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Contents

Introduction

The representations of quivers can be viewed as a formalization of some linear algebra problems. Symmetric quivers have been introduced by Derksen and Weyman in [DW2] to provide similar formalization for other classical groups.
In the recent years the quiver representations were used to prove interesting results related to general linear groups.
Derksen and Weyman in [DW1] gave a proof of saturation property for Littlewood-Richardson coefficients.
Magyar, Weyman and Zelevinsky in [MWZ1] classified products of flag varieties with finitely many orbits under the diagonal action of general linear groups. We hope that the representations of symmetric quivers are a tool to solve similar problems for classical groups.
Another interesting aspect and direction for future research is the connection with Cluster algebras (see [FZ1]). Igusa, Orr, Todorov and Weyman in [IOTW] generalized the semi-invariants of quivers to virtual representations of quivers. They associated, via virtual semi-invariants of quivers, a simplicial complex $\mathcal{T}(Q)$ with each quiver Q. In particular, if Q is of finite type, then the simplices of $\mathcal{T}(Q)$ correspond to tilting objects in a corresponding Cluster category (defined in [BMRRT]). It would be interesting to carry out a similar construction for symmetric quivers of finite type and to relate it to Cluster algebras (see [FZ2]).
The results of this thesis are first steps in this direction. We describe the ring of semi-invariants for symmetric quivers of finite and tame type.
A symmetric quiver is a pair (Q, σ) where Q is a quiver (called *underlying quiver of* (Q, σ)) and σ is a contravariant involution on the union of the set of arrows and the set of vertices of Q. The involution allows us to define a nondegenerate bilinear form $<,>$ on a representation V of Q. We call the pair $(V, <,>)$ orthogonal representation (respectively symplectic) of (Q, σ) if $<,>$ is symmetric (respectively skew-symmetric). We define $SpRep(Q, \beta)$ and $ORep(Q, \beta)$ to be respectively the space of symplectic β-dimensional representations and the space of orthogonal β-dimensional representations of (Q, σ). Moreover we can define an action of a product of classical groups, which we call $SSp(Q, \beta)$ in the symplectic case and $SO(Q, \beta)$ in the orthogonal case, on these space. We describe a set of generators of the ring of

semi-invariants of $ORep(Q,\beta)$

$$OSI(Q,\beta) = \mathbb{K}[ORep(Q,\beta)]^{SO(Q,\beta)} =$$

$$\{f \in \mathbb{K}[ORep(Q,\beta)] | g \cdot f = f \ \forall g \in SO(Q,\beta)\}$$

and of the ring of semi-invariants of $SpRep(Q,\alpha)$

$$SpSI(Q,\beta) = \mathbb{K}[SpRep(Q,\beta)]^{SSp(Q,\beta)} =$$

$$\{f \in \mathbb{K}[SpRep(Q,\beta)] | g \cdot f = f \ \forall g \in SSp(Q,\beta)\},$$

where $\mathbb{K}[ORep(Q,\beta)]$ is the ring of polynomial functions on $ORep(Q,\beta)$ and $\mathbb{K}[SpRep(Q,\beta)]$ is the ring of polynomial functions on $SpRep(Q,\beta)$.
Let (Q,σ) be a symmetric quiver and V a representation of the underlying quiver Q such that $\langle \underline{dim}\, V, \beta\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is the Euler form of Q. Let

$$0 \longrightarrow P_1 \xrightarrow{d^V} P_0 \longrightarrow V \longrightarrow 0$$

be the canonical projective resolution of V (see [R1]). We define the semi-invariant $c^V := det(Hom_Q(d^V,\cdot))$ of $OSI(Q,\beta)$ and $SpSI(Q,\beta)$ (see [DW1] and [S]).
Let τ be the Auslander-Reiten translation functor and let ∇ be the duality functor. We will prove in the symmetric case the following

Theorem 1. *Let (Q,σ) be a symmetric quiver of finite type or of tame type such that the underlying quiver Q is without oriented cycles and let β be a symmetric dimension vector. The ring $SpSI(Q,\beta)$ is generated by semi-invariants*

(i) c^V *if $V \in Rep(Q)$ is such that $\langle \underline{dim}\, V,\beta\rangle = 0$,*

(ii) $pf^V := \sqrt{c^V}$ *if $V \in Rep(Q)$ is such that $\langle \underline{dim}\, V,\beta\rangle = 0$, $\tau V = \nabla V$ and the almost split sequence $0 \to \nabla V \to Z \to V \to 0$ has the middle term Z in $ORep(Q)$.*

Theorem 2. *Let (Q,σ) be a symmetric quiver of finite type or of tame type such that the underlying quiver Q is without oriented cycles and let β be a symmetric dimension vector. The ring $OSI(Q,\beta)$ is generated by semi-invariants*

(i) c^V *if $V \in Rep(Q)$ is such that $\langle \underline{dim}\, V,\beta\rangle = 0$,*

(ii) $pf^V := \sqrt{c^V}$ *if $V \in Rep(Q)$ is such that $\langle \underline{dim}\, V,\beta\rangle = 0$, $\tau V = \nabla V$ and the almost split sequence $0 \to \nabla V \to Z \to V \to 0$ has the middle term Z in $SpRep(Q)$.*

The strategy of the proofs is the following. First we set the technique of reflection functors on the symmetric quivers. Then we prove that we can reduce theorems 1 and 2, by this technique, to particular orientations of the symmetric quivers. Finally, we check theorems 1 and 2 for these orientations.

In the first chapter we give general notions and results about symmetric quivers and their representations. First, we state main results 1 and 2. Next, we adjust to symmetric quivers the technique of reflection functors and we describe particular orientations for every symmetric quiver of finite type and tame type. Finally, we prove general results about semi-invariants of symmetric quivers and we check that we can reduce theorems 1 and 2 to these particular orientations.

In the second chapter, using classical invariant theory and the technique of Schur functors, we prove case by case theorems 1 and 2 for symmetric quivers of finite type with the orientations described in chapter 1.

In the third chapter we prove theorems 1 and 2 for symmetric quivers of tame type with the orientations described in chapter 1. First, we deal with symplectic and orthogonal representations of dimension $\beta = ph$, where $p \in \mathbb{N}$ and h is the homogeneous simple regular dimension vector. We give a proof of theorems 1 and 2 case by case. Next, we adjust to symmetric quivers some general results of Dlab and Ringel about regular representations of tame quivers (see [DR]) and we describe generic decomposition of dimension vectors of symplectic and orthogonal representations (see [K1] and [K2]). Finally, by these results, we describe case by case the ring of semi-invariants of symmetric quivers of tame type for any regular dimension vectors.

At last, in appendix *A* we recall some results of representations of general linear group and of invariant theory. In appendix *B* we recall general definitions and results about quiver representations and semi-invariants of quivers.

Chapter 1

Main results

1.1 Symmetric quivers

Throughout all this section, we use the notation of section B.1.

Definition 1.1.1. *A symmetric quiver is a pair (Q,σ) where Q is a quiver (called the underlying quiver of (Q,σ)) and σ is an involution from the disjoint union $Q_0 \coprod Q_1$ to itself, such that*

(i) $\sigma(Q_0) = Q_0$ *and* $\sigma(Q_1) = Q_1$,

(ii) $t\sigma(a) = \sigma(ha)$ *and* $h\sigma(a) = \sigma(ta)$ *for all* $a \in Q_1$,

(iii) $\sigma(a) = a$ *whenever* $a \in Q_1$ *and* $\sigma(ta) = ha$.

Definition 1.1.2. *Let (Q,σ) be a symmetric quiver and*

$$V = \left\{\{V(x)\}_{x\in Q_0}, \{V(a)\}_{a\in Q_1}\right\}$$

be a representation of the underlying quiver Q. We define the duality functor $\nabla : V \to V^*$ with $V^* = \left\{\{V^*(x)\}_{x\in Q_0}, \{V^*(a)\}_{a\in Q_1}\right\}$ where $V^*(x) := V(\sigma(x))^*$ for every $x \in Q_0$ and $V^*(a) := -V(\sigma(a))^*$ for every $a \in Q_1$. Moreover if W is another representation of Q and $f : V \to W$ is a morphism, then $\nabla f : \nabla W \to \nabla V$ is defined by $(\nabla f)(x) := f(\sigma(x))^* : W^*(x) \to V^*(x)$, for every $x \in Q_0$. We shall call V selfdual if $\nabla V = V$.

Definition 1.1.3. *An orthogonal (resp. symplectic) representation of a symmetric quiver (Q,σ) is a pair $(V, < \cdot,\cdot >)$, where V is a representation of the underlying quiver Q with a nondegenerate symmetric (resp. skew-symmetric) scalar product $< \cdot,\cdot >$ on $\bigoplus_{x\in Q_0} V(x)$ such that*

(i) the restriction of $< \cdot,\cdot >$ *to* $V(x) \times V(y)$ *is 0 if* $y \neq \sigma(x)$,

(ii) $< V(a)(v), w > + < v, V(\sigma(a))(w) >= 0$ *for all* $v \in V(ta)$ *and all* $w \in V(\sigma(ha))$.

By properties *(i)* and *(ii)* of definition 1.1.3, an orthogonal or symplectic representation $(V, < \cdot, \cdot >)$ of a symmetric quiver is selfdual.

Definition 1.1.4. *An orthogonal (respectively symplectic) representation is called indecomposable orthogonal (respectively indecomposable symplectic) if it cannot be expressed as a direct sum of orthogonal (respectively symplectic) representations.*

We denote Q_0^σ (respectively Q_1^σ) the set of vertices (respectively arrows) fixed by σ. Thus we have partitions

$$Q_0 = Q_0^+ \cup Q_0^\sigma \cup Q_0^-$$

$$Q_1 = Q_1^+ \cup Q_1^\sigma \cup Q_1^-$$

such that $Q_0^- = \sigma(Q_0^+)$ and $Q_1^- = \sigma(Q_1^+)$, satisfying:

i) $\forall a \in Q_1^+$, either $\{ta, ha\} \subset Q_0^+$ or one of the elements in $\{ta, ha\}$ is in Q_0^+ while the other is in Q_0^σ;

ii) $\forall x \in Q_0^+$, if $a \in Q_1$ with $ta = x$ or $ha = x$, then $a \in Q_1^+ \cup Q_1^\sigma$.

Definition 1.1.5. *Let (Q, σ) be a symmetric quiver. We define a linear map $\delta : \mathbb{Z}_{\geq 0}^{Q_0} \to \mathbb{Z}_{\geq 0}^{Q_0}$ by setting $\{\delta\alpha(i)\}_{i \in Q_0} = \{\alpha(\sigma(i))\}_{i \in Q_0}$ for every dimension vector α.*

Remark 1.1.6. *Since σ is an involution, also δ is one.*

Remark 1.1.7. *If V is a representation of dimension α then $\delta\alpha = \underline{dim}(\nabla V)$. In particular if V is an orthogonal or symplectic representation of (Q, σ) of dimension α, then $\delta\alpha = \alpha$. Such α is called symmetric dimension vector.*

Proposition 1.1.8. *Let $\delta : \mathbb{Z}_{\geq 0}^{Q_0} \to \mathbb{Z}_{\geq 0}^{Q_0}$ as in definition 1.1.5. If α and β are dimension vectors, then*

$$\langle \alpha, \beta \rangle = \langle \delta\beta, \delta\alpha \rangle. \tag{1.1}$$

Proof.

$$\begin{aligned} \langle \alpha, \beta \rangle = & \textstyle\sum_{i \in Q_0^+ \cup Q_0^\sigma} \alpha(i)\beta(i) + \sum_{i \in Q_0^+} \alpha(\sigma(i))\beta(\sigma(i)) \\ & + \textstyle\sum_{a \in Q_1^+ \cup Q_1^\sigma} \alpha(ta)\beta(ha) + \sum_{a \in Q_1^+} \alpha(t\sigma(a))\beta(h\sigma(a)) \end{aligned} \quad . \tag{1.2}$$

By definition of σ, we have

$$\langle \delta\beta, \delta\alpha \rangle =$$

$$\sum_{i \in Q_0^+ \cup Q_0^\sigma} \beta(\sigma(i))\alpha(\sigma(i)) + \sum_{i \in Q_0^+} \beta(\sigma(\sigma(i)))\alpha(\sigma(\sigma(i))) +$$

$$\sum_{a \in Q_1^+ \cup Q_1^\sigma} \beta(\sigma(ta))\alpha(\sigma(ha)) + \sum_{a \in Q_1^+} \beta(\sigma(t\sigma(a)))\alpha(\sigma(h\sigma(a))) =$$

$$\sum_{i\in Q_0^+} \beta(\sigma(i))\alpha(\sigma(i)) + \sum_{i\in Q_0^\sigma} \beta(i)\alpha(i) +$$

$$\sum_{i\in Q_0^+} \beta(i)\alpha(i) + \sum_{a\in Q_1^+} \beta(h\sigma(a))\alpha(t\sigma(a)) +$$

$$\sum_{a\in Q_1^\sigma} \beta(h\sigma(a))\alpha(t\sigma(a)) + \sum_{a\in Q_1^+} \beta(\sigma^2(ha))\alpha(\sigma^2(t\sigma(a))$$

which is the right hand side of (1.2), recalling that σ is an involution. □

The space of orthogonal α-dimensional representations of a symmetric quiver (Q,σ) can be identified with

$$ORep(Q,\alpha) = \bigoplus_{a\in Q_1^+} Hom(\mathbb{K}^{\alpha(ta)}, \mathbb{K}^{\alpha(ha)}) \oplus \bigoplus_{a\in Q_1^\sigma} \bigwedge^2 (\mathbb{K}^{\alpha(ta)})^*. \tag{1.3}$$

The space of symplectic α-dimensional representations can be identified with

$$SpRep(Q,\alpha) = \bigoplus_{a\in Q_1^+} Hom(\mathbb{K}^{\alpha(ta)}, \mathbb{K}^{\alpha(ha)}) \oplus \bigoplus_{a\in Q_1^\sigma} S_2(\mathbb{K}^{\alpha(ta)})^*. \tag{1.4}$$

We define the group

$$O(Q,\alpha) = \prod_{x\in Q_0^+} GL(\mathbb{K},\alpha(x)) \times \prod_{x\in Q_0^\sigma} O(\mathbb{K},\alpha(x)) \tag{1.5}$$

and the subgroup

$$SO(Q,\alpha) = \prod_{x\in Q_0^+} SL(\mathbb{K},\alpha(x)) \times \prod_{x\in Q_0^\sigma} SO(\mathbb{K},\alpha(x)). \tag{1.6}$$

Here $O(\mathbb{K},\alpha(x))$ is the group of orthogonal transformations for the symmetric form $< \cdot,\cdot >$ restricted to $V(x)$.

Assuming that $\alpha(x)$ is even for every $x \in Q_0^\sigma$, we define the group

$$Sp(Q,\alpha) = \prod_{x\in Q_0^+} GL(\mathbb{K},\alpha(x)) \times \prod_{x\in Q_0^\sigma} Sp(\mathbb{K},\alpha(x)) \tag{1.7}$$

and the subgroup

$$SSp(Q,\alpha) = \prod_{x\in Q_0^+} SL(\mathbb{K},\alpha(x)) \times \prod_{x\in Q_0^\sigma} Sp(\mathbb{K},\alpha(x)). \tag{1.8}$$

Here $Sp(\mathbb{K}, \alpha(x))$ is the group of isometric transformations for the skew-symmetric form $< \cdot, \cdot >$ restricted to $V(x)$.

The action of these groups is defined by

$$g \cdot V = \{g_{ha} V(a) g_{ta^{-1}}\}_{a \in Q_1^+ \cup Q_1^\delta}$$

where $g = (g_x)_{x \in Q_0} \in O(Q, \alpha)$ (respectively $g \in Sp(Q, \alpha)$) and $V \in ORep(Q, \alpha)$ (respectively in $SpRep(Q, \alpha)$). In particular we can suppose $g_{\sigma(x)} = (g_x^{-1})^t$ for every $x \in Q_0$.

Example 1.1.9. *(1) Consider the symmetric quiver* (Q, σ)

$$\circ \to \bullet \to \circ$$

where σ interchanges the antipodal nodes and fixes the closed node. An orthogonal representation of (Q, σ) is a quadruple $(V_1, V_2, \phi, \langle \cdot, \cdot \rangle)$ where V_1 and V_2 are vector spaces, $\phi : V_1 \to V_2$ is a linear map and $\langle \cdot, \cdot \rangle$ is a non-degenerate symmetric bilinear form on V_2. We also have the dual map $-\phi^* : V_2^* \cong V_2 \to V_1^*$ and so we have the following diagram:

$$V_1 \xrightarrow{\phi} V_2 \xrightarrow{-\phi^*} V_1^*.$$

Hence the isomorphism classes of orthogonal representations of (Q, σ) are the $GL(V_1) \times O(V_2)$-orbits in $Hom(V_1, V_2)$.

(2) Consider the symmetric quiver (Q, σ)

$$\circ \to \circ \to \circ \to \circ$$

where σ sends the first vertex to the last one and the second one to the third one. A symplectic representation of (Q, σ) is a quadriple (V_1, V_2, ϕ, ψ) where V_1 and V_2 are vector spaces, $\phi : V_1 \to V_2$ is linear map and $\psi \in S_2 V_2^*$. We also have the dual map $-\phi^* : V_2^* \to V_1^*$. We consider the following diagram:

$$V_1 \xrightarrow{\phi} V_2 \xrightarrow{\psi} V_2^* \xrightarrow{-\phi^*} V_1^*.$$

Hence the isomorphism classes of symplectic representations of (Q, σ) are the $GL(V_1) \times GL(V_2)$-orbits in $Hom(V_1, V_2) \oplus S_2 V_2^*$.

Definition 1.1.10. *(i) Let $\mathbb{K}[ORep(Q, \alpha)]$ be the ring of polynomial functions on $ORep(Q, \alpha)$.*

$$OSI(Q, \alpha) = \mathbb{K}[ORep(Q, \alpha)]^{SO(Q,\alpha)} =$$

$$\{f \in \mathbb{K}[ORep(Q, \alpha)] | g \cdot f = f \ \forall g \in SO(Q, \alpha)\} \tag{1.9}$$

is the ring of orthogonal semi-invariants of (Q, α).

(ii) *Let* $\mathbb{K}[SpRep(Q,\alpha)]$ *be the ring of polynomial functions on* $SpRep(Q,\alpha)$*,*

$$SpSI(Q,\alpha) = \mathbb{K}[SpRep(Q,\alpha)]^{SSp(Q,\alpha)} =$$

$$\{f \in \mathbb{K}[SpRep(Q,\alpha)] | g \cdot f = f \ \forall g \in SSp(Q,\alpha)\} \qquad (1.10)$$

is the ring of symplectic semi-invariants of (Q,α)*.*

1.1.1 Symmetric quivers of finite type

Definition 1.1.11. *A symmetric quiver is said to be of finite representation type if it has only finitely many indecomposable orthogonal (resp. symplectic) representations up to isomorphisms.*

We recall the following theorem proved by Derksen and Weyman in [DW2]

Theorem 1.1.12. *A symmetric quiver* (Q,σ) *is of finite type if and only if the underlying quiver* Q *is of type* A_n*.*

Proof. See [DW2, theorem 3.1 and proposition 3.3] □

1.1.2 Symmetric quivers of tame type

Definition 1.1.13. *A symmetric quiver is said to be of tame representation type if is not of finite representation type, but in every dimension vector the indecomposable orthogonal (symplectic) representations occur in families of dimension* ≤ 1*.*

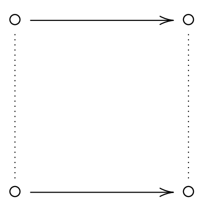
Theorem 1.1.14. *A symmetric quiver* (Q,σ) *with* Q *connected is tame if and only if the underlying quiver* Q *is an extended Dynkin quiver.*

Proof. See [DW2, theorem 4.1]. □

One can classify the symmetric tame quivers with connected underlying quiver.

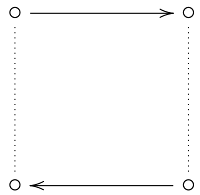
Proposition 1.1.15. *Let* (Q,σ) *be a symmetric tame quiver with* Q *connected. Then* (Q,σ) *is one of the following symmetric quivers.*

(1) Of type $\widetilde{A}_n^{2,0,1}$*:*



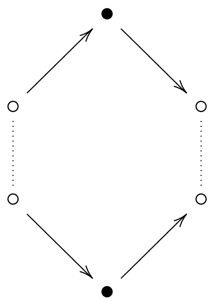
with arbitrary orientation reversed under σ *if* $Q = \widetilde{A}_{2n+1}$ (≥ 1)*. Here* σ *is a reflection with respect to a central vertical line (so* σ *fixes two arrows and no vertices).*

(2) *Of type* $\widetilde{A}_n^{2,0,2}$:



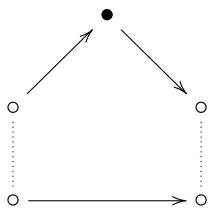
with arbitrary orientation reversed under σ *if* $Q = \widetilde{A}_{2n+1}$ (≥ 1). *Here* σ *is a reflection with respect to a central vertical line (so* σ *fixes two arrows and no vertices).*

(3) *Of type* $\widetilde{A}_n^{0,2}$:



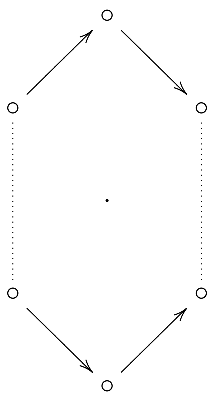
with arbitrary orientation reversed under σ *if* $Q = \widetilde{A}_{2n-1}$ $(n \leq 1)$. *Here* σ *is a reflection with respect to a central vertical line (so* σ *fixes two vertices and no arrows).*

(4) *Of type* $\widetilde{A}_n^{1,1}$:



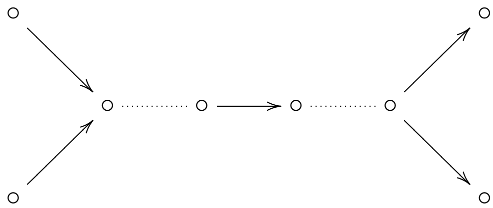
with arbitrary orientation reversed under σ *if* $Q = \widetilde{A}_{2n}$ $(n \geq 1)$. *Here* σ *is a reflection with respect to a central vertical line (so* σ *fixes one arrow and one vertex).*

(5) *Of type* $\widetilde{A}_n^{0,0}$*:*



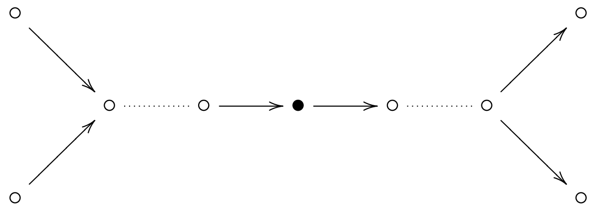
with arbitrary orientation reversed under σ *if* $Q = \widetilde{A}_{2n+1}$ $(n \geq 1)$*. Here* σ *is a central symmetry (so* σ *fixes neither arrows nor vertices).*

(6) *Of type* $\widetilde{D}_n^{1,0}$



with arbitrary orientation reversed under σ *if* $Q = \widetilde{D}_{2n}$ $(n \geq 2)$*. Here* σ *is a reflection with respect to a central vertical line (so* σ *fixes one arrow and no vertices).*

(7) *Of type* $\widetilde{D}_n^{0,1}$



with arbitrary orientation reversed under σ *if* $Q = \widetilde{D}_{2n-1}$ $(n \geq 2)$*. Here* σ *is a reflection with respect to a central vertical line (so* σ *fixes one vertex and no arrows).*

Proof. See [DW2, proposition 4.3]. □

1.2 The main results

In this thesis we describe the rings of semi-invariants of symmetric quivers in the finite type and in the tame cases. We also conjecture in general the following results. Below we use the notations of section B.4 and we conjecture the following theorems

Conjecture 1.2.1. *Let (Q,σ) a symmetric quiver such that the underlying quiver Q is without oriented cycles and let β be a symmetric dimension vector. The ring $SpSI(Q,\beta)$ is generated by semi-invariants*

(i) c^V *if* $V \in Rep(Q)$ *is such that* $\langle \underline{dim}\, V, \beta\rangle = 0$,

(ii) $pf^V := \sqrt{c^V}$ *if* $V \in Rep(Q)$ *is such that* $\langle \underline{dim}\, V, \beta\rangle = 0$, $V = \tau^- \nabla V$ *and the almost split sequence* $0 \to \nabla V \to Z \to V \to 0$ *has the middle term* Z *in* $ORep(Q)$.

Conjecture 1.2.2. *Let (Q,σ) a symmetric quiver such that the underlying quiver Q is without oriented cycles and let β be a symmetric dimension vector. The ring $OSI(Q,\beta)$ is generated by semi-invariants*

(i) c^V *if* $V \in Rep(Q)$ *is such that* $\langle \underline{dim}\, V, \beta\rangle = 0$,

(ii) $pf^V := \sqrt{c^V}$ *if* $V \in Rep(Q)$ *is such that* $\langle \underline{dim}\, V, \beta\rangle = 0$, $V = \tau^- \nabla V$ *and the almost split sequence* $0 \to \nabla V \to Z \to V \to 0$ *has the middle term* Z *in* $SpRep(Q)$.

We prove these conjectures for symmetric quivers of finite type (chapter 2) and for symmetric quivers of tame type and regular dimension vectors β (chapter 3).
We use the following strategy. First we adjust to symmetric quivers the technique of reflection functors. Next we prove with this technique that we can reduce the conjectures 1.2.1 and 1.2.2 to a particular orientation of the quiver. Then we state and prove conjectures 1.2.1 and 1.2.2 for these orientations.

Definition 1.2.3. *We will say that* $V \in Rep(Q)$ *satisfies property (Op) if*

(i) $V = \tau^- \nabla V$

(ii) the almost split sequence $0 \to \nabla V \to Z \to V \to 0$ *has the middle term* Z *in* $ORep(Q)$.

Similarly we will say that $V \in Rep(Q)$ *satisfies property (Spp) if*

(i) $V = \tau^- \nabla V$

(ii) the almost split sequence $0 \to \nabla V \to Z \to V \to 0$ *has the middle term* Z *in* $SpRep(Q)$.

1.3 Reflection functors for symmetric quivers

In this section we describe the technique of reflection functors for the symmetric quivers.

1.3.1 Admissible sink-source pairs

We use the notation of section B.3.

Definition 1.3.1. *Let (Q, σ) be a symmetric quiver. A sink (respectively source) $x \in Q_0$ is called admissible if there are no arrows connecting x and $\sigma(x)$.*

By definition of σ, x is a sink (respectively a source) if and only if $\sigma(x)$ is a source, so we can define the quiver $c_{\sigma(x)}c_x(Q)$. We shall call $(x, \sigma(x))$ *the admissible sink-source pair*. The corresponding reflection is denoted by $c_{(x,\sigma(x))} := c_{\sigma(x)}c_x$.

Lemma 1.3.2. *If (Q, σ) is a symmetric quiver and x is an admissible sink or source, then $(c_{(x,\sigma(x))}(Q), \sigma)$ is symmetric.*

Proof. Let $x \in Q_0$ be an admissible sink of (Q, σ). When we apply $c_{(x,\sigma(x))}$ to Q, the only arrows which we reverse are the arrows connecting to x and those connecting to $\sigma(x)$. Now in $c_{(x,\sigma(x))}(Q)$, x becomes a source and $\sigma(x)$ becomes a sink. So if a is an arrow connecting to x or to $\sigma(x)$ we have $\sigma(tc_{(x,\sigma(x))}(a)) = \sigma(ha) = t\sigma(a) = h\sigma(c_{(x,\sigma(x))}(a))$ and $\sigma(hc_{(x,\sigma(x))}(a)) = \sigma(ta) = h\sigma(a) = t\sigma(c_{(x,\sigma(x))}(a))$. Hence $c_{(x,\sigma(x))}(Q)$ is a symmetric quiver. One proves similarly if x is a source. □

Definition 1.3.3. *Let (Q, σ) be a symmetric quiver. A sequence $x_1, \ldots, x_m$ of vertices of Q is an admissible sequence of sinks (or sources) for admissible sink-source pairs if x_{i+1} is a sink such that there are no arrows linking x_{i+1} and $\sigma(x_{i+1})$ in $c_{(x_i,\sigma(x_i))} \cdots c_{(x_1,\sigma(x_1))}(Q)$ for $i = 1, \ldots, m-1$.*

Proposition 1.3.4. *Let (Q, σ) and (Q', σ) be two symmetric connected quivers, without cycles, with the same underlying graph and such that Q' differs from Q only by changing the orientation of some arrows. Then there exists a sequence $x_1, \ldots, x_m \in Q_0$ which is an admissible sequence of sinks (or sources) for admissible sink-source pairs such that*

$$Q' = c_{(x_m,\sigma(x_m))} \cdots c_{(x_1,\sigma(x_1))}(Q).$$

For the proof of proposition 1.3.4, we need a lemma.

Lemma 1.3.5. *If (Q, σ) is a symmetric quiver with $|\{x \to \sigma(x) | x \in Q_0\}| > 1$, then (Q, σ) has cycles or it is not connected.*

Proof of lemma 1.3.5. If there are more than one arrow $x \to \sigma(x)$ for the same x in Q then Q has cycles. Otherwise we suppose that Q is connected and that there are two arrows $x \xrightarrow{a} \sigma(x)$ and $y \xrightarrow{b} \sigma(y)$, with $x \neq y$ in Q. Since Q is connected, these two arrows have to be linked with a sequence of other arrows (this regarding their orientation). If there exists a sequence of arrows $a_1, \ldots, a_t$ from x to y then, by definition of σ, there exists a sequence of arrows $\sigma(a_1), \ldots, \sigma(a_t)$ from $\sigma(y)$ to $\sigma(x)$, reversed respect to $a_1, \ldots, a_t$. So $a_1 \cdots a_t a \sigma(a_t) \cdots \sigma(a_1) b$ is a cycle. By a similar reasoning for the other possible three links between $x \to \sigma(x)$ and $y \to \sigma(y)$ (from x to $\sigma(y)$, from y to $\sigma(x)$ and from $\sigma(x)$ to $\sigma(y)$), we obtain the same conclusion. □

Proof of proposition 1.3.4. By lemma 1.3.5 we can suppose that Q has at most one arrow $x \to \sigma(x)$ for some $x \in Q_0$. First of all we notice that the underlying graph of Q and Q', being a connected graph without cycles, is a tree, i.e. a graph where every vertex x has one parent and a several of children each connected by one edge to the vertex x. We define ancestor and descendants in obvious way and we call $x \in Q_0$ a vertex without children if there is only one edge connected to x. Let S be a set of vertices without children in Q.

We observe, by definition of σ, that if $Q \neq A_2$, in that case there are no admissible sink or source, and if S contains $x \in Q_0$ then it contains $\sigma(x)$.

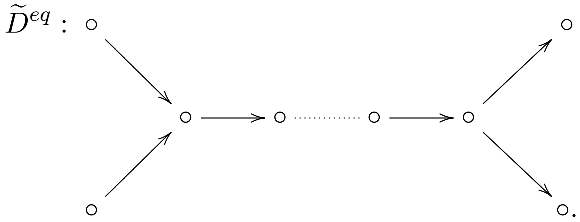
Observe that, using reflection of the admissible sink-source pair at the vertex without children x, we can change arbitrarily orientation of arrow connected to x and so of the arrow connected to $\sigma(x)$.

We proceed by induction on the number m of generations in the tree. If the number of generations is one, each vertex but one is without children, applying reflection at the admissible sink-source pairs we can pass from orientation of Q to orientation of Q', by which we observed before.

Assume proposition true for the trees with $m - 1$ generations. We remove all vertices without children from Q and Q', so the resulting quivers $\tilde{Q}$ and $\tilde{Q}'$ have $m-1$ generations and are symmetric. By inductive assumption, we can go from $\tilde{Q}$ to $\tilde{Q}'$ by a sequence of reflections at admissible sink-source pairs.

To pass from Q to Q' we use the same sequence of reflections at each point, adjusting the orientations of arrows incident to S, to get the next admissible sink-source pair if necessarily. □

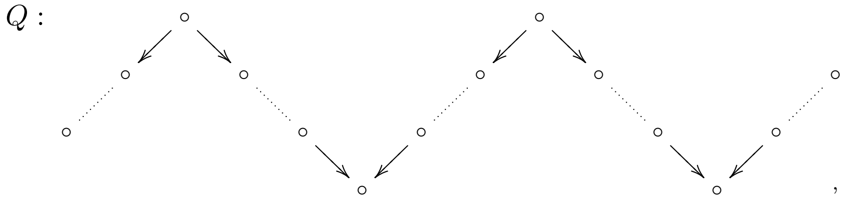
We prove some results on orientations of symmetric quivers of tame type. The underlying graph of $\widetilde{D}$ is a tree, so by proposition 1.3.4, we will consider a particular orientation of $\widetilde{D}$



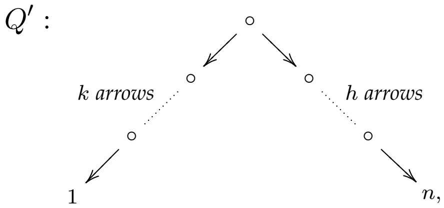
(1.11)

Applying a compositions of reflections at admissible sink-source pairs we can get any orientation of $\widetilde{D}$ from $\widetilde{D}^{eq}$.
Now we deal with orientation of symmetric quivers with underlying quiver of type $\widetilde{A}$. First we prove lemma about possible exchange of orientation of a quiver Q of type A_n, that does not involve reflections at the end points of Q. We denote vertices of Q with $\{1, \ldots, n\}$ from left to right.

Lemma 1.3.6. *Let*

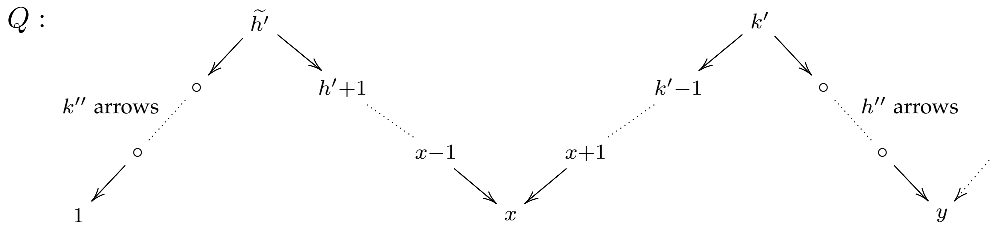


with k south-west arrows and h south-east arrows. Then there exists a sequence of admissible sinks $x_1, \ldots, x_l$ with $x_i \neq 1, n$ for every $i \in \{1, \ldots, l\}$, such that $c_{x_1} \cdots c_{x_l} Q$ is

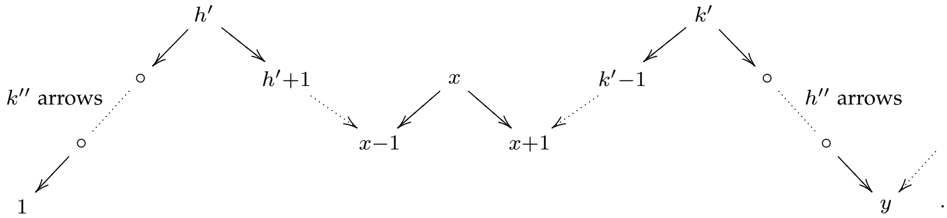


i.e. Q' has $1, n$ as only sinks, with k south-west arrows and with h south-east arrows.

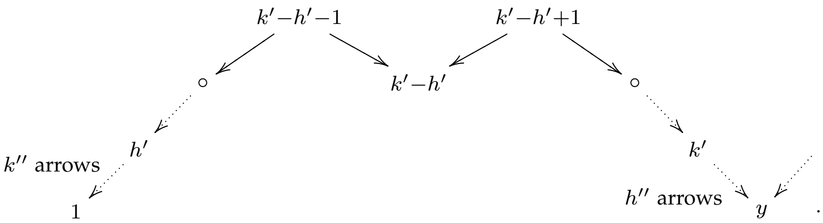
Proof. Let x and y be two sinks closest to 1.



