A fast algorithm to remove proper and homogenous pairs of cliques (while preserving some graph invariants)

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Abstract

We introduce a family of reductions for removing proper and homogeneous pairs of cliques from a graph G. This family generalizes some reductions presented in the literature, mostly in the context of claw-free graphs. We show that these reductions can be embedded in a simple algorithm that in at most $|E(G)|$ steps builds a new graph G' such that G' has no proper and homogeneous pairs of cliques, and G and G' agree on the value of some relevant invariant (or property).

1 Introduction

A pair of vertex disjoint cliques $\{K_1, K_2\}$ is *homogeneous* if every vertex that is neither in K_1, nor in K_2 is either adjacent to all vertices from K_1, or non-adjacent to all of them, and similarly for K_2. Homogeneous pairs of cliques seem to play a non-trivial role in combinatorial, structural and polyhedral properties of claw-free graphs. For instance, a well-known decomposition result by Chudnovsky and Seymour states the following:

Theorem 1 *[4] For every connected claw-free graph G with $\alpha(G) \geq 4$, if G does not admit a 1-join and there is no homogeneous pair of cliques in G, then either G is a circular interval graph, or G is a composition of linear interval strips, XX-strips, and antihat strips.*

We shall not provide the definition of graphs and operations involved in Theorem 1, since they are of no use for the present paper; the interested reader may refer to [4]. What is interesting for us is the fact that homogeneous pair of cliques is somehow an *annoying* structure: as it is written in [4], "There is also a ”fuzzy” version of this (*ie Theorem 1*), without the hypothesis that there is no homogeneous pair of cliques in G, but it is quite complicated and again we omit it." A structure theorem for the whole class of claw-free graphs was indeed presented by the same authors in [5], where they overcome (among others) the problem caused by homogeneous pairs of cliques. But the latter result extends over several papers and requires a suitable generalization of simple graphs (i.e. trigraphs). A similar situation can be found again in the structure theorem on Berge graphs by Chudnovsky et al. [3], which was the main tool to prove the strong perfect graphs conjecture.

An interesting class of homogeneous pair of cliques are those that are also *proper*. We say that a pair of cliques $\{K_1, K_2\}$ is *proper* if each vertex in K_1 is neither complete nor anticomplete to K_2, and each vertex in K_2 is neither complete nor anticomplete to K_1. Note that, given a graph $G(V, E)$, an $O(|V|^2|E|^2)$-time algorithm finds a proper and homogeneous pairs of cliques of G, if any [10, 14]. Several papers from the literature propose indeed *reduction techniques* for getting rid of proper and homogeneous pairs of cliques, while preserving graph invariants, such as chromatic number [9, 10] and stability number [12], or graph properties, such as the property of a graph of being quasi-line [2], fuzzy circular interval [13], or even facets of the stable set polytope [7].

In this paper we introduce a reduction operation that generalizes and unifies those different techniques. Our reduction essentially replaces a proper and homogeneous pair of cliques $\{K_1, K_2\}$

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with another pair of cliques $\{A_1, A_2\}$ that is homogeneous but non-proper, and such that $|A_i| \leq |K_i|$ for $i = 1, 2$. Note that a large number of pairs $\{A_1, A_2\}$ can be used in our reduction, and the choice of a particular pair is done depending on some invariant (or property) we want the reduction to preserve. Regardless of this choice, we can however embed our reduction in a fast algorithm, Algorithm 2, that iteratively replaces a proper and homogeneous pair of cliques $\{K_1^i, K_2^i\}$ with a non-proper and homogeneous one $\{A_1^i, A_2^i\}$. Our main result will be then the following:

Theorem 2 *Let $G(V, E)$ be any graph. Algorithm 2 builds a sequence of graphs $G = G^0, G^1, \dots, G^q$, with $q \leq m$, such that each G^i, $i < q$, has a proper and homogeneous pair of cliques $\{K_1^i, K_2^i\}$, while G^q has no proper and homogeneous pairs of cliques. The algorithm can be implemented as to run in $O(|V|^2|E| + \sum_{i=1}^{q} p(i))$-time, if, for $i = 1, \dots, q$, it takes $p(i)$-time to generate the pair of cliques $\{A_1^i, A_2^i\}$, from the knowledge of G^i, K_1^i and K_2^i.*

(A more precise statement of this theorem is given later, see Theorem 20.) Combining this theorem with a few results from the literature, we will show some more facts, among which:

- we can reduce in time $O(|V|^{\frac{5}{2}}|E|)$ the coloring problem (resp. the maximum clique problem) on a graph $G(V, E)$ to the same problem on G' without PH pairs of cliques;
- we can reduce in time $O(|V|^2|E|)$ the maximum weighted stable set problem on a graph $G(V, E)$ to the same problem on G' without PH pairs of cliques.

It is interesting to point out again that the state of the art complexity for recognizing whether a graph $G(V, E)$ has *some* proper and homogeneous pairs of cliques is $O(|V|^2|E|)$ ([10, 14]).

