in the r.h.s. of (2.20) as $\hat{\mathcal{V}}^{(h)}(A, \sqrt{Z_{h-1}}\psi) = \overline{\mathcal{L}}\mathcal{V}^{(h)}(A, \sqrt{Z_h}\psi) + \mathcal{R}\mathcal{V}^{(h)}(A, \sqrt{Z_h}\psi)$, where $\overline{\mathcal{L}} := \mathcal{L} - \mathcal{L}_\psi$, see (2.9). Let us graphically represent $\mathcal{V}^{(h)}$, $\overline{\mathcal{L}}\mathcal{V}^{(h)}$ and $\mathcal{R}\mathcal{V}^{(h)}$ as in the first line of Fig. 1, and let us represent Eq. (2.20) as in the second line of Fig. 1; in the second line, the node on scale h represents the action of $\mathcal{E}_h^T$. Iterating the graphical equation in Fig. 1 up to scale 0, we end up with a representation of $\mathcal{V}^{(h)}$ in terms of a sum over *Gallavotti–Nicolò* trees τ [3,15,16]:

$$\mathcal{V}^{(h)}(A^{(\leq h)}, \sqrt{Z_h}\,\psi^{(\leq h)}) = \sum_{N\geq 1}\sum_{\tau\in\mathcal{T}_{h,N}} \mathcal{V}^{(h)}(\tau), \tag{2.22}$$

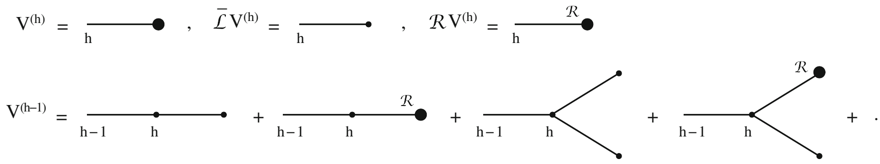

FIGURE 1. Graphical interpretation of Eq. (2.20). The graphical equations for $\overline{\mathcal{L}}\mathcal{V}^{(h-1)}$, $\mathcal{R}\mathcal{V}^{(h-1)}$ are obtained from the equation in the second line by putting an $\overline{\mathcal{L}}$, $\mathcal{R}$ label, respectively, over the vertices on scale h

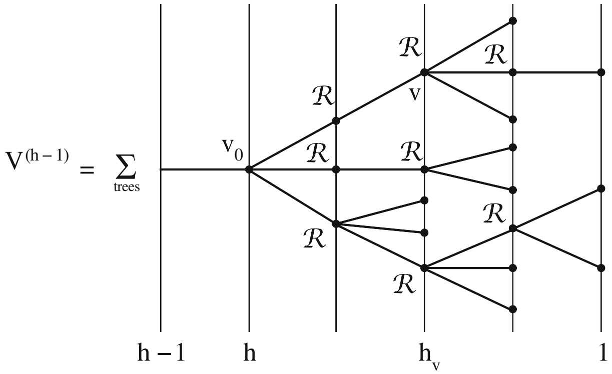

FIGURE 2. The effective potential $\mathcal{V}^{(h-1)}$ can be represented as a sum over *Gallavotti–Nicolò* trees. The *black dots* will be called *vertices* of the tree. All the vertices except the first (i.e. the one on scale h) have an $\mathcal{R}$ label attached, which means that they correspond to the action of $\mathcal{R}\mathcal{E}^T_{h_v}$, while the first represents $\mathcal{E}^T_h$. The endpoints correspond to the graph elements in Fig. 3 associated to the two terms in (2.14)

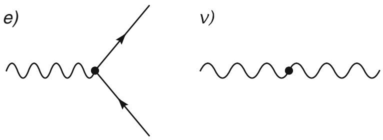

FIGURE 3. The two possible graph elements associated to the endpoints of a tree, corresponding to the two terms in the r.h.s. of (2.14)

where $\mathcal{T}_{h,N}$ is the set of rooted trees with *root* r on scale $h_r = h$ and N endpoints, see Fig. 2. The tree value $\mathcal{V}^{(h)}(\tau)$ can be evaluated in terms of a sum over connected Feynman diagrams, defined by the following rules.

With each endpoint v of τ we associate a graph element of type e or ν, corresponding to the two terms in the r.h.s. of (2.14), see Fig. 3. We introduce a *field label* f to distinguish the fields associated to the graph elements e and ν (any field label can be either of type A or of type ψ); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we call I_v the set of field labels associated with the endpoints following the vertex v on τ.

We start by looking at the graph elements corresponding to endpoints of scale 1: we group them in *clusters*, each cluster G_v being the set of endpoints attached to the same vertex v of scale 0, to be graphically represented by a box enclosing its elements. For any G_v of scale 0 (associated to a vertex v of scale 0 that is not an endpoint), we contract in pairs some of the fields in $\cup_{w\in G_v} I_w$, in such a way that after the contraction the elements of G_v are connected; each

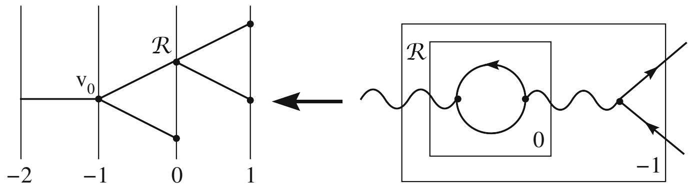

FIGURE 4. A possible Feynman diagram contributing to $V^{(-2)}$ and its cluster structure

contraction produces a propagator $g^{(0)}$ or $w^{(0)}$, depending on whether the two fields are of type ψ or of type A. We denote by $\mathcal{I}_v$ the set of contracted fields inside the box G_v and by $P_v = I_v \backslash \mathcal{I}_v$ the set of external fields of G_v; if v is not the vertex immediately following the root we attach a label $\mathcal{R}$ over the box G_v, which means that the $\mathcal{R}$ operator, defined after (2.4), acts on the value of the graph contained in G_v. Next, we group together the scale 0 clusters into scale-(-1) clusters, each scale-(-1) cluster G_v being a set of scale 0 clusters attached to the same vertex v of scale -1, to be graphically represented by a box enclosing its elements, see Fig. 4.

Again, for each v of scale -1 that is not an endpoint, if we denote by $v_1, \ldots, v_{s_v}$ the vertices immediately following v on τ, we contract some of the fields of $\cup_{i=1}^{s_v} P_{v_i}$ in pairs, in such a way that after the contraction the boxes associated to the scale 0 clusters contained in G_v are connected; each contraction produces a propagator $g^{(-1)}$ or $w^{(-1)}$. We denote by $\mathcal{I}_v$ the set of fields in $\cup_{i=1}^{s_v} P_{v_i}$ contracted at this second step and by $P_v = \cup_{i=1}^{s_v} P_{v_i} \backslash \mathcal{I}_v$ the set of fields external to G_v; if v is not the vertex immediately following the root we attach a label $\mathcal{R}$ over the box G_v (Table 1).

Now, we iterate the construction, producing a sequence of boxes into boxes, hierarchically arranged with the same partial ordering as the tree τ. Each box G_v is associated to many different Feynman (sub-)diagrams, constructed by contracting in pairs some of the lines external to G_{v_i}, with v_i, $i = 1, \ldots, s_v$, the vertices immediately following v on τ; the contractions are made in such a way that the clusters $G_{v_1}, \ldots, G_{v_{s_v}}$ are connected through propagators of scale h_v. We denote by P_v^A and by P_v^ψ the set of fields of type A and ψ, respectively, external to G_v. The set of connected Feynman diagrams compatible with this hierarchical cluster structure will be denoted by $\Gamma(\tau)$. Given these definitions, we can write:

$$\begin{aligned} \mathcal{V}^{(h)}(\tau) &= \sum_{\mathcal{G} \in \Gamma(\tau)} \int \prod_{f \in P_{v_0}^\psi} \frac{\mathrm{d}\mathbf{k}_f}{(2\pi)^3} \prod_{f \in P_{v_0}^A} \frac{d\mathbf{p}_f}{(2\pi)^3} \mathrm{Val}(\mathcal{G}), \\ \mathrm{Val}(\mathcal{G}) &= \left[\prod_{f \in P_{v_0}^A} A_{\mu(f), \mathbf{p}_f}^{(\leq h)} \right] \left[\prod_{f \in P_{v_0}^\psi} \sqrt{Z_{h-1}}\, \widetilde{\psi}_{\mathbf{k}_f, \rho(f)}^{(\leq h)} \right] \delta(v_0) \widehat{\mathrm{Val}}(\mathcal{G}), \end{aligned} \tag{2.23}$$

Table 1. List of the symbols introduced in Sects. 2.2, 2.4

Symbol	Description
τ	Gallavotti–Nicolò (GN) tree
r	Root label of the tree
v_0	First vertex of the tree, immediately following the root
h_v	Scale label of the tree vertex v
$\mathcal{T}_{h,N}$	Set of GN trees with root on scale $h_r = h$ and with N endpoints
I_v	Set of field labels associated with the endpoint of the tree v
G_v	Cluster associated with the tree vertex v
$\mathcal{I}_v$	Set of contracted fields inside the box corresponding to the cluster G_v
P_v	Set of external fields of G_v
v_i	i-th vertex immediately following v on the tree
s_v	Number of vertices immediately following the vertex v on the tree
$P_v^{\#}$	Set of fields of type $\# = A, \psi$ external to G_v
$\Gamma(\tau)$	Set of connected Feynman diagrams compatible with the hierarchical cluster structure of the tree τ
n_v^0	Number of propagators contained in G_v but not in any smaller cluster
m_v^ν	Number of end-points of type ν immediately following v on the tree
v'	Vertex immediately preceding v on the tree
$n_v^{\#}$	Number of vertices of type $\# = e, \nu$ following v on the tree
z_v	Improvement on the scaling dimension due to the renormalization

$$\widehat{\mathrm{Val}}(\mathcal{G}) = (-1)^\pi \int \prod_{v \text{ not e.p.}} \left(\frac{Z_{h_v-1}}{Z_{h_v-2}}\right)^{\frac{|P_v^\psi|}{2}} \frac{\mathcal{R}^{\alpha_v}}{s_v!} \\ \times \left[\left(\prod_{\ell\in v} g_\ell^{(h_v)}\right)\left(\prod_{\substack{v^* \text{ e.p.}\\ v^*>v,\\ h_{v^*}=h_v+1}} K_{v^*}^{(h_v)}\right)\right]$$

where $(-1)^\pi$ is the sign of the permutation necessary to bring the contracted fermionic fields next to each other; in the product over $f \in P_v^\psi$, $\widetilde{\psi}$ can be either $\overline{\psi}$ or ψ, depending on the specific field label f; $\delta(v_0) = \delta(\sum_{f\in P_{v_0}^A} \mathbf{p}_f - \sum_{f\in P_{v_0}^\psi} (-1)^{\varepsilon(f)}\mathbf{k}_f)$, where $\varepsilon(f) = \pm$ depending on whether $\widetilde{\psi}$ is equal to $\overline{\psi}$ or ψ; the integral in the third line runs over the independent loop momenta; s_v is the number of vertices immediately following v on τ; $\mathcal{R} = 1 - \mathcal{L}$ is the operator defined in (2.5) and preceding lines); $\alpha_v = 0$ if $v = v_0$, and otherwise $\alpha_v = 1$; $g_\ell^{(k)}$ is equal to $g^{(k)}$ or to $w^{(k)}$ depending on the fermionic or bosonic nature of the line ℓ, and $\ell \in v$ means that ℓ is contained in the box G_v but not in any other smaller box; finally, $K_{v^*}^{(k)}$ is the matrix associated to the endpoints v^* on scale $k+1$ (given by $ie_{0,k}\gamma_0$ if v^* is of type (a) with label $\rho = 0$, by $ie_{j,k}v_k\gamma_j$ if v^* is of type e with label $\rho = j \in \{1, 2\}$, or by $-M^k\nu_{\mu,k}$ if v^* is of type ν. In (2.23) it is understood that the operators $\mathcal{R}$ act in the order induced by the tree ordering (i.e., starting from the endpoints and moving toward the root); moreover, the matrix structure of $g_\ell^{(k)}$ is neglected, for simplicity of notations.

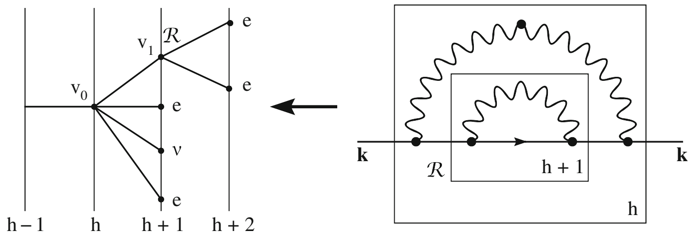

FIGURE 5. A possible Feynman diagram contributing to $\mathcal{V}^{(h-1)}$ and its cluster structure

2.3. An Example of Feynman Graph

To be concrete, let us apply the rules described above in the evaluation of a simple Feynman graph $\mathcal{G}$ arising in the tree expansion of $\mathcal{V}^{(h-1)}$. Let $\mathcal{G}$ be the diagram in Fig. 5, associated to the tree τ drawn in the left part of the figure; let us assume that the sets P_v of the external lines associated to the vertices of τ are all assigned. Setting

$$e_{0,h} := e_{0,h}\,, \quad \bar{e}_{j,h} := v_{h-1} e_{j,h}, \tag{2.24}$$

we can write:

$$\begin{aligned} \mathrm{Val}(\mathcal{G}) = & -\frac{1}{4!2!}\frac{Z_h}{Z_{h-1}}\frac{Z_{h-1}}{Z_{h-2}}\bar{e}^2_{\mu_1,h}\bar{e}^2_{\mu_2,h+1}M^h \nu_{\mu_1,h}\overline{\psi}_{\mathbf{k}} \\ & \times\left\{\int \frac{d\mathbf{p}}{(2\pi)^3}\,|w^{(h)}(\mathbf{p})|^2\gamma_{\mu_1} g^{(h)}(\mathbf{k}+\mathbf{p})\right. \\ & \times\mathcal{R}\left[\int \frac{d\mathbf{q}}{(2\pi)^3}\,\gamma_{\mu_2} g^{(h+1)}(\mathbf{k}+\mathbf{p}+\mathbf{q})\gamma_{\mu_2} w^{(h+1)}(\mathbf{q})\right] \\ & \left.\times\, g^{(h)}(\mathbf{k}+\mathbf{p})\gamma_{\mu_1}\right\}\psi_{\mathbf{k}}, \end{aligned} \tag{2.25}$$

where $\mathcal{R}\,[F(\mathbf{k}+\mathbf{p})] = F(\mathbf{k}+\mathbf{p}) - F(\mathbf{0}) - (\mathbf{k}+\mathbf{p})\cdot\nabla F(\mathbf{0}) \equiv \frac{1}{2}(k_\mu + p_\mu)(k_\nu + p_\nu)\partial_\mu\partial_\nu F(\mathbf{k}^*)$. Notice that the same Feynman graph appears in the evaluation of other trees, which are topologically equivalent to the one represented in the left part of Fig. 5 and that can be obtained from it by: (i) relabeling the fields in P_{v_1}, P_{v_0}, (ii) relabeling the endpoints of the tree, (iii) exchanging the relative positions of the topologically different subtrees with root v_0. If one sums over all these trees, the resulting value one obtains is the one in Eq. (2.25) times a combinatorial factor $2^2\cdot 3\cdot 4$ (2^2 is the number of ways for choosing the fields in P_{v_1} and in P_{v_0}; 3 is the number of ways in which one can associate the label ν to one of the endpoints of scale $h+1$; 4 is the number of distinct unlabeled trees that can be obtained by exchanging the positions of the subtrees with root v_0).

2.4. Dimensional Bounds

We are now ready to derive a general bound for the Feynman graphs produced by the multiscale integration. Let $W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}$ be the contribution from trees with

N end-points to the kernel $W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}$ in 2.4, i.e.

$$W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}(\{\mathbf{k}_i\},\{\mathbf{p}_j\}) = \sum_{N=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,N}}\sum^{*}_{\substack{\mathcal{G}\in\Gamma(\tau)\\ |P^A_{v_0}|=m,\\ |P^\psi_{v_0}|=2n}} \widehat{\mathrm{Val}}(\mathcal{G}) \equiv \sum_{N=1}^{\infty} W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}(\{\mathbf{k}_i\},\{\mathbf{p}_j\}), \quad (2.26)$$

where the $*$ on the sum indicates the constraints that: $\cup_{f\in P^A_{v_0}}\{\mathbf{p}_f\} = \cup_{j=1}^{m}\{\mathbf{p}_j\}$; $\cup_{f\in P^\psi_{v_0}}\{\mathbf{k}_f\} = \cup_{i=1}^{2n}\{\mathbf{k}_i\}$; $\cup_{f\in P^A_{v_0}}\{\mu(f)\} = \underline{\mu}$; $\cup_{f\in P^\psi_{v_0}}\{\rho(f)\} = \underline{\rho}$.