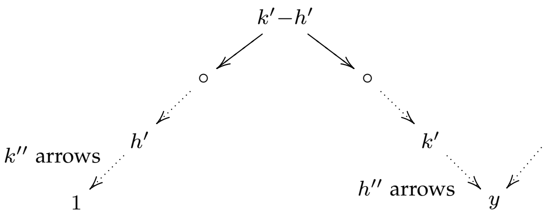
From 1 to y, Q has $k'+k''$ south-west arrows and $h'+h''$ south-east arrows. We remove x by applying only reflections at vertices with number smaller than y, as follows. We suppose $k' \geq h'$ (the other case is similar). Applying c_x we get



Now we can apply $c_{x-1}c_{x+1}$ and so on we obtain



Finally, applying $c_{k'-h'}$ we get



in which there are $(k'-h')+h'+k'' = k'+k''$ south-west arrows and $k'-(k'-h')+h'' = h'+h''$ south-east arrows. Removing internal sinks in this way proves lemma. □

Definition 1.3.7. *We will say that a symmetric quiver is of type* (s,t,k,l) *if*

(i) it is of type $\widetilde{A}$*,*

(ii) $|Q_1^\sigma| = s$ *and* $|Q_0^\sigma| = t$*,*

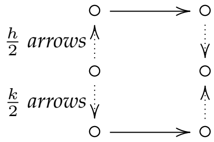
(iii) it has k *counterclockwise arrows and* l *clockwise arrows in* $Q_1^+ \sqcup Q_1^-$*.*

By proposition 1.1.15, $s,t \in \{0,1,2\}$ and if either s or t are not zero, then $s+t=2$. Moreover, by symmetry, we note that k and l have to be even.

Proposition 1.3.8. *Let* (Q,σ) *be a symmetric quiver of type* $\widetilde{A}$ *such that* Q *is without oriented cycles. Then there is an admissible sequence of sinks* $x_1,\dots,x_s$ *of* Q *for admissible sink-source pairs such that* $c_{(x_1,\sigma(x_1))}\cdots c_{(x_s,\sigma(x_s))}Q$ *is one of the quivers:*

(1)

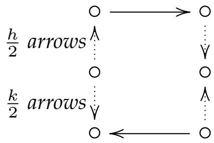
$$\widetilde{A}^{2,0,1}_{k,h}:$$



and

(2)

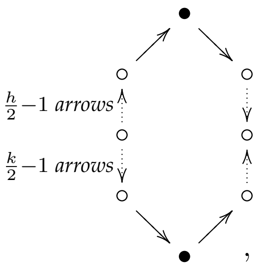
$$\widetilde{A}^{2,0,2}_{k,h}:$$



if (Q, σ) *is of type* $(2, 0, k, l)$*;*

(3)

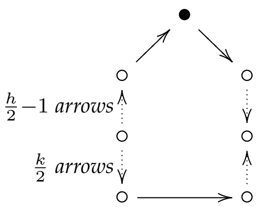
$$\widetilde{A}^{0,2}_{k,l}:$$



if (Q, σ) *is of type* $(0, 2, k, l)$*;*

(4)

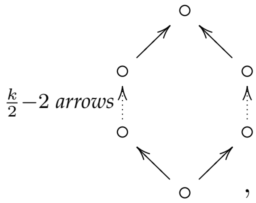
$$\widetilde{A}^{1,1}_{k,l}:$$



if (Q, σ) *is of type* $(1, 1, k, l)$*;*

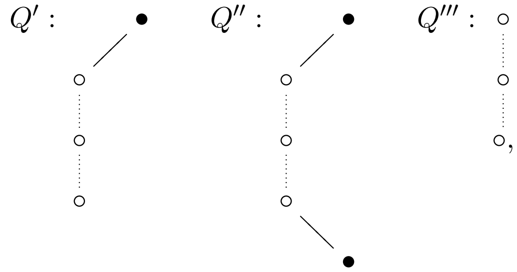
(5)

$$\widetilde{A}^{0,0}_{k,k}:$$

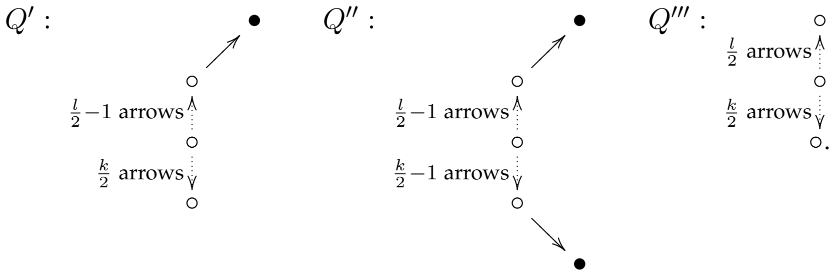


if (Q, σ) *if of type* $(0, 0, k, k)$*.*

Proof. For (Q,σ) of types $(2,0,k,l)$, $(0,2,k,l)$ and $(1,1,k,l)$ we apply lemma 1.3.6 respectively to the subquivers whose the underlying graphs are



i.e. the subquivers which have as first and last vertex respectively: the σ-fixed vertex and ta, where a is the σ-fixed arrow, for Q'; the σ-fixed vertices for Q''; ta and tb, where a and b are the σ-fixed arrows, for Q'''. We note that these three quivers have $\frac{k}{2}$ counterclockwise arrows and $\frac{l}{2}$ clockwise arrows. So for each one of Q', Q'' and Q''' there exists a sequence of sinks $x_1,\dots,x_s$ such that $c_{x_1}\cdots c_{x_s}Q'$, $c_{x_1}\cdots c_{x_s}Q''$ and $c_{x_1}\cdots c_{x_s}Q'''$ are respectively

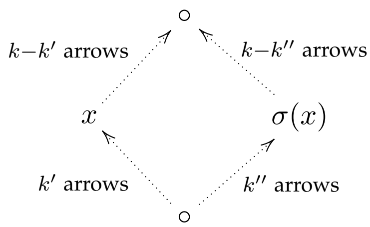


Hence, by symmetry, applying $c_{(x_1,\sigma(x_1))}\cdots c_{(x_s,\sigma(x_s))}$, we obtain the desired orientations.

For (Q,σ) of type $(0,0,k,k)$ we consider a sink x of Q and we apply lemma 1.3.6 to the subquiver Q' which has as first and last vertex respectively x and $\sigma(x)$. So there exists a sequence of sinks $x_1,\dots,x_s$ such that $c_{x_1}\cdots c_{x_s}Q'$ is

$$x \xleftarrow{k' \text{ arrows}} \circ \xrightarrow{k'' \text{ arrows}} \sigma(x).$$

Hence, by symmetry, applying $c_{(x_1,\sigma(x_1))}\cdots c_{(x_s,\sigma(x_s))}$ we obtain



i.e. the desired orientation. □

1.3.2 Reflection functors for symmetric quivers

Let (Q, σ) be a symmetric quiver, $(x, \sigma(x))$ a sink-source admissible pair. For every $V \in Rep(Q)$, we define the reflection functors

$$C^{+}_{(x,\sigma(x))}V := C^{-}_{\sigma(x)}C^{+}_{x}V$$

and

$$C^{-}_{(\sigma(x),x)}V := C^{-}_{x}C^{+}_{\sigma(x)}V.$$

We note that $C^{-}_{\sigma(x)}C^{+}_{x}V = C^{+}_{x}C^{-}_{\sigma(x)}V$ (respectively $C^{-}_{x}C^{+}_{\sigma(x)}V = C^{-}_{\sigma(x)}C^{+}_{x}V$) since there are no arrows connecting x and $\sigma(x)$.

Proposition 1.3.9. *Let (Q, σ) be a symmetric quiver and V be a representation of the underlying quiver.*

(i) If x is an admissible sink, then $\nabla C^{+}_{(x,\sigma(x))}V \cong C^{+}_{(x,\sigma(x))}\nabla V$.

(ii) If x is an admissible source, then $\nabla C^{-}_{(x,\sigma(x))}V \cong C^{-}_{(x,\sigma(x))}\nabla V$.

In particular for every x admissible sink and y admissible source we have

$$V = \nabla V \Leftrightarrow C^{+}_{(x,\sigma(x))}V = \nabla C^{+}_{(x,\sigma(x))}V \Leftrightarrow C^{-}_{(y,\sigma(y))}V = \nabla C^{-}_{(y,\sigma(y))}V.$$

Proof. We prove (i) (the proof of (ii) is similar). Recall that $x \neq \sigma(x)$, otherwise x is not a sink. Let $\{a_1, \ldots, a_k\}$ be the set of arrows whose head is x.

$$(\nabla C^{+}_{(x,\sigma(x))}V)_y = (C^{+}_{x}C^{-}_{\sigma(x)}V)^{*}_{\sigma(y)} =$$

$$\begin{cases} (V_{\sigma(y)})^{*} & \sigma(y) \neq \sigma(x), x \\ (Coker(V_{\sigma(x)} \xrightarrow{\tilde{h}} \bigoplus_{i=1}^{k} V_{h\sigma(a_i)}))^{*} & \sigma(y) = \sigma(x) \\ (Ker(\bigoplus_{i=1}^{k} V_{ta_i} \xrightarrow{h'} V_x))^{*} & \sigma(y) = x, \end{cases}$$

where $\tilde{h}(v) = (V(\sigma(a_1))(v), \ldots, V(\sigma(a_k))(v))$ with $v \in V_{\sigma(x)}$ and $h'(v_1, \ldots, v_k) = V(a_1)(v_1) + \cdots + V(a_k)(v_k)$ with $(v_1, \ldots, v_k) \in \bigoplus_{i=1}^{k} V_{ta_i}$.

$$(C^{+}_{(x,\sigma(x))}\nabla V)_y =$$

$$\begin{cases} (\nabla V_y)_y & y \neq \sigma(x), x \\ Coker((\nabla V)_{\sigma(x)} \xrightarrow{\tilde{h'}} \bigoplus_{i=1}^{k} (\nabla V)_{h\sigma(a_i)}) & y = \sigma(x) \\ Ker(\bigoplus_{i=1}^{k} (\nabla V)_{ta_i} \xrightarrow{h} (\nabla V)_x) & y = x, \end{cases}$$

where $\tilde{h'}(v) = (\nabla V(\sigma(a_1))(v), \ldots, \nabla V(\sigma(a_k))(v))$ with $v \in (\nabla V)_{\sigma(x)}$ and $h(v_1, \ldots, v_k) = \nabla V(a_1)(v_1) + \cdots + \nabla V(a_k)(v_k)$ with $(v_1, \ldots, v_k) \in \bigoplus_{i=1}^{k} (\nabla V)_{ta_i}$. Since $(\nabla V)_y = (V_{\sigma(y)})^{*}$ for every $y \in Q_0$ and $\nabla V(a) = -V(\sigma(a))^{*}$, we

have $h = -\tilde{h}^*$ and $h' = -\tilde{h'}^*$; moreover if φ is a linear map, in general we have $(Ker(\varphi))^* \cong Coker(\varphi^*)$ and $(Coker(\varphi))^* \cong Ker(\varphi^*)$, so $(\nabla C^+_{(x,\sigma(x))} V)_y \cong (C^+_{(x,\sigma(x))} \nabla V)_y$ for every $y \in Q_0$.
We note that

$$\sigma(c_{(x,\sigma(x))}a) = \sigma(c_x c_{\sigma(x)} a) = \begin{cases} c_{\sigma(x)}\sigma(a_i) & a = a_i \text{ with } i \in \{1,\dots,k\} \\ c_x a_i & a = \sigma(a_i) \text{ with } i \in \{1,\dots,k\} \\ \sigma(a) & a \neq a_i, \sigma(a_i) \text{ with } i \in \{1,\dots,k\}. \end{cases}$$

So we have

$$(\nabla C^+_{(x,\sigma(x))} V)(c_{(x,\sigma(x))}a) = -((C^+_x C^-_{\sigma(x)} V)(\sigma(c_x c_{\sigma(x)} a)))^* =$$

$$\begin{cases} -V(\sigma(a))^* & a \neq a_j, \sigma(a_j) \text{ with } j \in \{1,\dots,k\} \\ -(V_x \hookrightarrow \bigoplus_{i=1}^k V_{ta_i} \twoheadrightarrow V_{ta_j})^* & a = \sigma(a_j) \text{ with } j \in \{1,\dots,k\} \\ -(V_{h\sigma(a_j)} \hookrightarrow \bigoplus_{i=1}^k V_{h\sigma(a_i)} \twoheadrightarrow V_{\sigma(x)})^* & a = a_j \text{ with } j \in \{1,\dots,k\} \end{cases}$$

and

$$(C^+_{(x,\sigma(x))} \nabla V)(c_{(x,\sigma(x))}a) =$$

$$\begin{cases} \nabla V(a) & a \neq a_j, \sigma(a_j) \text{ with } j \in \{1,\dots,k\} \\ (\nabla V)_x \hookrightarrow \bigoplus_{i=1}^k (\nabla V)_{ta_i} \twoheadrightarrow (\nabla V)_{ta_j} & a = a_j \text{ with } j \in \{1,\dots,k\} \\ (\nabla V)_{h\sigma(a_j)} \hookrightarrow \bigoplus_{i=1}^k (\nabla V)_{h\sigma(a_i)} \twoheadrightarrow (\nabla V)_{\sigma(x)} & a = a_j \text{ with } j \in \{1,\dots,k\}. \end{cases}$$

Hence $\nabla C^+_{(x,\sigma(x))} V \cong C^+_{(x,\sigma(x))} \nabla V$. □

Corollary 1.3.10. *Let (Q,σ) and (Q',σ) be two symmetric quivers with the same underlying graph. We suppose there exists a sequence $x_1,\dots,x_m$ of admissible sinks for admissible sink-source pairs such that $Q' = c_{(x_m,\sigma(x_m))} \cdots c_{(x_1,\sigma(x_1))} Q$. Let $V \in Rep(Q)$ and $V' = C^+_{(x_m,\sigma(x_m))} \cdots C^+_{(x_1,\sigma(x_1))} V \in Rep(Q')$. Then*

$$V = \tau^- \nabla V \iff V' = \tau^- \nabla V'.$$

Proof. By proposition 1.3.9, we have

$$\tau^- \nabla V' = \tau^- \nabla C^+_{(x_m,\sigma(x_m))} \cdots C^+_{(x_1,\sigma(x_1))} V = \tau^- C^+_{(x_m,\sigma(x_m))} \cdots C^+_{(x_1,\sigma(x_1))} \nabla V =$$

$$\tau^- C^-_{\sigma(x_m)} C^+_{x_m} \cdots C^-_{\sigma(x_1)} C^+_{x_1} \tau^+ V = C^-_{\sigma(x_m)} \tau^- C^+_{x_m} \cdots C^-_{\sigma(x_1)} \tau^+ C^+_{x_1} V = \cdots =$$

$$C^-_{\sigma(x_m)} \cdots C^-_{\sigma(x_1)} \tau^- \tau^+ C^+_{x_m} \cdots C^+_{x_1} V = C^+_{(x_m,\sigma(x_m))} \cdots C^+_{(x_1,\sigma(x_1))} V = V'. \quad \square$$

Proposition 1.3.11. *Let (Q,σ) be a symmetric quiver and let x be an admissible sink. Then*

(i) V is a symplectic representation of (Q,σ) if and only if $C^+_{(x,\sigma(x))} V$ is a symplectic representation;

(ii) V is a orthogonal representation of (Q,σ) if and only if $C^+_{(x,\sigma(x))}V$ is a orthogonal representation.

Similarly if x is an admissible source then $C^-_{(x,\sigma(x))}$ sends symplectic representations to symplectic representations and orthogonal representations to orthogonal representations.

Proof. By proposition 1.3.9 we have $V = \nabla V$ if and only if $C^+_{(x,\sigma(x))}V = \nabla C^+_{(x,\sigma(x))}V$. To define an orthogonal (respectively symplectic) structure on $C^+_{(x,\sigma(x))}V$ the only problem could occur at the vertices fixed by σ. But, by definition of admissible sink and of the involution σ, fixed vertices and fixed arrows don't change under our reflection. The proof is similar for $C^-_{(x,\sigma(x))}$ with x an admissible source. □

Next we prove that the reflection functors for symmetric quivers preserve the rings of orthogonal and symplectic semi-invariants. We need some basic property of Grasmannians.

Definition 1.3.12. *Let W be a vector space of dimension n. Consider the set of all decomposable tensor $w_1 \wedge \ldots \wedge w_r$, with $w_1, \ldots, w_r \in W$, inside $\bigwedge^r W$. This set is an affine subvariety of the space vector $\bigwedge^r W$, called affine cone over the Grasmannian. It will be denoted by $\widetilde{Gr}(r,W)$.*

Definition 1.3.13. *The Grasmannian $Gr(r,W)$ is the projective subvariety of $\mathbb{P}(\bigwedge^r W)$ corresponding to $\widetilde{Gr}(r,W)$.*

This variety can be thought as the set of r-dimensional subspaces of W. The identification between $\bigwedge^r W$ and $\bigwedge^{n-r} W^*$ induces an identification between $\widetilde{Gr}(r,W)$ and $\widetilde{Gr}(n-r,W^*)$ and so between $Gr(r,W)$ and $Gr(n-r,W^*)$. By the first fundamental theorem (FFT) for $SL\,V$ (see [P, chapter 11 section 1.2]), it follows that

$$\mathbb{K}[V \otimes W]^{SL\,V} \cong \mathbb{K}[\widetilde{Gr}(r,W)],$$

where $r = dim(V)$.

Lemma 1.3.14. *If x is an admissible sink or source for a symmetric quiver (Q,σ) and α is a dimension vector such that $c_{(x,\sigma(x))}\alpha(x) \geq 0$, then*

i) if $c_{(x,\sigma(x))}\alpha(x) > 0$ there exist isomorphisms

$$SpSI(Q,\alpha) \xrightarrow{\varphi^{Sp}_{x,\alpha}} SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$$

and

$$OSI(Q,\alpha) \xrightarrow{\varphi^{O}_{x,\alpha}} OSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha),$$

ii) if $c_{(x,\sigma(x))}\alpha(x) = 0$ there exist isomorphisms

$$SpSI(Q,\alpha) \xrightarrow{\varphi^{Sp}_{x,\alpha}} SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)[y]$$

and

$$OSI(Q,\alpha) \xrightarrow{\varphi^{O}_{x,\alpha}} OSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)[y]$$

where $A[y]$ denotes a polynomial ring with coefficients in A.

Proof. We will prove the lemma for the symplectic case because the orthogonal case is similar. Let $x \in Q_0$ be an admissible sink. Put $r = \alpha(x)$ and $n = \sum_{ha=x}\alpha(ta)$. We note that $c_{(x,\sigma(x))}\alpha(x) = n - r$. Put $V = \mathbb{K}^r$, $V' = \mathbb{K}^{n-r}$ and $W = \bigoplus_{ha=x}\mathbb{K}^{\alpha(ta)} \cong \mathbb{K}^n$. We define

$$Z = \bigoplus_{\substack{a\in Q_1^+ \\ ha\neq x}} Hom(\mathbb{K}^{\alpha(ta)}, \mathbb{K}^{\alpha(ha)}) \oplus \bigoplus_{a\in Q_1^\sigma} S^2(\mathbb{K}^{\alpha(ta)})^*$$

and

$$G = \prod_{\substack{y\in Q_0^+ \\ y\neq x}} SL(\alpha(y)) \times \prod_{y\in Q_0^\sigma} Sp(\alpha(y)).$$

Proof of i). If $c_{(x,\sigma(x))}\alpha(x) > 0$ we have

$$SpSI(Q,\alpha) = \mathbb{K}[SpRep(Q,\alpha)]^{SSp(Q,\alpha)} =$$
$$\mathbb{K}[Z \times Hom(W,V)]^{G\times SL\,V} = (\mathbb{K}[Z]\otimes\mathbb{K}[Hom(W,V)]^{SL\,V}]^G =$$
$$(\mathbb{K}[Z]\otimes\mathbb{K}[\widetilde{Gr}(r,W^*)])^G$$

and

$$SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha) =$$
$$\mathbb{K}[SpRep(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)]^{SSp(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)} =$$
$$\mathbb{K}[Z \times Hom(V',W)]^{G\times SL\,V'} = (\mathbb{K}[Z]\otimes\mathbb{K}[Hom(V',W)]^{SL\,V'}]^G =$$
$$(\mathbb{K}[Z]\otimes\mathbb{K}[\widetilde{Gr}(n-r,W)])^G.$$

Since $\widetilde{Gr}(r,W^*)$ and $\widetilde{Gr}(n-r,W)$ are isomorphic as G-varieties, it follows that $SpSI(Q,\alpha)$ and $SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$ are isomorphic.
Proof ii). If $c_{(x,\sigma(x))}\alpha(x) = 0$, then $n = r$ and $V' = 0$. So $\widetilde{Gr}(0,W)$ is a point and hence

$$SpSI(Q,\alpha) = (\mathbb{K}[Z]\otimes\mathbb{K}[Hom(W,V)])^{G\times SL\,V} \tag{1.12}$$

is isomorphic to

$$SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha) = (\mathbb{K}[Z]\otimes\mathbb{K}[Hom(V',W)])^{G\times SL\,V} = \mathbb{K}[Z]^{G\times SL(V)}.$$

Now let $A = \{a \in Q_1^+ | \, ha = x\}$. Using theorem A.1.9, each summand of (1.12) contains $(\bigotimes_{a\in A} S_{\lambda(a)}V)^{SL\,V}$ as factor. By proposition A.2.8 each $\lambda(a)$, with $a \in A$, has to contain a column of height $\alpha(ta)$, hence $\lambda(a) = \mu(a) + (1^{\alpha(ta)})$, for some $\mu(a)$ in the set of partitions Λ. So as factor we have

$$\bigotimes_{a\in A}(S_{(1^{\alpha(ta)})}\mathbb{K}^{\alpha(ta)})^{SL\,V_{ta}} \otimes \left(\bigotimes_{a\in A} S_{(1^{\alpha(ta)})}V\right)^{SL\,V}$$

which is generated by $det(\bigoplus_{ha=x}\mathbb{K}^{\alpha(ta)} \to \mathbb{K}^{\alpha(x)})$. On the other hand we have $\mathbb{K}[Hom(W,V)])^{G\times SL\,V} = \mathbb{K}[det(\bigoplus_{ha=x}\mathbb{K}^{\alpha(ta)} \to \mathbb{K}^{\alpha(x)})]$ and so we have the statement *ii)*, with $y = det(\bigoplus_{ha=x}\mathbb{K}^{\alpha(ta)} \to \mathbb{K}^{\alpha(x)})$. □

1.4 Semi-invariants of symmetric quivers

In this section we prove some general results about semi-invariants of symmetric quivers with underlying quiver without oriented cycles.
We assume that (Q,σ) is a symmetric quiver with underlying quiver Q without oriented cycles for rest of the thesis.
We recall that, by definition, symplectic groups or orthogonal groups act on the spaces which are defined on the σ-fixed vertices, so we have

Definition 1.4.1. *Let V be a representation of the underlying quiver Q with $\underline{dim}V = \alpha$ such that $\langle\alpha,\beta\rangle = 0$ for some symmetric dimension vector β. The weight of c^V on $SpRep(Q,\beta)$ (respectively on $ORep(Q,\beta)$) is $\langle\alpha,\cdot\rangle - \sum_{x\in Q_0^\sigma}\varepsilon_{x,\alpha}$, where*

$$\varepsilon_{x,\alpha}(y) = \begin{cases} \langle\alpha,\cdot\rangle(x) & y = x \\ 0 & \textit{otherwise.} \end{cases} \tag{1.13}$$

In general we define an involution γ on the space of weights $\langle\alpha,\cdot\rangle$ with α dimension vector.

Definition 1.4.2. *Let α be the dimension vector of a representation V of the underlying quiver Q and let $\langle\alpha,\cdot\rangle = \chi = \{\chi(i)\}_{i\in Q_0}$ be the weight of c^V. We define $\gamma\chi = \{\gamma\chi(i)\}_{i\in Q_0}$ where $\gamma\chi(i) = -\chi(\sigma(i))$ for every $i \in Q_0$*

We number vertices in such way that $ta < ha$ for every $a \in Q_1$. We note that $\chi = \langle\alpha,\cdot\rangle = (\alpha(j) - \sum_{i<j} b_{i,j}\alpha(i))_{j\in Q_0}$,
where $b_{i,j} := |\{a \in Q_1 | ta = i, ha = j\}| = |\{a \in Q_1 | ta = \sigma(j), ha = \sigma(i)\}| =: b_{\sigma(j),\sigma(i)}$.

Lemma 1.4.3.

$$\gamma\chi = \langle\tau^-\delta\alpha,\cdot\rangle = \langle\underline{dim}(\tau^-\nabla V),\cdot\rangle, \tag{1.14}$$

i.e. $\gamma\chi$ is the weight of $c^{\tau^-\nabla V}$. Moreover γ is an involution.

Proof. By definition of γ, $\gamma\chi(j) = -\alpha(\sigma(j)) + \sum_{i<j} b_{i,j}\alpha(\sigma(i))$. Now it follows by theorem B.1.9 that $\langle \tau^- \delta\alpha, \cdot \rangle = -\langle \cdot, \delta\alpha \rangle$, thus, for every $j \in Q_0$, $\langle \tau^- \delta\alpha, \cdot \rangle(j) = -\langle \cdot, \delta\alpha \rangle(j) = -\delta\alpha(j) + \sum_{i<j} b_{i,j}\delta\alpha(i) = \gamma\chi(j)$. Hence $\gamma\chi = \langle \tau^- \delta\alpha, \cdot \rangle$. □

Moreover, since $\gamma\gamma\chi(i) = \gamma(-\chi(\sigma(i))) = \chi(\sigma\sigma(i)) = \chi(i)$ for every $i \in Q_0$, γ is an involution.
If β is the dimension vector of a representation W of the underlying quiver Q, we have

$$\langle \alpha, \beta \rangle = 0 \Leftrightarrow \langle \tau^- \delta\alpha, \delta\beta \rangle = 0. \tag{1.15}$$

Indeed, by theorem B.1.9,

$$\langle \alpha, \beta \rangle = \langle \delta\beta, \delta\alpha \rangle = -\langle \tau^- \delta\alpha, \delta\beta \rangle. \tag{1.16}$$

Since β is the dimension vector of an orthogonal or symplectic representation W, we have that β is a symmetric dimension vector and so

$$\langle \alpha, \beta \rangle = 0 \Leftrightarrow \langle \tau^- \delta\alpha, \beta \rangle = 0. \tag{1.17}$$

Lemma 1.4.4. *Let (Q, σ) be a symmetric quiver. For every representation V of the underlying quiver Q and for every orthogonal or symplectic representation W such that $\langle \underline{dim}(V), \underline{dim}(W) \rangle = 0$, we have*

$$c^V(W) = c^{\tau^- \nabla V}(W).$$

Proof. It follows directly from lemma B.5.3. □

Now we prove in general a crucial lemma which will be useful later. Let (Q, σ) be a symmetric quiver. If V is a representation of the underlying quiver Q such that $V = \tau^- \nabla V$ then, by the theorem B.1.11, there exists an almost split sequence $0 \to \nabla V \to Z \to V \to 0$ with $Z \in Rep(Q)$. Moreover for such $V \in Rep(Q)$ with $\underline{dim} V = \alpha$ we have $\alpha = \tau^- \delta\alpha$ and $\gamma\chi = \chi$, where $\chi = \langle \alpha, \cdot \rangle$. So $\chi(i) = \gamma\chi(i) = -\chi(\sigma(i))$ for every $i \in \{1, \ldots, n\}$, in particular $\chi(i) = 0$ if $\sigma(i) = i$.

Definition 1.4.5. *A weight χ such that $\gamma\chi = \chi$ is called a symmetric weight.*

Lemma 1.4.6. *Let (Q, σ) be a symmetric quiver of finite type or of tame type. Let d^V_{min} be the matrix of the minimal projective presentation of $V \in Rep(Q, \alpha)$ and let β be a symmetric dimension vector such that $\langle \alpha, \beta \rangle = 0$. Then*

(1) $Hom_Q(d^V_{min}, \cdot)$ is skew-symmetric on $SpRep(Q, \beta)$ if and only if V satisfies property (Op);

(2) $Hom_Q(d^V_{min}, \cdot)$ is skew-symmetric on $ORep(Q, \beta)$ if and only if V satisfies property (Spp).

Proof. We use notation of section B.2. We call (Q', σ) the symmetric quiver with the same underlying graph of (Q, σ) such that

(i) if Q is of type A, then Q' has all the arrows with the same orientations;

(ii) if Q is of type $\widetilde{A}$, then Q' is one of the quiver as in proposition 1.3.8 (it depends on which kind of quiver is Q);

(iii) if Q is of type $\widetilde{D}$, then Q' is $\widetilde{D}^{eq}$ (see picture (1.11)).

By propositions 1.3.4 and 1.3.8, there exists a sequence $x_1, \ldots, x_m$ of admissible sink for admissible sink-source pairs such that $c_{(x_m,\sigma(x_m))} \cdots c_{(x_1,\sigma(x_1))} Q = Q'$. We call $V' := C^+_{(x_m,\sigma(x_m))} \cdots C^+_{(x_1,\sigma(x_1))} V$ for every $V \in Rep(Q)$ and if $\alpha = \underline{dim}\, V$, then $\alpha' := c_{(x_m,\sigma(x_m))} \cdots c_{(x_1,\sigma(x_1))} \alpha$. We note that, by corollary 1.3.10 and proposition 1.3.11, V satisfies property *(Op)* (respectively property *(Spp)*) if and only if V' satisfies property *(Op)* (respectively property *(Spp)*). We prove only *(1)*, because the proof of *(2)* is similar.

Type *A*. Let (A_n, σ) be a symmetric quiver of type A. We enumerate vertices with $1, \ldots, n$ from left to right and we call a_i the arrow with i on the left and $i+1$ on the right. We define σ by $\sigma(i) = n - i + 1$ for every $i \in Q_0$ and $\sigma(a_i)$ for every $i \in \{1, \ldots, n-1\}$. Let $V' = V_{i,\sigma(i)-1}$, i.e. is the indecomposable of A_n such that

$$(\underline{dim} V_{i,\sigma(i)-1})_j = \begin{cases} 1 & j \in \{i, \ldots, \sigma(i)-1\} \\ 0 & \text{otherwise.} \end{cases}$$

We note that $\nabla V' = V_{i+1,\sigma(i)} = \tau^+ V'$ and $Z' = V_{i,\sigma(i)} \oplus V_{i+1,\sigma(i)-1}$. So, by definition 1.1.3, on Z' we can define a structure of orthogonal representation if n is odd and a structure of symplectic representation if n is even. So it's enough to check when $Hom_Q(d^V_{min}, \cdot)$ is skew symmetric and, for type A, we do it explicitly.
Let $\chi = \langle \alpha, \cdot \rangle - \sum_{x \in Q_0^\sigma} \varepsilon_{x,\alpha}$ be the symmetric weight associated to α. If m_1 is the first vertex such that $\chi(m_1) \neq 0$, in particular we suppose $\chi(m_1) = 1$, then the last vertex m_s such that $\chi(m_s) \neq 0$ is $m_s = \sigma(m_1)$ and $\chi(m_s) = -1$. Between m_1 and m_s, -1 and 1 alternate in correspondence respectively of sinks and of sources. Moreover, by definition of symmetric weight, we have $s = 2l$ for some $l \in \mathbb{N}$. We call $i_2, \ldots, i_l$ the sources, $j_1, \ldots, j_{l-1}$ the sinks, $i_1 = m_1$ and $j_l = m_s$. Hence we have $\sigma(i_t) = j_{l-t+1}$ and $i_1 < j_1 < \ldots < i_l < j_l$. Now the minimal projective resolution for V is

$$0 \longrightarrow \bigoplus_{j=j_1}^{j_l} P_j \overset{d^V_{min}}{\longrightarrow} \bigoplus_{i=i_1}^{i_l} P_i \longrightarrow V \longrightarrow 0 \tag{1.18}$$

and for the remark above we have

$$0 \longrightarrow \bigoplus_{j=j_1}^{j_l} P_j \xrightarrow{d^V_{min}} \bigoplus_{j=j_1}^{j_l} P_{\sigma(j)} \longrightarrow V \longrightarrow 0, \tag{1.19}$$

with

$$(d^V_{min})_{hk} = \begin{cases} -a_{i_{k+1},j_k} & \text{if} \quad h = l-k \\ a_{i_k,j_k} & \text{if} \quad h = l-k+1 \\ 0 & \text{otherwise,} \end{cases} \tag{1.20}$$

where $a_{i,j}$ is the oriented path from i to j.
Hence

$$Hom(d^V_{min}, W) : \bigoplus_{j=j_1}^{j_l} W(\sigma(j)) = \bigoplus_{j=j_1}^{j_l} W(j)^* \longrightarrow \bigoplus_{j=j_1}^{j_l} W(j) \tag{1.21}$$

where

$$(Hom(d^V_{min}, W))_{hk} = \begin{cases} -W(a_{i_{h+1},j_h}) & \text{if} \quad k = l-h \\ W(a_{i_h,j_h}) & \text{if} \quad k = l-h+1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.22}$$

Now W is orthogonal or symplectic, so for $k \neq h$, if $k = l-h+1$ we have

$$(Hom(d^V_{min}, W))_{hk} = W(a_{i_h,j_h}) = W(a_{\sigma(j_{l-h+1}),j_h}) = -W(a_{\sigma(j_h),j_{l-h+1}})^t =$$

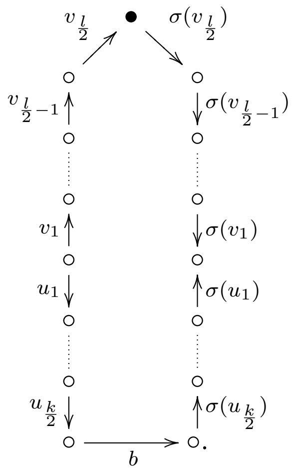
$$-W(a_{i_{l-h+1},j_{l-h+1}})^t = -W(a_{i_k,j_k})^t = -((Hom(d^V_{min}, W))_{kh})^t.$$

In a similar way it proves that if $k = l-h$ then $(Hom(d^V_{min}, W))_{hk} = -((Hom(d^V_{min}, W))_{kh})^t$.
Finally the only cases for which $(Hom(d^V_{min}, W))_{hh} \neq 0$ are when $h = l-h+1$ and $h = l-h$. In the first case (the second one is similar) we have $(Hom(d^V_{min}, W))_{hh} = W(a_{i_h,j_h}) = W(a_{\sigma(j_h),j_h})$ and $-((Hom(d^V_{min}, W))_{hh})^t = -W(a_{i_h,j_h})^t = -W(a_{\sigma(j_h),j_h})^t$. But $W(a_{\sigma(j_h),j_h}) = -W(a_{\sigma(j_h),j_h})^t$ for n even if and only if $W \in ORep(Q)$, for n odd if and only if $W \in SpRep(Q)$.

We consider the tame case. First we note, by Auslander-Reiten quiver of Q, that if (Q, σ) is a symmetric quiver of tame type, then the only representations $V \in Rep(Q)$ such that $\tau^- \nabla V = V$ are regular ones.

Type $\widetilde{A}$. We prove lemma only for Q of type $(1, 1, k, l)$ because for the other cases it proceeds similarly. We consider the following labelling for

$Q' = \widetilde{A}^{1,1}_{k,l}$:



The following indecomposable representations $V' \in Rep(Q')$ satisfy property *(Op)*. The other regular indecomposable representations of $Rep(Q')$ satisfying property *(Op)* are extensions of these.

(a) $V_{(0,1)}$; in this case $Z' = E^1_h \oplus E_{2,0}$ where E^1_h is the regular indecomposable representation of dimension $e_1 + h$ with socle E_1.

(b) $E_{i,j-1}$, with $1 \leq i < j \leq l+1$, such that $\nabla E_{i,j-1} = E_{i+1,j}$; in this case we have $Z' = E_{i+1,j-1} \oplus E_{i,j}$.