2 Preliminaries

Given a simple graph $G(V, E)$, let $n = |V|$ and $m = |E|$. We denote by uv an edge of G, while we denote by $\{u, v\}$ a pair of vertices $u, v \in V$. For a given $x \in V$, the *neighborhood* $N(x)$ is the set of vertices $\{v \in V : xv \in E\}$. We say that v is *universal* to $u \in V$ if v is adjacent to u and to every vertex in $N(u) \setminus \{v\}$. Let $S \subset V$, then $x \notin S$ is *complete* (resp. *anticomplete*) to S in G if $S \cap N(x) = S$ (resp. $S \cap N(x) = \emptyset$). Finally, we denote by $G[U]$ the subgraph induced on G by $U \subseteq V$; a C_4 is an induced chordless cycle on four vertices.

Definition 3 *Let G be a graph and $\{K_1, K_2\}$ be a pair of non-empty and vertex disjoint cliques. The pair $\{K_1, K_2\}$ is* homogeneous *if each vertex $z \notin (K_1 \cup K_2)$ is either complete or anti-complete to K_1 and either complete or anti-complete to K_2.*

Definition 4 *Let K be a clique of a graph G and let $v \notin K$. v is* proper *to K if v is neither complete nor anti-complete to K, and $P(K)$ is the set of vertices that are proper to K.*

Definition 5 *Let G be a graph and $\{K_1, K_2\}$ be a pair of non-empty and vertex disjoint cliques. The pair $\{K_1, K_2\}$ is* proper *if each vertex $u \in K_1$ (K_2, respectively) is proper to K_2 (K_1). A pair of vertex disjoint cliques that are proper and homogeneous is also called a* PH pair.

We skip the simple proof of the following lemma.

Lemma 6 *Let G be a graph and $\{K_1, K_2\}$ be a homogeneous pair of cliques. Then $\{K_1, K_2\}$ is proper if an only if, for each $k \in \{1, 2\}$ and $x \in K_i$, there exist $y_1, y_2 \in K_i$ (possibly $y_1 = y_2$) such that x is non-universal to y_1 and y_2 is non-universal to x.*

In fact, Chudnovsky and Seymour observed that for each clique K_i of a proper pair $\{K_1, K_2\}$ there always exist two vertices $x, y \in K_i$ that are non-universal to each other. Namely, we have the following:

Lemma 7 *[4] Let $\{K_1, K_2\}$ be a proper pair of cliques in a graph G. Then $G[K_1 \cup K_2]$ contains C_4 as an induced subgraph.*

Hence, when looking for a PH pair in a graph, one can start from a pair of vertices that are adjacent and not universal to each other, and then determine whether they have a *PH-embedding*, namely:

Definition 8 *Let u and v be two adjacent vertices of a graph G. We say that u and v have a* PH-embedding *if they are not universal to each other, and there exists a PH pair of cliques $\{K_1, K_2\}$ such that $u, v \in K_1$. We also denote by PH(G) the set of pairs of vertices of G that have a PH-embedding.*

The next lemma is therefore trivial.

Lemma 9 *If no pair of vertices of G have a PH-embedding, then G has no PH pairs of cliques.*

Given two adjacent vertices that are non-universal to each other, a simple algorithm recognizes in $O(n^2)$-time whether they have a PH embedding. This routine, which we report below, was independently proposed by King and Reed [10] and Pietropaoli [14] (see also [13]). Actually King and Reed designed an algorithm for a slightly different problem: call $\{K_1, K_2\}$ a *non-trivial homogeneous* (NTH) pair of cliques in G if $\{K_1, K_2\}$ is a homogeneous pair of cliques in G, and $G[K_1 \cup K_2]$ has an induced C_4. Lemma 7 implies that each PH pair of cliques is a NTH pair of cliques, and one can immediately check that the converse does not always hold. But given a NTH pair of cliques $\{K_1, K_2\}$, one can obtain a PH pair of cliques H_1, H_2 with $H_1 \subseteq K_1$, $H_2 \subseteq K_2$, by iteratively removing from $\{K_1, K_2\}$ vertices that are non-proper to the opposite clique. Thus, in order to find a NTH pair one can look for a PH pair: this is exactly what King and Reed do in [10] (see Section 3).

Algorithm 1 Finding a PH embedding

Require: A graph G, and a pair of adjacent vertices $\{u, v\}$ that are not universal to each other.
Ensure: A PH-embedding $\{K', K\}$ for $\{u, v\}$, if any.

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K' := {u, v}; K := P({u, v}).
while K is a clique and P(K) ≠ K' do
    K' := K, K := P(K)
end while
if K is not a clique then there is no PH-embedding for {u, v}
else P(K) = K' and {K, K'} is a PH-embedding for {u, v}: stop
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Theorem 10 *[10], [14] It is possible to implement Algorithm 1 as to run in $O(|V(G)|^2)$.*

Besides considering pairs of cliques that are proper and homogeneous, we will also consider pairs of cliques that are homogeneous but non-proper. This leads to the following:

Definition 11 *Let G be a graph and $\{A_1, A_2\}$ be a pair of non-empty and vertex disjoint cliques that are not complete to each other. The pair $\{A_1, A_2\}$ is C_4^{free} if $G[A_1 \cup A_2]$ has no induced C_4. A pair of cliques that are C_4^{free} and homogeneous is also called a C_4^{free}*H pair.

It follows from Lemma 7 that no pair of C_4^{free} cliques is proper. We skip the simple proof of the next lemma.