The N-th order contribution to the kernel of the effective potential admits the following bound.

Theorem 2.1. ($N!$ bound) *Let* $\bar{\varepsilon}_h = \max_{h<k\le 0}\{|e_{\mu,k}|,|\nu_{\mu,k}|\}$ *be small enough. If* $Z_k/Z_{k-1} \le e^{C\bar{\varepsilon}_h^2}$ *and* $C^{-1} \le v_{k-1} \le 1$, *for all* $h < k \le 0$ *and a suitable constant* $C > 0$, *then*

$$||W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}|| \le (\text{const.})^N \bar{\varepsilon}_h^N \binom{N}{2} M^{h(3-m-2n)}, \quad (2.27)$$

where $||W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}|| := \sup_{\{\mathbf{k}_i\},\{\mathbf{p}_j\}} |W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}(\{\mathbf{k}_i\},\{\mathbf{p}_j\})|$.

The factor $3 - 2n - m$ in (2.27) is referred to as the *scaling dimension* of the kernel with $2n$ external fermionic fields and m external bosonic fields; according to the usual RG terminology, kernels with positive, vanishing or negative scaling dimensions are called *relevant, marginal* or *irrelevant* operators, respectively. Notice that, if we tried to expand the effective potential in terms of the bare couplings e, ν_μ, the N-th order contributions in this "naive" perturbation series could *not be bounded uniformly in the scale* h as in (2.27), but rather by the r.h.s. of (2.27) times $|h|^N$, an estimate which blows up order by order as $h \to -\infty$ (Table 1).

Proof. Using the bounds

$$\begin{aligned} \left\|g^{(h)}(\mathbf{k})\right\| &\le \text{const}\cdot M^{-h}, & \int d\mathbf{k}\left\|g^{(h)}(\mathbf{k})\right\| &\le \text{const}\cdot M^{2h},\\ \left|w^{(h)}(\mathbf{k})\right| &\le \text{const}\cdot M^{-h}, & \int d\mathbf{k}\left|w^{(h)}(\mathbf{k})\right| &\le \text{const}\cdot M^{2h}, \end{aligned} \quad (2.28)$$

and the assumptions on v_{k-1} and Z_k/Z_{k-1} into (2.23), we find that, if $\tau \in \mathcal{T}_{h,N}$ and $\mathcal{G} \in \Gamma(\tau)$,

$$|\widehat{\mathrm{Val}}(\mathcal{G})| \le (\text{const.})^N \bar{\varepsilon}_h^N \prod_{v \text{ not e.p.}} \frac{e^{\frac{C}{2}\bar{\varepsilon}_h^2 |P^\psi_v|}}{s_v!} M^{-3h_v(s_v-1)} M^{2h_v n^0_v} M^{h_v m^\nu_v} \times \prod_{\substack{v \text{ not e.p.}\\ v>v_0}} M^{-z_v(h_v-h_{v'})}, \quad (2.29)$$

where n^0_v is the number of propagators $\ell \in v$, i.e., of propagators ℓ contained in the box G_v but not in any smaller cluster; s_v is the number of vertices immediately following v on τ; m_v^ν is the number of end-points of type ν immediately following v on τ (i.e., contained in G_v but not in any smaller cluster); v' is the vertex immediately preceding v on τ and $z_v = 2$ if $|P_v^\psi| = |P_v| = 2$, $z_v = 1$ is $|P_v^\psi| = 2|P_v^A| = 2$ and $z_v = 0$ otherwise. The last product in (2.29) is due to the action of $\mathcal{R}$ on the vertices $v > v_0$ that are not end-points. In fact, the operator $\mathcal{R}$, when acting on a kernel $W_{1,1}^{(h_v)}(\mathbf{p}, \mathbf{k})$ associated to a vertex v with $|P_v^\psi| = 2|P_v^A| = 2$, extracts from $W_{1,1}^{(h_v)}$ the rest of first order in its Taylor expansion around $\mathbf{p} = \mathbf{k} = \mathbf{0}$: if $|W_{1,1}^{(h_v)}(\mathbf{p}, \mathbf{k})| \leq C(v)$, then $|\mathcal{R}W_{1,1}^{(h_v)}(\mathbf{p}, \mathbf{k})| = \frac{1}{2}|(\mathbf{p}\partial_{\mathbf{p}} + \mathbf{k}\partial_{\mathbf{k}})W_{1,1}^{(h_v)}(\mathbf{p}^*, \mathbf{k}^*)| \leq (\text{const.})M^{-h_v + h_{v'}}C(v)$, where M^{-h_v} is a bound for the derivative with respect to momenta on scale h_v and $M^{h_{v'}}$ is a bound for the external momenta $\mathbf{p}$, $\mathbf{k}$; i.e., $\mathcal{R}$ is dimensionally equivalent to $M^{-(h_v - h_{v'})}$. The same is true if $\mathcal{R}$ acts on a kernel $W_{3,0}^{(h_v)}(\mathbf{p}_1, \mathbf{p}_2)$. Similarly, if $\mathcal{R}$ acts on a terms with $|P_v| = 2$, it extracts the rest of second order in the Taylor expansion around $\mathbf{k} = \mathbf{0}$, and it is dimensionally equivalent to $\mathbf{k}^2\partial_{\mathbf{k}}^2 \sim M^{-2(h_v - h_{v'})}$. As a result, we get (2.29).

Now, let n_v^e (n_v^ν) be the number of vertices of type e (of type ν) following v on τ. If we plug in (2.29) the identities

$$\begin{aligned} \sum_{v \text{ not e.p.}} (h_v - h)(s_v - 1) &= \sum_{v \text{ not e.p.}} (h_v - h_{v'})(n_v^e + n_v^\nu - 1) \\ \sum_{v \text{ not e.p.}} (h_v - h)n_v^0 &= \sum_{v \text{ not e.p.}} (h_v - h_{v'}) \left(\frac{3}{2}n_v^e + n_v^\nu - \frac{|P_v|}{2}\right) \\ \sum_{v \text{ not e.p.}} (h_v - h)m_v^\nu &= \sum_{v \text{ not e.p.}} (h_v - h_{v'})n_v^\nu \end{aligned} \tag{2.30}$$

we get the bound

$$\begin{aligned} &|\widehat{\text{Val}}(\mathcal{G})| \\ &\quad \leq (\text{const.})^N \bar{\varepsilon}_h^N \frac{1}{s_{v_0}!} M^{h(3-|P_{v_0}|)} \prod_{\substack{v \text{ not e.p.} \\ v > v_0}} \frac{e^{\frac{C}{2}\bar{\varepsilon}_h^2 |P_v^\psi|}}{s_v!} M^{(h_v - h_{v'})(3 - |P_v| - z_v)}. \end{aligned} \tag{2.31}$$

In the latter equation, $3 - |P_v|$ is the *scaling dimension* of the cluster G_v, and $3 - |P_v| - z_v$ is its renormalized scaling dimension. Notice that the renormalization operator $\mathcal{R}$ has been introduced precisely to guarantee that $3 - |P_v| - z_v < 0$ for all v, by construction. This fact allows us to sum over the scale labels $h \leq h_v \leq 1$, and to conclude that the perturbative expansion is well defined at any order N of the renormalized expansion. More precisely, the fact that the renormalized scaling dimensions are all negative implies, via a standard argument (see, e.g., [3, 16]), the following bound, valid for a suitable constant C (see (2.27) for a definition of the norm $\|\cdot\|$):

$$||W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}|| \leq (\text{const.})^N \bar{\varepsilon}_h^N \frac{1}{s_{v_0}!} M^{h(3-m-2n)}$$
$$\times \sum_{\tau\in\mathcal{T}_{h,N}} \sum_{\substack{\mathcal{G}\in\Gamma(\tau)\\ |P^A_{v_0}|=m,\\ |P^\psi_{v_0}|=2n}} \prod_{\substack{v \text{ not e.p.}\\ v>v_0}} \frac{e^{\frac{C}{2}\bar{\varepsilon}_h^2|P^\psi_v|}}{s_v!} M^{(h_v-h_{v'})(3-|P_v|-z_v)}, \quad (2.32)$$

from which, after counting the number of Feynman graphs contributing to the sum in (2.32), (2.27) follows. □

An immediate corollary of the proof leading to (2.27) is that contributions from trees $\tau \in \mathcal{T}_{h,N}$ with a vertex v on scale $h_v = k > h$ admit an improved bound with respect to (2.27), of the form $\leq (\text{const.})^N \bar{\varepsilon}_h^N (N/2)!\, M^{h(3-|P_{v_0}|)} M^{\theta(h-k)}$, for any $0 < \theta < 1$; the factor $M^{\theta(h-k)}$ can be thought of as a dimensional gain with respect to the "basic" dimensional bound in (2.27). This improved bound is usually referred to as the *short memory* property (i.e., long trees are exponentially suppressed); it is due to the fact that the renormalized scaling dimensions $d_v = 3 - |P_v| - z_v$ in (2.31) are all negative, and can be obtained by taking a fraction of the factors $M^{(h_v-h_{v'})d_v}$ associated to the branches of the tree τ on the path connecting the vertex on scale k to the one on scale h.

Remark. All the analysis above is based on the fact that the scaling dimension $3-|P_v|$ in (2.31) is independent of the number of endpoints of the tree τ; i.e., the model is *renormalizable*. A rather different situation is found in the case of instantaneous Coulomb interactions, in which case the bosonic propagator is given by $(2|\vec{p}|)^{-1}$ rather than by $(2|\mathbf{p}|)^{-1}$. In this case, choosing the bosonic single scale propagator as $w^{(h)}(\mathbf{p}) = \chi_0(\mathbf{p}) f_h(\vec{p})(2|\vec{p}|)^{-1}$, one finds that the last bound in (2.28) is replaced by $\int d\mathbf{p}\, \left|w^{(h)}(\mathbf{p})\right| \leq (\text{const.})\, M^h$ (dimensionally, this bound has a factor M^h missing). Repeating the steps leading to (2.31), one finds a general bound valid at all orders, in which the new scaling dimension is $3-|P_v|+n^e_v+n^\nu_v$; this (pessimistic) general bound assumes that at each scale the loop lines of the graph are all bosonic. Perhaps, this bound can be improved, by taking into account the explicit structure of the expansion; however, it shows that the renormalizability of the instantaneous case, *if true*, does not follow from purely dimensional considerations and its proof will require the implementation of suitable cancellations.

2.5. The Schwinger Functions

A similar analysis can be performed for the two-point function, see Appendix B. It turns out that, similarly to what we found above for the effective potentials, the two-point function can be written in terms of a renormalized perturbative expansion in the effective couplings $\{e_{\mu,k}, \nu_{\mu,k}\}_{k\leq 0}$ and in the renormalization constants $\{Z_k, v_k\}_{k\leq 0}$, with coefficients represented as sums of Feynman graphs, uniformly bounded as $|\Lambda| \to \infty$; in contrast, the graphs forming the naive expansion in e, ν_μ are plagued by logarithmic infrared divergences.

More explicitly, if $M^h \le |\mathbf{k}| \le M^{h+1}$, we get (see Eqs. B.10–B.13):

$$\langle \psi_{\mathbf{k}} \overline{\psi}_{\mathbf{k}} \rangle = \sum_{j=h}^{h+1} \frac{g^{(j)}(\mathbf{k})}{Z_{j-1}} \Big(1 + \tilde{B}(\mathbf{k})\Big), \tag{2.33}$$

where $\tilde{B}(\mathbf{k})$ is given by a formal power series in $\{e_{\mu,k}, \nu_{\mu,k}\}_{k\le 0}$ with coefficients depending on $\{Z_k, v_k\}_{k\le 0}$, and starting from second order; under the same hypothesis of Theorem 2.1, the N-th order contribution to $\tilde{B}(\mathbf{k})$ is bounded by $(\text{const.})^N (\bar{\varepsilon}_{-\infty})^N (N/2)!$ uniformly in $\mathbf{k}$. Eq. (2.33) is equivalent to Eq. (1.7) of the main result (see the end of Section 3.3 below for the explicit relation between Z_h, v_h and $Z(\mathbf{k}), v(\mathbf{k})$).

To prove our main result we need to control the flow of the effective charges at all orders in perturbation theory, and to do this we shall use Ward Identities, see Sect. 4. These are non-trivial relations for the three-point functions, which can be related to the renormalized charges in the following way. Consider a theory with a bosonic infrared cutoff M^{h_*}, that is assume that the bare bosonic propagator is given by (1.5) with $\chi_0(\mathbf{p})$ replaced by $\chi_{[h^*,0]}(\mathbf{p}) := \chi_0(\mathbf{p}) - \chi_0(M^{-h^*}\mathbf{p})$, which is vanishing for $|\mathbf{p}| \le M^{h_*}$ and it is equal to $\chi_0(\mathbf{p})$ for $|\mathbf{p}| \ge M^{h^*+1}$; denote by $\langle \ldots \rangle_{h^*}$ the expectation value in the presence of the bosonic infrared cutoff. As shown in Appendix B, setting $\bar{e}_{0,h} := e_{0,h}$, $\bar{e}_{1,h} = \bar{e}_{2,h} := v_{h-1} e_{1,h}$, and taking $|\mathbf{q}| = M^{h^*}$, $|\mathbf{q}+\mathbf{p}| \le M^{h^*}$, $|\mathbf{p}| \ll M^{h^*}$ (we will be interested in the limit $\mathbf{p} \to \mathbf{0}$), the following result holds (see Eqs. B.14–B.16):

$$\begin{aligned} &\langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{q}+\mathbf{p}} \overline{\psi}_{\mathbf{q}} \rangle_{h^*} \\ &\quad = i Z_{h^*-1} \frac{\bar{e}_{\mu,h^*}}{e} \langle \psi_{\mathbf{q}+\mathbf{p}} \overline{\psi}_{\mathbf{q}+\mathbf{p}} \rangle_{h^*} \left(\gamma_\mu + \bar{B}_{\mu,h^*}(\mathbf{p},\mathbf{q})\right) \langle \psi_{\mathbf{q}} \overline{\psi}_{\mathbf{q}} \rangle_{h^*}, \end{aligned} \tag{2.34}$$

where $\bar{B}_{\mu,h^*}$ is given by a formal power series in $\{e_{\mu,k}, \nu_{\mu,k}\}_{h^*<k\le 0}$, starting from second order and with the N-th order of the series admitting a bound proportional to $(\bar{\varepsilon}_{h^*})^N (N/2)!$, uniformly in $\mathbf{k}$. Eq. (2.34) is one of the two desired equations relating the three-point function to the two-point function and the effective charge e_{μ,h^*}. A second independent equation expressing the three-point function in terms of the two-point function and of the *bare* charge e will be derived in Sect. 4, see (4.4), using the (approximate) gauge invariance of the theory. Combining the two equations we will be able to relate e_{μ,h^*} to the bare charge e, for all $h^* < 0$, and this will allows us to control the flow of the effective couplings on all infrared scales. This procedure will be described in detail in the next two sections.

3. The Flow of the Effective Couplings

3.1. The Beta Function

A crucial point for the consistency of our approach is that the running coupling constants $e_{\mu,h}, \nu_{\mu,h}$ are small for all $h \le 0$, that the ratios Z_h/Z_{h-1} are close to 1, and the effective Fermi velocity v_h does not approach zero. Even if we do not prove the convergence of the series but only $N!$ bounds, we expect that our series gives meaningful information only as long as the running coupling constants satisfy these conditions. In this section we describe how to control their flow. We shall proceed by induction: we will first assume that $\bar{\varepsilon} = \max_{k\le 0}\{|e_{\mu,k}|\}$ is small, that $Z_h/Z_{h-1} \le e^{C\bar{\varepsilon}^2}$ and $C^{-1} \le v_h \le 1$ for all $h \le 0$ and a suitable constant $C > 0$, and we will show that, by properly choosing the values of the counterterms ν_μ in (1.2), the constants $\nu_{\mu,h}$ remain small: $\max_{h\le 0}\{|\nu_{\mu,h}|\} \le (\text{const.})\,\bar{\varepsilon}^2$. Next, once that the flow of $\nu_{\mu,h}$ is controlled, we will study the flow of Z_h and v_h under the assumption that the constants $e_{\mu,h}$ remain bounded and small for all $h \le 0$; we will show that, asymptotically as $h \to -\infty$, $Z_h \sim M^{-\eta h}$, with $\eta = O(e^2)$ a positive exponent, while v_h grows, approaching a limiting value v_{eff} close to the speed of light. Finally, we shall start to discuss the remarkable cancellations following from a *Ward Identity* that guarantee that the constants $e_{\mu,h}$ remain bounded and small for all $h \le 0$; the full proof of this fact will be postponed to Sect. 4 and Appendix D.