(c) $E'_{i,j-1}$, with $2 \leq j < i-1 \leq k+1$, such that $\nabla E'_{i,j-1} = E'_{i+1,j}$; in this case we have $Z' = E'_{i+1,j-1} \oplus E'_{i,j}$.

Let χ be the symmetric weight associated to α. We order vertices of Q clockwise from $tb = 1$ to $hb = k + l + 1$. We use the same notation of type A for vertices on which the components of χ are not zero.
Let W be a symplectic representation. We prove that $Hom_Q(d^V_{min}, W)$ is skew-symmetric for every regular indecomposable representation V of type (a), (b) and (c). First we observe that the associated to V symmetric weight χ have components equal to 0, 1 and -1. In particular, $\chi(m_1) = \pm 1 = -\chi(m_s)$ and $\chi(m_i) = 1, -1$, for every $i \in \{2, \dots, s-1\}$, respectively if m_i is a source or a sink. We note that, for every $Hom_Q(d^V_{min}, W)$ with V one representation of type (a), (b) and (c), we can restrict to the symmetric subquiver of type A which has first vertex m_1 and last vertex m_s and passing through the σ-fixed vertex of Q. Hence it proceeds as done for type A.
Finally, if V is the middle term of a short exact sequence $0 \to V^1 \to V \to V^2 \to 0$, with V^1 and V^2 one of the representations of type (a), (b) or (c), we have the blocks matrix

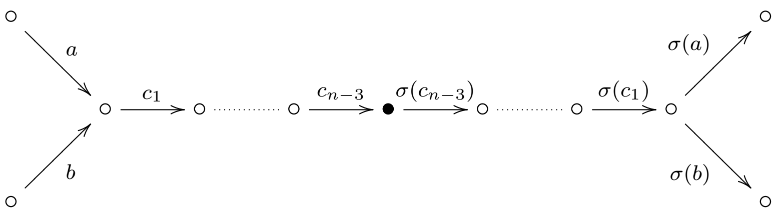
$$Hom_Q(d^V_{min}, \cdot) = \begin{pmatrix} Hom_Q(d^{V^1}_{min}, \cdot) & 0 \\ Hom_Q(B, \cdot) & Hom_Q(d^{V^2}, \cdot) \end{pmatrix}.$$

where $d_{min}^{V^1} : P_1^1 \to P_0^1$ is the minimal projective presentation of V^1, $d_{min}^{V^2} : P_1^2 \to P_0^2$ is the minimal projective presentation of V^2 and for some $B \in Hom_Q(P_1^2, P_0^1)$. In general for every blocks matrix we have $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} Id & 0 \\ -BA^{-1} & Id \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ if A is invertible. Hence using rows operations on $Hom_Q(d_{min}^V, \cdot)$, we obtain

$$Hom_Q(d_{min}^V, \cdot) \approx \begin{pmatrix} Hom_Q(d_{min}^{V^1}, \cdot) & 0 \\ 0 & Hom_Q(d_{min}^{V^2}, \cdot) \end{pmatrix}.$$

So it's enough to prove the skew-symmetry of $Hom_Q(d_{min}^V, \cdot)$ for V one of representations of type (a), (b) and (c).

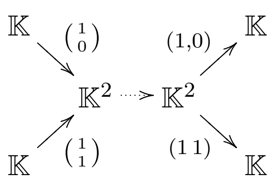
Type $\widetilde{D}$. We prove lemma only for $Q = \widetilde{D}_n^{0,1}$ because for the case $\widetilde{D}_n^{1,0}$ it proceeds similarly. We consider the following labelling for $(\widetilde{D}_n^{0,1})^{eq}$:

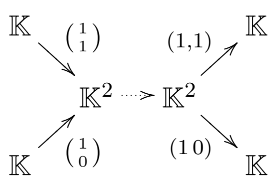


We consider again indecomposable representations $V' \in Rep(Q')$ satisfying property *(Op)*. The other regular indecomposable representations of $Rep(Q')$ satisfying property *(Op).* are extensions of these.

(a) $E_{i,j-1}$, with $1 \le i < j \le 2n-3$ or $2 \le j < i-1 \le 2n-4$, such that $\nabla E_{i,j-1} = E_{i+1,j}$; in this case we have $Z' = E_{i+1,j-1} \oplus E_{i,j}$.

(b) E_0'' and E_1''. We note that $\nabla E_0'' = E_1'' = \tau^+ E_0''$, $\nabla E_1'' = E_0'' = \tau^+ E_1''$ and the respective Z' are



and


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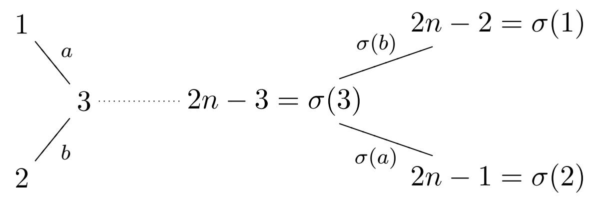
where linear maps defined on c_i, with $1 \le i \le n-3$, are identity maps.

(c) $V_{(0,1)}$ and $V_{(1,1)}$; respectively $Z' = E_h^{n-1} \oplus E_{0,2n-6}$ and $Z' = E_h^1 \oplus E_{2n-6,0}$ where E_h^1 and E_h^{n-1} are the regular indecomposable representations respectively of dimension $e_1 + h$ and $e_{n-1} + h$.

We consider the following labelling of vertices end arrows for $\widetilde{D}_n^{0,1}$:



and we call c_{i-2} the arrow such that $tc_{i-2} = i$.
Let χ be the symmetric weight associated to V. We use the same notation of type A for vertices from 3 to $2n-3$ on which the components of χ are not zero. Suppose that 1 and 2 are source (the other cases are similar). We check when $Hom_Q(d^V_{min}, \cdot)$ is skew-symmetric, for V of type (a), (b) and (c)

(a) Let V be one of representation of type (a). We note that either $\chi(1) = 0 = \chi(2)$ or $\chi(1) \neq 0 \neq \chi(2)$. If $\chi(1) = 0 = \chi(2)$, then we have $\chi(m_1) = \pm 1 = -\chi(m_s)$ and $\chi(m_i) = 1, -1$, for every $i \in \{2, \ldots, s-1\}$, respectively if m_i is a source or a sink. Hence it proceeds as in type A. If $\chi(1) \neq 0 \neq \chi(2)$ then $-\chi(2n-2) = \chi(1) = 1 = \chi(2) = -\chi(2n-1$ and we have $\chi(m_i) = 1, -1$, for every $i \in \{1, \ldots, s\}$, respectively if m_i is a source or a sink. Let $i_1 < \ldots < i_t$ be the sources from 3 to $2n-3$ and let $j_1 < \ldots < j_t$ be the sinks from 3 to $2n-3$. We also note that $j_1 < i_1 < \ldots < j_t < i_t$.
d^V_{min} is a matrix $(t+2) \times (t+2)$ whose entries are

$$(d^V_{min})_{h,k} = \begin{cases} -a_{i_{k+1},j_k} & h = t-k \text{ and } 1 \leq k \leq t-1 \\ a_{i_k,j_k} & h = t-k+1 \text{ and } 1 \leq k \leq t \\ -a_{i,j_1} & h = t+i \text{ and } k = 1 \text{ for } i = 1,2 \\ -\sigma(a_{i,j_1}) & h = 1 \text{ and } k = t+i \text{ for } i = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

where $a_{i,j}$ is oriented path from i to j.
Finally, as for the type A, we note that $Hom_Q(d^V_{min}, W)$ is skew-symmetric if and only if $W \in SpRep(\widetilde{D}_n^{0,1}, \beta)$.

(b) Let V be a representation of type (b). We note that if χ if the weight associated to E''_0, then $-\chi(2n-2) = \chi(1) = 1$ and $\chi(m_i) = 1, -1$, for every $i \in \{1, \ldots, s\}$, respectively if m_i is a source or a sink. So we can proceed as in type A.

(c) Let V be a representation of type (c). We use the same notation of part (a) of type $\widetilde{D}$. We note that $-\chi(2n-2) = \chi(1) = 1 = \chi(2) = -\chi(2n-1$ and we have $\chi(m_i) = 2, -2$, for every $i \in \{1, \ldots, s\}$, respectively if m_i is a source or a sink.
In the remainder of the proof, we use notation of section B.5. In this case, d^V_{min} is a blocks $(2t+2) \times (2t+2)$-matrix $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}$. Here

(i) A is a $2t \times 2t$-matrix with 2×2-blocks $A_{h,k}$, defined as follows

$$A_{h,k} = \begin{cases} (-a_{i_{k+1},j_k})_{Id_2} & h = t-k \text{ and } 1 \leq k \leq t-1 \\ (a_{i_k,j_k})_{Id_2} & h = t-k+1 \text{ and } 1 \leq k \leq t \\ 0 & \text{otherwise.} \end{cases}$$

(ii) B is a $2 \times 2t$-matrix, whose entries $b_{h,k}$ are

$$\begin{cases} (-1)^{h+k+1} a_{h,j_1} & h = 1,2 \text{ and } k = 1,2 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) C is a $2t \times 2$-matrix, whose entries $c_{h,k}$ are

$$\begin{cases} (-1)^{h+k+1} \sigma(a_{k,j_1}) & h = 1,2 \text{ and } k = 1,2 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, as for the type A, we note that $Hom_Q(d^V_{min}, W)$ is skew-symmetric if and only if $W \in SpRep(\widetilde{D}^{0,1}_n, \beta)$.

At last it remains to prove that lemma is true also for every V decomposable representation. But we note that if $V = V^1 \oplus V^2$, then

(i) V satisfies property *(Op)* if and only if V^1 and V^2 satisfy property *(Op)*;

(ii) $d^V_{min} = \begin{pmatrix} d^{V^1}_{min} & 0 \\ 0 & d^{V^2}_{min} \end{pmatrix}$.

This concludes the proof. □

1.5 Relations between semi-invariants of (Q, σ) and of $(c_{(x,\sigma(x))}(Q), \sigma)$

Let (Q, σ) be a symmetric quiver and let x be an admissible sink of (Q, σ). First we consider the action of $c_{(x,\sigma(x))}$ on the weights of semi-invariants

Lemma 1.5.1. *Let (Q, σ) be a symmetric quiver and let x be an admissible sink-source of Q. If $\chi = \langle \alpha, \cdot \rangle - \sum_{x \in Q^\sigma_0} \varepsilon_{x,\alpha}$ is a weight for some dimension vector α (see definition 1.4.1), then*

$$(c_{(x,\sigma(x))}\chi)(y) = \begin{cases} -\chi(x) & y = x \\ -\chi(\sigma(x)) & y = \sigma(x) \\ \chi(y) + b_{x,y}\chi(x) & y \notin Q^\sigma_0 \cup \{x\} \\ \chi(y) + b_{\sigma(x),y}\chi(x) & y \notin Q^\sigma_0 \cup \{\sigma(x)\} \\ 0 & \textit{otherwise}, \end{cases} \tag{1.23}$$

where $b_{x,y}$ is the number of arrows linking x and y.

Proof. First we note that, by definition, $\chi(y) = 0$ for every $y \in Q_0^\sigma$.

(i) If $y = x$, then $y \notin Q_0^\sigma$ and

$$(c_{(x,\sigma(x))}\chi)(x) = (c_{(x,\sigma(x))}\alpha)(x) = \sum_{\substack{a\in Q_1:\\ ha=x}} \alpha(ta) - \alpha(x) = -\chi(x).$$

Similarly one proves the case $y = \sigma(x)$.

(ii) If $y = ta \notin Q_0^\sigma \cup \{x\}$ such that $ha = x$ in Q, then $y = hc_{(x,\sigma(x))}a$ such that $tc_{(x,\sigma(x))}a = x$ in $c_{(x,\sigma(x))}Q$ and

$$(c_{(x,\sigma(x))}\chi)(y) =$$

$$(c_{(x,\sigma(x))}\alpha)(y) - \sum_{\substack{a\in c_{(x,\sigma(x))}Q_1:\\ ha=y \text{ and } ta\neq x}} (c_{(x,\sigma(x))}\alpha)(ta) - \sum_{\substack{a\in c_{(x,\sigma(x))}Q_1:\\ ha=y \text{ and } ta=x}} (c_{(x,\sigma(x))}\alpha)(x) =$$

$$\alpha(y) - \sum_{\substack{a\in Q_1:\\ ha=y}} \alpha(ta) + \sum_{\substack{a\in Q_1:\\ ha=x}} (\alpha(x) - \sum_{\substack{a\in Q_1:\\ ha=x}} \alpha(ta)) =$$

$$\chi(y) + b_{x,y}\chi(x).$$

Similarly one proves the case $y = h\sigma(a) \notin Q_0^\sigma \cup \{\sigma(x)\}$ such that $t\sigma(a) = x$ in Q.

(iii) Finally we have to consider y such that there are no arrows linking y and x (i.e. $b_{x,y} = 0$) and no arrows linking y and $\sigma(x)$. In this case

$$(c_{(x,\sigma(x))}\chi)(y) =$$

$$(c_{(x,\sigma(x))}\alpha)(y) - \sum_{\substack{a\in c_{(x,\sigma(x))}Q_1:\\ ha=y}} (c_{(x,\sigma(x))}\alpha)(ta) =$$

$$\alpha(y) - \sum_{\substack{a\in Q_1:\\ ha=y}} \alpha(ta) =$$

$$\chi(y).$$

Similarly one proves for $\sigma(x)$. □

Next we study the relation between $SpSI(Q,\alpha)$ and $SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$ (respectively between $OSI(Q,\alpha)$ and $OSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$) with the following lemmas

Lemma 1.5.2. *Let (Q,σ) be a symmetric quiver, let x be a sink and let α be the dimension vector of a symplectic representation.*

(i) If $V \in Rep(Q)$ is indecomposable, not projective, such that $C^+_{(x,\sigma(x))}V$ is not projective and $\langle \underline{dim}V, \alpha\rangle = 0$, then $c^V \in SpSI(Q,\alpha)$ and $c^{C^+_{(x,\sigma(x))}V} \in SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$.

(ii) If $V = S_x$ and $\langle \underline{dim}S_x, c_{(x,\sigma(x))}\alpha\rangle = 0$, then c^{S_x} and $c^{C^- S_{\sigma(x)}}$ in $SpSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$, where S_x and $S_{\sigma(x)}$ are considered as representation of $c_{(x,\sigma(x))}Q$, but c^{S_x} and $c^{C^- S_{\sigma(x)}}$ are zero for Q. Moreover $c^{S_x} = c^{C^- S_{\sigma(x)}}$.

(iii) If $V = C^- S_x$ and $\langle \underline{dim}C^- S_x, \alpha\rangle = 0$, then we have $c^{C^- S_x}, c^{S_{\sigma(x)}} \in SpSI(Q,\alpha)$ but they are zero for $c_{(x,\sigma(x))}Q$. Moreover $c^{S_{\sigma(x)}} = c^{C^- S_x}$.

Lemma 1.5.3. *Let (Q,σ) be a symmetric quiver, let x be a sink and let α be the vector dimension of an orthogonal representation.*

(i) If $V \in Rep(Q)$ is indecomposable, not projective and such that $C^+_{(x,\sigma(x))}V$ is not projective and $\langle \underline{dim}V, \alpha\rangle = 0$, then $c^V \in OSI(Q,\alpha)$ and $c^{C^+_{(x,\sigma(x))}V} \in OSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$.

(ii) If $V = S_x$ and $\langle \underline{dim}S_x, c_{(x,\sigma(x))}\alpha\rangle = 0$, then we have c^{S_x} and $c^{C^- S_{\sigma(x)}}$ in $OSI(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)$, where S_x and $S_{\sigma(x)}$ are considered as representation of $c_{(x,\sigma(x))}Q$, but c^{S_x} and $c^{C^- S_{\sigma(x)}}$ are zero for Q. Moreover $c^{S_x} = c^{C^- S_{\sigma(x)}}$.

(iii) If $V = C^- S_x$ and $\langle \underline{dim}C^- S_x, \alpha\rangle = 0$, then we have $c^{C^- S_x}, c^{S_{\sigma(x)}} \in OSI(Q,\alpha)$ but they are zero for $c_{(x,\sigma(x))}Q$. Moreover $c^{S_{\sigma(x)}} = c^{C^- S_x}$.

We prove only lemma 1.5.2 because the proof of lemma 1.5.3 is similar.

Proof. First of all we note that if x is an admissible sink, then $S_{\sigma(x)} \neq \tau^- \nabla S_{\sigma(x)}$ and $C^- S_x \neq \tau^- \nabla C^- S_x$ and so, by lemma 1.4.6, we can not define both $pf^{S_{\sigma(x)}}$ and $pf^{C^- S_x}$. It's enough to prove the first one because, by lemma B.3.9, $\tau^- \nabla C^- S_x = \tau^- \nabla \tau^- S_x = \nabla \tau^+ \tau^- S_x = \nabla S_x = S_{\sigma(x)}$. If $S_{\sigma(x)} = \tau^- \nabla S_{\sigma(x)}$, by theorem B.1.11 there exists an almost split sequence

$$0 \longrightarrow \nabla S_{\sigma(x)} = S_x \longrightarrow Z \longrightarrow S_{\sigma(x)} \longrightarrow 0. \tag{1.24}$$

Hence $(\underline{dim}\, Z)_y = \begin{cases} 1 & \text{if } y = x, \sigma(x) \\ 0 & \text{otherwise} \end{cases}$ and so either $Z = S_x \oplus S_{\sigma(x)}$ which is an absurd because (1.24) would be a split sequence, or Z is indecomposable and thus there is an arrow $\sigma(x) \to x$ which is not possible since x is an admissible sink.

We recall that $(\underline{dim}S_{\sigma(x)})_y = \begin{cases} 1 & \text{if } \sigma(x) = y \\ 0 & \text{otherwise} \end{cases}$, that $\alpha_x = \alpha_{\sigma(x)}$ for every $x \in Q_0$ and, by theorem B.1.9, that $\langle \underline{dim}C^- S_x, \alpha\rangle = -\langle \alpha, \underline{dim}S_x\rangle$. So, for a dimension vector α of a symplectic (respectively orthogonal) representation, $\langle \underline{dim} S_{\sigma(x)}, \alpha \rangle = \alpha_{\sigma(x)} - \sum_{a \in Q_1 : ha = x} \alpha_{\sigma(ta)} = \alpha_x - \sum_{a \in Q_1 : ha = x} \alpha_{ta} = \langle \alpha, \underline{dim} S_x \rangle = -\langle \underline{dim} C^- S_x, \alpha \rangle$. Similarly we have $\langle \underline{dim} S_x, c_{(x,\sigma(x))}\alpha \rangle = -\langle \underline{dim} C^- S_{\sigma(x)}, c_{(x,\sigma(x))}\alpha \rangle$. Hence, since x is a sink of Q and $\sigma(x)$ is a sink of $c_{(x,\sigma(x))}Q$, it's enough to apply lemma B.6.1 to both Q and $c_{(x,\sigma(x))}Q$. Finally $\tau^- \nabla S_{\sigma(x)} = \tau^- S_x = C^- S_x$ and $\tau^- \nabla C^- S_{\sigma(x)} = \tau^- \tau^+ \nabla S_{\sigma(x)} = S_x$, so, by lemma 1.4.4, $c^{S_x} = c^{C^- S_{\sigma(x)}}$ and $c^{S_{\sigma(x)}} = c^{C^- S_x}$. □

We observe that, by proposition 1.3.9, $\tau^- \nabla V = V$ if and only if $\tau^- \nabla C^+_{(x,\sigma(x))} V = C^+_{(x,\sigma(x))} V$. Let α be a symmetric dimension vector. We recall that $\alpha_y = c_{(x,\sigma(x))}\alpha_y$ for every $y \neq x, \sigma(x)$ and $(c_{(x,\sigma(x))}\alpha)_x = \sum_{a \in Q_1 : ha = x} \alpha_{ta} - \alpha_x = \sum_{a \in Q_1 : ha = x} \alpha_{\sigma(ta)} - \alpha_{\sigma(x)} = (c_{(x,\sigma(x))}\alpha)_{\sigma(x)}$, so we consider three cases.

(i) $0 \neq \alpha_x \neq \sum_{a \in Q_1 : ha = x} \alpha_{ta}$, i.e. $\langle \underline{dim} S_{\sigma(x)}, \alpha \rangle \neq 0$ and $\langle \underline{dim} S_x, c_{(x,\sigma(x))}\alpha \rangle \neq 0$.

(ii) $0 = \alpha_x \neq \sum_{a \in Q_1 : ha = x} \alpha_{ta}$, i.e. $\langle \underline{dim} S_{\sigma(x)}, \alpha \rangle \neq 0$ and $\langle \underline{dim} S_x, c_{(x,\sigma(x))}\alpha \rangle = 0$.

(iii) $0 \neq \alpha_x = \sum_{a \in Q_1 : ha = x} \alpha_{ta}$, i.e. $\langle \underline{dim} S_{\sigma(x)}, \alpha \rangle = 0$ and $\langle \underline{dim} S_x, c_{(x,\sigma(x))}\alpha \rangle \neq 0$.

We note that $0 = \alpha_x = \sum_{a \in Q_1 : ha = x} \alpha_{ta}$ is not possible, unless $\alpha_{ta} = 0$ for every a such that $ha = x$.

Proposition 1.5.4. *Let (Q, σ) be a symmetric quiver. Let α be a symmetric dimension vector, x be an admissible sink and $\varphi^{Sp}_{x,\alpha}$ be as defined in lemma 1.3.14. Then $\varphi^{Sp}_{x,\alpha}(c^V) = c^{C_{(x,\sigma(x))}V}$ and $\varphi^{Sp}_{x,\alpha}(pf^W) = pf^{C_{(x,\sigma(x))}W}$, where V and W are indecomposables of Q such that $\langle \underline{dim}\, V, \alpha \rangle = 0 = \langle \underline{dim}\, W, \alpha \rangle$ and W satisfies property (Op). In particular*

(i) if $0 = \alpha_x \neq \sum_{a \in Q_1 : ha = x} \alpha_{ta}$, then $(\varphi^{Sp}_{x,\alpha})^{-1}(c^{S_x}) = 0$;

(ii) if $0 \neq \alpha_x = \sum_{a \in Q_1 : ha = x} \alpha_{ta}$, then $\varphi^{Sp}_{x,\alpha}(c^{S_{\sigma(x)}}) = 0$.

Proof. We consider the same notation of proof of lemma 1.3.14. If x is an admissible sink of (Q, σ), then we have

$$C^-_{(x,\sigma(x))}(Z \times Hom(V', W)) = C^-_{(x,\sigma(x))}(SpRep(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)) =$$

$$SpRep(Q, \alpha) = Z \times Hom(W, V).$$

So, by definition,

$$C^-_{(x,\sigma(x))}|_Z(SpRep(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)) = Z$$

and

$$C^-_{(x,\sigma(x))}|_{Hom(V',W)}(SpRep(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)) = Hom(W, V).$$

Now $C^-_{(x,\sigma(x))}$ induces a ring morphism

$$\begin{array}{rcl} \phi^{Sp}_{x,\alpha} : \mathbb{K}[SpRep(Q,\alpha)] & \longrightarrow & \mathbb{K}[SpRep(c_{(x,\sigma(x))}Q, c_{(x,\sigma(x))}\alpha)] \\ f & \longmapsto & f \circ C^-_{(x,\sigma(x))} \end{array}$$

By proof of lemma 1.3.14, we note that

$$\mathbb{K}[C^-_{(x,\sigma(x))}Z \times C^-_{(x,\sigma(x))}Hom(V',W)]^{SSp(Q,\alpha)} = \mathbb{K}[Z \times Hom(W,V)]^{SSp(Q,\alpha)}$$

is isomorphic by $\varphi^{Sp}_{x,\alpha}$ to $\mathbb{K}[Z \times Hom(V',W)]^{SSp(c_{(x,\sigma(x))}Q,c_{(x,\sigma(x))}\alpha)}$. Hence $\varphi^{Sp}_{x,\alpha} = \phi^{Sp}_{x,\alpha}|_{SpSI(Q,\alpha)}$ and so for every representation Z of dimension vector α of (Q,σ) we have

$$\varphi^{Sp}_{x,\alpha}(c^V)(C^+_{(x,\sigma(x))}Z) = (c^V \circ C^-_{(x,\sigma(x))})(C^+_{(x,\sigma(x))}Z) = c^V(Z) \qquad (1.25)$$

and

$$\varphi^{Sp}_{x,\alpha}(pf^W)(C^+_{(x,\sigma(x))}Z) = (pf^W \circ C^-_{(x,\sigma(x))})(C^+_{(x,\sigma(x))}Z) = pf^W(Z). \qquad (1.26)$$

By lemma B.5.1 and B.5.2 we have $c^V(Z) = \lambda \cdot c^{C^+_{(x,\sigma(x))}V}(C^+_{(x,\sigma(x))}Z)$, for some $\lambda \in \mathbb{K}$. So, by (1.25), $\varphi^{Sp}_{x,\alpha}$ sends c^V to $c^{C^+_{(x,\sigma(x))}V}$ up to a constant in $\mathbb{K}$. Similarly for pf^W. Finally *(i)* and *(ii)* follow by lemma 1.5.2. □

Proposition 1.5.5. *Let (Q,σ) be a symmetric quiver. Let α be a symmetric dimension vector, x be an admissible sink and $\varphi^O_{x,\alpha}$ be as defined in lemma 1.3.14. Then $\varphi^O_{x,\alpha}(c^V) = c^{C_{(x,\sigma(x))}V}$ and $\varphi^O_{x,\alpha}(pf^W) = pf^{C_{(x,\sigma(x))}W}$, where V and W are indecomposables of Q such that $\langle \underline{dim}\, V,\alpha\rangle = 0 = \langle \underline{dim}\, W,\alpha\rangle$ and W satisfies property (Spp). In particular*

(i) if $0 = \alpha_x \neq \sum_{a\in Q_1:ha=x} \alpha_{ta}$, then $(\varphi^O_{x,\alpha})^{-1}(c^{S_x}) = 0$;

(ii) if $0 \neq \alpha_x = \sum_{a\in Q_1:ha=x} \alpha_{ta}$, then $\varphi^O_{x,\alpha}(c^{S_{\sigma(x)}}) = 0$.

Proof. It is similar to that one of proposition 1.5.4. □

By previous propositions and by lemma 1.3.14 it follows that if the conjectures 1.2.1 and 1.2.2 are true for a symmetric quiver (Q,σ), then they are true for $(c_{(x,\sigma(x))}Q,\sigma)$.

1.6 Composition lemmas

We conclude this chapter with general lemmas which will be useful in our proofs.

Lemma 1.6.1. *Let*

$$(Q,\sigma):\cdots y \xrightarrow{a} x \xrightarrow{b} z \cdots \sigma(z) \xrightarrow{\sigma(b)} \sigma(x) \xrightarrow{\sigma(a)} \sigma(y) \cdots$$

be a symmetric quiver. Assume the underlying quiver with n vertices. Also assume there exist only two arrows in Q_1^+ incident to $x \in Q_0^+$, $a: y \to x$ and $b: x \to z$ with $y, z \in Q_0^+ \cup Q_0^\sigma$. Let V be an orthogonal or symplectic representation with symmetric dimension vector $(\alpha_1, \ldots, \alpha_n) = \alpha$ such that $\alpha_x \geq max\{\alpha_y, \alpha_z\}$.
We define the symmetric quiver $Q' = ((Q_0', Q_1'), \sigma)$ with $n-2$ vertices such that $Q_0' = Q_0 \setminus \{x, \sigma(x)\}$ and $Q_1' = Q_1 \setminus \{a, b, \sigma(a), \sigma(b)\} \cup \{ba, \sigma(a)\sigma(b)\}$, i.e.

$$Q':\cdots y \xrightarrow{ba} z \cdots \sigma(z) \xrightarrow{\sigma(a)\sigma(b)} \sigma(y) \cdots,$$

and let α' be the dimension of V restricted to Q'.
We have:

(Sp) Assume V symplectic. Then

- *(a) if $\alpha_x > max\{\alpha_y, \alpha_z\}$ then $SpSI(Q,\alpha) = SpSI(Q',\alpha')$,*
- *(b) if $\alpha_x = \alpha_y > \alpha_z$ then $SpSI(Q,\alpha) = SpSI(Q',\alpha')[detV(a)]$,*
- *(b') if $\alpha_x = \alpha_z > \alpha_y$ then $SpSI(Q,\alpha) = SpSI(Q',\alpha')[detV(b)]$,*
- *(c) if $\alpha_x = \alpha_y = \alpha_z$ then $SpSI(Q,\alpha) = SpSI(Q',\alpha')[detV(a), detV(b)]$.*

(O) Assume V orthogonal. Then

- *(a) if $\alpha_x > max\{\alpha_y, \alpha_z\}$ then $OSI(Q,\alpha) = OSI(Q',\alpha')$,*
- *(b) if $\alpha_x = \alpha_y > \alpha_z$ then $OSI(Q,\alpha) = OSI(Q',\alpha')[detV(a)]$,*
- *(b') if $\alpha_x = \alpha_z > \alpha_y$ then $OSI(Q,\alpha) = OSI(Q',\alpha')[detV(b)]$,*
- *(c) if $\alpha_x = \alpha_y = \alpha_z$ then $OSI(Q,\alpha) = OSI(Q',\alpha')[detV(a), detV(b)]$.*

Proof. We use the notation of section A.1.
(Sp) Using Cauchy formula (theorem A.1.9) we have

$$SpSI(Q,\alpha) = \left(\bigoplus_{\substack{\lambda: Q_1^+ \to \Lambda \\ \mu: Q_1^\sigma \to ER\Lambda}} \bigotimes_{c \in Q_1^+} (S_{\lambda(c)} V_{tc} \otimes S_{\lambda(c)} V_{hc}^*) \otimes \left(\bigotimes_{d \in Q_1^\sigma} S_{\mu(d)} V_{td} \right) \right)^{SSp(Q,\alpha)}$$

where Λ is the set of all partitions and $ER\Lambda$ is the set of the partitions with even rows.
(a) If $\alpha_x > max\{\alpha_y, \alpha_z\}$, by theorem A.1.8,

$$S_{\lambda(a)} V_x^* = S_{(\underbrace{0,\ldots,0}_{\alpha_x - \alpha_y}, \underbrace{-\lambda(a)_{\alpha_y}, \ldots, -\lambda(a)_1}_{\alpha_y})} V_x,$$

where $\lambda(a) = (\lambda(a)_1, \dots, \lambda(a)_{\alpha_y})$. By proposition A.2.8, $\lambda(a)$ and $\lambda(b)$ have to satisfy the following equations

$$\begin{cases} \lambda(b)_i - \lambda(b)_{i+1} = 0, & i \in \{\alpha_y + 1, \dots, \alpha_x - 1\} \\ \lambda(b)_{\alpha_y} - \lambda(b)_{\alpha_y+1} = \lambda(a)_{\alpha_y} \\ \lambda(b)_{\alpha_y - i} - \lambda(b)_{\alpha_y - i+1} = \lambda(a)_{\alpha_y - i} - \lambda(a)_{\alpha_y - i+1}, & i \in \{1, \dots, \alpha_y - 1\}. \end{cases} \tag{1.27}$$

We call $\lambda(b)_i = k \geq 0$ for every $i \in \{\alpha_y + 1, \dots, \alpha_x\}$ and so

$$\lambda(b) = (\lambda(b)_1, \dots, \lambda(b)_{\alpha_x}) = (\underbrace{\lambda(a)_1 + k, \dots, \lambda(a)_{\alpha_y} + k}_{\alpha_y}, \underbrace{k, \dots, k}_{\alpha_x - \alpha_y}).$$

Now, by theorem A.1.8, $S_{\lambda(b)} V_z^* = 0$ unless $ht(\lambda(b)) \leq \alpha_z$. If $\alpha_y \leq \alpha_z$, then $S_{\lambda(b)} V_z^* = 0$ unless $\lambda(b)_{\alpha_z+1} = \dots = \lambda(b)_{\alpha_x} = 0$, i.e. $k = 0$, so $\lambda(b) = (\lambda(a)_1, \dots, \lambda(a)_{\alpha_y}, \underbrace{0, \dots, 0}_{\alpha_x - \alpha_y}) = \lambda(a)$. If $\alpha_z < \alpha_y$, then $S_{\lambda(b)} V_z^* = 0$ unless $\lambda(b)_{\alpha_z+1} = \dots = \lambda(b)_{\alpha_x} = 0$, i.e. $k = 0$ and $\lambda(a)_{\alpha_z+1} = \dots = \lambda(a)_{\alpha_y} = 0$, so $\lambda(b) = \lambda(a)$ again.