Lemma 12 *Let G be a graph and $\{A_1, A_2\}$ be a pair of non-empty and vertex disjoint cliques that are not complete to each other. Then $\{A_1, A_2\}$ is C_4^{free} if and only if the following holds: if u and $v \in A_1$ then either u is universal to v or v is universal to u (note that this property holds if and only if the same happens with the vertices of A_2).*

The next lemma analyzes the possible intersections between PH and C_4^{free}H pairs of cliques.

Lemma 13 *Let $G(V, E)$ be a graph with a PH pair of cliques $\{K_1, K_2\}$ and a C_4^{free}H pair of cliques $\{A_1, A_2\}$. Then either $K_1 \cap A_2 = K_2 \cap A_1 = \emptyset$ or $K_1 \cap A_1 = K_2 \cap A_2 = \emptyset$.*

Proof. We start with the following:

Claim 1 *Either $K_i \cap A_1 = \emptyset$ or $K_i \cap A_2 = \emptyset$, for $i = 1, 2$.*

Proof. Without loss of generality, suppose to the contrary that there exist $a \in A_1$ and $b \in A_2$ such that $a, b \in K_1$. Being K_1 proper to K_2, there exist $c, d \in K_2$ (possibly non-distinct) such that $ad, bc \notin E$. We first show that $c, d \notin A_1 \cup A_2$. Note that $d \notin A_1$ and $c \notin A_2$. Now suppose that $d \in A_2$; it follows that $d \neq c$. Since c is adjacent to d and not adjacent to b, and $\{A_1, A_2\}$ is a homogenous pair, it follows that $c \in A_1$. But then a, b, c, d induce a C_4 on $G[A_1 \cup A_2]$, and therefore neither a is universal to c nor c is universal to a, which is a contradiction to Lemma 12. We get an analogous contradiction if we assume that $c \in A_1$.

So $c, d \notin A_1 \cup A_2$; being $ad, bc \notin E$ and $\{A_1, A_2\}$ a homogeneous pair, c is anti-complete to A_2 and d is anti-complete to A_1. Since K_2 is a clique, it follows that $K_2 \cap (A_1 \cup A_2) = \emptyset$. Since $A_1 \cup A_2$ is not a clique, there exist $a' \in A_1$, $b' \in A_2$ such that $a'b' \notin E$. Note that $da' \notin E$ and that $a' \notin K_2$. We now show that $a' \notin K_1$. For, suppose the contrary; then $b' \neq b$ and $b' \notin K_1$, and so b' is proper to K_1 and therefore belongs to K_2, which is a contradiction, since we already argued that $K_2 \cap (A_1 \cup A_2) = \emptyset$.

Hence $a' \notin K_1 \cup K_2$. Since $\{K_1, K_2\}$ is a proper pair, there exists a vertex $e \in K_2$ such that $ea \in E$. Since $K_2 \cap (A_1 \cup A_2) = \emptyset$ and $\{A_1, A_2\}$ is a homogeneous pair, it follows that $ea' \in E$. On the other hand, we observed that $da' \notin E$. But then a' is proper to K_2, contradicting $a' \notin K_1$. (*End of the claim.*)

From the claim, we may assume without loss of generality that $K_1 \cap A_1 = \emptyset$. In this case, the statement follows if $K_2 \cap A_2 = \emptyset$, so suppose that there exists $v_2 \in K_2 \cap A_2$. It again follows from the previous claim that $K_2 \cap A_1 \neq \emptyset$; hence the statement follows if $K_1 \cap A_2 = \emptyset$. So suppose that there exists $v_1 \in K_1 \cap A_2$; since $\{K_1, K_2\}$ is a proper pair, it follows that v_1 and v_2 are not universal to each other, a contradiction. □

3 An algorithm for removing proper and homogeneous pairs

We now define an operation of *reduction* that is crucial for the paper. This operation essentially replaces a PH pair of cliques with a C_4^{free}H pair of cliques. The latter pair will be defined through a suitable graph that we call, for shortness, a *non-proper 2-clique*.

Definition 14 *A* non-proper 2-clique $H_{\{A_1,A_2\}}$ *is a graph with a* C_4^{free} *pair of cliques* $\{A_1, A_2\}$*, such that* $V(H_{\{A_1,A_2\}}) = A_1 \cup A_2$.

Definition 15 *Let* G *be a graph with a PH pair of cliques* $\{K_1, K_2\}$*. Also let* $H_{\{A_1,A_2\}}$ *be a non-proper 2-clique graph vertex disjoint from* G*. The* PH reduction *of* G *with respect to* $(K_1, K_2, H_{\{A_1,A_2\}})$ *returns a new graph* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *defined as follows:*

- $V(G|_{K_1,K_2,H_{\{A_1,A_2\}}}) = (V(G) \setminus (K_1 \cup K_2)) \cup (A_1 \cup A_2)$;
- *Let* $x, y \in V(G|_{K_1,K_2,H_{\{A_1,A_2\}}})$. $xy \in E(G|_{K_1,K_2,H_{\{A_1,A_2\}}})$ *if and only if one of the following holds:*
  - $xy \in E(G)$ *with* $x, y \notin K_1 \cup K_2$;
  - $xy \in E(H_{\{A_1,A_2\}})$ *with* $x, y \in A_1 \cup A_2$;
  - $y \in A_1$, $x \notin K_1 \cup K_2$ *and* x *is complete to* K_1;
  - $y \in A_2$, $x \notin K_1 \cup K_2$ *and* x *is complete to* K_2.