The renormalized parameters obey to recursive equations induced by the previous construction; i.e., (2.6), (2.7), (2.11), (2.13) imply the flow equations:

$$\frac{Z_{h-1}}{Z_h} = 1 + z_{0,h} := 1 + \beta^z_h, \quad v_{h-1} = \frac{Z_h}{Z_{h-1}}(v_h + z_{1,h}) := v_h + \beta^v_h \tag{3.1}$$

$$\nu_{\mu,h} = -M^{-h}\, W^{(h)}_{2,0,\mu,\mu}(\mathbf{0}) := M\nu_{\mu,h+1} + \beta^\nu_{\mu,h+1}, \tag{3.2}$$

$$e_{0,h} = \frac{Z_h}{Z_{h-1}}\lambda_{0,h} := e_{0,h+1} + \beta^e_{0,h+1}, \tag{3.3}$$

$$e_{1,h} = \frac{Z_h}{Z_{h-1}}\frac{\lambda_{1,h}}{v_{h-1}} := e_{1,h+1} + \beta^e_{1,h+1}, \tag{3.4}$$

and $e_{2,h} = e_{1,h}$. The *beta functions* appearing in the r.h.s. of flow equations are related, see (2.7), to the kernels $W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}$, so that they are expressed by series in the running coupling constants admitting the bound (2.27). For the explicit expressions of the one-loop contributions to the beta function, see below.

3.2. The Flow of $\nu_{\mu,h}$

Let us assume that $\bar{\varepsilon} = \max_{k\le 0}\{|e_{\mu,k}|\}$ is small, that $Z_h/Z_{h-1} \le e^{C\,\bar{\varepsilon}^2}$ and $C^{-1} \le v_h \le 1$ for a suitable constant C, for all $h \le 0$. Under these assumptions, the flow of $\nu_{\mu,h}$ can be controlled by suitably choosing the counterterms ν_μ; in fact, if ν_μ is chosen as

$$\nu_\mu = -\sum_{k=-\infty}^{0} M^{k-1}\beta^\nu_{\mu,k}, \tag{3.5}$$

then the effective coupling $\nu_{\mu,h}$ is

$$\nu_{\mu,h} = -\sum_{k=-\infty}^{h} M^{-h-1+k}\beta^\nu_{\mu,k}, \tag{3.6}$$

from which one finds that $\nu_{\mu,h}$ can be expressed by a series in $\{e_{\mu,k}\}_{k\le 0}$, starting at second order and with coefficients bounded uniformly in h. At lowest order, if $h < 0$ and setting $\xi_h := \frac{\sqrt{1-v_h^2}}{v_h}$ (see Appendix C):

$$\beta_{0,h}^{\nu,(2)} = -(M-1)\frac{e_{0,h}^2 v_h^{-2}}{\pi^2}\left[\frac{\xi_h - \arctan\xi_h}{\xi_h^3}\right]\int_0^\infty \mathrm{d}t\,\left(2\chi(t)-\chi^2(t)\right) \quad (3.7)$$

$$\beta_{1,h}^{\nu,(2)} = -(M-1)\frac{e_{1,h}^2}{2\pi^2}\left[\frac{\arctan\xi_h}{\xi_h} - \frac{\xi_h - \arctan\xi_h}{\xi_h^3}\right] \times\int_0^\infty \mathrm{d}t\,\left(2\chi(t)-\chi^2(t)\right). \quad (3.8)$$

By the above equations we see that lowest order contributions to ν_μ are positive, that is ν_μ can be interpreted as *bare* photon masses. Using the short memory property and symmetry considerations, one can also show that $\beta_{0,h}^\nu - \beta_{1,h}^\nu$ is a sum of graphs whose contributions are of the order $O(1-v_h)$ or $O(e_{0,h} - e_{1,h})$.

3.3. The Flow of Z_h and v_h

In this section we show that, under proper assumptions on the flow of the effective charges, the effective Fermi velocity v_h tend to a limit value $v_{\text{eff}} = v_{-\infty}$ and that both $v_{\text{eff}} - v_h$ and Z_h^{-1} vanish as $h \to -\infty$ with an anomalous power law.

Let us assume that the effective charges tend to a line of fixed points:

$$e_{\mu,h} = e_{\mu,-\infty} + O(e^3(v_{-\infty} - v_h)) + O(e^3 M^{\theta h}), \quad (3.9)$$

with $0 < \theta < 1$ and $e_{\mu,-\infty} = e + O(e^3)$; this is a remarkable property that will be proven order by order in perturbation theory using WIs, see the following section. Moreover, let ν_μ be fixed as in the previous subsection (under the proper inductive assumptions on Z_k and v_k).

We start by studying the flow of the Fermi velocity. At lowest order (see Appendix C), its beta function reads:

$$\beta_h^{v,(2)} = \frac{\log M}{4\pi^2}\left[\frac{e_{0,h}^2 v_h^{-1}}{2}\frac{\arctan\xi_h}{\xi_h} - \left(2e_{1,h}^2 v_h - \frac{e_{0,h}^2 v_h^{-1}}{2}\right)\frac{\xi_h - \arctan\xi_h}{\xi_h^3}\right]. \quad (3.10)$$

Note that if $e_{0,k} := e_{1,k}$, then the r.h.s. of (3.10) is strictly positive for all $\xi_h > 0$ and it vanishes quadratically in ξ_h at $\xi_h = 0$. The higher order contributions to β_h^v have similar properties. This can be proved as follows: we observe that the beta function β_h^v is a function of the renormalized couplings and of the Fermi velocities on scales $\geq h$, i.e.:

$$\beta_h^v = \beta_h^v\left(\{(e_{0,k}, e_{1,k}, e_{2,k}), (\nu_{0,k}, \nu_{1,k}, \nu_{2,k}), v_k\}_{k\geq h}\right). \quad (3.11)$$

We can rewrite β_h^v as $\beta_h^{v,rel} + \beta_h^{v,1} + \beta_h^{v,2} + \beta_h^{v,3}$, with:

$$\begin{aligned}
\beta_h^{v,rel} &= \beta_h^v\left(\{(e_{0,k}, e_{0,k}, e_{0,k}), (\nu_{0,k}, \nu_{0,k}, \nu_{0,k}), 1\}_{k\geq h}\right),\\
\beta_h^{v,1} &= \beta_h^v\left(\{(e_{0,k}, e_{0,k}, e_{0,k}), (\nu_{0,k}, \nu_{0,k}, \nu_{0,k}), v_k\}_{k\geq h}\right)\\
&\quad - \beta_h^v\left(\{(e_{0,k}, e_{0,k}, e_{0,k}), (\nu_{0,k}, \nu_{0,k}, \nu_{0,k}), 1\}_{k\geq h}\right),\\
\beta_h^{v,2} &= \beta_h^v\left(\{(e_{0,k}, e_{0,k}, e_{0,k}), (\nu_{0,k}, \nu_{1,k}, \nu_{2,k}), v_k\}_{k\geq h}\right)\\
&\quad - \beta_h^v\left(\{(e_{0,k}, e_{0,k}, e_{0,k}), (\nu_{0,k}, \nu_{0,k}, \nu_{0,k}), v_k\}_{k\geq h}\right),\\
\beta_h^{v,3} &= \beta_h^v\left(\{(e_{0,k}, e_{1,k}, e_{2,k}), (\nu_{0,k}, \nu_{1,k}, \nu_{2,k}), v_k\}_{k\geq h}\right)\\
&\quad - \beta_h^v\left(\{(e_{0,k}, e_{0,k}, e_{0,k}), (\nu_{0,k}, \nu_{1,k}, \nu_{2,k}), v_k\}_{k\geq h}\right).
\end{aligned} \tag{3.12}$$

By relativistic invariance it follows that $\beta_h^{v,rel} = 0$ and by the short memory property (see discussion after Eq. (2.32)) we get:

$$\begin{aligned}
\beta_h^{v,1} &= O\left(e_{0,h}^2(1 - v_h)\right),\\
\beta_h^{v,2} &= O\left(e_{0,h}^2(\nu_{0,h} - \nu_{1,h})\right),\\
\beta_h^{v,3} &= O\left(e_{0,h}(e_{0,h} - e_{1,h})\right).
\end{aligned} \tag{3.13}$$

Using (3.6) and an argument similar to the one leading to Eq. (3.13), we also find that $\nu_{0,h} - \nu_{1,h}$ can be written as a sum of contributions of order $e_{0,h}(e_{0,h} - e_{1,h})$ and of order $e_{0,h}^2(1 - v_h)$. Therefore, we can write:

$$\frac{v_{h-1}}{v_h} = 1 + \frac{\log M}{4\pi^2}\left[\frac{8}{5}e^2(1 - v_h)(1 + A'_h) + \frac{4}{3}e(1 + B'_h)(e_{0,h} - e_{1,h})\right], \tag{3.14}$$

where the numerical coefficients are obtained from the explicit lowest order computation (3.10); A'_h is a sum of contributions that are finite at all orders in the effective couplings, which are either of order two or more in the effective charges, or vanishing at $v_k = 1$; similarly, B'_h is a sum of contributions that are finite at all orders in the effective couplings, which are of order two or more in the effective charges. From (3.14) it is apparent that v_h tends as $h \to -\infty$ to a limit value

$$v_{\text{eff}} = 1 + \frac{5}{6e}(e_{0,-\infty} - e_{1,-\infty})(1 + C'_{-\infty}) \tag{3.15}$$

with $C'_{-\infty}$ a sum of contributions that are finite at all orders in the effective couplings, which are of order two or more in the effective charges. The fixed point (3.15) is found simply by requiring that in the limit $h \to -\infty$ the argument of the square brackets in (3.14) vanishes.

Using Eq. (3.9), we find that the expression in square brackets in the r.h.s. of (3.14) can be rewritten as $(8e^2/5)(v_{\text{eff}} - v_h + R'_h)(1 + A''_h)$, where (i) A''_h is a sum of contributions that are finite at all orders in the effective couplings, which are either of order two or more in the effective charges, or vanishing at $v_k = v_{\text{eff}}$; (ii) R'_h is a sum of contributions that are finite at all orders in the effective couplings, which are of order two or more in the effective charges and are bounded at all orders by $M^{\theta h}$, for some $0 < \theta < 1$. Therefore, (3.14) can be rewritten as

$$v_{\rm eff} - v_{h-1} = (v_{\rm eff} - v_h)\left(1 - v_h \frac{v_{\rm eff} - v_h + R'_h}{v_{\rm eff} - v_h} \log M \frac{2e^2}{5\pi^2}(1 + A''_h)\right), \tag{3.16}$$

from which, using the fact that $R'_h = O(e^2 M^{\theta h})$, we get that there exist two positive constants C_1, C_2 such that [1]:

$$C_1 M^{h\tilde\eta} \le \frac{v_{\rm eff} - v_h}{v_{\rm eff} - v} \le C_2 M^{h\tilde\eta}, \tag{3.17}$$

with

$$\tilde\eta = -\log_M\left[1 - v_{\rm eff} \log M \frac{2e^2}{5\pi^2}\left(1 + A''_{-\infty}\right)\right]; \tag{3.18}$$

at lowest order, Eq. (3.18) gives $\tilde\eta^{(2)} = 2e^2/(5\pi^2)$.

Similarly $C_1 M^{\eta h} \le Z_h \le C_2 M^{\eta h}$ for two suitable positive constants C_1, C_2, with $\eta = \lim_{h\to-\infty} \log_M(1 + z_{0,h})$; at lowest order we find (see Appendix C):

$$\beta_h^{z,(2)} = \frac{\log M}{4\pi^2}\left(2e_{1,h}^2 - e_{0,h}^2 v_h^{-2}\right)\frac{\xi_h - \arctan \xi_h}{\xi_h^3}, \tag{3.19}$$

so that $\eta^{(2)} = \frac{e^2}{12\pi^2}$.

Before we conclude this section, let us briefly comment about the relation between Z_h, v_h and the functions $Z(\mathbf{k})$ and $v(\mathbf{k})$ appearing in the main result, see (1.6). If $|\mathbf{k}| = M^h$, we define $Z(\mathbf{k}) = Z_h$ and $v(\mathbf{k}) = v_h$; for general $|\mathbf{k}| \le 1$, we let $Z(\mathbf{k})$ and $v(\mathbf{k})$ be smooth interpolations of these sequences. Of course, we can choose these interpolations in such a way that, if $M^h \le |\mathbf{k}| \le M^{h+1}$,

$$\begin{aligned} \left|\frac{Z(\mathbf{k})}{Z_h} - 1\right| &\le \left|\frac{Z_{h+1}}{Z_h} - 1\right| = O(\eta \log M), \\ \left|\frac{v(\mathbf{k}) - v_h}{v_{\rm eff} - v_h}\right| &\le \left|\frac{v_{h+1} - v_h}{v_{\rm eff} - v_h}\right| = O(\tilde\eta \log M). \end{aligned} \tag{3.20}$$

Therefore, we can replace in the leading part of the two-point Schwinger function (2.33) the wave function renormalization Z_j and the effective Fermi velocity v_j by $Z(\mathbf{k})$ and $v(\mathbf{k})$, provided that the correction term $\tilde B(\mathbf{k})$ in (2.33) is replaced by a quantity $B(\mathbf{k})$ defined so to take into account higher order corrections satisfying the bounds (3.20). This leads to the main result Eq. (1.6).