So let $\lambda(a) = \lambda(b) = \bar{\lambda}$. By proposition A.2.8, $S_{\bar{\lambda}} V_x^* \otimes S_{\bar{\lambda}} V_x$ contains a semi-invariant of weight zero, which is hence a $GL(V_x)$-invariant. Since $V_y^* \otimes V_x \oplus V_x^* \otimes V_z = V_x^{\alpha_y} \oplus (V_x^*)^{\alpha_z}$ and since $S_{\bar{\lambda}} V_x^* \otimes S_{\bar{\lambda}} V_x$ is a summand in the Cauchy formula of $\mathbb{K}[V_x^{\alpha_y} \oplus (V_x^*)^{\alpha_z}]$, using FFT for GL (theorem A.2.3) we obtain $SL(V)$ acts trivially on $S_{\bar{\lambda}} V_x^* \otimes S_{\bar{\lambda}} V_x$ and so $(S_{\lambda(a)} V_x^* \otimes S_{\lambda(b)} V_x)^{SL V_x} = \mathbb{K}$. So we have

$$SpSI(Q, \alpha) \cong SpSI(Q', \alpha').$$

(b) If $\alpha_x = \alpha_y > \alpha(z)$, by theorem A.1.8,

$$S_{\lambda(a)} V_x^* = S_{(-\lambda(a)_{\alpha_y = \alpha_x}, \dots, -\lambda(a)_1)} V_x.$$

By proposition A.2.8, $\lambda(a)$ and $\lambda(b)$ have to satisfy $\lambda(a)_i - \lambda(a_{i+1}) = \lambda(b)_i - \lambda(b)_{i+1}$ for every $i \in \{1, \dots, \alpha_x\}$ and moreover $S_{\lambda(b)} V_z^* = 0$ unless $ht(\lambda(b)) \leq \alpha_z < \alpha_x$. Hence we have

$$\begin{cases} \lambda(b)_i = 0 & i \in \{\alpha_{z+1}, \dots, \alpha_x\} \\ \lambda(a)_i - \lambda(a)_{i+1} = \lambda(b)_i - \lambda(b)_{i+1} & i \in \{1, \dots, \alpha_x - 1\} \end{cases} \tag{1.28}$$

and thus

$$\begin{cases} \lambda(a)_i - \lambda(a)_{i+1} = \lambda(b)_i - \lambda(b)_{i+1} & i \in \{1, \dots, \alpha_z - 1\} \\ \lambda(a)_{\alpha_z} = \lambda(a)_{\alpha_z+1} + \lambda(b)_{\alpha_z} \\ \lambda(a)_i = \lambda(a)_{i+1} & i \in \{\alpha_z + 1, \dots, \alpha_x - 1\}. \end{cases} \tag{1.29}$$

Hence $\lambda(a)$ contains a column of length $\alpha_x = \alpha_y$ for some $k \in \mathbb{N}$, so we have $\lambda(a) = (\lambda(b)_1 + k, \dots, \lambda(b)_{\alpha_z} + k, k, \dots, k)$ then $S_{\lambda(a)} V_y \otimes S_{\lambda(a)} V_x^* = S_{\lambda(b)} V_y \otimes (\bigwedge^{\alpha_y} V_y)^k \otimes (\bigwedge^{\alpha_x} V_x^*)^k \otimes S_{\lambda(b)} V_x^*$. Now $(\bigwedge^{\alpha_y} V_y)^k \otimes (\bigwedge^{\alpha_x} V_x^*)^k$ is spanned by

$(det\,V(a))^k$. So we have a semi-invariant f of the form $(detV(a))^k f'$ where f' is of weight zero, hence using theorem FFT for GL (A.2.3) as before and by lemma A.2.1, we have

$$SpSI(Q,\alpha) = SpSI(Q',\alpha')[detV(a)].$$

In the similar way we prove *(b′)*.
(c) If $\alpha(x) = \alpha(y) = \alpha(z)$, by theorem A.1.8,

$$S_{\lambda(a)}V_x^* = S_{(-\lambda(a)_{\alpha_y=\alpha_x},\ldots,-\lambda(a)_1)}V_x$$

and

$$S_{\lambda(b)}V_z^* = S_{(-\lambda(b)_{\alpha_x=\alpha_z},\ldots,-\lambda(b)_1)}V_z,$$

where $\lambda(a) = (\lambda(a)_1,\ldots,\lambda(a)_{\alpha_y})$ and $\lambda(b) = (\lambda(b)_1,\ldots,\lambda(b)_{\alpha_x})$. By proposition A.2.8, $\lambda(a)$ and $\lambda(b)$ have to satisfy the following equations

$$\lambda(a)_{i-1} - \lambda(a)_i = \lambda(b)_{i-1} - \lambda(b)_i \tag{1.30}$$

for every $i \in \{2,\ldots,\alpha_x = \alpha_y\}$. Thus $\lambda(a)_i = \lambda(b)_i - \lambda(b)_{\alpha_x} + \lambda(a)_{\alpha_x}$ for every $i \in \{1,\ldots,\alpha_x\}$. Hence if we set $\lambda(b)_{\alpha_x} = h$ and $\lambda(a)_{\alpha_x} = k$ we have

$$\lambda(a)_i = \lambda(b)_i - h + k \tag{1.31}$$

for every $i \in \{1,\ldots,\alpha_x\}$. So in our case $\lambda(a) = (\lambda(b) - (h^{\alpha_x})) + (k^{\alpha_x})$ and $\lambda(b) = (\lambda(a) - (k^{\alpha_x})) + (h^{\alpha_x})$. We call $\lambda(b) - (h^{\alpha_x}) = \lambda(b)'$ and $\lambda(a) - (k^{\alpha_x}) = \lambda(a)'$ and we note that $\lambda(a)' = \lambda(b)'$ by the system (1.31). Then $S_{\lambda(a)}V_y \otimes S_{\lambda(a)}V_x^* \otimes S_{\lambda(b)}V_x \otimes S_{\lambda(b)}V_z^* = S_{\lambda(b)'}V_y \otimes < (detV(a))^k > \otimes S_{\lambda(b)'}V_x^* \otimes S_{\lambda(a)'}V_x \otimes < (detV(b))^h > \otimes S_{\lambda(a)'}V_z^*$. So we have a semi-invariant f of the form $(detV(a))^k(detV(b))^h f'$ where f' is of weight zero, hence using theorem FFT for GL (A.2.3) as before and by lemma A.2.1, we have

$$SpSI(Q,\alpha) \cong SpSI(Q',\alpha')[detV(a), detV(b)].$$

(O) Using Cauchy formula we have

$$OSI(Q,\alpha) = \left(\bigoplus_{\substack{\lambda:Q_1^+\to\Lambda \\ \mu:Q_1^\sigma\to EC\Lambda}} \bigotimes_{c\in Q_1^+} (S_{\lambda(c)}V_{tc} \otimes S_{\lambda(c)}V_{hc}^*) \otimes \left(\bigotimes_{d\in Q_1^\sigma} S_{\mu(d)}V_{td} \right) \right)^{SO(Q,\alpha)}$$

where Λ is the set of all partitions and $EC\Lambda$ is the set of the partitions with even columns. The rest of the proof is similar of the symplectic case. □

Lemma 1.6.2. *Let (Q,σ) be a symmetric quiver with n vertices such that there exist only two arrows a and b incident to the vertex x in Q_0 and b is fixed by σ, i.e.*

$$Q : \cdots y \xrightarrow{a} x \xrightarrow{b} \sigma(x) \xrightarrow{\sigma(a)} \sigma(y) \cdots$$

Let
$$V: \cdots V_y \overset{V(a)}{\longrightarrow} V_x \overset{V(b)}{\longrightarrow} V_x^* \overset{-V(a)^t}{\longrightarrow} V_y^* \cdots$$
be an orthogonal or symplectic representation of (Q,σ) *with* $\underline{dim}V = \alpha$ *such that* $\alpha_x \geq \alpha_y$. *Moreover define the symmetric quiver* $(Q',\sigma) = ((Q'_0, Q'_1),\sigma)$ *with* $n-2$ *vertices such that* $Q'_0 = Q_0 \setminus \{x, \sigma(x)\}$ *and* $Q'_1 = Q_1 \setminus \{a, b, \sigma(a)\} \cup \{\sigma(a)ba\}$, *i.e*
$$Q': \cdots y \overset{\sigma(a)ba}{\longrightarrow} \sigma(y) \cdots .$$
Let α' *be the dimension of* V *restricted to* Q'.

(Sp) If V *is symplectic, then*

(i) $\alpha_x > \alpha_y \Longrightarrow SpSI(Q,\alpha) = SpSI(Q',\alpha')[detV(b)]$

(ii) $\alpha_x = \alpha_y \Longrightarrow SpSI(Q,\alpha) = SpSI(Q',\alpha')[detV(a)]$.

(O) If V *is orthogonal, then*

(i) $\alpha_x > \alpha_y$ *and* α_x *is even* $\Longrightarrow OSI(Q,\alpha) = OSI(Q',\alpha')[pfV(b)]$

(ii) $\alpha_x = \alpha_y \Longrightarrow OSI(Q,\alpha) = OSI(Q',\alpha')[detV(a)]$.

Proof. We consider again the Cauchy formulas.
(Sp) If $\alpha_y \leq \alpha_x$, by proposition A.2.8, $\lambda(a)$ and $\lambda(b)$ have to satisfy $\lambda(a)_{i-1} - \lambda(a)_i = \lambda(b)_{i-1} - \lambda(b)_i$ for every $i \in \{2, \ldots, \alpha_y\}$.
(i) Let $\alpha_y < \alpha_x$, we have
$$S_{\lambda(a)}V_x^* = S_{(0,\ldots,0,-\lambda(a)_{\alpha_y},\ldots,-\lambda(a)_1)}V_x$$
and so
$$\lambda(b) = (\overbrace{\lambda(a)_1, \ldots, \lambda(a)_{\alpha_y}, 0, \ldots, 0}^{\alpha_x}) + (\overbrace{2k, \ldots, 2k}^{\alpha_x}),$$
for some $k \in \mathbb{Z}_{\geq 0}$ and with $\lambda(a)_i$ even for every i. Then $S_{\lambda(a)}V_x^* \otimes S_{\lambda(b)}V_x = S_{\lambda(a)}V_x^* \otimes S_{\lambda(a)}V_x \otimes (\bigwedge^{\alpha_x} V_x)^{2k}$. Now $(\bigwedge^{\alpha_x} V_x)^{2k}$ is spanned by $(det\, V(b))^k$. So we have a semi-invariant f of the form $(detV(b))^k f'$ where f' is of weight zero, hence using theorem FFT for GL (A.2.3) as before and by lemma A.2.1, we have
$$SpSI(Q,\alpha) \cong SpSI(Q',\alpha')[det\ V(b)].$$
(ii) If $\alpha_x = \alpha_y$, the proof is similar to the part *(b)* of lemma 1.6.1.

(O) If $\alpha_y \leq \alpha_x$, by proposition A.2.8, $(S_{\lambda(a)}V_x^* \otimes S_{\lambda(b)}V_x)^{SL(V_x)} \neq 0$ if and only if $\lambda(a)_{i-1} - \lambda(a)_i = \lambda(b)_{i-1} - \lambda(b)_i$ for every $i \in \{2, \ldots, \alpha_y\}$.
Now the proof is similar to the symplectic case, recalling that $V(b)$, in this case, is skew-symmetric, so we can define $pf\ V(b)$. □

Chapter 2

Semi-invariants of symmetric quivers of finite type

In this chapter we prove conjectures 1.2.1 and 1.2.2 for the symmetric quivers of finite type. We recall that, by theorem 1.1.12, a symmetric quiver of finite type has the underlying quiver of type A_n. Throughout this chapter we enumerate vertices with $1, \dots, n$ from left to right and we call a_i the arrow with i on the left and $i+1$ on the right; moreover we define σ by $\sigma(i) = n - i + 1$, for every $i \in \{1, \dots, n\}$, and $\sigma(a_i) = a_{n-i}$, for every $i \in \{1, \dots, n-1\}$.
First we prove a lemma valid for $Q = A_n$, which is a particular case of lemma 1.4.6.

Lemma 2.0.3. *Let (A_n, σ) be a symmetric quiver of type A. Let $V \in Rep(Q)$ such that $V = \tau^- \nabla V$ and let W a selfdual representation such that $\langle \underline{dim} V, \underline{dim} W\rangle$=0, then we have the following.*

(i) If n is even, d^V_W is skew-symmetric if and only if $W \in ORep(Q, \underline{dim} W)$.

(ii) If n is odd d^V_W is skew-symmetric if and only if $W \in SpRep(Q, \underline{dim} W)$.

Proof. It checked in the proof of lemma 1.4.6. □

By proof of lemma 1.4.6 we noted also that an indecomposable representation V of A_n satisfies property *(Spp)* if n is even and it satisfies property *(Op)* if n is odd.
The conjectures 1.2.1 and 1.2.2 for symmetric quivers of finite type become

Theorem 2.0.4. *Let (Q, σ) be a symmetric quiver of finite type. Let α be the dimension vector of a symplectic representation. Then $SpSI(Q, \alpha)$ is generated by the following semi-invariants.*

(n even) c^V *with V indecomposable in $Rep(Q)$ such that $\langle \underline{dim} V, \alpha\rangle = 0$.*

(*n odd*) (i) c^V *with* V *indecomposable in* $Rep(Q)$ *such that* $\langle \underline{dim}V, \alpha\rangle = 0$;
(ii) pf^V *with* $V \in Rep(Q)$ *such that* $V = \tau^{-}\nabla V$.

Theorem 2.0.5. *Let* (Q, σ) *be a symmetric quiver of finite type. Let* α *be the dimension vector of an orthogonal representation. Then* $OSI(Q, \alpha)$ *is generated by the following semi-invariants.*

(*n odd*) c^V *with* V *indecomposable in* $Rep(Q)$ *such that* $\langle \underline{dim}V, \alpha\rangle = 0$.

(*n even*) (i) c^V *with* V *indecomposable in* $Rep(Q)$ *such that* $\langle \underline{dim}V, \alpha\rangle = 0$;
(ii) pf^V *with* $V \in Rep(Q)$ *such that* $V = \tau^{-}\nabla V$.

By proposition 1.3.4 and by propositions 1.5.4 and 1.5.5, it's enough to study the equioriented case, i.e. the case in which all the arrows have orientation from left to right.

Lemma 2.0.6. *Let* (Q, σ) *be a symmetric quiver of finite type. Then* $SpSI(Q, \beta)$ *and* $OSI(Q, \beta)$ *are polynomial rings, for every symmetric dimension vector* β.

Proof. Since the isomorphism classes of β-dimensional symplectic (resp. orthogonal) representations of (Q, σ) correspond to the orbits of the action of $Sp(Q, \beta)$ (resp. of $O(Q, \beta)$) on $SpRep(Q, \beta)$ (resp. on $ORep(Q, \beta)$), then lemma follows by definition of symmetric quiver finite type and by lemma A.2.5. □

2.1 Equioriented symmetric quivers of finite type

In this section we state and prove case by case theorems 2.0.4 and 2.0.5 for equioriented case. Throughout this section we call $V_{j,i}$ the indecomposable of A_n with dimension vector

$$(v_{j,i})_k = \begin{cases} 1 & \text{if } j \leq k \leq i \\ 0 & \text{otherwise.} \end{cases}$$

2.1.1 The symplectic case for A_{2n}

We rewrite theorem 2.0.4 in the following way

Theorem 2.1.1. *Let* (Q, σ) *be an equioriented symmetric quiver of type* A_{2n} *and let* α *be the dimension vector of a symplectic representation of* (Q, σ).
Then $SpSI(Q, \alpha)$ *is generated by the following indecomposable semi-invariants:*

(i) $c^{V_{j,i}}$ *of weight* $\langle \underline{dim}V_{j,i}, \cdot\rangle$ *for every* $1 \leq j \leq i \leq n-1$ *such that* $\langle \underline{dim}\, V_{j,i}, \alpha\rangle = 0$,

(ii) $c^{V_{i,2n-i}}$ *of weight* $\langle \underline{dim}V_{i,2n-i}, \cdot\rangle$ *for every* $i \in \{1, \ldots, n\}$.

The result follows from the following statement

Theorem 2.1.2. *Let (Q, σ) be an equioriented symmetric quiver of type A_{2n}, where*

$$Q = A_n^{eq} : 1 \xrightarrow{a_1} 2 \cdots n \xrightarrow{a_n} n+1 \cdots 2n-1 \xrightarrow{a_{2n-1}} 2n,$$

$\sigma(i) = 2n - i + 1$ and $\sigma(a_i) = a_{2n-i}$ for every $i \in \{1, \ldots, n\}$ and let V be a symplectic representation, $\underline{dim}(V) = (\alpha_1, \ldots, \alpha_n) = \alpha$.
Then $SpSI(Q, \alpha)$ is generated by the following indecomposable semi-invariants:

(i) $det(V(a_i) \cdots V(a_j))$ with $j \leq i \in \{1, \ldots, n-1\}$ if $min\{\alpha_{j+1}, \ldots, \alpha_i\} > \alpha_j = \alpha_{i+1}$;

(ii) $det(V(a_{2n-i}) \cdots V(a_i))$ with $i \in \{1, \ldots, n\}$ if $min\{\alpha_{i+1}, \ldots, \alpha_n\} > \alpha_i$.

Proof. First we recall that if V is a symplectic representation of dimension $\alpha = (\alpha_1, \ldots, \alpha_n)$ of a symmetric quiver of type A_{2n}, then we have

$$SpRep(Q, \alpha) = \bigoplus_{i=1}^{n-1} V(ta_i)^* \otimes V(ha_i) \oplus S_2 V_n^*.$$

We proceed by induction on n. For $n = 1$ we have the symplectic representation

$$V_1 \xrightarrow{V(a)} V_1^*$$

where V_1 is a vector space of dimension α and $V(a)$ is a linear map such that $V(a) = V(a)^t$. So

$$SpRep(Q, \alpha) = S^2 V_1^*$$

and by theorem A.1.9

$$SpSI(Q, \alpha) = \bigoplus_{\lambda \in ER\Lambda} (S_\lambda V_1)^{SL(V_1)},$$

where $ER\Lambda$ is the set of the partitions with even rows. By proposition A.2.7 and since $\lambda \in ER\Lambda$, $SpSI(Q, \alpha) \neq 0$ if and only if $\lambda = (\overbrace{2k, \ldots, 2k}^{\alpha})$ for some $k \in \mathbb{Z}_{\geq 0}$ and we have that $(S_\lambda V_1)^{SL(V_1)}$ is generated by a semi-invariant of weight $2k$. Since $g^k \cdot detV(a) = det((g^t)^k V(a) g^k) = (det\, g)^{2k} detV(a)$ for every $g \in GL(V)$, we note that $V(a) \in S_2 V_1^* \mapsto (detV(a))^k$ is a semi-invariant of weight $2k$. So $(detV(a))^k$ is a generator of $(S_\lambda V_1)^{SL(V_1)}$ and thus $SpSI(Q, \alpha) = \mathbb{K}[detV(a)]$.
Now we prove the induction step. By theorem A.1.9 we obtain

$$SpSI(Q, \alpha) = \left(\mathbb{K}[X]\right)^{SL(V)} =$$

$$\bigoplus_{\substack{\lambda(a_1), \ldots, \lambda(a_{n-1}) \text{ and} \\ \lambda(a_n) \in ER\Lambda}} (S_{\lambda(a_1)} V_1)^{SL(V_1)} \otimes (S_{\lambda(a_1)} V_2^* \otimes S_{\lambda(a_2)} V_2)^{SL(V_2)} \otimes$$

$$\cdots \otimes (S_{\lambda(a_{n-1})}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)}.$$

where $SL(V) = SL(V_1) \times \cdots \times SL(V_n)$. We suppose that there exists $i \in \{1, \dots, n-2\}$ such that $\alpha_1 \leq \cdots \leq \alpha_i > \alpha_{i+i}$. By lemma 1.6.1,

$$SpSI(Q, \alpha) = SpSI(Q^1, \alpha^1)$$

where Q^1 is the smaller quiver $1 \longrightarrow 2 \cdots i-1 \longrightarrow i+1 \cdots 2n-i+1 \longrightarrow 2n-i+3 \cdots 2n-1 \longrightarrow 2n$ and α^1 is the restriction of α in Q^1.
If i does't exist, we have $\alpha_1 \leq \cdots \leq \alpha_{n-1}$. So, by lemma 1.6.1, we have the generators $det\, V(a_i) = det\, V(\sigma(a_i))$ if $\alpha_i = \alpha_{i+1}$, $1 \leq i \leq n-2$.
We note that, by proposition A.2.7,

$$\lambda(a_1) = (\overbrace{k_1, \dots, k_1}^{\alpha_1})$$

is a rectangle with k_1 columns of height α_1, for some $k_1 \in \mathbb{Z}_{\geq 0}$. Since $\alpha_1 \leq \cdots \leq \alpha_{n-1}$, by proposition A.2.8, we obtain that there exist $k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda(a_i) = (\overbrace{k_i + \cdots + k_1, \dots, k_i + \cdots + k_1}^{\alpha_1}, \dots, \overbrace{k_i, \dots, k_i}^{\alpha_i - \alpha_{i-1}}),$$

for every $i \in \{1, \dots, n-1\}$. We also know that λ_n must have even rows. If $\alpha_n = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_{n-1}$ for some $j \in \{1, \dots, n-1\}$ then $S_{\lambda_{n-1}}V_n^* = 0$ unless $k_{n-1} + \cdots + k_{j+1} = 0$, so $\lambda(a_{n-1}) = \cdots = \lambda(a_{j+1}) = \lambda(a_j)$. By proposition A.2.8, $(S_{\lambda(a_{n-1})}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)} = (S_{\lambda(a_j)}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)}$ contains a semi-invariant if and only if

$$\lambda(a_n) = (\overbrace{k_n + k_{j-1} + \cdots + k_1, \dots, k_n + k_{j-1} + \cdots + k_1}^{\alpha_1}, \dots, \overbrace{k_n, \dots, k_n}^{\alpha_n - \alpha_{j-1}}),$$

but $k_n + k_{j-1} + \cdots + k_1, k_n + k_{j-1} + \cdots + k_2, \dots, k_n$ have to be even and then $k_n, k_{j-1}, \dots, k_1$ have to be even. As before, by lemma 1.6.1, we can consider the smaller quiver $Q^2 : 1 \longrightarrow 2 \cdots j \longrightarrow n \longrightarrow n+1 \longrightarrow 2n-j+1 \cdots 2n-1 \longrightarrow 2n$ and then

$$SpSI(Q, \alpha) \cong SpSI(Q^2, \alpha^2) =$$

$$(S_{\lambda(a_1)}V_1)^{SL(V_1)} \otimes \cdots \otimes (S_{\lambda(a_{j-1})}V_j^* \otimes S_{\lambda(a_j)}V_j)^{SL(V_j)} \otimes (S_{\lambda(a_j)}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)}.$$

Now to complete the proof it's enough to find the generators of $SpSI(Q^2, \alpha^2)$ for $\alpha_n = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_{n-1}$.

(a) By proposition A.2.8, for every $l \in \{1, \dots, j\}$, $(S_{\lambda(a_{l-1})}V_l^* \otimes S_{\lambda(a_l)}V_l)^{SL(V_l)}$ is generated by a semi-invariant of weight $(0, \dots, 0, k_l, 0, \dots, 0)$ where $k_l = 2h$ with $h \in \mathbb{Z}_{\geq 0}$, is l-th component. Since $g^h \cdot det(V(a_{2n-l}) \cdots V(a_l)) = det((g_{\sigma(l)}^{-1})^h V(a_{2n-l}) \cdots V(a_l)(g_l)^h) = det((g_l^t)^h V(a_{2n-l}) \cdots V(a_l)(g_l)^h) =$

$(det\, g_l)^{2h} det(V(a_{2n-l})\cdots V(a_l))$ for every $g = \{g_i\}_{i\in Q_0} \in GL(V)$, we note that $V(a_{2n-l})\cdots V(a_l) \in SpSI(Q,\alpha) \mapsto (det(V(a_{2n-l})\cdots V(a_l)))^h$ is a semi-invariant of weight $(0,\ldots,0,k_l,0,\ldots,0)$, so it generates $(S_{\lambda(a_{l-1})}V_l^* \otimes S_{\lambda(a_l)}V_l)^{SL(V_l)}$. Now $\lambda(a_l) = \lambda(a_{l-1}) + (k_l^{\alpha_l})$ hence, using lemma A.2.1, $det(V(a_{2n-l})\cdots V(a_l))$ is a generator of $SpSI(Q,\alpha)$.

(b) In the summand of $SpSI(Q,\alpha)$ indexed by the families of partitions in which $\lambda(a_j) = (\overbrace{k_j,\ldots,k_j}^{\alpha_j=\alpha_n})$, with $k_j \in \mathbb{Z}_{\geq 0}$, we have that $(S_{\lambda(a_j)}V_j)^{SL(V_j)} \otimes (S_{\lambda(a_j)}V_n^*)^{SL(V_n)}$ is generated by a semi-invariant of weight $(0,\ldots,0,k_j,0,\ldots,0,-k_j)$ where k_j and $-k_j$ are respectively the j-th and the n-th component and we note, as before, that $(det(V(a_{n-1})\cdots V(a_j)))^{k_j}$ is a semi-invariant of weight $(0,\ldots,0,k_j,0,\ldots,0,-k_j)$. Since $\lambda(a_j) = \lambda(a_{j-1}) + (k_j^{\alpha_j=\alpha_n})$, $det(V(a_{n-1})\cdots V(a_j))$ is a generator of $SpSI(Q,\alpha)$;

(c) in the summand of $SpSI(Q,\alpha)$ indexed by the families of partitions in which $\lambda(a_n) = (\overbrace{k_n,\ldots,k_n}^{\alpha_n})$ with $k_n \in 2\mathbb{Z}_{\geq 0}$, we note again that $(S_{\lambda(a_n)}V_n)^{SL(V_n)}$ is generated by $(det(V(a_n)))^{k_n}$ of weight $(0,\ldots,0,k_n)$ where n-th component k_n is even. Since $\lambda(a_n) = \lambda(a_{j-1}) + (k_n^{\alpha_n})$, $det(V(a_n))$ is a generator of $SpSI(Q,\alpha)$. □

Proof theorem 2.1.1. First we note that $det(V(a_i)\cdots V(a_j)) = det(V_j \to V_{i+1}) = c^{V_{j,i}}(V)$ and $\alpha_j = \alpha_{i+1}$ is equivalent to $\langle \underline{dim}\, V_{j,i}, \underline{dim}\, V\rangle = 0$. We recall, in fact, that the definition of $c^{V_{j,i}}$ doesn't depend to the choose of projective resolution of $V_{j,i}$. If we consider the minimal projective resolution of $V_{j,i}$, we have

$$0 \longrightarrow P_{i+1} \stackrel{a_i\cdots a_j}{\longrightarrow} P_j \longrightarrow V_{j,i} \longrightarrow 0$$

and applying the Hom-functor we have

$$Hom(a_i\cdots a_j, V) : Hom(P_j,V) = V_j \stackrel{V(a_i\cdots a_j)}{\longrightarrow} V_{i+1} = Hom(P_{i+1},V).$$

In the same way one proves that $det(V(a_{2n-i})\cdots V(a_i)) = det(V_i \longrightarrow V_{2n-i+1} = V_i^*) = c^{V_{i,2n-i}}(V)$, but in this case, since $\underline{dim}V = \underline{dim}\nabla V$, we have $\alpha_i = \alpha_{2n-i+1}$ and so $\langle \underline{dim}V_{i,2n-i}, \underline{dim}V\rangle = 0$ for every $i \in \{1,\ldots,n\}$. Moreover we note that

(i) $c^{V_{2n-i,2n-j}}(V) = c^{V_{j,i}}(V)$, by lemma 1.4.4, since $\tau^-\nabla V_{j\,i} = V_{2n-i\,2n-j}$;

(ii) for every $j \in \{1,\ldots,n-1\}$ and for every $i \in \{n+1,\ldots,2n-1\} \setminus \{2n-j\}$ there exists $j < k \in \{1,\ldots,n-1\}$ such that $2n-k = i$ and so $c^{V_{j,i}}(V) = c^{V_{j,k-1}}(V)\cdot c^{V_{k,2n-k}}(V)$.

Now, using theorem 2.1.2, we obtain the statement of the theorem. □

2.1.2 The orthogonal case for A_{2n}

We rewrite theorem 2.0.5 in the following way

Theorem 2.1.3. *Let (Q,σ) be an equioriented symmetric quiver of type A_{2n} and let α be the dimension vector of an orthogonal representation of (Q,σ).*
Then $OSI(Q,\alpha)$ is generated by the following indecomposable semi-invariants:

(i) $c^{V_{j,i}}$ *of weight* $\langle \underline{dim} V_{j,i}, \cdot\rangle$ *for every* $1 \leq j \leq i \leq n-1$ *such that* $\langle \underline{dim}\, V_{j,i}, \alpha\rangle = 0$,

(ii) $pf^{V_{i,2n-i}}$ *of weight* $\frac{\langle \underline{dim} V_{i,2n-i}, \cdot\rangle}{2}$ *for every* $i \in \{1,\ldots,n\}$ *such that* α_i *is even.*

The result follows from the following statement

Theorem 2.1.4. *Let (Q,σ) be an equioriented symmetric quiver of type A_{2n}, where*

$$Q = A_n^{eq} : 1 \xrightarrow{a_1} 2 \cdots n \xrightarrow{a_n} n+1 \cdots 2n-1 \xrightarrow{a_{2n-1}} 2n,$$

$\sigma(i) = 2n-i+1$ and $\sigma(a_i) = a_{2n-i}$ for every $i \in \{1,\ldots,n\}$ and let V be an orthogonal representation, $\underline{dim}(V) = (\alpha_1,\ldots,\alpha_n) = \alpha$.
Then $OSI(Q,\alpha)$ is generated by the following indecomposable semi-invariants:

(i) $det(V(a_i)\cdots V(a_j))$ *with* $j \leq i \in \{1,\ldots,n-1\}$ *if* $min(\alpha_{j+1},\ldots,\alpha_i) > \alpha_j = \alpha_{i+1}$;

(ii) $pf(V(a_{2n-i})\cdots V(a_i))$ *with* $i \in \{1,\ldots,n\}$ *if* $min(\alpha_{i+1},\ldots,\alpha_n) > \alpha_i$*and* α_i *is even.*

Proof. First we recall that if V is a orthogonal representation of dimension $\alpha = (\alpha_1,\ldots,\alpha_n)$ of a symmetric quiver of type A_{2n}, then

$$ORep(Q,\alpha) = \bigoplus_{i=1}^{n-1} V(ta_i)^* \otimes V(ha_i) \oplus \bigwedge^2 V_n^*.$$

We proceed by induction on n. For $n=1$ we have the orthogonal representation

$$V_1 \xrightarrow{V(a)} V_1^*$$

where V_1 is a vector space of dimension α and $V(a)$ is a linear map such that $V(a) = -V(a)^t$.

$$ORep(Q,\alpha) = \bigwedge^2 V_1^*$$

and by theorem A.1.9

$$OSI(Q,\alpha) = \bigoplus_{\lambda\in EC\Lambda} (S_\lambda V_1)^{SL(V_1)}$$

where with $EC\Lambda$ we denote the set of partitions with even columns. By proposition A.2.7 since $\lambda \in EC\Lambda$, $OSI(Q,\alpha) \neq 0$ if and only if $\lambda = (\overbrace{k,\dots,k}^{\alpha})$ with α even, for some k. Since for every $g \in GL(V_1)$, $g^k \cdot pfV(a) = g^k \cdot \sqrt{detV(a)} = \sqrt{det((g^t)^{\frac{k}{2}}V(a)g^{\frac{k}{2}})} = (det\, g)^k pfV(a)$, we note that $V(a) \in \bigwedge^2 V^* \mapsto (pfV(a))^k$ is a semi-invariant of weight k so $(S_\lambda V_1)^{SL(V_1)}$ is generated by the semi-invariant $(pf\, V(a))^k$ if α is even and $OSI(Q,\alpha) = \mathbb{K}[pfV(a)]$.
Now we prove the induction step. Let $X = ORep(Q,\alpha)$ and by theorem A.1.9 we obtain

$$OSI(Q,\alpha) = \big(\mathbb{K}[X]\big)^{SL(V)} =$$

$$\bigoplus_{\substack{\lambda(a_1),\dots,\lambda(a_{n-1})\ and \\ \lambda(a_n)\in EC\Lambda}} (S_{\lambda(a_1)}V_1)^{SL(V_1)} \otimes (S_{\lambda(a_1)}V_2^* \otimes S_{\lambda(a_2)}V_2)^{SL(V_2)} \otimes$$

$$\cdots \otimes (S_{\lambda(a_{n-1})}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)},$$

where $SL(V) = SL(V_1) \times \cdots \times SL(V_n)$.
The proof of this theorem is the same of the proof of the theorem 2.1.2 up to when we have to consider α_n. As in the previous proof we can suppose $\alpha_1 \leq \cdots \leq \alpha_{n-1}$, otherwise, by induction, we can reduce to a smaller quiver.
By lemma 1.6.1, we have the generators $det\, V(a_i) = det\, V(\sigma(a_i))$ if $\alpha_i = \alpha_{i+1}$, $1 \leq i \leq n-2$.
By proposition A.2.8, we obtain that there exist $k_1,\dots,k_{n-1} \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda(a_i) = (\overbrace{k_i+\cdots+k_1,\dots,k_i+\cdots+k_1}^{\alpha_1},\dots,\overbrace{k_i,\dots,k_i}^{\alpha_i-\alpha_{i-1}}),$$

for every $i \in \{1,\dots,n-1\}$.
Now we consider the hypothesis on $\lambda(a_n)$ by which it must have even columns. If $\alpha_n = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_{n-1}$ for some $j \in \{1,\dots,n-1\}$ then $S_{\lambda(a_{n-1})}V_n^* = 0$ unless $k_{n-1}+\cdots+k_{j+1} = 0$, so $\lambda(a_{n-1}) = \cdots = \lambda(a_{j+1}) = \lambda(a_j)$. By proposition A.2.8, $(S_{\lambda(a_{n-1})}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)} = (S_{\lambda(a_j)}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)}$ contains a semi-invariant if and only if

$$\lambda(a_n) = (\overbrace{k_n+k_{j-1}+\cdots+k_1,\dots,k_n+k_{j-1}+\cdots+k_1}^{\alpha_1},\dots,\overbrace{k_n,\dots,k_n}^{\alpha_n-\alpha_{j-1}}),$$

but $\alpha_1, \alpha_2-\alpha_1,\dots,\alpha_n-\alpha_{j-1}$ have to be even and then $\alpha_1,\dots,\alpha_{j-1},\alpha_n$ have to be even. As before, by lemma 1.6.1, we can consider the smaller quiver $Q^1 : 1 \longrightarrow 2\cdots j \longrightarrow n \longrightarrow n+1 \longrightarrow 2n-j+1 \cdots 2n-1 \longrightarrow 2n$ and then

$$OSI(Q,\alpha) \cong (S_{\lambda(a_1)}V_1)^{SL(V_1)} \otimes \cdots \otimes (S_{\lambda(a_{j-1})}V_j^* \otimes S_{\lambda(a_j)}V_j)^{SL(V_j)} \otimes$$

$$(S_{\lambda(a_j)}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)}.$$

Now to complete the proof it's enough to find the generator of this algebra for $\alpha_n = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_{n-1}$.

(a) By proposition A.2.8, for every $l \in \{1, \ldots, j\}$ such that α_l is even, $(S_{\lambda(a_{l-1})}V_l^* \otimes S_{\lambda(a_l)}V_l)^{SL(V_l)}$ is generated by a semi-invariant of weight $(0, \ldots, 0, k_l, 0, \ldots, 0)$ where $k_l \in \mathbb{Z}_{\geq 0}$, is l-th component. Since $g^k \cdot pf(V(a_{2n-l}) \cdots V(a_l)) = \sqrt{det((g_{\sigma(l)}^{-1})^{\frac{k}{2}} V(a_{2n-l}) \cdots V(a_l)(g_l)^{\frac{k}{2}})} = (det\, g_l)^k pf(V(a_{2n-l}) \cdots V(a_l))$ for every $g = \{g_i\}_{i \in Q_0} \in GL(V)$, we note that $V(a_{2n-l}) \cdots V(a_l) \in OSI(Q, \alpha) \mapsto (pf(V(a_{2n-l}) \cdots V(a_l)))^{k_l}$ is a semi-invariant of weight $(0, \ldots, 0, k_l, 0, \ldots, 0)$, so it generates $(S_{\lambda(a_{l-1})}V_l^* \otimes S_{\lambda(a_l)}V_l)^{SL(V_l)}$. Since $\lambda(a_l) = \lambda(a_{l-1}) + (k_l^{\alpha_l})$, $pf(V(a_{2n-l}) \cdots V(a_l))$ is a generator of $OSI(Q, \alpha)$.