We skip the trivial proof of the following lemma.

Lemma 16 *The graph* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *is such that the following properties hold:*

- $\{A_1, A_2\}$ *is a* $C_4^{free}H$ *pair of cliques.*
- *If* $x, y \in A_1$ *(resp.* $x, y \in A_2$*), then either* x *is universal to* y *or* y *is universal to* x.
- *If* $|K_1| \geq |A_1|$ *and* $|K_2| \geq |A_2|$*, then* $|V(G|_{K_1,K_2,H_{\{A_1,A_2\}}})| \leq |V(G)|$ *and* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *can be built from* G *in time* $O(|V(G)|^2)$.

The following crucial lemma shows that all the PH pairs of $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ are "inherited" by the input graph G.

Lemma 17 *Let* $\{w_1, w_2\}$ *be a pair of adjacent vertices of* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *with a PH embedding. Then:*

1. w_1 *and* w_2 *do not both belong to* $A_1 \cup A_2$;
2. *if* $w_1, w_2 \notin A_1 \cup A_2$*, then* $\{w_1, w_2\}$ *also admits a PH embedding in* G;
3. *if* $w_1 \in A_1$ *(resp.* $w_1 \in A_2$*) and* $w_2 \notin A_1 \cup A_2$*, then, for each* $a \in K_1$ *(resp.* $a \in K_2$*),* $\{a, w_2\}$ *admits a PH embedding in* G.

Proof. Throughout the proof, when referring to vertices of $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$, we call *artificial* the vertices of $A_1 \cup A_2$, and *non-artificial* the others. Moreover, we let $G' = G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ and let $\{K'_1, K'_2\}$ be a PH embedding for $\{w_1, w_2\}$ in G'.

It follows from Lemma 16 that $\{A_1, A_2\}$ is a C_4^{free}H pair of cliques of G'. Therefore it follows from Lemma 13 that either $K'_1 \cap A_2 = K'_2 \cap A_1 = \emptyset$ or $K'_1 \cap A_1 = K'_2 \cap A_2 = \emptyset$. Now suppose that $w_1, w_2 \in A_1 \cup A_2$, and recall that, by definition, $w_1, w_2 \in K'_1$. It follows that either $w_1, w_2 \in A_1$, or $w_1, w_2 \in A_2$: thus, there exist two vertices of A_1 (resp. A_2) that are non-universal to each other, contradicting Lemma 16. Therefore w_1 and w_2 do not both belong to $A_1 \cup A_2$, i.e. statement 1 holds.

W.l.o.g. in the following we assume that $K'_1 \cap A_2 = K'_2 \cap A_1 = \emptyset$. Now define the sets H_1, H_2 of vertices in G as follows: for $i = 1, 2$, if K'_i has no artificial vertices, define $H_i = K'_i$; otherwise $H_i = (K'_i \cap V(G)) \cup K_i$. Note that this implies that $H_1 \cap K_2 = H_2 \cap K_1 = \emptyset$ and that H_1 and H_2 are cliques.

Claim 2 *Let $u, v \in K'_1$ (respectively K'_2) be two non-artificial vertices of G' such that u is non-universal to v in G'. Then $u, v \in H_1$ (respectively H_2) and u is non-universal to v in G.*

Proof. We prove the statement for $u, v \in K'_1$. Since u, v are non-artificial, $u, v \in H_1$ by definition. By hypothesis, there exists $z \in K'_2$ s.t. $uz \notin E(G')$, $vz \in E(G')$. If z is non-artificial, $z \in H_2$ by definition, thus u is non-universal to v in G. Suppose now z is artificial, then $z \in A_2$, since $K'_2 \cap A_1 = \emptyset$. Then by construction v is complete and u anticomplete to K_2 in G, thus u is non-universal to v in G. (*End of the claim.*)

Claim 3 *let $u, v \in K'_1$ (respectively K'_2), and suppose u is artificial and v is not. Then $\{v\} \cup K_1 \subseteq H_1$ (resp. $\{v\} \cup K_2 \subseteq H_2$). Furthermore:*

1. *If u is non-universal to v, then a is non-universal to v for each $a \in K_1$ (respectively K_2).*
2. *If v is non-universal to u, then v is non-universal to a, for each $a \in K_1$ (resp. K_2).*

Proof. We prove the statement for $u, v \in K'_1$. We are assuming that $K'_1 \cap A_2 = \emptyset$, hence $u \in A_1$. So by definition, $\{v\} \cup K_1 \subseteq H_1$. Suppose u is non-universal to v: there exists $z \in K'_2$ s.t. $uz \notin E(G')$, $vz \in E(G')$. If z is an artificial vertex, then $z \in A_2$, which implies that v is complete to K_2, while each vertex $a \in K_1$ is proper to K_2. If z is non-artificial, then by construction z is anticomplete to K_1 while $vz \in E(G)$. This shows 1. Now suppose that v is non-universal to u, i.e. there exists $z \in K'_2$ such that $uz \in E(G'), vz \notin E(G')$. If z is an artificial vertex, then $K_2 \subseteq H_2$ and v is anticomplete to K_2; since each vertex $a \in K_1$ is proper to K_2, v is non-universal to a. If z is non-artificial, then z is complete to K_1 in G, while $zv \notin E(G)$; thus, v is non-universal to $a \in K_1$.(*End of the claim.*)

Claim 4 *$\{H_1, H_2\}$ is a PH pair of cliques in G.*

Proof. We already observed that H_1 and H_2 are cliques, and it is straightforward to see that $\{H_1, H_2\}$ is a homogeneous pair. So we conclude the proof by showing that H_1 is proper to H_2 (the other case following by symmetry).