3.4. The Flow of the Effective Charges at Lowest Order

The physical behavior of the system is driven by the flow of $e_{\mu,h}$; in the following section, using a WI relating the three- and two-point functions, we will show that $e_{\mu,h}$ remain close to their initial values for all scales $h \le 1$ and $\lim_{h\to-\infty} e_{\mu,h} = e_{\mu,-\infty} = e + F_\mu$, where F_μ can be expressed as series in the renormalized couplings starting at third order in the effective charges. In

[1] Equation (3.17) must be understood as an order by order inequality: if we truncate the theory at order N in the bare coupling e, both sides of the inequality in Eq. (3.17) are verified asymptotically as $e \to 0$, for all $N \ge 1$.

perturbation theory, this fact follows from non-trivial cancellations that are present at all orders. For illustrative purposes, here we perform the lowest order computation in non-renormalized perturbation theory, *in the presence of an infrared cutoff on the bosonic propagator* $\chi_{[h,0]}(\mathbf{p}) = \chi_0(\mathbf{p}) - \chi_0(M^{-h}\mathbf{p})$; at this lowest order, such a "naive" computation gives the same result as the renormalized one; for the full computation, see next section and Appendix D. If

$$(\bar\gamma_0, \bar\gamma_1, \bar\gamma_2) := (\gamma_0, v\gamma_1, v\gamma_2), \quad (\bar k_0, \bar k_1, \bar k_2) := (k_0, vk_1, vk_2), \tag{3.21}$$

the effective charges on scale h at third order are given by:

$$\begin{aligned}&\left(e^{(3)}_{\mu,h} - e\right)\gamma_\mu \\ &\quad = ie^3 \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{\chi_{[h,0]}(\mathbf{k})}{2|\mathbf{k}|}\left[\bar\gamma_\nu g^{(\le 0)}(\mathbf{k}) i\gamma_\mu g^{(\le 0)}(\mathbf{k})\bar\gamma_\nu + \bar\gamma_\nu \partial_{\bar k_\mu} g^{(\le 0)}(\mathbf{k})\bar\gamma_\nu\right]\end{aligned} \tag{3.22}$$

where the first term in square brackets is the vertex renormalization, while the second term is due to the wave function and velocity renormalizations. Note that both integrals are well defined in the ultraviolet (thanks to the presence of an ultraviolet cutoff in the propagators), while for $h \to -\infty$ they are logarithmically divergent in the infrared. However, a remarkable cancellation takes place between the two integrals; in fact:

$$\begin{aligned}&g^{(\le 0)}(\mathbf{k}) i\gamma_\mu g^{(\le 0)}(\mathbf{k}) + \partial_{\bar k_\mu} g^{(\le 0)}(\mathbf{k}) \\ &\quad = \frac{\partial_{\bar k_\mu}\chi_0(\mathbf{k})}{i\not{\mathbf{k}}} + \chi_0(\mathbf{k})\left(\chi_0(\mathbf{k}) - 1\right)\frac{1}{i\not{\mathbf{k}}} i\gamma_\mu \frac{1}{i\not{\mathbf{k}}},\end{aligned} \tag{3.23}$$

with $\not{\mathbf{k}} := k_0\gamma_0 + v\vec{k}\cdot\vec{\gamma}$, so that

$$\begin{aligned}e^{(3)}_{0,h} &= e + ie^3 \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\bar\gamma_\mu \frac{1}{i\not{\mathbf{k}}}\bar\gamma_\mu \frac{k_0}{2|\mathbf{k}|^2}\chi_0'(\mathbf{k})\chi_{[h,0]}(\mathbf{k}) + O\left(e^3(M-1)\right) \\ e^{(3)}_{1,h} &= e + \frac{ie^3}{v}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\bar\gamma_\mu \frac{1}{i\not{\mathbf{k}}}\bar\gamma_\mu \frac{k_1}{2|\mathbf{k}|^2}\chi_0'(\mathbf{k})\chi_{[h,0]}(\mathbf{k}) + O\left(e^3(M-1)\right).\end{aligned} \tag{3.24}$$

Notice that the cancellation *does not* depend on the presence of the bosonic IR cutoff; this fact will play an important role in the analysis at all orders of the flow of the effective charges, see next section. An explicit computation of (3.24) says that, at third order in e,

$$e^{(3)}_{\mu,-\infty} = e + e\alpha^{(2)}_\mu, \tag{3.25}$$

where

$$\alpha^{(2)}_0 = \frac{e^2}{8\pi^2}\left(2 - v^{-2}\right)\left(\frac{\xi_0 - \arctan\xi_0}{\xi_0^3}\right) + O(e^2(M-1)), \tag{3.26}$$

$$\alpha^{(2)}_1 = \frac{e^2}{16\pi^2}\frac{1}{v^2}\left(\frac{\arctan\xi_0}{\xi_0} - \frac{\xi_0 - \arctan\xi_0}{\xi_0^3}\right) + O(e^2(M-1)); \tag{3.27}$$

the correction terms $O(e^2(M-1))$ can be made as small as desired, by choosing $0 < M - 1 \ll 1$. Note that the two effective charges are different:

$$e^{(3)}_{0,-\infty} - e^{(3)}_{1,-\infty} = -\frac{e^3}{5\pi^2} F(v) + O\left(e^3(M-1)\right), \tag{3.28}$$

where $F(v)$ the function defined in (1.9). Combining (3.28) with (3.15) gives the last equation of (1.8)

Of course, the one-loop computation that we just described does not say much: if we could not guarantee that a similar cancellation takes place at all orders there would always be the possibility that higher orders produce a completely different behavior, e.g. a vanishing or diverging flow for $e_{\mu,h}$, corresponding to completely different physical properties of the system. In order to obtain a control at all orders on $e_{\mu,h}$ one needs to combine the multiscale evaluation of the effective potentials with Ward Identities. This is not a trivial task: Wilsonian RG methods are based on a multiscale momentum decomposition which breaks the local gauge invariance, which Ward Identities are based on. In Sect. 4 below, following a strategy recently proposed and developed in [6], we will prove (3.25).

Remark. Note the unusual feature that $e_{0,h} \neq e_{1,h}$, an effect due to the presence of the momentum cut-off and the fact that $v \neq 1$. The discussion of this and previous sections can be repeated in the case that the bare interaction involves two different charges, e_0 and e_1, describing the couplings of the photon field with the temporal and spatial components of the current. If $e = (e_0 + e_1)/2$ and $e_0 - e_1 = O(e^3)$, the conclusion is that $v_{\text{eff}} = 1 - (e^2/6\pi^2)F(v) + (5/6)(e_0 - e_1)/e + O(e^4)$ and it is of course possible to fine tune the bare parameters e_0 and e_1 in such a way that $e_{0,-\infty} = e_{1,-\infty}$ and $v_{\text{eff}} = 1$. Note that, in a more realistic model for graphene, describing tight binding electrons on the honeycomb lattice coupled with a 3D photon field via a lattice gauge invariant coupling, one expects that $e_{0,-\infty} = e_{1,-\infty}$ and $v_{\text{eff}} = 1$.

4. Ward Identities

In this section we prove that order by order in perturbation theory the effective charges $e_{\mu,h}$ remain close to their original values $e_{\mu,0} = e$; moreover, we prove that asymptotically as $h \to -\infty$, $e_{0,h} \neq e_{1,h}$, see (3.25)–(3.28). The proof is based on a suitable combination of the RG methods described in the previous sections together with Ward Identities; even though the momentum regularization breaks the local gauge invariance needed to formally derive the WIs, we will be able, following the strategy of [6], to rigorously take into account the effects of cutoffs, and to control the corrections generated by their presence.

As anticipated at the end of Sect. 2, we consider a sequence of models, to be called *reference models* in what follows, with different infrared bosonic cutoffs on scale h, *i.e.* with bosonic propagator given by:

$$w^{[h,0]}(\mathbf{p}) \equiv \frac{\chi_{[h,0]}(\mathbf{p})}{2|\mathbf{p}|}, \quad \chi_{[h,0]}(\mathbf{p}) \equiv \chi_0(\mathbf{p}) - \chi_0(M^{-h}\mathbf{p}) \tag{4.1}$$

(the idea of introducing an infrared cutoff only in the bosonic sector is borrowed from Adler and Bardeen [1], who used a similar regularization scheme in order to understand anomalies in quantum field theory). The generating functional $\mathcal{W}_{[h,0]}(J,\phi)$ of the correlations of the reference model can be evaluated following an iterative procedure similar to the one described in Sect. 2 (see Appendix B for details), with the important difference that after the integration of the scale h we are left with a purely fermionic theory, which is *superrenormalizable*: in fact, setting $m = 0$ in the formula for the scaling dimension (see lines following (2.27) and recall that for scales smaller than h the reference model has no bosonic lines) we recognize that the scaling dimension of this fermionic theory is $3-2n$, which is always negative once that the two-legged subdiagrams have been renormalized, see [18,19]. Let us denote by $\{e^{[h]}_{\mu,k}\}_{k\geq h}$ the effective couplings of the reference model; of course, if $k \geq h$

$$e^{[h]}_{\mu,k} = e_{\mu,k}, \tag{4.2}$$

where $\{e_{\mu,k}\}_{k\leq 0}$ are the running coupling constants of the original model. On the other hand, as proven in Appendix B, the vertex functions $\langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h$ of the reference model with bosonic cutoff on scale h computed at external momenta $\mathbf{k}$, $\mathbf{k}+\mathbf{p}$ such that $|\mathbf{k}+\mathbf{p}|, |\mathbf{k}| \simeq M^h$ and $|\mathbf{p}| \ll M^h$ are proportional to the charges $e^{[h]}_{\mu,h} = e_{\mu,h}$, see (2.34); therefore, if we get informations on the vertex functions of the reference models, we automatically infer informations on the effective couplings of the original model.

Such informations are provided by Ward Identities; by performing the change of variables $\psi_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$, $\overline{\psi}_{\mathbf{x}} \to e^{-i\alpha_{\mathbf{x}}}\overline{\psi}_{\mathbf{x}}$ in the generating functional $\mathcal{W}_{[h,0]}(J,\phi)$ of the reference model and using that the Jacobian of this transformation is equal to 1, see [6], we get:

$$e^{\mathcal{W}_{[h,0]}(J,\phi)} = \int P(\mathrm{d}\psi)P_{[h,0]}(dA)e^{-\int d\mathbf{x}\,\overline{\psi}_{\mathbf{x}}(e^{-i\alpha_{\mathbf{x}}}De^{i\alpha_{\mathbf{x}}}-D)\psi_{\mathbf{x}}+V(A,\psi)+B(J,\phi e^{-i\alpha})}, \tag{4.3}$$

where $P_{[h,0]}(\mathrm{d}A)$ is the gaussian integration with propagator (4.1) and, if $\not{\mathbf{k}} = \gamma_0 k_0 + v\vec{\gamma}\cdot\vec{k}$, the pseudo-differential operator D is defined by:

$$(D\psi)_{\mathbf{x}} = \int_{\chi(\mathbf{k})>0} \frac{d\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\mathbf{x}}}{\chi_0(\mathbf{k})} i\not{\mathbf{k}}\,\psi_{\mathbf{k}}.$$

If we derive (4.3) with respect to α, $\overline{\phi}$ and ϕ and then set $\alpha = \phi = J = 0$, we get the following identity:

$$p_\mu\langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h = \langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_h - \langle\psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}+\mathbf{p}}\rangle_h + \Delta_h(\mathbf{k},\mathbf{p}) \tag{4.4}$$

where

$$\Delta_h(\mathbf{k},\mathbf{p}) = \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} \langle \overline{\psi}_{\mathbf{k}'+\mathbf{p}} C(\mathbf{k}',\mathbf{p})\psi_{\mathbf{k}'}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h \tag{4.5}$$

and

$$C(\mathbf{k},\mathbf{p}) = i\not{k}\left(\chi_0(\mathbf{k})^{-1}-1\right) - i(\not{k}+\not{p})\left(\chi_0(\mathbf{k}+\mathbf{p})^{-1}-1\right). \tag{4.6}$$

The correction term $\Delta_h(\mathbf{k},\mathbf{p})$ in (4.4) is due to the presence of the ultraviolet momentum cut-off, and it can be computed by following a strategy analogous to the one used to prove the vanishing of the beta function in one-dimensional Fermi systems [6]. We can write

$$\Delta_h(\mathbf{k},\mathbf{p}) = \alpha_\mu p_\mu \langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h + \frac{p_\mu}{Z_h} R_{\mu,h}(\mathbf{k},\mathbf{p}), \tag{4.7}$$

where the correction $R_{\mu,h}(\mathbf{k},\mathbf{p})$ is dimensionally negligible with respect to the first term, see Appendix D. More precisely, in Appendix D it is shown that: (i) $R_{\mu,h}(\mathbf{k},\mathbf{p})$ can be written as a sum over trees with N endpoints of contributions $R^{(N)}_{\mu,h}(\mathbf{k},\mathbf{p})$; (ii) it is possible to choose α_μ in such a way that, under the same conditions of Theorem 2.1 and if $|\mathbf{k}| = M^h$, $|\mathbf{k}+\mathbf{p}| \le M^h$ and $|\mathbf{p}| \ll M^h$,

$$|R^{(N)}_{\mu,h}(\mathbf{k},\mathbf{p})| \le (\mathrm{const.})^N \left(\frac{N}{2}\right)! M^{-2h} M^{\frac{h}{2}} \bar{\varepsilon}_h^N. \tag{4.8}$$

An explicit computation, see Appendix D, shows that at lowest order $\alpha^{(2)}_\mu$ is given by (3.26)–(3.27).

Let us now show how to use the previous relations in order to derive bounds on the effective charges. Let us pick $|\mathbf{k}| = M^h$ and $|\mathbf{p}| \ll M^h$; using Eqs. (2.33)–(2.34) and the fact that

$$g^{(h)}(\mathbf{k}) - g^{(h)}(\mathbf{k}+\mathbf{p}) = g^{(h)}(\mathbf{k}+\mathbf{p})(ip_0\gamma_0 + iv_{h-1}\vec{p}\cdot\vec{\gamma})g^{(h)}(\mathbf{k}) + p_\mu \hat{r}_\mu(\mathbf{k},\mathbf{p}), \tag{4.9}$$

with $\hat{r}_\mu(\mathbf{k},\mathbf{p}) = O(|\mathbf{p}|M^{-3h})$, we find that

$$\begin{aligned}\langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_h - \langle\psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}+\mathbf{p}}\rangle_h &= \frac{1}{Z_{h-1}} g^{(h)}(\mathbf{k}+\mathbf{p})(ip_0\gamma_0 + iv_{h-1}\vec{p}\cdot\vec{\gamma})g^{(h)}(\mathbf{k}) \\ &\quad + \frac{p_\mu}{Z_{h-1}}\left(\tilde{r}_\mu(\mathbf{k},\mathbf{p}) + \hat{r}_\mu(\mathbf{k},\mathbf{p})\right), \end{aligned} \tag{4.10}$$

$$\begin{aligned}p_\mu\langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h &= \frac{1}{eZ_{h-1}} g^{(h)}(\mathbf{k}+\mathbf{p})(ie_{0,h}p_0\gamma_0 + iv_{h-1}e_{1,h}\vec{p}\cdot\vec{\gamma}) \\ &\quad \times g^{(h)}(\mathbf{k}) + \frac{p_\mu}{Z_{h-1}} r_\mu(\mathbf{k},\mathbf{p}),\end{aligned} \tag{4.11}$$

with $|r_\mu(\mathbf{k},\mathbf{p})|, |\tilde{r}_\mu(\mathbf{k},\mathbf{p})|$ expressed by sums over trees of order $N \ge 2$ of contributions $r^{(N)}_\mu(\mathbf{k},\mathbf{p}), \tilde{r}^{(N)}_\mu(\mathbf{k},\mathbf{p})$ bounded by (see Appendix B, formulas (B.13) and (B.16))

$$|r^{(N)}_\mu(\mathbf{k},\mathbf{p})| + |\tilde{r}^{(N)}_\mu(\mathbf{k},\mathbf{p})| \le (\mathrm{const.})^N \bar{\varepsilon}_h^N \left(\frac{N}{2}\right)! \, M^{-2h}. \tag{4.12}$$

Now, if we plug (4.7) into the Ward identity (4.4), and we use the relations (4.10)–(4.11), we get an identity that, computed at $\mathbf{k} = \mathbf{k}_0 := (M^h, \vec{0})$ and $\mathbf{p} = \mathbf{p}_0 := (p, \vec{0})$, after taking the limit $p \to 0$, reduces to:

$$\begin{aligned}\frac{e_{0,h}}{e}(1-\alpha_0) &= 1 + iM^{2h}\left[\tilde{r}_0(\mathbf{k}_0,\mathbf{0}) + R_{0,h}(\mathbf{k}_0,\mathbf{0}) - (1-\alpha_0)r_0(\mathbf{k}_0,\mathbf{0})\right]\gamma_0 \\ &\equiv 1 + A_{0,h}, \end{aligned} \tag{4.13}$$

with $A_{0,h}$ a sum of contributions associated to trees of order $N \geq 2$ bounded at the N-th order by $(\text{const.})^N(\bar{\varepsilon}_h)^N(N/2)!$, as it follows from the estimates on $R_{0,h}, r_0, \tilde{r}_0$ (note the crucial point that such estimate is proportional to $(\bar{\varepsilon}_h)^N$ rather than to $(\bar{\varepsilon}_{-\infty})^N$; this is the main reason why we chose to introduce the infrared cutoff on the bosonic propagator, see the end of Sect. 2.5 and the beginning of this section). Eq. (4.13) combined with (3.26) implies, as desired, that *the effective charge* $e_{0,h}$ *remains close to* $e_{0,0} = e$ *at all orders in renormalized perturbation theory*. Moreover, proceeding as in the derivation of Eq. (3.13), we find that $|A_{0,h} - A_{0,-\infty}| = O(e^2(v_{\text{eff}} - v_h)) + O(e^2 M^{\theta h})$, for some $0 < \theta < 1$, from which we get Eq. (3.9) for $\mu = 0$. Similarly, if $\mathbf{k}_1 := (0, M^h, 0)$, we get:

$$\begin{aligned}\frac{e_{1,h}}{e}(1-\alpha_1) &= \frac{v_{h-1}}{v_h} + iM^{2h}v_{h-1} \\ &\quad\times\left[\tilde{r}_1(\mathbf{k}_1,\mathbf{0}) + R_{1,h}(\mathbf{k}_1,\mathbf{0}) - (1-\alpha_1)r_1(\mathbf{k}_1,\mathbf{0})\right]\gamma_1 \\ &\equiv 1 + A_{1,h}, \end{aligned} \tag{4.14}$$

with $A_{1,h}$ a sum of contributions associated to trees of order $N \geq 2$ bounded at the N-th order by $(\text{const.})^N(\bar{\varepsilon}_h)^N(N/2)!$, which implies that *the effective charge* $e_{1,h}$ *remains close to* $e_{1,0} = e$ *at all orders in renormalized perturbation theory*. Moreover, as in the $\mu = 0$ case, $|A_{1,h} - A_{1,-\infty}| = O(e^2(v_{\text{eff}} - v_h)) + O(e^2 M^{\theta h})$, for some $0 < \theta < 1$, from which we get Eq. (3.9) for $\mu = 1$.