(b) In the summand of $OSI(Q, \alpha)$ indexed by the families of partitions in which $\lambda(a_j) = (\overbrace{k_j, \ldots, k_j}^{\alpha_j = \alpha_n})$, with $k_j \in \mathbb{Z}_{\geq 0}$, we have that $(S_{\lambda(a_j)}V_j)^{SL(V_j)} \otimes (S_{\lambda(a_j)}V_n^*)^{SL(V_n)}$ is generated by a semi-invariant of weight $(0, \ldots, 0, k_j, 0, \ldots, 0, -k_j)$ where k_j and $-k_j$ are respectively the j-th and the n-th component and we note, as before, that $(det(V(a_{n-1}) \cdots V(a_j)))^{k_j}$ is a semi-invariant of weight $(0, \ldots, 0, k_j, 0, \ldots, 0, -k_j)$. Since $\lambda(a_j) = \lambda(a_{j-1}) + (k_j^{\alpha_j = \alpha_n})$, $det(V(a_{n-1}) \cdots V(a_j))$ is a generator of $OSI(Q, \alpha)$;

(c) in the summand of $OSI(Q, \alpha)$ indexed by the families of partitions in which $\lambda(a_n) = (\overbrace{k_n, \ldots, k_n}^{\alpha_n})$ with $k_n \in \mathbb{Z}_{\geq 0}$, we note again that if α_n is even $(S_{\lambda(a_n)}V_n)^{SL(V_n)}$ is generated by $(pf(V(a_n)))^{k_n}$ of weight $(0, \ldots, 0, k_n)$. Since $\lambda(a_n) = \lambda(a_{j-1}) + (k_n^{\alpha_n})$, $pf(V(a_n))$ is a generator of $SpSI(Q, \alpha)$. □

Proof of theorem 2.1.3. By lemma 2.0.3, we can define pf^V if $V = \tau^- \nabla V$, since we are dealing with orthogonal case. Moreover we note that $V_{i,2n-i} = \tau^- \nabla V_{i,2n-i}$. Hence using the theorem 2.1.4, the proof is similar to the proof of theorem 2.1.1. □

2.1.3 The symplectic case for A_{2n+1}

We rewrite theorem 2.0.4 in the following way

Theorem 2.1.5. *Let (Q, σ) be an equioriented symmetric quiver of type A_{2n+1} and let α be the dimension vector of an symplectic representation of (Q, σ). Then $SpSI(Q, \alpha)$ is generated by the following indecomposable semi-invariants:*

(i) $c^{V_{j,i}}$ *of weight* $\langle \underline{dim} V_{j,i}, \cdot \rangle - \varepsilon_{n+1, v_{j,i}}$ *for every* $1 \leq j \leq i \leq n$ *such that* $\langle \underline{dim}\, V_{j,i}, \alpha \rangle = 0$, *where*

$$\varepsilon_{n+1, v_{j,i}}(h) = \begin{cases} \langle \underline{dim} V_{j,i}, \cdot \rangle (n+1) & \text{if} \quad h = n+1 \\ 0 & \text{otherwise}, \end{cases}$$

(ii) $pf^{V_{i,2n+1-i}}$ *of weight* $\frac{\langle \underline{dim} V_{i,2n+1-i}, \cdot \rangle}{2}$ *for every* $i \in \{1, \ldots, n\}$ *such that* α_i *is even.*

The result follows from the following statement

Theorem 2.1.6. *Let* (Q, σ) *be an equioriented symmetric quiver of type* A_{2n+1}*, where*

$$Q : 1 \xrightarrow{a_1} 2 \cdots n \xrightarrow{a_n} n+1 \xrightarrow{a_{n+1}} n+2 \cdots 2n \xrightarrow{a_{2n}} 2n+1,$$

$\sigma(i) = 2n - i + 2$ *and* $\sigma(a_i) = a_{2n-i+1}$ *for every* $i \in \{1, \ldots, n+1\}$ *and let* V *be an symplectic representation,* $\underline{dim}(V) = (\alpha_1, \ldots, \alpha_{n+1}) = \alpha$*.*
Then $SpSI(Q, \alpha)$ *is generated by the following indecomposable semi-invariants:*

(i) $det(V(a_i) \cdots V(a_j))$ *with* $j \leq i \in \{1, \ldots, n+1\}$ *if* $min(\alpha_{j+1}, \ldots, \alpha_i) > \alpha_j = \alpha_{i+1}$*;*

(ii) $pf(V(a_{2n-i+1}) \cdots V(a_i))$ *with* $i \in \{1, \ldots, n\}$ *if* $min(\alpha_{i+1}, \ldots, \alpha_{n+1}) > \alpha_i$ *and* α_i *is even.*

Proof. First we recall that if V is a symplectic representation of dimension $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ of a symmetric quiver of type A_{2n+1}, in the symplectic case, $V_{n+1} = V_{n+1}^*$ is a symplectic space, so if $V_{n+1} \neq 0$ then $dim\, V_{n+1}$ has to be even. We proceed by induction on n. For $n = 1$ we have the symplectic representation

$$V_1 \xrightarrow{V(a)} V_2 = V_2^* \xrightarrow{-V(a)^t} V_1^*.$$

By theorem A.1.9

$$SpSI(Q, \alpha) = \bigoplus_{\lambda \in \Lambda} (S_\lambda V_1)^{SL(V_1)} \otimes (S_\lambda V_2)^{Sp(V_2)}.$$

By proposition A.2.7 and proposition A.2.9, $SpSI(Q, \alpha) \neq 0$ if and only if $\lambda = (\overbrace{k, \ldots, k}^{\alpha_1})$, for some k, and $ht(\lambda)$ has to be even. If $\alpha_1 > \alpha_2$ then $S_\lambda V_2 = 0$ unless $\lambda = 0$ and in this case $SpSI(Q, \alpha) = \mathbb{K}$. If $\alpha_1 = \alpha_2$ then $ht(\lambda) = \alpha_1 = \alpha_2$. For every $(g_1, g_2) \in GL(V_1) \times Sp(V_2)$, $(g_1, g_2)^k \cdot detV(a) = det(g_1)^k det(g_2^{-1})^k detV(a) = det(g_1)^k detV(a)$, because $g_2 \in Sp(V_2)$ so we note that $detV(a)^k$ is a semi-invariant of weight $(k, 0)$. Hence $(S_\lambda V_1)^{SL(V_1)} \otimes (S_\lambda V_2)^{Sp(V_2)}$ is generated by the semi-invariant $detV(a)^k$, so $SpSI(Q, \alpha) = \mathbb{K}[detV(a)]$. Finally if $\alpha_1 < \alpha_2$ then $ht(\lambda) = \alpha_1$ has to be even. We recall that in the symplectic case $-V(a)^t V(a)$ is skew-symmetric. Since for every $(g_1, g_2) \in GL(V_1) \times Sp(V_2)$, $(g_1, g_2)^k \cdot pf(-V(a)^t V(a)) = (g_1, g_2)^k \cdot \sqrt{det(-V(a)^t V(a))} =$
$\sqrt{det((g_1^t)^{\frac{k}{2}}(-V(a)^t)(g_2)^{\frac{k}{2}}(g_2^{-1})^{\frac{k}{2}}(V(a))g_1^{\frac{k}{2}})} = (det\, g_1)^k pf(-V(a)^t V(a))$, we note that $pf(-V(a)^t V(a))^k$ is a semi-invariant of weight $(k, 0)$ so $(S_\lambda V_1)^{SL(V_1)} \otimes$

$(S_\lambda V_2)^{Sp(V_2)}$ is generated by the semi-invariant $pf(-V(a)^tV(a))^k$ if α_1 is even and thus $SpSI(Q,\alpha) = \mathbb{K}[pf(-V(a)^tV(a))]$.
Now we prove the induction step. Let $X = SpRep(Q,\alpha)$ and by theorem A.1.9 we obtain

$$SpSI(Q,\alpha) = \big(\mathbb{K}[X]\big)^{SSp(V)} =$$

$$\bigoplus_{\lambda(a_1),\dots,\lambda(a_n)\in\Lambda} (S_{\lambda(a_1)}V_1)^{SL(V_1)} \otimes (S_{\lambda(a_1)}V_2^* \otimes S_{\lambda(a_2)}V_2)^{SL(V_2)} \otimes$$

$$\cdots \otimes (S_{\lambda(a_{n-1})}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)} \otimes (S_{\lambda(a_n)}V_{n+1})^{Sp(V_{n+1})},$$

where $SSp(V) = SL(V_1) \times \cdots \times SL(V_n) \times Sp(V_{n+1})$.
The proof of this theorem is the same of the proof of the theorem 2.1.2 up to when we have to consider α_{n+1}. As in the proof of theorem 2.1.2 we can suppose $\alpha_1 \leq \cdots \leq \alpha_n$, otherwise, by induction, we can reduce to a smaller quiver.
By lemma 1.6.1, we have the generators $det\, V(a_i) = det\, V(\sigma(a_i))$ if $\alpha_i = \alpha_{i+1}$, $1 \leq i \leq n-1$.
By proposition A.2.8, we obtain that there exist $k_1,\dots,k_n \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda(a_i) = (\overbrace{k_i + \cdots + k_1, \dots, k_i + \cdots + k_1}^{\alpha_1}, \dots, \overbrace{k_i, \dots, k_i}^{\alpha_i - \alpha_{i-1}}),$$

for every $i \in \{1,\dots,n\}$.
Now, by proposition A.2.9, $\lambda(a_n)$ must have even columns. If $\alpha_{n+1} = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_n$ for some $j \in \{1,\dots,n\}$ then $S_{\lambda(a_n)}V_{n+1}^* = 0$ unless $k_n + \cdots + k_{j+1} = 0$, so $\lambda(a_n) = \cdots = \lambda(a_{j+1}) = \lambda(a_j)$. As before, by lemma 1.6.1, we can consider the smaller quiver $Q^1 : 1 \longrightarrow 2 \cdots j \longrightarrow n+1 \longrightarrow 2n-j+2 \cdots 2n \longrightarrow 2n+1$ and then

$$SpSI(Q,\alpha) \cong (S_{\lambda(a_1)}V_1)^{SL(V_1)} \otimes \cdots \otimes (S_{\lambda(a_{j-1})}V_j^* \otimes S_{\lambda(a_j)}V_j)^{SL(V_j)} \otimes$$

$$(S_{\lambda(a_{j-1})}V_n^* \otimes S_{\lambda(a_j)}V_n)^{SL(V_n)} \otimes (S_{\lambda(a_j)}V_{n+1})^{Sp(V_{n+1})}, \qquad (2.1)$$

where

$$\lambda(a_j) = (\overbrace{k_j + \cdots + k_1, \dots, k_j + \cdots + k_1}^{\alpha_1}, \dots, \overbrace{k_j, \dots, k_j}^{\alpha_{n+1} - \alpha_{j-1}}),$$

and $\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_{n+1} - \alpha_{j-1}$ have to be even otherwise, by proposition A.2.9, $(S_{\lambda(a_j)}V_{n+1})^{Sp(V_{n+1})} = 0$. Now to complete the proof it's enough to find the generators of the algebra (2.1) for $\alpha_{n+1} = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_n$.

(a) By proposition A.2.8, for every $l \in \{1,\dots,j\}$ such that α_l is even, $(S_{\lambda(a_{l-1})}V_l^* \otimes S_{\lambda(a_l)}V_l)^{SL(V_l)}$ is generated by a semi-invariant of weight

$(0,\dots,0,k_l,0,\dots,0)$ where $k_l \in \mathbb{Z}_{\geq 0}$, is l-th component. Since $g^k \cdot pf(V(a_{2n-l+1})\cdots V(a_l)) = \sqrt{det((g^{-1}_{\sigma(l)})^{\frac{k}{2}} V(a_{2n-l+1})\cdots V(a_l)(g_l)^{\frac{k}{2}})} =$

$(det\, g_l)^k pf(V(a_{2n-l+1})\cdots V(a_l))$ for every $g = \{g_i\}_{i\in Q_0} \in Sp(V)$, we note that $V(a_{2n-l+1})\cdots V(a_l) \in SpSI(Q,\alpha) \mapsto (pf(V(a_{2n-l+1})\cdots V(a_l)))^{k_l}$ is a semi-invariant of weight $(0,\dots,0,k_l,0,\dots,0)$, so it generates $(S_{\lambda(a_{l-1})}V_l^* \otimes S_{\lambda(a_l)}V_l)^{SL(V_l)}$. Since $\lambda(a_l) = \lambda(a_{l-1}) + (k_l)^{\alpha_l}$, then $pf(V(a_{2n-l+1})\cdots V(a_l))$ is a generator of $SpSI(Q,\alpha)$.

(b) In the summand of $SpSI(Q,\alpha)$ indexed by the families of partitions in which $\lambda(a_j) = (\overbrace{k_j,\dots,k_j}^{\alpha_j=\alpha_{n+1}})$, with $k_j \in \mathbb{Z}_{\geq 0}$, we have that $(S_{\lambda(a_j)}V_j)^{SL(V_j)} \otimes (S_{\lambda(a_j)}V_{n+1})^{Sp(V_{n+1})}$ is generated by a semi-invariant of weight $(0,\dots,0,k_j,0,\dots,0,0)$ where k_j is the j-th component and we note, as before, that $(det(V(a_n)\cdots V(a_j)))^{k_j}$ is a semi-invariant of weight $(0,\dots,0,k_j,0,\dots,0,0)$. Since $\lambda(a_j) = \lambda(a_{j-1}) + (k_j)^{\alpha_j=\alpha_{n+1}}$, $det(V(a_n)\cdots V(a_j))$ is a generator of $SpSI(Q,\alpha)$. □

Proof of theorem 2.1.5. By lemma 2.0.3, we can define pf^V if $V = \tau^-\nabla V$, since we are dealing with symplectic case. Moreover we note that $V_{i,2n+1-i} = \tau^-\nabla V_{i,2n+1-i}$, for every $i \in \{1,\dots,n\}$. Hence using the theorem 2.1.6, the proof is similar to the proof of theorem 2.1.1. □

2.1.4 The orthogonal case for A_{2n+1}

We rewrite the theorem 2.0.5 in the following way

Theorem 2.1.7. *Let (Q,σ) be an equioriented symmetric quiver of type A_{2n+1} and let α be the dimension vector of an orthogonal representation of (Q,σ). Then $OSI(Q,\alpha)$ is generated by the following indecomposable semi-invariants:*

(i) $c^{V_{j,i}}$ *of weight* $\langle \underline{dim} V_{j,i},\cdot\rangle - \varepsilon_{n+1,v_{j,i}}$ *for every* $1 \leq j \leq i \leq n$ *such that* $\langle \underline{dim}\, V_{j,i},\alpha\rangle = 0$, *where*
$$\varepsilon_{n+1,v_{j,i}}(h) = \begin{cases} \langle \underline{dim} V_{j,i},\cdot\rangle(n+1) & \text{if} \quad h = n+1 \\ 0 & \text{otherwise,} \end{cases}$$

(ii) $c^{V_{i,2n+1-i}}$ *of weight* $\langle \underline{dim} V_{i,2n+1-i},\cdot\rangle$ *for every* $i \in \{1,\dots,n\}$.

The result follows from the following statement

Theorem 2.1.8. *Let (Q,σ) be an equioriented symmetric quiver of type A_{2n+1}, where*
$$Q : 1 \xrightarrow{a_1} 2 \cdots n \xrightarrow{a_n} n+1 \xrightarrow{a_{n+1}} n+2 \cdots 2n \xrightarrow{a_{2n}} 2n+1,$$
$\sigma(i) = 2n-i+2$ and $\sigma(a_i) = a_{2n-i+1}$ for every $i \in \{1,\dots,n+1\}$ and let V be an orthogonal representation, $\underline{dim}(V) = (\alpha_1,\dots,\alpha_{n+1}) = \alpha$. Then $OSI(Q,\alpha)$ is generated by the following indecomposable semi-invariants:

(i) $det(V(a_i)\cdots V(a_j))$ *with* $j \leq i \in \{1,\ldots,n+1\}$ *if* $min(\alpha_{j+1},\ldots,\alpha_i) > \alpha_j = \alpha_{i+1}$*;*

(ii) $det(V(a_{2n-i+1})\cdots V(a_i))$ *with* $i \in \{1,\ldots,n\}$ *if* $min(\alpha_{i+1},\ldots,\alpha_{n+1}) > \alpha_i$*.*

Proof. First we recall that if V is a orthogonal representation of dimension $\alpha = (\alpha_1,\ldots,\alpha_{n+1})$ of a symmetric quiver of type A_{2n+1}, in the orthogonal case, $V_{n+1} = V_{n+1}^*$ is a orthogonal space. We proceed by induction on n. For $n = 1$ we have the orthogonal representation

$$V_1 \xrightarrow{V(a)} V_2 = V_2^* \xrightarrow{-V(a)^t} V_1^*$$

where V_1 is a vector space of dimension α_1, V_2 is a orthogonal space of dimension α_2 and $V(a)$ is a linear map. By theorem A.1.9

$$OSI(Q,\alpha) = \bigoplus_{\lambda\in\Lambda}(S_\lambda V_1)^{SL(V_1)} \otimes (S_\lambda V_2)^{SO(V_2)}.$$

By proposition A.2.7 and proposition A.2.9, $OSI(Q,\alpha) \neq 0$ if and only if $\lambda = (\overbrace{k,\ldots,k}^{\alpha_1})$, for some $k \in 2\mathbb{Z}$. If $\alpha_1 > \alpha_2$ then $S_\lambda V_2 = 0$ unless $\lambda = 0$ and in this case $OSI(Q,\alpha) = \mathbb{K}$. If $\alpha_1 = \alpha_2$ then $ht(\lambda) = \alpha_1 = \alpha_2$. For every $(g_1,g_2) \in GL(V_1)\times SO(V_2)$, $(g_1,g_2)^k \cdot detV(a) = det(g_1)^k det(g_2^{-1})^k detV(a) = det(g_1)^k detV(a)$, because $g_2 \in SO(V_2)$ so we note that $detV(a)^k$ is a semi-invariant of weight $(k,0)$. Hence $(S_\lambda V_1)^{SL(V_1)} \otimes (S_\lambda V_2)^{SO(V_2)}$ is generated by the semi-invariant $detV(a)^k$, so $OSI(Q,\alpha) = \mathbb{K}[detV(a)]$. Finally if $\alpha_1 < \alpha_2$ for every $(g_1,g_2) \in GL(V_1)\times SO(V_2)$, $(g_1,g_2)^k \cdot det(-V(a)^tV(a)) = (g_1,g_2)^k \cdot det(-V(a)^tV(a)) =$
$det((g_1^t)^k(-V(a)^t)(g_2)^k(g_2^{-1})^k(V(a))g_1^k) = (det\, g_1)^k det(-V(a)^tV(a))$, we note that $det(-V(a)^tV(a))^k$ is a semi-invariant of weight $(k,0)$ so $(S_\lambda V_1)^{SL(V_1)} \otimes (S_\lambda V_2)^{SO(V_2)}$ is generated by the semi-invariant $det(-V(a)^tV(a))^k$ and thus $OSI(Q,\alpha) = \mathbb{K}[det(-V(a)^tV(a))]$.
Now we prove the induction step. Let $X = ORep(Q,\alpha$ and by theorem A.1.9 we obtain

$$OSI(Q,\alpha) = \big(\mathbb{K}[X]\big)^{SO(V)} =$$

$$\bigoplus_{\lambda(a_1),\ldots,\lambda(a_n)\in\Lambda}(S_{\lambda(a_1)}V_1)^{SL(V_1)} \otimes (S_{\lambda(a_1)}V_2^* \otimes S_{\lambda(a_2)}V_2)^{SL(V_2)} \otimes$$

$$\cdots \otimes (S_{\lambda(a_{n-1})}V_n^* \otimes S_{\lambda(a_n)}V_n)^{SL(V_n)} \otimes (S_{\lambda(a_n)}V_{n+1})^{SO(V_{n+1})},$$

where $SO(V) = SL(V_1)\times\cdots\times SL(V_n)\times SO(V_{n+1})$.
The proof of this theorem is the same of the proof of the theorem 2.1.2 up to when we have to consider α_{n+1}. As in the proof of theorem 2.1.2 we can

suppose $\alpha_1 \leq \cdots \leq \alpha_n$, otherwise, by induction, we can reduce to a smaller quiver.
By lemma 1.6.1, we have the generators $det\, V(a_i) = det\, V(\sigma(a_i))$ if $\alpha_i = \alpha_{i+1}$, $1 \leq i \leq n-2$.
By proposition A.2.8, we obtain that there exist $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda(a_i) = (\overbrace{k_i + \cdots + k_1, \ldots, k_i + \cdots + k_1}^{\alpha_1}, \ldots, \overbrace{k_i, \ldots, k_i}^{\alpha_i - \alpha_{i-1}}),$$

for every $i \in \{1, \ldots, n\}$.
Now, by proposition A.2.9, $\lambda(a_n)$ must have even rows. If $\alpha_{n+1} = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_n$ for some $j \in \{1, \ldots, n\}$ then $S_{\lambda(a_n)} V_{n+1}^* = 0$ unless $k_n + \cdots + k_{j+1} = 0$, so $\lambda(a_n) = \cdots = \lambda(a_{j+1}) = \lambda(a_j)$. As before, by lemma 1.6.1, we can consider the smaller quiver $Q^1 : 1 \longrightarrow 2 \cdots j \longrightarrow n+1 \longrightarrow 2n-j+2 \cdots 2n \longrightarrow 2n+1$ and then

$$OSI(Q,\alpha) \cong (S_{\lambda(a_1)} V_1)^{SL(V_1)} \otimes \cdots \otimes (S_{\lambda(a_{j-1})} V_j^* \otimes S_{\lambda(a_j)} V_j)^{SL(V_j)} \otimes$$
$$(S_{\lambda(a_{j-1})} V_n^* \otimes S_{\lambda(a_j)} V_n)^{SL(V_n)} \otimes (S_{\lambda(a_j)} V_{n+1})^{SO(V_{n+1})}, \qquad (2.2)$$

where

$$\lambda(a_j) = (\overbrace{k_j + \cdots + k_1, \ldots, k_j + \cdots + k_1}^{\alpha_1}, \ldots, \overbrace{k_j, \ldots, k_j}^{\alpha_{n+1} - \alpha_{j-1}}),$$

and $k_j + \cdots + k_1, \ldots, k_j$ have to be even otherwise, by proposition A.2.9, $(S_{\lambda(a_j)} V_{n+1})^{SO(V_{n+1})} = 0$. Hence k_l has to be even for every $l \in \{1, \ldots, j\}$.
Now to complete the proof it's enough to find the generators of the algebra (2.2) for $\alpha_{n+1} = \alpha_j \leq \alpha_{j+1} \leq \cdots \leq \alpha_n$.

(a) By proposition A.2.8, for every $l \in \{1, \ldots, j\}$, $(S_{\lambda(a_{l-1})} V_l^* \otimes S_{\lambda(a_l)} V_l)^{SL(V_l)}$ is generated by a semi-invariant of weight $(0, \ldots, 0, k_l, 0, \ldots, 0)$ where $k_l \in 2\mathbb{Z}_{\geq 0}$, is l-th component. Since $g^k \cdot det(V(a_{2n-l+1}) \cdots V(a_l)) = det((g_{\sigma(l)}^{-1})^k V(a_{2n-l+1}) \cdots V(a_l)(g_l)^k) =$
$(det\, g_l)^k det(V(a_{2n-l+1}) \cdots V(a_l))$ for every $g = \{g_i\}_{i \in Q_0} \in SO(V)$, we note that

$$V(a_{2n-l+1}) \cdots V(a_l) \in OSI(Q,\alpha) \mapsto (det(V(a_{2n-l+1}) \cdots V(a_l)))^{\frac{k_l}{2}}$$

is a semi-invariant of weight $(0, \ldots, 0, k_l, 0, \ldots, 0)$, so it generates $(S_{\lambda(a_{l-1})} V_l^* \otimes S_{\lambda(a_l)} V_l)^{SL(V_l)}$. Since $\lambda(a_l) = \lambda(a_{l-1}) + (k_l)^{\alpha_l}$, then $det(V(a_{2n-l+1}) \cdots V(a_l))$ is a generator of $OSI(Q,\alpha)$.

(b) In the summand of $OSI(Q,\alpha)$ indexed by the families of partitions in which $\lambda(a_j) = (\overbrace{k_j, \ldots, k_j}^{\alpha_j = \alpha_{n+1}})$, with $k_j \in 2\mathbb{Z}_{\geq 0}$, we have that $(S_{\lambda(a_j)} V_j)^{SL(V_j)} \otimes$

$(S_{\lambda(a_j)} V_{n+1})^{SO(V_{n+1})}$ is generated by a semi-invariant of weight $(0, \ldots, 0, k_j, 0, \ldots, 0, 0)$ where k_j is the j-th component and we note, as before, that $(det(V(a_n) \cdots V(a_j)))^{k_j}$ is a semi-invariant of weight $(0, \ldots, 0, k_j, 0, \ldots, 0, 0)$. Since $\lambda(a_j) = \lambda(a_{j-1}) + (k_j)^{\alpha_j = \alpha_{n+1}}$, $det(V(a_n) \cdots V(a_j))$ is a generator of $OSI(Q, \alpha)$. □

Proof of theorem 2.1.7. Using the theorem 2.1.8, the proof is similar to the proof of theorem 2.1.1. □

Chapter 3

Semi-invariants of symmetric quivers of tame type

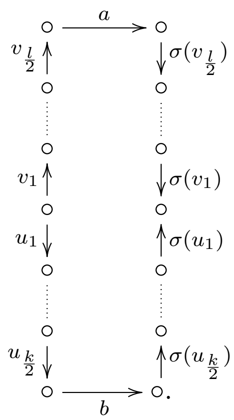
In this chapter we prove conjectures 1.2.1 and 1.2.2 for the symmetric quivers of tame type. We recall that the underlying quiver of a symmetric quiver of tame type is either $\widetilde{A}$ or $\widetilde{D}$ as in proposition 1.1.15. As done for the finite case we again reduce the proof to particular orientations (orientations in proposition 1.3.8 for $\widetilde{A}$ and orientation of $\widetilde{D}^{eq}$ for $\widetilde{D}$). In section 3.1, we prove the conjectures for dimension vector ph (for definition, see proposition B.2.2). In section 3.2, we treat the other regular dimension vectors.

3.1 Semi-invariants of symmetric quivers of tame type for dimension vector ph

In this section we deal with dimension vector ph. By lemma 1.3.14 and proposition 1.5.4 and 1.5.5, it's enough to consider particular orientations of symmetric quivers of type $\widetilde{A}$ in proposition 1.3.8 and orientation of symmetric quiver $\widetilde{D}^{eq}$. First we prove case by case some theorems by which conjectures 1.2.1 and 1.2.2 follow. Finally, in section 3.1.8, we conclude proofs of conjectures 1.2.1 and 1.2.2. We note that h is preserved under reflection functor.

3.1.1 $\widetilde{A}^{2,0,1}_{k,l}$ for dimension vector ph

Theorem 3.1.1. *Let (Q, σ) be a symmetric quiver of type $(2, 0, k, l)$ of orientation*



Then
Sp) $SpSI(Q, ph)$ is generated by the following indecomposable semi-invariants:

a) $det\,V(u_j)$ with $j \in \{1, \ldots, \frac{k}{2}\}$;

b) $det\,V(v_j)$ with $j \in \{1, \ldots, \frac{l}{2}\}$;

c) $det\,V(a)$ and $det\,V(b)$;

d) the coefficients c_i of $\varphi^{p-i}\psi^i$, $0 \le i \le p$, in $det(\psi V(\bar{a}) + \varphi V(\bar{b}))$, where $\bar{a} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1$ and $\bar{b} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1$.

O) $OSI(Q, ph)$ is generated by the following indecomposable semi-invariants:
if p is even,

a) $det\,V(u_j)$ with $j \in \{1, \ldots, \frac{k}{2}\}$;

b) $det\,V(v_j)$ with $j \in \{1, \ldots, \frac{l}{2}\}$;

c) $pf\,V(a)$ and $pf\,V(b)$;

d) the coefficients c_i of $\varphi^{p-2i}\psi^{2i}$, $0 \le i \le \frac{p}{2}$, in $pf(\psi V(\bar{a}) + \varphi V(\bar{b}))$, where $\bar{a} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1$ and $\bar{b} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1$;

if p is odd,

a) $det\,V(u_j)$ with $j \in \{1, \ldots, \frac{k}{2}\}$;

b) $det\,V(v_j)$ with $j \in \{1, \ldots, \frac{l}{2}\}$.

Proof. We proceed by induction on $\frac{k}{2} + \frac{l}{2}$. The smallest case is $\widetilde{A}_{0,0}^{2,0,1}$

$$1 \underset{b}{\overset{a}{\Longrightarrow}} \sigma(1).$$

The induction step follows by lemma 1.6.2, so it's enough to prove the theorem for $\widetilde{A}_{0,0}^{2,0,1}$.
Let V be a representation of $\widetilde{A}_{0,0}^{2,0,1}$ of dimension ph for some $p \in \mathbb{Z}_{\geq 0}$, in this case $h = 1$.
Sp) The ring of symplectic semi-invariants is

$$SpSI(\widetilde{A}_{0,0}^{2,0,1}, ph) = \bigoplus_{\lambda(a),\lambda(b) \in ER\Lambda} (S_{\lambda(a)}V \otimes S_{\lambda(b)}V)^{SLV}.$$

By proposition A.2.8 we have

$$\lambda(a)_j + \lambda(b)_{p+j-1} = t \tag{3.1}$$

for some $t \in \mathbb{Z}_{\geq 0}$ and for every $0 \leq j \leq p$.
We consider the summand in which $t = 2$ because the other ones are generated by products of powers of the generators of this summand. The solutions of (3.1) are $\lambda(a) = (2^i)$ and $\lambda(b) = (2^{p-i})$ for every $0 \leq i \leq p$. So the considered summand $\bigoplus_{i=0}^{p}(S_{(2^i)}V \otimes S_{(2^{p-i})}V)^{SLV}$ is generated by semi-invariants of weight 2, i.e. the coefficients c_i of $\varphi^{p-i}\psi^i$ in $det(\psi V(a) + \varphi V(b))$ (see [R2]). In particular we have $c_0 = det\, V(b)$ and $c_p = det\, V(a)$.
O) The ring of orthogonal semi-invariants is

$$OSI(\widetilde{A}_{0,0}^{2,0,1}, ph) = \bigoplus_{\lambda(a),\lambda(b) \in EC\Lambda} (S_{\lambda(a)}V \otimes S_{\lambda(b)}V)^{SLV}.$$

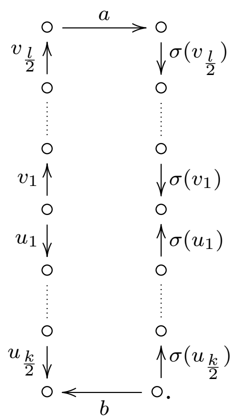
By proposition A.2.8 we have

$$\lambda(a)_j + \lambda(b)_{p+j-1} = t \tag{3.2}$$

for some $t \in \mathbb{Z}_{\geq 0}$ and for every $0 \leq j \leq p$.
We consider the summand in which $t = 1$ because the other ones are generated by products of powers of the generators of this summand. Let p be odd. $(\lambda(a)')_1$ and $(\lambda(b)')_p$ have to be even but $(\lambda(a)')_1 + (\lambda(b)')_p = p$ is odd, this is an absurd, and so $OSI(\widetilde{A}_{0,0}^{2,0,1}, ph) = \mathbb{K}$.
Let p be even. The solutions of (3.2) are $\lambda(a) = (1^{2i})$ and $\lambda(b) = (1^{p-2i})$ for every $0 \leq i \leq \frac{p}{2}$. So the considered summand $\bigoplus_{i=0}^{p}(S_{(1^{2i})}V \otimes S_{(1^{p-2i})}V)^{SLV}$ is generated by semi-invariants of weight 1, i.e. the coefficients c_i of $\varphi^{p-2i}\psi^{2i}$ in $pf(\psi V(a) + \varphi V(b))$. In particular we have $c_0 = pf\, V(b)$ and $c_{\frac{p}{2}} = pf\, V(a)$.
□

3.1.2 $\widetilde{A}^{2,0,2}_{k,l}$ for dimension vector ph

Theorem 3.1.2. *Let (Q,σ) be a symmetric quiver of type $(2,0,k,l)$ with orientation*



Then
Sp) $SpSI(Q,ph)$ is generated by the following indecomposable semi-invariants:

a) *$\det V(u_j)$ with $j \in \{1,\dots,\frac{k}{2}\}$;*

b) *$\det V(v_j)$ with $j \in \{1,\dots,\frac{l}{2}\}$;*

c) *$\det V(a)$ and $\det V(b)$;*

d) *the c_i coefficients of $\varphi^i\psi^i$, $0 \le i \le p$, in $\det \begin{pmatrix} \varphi V(\bar{a}) & V(c) \\ V(\sigma(c)) & \psi V(b) \end{pmatrix}$, where $\bar{a} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1$ and $c = u_{\frac{k}{2}}\cdots u_1$.*

O) $OSI(Q,ph)$ is generated by the following indecomposable semi-invariants:
if p is even,

a) *$\det V(u_j)$ with $j \in \{1,\dots,\frac{k}{2}\}$;*

b) *$\det V(v_j)$ with $j \in \{1,\dots,\frac{l}{2}\}$;*

c) *$pf\, V(a)$ and $pf\, V(b)$;*

d) *the coefficients c_i of $\varphi^i\psi^i$, $0 \le i \le \frac{p-1}{2}$, in $pf \begin{pmatrix} \varphi V(\bar{a}) & V(c) \\ V(\sigma(c)) & \psi V(b) \end{pmatrix}$, where $\bar{a} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1$ and $c = u_{\frac{k}{2}}\cdots u_1$.*

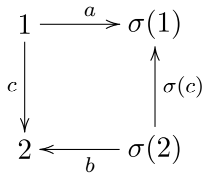
if p is odd,

a) *$\det V(u_j)$ with $j \in \{1,\dots,\frac{k}{2}\}$;*

b) *$\det V(v_j)$ with $j \in \{1,\dots,\frac{l}{2}\}$;*

c) the coefficients c_i of $\varphi^i\psi^i$, $0 \le i \le \frac{p-1}{2}$, in $pf\begin{pmatrix} \varphi V(\bar{a}) & V(c) \\ V(\sigma(c)) & \psi V(b)) \end{pmatrix}$, *where $\bar{a} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1$ and $c = u_{\frac{k}{2}}\cdots u_1$.*

Proof. We proceed by induction on $\frac{k}{2}+\frac{l}{2}$. The smallest case is $\widetilde{A}_{2,0}^{2,0,2}$



and so it's enough to study the semi-invariants of $\widetilde{A}_{2,0}^{2,0,2}$.
The induction step follows by lemma 1.6.2 and by lemma 1.6.1, so it's enough to prove the theorem for $\widetilde{A}_{2,0}^{2,0,2}$.
Sp) The ring of symplectic semi-invariants is

$$SpSI(\widetilde{A}_{2,0}^{2,0,2}, ph) = \bigoplus_{\substack{\lambda(a),\lambda(b)\in ER\Lambda \\ \lambda(c)\in\Lambda}} (S_{\lambda(a)}V_1 \otimes S_{\lambda(c)}V_1)^{SL\,V_1} \otimes (S_{\lambda(b)}V_2^* \otimes S_{\lambda(c)}V_2^*)^{SL\,V_2}.$$

By proposition A.2.8 we have

$$\begin{cases} \lambda(a)_j + \lambda(c)_{p+j-1} = k_1 \\ \lambda(b)_j + \lambda(c)_{p+j-1} = k_2 \end{cases} \tag{3.3}$$

for some $k_1, k_2 \in \mathbb{Z}_{\ge 0}$ and for every $0 \le j \le p$.
We consider the summands in which $k_1 = 0, 1, 2$ and $k_2 = 0, 1, 2$ because the other ones are generated by products of powers of the generators of this summands. If $k_1 = 2$ and $k_2 = 0$ we have $\lambda(b) = 0 = \lambda(c)$ and so the summand is $(S_{(2^p)}V_1)^{SL\,V_1}$ which is generated by a semi-invariant of weight $(2,0)$, i.e. $det\,V(a)$. If $k_1 = 0$ and $k_2 = 2$ as before we obtain the generator of ring of semi-invariant $det\,V(b)$ of weight $(0,-2)$. The summand in which $k_1 = 1$ and $k_2 = 0$ (respectively $k_1 = 0$ and $k_2 = 1$) doesn't exist because otherwise we have $\lambda(a)$ (respectively $\lambda(b)$) with odd columns. If $k_1 = 1 = k_2$ we have $\lambda(a) = 0 = \lambda(b)$ and $\lambda(c) = (1^p)$ and so the summand is $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2^*)^{SL\,V_2}$ which is generated by a semi-invariant of weight $(1,-1)$ which is $det\,V(c) = det\,V(\sigma(c))$. If $k_1 = 2 = k_2$, the solutions of (3.3) are $\lambda(a) = (2^i) = \lambda(b)$ and $\lambda(c) = (2^{p-i})$. The corresponding summand is $\bigoplus_{i=0}^{p}(S_{(2^i)}V_1 \otimes S_{(2^{p-i})}V_1)^{SL\,V_1} \otimes (S_{(2^i)}V_2^* \otimes S_{(2^{p-i})}V_2)^{SL\,V_2^*}$ and it is spanned by the coefficients of $\varphi^i\psi^i$ in

$$det\begin{pmatrix} \varphi V(a) & V(c) \\ V(\sigma(c)) & \psi V(b) \end{pmatrix},$$

semi-invariants of weight $(2,-2)$. In particular for $i = 0$ we have $(det\,V(c))^2$ and for $i = p$ we have $det\,V(a) \cdot det\,V(b)$.