We need to show that each vertex $x \in H_1$ has at least one neighbor and at least one non-neighbor in H_2. Recall that $x \notin K_2$. Suppose first that $x \in K_1$; then by construction $K_1 \subseteq H_1$ and K'_1 has at least one artificial vertex, say a. Since $\{K'_1, K'_2\}$ is a proper pair, it follows from Lemma 6 that there exist a vertex $t_1 \in K'_1$ to which a is non-universal, and a vertex $t_2 \in K'_1$ which is non-universal to a. If either t_1 or t_2 is artificial, then K'_2 intersects A_2 (recall that $a, t_1, t_2 \in A_1$ have the same neighborhood outside K'_2) and consequently, by construction, $K_2 \subseteq H_2$; then the statement follows since $\{K_1, K_2\}$ is a proper pair of cliques. Conversely, if both t_1 and t_2 are non-artificial, then, using Claim 3, we conclude that in G x is non-universal to t_1 and that t_2 is non-universal to x, and therefore x has at least one neighbor and at least one non-neighbor in H_2.

Suppose now $x \notin K_1$: then, x is a non-artificial vertex of K'_1, and since $\{K'_1, K'_2\}$ is proper, it follows again from Lemma 6 that there exist a vertex $t_1 \in K'_1$ to which x is non-universal, and a vertex $t_2 \in K'_1$ which is non-universal to x. If both t_1 and t_2 are non-artificial, then also in G we have that x is non-universal to t_1 and t_2 is non-universal to x. If either t_1 or t_2 is artificial, then thanks to Claim 3, we may suitably replace t_1 or t_2 with vertices from K_1 as to get the same conclusion. (*End of the claim.*)

We conclude the proof of the lemma: part 2 holds by Claims 2 and 4, while part 3 holds by Claims 3 and 4. □

As we show in the following, if we iterate the reduction of Definition 15, we end up, in at most $|E(G)|$ steps, with a graph without PH pairs of cliques. We first need a definition and a simple lemma, going along the same lines of Definition 15 and Lemma 17. For a graph G, we denote by $\binom{V(G)}{2}$ the set of unordered pairs of vertices of $V(G)$.

Definition 18 *Let G and $G' := G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ be as in Definition 15, and let $S \subseteq \binom{V(G)}{2}$. The set $S|_{K_1,K_2,H_{\{A_1,A_2\}}} \subseteq \binom{V(G')}{2}$ is the set of pairs $\{x,y\}$ such that one of the following hold:*

- *$\{x,y\} \in S$ and $x, y \notin A_1 \cup A_2$;*
- *$x \in A_1$, $y \notin A_1 \cup A_2$ and $\{\{a,y\}, a \in K_1\} \subseteq S$;*
- *$y \in A_2$, $x \notin A_1 \cup A_2$ such that $\{\{x,a\}, a \in K_2\} \subseteq S$;*

Corollary 19 *Let G, $G' := G|_{K_1,K_2,H_{\{A_1,A_2\}}}$, S and $S' := S|_{K_1,K_2,H_{\{A_1,A_2\}}}$ be as in Definition 15 and Definition 18.*

(i) If S is a superset of $PH(G)$, then S' is a superset of $PH(V(G'))$.

(ii) If $|K_1| \geq |A_1|$ and $|K_2| \geq |A_2|$, then $|S'| < |S|$ and S' can be built from S in time $O(|V(G)|^2)$.

Proof. (i) Pick any pair $\{w_1, w_2\}$ of vertices of G' which admit a PH embedding in G': by part (1) of Lemma 17, they cannot both belong to $A_1 \cup A_2$. Suppose that $w_1, w_2 \notin A_1 \cup A_2$. Then, by part (2) of Lemma 17, $\{w_1, w_2\}$ also have a PH embedding in G and thus $\{w_1, w_2\} \in S$. Then, by construction, $\{w_1, w_2\} \in S'$. Now, suppose that exactly one of them belongs to $A_1 \cup A_2$, w.l.o.g. w_1, and let first $w_1 \in A_1$; then by part (3) of Lemma 17, for each $a \in K_1$, $\{a, w_2\}$ is a pair of vertices with a PH embedding in G, i.e. $\{\{a, w_2\}, a \in K_1\} \subseteq PH(G) \subseteq S$. Then, by construction, $\{w_1, w_2\} \in S'$. A similar argument works for $w_1 \in A_2$. (ii) The statements holds easily by construction. □

We are now ready to give our algorithm, see the following. Note that the algorithm is fully determined, but for the choice of the non-proper 2-clique graph $H_{\{A_1^i, A_2^i\}}$ to be used in each iteration i. In fact, the definition of $H_{\{A_1^i, A_2^i\}}$ will depend on G^i, K_1^i and K_2^i: this will be discussed in the next section. Regardless of this, we can anyhow state the Theorem 20.