Equations (4.13) and (4.14) not only imply the boundedness of the effective charges $e_{\mu,h}$ but they also allow us to compute the difference $e_{0,h} - e_{1,h}$, asymptotically as $h \to -\infty$, at all orders in renormalized perturbation theory. At lowest order, $e^{(3)}_{0,h} - e^{(3)}_{1,h} = e(\alpha^{(2)}_0 - \alpha^{(2)}_1)$, as anticipated in previous section.

5. Conclusions

We considered an effective continuum model for the low energy physics of single-layer graphene, first introduced by Gonzalez et al. [20]. We analyzed it by *constructive Renormalization Group* methods, which have already been proved effective in the *non-perturbative* study of several low-dimensional fermionic models, such as one-dimensional interacting fermions [6], or the Hubbard model on the honeycomb lattice [18,19]. While in the present case we are not able yet to prove the convergence of the renormalized expansion, we can prove that it is *order by order finite*, see Theorem 2.1 above. Note that, on the contrary, the power series expansion in the bare couplings is plagued by *logarithmic divergences* and, therefore, informations obtained from it by lowest order truncation are quite unreliable. In perspective, the proof of convergence of the renormalized expansion appears to be much more difficult than the one in [6] or [18,19], due to the simultaneous presence of bosons and fermions, but it should be feasible (using determinant bounds for the fermionic sector and cluster expansion techniques for the boson sector).

A key point of our analysis is the control at all orders of the flow of the effective couplings: this is obtained via Ward Identities relating three- and two-point functions, using a technique developed in [6] for the analysis of Luttinger liquids, in cases where bosonization cannot be applied (like in the presence of an underlying lattice or of non-linear bands). The Ward Identities have *corrections* with respect to the formal ones, due to the presence of a fermionic ultraviolet cut-off. Remarkably, these corrections can be rigorously bounded at all orders in renormalized perturbation theory (see Sect. 4).

Several questions remain to be understood. First of all, the effective model we considered is clearly not fundamental: a more realistic model for graphene should be obtained by considering electrons on the honeycomb lattice coupled to an electromagnetic field living in the 3D continuum. We believe that a Renormalization Group analysis, similar to the one we performed here, is possible also for the lattice model, by combining the techniques and results of [18,19] with those of the present paper; we expect that the lattice model is asymptotic to the continuum one considered here, provided that the bare parameters of the continuum model are properly tuned. Another important open problem is to understand the behavior of the system in the case of static Coulomb interactions; this case can be obtained by taking the limit $c \to \infty$ together with a proper rescaling of the electronic charge in the model with retarded interactions. However, as discussed in the Remark at the end of Sect. 2.4, the static case seems to be much more subtle than the one considered in this paper, since it apparently requires cancellations even to prove renormalizability of the theory at all orders. We plan to come back to this case in a future publication.

Acknowledgements

A.G. and V.M. gratefully acknowledge financial support from the ERC Starting Grant CoMBoS-239694. We thank D. Haldane and M. Vozmediano for many valuable discussions.

Appendix A. Symmetries

In this Appendix we prove formulas (2.6) and (2.7); to do this, we exploit suitable symmetry transformations. We use the following explicit representation of the euclidean gamma matrices:

$$\gamma_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}, \tag{A.1}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. It is also useful to define: $\gamma_3 = \begin{pmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix}$, with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the corresponding fifth gamma matrix

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{A.2}$$

which anticommutes with all the other gamma matrices: $\{\gamma_\mu, \gamma_5\} = 0$, $\forall \mu = 0, \ldots, 3$. Finally, given $\omega \in \{+, -\}$, we define the chiral projector $P_\omega := (1 + \omega\gamma_5)/2$.

It is straightforward to check that both the gaussian integrations $P(\mathrm{d}\psi)$, $P(\mathrm{d}A)$ and the interaction $V(A, \psi)$ are invariant under the following symmetry transformations, which are preserved by the multiscale integration:

(1) *Chirality*: $P_\omega\psi_{\mathbf{k}} \to e^{-i\alpha_\omega} P_\omega\psi_{\mathbf{k}}$, $\overline{\psi}_{\mathbf{k}} P_{-\omega} \to \overline{\psi}_{\mathbf{k}} P_{-\omega} e^{+i\alpha_\omega}$, with $\alpha_\omega \in \mathbb{R}$ independent of $\mathbf{k}$.

(2) *Spatial rotations*: $\psi_{\mathbf{k}} \to e^{\frac{\theta}{4}[\gamma_1,\gamma_2]} \psi_{R^{[1,2]}_{-\theta}\mathbf{k}}$, $\overline{\psi}_{\mathbf{k}} \to \overline{\psi}_{R^{[1,2]}_{-\theta}\mathbf{k}} e^{-\frac{\theta}{4}[\gamma_1,\gamma_2]}$ and $A_{\mu,\mathbf{p}} \to \left[R^{[1,2]}_\theta A_{\cdot, R^{[1,2]}_{-\theta}\mathbf{p}}\right]_\mu$, with

$$R^{[1,2]}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{A.3}$$

The invariance of the model under (2) is a simple consequence of the fact that

$$\begin{aligned} & e^{-\frac{\theta}{4}[\gamma_1,\gamma_2]} \left(\gamma_0, \gamma_1, \gamma_2\right) e^{\frac{\theta}{4}[\gamma_1,\gamma_2]} \\ & \quad = (\gamma_0, \gamma_1\cos\theta - \gamma_2\sin\theta, \gamma_2\cos\theta + \gamma_1\sin\theta). \end{aligned} \tag{A.4}$$

(3) *Complex conjugation*: $\psi_{\mathbf{k}} \to (-i\gamma_2)\psi_{-\mathbf{k}}$, $\overline{\psi}_{\mathbf{k}} \to \overline{\psi}_{-\mathbf{k}}(i\gamma_2)$, $A_{\mu,\mathbf{k}} \to -A_{\mu,-\mathbf{k}}$ and $\kappa \to \kappa^*$, where κ is a generic constant appearing in $P(\mathrm{d}\psi)$, $P(\mathrm{d}A)$ and/or in $V(A,\psi)$.

(4.a) *Horizontal reflections*: $\psi_{\mathbf{k}} \to (i\gamma_3\gamma_1)\psi_{\tilde{\mathbf{k}}}$, $\overline{\psi}_{\mathbf{k}} \to \overline{\psi}_{\tilde{\mathbf{k}}}(-i\gamma_1\gamma_3)$ and $A_{\mu,\mathbf{p}} \to (-1)^\mu A_{\mu,\tilde{\mathbf{p}}}$, where $\tilde{\mathbf{k}} = (k_0, -k_1, k_2)$.

(4.b) *Vertical reflections*: $\psi_{\mathbf{k}} \to (-i\gamma_2)\psi_{\tilde{\mathbf{k}}}$, $\overline{\psi}_{\mathbf{k}} \to \overline{\psi}_{\tilde{\mathbf{k}}}(i\gamma_2)$ and $A_{\mu,\mathbf{p}} \to (-1)^{\delta_{\mu,2}} A_{\mu,\tilde{\mathbf{p}}}$, where $\tilde{\mathbf{k}} = (k_0, k_1, -k_2)$.

(5) *Particle-hole*: $\psi_{\mathbf{k}} \to (-\gamma_0\gamma_2)\overline{\psi}^T_{\tilde{\mathbf{k}}}$, $\overline{\psi}_{\mathbf{k}} \to \psi^T_{\tilde{\mathbf{k}}}\gamma_2\gamma_0$, and $A_{\mu,\mathbf{p}} \to (-1)^{1-\delta_{\mu,0}} A_{\mu,\tilde{\mathbf{p}}}$, where $\tilde{\mathbf{k}} = (k_0, -\vec{k})$.

(6) *Inversion*: $\psi_{\mathbf{k}} \to \gamma_0\gamma_3\psi_{\tilde{\mathbf{k}}}$, $\overline{\psi}_{\mathbf{k}} \to \overline{\psi}_{\tilde{\mathbf{k}}}\gamma_3\gamma_0$ and $A_{\mu,\mathbf{k}} \to (-1)^{\delta_{\mu,0}} A_{\mu,\tilde{\mathbf{p}}}$, where $\tilde{\mathbf{k}} = (-k_0, \vec{k})$.

In addition to the previous symmetries, if $v = c = 1$ the theory has an additional space–time invariance, namely:

(7) *Relativistic invariance*: $\psi_{\mathbf{k}} \to e^{\frac{\theta}{4}[\gamma_0,\gamma_1]}\psi_{R^{[0,1]}_{-\theta}\mathbf{k}}$, $\overline{\psi}_{\mathbf{k}} \to \overline{\psi}_{R^{[0,1]}_{-\theta}\mathbf{k}} e^{-\frac{\theta}{4}[\gamma_0,\gamma_1]}$ and $A_{\mu,\mathbf{p}} \to \left[R_\theta A_{\cdot,R_\theta^{-1}\mathbf{p}}\right]_\mu$, with

$$R^{[0,1]}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.5}$$

The invariance of the model under (7) is a simple consequence of the remark that

$$e^{-\frac{\theta}{4}[\gamma_0,\gamma_1]}\,(\gamma_0,\gamma_1,\gamma_2)\,e^{\frac{\theta}{4}[\gamma_0,\gamma_1]} = (\gamma_0\cos\theta - \gamma_1\sin\theta, \gamma_1\cos\theta + \gamma_0\sin\theta, \gamma_2). \tag{A.6}$$

It is now straightforward to check that these symmetries imply (2.6), (2.7). In fact, the first two identities in the first line of (2.6) and the second identity in the second line of (2.6) easily follow from (4.a) + (4.b) + (6). Using (4.a) + (4.b) + (6) we also find that

$$W^{(h)}_{2,0,\mu,\nu}(\mathbf{0}) = \delta_{\mu\nu} W^{(h)}_{2,0,\mu,\mu}(\mathbf{0}), \tag{A.7}$$

while, from (2) + (3), we get

$$W^{(h)}_{2,0,1,1}(\mathbf{0}) = W^{(h)}_{2,0,2,2}(\mathbf{0}), \quad W^{(h)}_{2,0,\mu,\mu}(\mathbf{0}) \in \mathbb{R}, \tag{A.8}$$

which imply the first identity in the second line of (2.6) (notice that, if $v = c = 1$, from (7) we also get that $W^{(h)}_{2,0,0,0}(\mathbf{0}) = W^{(h)}_{2,0,1,1}(\mathbf{0})$).

Let us now consider the combination $\overline{\psi}_{\mathbf{k}} W^{(h)}_{0,1}(\mathbf{k})\psi_{\mathbf{k}}$. Using the fact that $\{I, \gamma_5, \{\gamma_j\}_{0\le j\le 3}, \{\gamma_j\gamma_5\}_{0\le j\le 3}, \{\gamma_{j_1}\gamma_{j_2}\}_{0\le j_1<j_2\le 3}\}$ is a complete basis for the space of complex 4×4 matrices, we can rewrite it as:

$$\overline{\psi}_{\mathbf{k}} W^{(h)}_{0,1}(\mathbf{k})\psi_{\mathbf{k}} = \overline{\psi}_{\mathbf{k}} \Bigg\{ c_0(\mathbf{k})I + \sum_{j=0}^{3}\left[c_1^j(\mathbf{k})\gamma_j + c_{15}^j(\mathbf{k})\gamma_j\gamma_5\right] + \sum_{0\le j_1<j_2\le 5} c_2^{j_1 j_2}(\mathbf{k})\gamma_{j_1}\gamma_{j_2} + c_5(\mathbf{k})\gamma_5 \Bigg\}\psi_{\mathbf{k}}. \tag{A.9}$$

Now, using the invariance under (1), we find that, e.g.,

$$\overline{\psi}_{\mathbf{k}} c_0(\mathbf{k}) I \psi_{\mathbf{k}} = \sum_{\omega=\pm} \overline{\psi}_{\mathbf{k}} P_\omega c_0(\mathbf{k}) I P_\omega \psi_{\mathbf{k}} = e^{-2i\alpha_\omega} \sum_{\omega=\pm} \overline{\psi}_{\mathbf{k}} P_\omega c_0(\mathbf{k}) I P_\omega \psi_{\mathbf{k}}, \tag{A.10}$$

for all $\alpha_\omega \in \mathbb{R}$, which implies that $c_0(\mathbf{k}) = 0$; similarly, using the invariance under (1) and the fact that $[\gamma_5, P_\omega] = [\gamma_{j_1}\gamma_{j_2}, P_\omega] = 0$, $\forall 0 \le j_1 < j_2 \le 3$, we find that $c_5(\mathbf{k}) = 0$ and $c_2^{j_1 j_2}(\mathbf{k}) = 0$, $\forall 0 \le j_1 < j_2 \le 3$. Therefore,

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \overline{\psi}_{\mathbf{k}} W_{0,1}^{(h)}(\mathbf{0}) \psi_{\mathbf{k}} = \int \sum_{j=0}^{3} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \overline{\psi}_{\mathbf{k}} \left[c_1^j(\mathbf{0})\gamma_j + c_{15}^j(\mathbf{0})\gamma_j\gamma_5 \right] \psi_{\mathbf{k}}, \tag{A.11}$$

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \overline{\psi}_{\mathbf{k}} \mathbf{k}\partial_{\mathbf{k}} W_{0,1}^{(h)}(\mathbf{0}) \psi_{\mathbf{k}} = \sum_{j=0}^{3} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \overline{\psi}_{\mathbf{k}} \left[\mathbf{k}\partial_{\mathbf{k}} c_1^j(\mathbf{0})\gamma_j + \mathbf{k}\partial_{\mathbf{k}} c_{15}^j(\mathbf{0})\gamma_j\gamma_5 \right] \psi_{\mathbf{k}}. \tag{A.12}$$

Let us first look at (A.11). Using the invariance under (4.a), we find that $[c_1^j(\mathbf{0})\gamma_j + c_{15}^j(\mathbf{0})\gamma_j\gamma_5] = \gamma_3\gamma_1[c_1^j(\mathbf{0})\gamma_j + c_{15}^j(\mathbf{0})\gamma_j\gamma_5]\gamma_1\gamma_3$, which implies that $c_1^1(\mathbf{0}) = c_1^3(\mathbf{0}) = c_{15}^1(\mathbf{0}) = c_{15}^3(\mathbf{0}) = 0$. Using (2), we find that also $c_1^2(\mathbf{0}) = c_{15}^2(\mathbf{0}) = 0$; finally, using (6), we find that $c_1^0(\mathbf{0}) = c_{15}^0(\mathbf{0}) = 0$. This concludes the proof of the third identity in the first line of (2.6).

Let us now look at (A.12). The terms proportional to k_0 in the r.h.s. of (A.12) are invariant under (2) + (4.a) + (4.b), which implies that $\partial_{k_0} c_1^1(\mathbf{0}) = \partial_{k_0} c_1^2(\mathbf{0}) = \partial_{k_0} c_1^3(\mathbf{0}) = \partial_{k_0} c_{15}^j(\mathbf{0}) = 0$. The terms proportional to k_1 are invariant under (4.b) + (6), while the terms proportional to k_2 are invariant under (4.a) + (6); combining these transformations with (2), we find that $\partial_{k_1} c_1^0(\mathbf{0}) = \partial_{k_1} c_1^2(\mathbf{0}) = \partial_{k_1} c_1^3(\mathbf{0}) = \partial_{k_1} c_{15}^j(\mathbf{0}) = 0$, that $\partial_{k_2} c_1^0(\mathbf{0}) = \partial_{k_2} c_1^1(\mathbf{0}) = \partial_{k_2} c_1^3(\mathbf{0}) = \partial_{k_2} c_{15}^j(\mathbf{0}) = 0$, and that $\partial_{k_1} c_1^1(\mathbf{0}) = \partial_{k_2} c_1^2(\mathbf{0})$. Therefore,

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \overline{\psi}_{\mathbf{k}} \mathbf{k}\partial_{\mathbf{k}} W_{0,1}^{(h)}(\mathbf{0}) \psi_{\mathbf{k}} = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \overline{\psi}_{\mathbf{k}} \left[a_0 k_0 \gamma_0 + a_1 \vec{k} \cdot \vec{\gamma} \right] \psi_{\mathbf{k}}, \tag{A.13}$$

for two suitable constants a_0, a_1. Using the invariance under (3), we find that $a_0 = iz_{0,h}$ and $a_1 = iz_{1,h}$, with $z_{\mu,h} \in \mathbb{R}$, which concludes the proof of the first line of (2.7) (of course, if $v = c = 1$, then from (7) we also get that $z_{0,h} = z_{1,h}$, that is *the speed of light is not renormalized*).