O) The ring of orthogonal semi-invariants is

$$OSI(\widetilde{A}^{2,0,2}_{2,0}, ph) = \bigoplus_{\substack{\lambda(a),\lambda(b)\in EC\Lambda \\ \lambda(c)\in\Lambda}} (S_{\lambda(a)}V_1 \otimes S_{\lambda(c)}V_1)^{SL\,V_1} \otimes (S_{\lambda(b)}V_2^* \otimes S_{\lambda(c)}V_2^*)^{SL\,V_2}.$$

By proposition A.2.8 we have

$$\begin{cases} \lambda(a)_j + \lambda(c)_{p+j-1} = k_1 \\ \lambda(b)_j + \lambda(c)_{p+j-1} = k_2 \end{cases} \tag{3.4}$$

for some $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and for every $0 \leq j \leq p$.
We consider the summands in which $k_1 = 0, 1$ and $k_2 = 0, 1$ because the other ones are generated by the monomials of these. Let p be even. If $k_1 = 1$ and $k_2 = 0$ we have $\lambda(b) = 0 = \lambda(c)$ and so the summand is $(S_{(1^p)}V_1)^{SL\,V_1}$ which is generated by a semi-invariant of weight $(1, 0)$, i.e. $pf\,V(a)$. If $k_1 = 0$ and $k_2 = 1$ as before we obtain the generator of ring of semi-invariant $pf\,V(b)$ of weight $(0, -1)$. If $k_1 = 1 = k_2$, the solutions of (3.4) are $\lambda(a) = (1^{2i}) = \lambda(b)$ and $\lambda(c) = (1^{p-2i})$ with $0 \leq i \leq \frac{p}{2}$. So the summand is $\bigoplus_{i=0}^{\frac{p}{2}}(S_{(1^{2i})}V_1 \otimes S_{(1^{p-2i})}V_1)^{SL\,V_1} \otimes (S_{(1^{2i})}V_2^* \otimes S_{(1^{p-2i})}V_2^*)^{SL\,V_2}$ which is generated by the coefficients of $\varphi^i\psi^i$ in

$$pf \begin{pmatrix} \varphi V(a) & V(c) \\ V(\sigma(c)) & \psi V(b) \end{pmatrix},$$

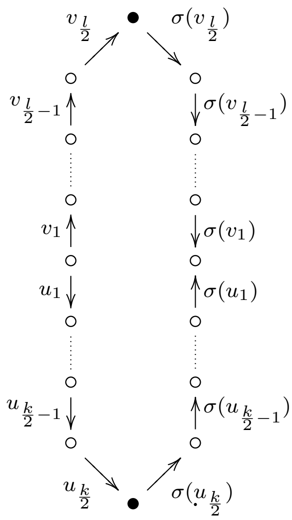
semi-invariants of weight $(1, -1)$. In particular for $i = 0$ we have $det\,V(c) = det\,V(\sigma(c))$ and for $i = p$ we have $pf\,V(a) \cdot pf\,V(b)$. Let p be odd. In this case the summand $(S_{(1^p)}V_1)^{SL\,V_1}$ (respectively $(S_{(1^p)}V_2)^{SL\,V_2}$) doesn't exist since $\lambda(a)$ (respectively $\lambda(b)$) must have even columns. If $k_1 = 1 = k_2$, the solutions of 3.4 are $\lambda(a) = (1^{2i}) = \lambda(b)$ and $\lambda(c) = (1^{p-2i})$ with $0 \leq i \leq \frac{p-1}{2}$. So the summand is $\bigoplus_{i=0}^{\frac{p-1}{2}}(S_{(1^{2i})}V_1 \otimes S_{(1^{p-2i})}V_1)^{SL\,V_1} \otimes (S_{(1^{2i})}V_2^* \otimes S_{(1^{p-2i})}V_2^*)^{SL\,V_2}$ which is generated by the coefficients of $\varphi^i\psi^i$ in

$$pf \begin{pmatrix} \varphi V(a) & V(c) \\ V(\sigma(c)) & \psi V(b) \end{pmatrix},$$

semi-invariants of weight $(1, -1)$. In particular for $i = 0$ we get $det\,V(c) = det\,V(\sigma(c))$. $\square$

3.1.3 $\widetilde{A}_{k,l}^{0,2}$ for dimension vector ph

Theorem 3.1.3. *Let (Q,σ) be a symmetric quiver of type $(0,2,k,l)$ with orientation*



Then

O) $OSI(Q,ph)$ is generated by the following indecomposable semi-invariants:

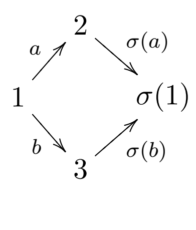
- *a) $\det V(u_j)$ with $j \in \{1,\dots,\frac{k}{2}\}$;*
- *b) $\det V(v_j)$ with $j \in \{1,\dots,\frac{l}{2}\}$;*
- *c) the coefficients c_i of $\varphi^{p-i}\psi^i$, $0 \le i \le p$, in $det(\psi V(\sigma(\bar{a})\bar{a}) + \varphi V(\sigma(\bar{b})\bar{b}))$, where $\bar{a} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1$ and $\bar{b} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})u_{\frac{k}{2}}\cdots u_1$.*

Sp) $SpSI(Q,ph)$ is generated by the following indecomposable semi-invariants: if p is even,

- *a) $\det V(u_j)$ with $j \in \{1,\dots,\frac{k}{2}\}$;*
- *b) $\det V(v_j)$ with $j \in \{1,\dots,\frac{l}{2}\}$;*
- *c) the coefficients c_i of $\varphi^{\frac{p}{2}-i}\psi^i$, $0 \le i \le \frac{p}{2}$, in $pf(\psi V(\sigma(\bar{a})\bar{a}) + \varphi V(\sigma(\bar{b})\bar{b}))$, where $\bar{a} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1$ and $\bar{b} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})u_{\frac{k}{2}}\cdots u_1$;*

if p is odd, $SpSI(Q,ph) = \mathbb{K}$.

Proof. We proceed by induction on $\frac{k}{2}+\frac{l}{2}$. The smallest case is $\widetilde{A}_{2,2}^{0,2}$



and so it's enough to study the semi-invariants of $\widetilde{A}_{2,2}^{0,2}$.
The induction step follows by lemma 1.6.1, so it's enough to prove the theorem for $\widetilde{A}_{2,2}^{0,2}$.
O) The ring of orthogonal semi-invariants is

$$\bigoplus_{\lambda(a),\lambda(b)\in\Lambda} (S_{\lambda(a)}V_1 \otimes S_{\lambda(b)}V_1)^{SL\,V_1} \otimes (S_{\lambda(a)}V_2)^{SO\,V_2} \otimes (S_{\lambda(b)}V_3)^{SO\,V_3}.$$

By proposition A.2.8 we have

$$\lambda(a)_j + \lambda(b)_{p-j+1} = k_1 \tag{3.5}$$

for every $0 \leq j \leq p$ and for some $k_1 \in \mathbb{Z}_{\geq 0}$. By proposition A.2.9 we have $\lambda(a) = 2\mu + (l^p)$ and $\lambda(b) = 2\nu + (m^p)$ for some $\mu, \nu \in \Lambda$ and for some $l, m \in \mathbb{Z}_{\geq 0}$. We consider the summands in which $k_1 = 1, 2$ because the other ones are generated by products of powers of the generators of this summands. If $k_1 = 1$ the only solutions of (3.5) are $\lambda(a) = (1^p)$, $\lambda(b) = 0$ and $\lambda(a) = 0$, $\lambda(b) = (1^p)$. Respectively, the summand $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2)^{SO\,V_2}$ is generated by a semi-invariant of weight $(1, 0, 0)$, i.e $det\,V(a) = det\,V(\sigma(a))$, and the summand $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_3)^{SO\,V_3}$ is generated by a semi-invariant of weight $(1, 0, 0)$, i.e $det\,V(b) = det\,V(\sigma(b))$. If $k_1 = 2$, the solutions of (3.5) are $\lambda(a) = (2^i)$, $\lambda(b) = (2^{p-i})$ with $0 \leq i \leq p$. So the summand is

$$\bigoplus_{i=0}^{p}(S_{(2^i)}V_1 \otimes S_{(2^{p-i})}V_1)^{SL\,V_1} \otimes (S_{(2^i)}V_2)^{SO\,V_2} \otimes (S_{(2^{p-i})}V_3)^{SO\,V_3}$$

which is generated by the coefficients of $\varphi^{p-i}\psi^i$ in $det(\psi V(\sigma(a)a) + \varphi V(\sigma(b)b))$, semi-invariants of weight $(2, 0, 0)$. In particular for $i = 0$ we have $det\,V(\sigma(b)b)$ and for $i = p$ we have $det\,V(\sigma(a)a)$.
Sp) The ring of symplectic semi-invariants is

$$\bigoplus_{\lambda(a),\lambda(b)\in\Lambda} (S_{\lambda(a)}V_1 \otimes S_{\lambda(b)}V_1)^{SL\,V_1} \otimes (S_{\lambda(a)}V_2)^{Sp\,V_2} \otimes (S_{\lambda(b)}V_3)^{Sp\,V_3}.$$

By proposition A.2.8 we have

$$\lambda(a)_j + \lambda(b)_{p-j+1} = k_1 \tag{3.6}$$

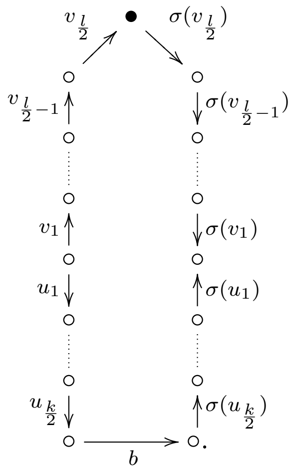
for every $0 \leq j \leq p$ and for some $k_1 \in \mathbb{Z}_{\geq 0}$. By proposition A.2.9 $\lambda(a)$ and $\lambda(b)$ have to be in $EC\Lambda$.
Let p be even. We consider the summands in which $k_1 = 1$ because the other ones are generated by products of powers of the generators of this summands. The solutions of (3.6) are $\lambda(a) = (1^{2i})$, $\lambda(b) = (1^{p-2i})$ with $0 \leq i \leq \frac{p}{2}$. So the summand is

$$\bigoplus_{i=0}^{\frac{p}{2}}(S_{(1^{2i})}V_1 \otimes S_{(1^{p-2i})}V_1)^{SL\,V_1} \otimes (S_{(1^{2i})}V_2)^{SO\,V_2} \otimes (S_{(1^{p-2i})}V_3)^{SO\,V_3}$$

which is generated by the coefficients of $\varphi^{\frac{p}{2}-i}\psi^i$ in $pf(\psi V(\sigma(a)a)+\varphi V(\sigma(b)b))$, semi-invariants of weight $(1,0,0)$. In particular for $i=0$ we have $pf\,V(\sigma(b)b) = \sqrt{det\,V(\sigma(b)b)} = \sqrt{det\,V(\sigma(b)) \cdot det\,V(b)} = \sqrt{(det\,V(b))^2} = det\,V(b)$ and for $i=\frac{p}{2}$ we have $pf\,V(\sigma(a)a) = det\,V(a)$.
If p is odd there not exist any non-trivial symplectic representations because a symplectic space of dimension odd doesn't exist. So we have $SpSI(Q,ph) = \mathbb{K}$. □

3.1.4 $\widetilde{A}^{1,1}_{k,l}$ for dimension vector ph

Theorem 3.1.4. *Let (Q,σ) be a symmetric quiver of type $(1,1,k,l)$ with orientation*



Then
O) $OSI(Q,ph)$ is generated by the following indecomposable semi-invariants:
if p is even,

a) $det\,V(u_j)$ *with* $j \in \{1,\dots,\frac{k}{2}\}$;

b) $det\,V(v_j)$ *with* $j \in \{1,\dots,\frac{l}{2}\}$;

c) $pf\,V(b)$

d) *the coefficients c_i of $\varphi^{p-2i}\psi^{2i}$, $0 \le i \le \frac{p}{2}$, in $det(\psi V(\sigma(\bar{a})\bar{a}) + \varphi V(\bar{b}))$, where $\bar{a} = v_{\sigma(1)}\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1$ and $\bar{b} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1$;*

if p is odd,

a) $det\,V(u_j)$ *with* $j \in \{1,\dots,\frac{k}{2}\}$;

b) $det\,V(v_j)$ *with* $j \in \{1,\dots,\frac{l}{2}\}$;

c) *the coefficients c_i of $\varphi^{p-2i}\psi^{2i}$, $0 \le i \le \frac{p-1}{2}$, in $det(\psi V(\sigma(\bar{a})\bar{a}) + \varphi V(\bar{b}))$, where $\bar{a} = v_{\sigma(1)}\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1$ and $\bar{b} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1$.*

Sp) $SpSI(Q, ph)$ is generated by the following indecomposable semi-invariants: if p is even,

a) $det\, V(u_j)$ with $j \in \{1, \ldots, \frac{k}{2}\}$;

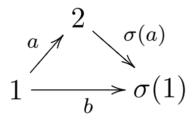
b) $det\, V(v_j)$ with $j \in \{1, \ldots, \frac{l}{2}\}$;

c) $det\, V(b)$

d) the coefficients c_i of $\varphi^{p-2i}\psi^{2i}$, $0 \leq i \leq \frac{p}{2}$, in $det(\psi V(\sigma(\bar{a})\bar{a}) + \varphi V(\bar{b}))$, where $\bar{a} = v_{\sigma(1)} \cdots \sigma(v_{\frac{l}{2}})v_{\frac{l}{2}} \cdots v_1$ and $\bar{b} = \sigma(u_1) \cdots \sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}} \cdots u_1$;

if p is odd, $SpSI(Q, ph) = \mathbb{K}$.

Proof. We proceed by induction on $\frac{k}{2} + \frac{l}{2}$. The smallest case is $\widetilde{A}^{1,1}_{0,2}$



and so it's enough to study the semi-invariants of $\widetilde{A}^{1,1}_{0,2}$.
The induction step follows by lemma 1.6.2 and by lemma 1.6.1, so it's enough to prove the theorem for $\widetilde{A}^{1,1}_{0,2}$.
O) The ring of orthogonal semi-invariants is

$$\bigoplus_{\substack{\lambda(a)\in\Lambda \\ \lambda(b)\in EC\Lambda}} (S_{\lambda(a)}V_1 \otimes S_{\lambda(b)}V_1)^{SL\,V_1} \otimes (S_{\lambda(a)}V_2)^{SO\,V_2}.$$

By proposition A.2.8 we have

$$\lambda(a)_j + \lambda(b)_{p-j+1} = k_1 \tag{3.7}$$

for every $0 \leq j \leq p$ and for some $k_1 \in \mathbb{Z}_{\geq 0}$. By proposition A.2.9 we have $\lambda(a) = 2\mu + (l^p)$ for some $\mu \in \Lambda$ and for some $l \in \mathbb{Z}_{\geq 0}$. We consider the summands in which $k_1 = 1, 2$ because the other ones are generated by products of powers of the generators of this summands. Let p be even. If $k_1 = 1$ the only solutions of (3.7) are $\lambda(a) = (1^p)$, $\lambda(b) = 0$ and $\lambda(a) = 0$, $\lambda(b) = (1^p)$. Respectively, the summand $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2)^{SO\,V_2}$ is generated by a semi-invariant of weight $(1, 0)$, i.e $det\, V(a) = det\, V(\sigma(a))$, and the summand $(S_{(1^p)}V_1)^{SL\,V_1}$ is generated by a semi-invariant of weight $(1, 0)$, i.e $pf\, V(b)$. If $k_1 = 2$, the solutions of (3.7) are $\lambda(a) = (2^{2i})$, $\lambda(b) = (2^{p-2i})$ with $0 \leq i \leq \frac{p}{2}$. So the summand is

$$\bigoplus_{i=0}^{\frac{p}{2}} (S_{(2^{2i})}V_1 \otimes S_{(2^{p-2i})}V_1)^{SL\,V_1} \otimes (S_{(2^{2i})}V_2)^{SO\,V_2}$$

which is generated by the coefficients of $\varphi^{p-2i}\psi^{2i}$ in $det(\psi V(\sigma(a)a)+\varphi V(b))$, semi-invariants of weight $(2,0)$. In particular for $i=0$ we have $det\, V(b)$ and for $i=\frac{p}{2}$ we have $det\, V(\sigma(a)a)$.
Let p be odd. If $k_1 = 1$ the only solutions of (3.7) are $\lambda(a) = (1^p)$, $\lambda(b) = 0$. The summand $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2)^{SO\,V_2}$ is generated by a semi-invariant of weight $(1,0)$, i.e $det\, V(a) = det\, V(\sigma(a))$. If $k_1 = 2$, the solutions of (3.7) are $\lambda(b) = (2^{2i})$, $\lambda(a) = (2^{p-2i})$ with $0 \le i \le \frac{p-1}{2}$. So the summand is

$$\bigoplus_{i=0}^{\frac{p-1}{2}} (S_{(2^{p-2i})}V_1 \otimes S_{(2^{2i})}V_1)^{SL\,V_1} \otimes (S_{(2^{p-2i})}V_2)^{SO\,V_2}$$

which is generated by the coefficients of $\varphi^{2i}\psi^{p-2i}$ in $det(\psi V(\sigma(a)a)+\varphi V(b))$, semi-invariants of weight $(2,0)$. In particular for $i=\frac{p-1}{2}$ we have $det\, V(\sigma(a)a)$.
Sp) The ring of symplectic semi-invariants is

$$\bigoplus_{\substack{\lambda(a)\in\Lambda \\ \lambda(b)\in ER\Lambda}} (S_{\lambda(a)}V_1 \otimes S_{\lambda(b)}V_1)^{SL\,V_1} \otimes (S_{\lambda(a)}V_2)^{Sp\,V_2}.$$

By proposition A.2.8 we have

$$\lambda(a)_j + \lambda(b)_{p-j+1} = k_1 \tag{3.8}$$

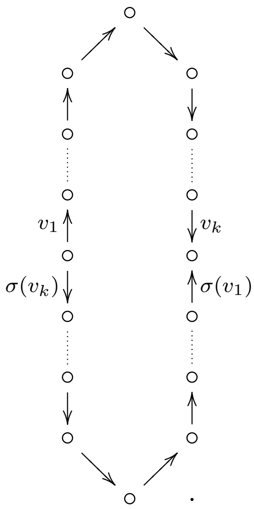
for every $0 \le j \le p$ and for some $k_1 \in \mathbb{Z}_{\ge 0}$. By proposition A.2.9 we have $\lambda(a) \in EC\Lambda$. We consider the summands in which $k_1 = 1, 2$ because the other ones are generated by products of powers of the generators of this summands. Let p be even. If $k_1 = 1$ the only solutions of (3.8) are $\lambda(a) = (1^p)$, $\lambda(b) = 0$. The summand $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2)^{Sp\,V_2}$ is generated by a semi-invariant of weight $(1,0)$, i.e $det\, V(a) = det\, V(\sigma(a)) = pf\, V(\sigma(a)a)$. If $k_1 = 2$, the solutions of (3.8) are $\lambda(a) = (2^{2i})$, $\lambda(b) = (2^{p-2i})$ with $0 \le i \le \frac{p}{2}$. So the summand is

$$\bigoplus_{i=0}^{\frac{p}{2}} (S_{(2^{2i})}V_1 \otimes S_{(2^{p-2i})}V_1)^{SL\,V_1} \otimes (S_{(2^{2i})}V_2)^{Sp\,V_2}$$

which is generated by the coefficients of $\varphi^{p-2i}\psi^{2i}$ in $det(\psi V(\sigma(a)a)+\varphi V(b))$, semi-invariants of weight $(2,0)$. In particular for $i=0$ we have $det\, V(b)$ and for $i=\frac{p}{2}$ we have $det\, V(\sigma(a)a)$.
If p is odd there not exist any non-trivial symplectic representations because a symplectic space of dimension odd doesn't exist. So we have $SpSI(Q, ph) = \mathbb{K}$. □

3.1.5 $\widetilde{A}^{0,0}_{k,k}$ for dimension vector ph

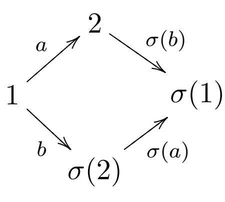
Theorem 3.1.5. *Let (Q,σ) be a symmetric quiver of type $(0,0,k,k)$ with orientation*



Then
$OSI(Q,ph) = SpSI(Q,ph)$ is generated by the following indecomposable semi-invariants:

- *a) $det\,V(v_j)$ with $j \in \{1,\dots,k\}$;*
- *b) $pf(V(\bar{a}) + V(\sigma(\bar{a})))$*
- *c) the coefficients c_i of $\varphi^{p-i}\psi^i$, $0 \le i \le p$, in $det(\psi V(\bar{a}) + \varphi V(\sigma(\bar{a})))$, where $\bar{a} = v_k \cdots v_1$.*

Proof. We proceed by induction on $\frac{k}{2} + \frac{h}{2}$. The smallest case is $\widetilde{A}^{0,0}_{2,2}$



and so it's enough to study the semi-invariants of $\widetilde{A}^{0,0}_{2,2}$.
The induction step follows by lemma 1.6.1, so it's enough to prove the theorem for $\widetilde{A}^{0,0}_{2,2}$.
In this case we have $ORep(Q,ph) = SpRep(Q,ph)$ and so $OSI(Q,ph) = SpSI(Q,ph)$. The ring of semi-invariants is

$$\bigoplus_{\lambda(a),\lambda(b)\in\Lambda} (S_{\lambda(a)}V_1 \otimes S_{\lambda(b)}V_1)^{SL\,V_1} \otimes (S_{\lambda(a)}V_2^* \otimes S_{\lambda(b)}V_2)^{SL\,V_2}.$$

By proposition A.2.8 we have

$$\begin{cases} \lambda(a)_j + \lambda(b)_{p-j+1} = k_1 \\ \lambda(a)_j = \lambda(b)_j + k_2 \end{cases} \tag{3.9}$$

for every $0 \leq j \leq p$ and for some $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. We consider the summands in which $k_1 = 1, 2$ and $k_2 = 0, 1$ because the other ones are generated by products of powers of the generators of this summands. Let p even. If $k_1 = 1$ and $k_2 = 0$ the only solution of (3.9) are $\lambda(a) = (1^{\frac{p}{2}})$, $\lambda(b) = (1^{\frac{p}{2}})$. The summand $(S_{(1^{\frac{p}{2}})}V_1 \otimes S_{(1^{\frac{p}{2}})}V_1)^{SL\,V_1} \otimes (S_{(1^{\frac{p}{2}})}V_2^* \otimes S_{(1^{\frac{p}{2}})}V_2)^{SL\,V_2}$ is generated by a semi-invariant of weight $(1, 0)$, i.e. $pf(V(\sigma(b)a) + V(\sigma(a)b))$. If $k_1 = 2$ and $k_2 = 0$, the solutions of (3.9) are $\lambda(a) = (2^i, 1^{p-2i})$, $\lambda(b) = (2^i, 1^{p-2i})$ with $0 \leq i \leq \frac{p}{2}$. So the summand is

$$\bigoplus_{i=0}^{\frac{p}{2}} (S_{(2^i,1^{p-2i})}V_1 \otimes S_{(2^i,1^{p-2i})}V_1)^{SL\,V_1} \otimes (S_{(2^i,1^{p-2i})}V_2^* \otimes S_{(2^i,1^{p-2i})}V_2)^{SL\,V_2}$$

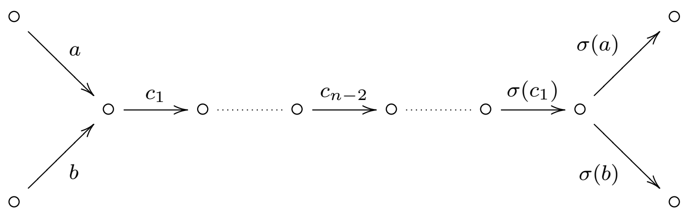
which is generated by the coefficients of $\varphi^{p-i}\psi^i$ with $0 \leq i \leq \frac{p}{2}$ in $det(\psi V(\sigma(b)a) + \varphi V(\sigma(a)b))$, semi-invariants of weight $(2, 0)$. In particular for $i = 0$ we have $det\,V(\sigma(b)a) = det\,V(\sigma(a)b)$. Let p be odd. If $k_1 = 1$ and $k_2 = 0$ we don't have any solutions of (3.9). If $k_1 = 2$ and $k_2 = 0$, the solutions of (3.9) are $\lambda(a) = (2^i, 1^{p-2i})$, $\lambda(b) = (2^i, 1^{p-2i})$ with $0 \leq i \leq \frac{p-1}{2}$. So the summand is

$$\bigoplus_{i=0}^{\frac{p-1}{2}} (S_{(2^i,1^{p-2i})}V_1 \otimes S_{(2^i,1^{p-2i})}V_1)^{SL\,V_1} \otimes (S_{(2^i,1^{p-2i})}V_2^* \otimes S_{(2^i,1^{p-2i})}V_2)^{SL\,V_2}$$

which is generated by the coefficients of $\varphi^{p-i}\psi^i$ with $0 \leq i \leq \frac{p-1}{2}$ in $det(\psi V(\sigma(b)a) + \varphi V(\sigma(a)b))$, semi-invariants of weight $(2, 0)$. In particular for $i = 0$ we have $det\,V(\sigma(b)a) = det\,V(\sigma(a)b)$.
If $k_2 = 1$, in both cases p even or odd, k_1 can't be 0 otherwise we have $\lambda(b)_j + \lambda(b)_{p-j+1} = -1$ but this is impossible. So $k_1 = 1$ and the only solutions of (3.9) are $\lambda(a) = (1^p)$, $\lambda(b) = 0$ and $\lambda(a) = 0$, $\lambda(b) = (1^p)$; respectively we have the summand $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2^*)^{SL\,V_2}$ generated by the semi-invariant $det\,V(a)$ of weight $(1, -1)$ and the summand $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2)^{SL\,V_2}$ generated by the semi-invariant $det\,V(b)$ of weight $(1, -1)$. □

3.1.6 $\widetilde{D}_n^{1,0}$ for dimension vector ph

Theorem 3.1.6. *Let (Q,σ) be a symmetric quiver of type $\widetilde{D}_n^{1,0}$ with orientation*



and let $\bar{c} = \sigma(c_1)\cdots c_{n-2}\cdots c_1$. Then
Sp) $SpSI(Q, ph)$ is generated by the following indecomposable semi-invariants:

a) $det\, V(c_j)$ *with* $j \in \{1, \dots, n-2\}$

b) $det\,(V(a), V(b)) = det \begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$

c) $det\, V(\sigma(a)\bar{c}a)$

d) $det\, V(\sigma(b)\bar{c}b)$

e) $det\, V(\sigma(b)\bar{c}a) = det\, V(\sigma(a)\bar{c}b)$

f) *the coefficients* c_i *of* $\varphi^i\psi^i$, $0 \le i \le p$, *in*

$$det \begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix}.$$

O) $OSI(Q, ph)$ is generated by the following indecomposable semi-invariants:
if p is even,

a) $det\, V(c_j)$ *with* $j \in \{1, \dots, n-2\}$;

b) $det\,(V(a), V(b)) = det \begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$

c) $pf\, V(\sigma(a)\bar{c}a)$

d) $pf\, V(\sigma(b)\bar{c}b)$

e) *the coefficients* c_i *of* $\varphi^i\psi^i$, $0 \le i \le \frac{p}{2}$, *in*

$$pf \begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & V(\sigma(b)\bar{c}b) \end{pmatrix};$$

if p is odd,

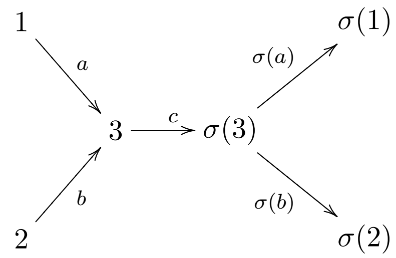
a) det $V(c_j)$ *with* $j \in \{1, \dots, n-2\}$

b) det $(V(a), V(b)) = det \begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$

c) the coefficients c_i *of* $\varphi^i \psi^i$, $0 \le i \le \frac{p-1}{2}$, *in*

$$pf \begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix}.$$

Proof. We proceed by induction on n. The smallest case is $(\widetilde{D}_3^{1,0})^{eq}$



The induction step follows by lemma 1.6.2, so it's enough to prove the theorem for $(\widetilde{D}_3^{1,0})^{eq}$.
Let V be a representation of $(\widetilde{D}_3^{1,0})^{eq}$ of dimension ph for some $p \in \mathbb{Z}_{\ge 0}$, in this case $h = (1,1,2)$.
Sp) The ring of symplectic semi-invariants is

$$SpSI(\widetilde{D}_3^{1,0}, ph) = \bigoplus_{\substack{\lambda(a),\lambda(b)\in\Lambda \\ \lambda(c)\in ER\Lambda}} (S_{\lambda(a)}V_1)^{SL\,V_1} \otimes S_{\lambda(b)}V_2)^{SL\,V_2} \otimes$$

$$(S_{\lambda(a)}V_3^* \otimes S_{\lambda(b)}V_3^* \otimes S_{\lambda(c)}V_3)^{SL\,V_3}.$$

By proposition A.2.7 we have $\lambda(a) = (k_1^p)$, $\lambda(b) = (k_2^p)$, for some $k_1, k_2 \in \mathbb{Z}_{\ge 0}$, and by proposition A.1.12 we have

$$S_{(k_1^p)}V_3^* \otimes S_{(k_2^p)}V_3^* = \bigoplus_{i=0}^{p} S_{\nu_i}V_3^* \tag{3.10}$$

where

$$v_i = (k_1 + \lambda_1, \dots, k_1 + \lambda_{p-i}, \underbrace{k_1, \dots, k_1}_{i}, \underbrace{k_2, \dots, k_2}_{i}, k_2 - \lambda_{p-i}, \dots, k_2 - \lambda_1)$$

with $0 \le \lambda_{p-i} \le \dots \le \lambda_1 \le k_2$ and for every $0 \le i \le p$. Moreover we have

$$(S_{\nu_i}V_3^* \otimes S_{\lambda(c)}V_3)^{SL\,V_3} \ne 0 \Leftrightarrow \lambda(c) = v_i + (k_3^{2p})$$

for some $k_3 \in \mathbb{Z}_{\geq 0}$.
We consider the summands in which $k_1 = 0, 1, 2$ and $k_2 = 0, 1, 2$ because the other ones are generated by products of powers of the generators of these summands.
If $\lambda(c) = 0$, then $\lambda(a) = (k_1^p) \neq 0 \neq \lambda(b) = (k_2^p)$ because otherwise if for example $\lambda(a) = 0$ we have $(S_{(k_2^p)} V_3^*)^{SL\,V_3} = 0$. We consider the summand in which $\lambda(c) = 0$ and $k_1 = 1 = k_2$, the only ν_i such that $(S_{\nu_i} V_3^*)^{SL\,V_3} \neq 0$ is $\nu_p = (1^{2p})$. So $(S_{(1^p)} V_1)^{SL\,V_1} \otimes (S_{(1^p)} V_2)^{SL\,V_2} \otimes (S_{(1^{2p})} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(1, 1, -1)$, i.e. $det\,(V(a), V(b)) = det \begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$.
Now we suppose $\lambda(c) \neq 0$. We can't consider $k_1 = 1$, $k_2 = 0$ and $k_1 = 0$, $k_2 = 1$ because otherwise we haven't $\lambda(c)$ with even rows. If $k_1 = 2$, $k_2 = 0$ and $k_3 = 0$ the summand $(S_{(2^p)} V_1)^{SL\,V_1} \otimes (S_{(2^p)} V_3^* \otimes S_{(2^p)} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(2, 0, 0)$, i.e. $det\, V(\sigma(a)\bar{c}a)$. If $k_1 = 0$, $k_2 = 2$ and $k_3 = 0$ the summand $(S_{(2^p)} V_2)^{SL\,V_2} \otimes (S_{(2^p)} V_3^* \otimes S_{(2^p)} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(0, 2, 0)$, i.e. $det\, V(\sigma(b)\bar{c}b)$. If $k_1 = 0 = k_2$, then k_3 has to be even. So, considering $k_3 = 2$, $(S_{(2^{2p})} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(0, 0, 2)$, i.e. $det\, V(c)$. If $k_1 = k_2 = 1$, by (3.10), $\lambda(c) = (2^p)$. So $(S_{(1^p)} V_1)^{SL\,V_1} \otimes (S_{(1^p)} V_2)^{SL\,V_2} \otimes (S_{(2^p)} V_3^* \otimes S_{(2^p)} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(1, 1, 0)$, i.e. $det\, V(\sigma(b)\bar{c}a) = det\, V(\sigma(a)\bar{c}b)$. Finally if $k_1 = k_2 = 2$, considering $k_3 = 0$, the summand is

$$(S_{(2^p)} V_1)^{SL\,V_1} \otimes (S_{(2^p)} V_2)^{SL\,V_2} \otimes (\bigoplus_{i=0}^{p} S_{(4^{p-2i}, 2^{4i})} V_3^* \otimes S_{(4^{p-2i}, 2^{4i})} V_3)^{SL\,V_3}$$

which is generated by the coefficients of $\varphi^i \psi^i$ in

$$det \begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix},$$

semi-invariants of weight $(2, 2, 0)$. In particular for $i = 0$ we have $(det\, V(\sigma(b)\bar{c}a))^2$ and for $i = p$ we have $det\, V(\sigma(a)\bar{c}a) \cdot det\, V(\sigma(b)\bar{c}b)$.
O) The ring of orthogonal semi-invariants is

$$SpSI(\widetilde{D}_3^{1,0}, ph) = \bigoplus_{\substack{\lambda(a),\lambda(b)\in\Lambda \\ \lambda(c)\in EC\Lambda}} (S_{\lambda(a)} V_1)^{SL\,V_1} \otimes S_{\lambda(b)} V_2)^{SL\,V_2} \otimes$$

$$(S_{\lambda(a)} V_3^* \otimes S_{\lambda(b)} V_3^* \otimes S_{\lambda(c)} V_3)^{SL\,V_3}.$$

By proposition A.2.7 we have $\lambda(a) = (k_1^p)$, $\lambda(b) = (k_2^p)$, for some $k_1, k_2 \in \mathbb{Z}_{\geq 0}$, and by proposition A.1.12 we have

$$S_{(k_1^p)} V_3^* \otimes S_{(k_2^p)} V_3^* = \bigoplus_{i=0}^{p} S_{\nu_i} V_3^* \qquad (3.11)$$

where

$$v_i = (k_1 + \lambda_1, \ldots, k_1 + \lambda_{p-i}, \underbrace{k_1, \ldots, k_1}_{i}, \underbrace{k_2, \ldots, k_2}_{i}, k_2 - \lambda_{p-i}, \ldots, k_2 - \lambda_1)$$

with $0 \leq \lambda_{p-i} \leq \ldots \leq \lambda_1 \leq k_2$ and for every $0 \leq i \leq p$. Moreover we have

$$(S_{\nu_i} V_3^* \otimes S_{\lambda(c)} V_3)^{SL\,V_3} \neq 0 \Leftrightarrow \lambda(c) = v_i + (k_3^{2p})$$

for some $k_3 \in \mathbb{Z}_{\geq 0}$. Since $\lambda(c) \in EC\Lambda$, also $\nu_i \in EC\Lambda$ for every i.
We consider the summands in which $k_1 = 0, 1$ and $k_2 = 0, 1$ because the other ones are generated by products of powers of the generators of these summands.
As before if $\lambda(c) = 0$, the only ν_i such that $(S_{\nu_i} V_3^*)^{SL\,V_3} \neq 0$ is $\nu_p = (1^{2p})$. So $(S_{(1^p)} V_1)^{SL\,V_1} \otimes (S_{(1^p)} V_2)^{SL\,V_2} \otimes (S_{(1^{2p})} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(1, 1, -1)$, i.e. $det\,(V(a), V(b)) = det \begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$. Now we suppose $\lambda(c) \neq 0$.
Let p be even. If $k_1 = 1$, $k_2 = 0$ and $k_3 = 0$ the summand $(S_{(1^p)} V_1)^{SL\,V_1} \otimes (S_{(1^p)} V_3^* \otimes S_{(1^p)} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(1, 0, 0)$, i.e. $pf\, V(\sigma(a)ca)$. If $k_1 = 0$, $k_2 = 1$ and $k_3 = 0$ the summand $(S_{(1^p)} V_2)^{SL\,V_2} \otimes (S_{(1^p)} V_3^* \otimes S_{(1^p)} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(0, 1, 0)$, i.e. $pf\, V(\sigma(b)cb)$. If $k_1 = 0 = k_2$, then k_3 has to be not zero. So, considering $k_3 = 1$, $(S_{(1^{2p})} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(0, 0, 1)$, i.e. $pf\, V(c)$. Finally if $k_1 = k_2 = 1$, considering $k_3 = 0$, the summand is

$$(S_{(1^p)} V_1)^{SL\,V_1} \otimes (S_{(1^p)} V_2)^{SL\,V_2} \otimes (\bigoplus_{i=0}^{\frac{p}{2}} S_{(2^{p-2i}, 1^{4i})} V_3^* \otimes S_{(2^{p-2i}, 1^{4i})} V_3)^{SL\,V_3}$$

which is generated by the coefficients of $\varphi^i \psi^i$ in

$$pf \begin{pmatrix} \varphi V(\sigma(a)ca) & V(\sigma(b)ca) \\ V(\sigma(a)cb) & \psi V(\sigma(b)cb) \end{pmatrix},$$

semi-invariants of weight $(1, 1, 0)$. In particular for $i = 0$ we have $det\, V(\sigma(b)ca) = det\, V(\sigma(a)cb)$ and for $i = \frac{p}{2}$ we have $pf\, V(\sigma(a)ca) \cdot pf\, V(\sigma(b)cb)$.
Let p be odd. In this case we can't consider $k_1 = 1$, $k_2 = 0$, $k_3 = 0$ and $k_1 = 0$, $k_2 = 1$, $k_3 = 0$ because otherwise we have $\lambda(c) = (1^p)$ with p odd but $\lambda(c)$ has to be in $EC\Lambda$. As before, if $k_1 = 0 = k_2$, then k_3 has to be not zero. So, considering $k_3 = 1$, $(S_{(1^{2p})} V_3)^{SL\,V_3}$ is generated by a semi-invariant of weight $(0, 0, 1)$, i.e. $pf\, V(c)$. Finally if $k_1 = k_2 = 1$, considering $k_3 = 0$, the summand is

$$(S_{(1^p)} V_1)^{SL\,V_1} \otimes (S_{(1^p)} V_2)^{SL\,V_2} \otimes (\bigoplus_{i=0}^{\frac{p-1}{2}} S_{(2^{p-(2i+1)}, 1^{4i+2})} V_3^* \otimes S_{(2^{p-(2i+1)}, 1^{4i+2})} V_3)^{SL\,V_3}$$