Theorem 20 *Algorithm 2 builds a sequence of graphs $G = G^0, G^1, \ldots, G^q$ such that each G^i is obtained from G^{i-1} by a PH reduction, G^q has no PH pairs of cliques and $q \leq m$, where $m = |E(G)|$. Moreover, it can be implemented as to run in $O(n^2 m + \sum_{i=1}^q p(i))$-time, if $n = |V(G)|$ and, for $i = 1, \ldots, q$, it takes $p(i)$-time to generate $H_{\{A_1^i, A_2^i\}}$ from the knowledge of G^i, K_1^i and K_2^i.*

The proof of this theorem follows easily from our previous arguments, so we skip it. Let us remark here that in the algorithm we start with a set $S^0 = E(G)$, since we assumed no prior knowledge on the pair of vertices of G candidate to have a PH embedding is available. For specific graphs we may have a better knowledge of those, and consequently start from a set S^0 smaller in size. This may lead to asymptotically faster implementation of Algorithm 2.

4 Preserving some graphical invariant or property

In this section we show that suitable PH reductions preserve graph invariants, such as chromatic number, stability number, clique number, or graph properties, such as perfection, or the property of a graph of being fuzzy circular interval. Most of these reductions were in fact proposed by several

Algorithm 2 Eliminating all proper and homogeneous pairs of cliques

Require: A graph G.

Ensure: A graph G^q, without PH pairs of cliques, that is obtained from G by successive PH reductions.

```
i := 0; G^0 := G; S^0 = E(G);
while S^i is non-empty do
  pick a pair {u,v} ∈ S^i ;
  using Algorithm 1 check whether the pair {u,v} ∈ S^i has PH embedding in G^i;
  if u,v have a PH embedding {K_1^i, K_2^i} then
    let H_{A_1^i,A_2^i} be a non-proper 2-clique graph vertex disjoint from V(G^0)∪V(G^1)∪...∪V(G^i)
    and such that |K_1^i| ≥ |A_1^i| and |K_2^i| ≥ |A_2^i|;
    G^{i+1} = G^i|_{K_1^i,K_2^i,H_{A_1^i,A_2^i}} (see Definition 15);
    S^{i+1} = S^i|_{K_1^i,K_2^i,H_{A_1^i,A_2^i}} (see Definition 18);
  else
    remove the pair {u,v} from S^i;
    G^{i+1} = G^i, S^{i+1} = S^i;
  end if
  i := i + 1;
end while
q := i.
return G^q.
```

authors in different contexts, but as we show in the following they can be easily embedded in the unifying setting of PH reductions.

In some cases [7, 10, 13] the reductions that were used have the following form: take a PH pair of cliques $\{K_1, K_2\}$ and remove some suitable set of edges between vertices of K_1 and vertices of K_2 so that, in particular, in the resulting graph $K_1 \cup K_2$ does not induce any C_4. In another case [12] the reduction has the following form: take a PH pair of cliques $\{K_1, K_2\}$ and add all possible edges between vertices of K_1 and vertices of K_2 but one. It is straightforward to see that both types of reductions can be interpreted in terms of our PH reduction; therefore, they can be embedded into the iterative framework of Algorithm 2, and one may rely on the complexity bound given by Theorem 20.

We begin with a reduction introduced by King and Reed [9, 10] for removing edges in a PH pair of cliques while preserving the chromatic number. Recall that $\chi(G)$ denotes the chromatic number, while $\chi_f(G')$ denotes the fractional one.

Lemma 21 *[9] Let G be a graph and suppose that we are given a PH pair of cliques $\{K_1, K_2\}$ of G. Also, let X be a maximum clique in $G[K_1 \cup K_2]$, and let G' be the graph obtained from G by removing each edge $uv \in E(G)$ such that: $u \in K_1$; $v \in K_2$; $\{u, v\} \not\subseteq X$. Then:*

(i) G' can be built in time $O(|V(G)|^{\frac{5}{2}})$ (from the knowledge of G, K_1 and K_2);

(ii) $\chi(G) = \chi(G')$, $\chi_f(G) = \chi_f(G')$ and each k-coloring of G' can be extended into a k-coloring of G of in time $O(|V(G)|^{\frac{5}{2}})$.

(iii) $\omega(G) = \omega(G')$, and each clique of G' is also a clique of G.

(iv) If G is claw-free (resp. quasi-line; perfect), then G' is claw-free (resp. quasi-line; perfect).

(One should mention that Lemma 21 can be extended to the case where $\{K_1, K_2\}$ is a *nonskeletal* and homogeneous pair of cliques [9]. Also, Andrew King [8] pointed us that this lemma is non-trivially implied by some proofs in [2]. In that paper, Chudnovsky and Ovetsky introduce another reduction for PH pairs of cliques, which is quite similar to the one above. This reduction preserves quasi-liness, while not increasing the clique number of G. It is a simple exercise to show that the reduction in [2] can be interpreted in terms of our PH reduction. Finally, we mention that proposition (*iii*) of Lemma 21 is not stated in [9], but it is almost straightforward.)