A completely analogous discussion can be repeated for the second line of (2.7), but we will not belabor the details here.

Appendix B. Multiscale Integration for the Correlation Functions

The multiscale integration used to compute the partition function $\mathcal{W}(0,0)$, described in Sect. 2, can be suitably modified in order to compute the two and three-point correlation functions in the reference model with bosonic infrared cutoff on scale h, see (4.1). We start by rewriting the two and three point Schwinger functions in the following way:

$$\begin{aligned} \langle \psi_{\mathbf{k}} \overline{\psi}_{\mathbf{k}} \rangle_{h^*} &= \frac{\partial^2}{\partial \overline{\phi}_{\mathbf{k}} \partial \phi_{\mathbf{k}}} \mathcal{W}_{[h^*,0]}(J,\phi)\big|_{J=\phi=0}, \\ \langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}} \overline{\psi}_{\mathbf{k}} \rangle_{h^*} &= \frac{\partial^3}{\partial J_{\mu,\mathbf{p}} \partial \overline{\phi}_{\mathbf{k}+\mathbf{p}} \partial \phi_{\mathbf{k}}} \mathcal{W}_{[h^*,0]}(J,\phi)\big|_{J=\phi=0} \end{aligned} \tag{B.1}$$

where $\mathcal{W}_{[h^*,0]}(J,\phi)$ is the generating function of the reference model. The two point Schwinger function $\langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle$ appearing in our main result is obtained as $\lim_{h^*\to-\infty}\langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_{h^*}$.

In order to compute $\mathcal{W}_{[h^*,0]}(J,\phi)$, we proceed in a way analogous to the one described in Sect. 2. We iteratively integrate the fields $\psi^{(0)}, A^{(0)}, \ldots, \psi^{(h+1)}, A^{(h+1)}, \ldots$, and after the integration of the first $|h|$ scales we are left with a functional integral similar to (2.2), but now involving new terms depending on J,ϕ. Let us first consider the case $h\geq h^*$; the regime $h<h^*$ will be discussed later.

Case $h\geq h^*$. We want to inductively prove that

$$\begin{aligned}
&e^{\mathcal{W}_{[h^*,0]}(J,\phi)}\\
&\quad = e^{|\Lambda|E_h+\mathcal{S}^{(\geq h)}(J,\phi)}\int P(\mathrm{d}\psi^{(\leq h)})P_{[h^*,0]}(dA^{(\leq h)})e^{\mathcal{V}^{(h)}(A^{(\leq h)}+G_AJ,\sqrt{Z_h}\psi^{(\leq h)})}\\
&\qquad \times e^{\mathcal{B}^{(h)}_{\phi}(A^{(\leq h)}+G_AJ,\sqrt{Z_h}\psi^{(\leq h)},\phi)+W^{(h)}_R(A^{(\leq h)}+G_AJ,\sqrt{Z_h}\psi^{(\leq h)},\phi)}, \qquad \text{(B.2)}
\end{aligned}$$

where $\mathcal{S}^{(\geq h)}(J,\phi)$ is independent of (A,ψ), $W^{(h)}_R$ contains terms explicitly depending on (A,ψ) and of order ≥ 2 in ϕ, while $\mathcal{B}^{(h)}_{\phi}$ is given by:

$$\begin{aligned}
\mathcal{B}^{(h)}_{\phi}(A,\sqrt{Z_h}\psi,\phi) &= \int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\left[\overline{\phi}_{\mathbf{k}}[Q^{(h+1)}(\mathbf{k})]^{\dagger}\psi_{\mathbf{k}}+\overline{\psi}_{\mathbf{k}}Q^{(h+1)}(\mathbf{k})\phi_{\mathbf{k}}\right]\\
&\quad+\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\left[\overline{\phi}_{\mathbf{k}}[G^{(h+1)}_{\psi}(\mathbf{k})]^{\dagger}\frac{\partial}{\partial\overline{\psi}_{\mathbf{k}}}\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)\right.\\
&\quad\left.+\frac{\partial}{\partial\psi_{\mathbf{k}}}\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)G^{(h+1)}_{\psi}(\mathbf{k})\phi_{\mathbf{k}}\right]. \qquad \text{(B.3)}
\end{aligned}$$

Moreover, the functions G_A, $Q^{(h)}$, $G^{(h)}_{\psi}$ are defined by the following relations:

$$\begin{aligned}
&eG_{A,\mu}(\mathbf{p}) := 1+\nu_{\mu}w^{[h^*,0]}(\mathbf{p}),\quad G^{(h)}_{\psi}(\mathbf{k}) := \sum_{i=h}^{0}\frac{g^{(i)}(\mathbf{k})}{Z_{i-1}}Q^{(i)}(\mathbf{k}),\\
&Q^{(h)}(\mathbf{k}) := Q^{(h+1)}(\mathbf{k})-iZ_hz_{\mu,h}k_{\mu}\gamma_{\mu}G^{(h+1)}_{\psi}(\mathbf{k}),
\end{aligned} \qquad \text{(B.4)}$$

with $Q^{(1)}(\mathbf{k})\equiv 1$, $G^{(1)}(\mathbf{k})\equiv 0$. Note that, if $\mathbf{k}$ is in the support of $g^{(h)}(\mathbf{k})$,

$$\begin{aligned}
Q^{(h)}(\mathbf{k}) &= 1-iz_{\mu,h}k_{\mu}\gamma_{\mu}g^{(h+1)}(\mathbf{k}),\\
G^{(h)}_{\psi}(\mathbf{k}) &= \frac{g^{(h)}(\mathbf{k})}{Z_{h-1}}Q^{(h)}(\mathbf{k})+\frac{g^{(h+1)}(\mathbf{k})}{Z_h},
\end{aligned} \qquad \text{(B.5)}$$

that is $||Q^{(h)}(\mathbf{k})-1||\leq(\text{const.})\,\bar{\varepsilon}^2_h$ and $||G^{(h)}_{\psi}(\mathbf{k})||\leq(\text{const.})\,Z_h^{-1}M^{-h}$. Moreover, by the compact support properties of $w^{[h^*,0]}(\mathbf{p})$, $G_{A,\mu}(\mathbf{p})\equiv e^{-1}$ for all $|\mathbf{p}|\leq M^{h^*}$.

In order to prove (B.2)–(B.4) by induction, let us first check them at the first step. The generating functional of the correlations is defined as (see (1.1)–(1.2))

$$e^{\mathcal{W}_{[h^*,0]}(J,\phi)} = \int P(\mathrm{d}\psi^{(\leq 0)})P_{[h^*,0]}(dA^{(\leq 0)})e^{\int \frac{d\mathbf{p}}{(2\pi)^3}(eA^{(\leq 0)}_{\mu,\mathbf{p}}+J_{\mu,\mathbf{p}})j^{(\leq 0)}_{\mu,-\mathbf{p}}-\nu_\mu(A^{(\leq 0)}_\mu,\,A^{(\leq 0)}_\mu)+B(0,\phi)} \tag{B.6}$$

which, under the change of variables

$$A^{(\leq 0)}_{\mu,\mathbf{p}} \to A^{(\leq 0)}_{\mu,\mathbf{p}} + e^{-1}\nu_\mu w^{[h^*,0]}(\mathbf{p})J_{\mu,\mathbf{p}}; \tag{B.7}$$

can be rewritten in the form (B.2), with

$$\begin{aligned} E_h = 0, \quad & e^2\mathcal{S}^{(\geq 0)} = \nu_\mu(J_\mu,\,J_\mu) + \nu_\mu^2(J_\mu,\,w^{[h^*,0]}J_\mu), \\ W_R^{(0)} = 0, \quad & \mathcal{V}^{(0)} = V, \quad \mathcal{B}^{(0)}_\phi(A+G_AJ,\psi,\phi) = B(0,\phi). \end{aligned} \tag{B.8}$$

Let us now assume that (B.2)–(B.4) are valid at scales $\geq h$, and let us prove that the inductive assumption is reproduced at scale $h-1$. We proceed as in Sect. 2; first, we renormalize the free measure by reabsorbing into $\widetilde{P}(\mathrm{d}\psi^{(\leq h)})$ the term $\exp\{\mathcal{L}_\psi\mathcal{V}^{(h)}\}$, see (2.8)–(2.11), and then we rescale the fields as in (2.12). Similarly, in the definition of $\mathcal{B}^{(h)}_\phi$, Eq. (B.3), we rewrite $\mathcal{V}^{(h)} = \mathcal{L}_\psi\mathcal{V}^{(h)} + \hat{\mathcal{V}}^{(h)}$, combine the terms proportional to $\mathcal{L}_\psi\mathcal{V}^{(h)}$ with those proportional to $Q^{(h+1)}$, and rewrite

$$\begin{aligned} \mathcal{B}^{(h)}(A,\sqrt{Z_h}\psi,\phi) &= \hat{\mathcal{B}}^{(h)}(A,\sqrt{Z_{h-1}}\psi,\phi) \\ &:= \int \frac{d\mathbf{k}}{(2\pi)^3}\left[\overline{\phi}_\mathbf{k}[Q^{(h)}(\mathbf{k})]^\dagger\psi_\mathbf{k} + \overline{\psi}_\mathbf{k}Q^{(h)}(\mathbf{k})\phi_\mathbf{k}\right] \\ &\quad + \int \frac{d\mathbf{k}}{(2\pi)^3}\left[\overline{\phi}_\mathbf{k}[G^{(h+1)}_\psi(\mathbf{k})]^\dagger \frac{\partial}{\partial\overline{\psi}_\mathbf{k}}\hat{\mathcal{V}}^{(h)}(A,\sqrt{Z_{h-1}}\psi)\right. \\ &\quad \left. + \frac{\partial}{\partial\psi_\mathbf{k}}\hat{\mathcal{V}}^{(h)}(A,\sqrt{Z_{h-1}}\psi)G^{(h+1)}_\psi(\mathbf{k})\phi_\mathbf{k}\right], \end{aligned}$$

with $Q^{(h)}$ defined by (B.4). Finally, we rescale $W_R^{(h)}$, by defining

$$\hat{W}_R^{(h)}(A+G_AJ,\sqrt{Z_{h-1}}\psi) := W_R^{(h)}(A+G_AJ,\sqrt{Z_h}\psi),$$

and perform the integration on scale h:

$$\begin{aligned} &\int P(\mathrm{d}\psi^{(h)})P(\mathrm{d}A^{(h)})e^{\hat{\mathcal{V}}^{(h)}(A^{(\leq h)}+G_AJ,\sqrt{Z_{h-1}}\psi^{(\leq h)})} \\ &\qquad \times e^{\hat{\mathcal{B}}^{(h)}_\phi(A^{(\leq h)}+G_AJ,\sqrt{Z_{h-1}}\psi^{(\leq h)})+\hat{W}_R^{(h)}} \\ &\equiv e^{|\Lambda|\tilde{E}_h+\mathcal{S}^{(h-1)}(J,\phi)+\mathcal{V}^{(h-1)}(A^{(\leq h-1)}+G_AJ,\sqrt{Z_{h-1}}\psi^{(\leq h-1)})} \\ &\qquad \times e^{\mathcal{B}^{(h-1)}_\phi(A^{(\leq h-1)}+G_AJ,\sqrt{Z_{h-1}}\psi^{(\leq h-1)})+W_R^{(h-1)}}, \end{aligned}$$

where $\mathcal{S}^{(h-1)}(J,\phi)$ contains terms depending on (J,ϕ) but independent of $A^{(\leq h-1)}$, $\psi^{(\leq h-1)}$. Defining $\mathcal{S}^{(\geq h-1)} := \mathcal{S}^{(h-1)} + \mathcal{S}^{(\geq h)}$, we immediately see that the inductive assumption is reproduced on scale $h-1$.

Case $h < h^*$. For scales smaller than h^*, there are no more bosonic fields to be integrated out, and we are left with a purely fermionic theory, with scaling dimensions $3-2n$, $2n$ being the number of external fermionic legs, see Theorem 2.1 and following lines. Therefore, once that the two-legged subdiagrams have been renormalized and step by step reabsorbed into the free fermionic measure, we are left with a superrenormalizable theory, as in [18,19]. In particular, the four fermions interaction is irrelevant, while the wave function renormalization and the Fermi velocity are modified by a finite amount with respect to their values at h^*; that is, if $\bar\varepsilon_{h^*} = \max_{k\ge h^*}\{|e_{\mu,k}|, |\nu_{\mu,k}|\}$:

$$Z_h = Z_{h^*}(1+O(\bar\varepsilon_{h^*}^2)), \quad v_h = v_{h^*}(1+O(\bar\varepsilon_{h^*}^2)). \tag{B.9}$$

Tree expansion for the two-point function. As for the partition function (see Sect. 2), the kernels of the effective potentials produced by the multiscale integration of $\mathcal{W}_{[h^*,0]}(J,\phi)$ can be represented as sums over trees, which in turn can be evaluated as sums over Feynman graphs. Let us consider first the expansion for the two-point Schwinger function. After having taken functional derivatives with respect to $\phi_{\mathbf{k}}$, $\overline{\phi}_{\mathbf{k}}$ and after having set $J=\phi=0$, we get an expansion in terms of a new class of trees $\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}$, with $\bar k\in(-\infty,-1]$ the scale of the root and $\bar h>\bar k$; these trees are similar to the ones described in section 2, up to the following differences.

(1) There are $N+2$ end–points and two of them, called v_1, v_2, are special and, respectively, correspond to $\left[Q^{(h_{v_1}-1)}(\mathbf{k})\right]^\dagger \psi^{(\le h_{v_1}-1)}_{\mathbf{k}}$ or to $\overline{\psi}^{(\le h_{v_2}-1)}_{\mathbf{k}} Q^{(h_{v_2}-1)}(\mathbf{k})$.
(2) The first vertex whose cluster contains both v_1, v_2, denoted by $\bar v$, is on scale $\bar h$. No $\mathcal{R}$ operation is associated to the vertices on the line joining $\bar v$ to the root.
(3) There are no lines external to the cluster corresponding to the root.
(4) There are no bosonic lines external to clusters on scale $h<h^*$.

In terms of the new trees, we can expand the two-point Schwinger function as:

$$\begin{aligned}\langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_{h^*} &= \sum_{j=h_{\mathbf{k}}}^{h_{\mathbf{k}}+1}[Q^{(j)}_\psi(\mathbf{k})]^\dagger\frac{g^{(j)}(\mathbf{k})}{Z_{j-1}}Q^{(j)}(\mathbf{k})\\ &\quad+\sum_{N=2}^{\infty}\sum_{\bar h=-\infty}^{0}\sum_{\bar k=-\infty}^{\bar h-1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}}\mathcal{S}_2(\tau;\mathbf{k}),\end{aligned} \tag{B.10}$$

where $h_{\mathbf{k}}<0$ is the integer such that $M^{h_{\mathbf{k}}}\le|\mathbf{k}|<M^{h_{\mathbf{k}}+1}$, and $\mathcal{S}_2(\tau;\mathbf{k})$ is defined in a way similar to $\mathcal{V}^{(h)}(\tau)$ in (2.23), modulo the modifications described in items (1)–(4) above. Using the bounds described immediately after (B.5), which are valid for $\mathbf{k}$ belonging to the support of $g^{(h)}(\mathbf{k})$, and proceeding as in Sect. 2.4, we get bounds on $\mathcal{S}_2(\tau;\mathbf{k})$, which are the analogues of Theorem 2.1:

$$\sum_{\bar h=-\infty}^{0}\sum_{\bar k=-\infty}^{\bar h-1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}}||\mathcal{S}_2(\tau;\mathbf{k})||\le(\text{const.})^N\bar\varepsilon^N_{h^*}\left(\frac{N}{2}\right)!\frac{M^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}}. \tag{B.11}$$

Notice that the result (B.10) and the bound (B.11) are true for any $\mathbf{k}$ such that $|\mathbf{k}| \geq M^{h^*}$; being the bound (B.11) uniform in h^*, our result on the two point function (1.6) and (2.33) is obtained by fixing $\mathbf{k}$ and taking the limit $h^* \to -\infty$ in (B.10).