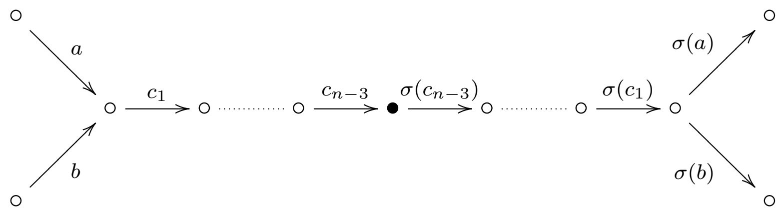
which is generated by the coefficients of $\varphi^i\psi^i$ in

$$pf\begin{pmatrix} \varphi V(\sigma(a)ca) & V(\sigma(b)ca) \\ V(\sigma(a)cb) & \psi V(\sigma(b)cb) \end{pmatrix},$$

semi-invariants of weight $(1,1,0)$. In particular for $i=0$ we have $det\, V(\sigma(b)ca) = det\, V(\sigma(a)cb)$. $\square$

3.1.7 $\widetilde{D}_n^{0,1}$ for dimension vector ph

Theorem 3.1.7. *Let (Q,σ) be a symmetric quiver of type $\widetilde{D}_n^{0,1}$ with orientation*



and let $\bar{c} = \sigma(c_1)\cdots\sigma(c_{n-3})c_{n-3}\cdots c_1$. Then
O) $OSI(Q,ph)$ is generated by the following indecomposable semi-invariants:

a) $det\, V(c_j)$ with $j \in \{1,\ldots,n-3\}$

b) $det\,(V(a),V(b)) = det\begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$

c) $det\, V(\sigma(a)\bar{c}a)$

d) $det\, V(\sigma(b)\bar{c}b)$

e) $det\, V(\sigma(b)\bar{c}a) = det\, V(\sigma(a)\bar{c}b)$

f) the coefficients c_i of $\varphi^i\psi^i$, $0 \le i \le p$, in

$$det\begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix}.$$

Sp) $SpSI(Q,ph)$ is generated by the following indecomposable semi-invariants:
if p is even,

a) $det\, V(c_j)$ with $j \in \{1,\ldots,n-2\}$;

b) $det\,(V(a),V(b)) = det\begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$

c) $pf\, V(\sigma(a)\bar{c}a)$

d) $pf\,V(\sigma(b)\bar{c}b)$

e) $det\,V(\sigma(b)\bar{c}a) = det\,V(\sigma(a)\bar{c}b)$

f) *the coefficients* c_i *of* $\varphi^i\psi^i$, $0 \leq i \leq \frac{p}{2}$, *in*

$$pf\begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix};$$

if p *is odd,*

a) $det\,V(c_j)$ *with* $j \in \{1, \ldots, n-2\}$

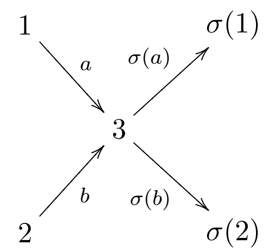
b) $det\,(V(a), V(b)) = det\begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$

c) $det\,V(\sigma(b)\bar{c}a) = det\,V(\sigma(a)\bar{c}b)$

d) *the coefficients* c_i *of* $\varphi^i\psi^i$, $0 \leq i \leq \frac{p-1}{2}$, *in*

$$pf\begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix}.$$

Proof. We proceed by induction on n. The smallest case is $(\widetilde{D}_3^{0,1})^{eq}$



The induction step follows by lemma 1.6.1, so it's enough to prove the theorem for $(\widetilde{D}_3^{0,1})^{eq}$.
Let V be a representation of $(\widetilde{D}_3^{0,1})^{eq}$ of dimension ph for some $p \in \mathbb{Z}_{\geq 0}$, in this case $h = (1, 1, 2)$.
O) The ring of orthogonal semi-invariants is

$$OSI(\widetilde{D}_3^{0,1}, ph) = \bigoplus_{\lambda(a),\lambda(b)\in\Lambda} (S_{\lambda(a)}V_1)^{SL\,V_1} \otimes (S_{\lambda(b)}V_2)^{SL\,V_2} \otimes (S_{\lambda(a)}V_3^* \otimes S_{\lambda(b)}V_3^*)^{SO\,V_3}.$$

By proposition A.2.7 we have $\lambda(a) = (k_1^p)$, $\lambda(b) = (k_2^p)$, for some $k_1, k_2 \in \mathbb{Z}_{\geq 0}$, and by proposition A.1.12 we have

$$S_{(k_1^p)}V_3^* \otimes S_{(k_2^p)}V_3^* = \bigoplus_{i=0}^{p} S_{\nu_i}V_3^* \qquad (3.12)$$

where

$$v_i = (k_1+\lambda_1, \dots, k_1+\lambda_{p-i}, \underbrace{k_1,\dots,k_1}_{i}, \underbrace{k_2,\dots,k_2}_{i}, k_2-\lambda_{p-i},\dots,k_2-\lambda_1)$$

with $0 \le \lambda_{p-i} \le \ldots \le \lambda_1 \le k_2$ and for every $0 \le i \le p$. Moreover we have

$$(S_{\nu_i}V_3^*)^{SO\,V_3} \neq 0 \Leftrightarrow v_i = 2\mu_i + (k_3^{2p}) \tag{3.13}$$

for some $k_3 \in \mathbb{Z}_{\geq 0}$ and for some $\mu_i \in \Lambda$.
We consider the summands in which $k_1 = 0,1,2$ and $k_2 = 0,1,2$ because the other ones are generated by products of powers of the generators of these summands.
We can't consider $k_1 = 1$, $k_2 = 0$ and $k_1 = 0$, $k_2 = 1$ because otherwise we haven't $v_i = 2\mu_i + (k_3^{2p})$. If $k_1 = 2$, $k_2 = 0$ the summand $(S_{(2^p)}V_1)^{SL\,V_1} \otimes (S_{(2^p)}V_3^*)^{SO\,V_3}$ is generated by a semi-invariant of weight $(2,0,0)$, i.e. $det\,V(\sigma(a)a)$. If $k_1 = 0$, $k_2 = 2$ the summand $(S_{(2^p)}V_2)^{SL\,V_2} \otimes (S_{(2^p)}V_3^*)^{SO\,V_3}$ is generated by a semi-invariant of weight $(0,2,0)$, i.e. $det\,V(\sigma(b)b)$. If $k_1 = k_2 = 1$, by (3.12) and by (3.13), $\nu_i = (2^p)$ or $\nu_i = (1^{2p})$. So we have $(S_{(1^p)}V_3^* \otimes S_{(1^p)}V_3^*)^{SO\,V_3} = (S_{(2^p)}V_3^* \oplus S_{(1^{2p})}V_3^*)^{SO\,V_3}$. Now

$$(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2)^{SL\,V_2} \otimes (S_{(1^{2p})}V_3^*)^{SO\,V_3}$$

is generated by a semi-invariant of weight $(1,1,0)$, i.e. $det\,(V(a),V(b)) = det\begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix}$ and $(S_{(1^p)}V_1)^{SL\,V_1} \otimes (S_{(1^p)}V_2)^{SL\,V_2} \otimes (S_{(2^p)}V_3^*)^{SO\,V_3}$ is generated by a semi-invariant of weight $(1,1,0)$, i.e. $det\,V(\sigma(b)a) = det\,V(\sigma(a)b)$.
Finally if $k_1 = k_2 = 2$ the summand is

$$(S_{(2^p)}V_1)^{SL\,V_1} \otimes (S_{(2^p)}V_2)^{SL\,V_2} \otimes (\bigoplus_{i=0}^{p} S_{(4^{p-2i},2^{4i})}V_3^*)^{SO\,V_3}$$

which is generated by the coefficients of $\varphi^i\psi^i$ in

$$det\begin{pmatrix} \varphi V(\sigma(a)a) & V(\sigma(b)a) \\ V(\sigma(a)b) & \psi V(\sigma(b)b) \end{pmatrix},$$

semi-invariants of weight $(2,2,0)$. In particular for $i = 0$ we have $(det\,V(\sigma(b)a))^2$ and for $i = p$ we have $det\,V(\sigma(a)a) \cdot det\,V(\sigma(b)b)$.
Sp) The ring of symplectic semi-invariants is

$$SpSI(\widetilde{D}_3^{0,1}, ph) = \bigoplus_{\lambda(a),\lambda(b)\in\Lambda} (S_{\lambda(a)}V_1)^{SL\,V_1} \otimes (S_{\lambda(b)}V_2)^{SL\,V_2} \otimes (S_{\lambda(a)}V_3^* \otimes S_{\lambda(b)}V_3^*)^{Sp\,V_3}.$$

By proposition A.2.7 we have $\lambda(a) = (k_1^p)$, $\lambda(b) = (k_2^p)$, for some $k_1, k_2 \in \mathbb{Z}_{\geq 0}$, and by proposition A.1.12 we have

$$S_{(k_1^p)}V_3^* \otimes S_{(k_2^p)}V_3^* = \bigoplus_{i=0}^{p} S_{\nu_i}V_3^* \tag{3.14}$$

where

$$v_i = (k_1 + \lambda_1, \ldots, k_1 + \lambda_{p-i}, \underbrace{k_1, \ldots, k_1}_{i}, \underbrace{k_2, \ldots, k_2}_{i}, k_2 - \lambda_{p-i}, \ldots, k_2 - \lambda_1)$$

with $0 \leq \lambda_{p-i} \leq \ldots \leq \lambda_1 \leq k_2$ and for every $0 \leq i \leq p$. Moreover we have

$$(S_{\nu_i} V_3^*)^{Sp\, V_3} \neq 0 \Leftrightarrow v_i \in EC\Lambda. \tag{3.15}$$

We consider the summands in which $k_1 = 0, 1$ and $k_2 = 0, 1$ because the other ones are generated by products of powers of the generators of these summands.
Let p be even. If $k_1 = 1$, $k_2 = 0$ the summand $(S_{(1^p)} V_1)^{SL\, V_1} \otimes (S_{(1^p)} V_3^*)^{Sp\, V_3}$ is generated by a semi-invariant of weight $(1, 0, 0)$, i.e. $pf\, V(\sigma(a)a)$. If $k_1 = 0$, $k_2 = 1$ the summand $(S_{(1^p)} V_2)^{SL\, V_2} \otimes (S_{(1^p)} V_3^*)^{Sp\, V_3}$ is generated by a semi-invariant of weight $(0, 1, 0)$, i.e. $pf\, V(\sigma(b)b)$. Finally if $k_1 = k_2 = 1$, the summand is

$$(S_{(1^p)} V_1)^{SL\, V_1} \otimes (S_{(1^p)} V_2)^{SL\, V_2} \otimes (\bigoplus_{i=0}^{\frac{p}{2}} S_{(2^{p-2i}, 1^{4i})} V_3^*)^{Sp\, V_3}$$

which is generated by the coefficients of $\varphi^i \psi^i$ in

$$pf \begin{pmatrix} \varphi V(\sigma(a)a) & V(\sigma(b)a) \\ V(\sigma(a)b) & \psi V(\sigma(b)b) \end{pmatrix},$$

semi-invariants of weight $(1, 1, 0)$. In particular for $i = 0$ we have $det\, V(\sigma(b)a) = det\, V(\sigma(a)b)$ and for $i = \frac{p}{2}$ we have $pf\, V(\sigma(a)a) \cdot pf\, V(\sigma(b)b)$.
Let p be odd. In this case we can't consider $k_1 = 1$, $k_2 = 0$ and $k_1 = 0$, $k_2 = 1$ because otherwise, by (3.15), $(S_{(1^p)} V_3)^{Sp\, V_3} = 0$. Finally if $k_1 = k_2 = 1$, the summand is

$$(S_{(1^p)} V_1)^{SL\, V_1} \otimes (S_{(1^p)} V_2)^{SL\, V_2} \otimes (\bigoplus_{i=0}^{\frac{p-1}{2}} S_{(2^{p-(2i+1)}, 1^{4i+2})} V_3^*)^{SL\, V_3}$$

which is generated by the coefficients of $\varphi^i \psi^i$ in

$$pf \begin{pmatrix} \varphi V(\sigma(a)a) & V(\sigma(b)a) \\ V(\sigma(a)b) & \psi V(\sigma(b)b) \end{pmatrix},$$

semi-invariants of weight $(1, 1, 0)$. In particular for $i = 0$ we have $det\, V(\sigma(b)a) = det\, V(\sigma(a)b)$. $\square$

3.1.8 End of the proof of conjecture 1.2.1 and 1.2.2 for dimension vector ph

First of all we note that, by definition of c^W and pf^W, when we have it, are not zero if $0 = \langle \underline{dim}\, W, ph\rangle = p\langle \underline{dim}\, W, h\rangle = -p\langle h, \underline{dim}\, W\rangle$, so we have to consider only regular representations W. Moreover it is enough to consider only simple regular representations W, because the other regular representations are extensions of simple regular ones and so, by lemma B.4.7, we obtain the c^W and pf^W with non-simple regular W as products of those with simple regular W. Now we check only for $\widetilde{A}^{2,0,1}_{k,l}$ and $\widetilde{D}^{1,0}_n$ that the generators found for $SpSI(Q, ph)$ and $OSI(Q, d)$ are of type c^W, for some simple regular W, and pf^W, for some simple regular W satisfying property *(Op)* in symplectic case and *(Spp)* in orthogonal case (see lemma 1.4.6). For the other types of quivers it is similar (see also [D, section 4.1]).
We use notation of section B.2. For $\widetilde{A}^{2,0,1}_{k,l}$, by definition of c^W and pf^W,

- **Sp)** if V is a symplectic representation, we have $c^{E_0}(V) = det(V(v_{\frac{l}{2}})) = det(V(v_1)) = c^{E_1}(V)$, $c^{E_i}(V) = det(V(v_i)) = det(V(v_{\sigma(i)})) = c^{E_{\sigma(i)}}(V)$ for every $i \in \{2, \dots, l\} \setminus \{\frac{l}{2}+1\}$, $c^{E_{\frac{l}{2}+1}}(V) = det(V(a))$, $c^{E'_0}(V) = det(V(u_{\frac{k}{2}})) = det(V(u_1)) = c^{E'_1}(V)$, $c^{E'_i}(V) = det(V(u_i)) = det(V(u_{\sigma(i)})) = c^{E'_{\sigma(i)}}(V)$ for every $i \in \{2, \dots, k\} \setminus \{\frac{k}{2}+1\}$, $c^{E'_{\frac{k}{2}+1}}(V) = det(V(b))$ and $c_{V_{(\varphi,\psi)}}(V) = det(\psi V(a) + \varphi V(b)$;
- **O)** if V is an orthogonal representation, the only differences with the symplectic case are, when p is even, we have $pf^{E_{\frac{l}{2}+1}}(V) = pf(V(a))$, $pf^{E'_{\frac{k}{2}+1}}(V) = pf(V(b))$ and $pf^{V_{(\varphi,\psi)}}(V) = pf(\psi V(a) + \varphi V(b)$, in fact $E_{\frac{l}{2}+1}$, $E'_{\frac{k}{2}+1}$ and $V_{(\varphi,\psi)}$ satisfy property *(Spp)*.

For $\widetilde{D}^{1,0}_n$, by definition of c^W and pf^W,

- **Sp)** if V is a symplectic representation, we have $c^{E_0}(V) = det\begin{pmatrix} V(\sigma(a)) \\ V(\sigma(b)) \end{pmatrix} = det(V(a), V(b)) = c^{E_1}(V)$, $c^{E_i}(V) = det(V(c_{i-1})) = det(V(c_{\sigma(i-1)})) = c^{E_{\sigma(i)}}(V)$ for every $i \in \{2, \dots, 2n-3\}$, $c^{E'_0}(V) = det(V(\sigma(b)\bar{c}a)) = det(V(\sigma(a)\bar{c}b)) = c^{E'_1}(V)$, $c^{E''_0}(V) = det(V(\sigma(a)\bar{c}a))$, $c^{E''_1}(V) = det(V(\sigma(b)\bar{c}b))$ and
$$c^{V_{(\varphi,\psi)}}(V) = det\begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix};$$
- **O)** if V is an orthogonal representation, the only differences with the symplectic case are that we have
$$pf^{V_{(\varphi,\psi)}}(V) = pf\begin{pmatrix} \varphi V(\sigma(a)\bar{c}a) & V(\sigma(b)\bar{c}a) \\ V(\sigma(a)\bar{c}b) & \psi V(\sigma(b)\bar{c}b) \end{pmatrix},$$

since $V_{(\varphi,\psi)}$ satisfies property *(Spp)* and $c^{E'_0}(V) = det(V(\sigma(b)\bar{c}a)) = det(V(\sigma(a)\bar{c}b)) = c^{E'_1}(V)$ is the coefficient of $\varphi^0\psi^0 = 1$ in $pf^{V_{(\varphi,\psi)}}(V)$; moreover, if p is even, we have $pf^{E_{n-1}}(V) = pf(V(c_{n-2}))$, $pf^{E''_0}(V) = pf(V(\sigma(a)\bar{c}a))$ and $pf^{E''_1}(V) = pf(V(\sigma(b)\bar{c}b))$, because E_{n-1}, E''_0 and E''_1 satisfy property *(Spp)*.

3.2 Semi-invariants of symmetric quivers of tame type for any regular dimension vector

In this section we prove theorems 1.2.1 and 1.2.2 for symmetric quiver of tame type and any regular symmetric dimension vector d.
We will use the same notation of section 3.1. For the type $\widetilde{A}$ we call $a_0 = tv_1 = tu_1$, $x_i = hv_i$ for every $i \in \{1, \dots, \frac{l}{2}\}$ and $y_i = hv_i$ for every $i \in \{1, \dots, \frac{k}{2}\}$. For the type $\widetilde{D}$ we call $t_1 = ta$, $t_2 = tb$ and $z_i = tc_i$ for every i such that $c_i \in (Q_1^+ \sqcup Q_1^\sigma) \setminus \{a, b\}$.
First we consider the canonical decomposition of d for the symmetric quivers.
Let (Q, σ) be a symmetric quiver of tame type and let $\Delta = \{e_i | \, i \in I = \{0, \dots, u\}\}$, $\Delta' = \{e'_i | \, i \in I' = \{0, \dots, v\}\}$ and $\Delta'' = \{e''_i | \, i \in I'' = \{0, \dots, w\}\}$ be the three τ^+-orbits of nonhomogeneous simple regular representations of the underlying quiver Q (see proposition B.2.7).
We shall call $I_\delta = \{i \in I | \, e_i = \delta e_i\}$ (respectively I'_δ and I''_δ).

Lemma 3.2.1. *Let $[x] := max\{z \in \mathbb{N} | \, z \leq x\}$ is the floor of $x \in \mathbb{R}$.*
(1) For $\widetilde{A}^{2,0,1}_{k,l}$, we have:

(1.1) decomposition $I = I_+ \sqcup I_\delta \sqcup I_-$ where $I_+ = \{2, \dots, \frac{l}{2} + 1\}$, $I_\delta = \{1\}$ and $I_- = I \setminus (I_+ \sqcup I_\delta)$;

(1.2) decomposition $I' = I'_+ \sqcup I'_\delta \sqcup I'_-$ where $I'_+ = \{2, \dots, \frac{k}{2} + 1\}$, $I'_\delta = \{1\}$ and $I'_- = I' \setminus (I'_+ \sqcup I'_\delta)$;

(1.3) $I'' = \emptyset$.

(2) For $\widetilde{A}^{2,0,2}_{k,l}$, we have:

(2.1) decomposition $I = I_+ \sqcup I_\delta \sqcup I_-$ where $I_+ = \{2, \dots, [\frac{l+1}{2}] + 2\}$, $I_\delta = \emptyset$ and $I_- = I \setminus I_+$;

(2.2) decomposition $I' = I'_+ \sqcup I'_\delta \sqcup I'_-$ where $I'_+ = \{2, \dots, [\frac{k-1}{2}] + 1\}$, $I'_\delta = \{1, [\frac{k-1}{2}] + 2\}$ and $I'_- = I' \setminus (I'_+ \sqcup I'_\delta)$;

(2.3) $I'' = \emptyset$.

(3) For $\widetilde{A}^{0,2}_{k,l}$, we have:

(3.1) decomposition $I = I_+ \sqcup I_\delta \sqcup I_-$ *where* $I_+ = \{2, \ldots, [\frac{l-1}{2}] + 1\}$, $I_\delta = \{1, [\frac{l-1}{2}] + 2\}$ *and* $I_- = I \setminus (I_+ \sqcup I_\delta)$;

(3.2) decomposition $I' = I'_+ \sqcup I'_\delta \sqcup I'_-$ *where* $I'_+ = \{2, \ldots, [\frac{k-1}{2}] + 1\}$, $I'_\delta = \{1, [\frac{k-1}{2}] + 2\}$ *and* $I'_- = I' \setminus (I'_+ \sqcup I'_\delta)$;

(3.3) $I'' = \emptyset$.

(4) For $\widetilde{A}^{1,1}_{k,l}$, *we have:*

(4.1) decomposition $I = I_+ \sqcup I_\delta \sqcup I_-$ *where* $I_+ = \{2, \ldots, [\frac{l-1}{2}] + 1\}$, $I_\delta = \{1, [\frac{l-1}{2}] + 2\}$ *and* $I_- = I \setminus (I_+ \sqcup I_\delta)$;

(4.2) decomposition $I' = I'_+ \sqcup I'_\delta \sqcup I'_-$ *where* $I'_+ = \{2, \ldots, \frac{k}{2} + 1\}$, $I'_\delta = \{1\}$ *and* $I'_- = I' \setminus (I'_+ \sqcup I'_\delta)$;

(4.3) $I'' = \emptyset$.

(5) For $\widetilde{A}^{0,0}_{k,k'}$ *we have:*

(5.1) $\Delta = \delta\Delta'$ *and so* $I = I'$;

(5.2) $I'' = \emptyset$.

(6) For $(\widetilde{D}^{1,0}_n)^{eq}$, *we have:*

(6.1) decomposition $I = I_+ \sqcup I_\delta \sqcup I_-$ *where* $I_+ = \{2, \ldots, [\frac{2n-4}{2}] + 1\}$, $I_\delta = \{1\}$ *and* $I_- = I \setminus (I_+ \sqcup I_\delta)$;

(6.2) $I' = I'_\delta = \{0, 1\}$ *and* $I'_- = I'_+ = \emptyset$;

(6.3) decomposition $I'' = I''_+ \sqcup I''_-$ *where* $I''_+ = \{0\}$, $I''_\delta = \emptyset$ *and* $I''_- = I'' \setminus I''_+$.

(7) For $(\widetilde{D}^{0,1}_n)^{eq}$, *we have:*

(7.1) decomposition $I = I_+ \sqcup I_\delta \sqcup I_-$ *where* $I_+ = \{2, \ldots, [\frac{2n-5}{2}] + 1\}$, $I_\delta = \{1, [\frac{2n-5}{2}] + 2\}$ *and* $I_- = I \setminus (I_+ \sqcup I_\delta)$;

(7.2) $I' = I'_\delta = \{0, 1\}$ *and* $I'_- = I'_+ = \emptyset$;

(7.3) decomposition $I'' = I''_+ \sqcup I''_-$ *where* $I''_+ = \{0\}$, $I''_\delta = \emptyset$ *and* $I''_- = I'' \setminus I''_+$.

Proof. We prove *(1)*, *(2)*, *(3)*, *(4)*, *(6)* and *(7)*. By [DR, section 6, page 40] and by [DR, section 6, pages 40 and 46] we note type by type that we have $|I_\delta| = 0, 1, 2$ (respectively $|I'_\delta| = 0, 1, 2$ and $|I''_\delta| = 0$). Now

i) if $|I_\delta| = 0$ we have $e_3 = \delta e_0$, $e_2 = \delta e_1$ and $e_i = \delta e_{u-i+4}$ for every $i \in \{4, \ldots, [\frac{u}{2}] + 2\}$,

ii) if $|I_\delta| = 1$ we have $e_2 = \delta e_0$, $e_1 = \delta e_1$ and $e_i = \delta e_{u-i+3}$ for every $i \in \{3, \ldots, [\frac{u}{2}] + 1\}$,

iii) if $|I_\delta| = 2$ we have $e_2 = \delta e_0$, $e_1 = \delta e_1$, $e_i = \delta e_{u-i+3}$ for every $i \in \{3, \dots, [\frac{u}{2}] + 1\}$ and $e_{[\frac{u}{2}]+2} = \delta e_{[\frac{u}{2}]+2}$

We define $I_+ \subseteq I$ such that

i) $I_+ = \{2, \dots, [\frac{u}{2}] + 2\} \Leftrightarrow |I_\delta| = 0$,

ii) $I_+ = \{2, \dots, [\frac{u}{2}] + 1\} \Leftrightarrow |I_\delta| = 1$,

iii) $I_+ = \{2, \dots, [\frac{u}{2}] + 1\} \Leftrightarrow |I_\delta| = 2$.

So respectively decompositions of I of the statement follow. One proceeds similarly for I' and I''.
(5) follows by the symmetry and considering [DR, section 6]. □

We note that in part *(5)* of previous lemma we can consider $I_\delta = I_- = I'_\delta = I'_- = I''_\delta = I''_- = I''_+ = \emptyset$ and so $I_+ = I = I' = I'_+$.

Proposition 3.2.2. *Let (Q, σ) be a symmetric quiver of tame type and let I_+, I_δ, I'_+, I'_δ, I''_+ and I''_δ be as above. Any regular symmetric dimension vector can be written uniquely in the following form:*

$$d = ph + \sum_{i \in I_+} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta} p_i e_i + \sum_{i \in I'_+} p'_i(e'_i + \delta e'_i) + \sum_{i \in I'_\delta} p'_i e'_i + \sum_{i \in I''_+} p''_i(e''_i + \delta e''_i) \tag{3.16}$$

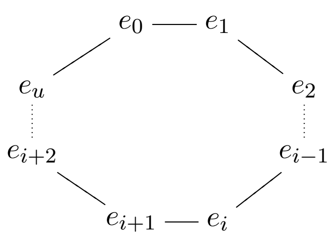
for some non-negative p, p_i, p'_i, p''_i with at least one coefficient in each family $\{p_i | \ i \in I_+ \sqcup I_\delta\}$, $\{p'_i | \ i \in I'_+ \sqcup I'_\delta\}$, $\{p''_i | \ i \in I''_+\}$ being zero. In particular, in the symplectic case,

i) if Q has one σ-fixed vertex and one σ-fixed arrow (i.e. $Q = \widetilde{A}^{1,1}_{k,l}$), then $p_{[\frac{l-1}{2}]+2}$ and p'_1 have to be even,

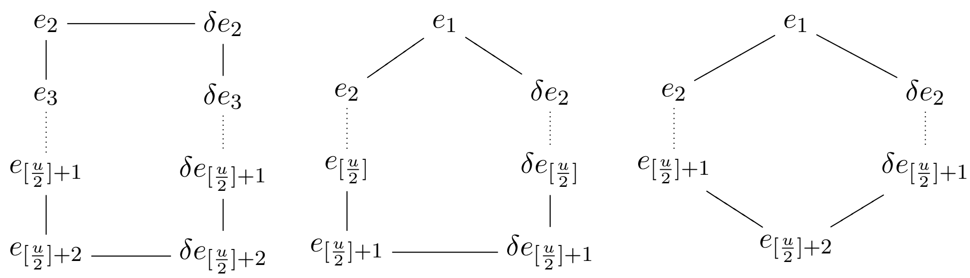
ii) if Q has one or two σ-fixed vertices and it has not any σ-fixed arrows (i.e. $Q = \widetilde{A}^{0,2}_{k,l}$ or $\widetilde{D}^{0,1}_n$), then both p_i's and p'_j's, with $i \in I_\delta$ and $j \in I'_\delta$, have to be even.

Proof. It follows by lemma 3.2.1 and by decomposition of any regular dimension vector of the underlying quiver of (Q, σ). In particular, since symplectic spaces with odd dimension don't exist, it implies *i)* and *ii)*. □

Graphically we can represent Δ (similarly Δ' and Δ'') as the polygons



if $Q = \widetilde{A}^{0,0}_{k,k}$ and



with a reflection respect to a central vertical line, in the other cases.

Definition 3.2.3. *We define an involution σ_I on the set of indices I such that $e_{\sigma_I(i)} = \delta e_i$ for every $i \in I$. Hence $\sigma_I(I) = I'$ for $\widetilde{A}^{0,0}_{k,k}$ and $\sigma_I I_+ = I_-$, $\sigma_I I_\delta = I_\delta$ for the other cases. Similarly we define an involution $\sigma_{I'}$ and an involution $\sigma_{I''}$ respectively on I' and on I''.*

Lemma 3.2.4. *(1) For $\widetilde{A}^{2,0,1}_{k,l}$, no one indecomposable regular representation is orthogonal. The following indecomposable regular representations are symplectic*

(1.1) $E_{i,\sigma_I(i)}$ such that $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$ and $E_{i,\sigma_I(i)}$ of dimension h containing $E_{\frac{l}{2}+1}$.

(1.2) $E'_{i,\sigma'_I(i)}$ such that $\sum_{k=i}^{\sigma_{I'}(i)} e'_k \neq h$ and $E'_{i,\sigma'_I(i)}$ of dimension h containing $E'_{\frac{k}{2}+1}$.

(2) For $\widetilde{A}^{2,0,2}_{k,l}$, no one indecomposable regular representation is orthogonal. The following indecomposable regular representations are symplectic

(2.1) $E_{i,\sigma_I(i)}$ such that $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$, $E_{i,\sigma_I(i)}$ of dimension h containing E_0 and $E_{i,\sigma_I(i)}$ of dimension h containing $E_{[\frac{l+1}{2}]+1}$.

(2.2) $E'_{i,\sigma_{I'}(i)}$ such that $\sum_{k=i}^{\sigma_{I'}(i)} e'_k \neq h$.

(3) For $\widetilde{A}^{0,2}_{k,l}$, no one indecomposable regular representations is symplectic. The following indecomposable regular representations are orthogonal

(3.1) $E_{i,\sigma_I(i)}$ such that $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$.

(3.2) $E'_{i,\sigma_{I'}(i)}$ such that $\sum_{k=i}^{\sigma_{I'}(i)} e'_k \neq h$.

(4) For $\widetilde{A}^{0,2}_{k,l}$, the following indecomposable regular representations are orthogonal

(4.1.1) $E_{i,\sigma_I(i)}$, with $i \leq \sigma_I(i)$, such that $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$.

(4.1.2) $E'_{i,\sigma_{I'}(i)}$*, with* $i \geq \sigma_{I'}(i)$*, such that* $\sum_{k=i}^{\sigma_{I'}(i)} e'_k \neq h$*.*

The following indecomposable regular representations are symplectic

(4.2.1) $E_{i,\sigma_I(i)}$*, with* $i \geq \sigma_I(i)$*, such that* $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$*.*

(4.2.2) $E'_{i,\sigma_{I'}(i)}$*, with* $i \leq \sigma_{I'}(i)$*, such that* $\sum_{k=i}^{\sigma_{I'}(i)} e'_k \neq h$ *and* $E'_{i,\sigma_{I'}(i)}$*, with* $i \leq \sigma_{I'}(i)$*, of dimension* h *containing* $E'_{\frac{k}{2}+1}$*.*

(5) For $\widetilde{A}^{0,0}_{k,k'}$*, no one indecomposable regular representation is symplectic or orthogonal.*

(6) For $(\widetilde{D}^{1,0}_n)^{eq}$*, no one indecomposable regular representation is othogonal. The following indecomposable regular representations are symplectic*

(6.1) $E_{i,\sigma_I(i)}$ *such that* $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$ *and* $E_{i,\sigma_I(i)}$ *of dimension* h *containing* E_{n-1}*.*

(6.2) E'_0 *and* E'_1*.*

(6.3) $E''_{0,1}$ *and* $E''_{1,0}$*.*

(7) For $(\widetilde{D}^{0,1}_n)^{eq}$*, no one indecomposable regular representation is symplectic. The orthogonal indecomposable regular representations are*

(7.1) $E_{i,\sigma_I(i)}$ *such that* $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$*.*

(7.2) E'_0 *and* E'_1*.*

(7.3) $E''_{0,1}$ *and* $E''_{1,0}$*.*

Proof. We check only part (1.1), similarly one proves the other parts. Let $Q = \widetilde{A}^{2,0,1}_{k,l}$. The only $E_{i,j}$ such that $\delta \underline{dim} E_{i,j} = \underline{dim} E_{i,j}$ are $E_{i,\sigma_I(i)}$. We have three cases.

(i) If $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$ and $i < \sigma_I(i)$ then we have for $j \in Q_0$

$$E_{i,\sigma_I(i)}(j) = \begin{cases} \mathbb{K} & j = x_s, \sigma(x_s) \text{ with } i-1 \leq s \leq \frac{l}{2} \\ 0 & \text{otherwise} \end{cases}$$

and for $c \in Q_1$

$$E_{i,\sigma_I(i)}(c) = \begin{cases} Id & c = v_s, \sigma(v_s), a \text{ with } i \leq s \leq \frac{l}{2} \\ 0 & \text{otherwise.} \end{cases}$$

So we note that we can define on such $E_{i,\sigma_I(i)}$ a symplectic structure.

(ii) If $\sum_{k=i}^{\sigma_I(i)} e_k \neq h$ and $i \geq \sigma_I(i)$ then we have for $j \in Q_0$

$$E_{i,\sigma_I(i)}(j) = \begin{cases} 0 & j = x_s, \sigma(x_s) \text{ with } i \leq s \leq \frac{l}{2} \\ \mathbb{K} & \text{otherwise} \end{cases}$$

and for $c \in Q_1$

$$E_{i,\sigma_I(i)}(c) = \begin{cases} 0 & c = v_s, \sigma(v_s), a \text{ with } i \leq s \leq \frac{l}{2} \\ Id & \text{otherwise.} \end{cases}$$

So we note that we can define on such $E_{i,\sigma_I(i)}$ a symplectic structure.