As already pointed out above, since $G'[K_1 \cup K_2]$ is C_4^{free}, the above reduction can be easily interpreted in terms of our PH reduction. Therefore, by embedding Lemma 21 in the iterative framework of Algorithm 2, we can reduce in polynomial time the problem of computing the chromatic (resp. clique) number on a given graph G to the same problem on a graph G' without PH pairs of cliques.

Corollary 22 *From a graph G one can obtain in time $O(|V(G)|^{\frac{5}{2}}|E(G)|)$ a graph G' without PH pairs of cliques such that $\chi(G) = \chi(G')$ and $\omega(G) = \omega(G')$. Moreover, one can derive an optimal coloring of G from an optimal coloring in G' in time $O(|V(G)|^{\frac{5}{2}}|E(G)|)$, while a maximum clique in G' is also a maximum clique in G.*

As argued by Li and Zang [11], the maximum weighted clique problem in the complement of a bipartite graph can be reduced to maximum flow, and hence solved in time $O(n^3)$. By building on the latter fact (and slightly increasing the complexity), Corollary 22 can be extended to the computation of a graph G' without PH cliques that preserves the maximum weighted clique and its value: we skip the details.

Consider now the maximum weighted stable set problem. The following lemma is given in [12]:

Lemma 23 *[12] Let $G(V, E)$ be a graph with a weight function $w : V \mapsto \mathbb{R}$ defined on its vertices. Let $\{K_1, K_2\}$ be a homogeneous pair of cliques, and let $\{a, b\}$ be a maximum weighted stable set of size two of $G[K_1 \cup K_2]$. Adding any edge different from ab between a vertex of K_1 and vertices of K_2 preserves the maximum weight of a stable set, moreover, every stable set of the new graph is a stable set of G with the same weight.*

(One should mention that Lemma 23 is stated by Oriolo, Pietropaoli and Stauffer for the more general class of *semi-homogeneous* pairs of cliques.)

Again, it is quite simple to view the above operation in terms of our PH reduction, and we can claim the following Lemma (we skip the details of its proof):

Corollary 24 *Let $G(V, E)$ be a graph with a weight function $w : V \mapsto \mathbb{R}$ defined on its vertices. In time $O(|V(G)|^2|E(G)|)$ one can build a graph G' without PH pairs of cliques such that a maximum weighted stable set of G' is also a maximum weighted stable set of G.*

Interestingly, if we now move from the maximum weighted stable set problem to the stable set polytope $STAB(G)$ of a graph G, we can also embed a result in [7] in our framework. Eisenbrand et al. show – see the remark following Lemma 5 in [7] – that each facet of the stable set polytope $STAB(G)$ is also a facet of another graph G' (obtained from G by removing edges) that does not contain any PH pair of cliques. As one easily checks (cfr. the proof of Lemma 5 in [7]), also their result can be phrased in the framework of Algorithm 2; we skip the details.

We now move from graph invariants to graph properties. First, Oriolo, Pietropaoli and Stauffer [13] show that a suitable reduction of PH pairs of cliques preserves the property of a graph of being, or not being, a fuzzy circular interval graph, and they exploit this fact in an algorithm for recognizing fuzzy circular interval graphs. Once again it is straightforward to view their reduction as a PH reduction. In fact, Theorem 20 is already used in [13] for bounding the complexity of the recognition algorithm.

Finally, every PH reduction preserves perfection, and under very general conditions it does not turn a non-perfect graph into a perfect one.

Lemma 25 *Let G be a perfect graph with a PH pair of cliques $\{K_1, K_2\}$. Also let $H_{\{A_1,A_2\}}$ be a non-proper 2-clique graph vertex disjoint from G. Then the graph $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ is perfect. The converse implication holds true if A_1 is not anticomplete to A_2.*

The proof of the previous lemma is standard so we omit it. We conclude by pointing out that, with the exception of the reduction from Lemma 21 (since $X \subseteq K_1$ or $X \subseteq K_2$ may happen), all the reductions from the current section do not turn an imperfect graph into a perfect one.

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Note While preparing this manuscript, we became aware that M. Chudnovsky and A. King independently found a result similar to Theorem 20, even though they reduce all proper and homogeneous pairs of cliques at once [1].

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A Proof of Theorem 20

Proof. The algorithm defines a sequence of graphs $G^0 = G, G^1, \ldots, G^q$. The sequence is such that, for each $i = 0, \ldots, q-1$, $G^{i+1} = G^i|_{K_1^i, K_2^i, H_{\{A_1^i, A_2^i\}}}$, where $\{K_1^i, K_2^i\}$ is a PH embedding of the current pair $\{u, v\}$ in G^i, or $G^{i+1} = G^i$. We first show that the algorithm is correct, i.e. G^q is indeed a graph without PH pairs of cliques. In order to prove that, we assume that the algorithm terminates, i.e. q is finite; we will show later that, in fact, $q \leq m$.