In order to understand (B.11), it is enough to notice that, as far as dimensional bounds are concerned, the vertices v_1 and v_2 play the role of two ν vertices with an external line (the ϕ line) and an extra $Z_{h_\mathbf{k}}^{-1/2} M^{-h_\mathbf{k}}$ factor each. Moreover, since the vertices on the path $\mathcal{P}_{r,\bar{v}}$ connecting the root with $\bar{v}$ are not associated with any $\mathcal{R}$ operation, we need to multiply the value of the tree $\tau \in \mathcal{T}^{(h^*)}_{\bar{k},\bar{h},N}$ by $M^{(1/2)(\bar{h}-\bar{k})} M^{(1/2)(\bar{k}-\bar{h})}$, and to exploit the factor $M^{(1/2)(\bar{k}-\bar{h})}$ in order to renormalize all the clusters in $\mathcal{P}_{r,\bar{v}}$. Therefore,

$$\sum_{\bar{h}=-\infty}^{0} \sum_{\bar{k}=-\infty}^{\bar{h}-1} \sum_{\tau \in \mathcal{T}^{(h^*)}_{\bar{k},\bar{h},N}} ||\mathcal{S}_2(\tau;\mathbf{k})|| \leq (\text{const.})^N \left(\frac{N}{2}\right)! \times \frac{\bar{\varepsilon}^N_{h^*}}{Z_{h_\mathbf{k}}} \sum_{\bar{h} \leq h_\mathbf{k}} \sum_{\bar{k} \leq \bar{h}} M^{\bar{k}} M^{\bar{h}-h_\mathbf{k}} M^{(1/2)(\bar{h}-\bar{k})} M^{-2h_\mathbf{k}} \tag{B.12}$$

where the factor $M^{\bar{k}}$ is due to the fact that graphs associated to the trees $\tau \in \mathcal{T}^{(h^*)}_{\bar{k},\bar{h},N}$ have two external lines; the factor $M^{\bar{h}-h_\mathbf{k}}$ is given by the product of the two short memory factors associated to the two paths connecting $\bar{v}$ with v_1 and v_2, respectively; the "bad" factor $M^{(1/2)(\bar{h}-\bar{k})}$ is the price to pay to renormalize the vertices in $\mathcal{P}_{r,\bar{v}}$; the $Z^{-1}_{h_\mathbf{k}}$ and the last $M^{-2h_\mathbf{k}}$ are due to the fact that v_1, v_2 behave dimensionally as ν vertices times an extra $Z_{h_\mathbf{k}}^{-1/2} M^{-h_\mathbf{k}}$ factor. Performing the summation over $\bar{k}$ and $\bar{h}$ in (B.12), we get (B.11). Note also that, if $\mathbf{k}$ is on scale $h_\mathbf{k} \simeq h^*$, then the derivatives of $||\mathcal{S}_2(\tau;\mathbf{k})||$ can be dimensionally bounded as

$$\sum_{\bar{h}=-\infty}^{0} \sum_{\bar{k}=-\infty}^{\bar{h}-1} \sum_{\tau \in \mathcal{T}^{(h^*)}_{\bar{k},\bar{h},N}} ||\partial^n_\mathbf{k} \mathcal{S}_2(\tau;\mathbf{k})|| \leq (\text{const.})^N \bar{\varepsilon}^N_{h^*} \left(\frac{N}{2}\right)! \frac{M^{-(1+n)h_\mathbf{k}}}{Z_{h_\mathbf{k}}}, \tag{B.13}$$

from which the bound on $\tilde{r}^{(N)}_\mu(\mathbf{k},\mathbf{p})$ stated in (4.12) immediately follows.

Tree expansion for the three-point function. Let us pick $|\mathbf{k}| = M^{h^*}$, $|\mathbf{k}+\mathbf{p}| \leq M^{h^*}$ and $|\mathbf{p}| \ll M^{h^*}$, which is the condition that we need in order to apply Ward Identities in the form described in Sect. 4. In this case, the expansion of three-point function $\langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}} \overline{\psi}_\mathbf{k} \rangle_{h^*}$ is very similar to the one just described for the two-point function. The result can be written in the form

$$\langle j_{\mu,-\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}} \overline{\psi}_\mathbf{k} \rangle_{h^*} = i \frac{\bar{e}_{\mu,h^*}}{e} [G^{(h^*-1)}_\psi(\mathbf{k}+\mathbf{p})]^\dagger \gamma_\mu g^{(h^*)}(\mathbf{k}) Q^{(h^*)}(\mathbf{k}) + \sum_{\substack{N\geq 1,\\ \bar{h}\leq h^*}} \sum_{\substack{\bar{k}<\bar{h},\\ h_{v_3}>h^*}} \sum_{\tau \in \mathcal{T}^{(h^*)}_{\bar{k},\bar{h},h_{v_3},N}} \mathcal{S}_3(\tau;\mathbf{k},\mathbf{p}) \tag{B.14}$$

where $\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},h_{v_3},N}$ is a new class of trees, with $\bar{k} < 0$ the scale of the root, similar to the trees in $\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},N}$, up to the fact that they have $N+3$ endpoints rather than $N+2$ (see item (1) in the list preceding (B.10)); three of them are special: v_1 and v_2 are associated to the same contributions described in item (1) above, while v_3 is associated to a contribution $Z_{h_{\bar{v}_3}-1}\,(e_{\mu,h_{\bar{v}_3}}/e) j^{(\le h_{\bar{v}_3})}_{\mu,-\mathbf{p}} - M^{h_{\bar{v}_3}}(\nu_{\mu,h_{\bar{v}_3}}/e) A_{\mu,-\mathbf{p}}$, with $\bar{v}_3$ the vertex immediately preceding v_3 on τ (which the endpoint v_3 is attached to) and $h_{v_3} > h^*$. The value of the tree, $\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})$, is defined in a way similar to $\mathcal{S}_2(\tau;\mathbf{k})$, modulo the modifications described above. $\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})$ admits bounds analogous to (B.11)–(B.12); recalling that $|\mathbf{k}| = M^{h^*}$, $|\mathbf{k}+\mathbf{p}| \le M^{h^*}$ and $|\mathbf{p}| \ll M^{h^*}$, we find:

$$\sum_{\bar{h}=-\infty}^{h^*} \sum_{\bar{k}=-\infty}^{\bar{h}-1} \sum_{h_{v_3}=h^*+1}^{1} \sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},h_{v_3},N}} ||\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})||$$
$$\le (\text{const.})^N \left(\frac{N}{2}\right)!\, \bar{\varepsilon}^N_{h^*} \frac{1}{Z_{h^*-1}} \times \sum_{\substack{\bar{h}\le h^* \\ \bar{k}<\bar{h} \\ h_{v_3}>h^*}} M^{(1/2)(\bar{k}-\bar{h})} M^{\bar{h}-h^*} M^{(1/2)(h^*-h_{v_3})} M^{-2h^*}, \tag{B.15}$$

where $M^{(1/2)(\bar{k}-\bar{h})}$ is the short memory factor associated to the path between the root and $\bar{v}$; $M^{\bar{h}-h^*}$ is the product of the two short memory factors associated to the paths connecting $\bar{v}$ with v_1 and v_2, respectively; $M^{(1/2)(h^*-h_{v_3})}$ is the short memory factor associated to a path between h^* and v_3; M^{-2h^*}/Z_{h^*-1} is the product of two factors $M^{-h_{\mathbf{k}}} Z^{-1/2}_{h_{\mathbf{k}}-1}$ associated to the vertices v_1 and v_2 (see the discussion following (B.11) and recall that in this case $h_{\mathbf{k}} = h^*$). We remark that in this case, contrary to the case of the two-point function, the fact that there is no $\mathcal{R}$ operator acting on the vertices on the path between the root and $\bar{v}$ does not create any problem, since those vertices are automatically irrelevant (they behave as vertices with at least five external lines, i.e., J, ϕ, $\bar{\phi}$ and at least two fermionic lines) and, therefore, $\mathcal{R} = 1$ on them. Note also that the vertices of type $J\phi\psi$, which have an $\mathcal{R}$ operator acting on, can only be on scale h^*-1 or h^* (by conservation of momentum) and, therefore, the action of the $\mathcal{R}$ operator on such vertices automatically gives the usual dimensional gain of the form const. $M^{h_v-h_{v'}}$. Performing the summations over $\bar{k}, \bar{h}, h_{v_3}$ in (B.15), we find the analogue of (B.11):

$$\sum_{\bar{h}=-\infty}^{h^*} \sum_{\bar{k}=-\infty}^{\bar{h}-1} \sum_{h_{v_3}=h^*+1}^{1} \sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},h_{v_3},N}} ||\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})|| \le (\text{const.})^N \left(\frac{N}{2}\right)!\, \bar{\varepsilon}^N_{h^*} \frac{M^{-2h^*}}{Z_{h^*-1}}, \tag{B.16}$$

from which the bound on $r^{(N)}_\mu(\mathbf{k},\mathbf{p})$ stated in (4.12).

Appendix C. Lowest Order Computations

In this Appendix we reproduce the details of the second order computations leading to (3.7)–(3.10).

C.1. Computation of $\beta_h^{z,(2)}$

By definition, see (2.7) and (3.1), $\beta_h^z = z_{0,h} = -i\gamma_0 \partial_{k_0} W_{0,1}^{(h)}(\mathbf{0})$. At one-loop, defining $\bar{e}_{0,h} = e_{0,h}$ and $\bar{e}_{1,h} = v_{h-1} e_{1,h}$, we find:

$$\begin{aligned}\beta_h^{z,(2)} = z_{0,h}^{(2)} &= -i\gamma_0 \bar{e}_{\mu,h+1}^2 \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \partial_{p_0} \left(\frac{f_{h+1}(\mathbf{p})}{2|\mathbf{p}|} \right) \gamma_\mu g^{(h+1)}(\mathbf{p}) \gamma_\mu \\ &\quad - i\gamma_0 \bar{e}_{\mu,h+2}^2 \left(\frac{Z_{h+1}}{Z_h} \right)^2 \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \partial_{p_0} \left(\frac{f_{h+2}(\mathbf{p})}{2|\mathbf{p}|} \right) \gamma_\mu g^{(h+1)}(\mathbf{p}) \gamma_\mu \\ &\quad - i\gamma_0 \bar{e}_{\mu,h+2}^2 \frac{Z_{h+1}}{Z_h} \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \partial_{p_0} \left(\frac{f_{h+1}(\mathbf{p})}{2|\mathbf{p}|} \right) \gamma_\mu g^{(h+2)}(\mathbf{p}) \gamma_\mu. \end{aligned} \tag{C.1}$$

Using inductively the beta function equations for $Z_{h+1}, v_{h+1}, e_{\mu,h+2}$, and neglecting higher order terms, we can rewrite (C.1) as

$$\begin{aligned} z_{0,h}^{(2)} = i\gamma_0 \bar{e}_{\mu,h+1}^2 \frac{1}{2} \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \frac{p_0^2}{|\mathbf{p}|^3} \frac{i\gamma_\mu \gamma_0 \gamma_\mu}{p_0^2 + v_h^2 |\vec{p}|^2} \\ \times \left[(f_{h+1}(\mathbf{p}) - |\mathbf{p}| f'_{h+1}(\mathbf{p}))(f_{h+1}(\mathbf{p}) + f_{h+2}(\mathbf{p})) \right. \\ \left. + (f_{h+2}(\mathbf{p}) - |\mathbf{p}| f'_{h+2}(\mathbf{p})) f_{h+1}(\mathbf{p}) \right]. \end{aligned} \tag{C.2}$$

Passing to radial coordinates, $\mathbf{p} = p(\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$, and using the fact that $\int \mathrm{d}p (f'_{h+1} f_{h+1} + f'_{h+1} f_{h+2} + f'_{h+2} f_{h+1}) = 0$, we find:

$$\begin{aligned} z_{0,h}^{(2)} = (2v_h^2 e_{1,h}^2 - e_{0,h}^2) \frac{1}{8\pi^2} \left[\int_0^\infty \frac{\mathrm{d}p}{p} (f_{h+1}^2 + 2 f_{h+1} f_{h+2}) \right] \\ \times \left[\int_{-1}^{1} \mathrm{d}\cos\theta \, \frac{\cos^2\theta}{\cos^2\theta + v_h^2 \sin^2\theta} \right]. \end{aligned} \tag{C.3}$$

The integral over the radial coordinate p can be computed using the definition (2.1):

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{d}p}{p} (f_{h+1}^2 + 2 f_{h+1} f_{h+2}) \\ &\quad = \int_0^\infty \frac{\mathrm{d}p}{p} [2(\chi(p) - \chi(Mp)) - (\chi^2(p) - \chi^2(Mp))] = \lim_{\varepsilon \to 0} \int_\varepsilon^{M\varepsilon} \frac{\mathrm{d}p}{p} = \log M. \end{aligned} \tag{C.4}$$

Finally, an explicit evaluation of the integral over $\mathrm{d}\cos\theta$ leads to (3.19).

C.2 Computation of $z^{(2)}_{1,h}$

By definition, see formulas (2.7) and (3.1), $\beta^{v,(2)}_h = z^{(2)}_{1,h} - v_h z^{(2)}_{0,h}$, with $z_{1,h} = -i\gamma_1 \partial_{k_1} W^{(h)}_{0,1}(\mathbf{0})$. At second order, proceeding as in the derivation of (C.2), we find:

$$\begin{aligned} z^{(2)}_{1,h} &= i\gamma_1 \bar{e}^2_{\mu,h+1}\frac{1}{2} \\ &\quad \times \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\frac{p_1^2}{|\mathbf{p}|^3}\frac{i\gamma_\mu v_h \gamma_1 \gamma_\mu}{p_0^2 + v_h^2|\vec{p}|^2}\left[(f_{h+1}(\mathbf{p}) - |\mathbf{p}| f'_{h+1}(\mathbf{p}))(f_{h+1}(\mathbf{p}) + f_{h+2}(\mathbf{p}))\right. \\ &\quad \left. + (f_{h+2}(\mathbf{p}) - |\mathbf{p}| f'_{h+2}(\mathbf{p})) f_{h+1}(\mathbf{p})\right] \\ &= e^2_{0,h} v_h \frac{1}{16\pi^2}\left[\int_0^\infty \frac{\mathrm{d}p}{p}(f^2_{h+1} + 2 f_{h+1} f_{h+2})\right] \\ &\quad \times \left[\int_{-1}^1 \mathrm{d}\cos\theta\, \frac{\sin^2\theta}{\cos^2\theta + v_h^2 \sin^2\theta}\right]. \end{aligned} \tag{C.5}$$

An explicit evaluation of the integral leads to

$$z^{(2)}_{1,h} = e^2_{0,h} v_h^{-1} \frac{\log M}{8\pi^2}\left(\frac{\arctan \xi_h}{\xi_h} - \frac{\xi_h - \arctan \xi_h}{\xi_h^3}\right), \tag{C.6}$$

which, combined with $\beta^{v,(2)}_h = z^{(2)}_{1,h} - v_h z^{(2)}_{0,h}$, leads to (3.19).