We claim that, at each iteration $i = 0, \ldots, q$, the set S^i, that is maintained by the algorithm, is a superset of PH(G^i). This is trivial for (S^0, G^0), since $S^0 = E(G)$ is a superset of PH(G). By induction, suppose that, for some i between 0 and $q-1$, S^i is a superset of PH(G^i). If $\{u, v\}$ is not a pair of vertices with a PH embedding, then the statement is trivial; otherwise, $G^{i+1} = G^i|_{K_1^i, K_2^i, H_{\{A_1^i, A_2^i\}}}$ and $S^{i+1} = S^i|_{K_1^i, K_2^i, H_{\{A_1^i, A_2^i\}}}$, and the statement follows from Corollary 19. Finally observe that, since S^q is a superset of PH(G^q) and the termination of the algorithm implies $S^q = \emptyset$, it follows from Lemma 9 that G^q is indeed a graph without PH pairs of cliques.

We now show that the algorithm terminates and, in particular, $q \leq m$. We start with $|S^0| = m$. While S^i is non-empty, either set $S^{i+1} = S^i|_{K_1^i, K_2^i, H_{\{A_1^i, A_2^i\}}}$, or $S^{i+1} = S^i \setminus \{\{u, v\}\}$. In both cases, the size of S decreases by at least 1 (see Corollary 19), and the statement follows.

Last, we focus on complexity issues. Note that from Lemma 16 we have that $|V(G^0)| \geq |V(G^1)| \geq \ldots \geq |V(G^q)|$. Thus, finding a PH embedding of two given vertices can be done in $O(n^2)$-time by Algorithm 1, and analogously we can generate S^{i+1} from S^i in $O(n^2)$-time, from Corollary 19. The global bound $O(n^2 m + \sum_{i=1}^{q} p(i))$ follows from the fact that $q \leq m$. □

B Proof of Lemma 25

Proof. Recall that a graph G is perfect if and only if it contains neither long odd holes, nor long odd anti-holes, *long* meaning of length at least 5 [3].

We start with proving the following fact: if $\{Q_1, Q_2\}$ is a homogeneous pair of cliques, then each (induced) long odd-hole (resp. each long odd anti-hole) takes at most one vertex from Q_1 and at most one vertex from Q_2. First suppose that H is a long odd hole of G. Suppose e.g. that $|V(H) \cap Q_1| \geq 2$. Then since $\{Q_1, Q_2\}$ is a homogeneous pair of clique and $k \geq 5$, it follows that two non-adjacent vertices of H must belong to Q_2, a contradiction.

Now let H be a long odd anti-hole of G. Suppose that either $|V(H) \cap Q_1| \geq 2$ or $|V(H) \cap Q_2| \geq 2$. Note that, for a given pair of vertices $\{h, h'\}$ such that h and $h' \in V(H) \cap Q_1$ or h and $h' \in V(H) \cap Q_2$, we can always label the vertices of H as $h_0, \ldots, h_{k-1}$ so that each h_j is not adjacent to h_{j-1} and h_{j+1} (computation here is taken modulo k), and, moreover, $h = h_0$ and $h' = h_i$, with i odd. Of all pairs $\{h, h'\}$ such that h and $h' \in V(H) \cap Q_1$ or h and $h' \in V(H) \cap Q_2$ and of all labellings of vertices of H with the properties above, choose one that minimizes i. We can suppose w.l.o.g. that $h_0, h_i \in Q_1$. Note that $i \geq 2$, since Q_1 is a clique. Actually, $i \geq 3$, since i is odd. Then h_1 is adjacent to h_i but non-adjacent to h_0 and, conversely, h_{i-1} is adjacent to h_0 but non-adjacent to h_i. Since $\{Q_1, Q_2\}$ is a homogeneous pair, both h_1 and h_{i-1} belong to Q_2. But since $i-1$ is even, this is in contradiction with our choice of h, h' and of the labelling.

We now show that $G|_{K_1, K_2, H_{\{A_1, A_2\}}}$ is perfect. Suppose it is not, i.e. in $G|_{K_1, K_2, H_{\{A_1, A_2\}}}$ there is an induced subgraph H' that is either a long odd-hole or a long anti-hole. Recall that $\{A_1, A_2\}$ is a homogeneous pair of cliques of $G|_{K_1, K_2, H_{\{A_1, A_2\}}}$ (see Lemma 16). It follows from above that $|V(H') \cap A_1| \leq 1$ and $|V(H') \cap A_2| \leq 1$. First suppose that both $V(H') \cap A_1$ and $V(H') \cap A_2$ are non-empty, and let $k_1 \in A_1 \cap V(H')$ and $k_2 \in A_2 \cap V(H')$. In this case, we choose a vertex $a_1 \in K_1$

and a vertex $a_2 \in K_2$ such that a_1 and a_2 are adjacent if and only if k_1 and k_2 are adjacent. Then it is straightforward to see that $H := G[(V(H) \setminus \{k_1, k_2\}) \cup \{a_1, a_2\}]$ is either an induced long odd hole or an induced long odd anti-hole of G, a contradiction. The cases where either $V(H') \cap A_1$ or $V(H') \cap A_2$ is empty can be managed along the same lines.

Analogously, one may show that, if H is either an induced long odd hole or an induced long odd anti-hole of G, and A_1 is not anticomplete to A_2, then also in $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ there is either an induced long odd hole or an induced long odd anti-hole. □