C.3 Computation of $\beta^{\nu,(2)}_{\mu,h}$

By definition, see (2.6) and (3.1), $\beta^{\nu}_{\mu,h} = -M^{-h+1} W^{(h-1)}_{2,0,\mu,\mu}(\mathbf{0}) - M\nu_{\mu,h}$. At second order, we find:

$$\begin{aligned} \beta^{\nu,(2)}_{\mu,h} &= -M^{-h+1}\frac{\bar{e}^2_{\mu,h}}{2}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\mathrm{Tr}\left(\gamma_\mu g^{(h)}(\mathbf{p})\gamma_\mu g^{(h)}(\mathbf{p})\right) \\ &\quad - M^{-h+1}\bar{e}^2_{\mu,h+1}\frac{Z_h}{Z_{h-1}}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\mathrm{Tr}\left(\gamma_\mu g^{(h+1)}(\mathbf{p})\gamma_\mu g^{(h)}(\mathbf{p})\right). \end{aligned} \tag{C.7}$$

Using inductively the beta function equations for $e_{\mu,h}, Z_{h-1}, v_{h-1}$, and neglecting higher orders, we can rewrite (C.7) as

$$\begin{aligned} \beta^{\nu,(2)}_{0,h} &= -2M^{-h+1} e^2_{0,h}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\frac{f_h(\mathbf{p})^2 + 2 f_h(\mathbf{p}) f_{h+1}(\mathbf{p})}{(p_0^2 + v_h^2|\vec{p}|^2)^2}(-p_0^2 + v_h^2|\vec{p}|^2), \\ \beta^{\nu,(2)}_{1,h} &= -2M^{-h+1} \bar{e}^2_{1,h}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\frac{f_h(\mathbf{p})^2 + 2 f_h(\mathbf{p}) f_{h+1}(\mathbf{p})}{(p_0^2 + v_h^2|\vec{p}|^2)^2}p_0^2, \end{aligned} \tag{C.8}$$

where we used that $\mathrm{Tr}\,(\gamma_\mu\gamma_\alpha\gamma_\mu\gamma_\alpha) = -4$ if $\mu \neq \alpha$ and 4 otherwise; passing to radial coordinates we find

$$\begin{aligned}\beta^{\nu,(2)}_{0,h} &= \frac{-2}{(2\pi)^2} M^{-h+1} e^2_{0,h} \left[\int_0^\infty \mathrm{d}p\, \left(f_h^2 + 2 f_h f_{h+1}\right)\right] \\ &\qquad \times \int_{-1}^{1} \mathrm{d}\cos\theta\, \frac{-\cos^2\theta + v_h^2 \sin^2\theta}{(\cos^2\theta + v_h^2\sin^2\theta)^2}, \\ \beta^{\nu,(2)}_{1,h} &= \frac{-2}{(2\pi)^2} M^{-h+1} \bar{e}^2_{1,h} \left[\int_0^\infty \mathrm{d}p\, \left(f_h^2 + 2 f_h f_{h+1}\right)\right] \\ &\qquad \times \int_{-1}^{1} \mathrm{d}\cos\theta\, \frac{\cos^2\theta}{(\cos^2\theta + v_h^2\sin^2\theta)^2}.\end{aligned} \tag{C.9}$$

The integral over the radial coordinate p can be rewritten as, using the definition (2.1):

$$\int_0^\infty \mathrm{d}p\, \left(f_h^2 + 2f_h f_{h+1}\right) = M^{h-1}(M-1)\int_0^\infty \mathrm{d}p\, \left(2\chi(p) - \chi^2(p)\right). \tag{C.10}$$

Finally, an explicit evaluation of the integral over $d\cos\theta$ leads to (3.7).

Appendix D. Multiscale Integration of the Correction Term to the WI

In this Appendix we prove (4.7) and the bound (4.8). We assume that $|\mathbf{k}| = M^h$ and $|\mathbf{p}| \ll M^h$. We start by rewriting

$$\frac{p_\mu}{Z_h} R_{\mu,h}(\mathbf{k},\mathbf{p}) = \frac{\partial^3}{\partial \tilde{J}_{\mathbf{p}} \partial \overline{\phi}_{\mathbf{k}+\mathbf{p}} \partial \phi_{\mathbf{k}}} \widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)\big|_{\tilde{J}=\phi=0}, \tag{D.1}$$

with $\widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)$ defined as:

$$e^{\widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)} := \int P(\mathrm{d}\psi) P_{[h,0]}(\mathrm{d}A)\, e^{V(A,\psi) + \widetilde{B}(\tilde{J},\phi)}, \tag{D.2}$$

and

$$\begin{aligned}\widetilde{B}(\tilde{J},\phi) &= \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\, \tilde{J}_{\mathbf{p}} \left[\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\, \overline{\psi}_{\mathbf{k}+\mathbf{p}} C(\mathbf{k},\mathbf{p}) \psi_{\mathbf{k}} - \alpha_\mu p_\mu j_{\mu,-\mathbf{p}}\right] \\ &\quad + \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \left[\phi_{\mathbf{k}} \overline{\psi}_{\mathbf{k}} + \overline{\phi}_{\mathbf{k}} \psi_{\mathbf{k}}\right].\end{aligned} \tag{D.3}$$

The main difference with respect to the generating functional of the correlation functions is the presence of the correction term proportional to $C(\mathbf{k},\mathbf{p})$, see (4.6) for a definition. Equation (D.2) can again be studied by RG methods, see

[6] for further details. A crucial role is played by the properties of the function $C(\mathbf{k},\mathbf{p})$; it is easy to verify that

$$g^{(i)}(\mathbf{k}+\mathbf{p})C(\mathbf{k},\mathbf{p})g^{(j)}(\mathbf{k}) \tag{D.4}$$

is non-vanishing only if at least one of the indices i, j is equal to 0; moreover, when it is non-vanishing, it is dimensionally bounded from above by $(\text{const.})|\mathbf{p}|M^{-i-j}$.

We start by integrating the scale 0, and we find:

$$e^{\widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)} = e^{|\Lambda|E_{-1}+\widetilde{\mathcal{S}}^{(\geq -1)}(\tilde{J},\phi)} \times \int P(\mathrm{d}\psi^{(\leq -1)})P_{[h,-1]}(dA^{(\leq -1)})\, e^{\mathcal{V}^{(-1)}(A^{(\leq -1)},\sqrt{Z_{-1}}\psi^{(\leq -1)})+\widetilde{\mathcal{B}}^{(-1)}}, \tag{D.5}$$

where $\widetilde{\mathcal{S}}^{(\geq -1)}$ collects the terms depending on $\tilde{J}, \phi$ but independent of A, ψ, and

$$\widetilde{\mathcal{B}}^{(-1)}(A,\psi) = \widetilde{\mathcal{B}}_J^{(-1)}(A,\psi) + \mathcal{B}_\phi^{(-1)}(A,\psi) + \widetilde{W}_R^{(-1)}, \tag{D.6}$$

with: $\widetilde{\mathcal{B}}_J^{(-1)}(A,\psi)$ linear in $\tilde{J}$ and independent of ϕ; $\mathcal{B}_\phi^{(-1)}(A,\psi)$ given by (B.3); $\widetilde{W}_R^{(-1)}$ the rest, which is at least quadratic in $(\tilde{J},\phi)$. With respect to the computation of $\mathcal{W}_{[h,0]}(\tilde{J},\phi)$, we now have new marginal terms of the form $\tilde{J}\,\bar{\psi}\psi$, which are contained in $\widetilde{\mathcal{B}}_J^{(-1)}(A,\psi)$ and need to be renormalized. Let us symbolically represent by $\widetilde{W}_{m,n}^{(-1)}(\{\mathbf{k}_i\},\{\mathbf{q}_i\},\mathbf{p})$ the generic non-trivial kernel appearing in $\widetilde{\mathcal{B}}_J^{(-1)}(A,\psi)$; m is the number of bosonic external lines while $2n$ is the number of ψ fields; $\{\mathbf{k}_i\}, \{\mathbf{q}_i\}$ are, respectively, the fermionic/bosonic momenta and $\mathbf{p}$ is the momentum flowing through $\tilde{J}$. As usual, these new kernels can be represented as sums over Feynman graphs. The $\tilde{J}$ external line can be attached to a simple vertex, corresponding to the monomial $-\alpha_\mu p_\mu \tilde{J}_\mathbf{p} j^{(\leq 0)}_{\mu,-\mathbf{p}}$, or to a "thick" vertex, representing $\tilde{J}_\mathbf{p}\overline{\psi}_{\mathbf{k}+\mathbf{p}}C(\mathbf{k},\mathbf{p})\psi_\mathbf{k}$ (the "small circle" associated to the vertex represents the matrix kernel $C(\mathbf{k},\mathbf{p})$, see Fig. 6). Let us denote by $W_{m,n}^{(-1),C}$ the contribution to $\widetilde{W}_{m,n}^{(-1)}$ coming from graphs with the $\tilde{J}$ line attached to a thick vertex, see Fig. 6. By the properties of the $C(\mathbf{k},\mathbf{p})$ function, see [6] for details, it follows that $\widetilde{W}_{m,n}^{(-1)}(\{\mathbf{k}_i\},\{\mathbf{q}_i\},\mathbf{p}) =: p_\mu \bar{W}_{m,n,\mu}^{(-1)}(\{\mathbf{k}_i\},\{\mathbf{q}_i\},\mathbf{p})$, with $\bar{W}_{m,n,\mu}^{(-1)}$ dimensionally bounded as a $W_{m+1,n}^{(-1)}$ kernel, uniformly in $\mathbf{p}$. We define the action of the $\mathcal{R} \equiv 1-\mathcal{L}$ operator on $\bar{W}_{m,n,\mu}^{(-1)}$ in a way similar to (2.5)–(2.7). In particular, $\mathcal{L}\bar{W}_{0,1,\mu}^{(-1)}(\mathbf{k},\mathbf{p}) := \bar{W}_{0,1,\mu}^{(-1)}(\mathbf{0},\mathbf{0})$ and, by symmetry,

$$Z_{-1}\tilde{J}_\mathbf{p}\overline{\psi}_{\mathbf{k}+\mathbf{p}}\mathcal{L}\bar{W}_{0,1,\mu}^{(-1)}\psi_\mathbf{k} = -Z_{-2}\tilde{J}_\mathbf{p}\alpha_{\mu,-1}j_{\mu-\mathbf{p}}^{(\leq -1)}, \tag{D.7}$$

for a real constant $\alpha_{\mu,-1}$, which is by definition the effective α-coupling on scale -1. Note that the last two graphs in Fig. 6 do not contribute to $\alpha_{\mu,-1}$ simply because they are one-particle reducible and, therefore, they are vanishing at zero external momenta.

We now iterate the same procedure, and step by step the local parts of the kernels of type $\tilde{J}\bar{\psi}\psi$ are collected together to form a new running coupling constant, $\alpha_{\mu,k}$; in order to show that $R_{\mu,h}$ is dimensionally negligible as

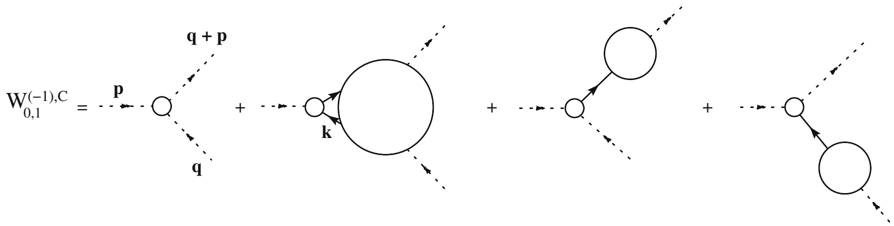

FIGURE 6. Schematic representation of the expansion for $W^{(-1),C}_{0,1}$; the *small circle* represents $C(\mathbf{k},\mathbf{p})$

$h\to-\infty$, we need to show that it is possible to fix the initial data $\alpha_\mu=\alpha_{\mu,0}$ in such a way that $\alpha_{\mu,h}$ goes exponentially to zero as $h\to-\infty$, which is proved in the following.

The flow of $\alpha_{\mu,k}$. The new marginal running coupling constants $\alpha_{\mu,h}$ evolve according to the flow equation: $\alpha_{\mu,k-1}=\alpha_{\mu,k}+\beta^{\alpha}_{\mu,k}$, where $\alpha_{\mu,0}=\alpha_\mu$ are the counterterms appearing in the bare interaction (D.3). The beta function $\beta^\alpha_{\mu,h}$ can be split as

$$\beta^{\alpha}_{\mu,k}=\beta^{\alpha,1}_{\mu,k}+\beta^{\alpha,2}_{\mu,k}, \tag{D.8}$$

where $\beta^{\alpha,1}_{\mu,k}$ collects the contributions independent of $\alpha_{\mu,k'}$ (which, therefore, are associated to graphs with the $\tilde J$ external line emerging from the thick vertex representing $C(\mathbf{k},\mathbf{p})$), and $\beta^{\alpha,2}_{\mu,k}$ collects the terms from graphs with one vertex of type $\alpha_{\mu,k'}$ for some $k'>k$. It is crucial to recall that by the properties of $C(\mathbf{k},\mathbf{p})$, the graphs contributing to $\beta^{\alpha,1}_{\mu,k}$ have at least one propagator on scale 0 or -1; by the short memory property, this means that they can be dimensionally bounded by (const.)$\bar\varepsilon_k^2M^{\theta k}$, for any $0<\theta<1$. Similarly, the contributions to $\beta^{\alpha,2}_{\mu,k}$ associated to graphs with at least one vertex of type $\alpha_{\mu,k'}$ for some $k'>k$ can be bounded by (const.)$\bar\varepsilon_k^2|\alpha_{\mu,k'}|M^{\theta(k-k')}$. The counterterms α_μ are fixed in such a way that $\alpha_{\mu,-\infty}=0$, i.e., $\alpha_\mu=-\sum_{k=-\infty}^{0}(\beta^{\alpha,1}_{\mu,k}+\beta^{\alpha,2}_{\mu,k})$. Finally, using the fact that $|\beta^{\alpha,1}_{\mu,k}|\le$ (const.)$\bar\varepsilon_k^2M^{\theta k}$ and $|\beta^{\alpha,2}_{\mu,k}|\le$ (const.)$\sum_{k'>k}\bar\varepsilon_k^2|\alpha_{\mu,k'}|M^{\theta(k-k')}$, we find that $|\alpha_{\mu,h}|\le$ (const.)$\bar\varepsilon_{h^*}^2M^{(\theta/2)h}$. This dimensional estimate on $\alpha_{\mu,h}$ easily implies the desired estimate on $R_{\mu,h}(\mathbf{k},\mathbf{p})$ stated in (4.8) and we will not belabor the details here.

Lowest order computation of α_μ. At lowest order, $\alpha^{(2)}_\mu=-\sum_{k\le0}\beta^{\alpha,1,(2)}_{\mu,k}$, where $\beta^{\alpha,1,(2)}_{\mu,k}$ is the one-loop contribution to $\beta^{\alpha,1}_{\mu,k}$. Moreover, $\beta^{\alpha,1,(2)}_{\mu,k}=0$ for all $k\le-1$. Therefore, neglecting higher order terms, we find $\alpha^{(2)}_\mu=-\beta^{\alpha,1,(2)}_{\mu,0}$,

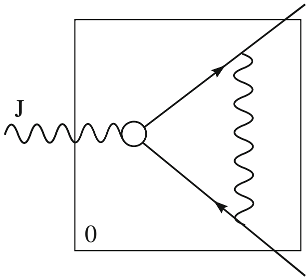

FIGURE 7. Lowest order contribution to α_0, α_1

that is (see Fig. 7):

$$\alpha_0^{(2)} = -i\gamma_0 \bar{e}_{\nu,0}^2 \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \gamma_\nu \partial_{p_0} \left[g^{(0)}(\mathbf{k}+\mathbf{p}) C(\mathbf{k},\mathbf{p}) g^{(0)}(\mathbf{k}) \right]_{\mathbf{p}=\mathbf{0}} \times \gamma_\nu w^{(0)}(\mathbf{k}), \tag{D.9}$$

$$\alpha_1^{(2)} = -\frac{i\gamma_1}{v} \bar{e}_{\nu,0}^2 \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \gamma_\nu \partial_{p_1} \left[g^{(0)}(\mathbf{k}+\mathbf{p}) C(\mathbf{k},\mathbf{p}) g^{(0)}(\mathbf{k}) \right]_{\mathbf{p}=\mathbf{0}} \times \gamma_\nu w^{(0)}(\mathbf{k}). \tag{D.10}$$

After a straightforward computation, using the fact that

$$\partial_{p_\mu} \left[g^{(0)}(\mathbf{k}+\mathbf{p}) C(\mathbf{k},\mathbf{p}) g^{(0)}(\mathbf{k}) \right]_{\mathbf{p}=\mathbf{0}} = \frac{1}{i\not{\mathbf{k}}} \left[-i\bar{\gamma}_\mu \chi_0(\mathbf{k}) \left(1-\chi_0(\mathbf{k})\right) + i\not{\mathbf{k}} \partial_\mu \chi_0(\mathbf{k}) \right] \frac{1}{i\not{\mathbf{k}}},$$

where $\not{\mathbf{k}} = k_0\gamma_0 + v\vec{k}\cdot\vec{\gamma}$ and $(\bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2) = (\gamma_0, v\gamma_1, v\gamma_2)$, we finally get (3.26)–(3.27).

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Communicated by Jean Bellissard.
Received: June 9, 2010.
Accepted: October 5, 2010.