(iii) If $\sum_{k=i}^{\sigma_I(i)} e_k = h$ and $E_{i,\sigma_I(i)}$ contains $E_{\frac{l}{2}+1}$, then, by AR quiver of Q, we note the following almost split sequence

$$0 \longrightarrow E_{\frac{l}{2}+1,\sigma_I(\frac{l}{2})} \longrightarrow E_{i,\sigma_I(i)} \oplus E_{\frac{l}{2},\sigma_I(\frac{l}{2})} \longrightarrow E_{\frac{l}{2},\sigma(\frac{l}{2})-1} \longrightarrow 0.$$

So we have for every $j \in Q_0$, $E_{i,\sigma_I(i)}(j) = \mathbb{K}$ and for $c \in Q_1$

$$E_{i,\sigma_I(i)}(c) = \begin{cases} 0 & c = a \\ Id & \text{otherwise.} \end{cases}$$

Finally, we note that we can define on such $E_{i,\sigma_I(i)}$ a symplectic structure. □

In the remainder of the section, we shall call

$$d' = \sum_{i \in I_+} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta} p_i e_i + \sum_{i \in I'_+} p'_i(e'_i + \delta e'_i) + \sum_{i \in I'_\delta} p'_i e'_i + \sum_{i \in I''_+} p''_i(e''_i + \delta e''_i). \tag{3.17}$$

Proposition 3.2.5. *If d is regular with decomposition (3.16) such that $d = d'$ or d is not regular then $SpRep(Q,d)$ (respectively $ORep(Q,d)$) has an open $Sp(Q,d)$-orbit (respectively $O(Q,d)$-orbit).*

Proof. If $d = d'$, we have no indecomposable of dimension vector ph and so there are finitely many orbits. If d is not regular, it follows from [R2, theorem 3.2]. □

In the next d shall be a regular symmetric dimension vector with decomposition (3.16) with $p \geq 1$ and $p \neq 0$. Now we shall describe the generators of $SpSI(Q,d)$ and $OSI(Q,d)$. To do this the following theorem, which we prove later, is useful.

Theorem 3.2.6. *Let (Q,σ) be a symmetric quiver of tame type and the decomposition (3.16) of a regular symmetric dimension vector with $p \geq 1$ and $d' \neq 0$. There exist isomorphisms of algebras*

$$SpSI(Q,d) \stackrel{\Phi_d}{\to} \bigoplus_{\chi \in char(Sp(Q,d))} SpSI(Q,ph)_\chi \otimes SpSI(Q,d')_{\chi'} \tag{3.18}$$

and

$$OSI(Q,d) \stackrel{\Psi_d}{\to} \bigoplus_{\chi \in char(O(Q,d))} OSI(Q,ph)_\chi \otimes OSI(Q,d')_{\chi'}, \tag{3.19}$$

where $\chi' = \chi|_{d'}$*, i.e. the restriction of the weight* χ *to the support of* d'*.*

By proposition 3.2.5 $Sp(Q,d')$ (respectively $O(Q,d')$) acting on $SpRep(Q,d')$ (respectively on $ORep(Q,d')$) has an open orbit so , by lemma A.2.5, dimension of $SpSI(Q,d')_{\chi'}$ (respectively dimension of $OSI(Q,d')_{\chi'}$) is 0 or 1. This allows us to identify one non-zero element of $SpSI(Q,d)_\chi$ (respectively of $OSI(Q,d)_\chi$) with the element of $SpSI(Q,ph)_\chi$ (respectively of $OSI(Q,ph)_\chi$) to which it restricts.
We proceed now to describe the generators of the algebra $SpSI(Q,d)$ (respectively $OSI(Q,d)$). If the corresponding I, I', I'' are not empty, we label the vertices e_i, e'_i, e''_i of the polygons Δ, Δ', Δ'' with the coefficients p_i, p'_i, p''_i. We recall that

a) we have to label with p_i (respectively with p'_i and p''_i) both vertices e_i and δe_i, i.e $p_i = p_{\sigma_I(i)}$ (respectively $p'_i = p'_{\sigma''_I(i)}$ and $p''_i = p''_{\sigma''_I(i)}$), if $e_i \neq \delta e_i$.

and in the symplectic case, by *i)* and *ii)* of proposition 3.2.2

b) for $\widetilde{A}^{1,1}_{k,l}$, $p_{[\frac{u}{2}]+2}$ and p'_1 have to be even,

c) for $\widetilde{A}^{0,2}_{k,l}$ and $\widetilde{D}^{0,1}_n$, $p_i \in I_\delta$ and $p'_i \in I'_\delta$ have to be even.

We shall call these labelled polygons respectively $\Delta(d)$, $\Delta'(d)$, $\Delta''(d)$.

Definition 3.2.7. *We shall say that the labelled arc* $p_i — \cdots — p_j$ *(in clockwise orientation) of the labelled polygon* $\Delta(d)$ *is admissible if* $p_i = p_j$ *and* $p_i < p_k$ *for every its interior labels* p_k*. We denote such a labelled arc* $p_i — \cdots — p_j$ *by* $[i,j]$*, and we define* $p_i = p_j$ *the index* $ind[i,j]$ *of* $[i,j]$*. Similarly we define admissible arcs and their indexes for the labelled polygons* $\Delta'(d)$ *and* $\Delta''(d)$*.*

We denote by $\mathcal{A}(d)$, $\mathcal{A}'(d)$, $\mathcal{A}''(d)$ the sets of all admissible labelled arcs in the polygons $\Delta(d)$, $\Delta'(d)$, $\Delta''(d)$ respectively. In particular we note that if $d = ph$, then the polygons $\Delta(d)$, $\Delta'(d)$, $\Delta''(d)$ are labelled by zeros and so $\mathcal{A}(d)$, $\mathcal{A}'(d)$, $\mathcal{A}''(d)$ consist of all edges of respective polygons. With these notations we have the following

Proposition 3.2.8. *For each arc* $[i,j]$ *from* $\mathcal{A}(d)$ *(respectively* $\mathcal{A}'(d)$ *and* $\mathcal{A}''(d)$*) there exists in* $SpSI(Q,d)$ *and in* $OSI(Q,d)$ *a non zero semi-invariant*

(i) of type $c^{E_{i,j-1}}$ *(respectively* $c^{E'_{i,j-1}}$ *and* $c^{E''_{i,j-1}}$*) or of type* $c^{V_{(\varphi,\psi)}}$*, with* $(\varphi,\psi) \in \{(1,0),(0,1),(1,1)\}$*;*

(ii) *of type* $pf^{E_{i,j-1}}$ *(respectively* $pf^{E'_{i,j-1}}$ *and* $pf^{E''_{i,j-1}}$*) or of type* $pf^{V_{(\varphi,\psi)}}$*, with* $(\varphi,\psi) \in \{(1,0),(0,1),(1,1)\}$*, if* $E_{i,j-1}$*,* $E'_{i,j-1}$*,* $E''_{i,j-1}$ *and* $V_{(\varphi,\psi)}$ *satisfy property (Op) in the symplectic case and property (Spp) in the orthogonal case.*

Let $c_0,\dots,c_t$, with $t = \frac{p-1}{2}$, $\frac{p}{2}$ and p, defined case by case in section 3.1. The generators of algebras $SpSI(Q,d)$ and $OSI(Q,d)$ are described by the following theorem

Theorem 3.2.9. *Let* (Q,d) *a symmetric quiver of tame type and* $d = ph + d'$ *the decomposition of a regular symmetric dimension vector* d *with* $p \geq 1$*. Then* $SpSI(Q,d)$ *(respectively* $OSI(Q,d)$*) is generated by*

(i) $c_0,\dots,c_t$*;*

(ii) $c^{E_{i,j-1}}$*,* $c^{E'_{r,s-1}}$*,* $c^{E''_{t,m-1}}$ *and* $c^{V_{(\varphi,\psi)}}$ *with* $[i,j] \in \mathcal{A}(d)$*,* $[r,s] \in \mathcal{A}'(d)$*,* $[t,m] \in \mathcal{A}''(d)$ *and* $(\varphi,\psi) \in \{(1,0),(0,1),(1,1)\}$*;*

(iii) $pf^{E_{i,j-1}}$*,* $pf^{E'_{r,s-1}}$*,* $pf^{E''_{t,m-1}}$ *and* $pf^{V_{(\varphi,\psi)}}$ *with* $[i,j] \in \mathcal{A}(d)$*,* $[r,s] \in \mathcal{A}'(d)$*,* $[t,m] \in \mathcal{A}''(d)$ *and* $(\varphi,\psi) \in \{(1,0),(0,1),(1,1)\}$*, if* $E_{i,j-1}$*,* $E'_{i,j-1}$*,* $E''_{i,j-1}$ *and* $V_{(\varphi,\psi)}$ *satisfy property (Op) (respectively property (Spp)).*

First we note that $\langle h,d\rangle = 0$ and further we have the following

Lemma 3.2.10. *For every regular dimension vector* d

$$\langle \underline{dim} E_{i,j-1}, d\rangle = 0 \Leftrightarrow p_i = p_j.$$

Proof. See [D, section 4.3]. □

So theorem 3.2.9 is equivalent to conjectures 1.2.1 and 1.2.2.

3.2.1 Proof of theorem 3.2.9 and 3.2.6

In this section we prove the theorem 3.2.9 and theorem 3.2.6. For theorem 3.2.9, by proposition 1.3.8, proposition 1.3.4 and lemma 1.3.14, we can reduce the proof to the orientation of $\widetilde{A}$ as in proposition 1.3.8 and to the equiorientation for $\widetilde{D}$. In the proof we use the notion of generic decomposition of the symmetric dimension vector d (see [K1], [K2], [KR]).

Definition 3.2.11. *A decomposition* $\alpha = \beta_1 \oplus \cdots \oplus \beta_q$ *of a dimension vector* α *is called generic if there is a Zariski open subset* $\mathcal{U}$ *of* $Rep(Q,\alpha)$ *such that each* $U \in \mathcal{U}$ *decomposes in* $U = \bigoplus_{i=1}^q U_i$ *with* U_i *indecomposable representation of dimension* β_i*, for every* $i \in \{1,\dots,q\}$*.*

Definition 3.2.12. *(1) A decomposition* $\alpha = \beta_1 \oplus \cdots \oplus \beta_q$ *of a symmetric dimension vector* α *is called symplectic generic if there is a Zariski open subset* $\mathcal{U}$ *of* $SpRep(Q,\alpha)$ *such that each* $U \in \mathcal{U}$ *decomposes in* $U = \bigoplus_{i=1}^q U_i$ *with* U_i *indecomposable symplectic representation of dimension* β_i*, for every* $i \in \{1,\dots,q\}$*.*

(2) *A decomposition* $\alpha = \beta_1 \oplus \cdots \oplus \beta_q$ *of a symmetric dimension vector* α *is called orthogonal generic if there is a Zariski open subset* $\mathcal{U}$ *of* $ORep(Q, \alpha)$ *such that each* $U \in \mathcal{U}$ *decomposes in* $U = \bigoplus_{i=1}^{q} U_i$ *with* U_i *indecomposable orthogonal representation of dimension* β_i*, for every* $i \in \{1, \ldots, q\}$.

For tame quivers the generic decomposition of any regular dimension vector is given by results of [R2, section 3].
We describe this decomposition explicitly for a symmetric regular dimension vector d with decomposition (3.16).
In the remainder of this section we set

$$\bar{d} = \sum_{i \in I_+} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta} p_i e_i, \tag{3.20}$$

$$\bar{d}' = \sum_{i \in I'_+} p'_i(e'_i + \delta e'_i) + \sum_{i \in I'_\delta} p'_i e'_i \tag{3.21}$$

$$\bar{d}'' = \sum_{i \in I''_+} p''_i(e''_i + \delta e''_i). \tag{3.22}$$

Remark 3.2.13. *(i) We remember that at least one coefficient in each family* $\{p_i |\ i \in I_+ \sqcup I_\delta\}$, $\{p'_i |\ i \in I'_+ \sqcup I'_\delta\}$, $\{p''_i |\ i \in I''_+\}$ *is zero.*

(ii) We can assume $p_i = 0$ *for* $i \in I_\delta$ *or* $p_i = 0$*, for* $i \in I_+$*, and so* $p_{\sigma_I(i)} = 0$.

Definition 3.2.14. *We divide the polygon* $\Delta(\bar{d})$ *in two parts:*

(i) the up part $\Delta_{up}(\bar{d})$ *is the part of* $\Delta(\bar{d})$ *from* p_{i-1} *to* $p_{\sigma_I(i-1)}$*;*

(ii) the down part $\Delta_{down}(\bar{d})$ *is the part of* $\Delta(\bar{d})$ *from* p_{i+1} *to* $p_{\sigma_I(i+1)}$.

Similarly for Δ' *and* Δ''.

Remark 3.2.15. *We note that if* $p_i = 0$ *with* $i \in I_\delta$*, then we have only the part* Δ_{up} *or the part* Δ_{down}.

We consider Δ, similarly one proceeds for Δ' and Δ''.

Definition 3.2.16. *We shall call symmetric arc, an arc invariant under* σ_I*, i.e. an arc of type* $[i, \sigma_I(i)]$.

Remark 3.2.17. *By the division of* Δ *in* Δ_{up} *and* Δ_{down}*, we note that all symmetric arcs pass through the same* σ_I*-fixed vertex of* Δ *or through the same* σ_I*-fixed edge of* Δ.

Lemma 3.2.18. *Let* (Q, σ) *be a symmetric quiver of tame type.*

(i) If $n = \sigma_I(n)$ *then either there exists unique* $x \in Q_0^\sigma$ *such that* $e_n(x) \neq 0$ *or there exists unique* $a \in Q_1^\sigma$ *such that* $e_n(ta) \neq 0$.

(ii) If $n - \sigma_I(n)$ is a σ_I-fixed edge in Δ, then there exists unique $a \in Q_1^\sigma$ such that $e_n(ta) \neq 0$.

Proof. One proceeds type by type. We consider $Q = \widetilde{A}_{k,l}^{2,0,1}$ since for the other types one proves similarly.
(i) By lemma 3.2.1, the only σ_I-fixed vertex of Δ is 1 and b is the unique arrow in Q_1^σ such that $e_1(tb) \neq 0$.
(ii) The only σ_I-fixed edge of Δ is $\frac{l}{2}+1 - \sigma_I(\frac{l}{2}+1)$ and a is the unique arrow in Q_1^σ such that $e_{\frac{l}{2}+1}(ta) \neq 0$. □

Definition 3.2.19. *(i) If $n = \sigma_I(n)$, we call $x(n)$ the unique $x \in Q_0^\sigma$ such that $e_n(x) \neq 0$.*

(ii) If $n = \sigma_I(n)$ or $n - \sigma_I(n)$ is a σ_I-fixed edge in Δ, we call $a(n)$ the unique $a \in Q_1^\sigma$ such that $e_n(ta) \neq 0$.

Definition 3.2.20. *For every arc $[i,j]$ in Δ, we define*

$$e_{[i,j]} = \sum_{k \in [i,j]} e_k.$$

Definition 3.2.21. *(i) $\mathcal{A}_+(\bar{d}) := \{[i,j] \in \mathcal{A}(\bar{d}) | \ [i,j] \subset I_+\}$*

(ii) $\mathcal{A}_+^k(\bar{d}) := \{[i,j] \in \mathcal{A}(\bar{d}) | \ [i,j] \subset I_+, ind[i,j] = k\}$.

(iii) $\mathcal{A}_{\sigma_I}^k(\bar{d}) = \{[i,j] = \sigma_I[i,j] \in \mathcal{A}(\bar{d}) | \ ind[i,j] = k\}$.

Remark 3.2.22. *$[i,j] \subset I_+$ if and only if $[\sigma_I(j), \sigma_I(i)] \subset I_-$ and $ind[i,j] = ind[\sigma_I(j), \sigma_I(i)]$.*

First we consider all the admissible arcs in $\mathcal{A}_{\sigma_I}^r(\bar{d}) \cup \mathcal{A}_+^r(\bar{d})$ such that $r = max\{p_k\}$. So we get

$$= \sum_{i \in I_+} \tilde{p}_i(\tilde{e}_i + \delta\tilde{e}_i) + \sum_{i \in I_\delta} \tilde{p}_i \tilde{e}_i =$$

$$\sum_{i \in I_+} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta} p_i e_i - \left(\bigoplus_{[i,j] \in \mathcal{A}_+^r(\bar{d})} (e_{[i,j]} + \delta e_{[i,j]}) + \bigoplus_{[i,\sigma_I(i)] \in \mathcal{A}_{\sigma_I}^r(\bar{d})} e_{[i,\sigma_I(i)]} \right), \tag{3.23}$$

where $max(\bar{p}_i) = r - 1$. Then we repeat the procedure for (3.23) and so on we have

$$\sum_{i \in I_+} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta} p_i e_i =$$

$$\bigoplus_{k=1}^{r} \left(\bigoplus_{[i,j] \in \mathcal{A}_+^k(\bar{d})} (e_{[i,j]} + \delta e_{[i,j]}) + \bigoplus_{[i,\sigma_I(i)] \in \mathcal{A}_{\sigma_I}^k(\bar{d})} e_{[i,\sigma_I(i)]} \right). \tag{3.24}$$

Remark 3.2.23. *(i) If $[i,j]$ and $[i',j']$ are two admissible arcs in $\mathcal{A}(\bar{d})$ such that $[i,j] \supseteq [i',j']$, then $ind[i,j] \leq ind[i',j']$.*

(ii) If there not exists $[i,j] \in \mathcal{A}^h_{\sigma_I}(\bar{d}) \cup \mathcal{A}^h_+(\bar{d})$ such that $[i,j] \supseteq [i',j']$ for some $[i',j'] \in \mathcal{A}^k_{\sigma_I}(\bar{d}) \cup \mathcal{A}^k_+(\bar{d})$, then the symmetric dimension vector corresponding to $[i',j']$ appears k-times in the decomposition (3.24), with $1 \leq h < k$.

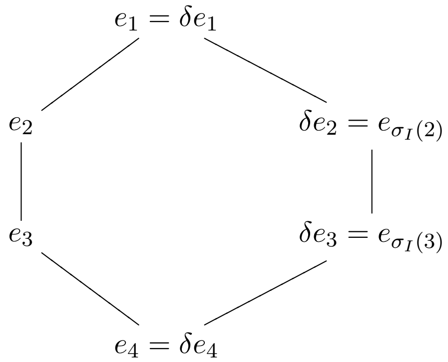
Definition 3.2.24. *Let $[i_1,j_1],\dots,[i_k,j_k]$ be the admissible arcs such that $[i_1,j_1] \supseteq \cdots \supseteq [i_k,j_k]$, with $k \geq 1$. We define $q_{[i_h,j_h]} = ind[i_h,j_h] - ind[i_{h-1},j_{h-1}]$ for every $1 \leq h \leq k$, where $ind[i_0,i_0] = 0$.*

We note that for every $[i,j] \in \mathcal{A}^k_{\sigma_I}(\bar{d}) \cup \mathcal{A}^k_+(\bar{d})$, $q_{[i,j]}$ is the multiplicity of the symmetric dimension vector corresponding to $[i,j]$ in the decomposition (3.24).
Finally we have

$$\sum_{i\in I_+} p_i(e_i+\delta e_i) + \sum_{i\in I_\delta} p_i e_i =$$

$$\bigoplus_{[i,j]\in\mathcal{A}_+(\bar{d})} (e_{[i,j]}+\delta e_{[i,j]})^{\oplus q_{[i,j]}} + \bigoplus_{[i,\sigma_I(i)]\in\mathcal{A}(\bar{d})} (e_{[i,\sigma_I(i)]})^{\oplus q_{[i,\sigma_I(i)]}}. \qquad (3.25)$$

Example 3.2.25. *If Δ is of the form*



(3.26)

and $p_1 = 2$, $p_2 = 3$, $p_3 = 0$ and $p_4 = 2$, then $[2,\sigma_I(2)] = \{2,1,\sigma_I(2)\} \subset I_+ \sqcup I_\delta \sqcup I_-$ with $q_{[2,\sigma_I(2)]} = ind[2,\sigma_I(2)] = 2$, $[2,2] = \{2\} \in I_+$ with $q_{[2,2]} = ind[2,2] - ind[2,\sigma_I(2)] = 1$ and $[4,4] = \{4\} \in I_\delta$ with $q_{[4,4]} = ind[4,4] = 2$. So we have

$$\sum_{i\in I_+} p_i(e_i+\delta e_i) + \sum_{i\in I_\delta} p_i e_i = ((e_2+\delta e_2)+e_1)^{\oplus 2} \oplus (e_2+\delta e_2) \oplus (e_4)^{\oplus 2}.$$

Similarly we proceed with the decomposition of $\bar{d}'$ and $\bar{d}''$. So we have the following

Proposition 3.2.26. *Let (Q,σ) be a symmetric quiver of tame type and let d be a symmetric dimension vector of a representation of the underlying quiver Q with decomposition (3.16). Then*

$$
\begin{aligned}
d = \bigoplus_{i=1}^{p} h + \bigoplus_{[i,j]\in\mathcal{A}_+(\bar d)} (e_{[i,j]}+\delta e_{[i,j]})^{\oplus q_{[i,j]}} + \bigoplus_{[i,\sigma_I(i)]\in\mathcal{A}(\bar d)} (e_{[i,\sigma_I(i)]})^{\oplus q_{[i,\sigma_I(i)]}} + \\
\bigoplus_{[i,j]\in\mathcal{A}'_+(\bar d')} (e'_{[i,j]}+\delta e'_{[i,j]})^{\oplus q'_{[i,j]}} + \bigoplus_{[i,\sigma_{I'}(i)]\in\mathcal{A}'(\bar d')} (e'_{[i,\sigma_{I'}(i)]})^{\oplus q'_{[i,\sigma_{I'}(i)]}} + \\
\bigoplus_{[i,j]\in\mathcal{A}''_+(\bar d'')} (e''_{[i,j]}+\delta e''_{[i,j]})^{\oplus q''_{[i,j]}} + \bigoplus_{[i,\sigma_{I''}(i)]\in\mathcal{A}''(\bar d'')} (e''_{[i,\sigma_{I''}(i)]})^{\oplus q''_{[i,\sigma_{I''}(i)]}}
\end{aligned}
\tag{3.27}
$$

is the generic decomposition of d.

We restrict to dimension vectors of regular symplectic representations and of regular orthogonal representations. We modify generic decomposition (3.27) of $d=(d_i)_{i\in Q_0}$ to get symplectic generic decomposition of d or orthogonal generic decomposition of d.
Let $[i,j]$ be an arc in Δ_{up} and let $[h,k]$ be an arc in Δ_{down}. If $E_{[i,j]}$ is the regular indecomposable symplectic (respectively orthogonal) representation of (Q,σ) corresponding to $[i,j]$ and $E_{[h,k]}$ is the regular indecomposable symplectic (respectively orthogonal) representation of (Q,σ) corresponding to $[h,k]$, then

$$Hom_Q(E_{[i,j]},E_{[h,k]}) = 0 = Hom_Q(E_{[h,k]},E_{[i,j]})$$

and

$$Ext^1_Q(E_{[i,j]},E_{[h,k]}) = 0 = Ext^1_Q(E_{[h,k]},E_{[i,j]}).$$

So we deal separately with Δ_{up} and Δ_{down}. We consider $I = I^{up}\sqcup I^{down}$, $I_+ = I_+^{up}\sqcup I_+^{down}$ and $I_\delta = I_\delta^{up}\sqcup I_\delta^{down}$. We have the decomposition $\bar d = \bar d_{up}+\bar d_{down}$, where

$$\bar d_{up} = \sum_{i\in I_+^{up}} p_i(e_i+\delta e_i) + \sum_{i\in I_\delta^{up}} p_i e_i \tag{3.28}$$

and

$$\bar d_{down} = \sum_{i\in I_+^{down}} p_i(e_i+\delta e_i) + \sum_{i\in I_\delta^{down}} p_i e_i. \tag{3.29}$$

By what has be said, the symplectic (respectively orthogonal) generic decomposition of $\bar d$ is direct sum of the symplectic (respectively orthogonal) generic decomposition of $\bar d_{up}$ and the symplectic (respectively orthogonal) generic decomposition of $\bar d_{down}$.

Remark 3.2.27. *(i) In the symplectic case, since $\bar{d}_x$ has to be even for every $x \in Q_0^\sigma$, we have to modify the symmetric dimension vectors corresponding to the arcs passing through the σ_I-fixed vertex n such that there exists $x = x(n) \in Q_0^\sigma$ such that $e_n(x) \neq 0$.*

(ii) In the orthogonal case, we have to modify the symmetric dimension vectors corresponding to the arcs passing through the σ_I-fixed vertex n such that $\bar{d}_{ta(n)}$ is even and those corresponding to the arcs passing through the σ_I-fixed edge $n \longrightarrow \sigma_I(n)$ such that $\bar{d}_{ta(n)}$ is even.

(iii) We have to modify also $ph+e_{[i,\sigma_I(i)]}$, with p odd, if $[i, \sigma_I(i)]$ is like in part (i) (respectively part (ii)), since $h + e_{[i,\sigma_I(i)]}$ is the dimension vector of regular indecomposable symplectic (respectively orthogonal) representation.

Definition 3.2.28. *(i)* $\mathcal{A}^{up}(\bar{d}) = \{[i,j] \in \mathcal{A}(\bar{d}) | \ [i,j] \subset I^{up}\}$.

(ii) $\mathcal{A}_+^{up}(\bar{d}) = \{[i,j] \in \mathcal{A}(\bar{d}) | \ [i,j] \subset I_+^{up}\}$.

(iii) $\mathcal{A}^{down}(\bar{d}) = \{[i,j] \in \mathcal{A}(\bar{d}) | \ [i,j] \subset I^{down}\}$.

(iv) $\mathcal{A}_+^{down}(\bar{d}) = \{[i,j] \in \mathcal{A}(\bar{d}) | \ [i,j] \subset I_+^{down}\}$.

Let $\bar{d} = \bar{d}_{up} + \bar{d}_{down}$ be a regular symplectic dimension vector. We consider Δ_{up}. Δ_{up} contains either a σ_I-fixed vertex n_{up} or a σ_I-fixed edge $n_{up} \longrightarrow \sigma_I(n_{up})$. Starting from generic decomposition (3.27) of $\bar{d}_{up}$ we modify it as follows.

(1) We keep the summands $(e_{[i,j]} + \delta e_{[i,j]})^{\oplus q_{[i,j]}}$ corresponding to the arc $[i,j] \subset I_+^{up}$.

(2) If n_{up} is such that there exists $a = a(n_{up}) \in Q_1^\sigma$, then we keep the summands $(e_{[i,\sigma_I(i)]})^{\oplus q_{[i,\sigma_I(i)]}}$ corresponding to the symmetric arcs $[i, \sigma_I(i)]$ of Δ_{up}.

(3) If n_{up} is such that there exists $x = x(n_{up}) \in Q_0^\sigma$, we have the symmetric dimension vectors

$$e_{[i_1,\sigma_I(i_1)]}, \dots, e_{[i_{2s},\sigma_I(i_{2s})]}$$

corresponding to the arcs $[i_1, \sigma_I(i_1)], \dots, [i_{2s}, \sigma_I(i_{2s})]$ such that $[i_1, \sigma_I(i_1)] \supseteq \cdots \supseteq [i_{2s}, \sigma_I(i_{2s})]$. Then we divide them into pairs

$$([i_{2k}, \sigma_I(i_{2k})], [i_{2k-1}, \sigma_I(i_{2k-1})]),$$

with $1 \le k \le s$. For each pair we consider $[i_{2k}, \sigma_I(i_{2k-1})] \cup [i_{2k-1}, \sigma_I(i_{2k})]$ and we substitute $e_{[i_{2k},\sigma_I(i_{2k})]} \oplus e_{[i_{2k-1},\sigma_I(i_{2k-1})]}$ for

$$e_{[i_{2k},\sigma_I(i_{2k-1})]} + e_{[i_{2k-1},\sigma_I(i_{2k})]}.$$

So, by equation 3.25, in the symplectic case we get

(i) if n_{up} is such that there exists $a = a(n_{up}) \in Q_1^\sigma$,

$$\bar{d}_{up} = \bigoplus_{[i,j]\in\mathcal{A}_+^{up}(\bar{d})} (e_{[i,j]} + \delta e_{[i,j]})^{\oplus q_{[i,j]}} + \bigoplus_{[i,\sigma_I(i)]\in\mathcal{A}^{up}(\bar{d})} (e_{[i,\sigma_I(i)]})^{\oplus q_{[i,\sigma_I(i)]}}; \tag{3.30}$$

(ii) If n_{up} is such that there exists $x = x(n_{up}) \in Q_0^\sigma$,

$$\bar{d}_{up} = \bigoplus_{[i,j]\in\mathcal{A}_+^{up}(\bar{d})} (e_{[i,j]} + \delta e_{[i,j]})^{\oplus q_{[i,j]}} + \bigoplus_{k=1}^{s} (e_{[i_{2k},\sigma_I(i_{2k-1})]} + e_{[i_{2k-1},\sigma_I(i_{2k})]}). \tag{3.31}$$

Similarly one proceeds for Δ_{down}.
Finally we have to modify like in (3) the dimension vector $ph + e_{[i,\sigma_I(i)]}$ if p is odd and $[i, \sigma_I(i)]$ passes through n_{up} such that there exists $x = x(n_{up}) \in Q_0^\sigma$.

Example 3.2.29. *Let (Q, σ) be the symmetric quiver $\widetilde{A}_{0,6}^{1,1}$. We recall that $x_{\frac{l}{2}} = \sigma(x_{\frac{l}{2}})$. Δ has the form (3.26).*
As in example 3.2.25, let $p_1 = 2$, $p_2 = 3$, $p_3 = 0$ and $p_4 = 2$. The σ_I-fixed vertex 4 is such that $e_4(x_{\frac{l}{2}}) \neq 0$. The only symmetric arc passing through 4 is $[4, 4]$. Thus we substitute $(e_4)^{\oplus 2}$ for $2e_4$. So, in the symplectic case we get

$$\sum_{i\in I_+} p_i(e_i + \delta e_i) + \sum_{i\in I_\delta} p_i e_i = ((e_2 + \delta e_2) + e_1)^{\oplus 2} \oplus (e_2 + \delta e_2) \oplus 2e_4.$$

Similarly we proceed with the decomposition of $\bar{d}'$ and $\bar{d}''$.
Let $\bar{d} = \bar{d}_{up} + \bar{d}_{down}$ be a regular orthogonal dimension vector. We consider Δ_{up}. Starting from generic decomposition (3.27) of $\bar{d}_{up}$ we modify it as follows.

(1) We keep the summands $(e_{[i,j]} + \delta e_{[i,j]})^{\oplus q_{[i,j]}}$ corresponding to the arc $[i, j] \subset I_+^{up}$.

(2) If n_{up} is such that there exists $a = a(n_{up}) \in Q_1^\sigma$ such that $\bar{d}_{ta}$ is odd or n_{up} is such that there exist $x = x(n_{up}) \in Q_0^\sigma$, then we keep the summands $(e_{[i,\sigma_I(i)]})^{\oplus q_{[i,\sigma_I(i)]}}$ corresponding to the symmetric arcs $[i, \sigma_I(i)]$ of Δ_{up}.

(3) If n_{up} is such that there exists $a = a(n_{up}) \in Q_1^\sigma$ such that $\bar{d}_{ta}$ is even, we have the symmetric dimension vectors

$$e_{[i_1,\sigma_I(i_1)]}, \ldots, e_{[i_{2s},\sigma_I(i_{2s})]}$$

corresponding to the arcs $[i_1, \sigma_I(i_1)], \ldots, [i_{2s}, \sigma_I(i_{2s})]$ such that $[i_1, \sigma_I(i_1)] \supseteq \cdots \supseteq [i_{2s}, \sigma_I(i_{2s})]$. Then we divide them into pairs

$$([i_{2k}, \sigma_I(i_{2k})], [i_{2k-1}, \sigma_I(i_{2k-1})]),$$

with $1 \leq k \leq s$. For each pair we consider $[i_{2k}, \sigma_I(i_{2k-1})] \cup [i_{2k-1}, \sigma_I(i_{2k})]$ and we substitute $e_{[i_{2k},\sigma_I(i_{2k})]} \oplus e_{[i_{2k-1},\sigma_I(i_{2k-1})]}$ for

$$e_{[i_{2k},\sigma_I(i_{2k-1})]} + e_{[i_{2k-1},\sigma_I(i_{2k})]}.$$

So, by equation 3.25, in the orthogonal case we get

(i) if n_{up} is such that there exists $a = a(n_{up}) \in Q_1^\sigma$ such that $\bar{d}_{ta}$ is odd or n_{up} is such that there exist $x = x(n_{up}) \in Q_0^\sigma$,

$$\bar{d}_{up} = \bigoplus_{[i,j] \in \mathcal{A}_+^{up}(d')} (e_{[i,j]} + \delta e_{[i,j]})^{\oplus q_{[i,j]}} + \bigoplus_{[i,\sigma_I(i)] \in \mathcal{A}^{up}(d')} (e_{[i,\sigma_I(i)]})^{\oplus q_{[i,\sigma_I(i)]}}; \tag{3.32}$$

(ii) if n_{up} is such that there exists $a = a(n_{up}) \in Q_1^\sigma$ such that $\bar{d}_{ta}$ is even,

$$\bar{d}_{up} = \bigoplus_{[i,j] \in \mathcal{A}_+^{up}(d')} (e_{[i,j]} + \delta e_{[i,j]})^{\oplus q_{[i,j]}} + \bigoplus_{k=1}^{s} (e_{[i_{2k},\sigma_I(i_{2k-1})]} + e_{[i_{2k-1},\sigma_I(i_{2k})]}). \tag{3.33}$$

Similarly one proceeds for Δ_{down}.
Finally we have to modify like in (3) the dimension vector $ph + e_{[i,\sigma_I(i)]}$ if p is odd and $[i, \sigma_I(i)]$ passes through n_{up} such that there exists $a = a(n_{up}) \in Q_1^\sigma$ such that $\bar{d}_{ta}$ is even.

Example 3.2.30. *Let (Q, σ) be the symmetric quiver $\widetilde{A}_{0,6}^{1,1}$. We recall that $b = \sigma(b)$. Δ has the form (3.26).*
As in example 3.2.25, let $p_1 = 2$, $p_2 = 3$, $p_3 = 0$ and $p_4 = 2$. The σ_I-fixed vertex 1 is such that $e_1(tb) \neq 0$ and $\bar{d}_{tb}$ is 2. The only symmetric arc passing through 1 is $[2, \sigma_I(2)]$. Thus we substitute $((e_2 + \delta e_2)) + e_1)^{\oplus 2}$ for $2((e_2 + \delta e_2)) + e_1)$. So, in the orthogonal case we get

$$\sum_{i \in I_+} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta} p_i e_i = (e_4)^{\oplus 2} \oplus (e_2 + \delta e_2) \oplus 2((e_2 + \delta e_2) + e_1).$$

Similarly we proceed with the decomposition of $\bar{d}'$ and $\bar{d}''$.
In general we have

Proposition 3.2.31. *Let (Q, σ) be a symmetric quiver of tame type.*

(1) *If d is a regular symplectic dimension vector with decomposition (3.16). Then*

$$d = \bigoplus_{i=1}^{p} h \oplus \bar{d}_{up} \oplus \bar{d}_{down} \oplus \bar{d}'_{up} \oplus \bar{d}'_{down} \oplus \bar{d}''_{up} \oplus \bar{d}''_{down} \tag{3.34}$$

is the symplectic generic decomposition of d.

(2) *If d is a regular orthogonal dimension vector with decomposition (3.16). Then (3.16). Then*

$$d = \bigoplus_{i=1}^{p} h \oplus \bar{d}_{up} \oplus \bar{d}_{down} \oplus \bar{d}'_{up} \oplus \bar{d}'_{down} \oplus \bar{d}''_{up} \oplus \bar{d}''_{down} \tag{3.35}$$

is the orthogonal generic decomposition of d.

For the proof, we need two propositions. We state and prove these propositions only for regular indecomposable symplectic (respectively orthogonal) representations related to polygon Δ, because for those related to polygon Δ' and to polygon Δ'' the statement and the proof are similar.

Proposition 3.2.32. *Let (Q, σ) be a symmetric quiver of tame tape. Let $V_1 \neq V_2$ be two regular indecomposable symplectic (respectively orthogonal) representations of (Q, σ) with symmetric dimension vector corresponding respectively to the arc $[i, j]$ and the arc $[h, k]$ of Δ (Δ' or Δ''). Moreover we suppose that $[i, j]$ and $[h, k]$ don't satisfy the following properties*

(i) $[i, j] \cap [h, k] \neq \emptyset$ and $[i, j]$ doesn't contain $[h, k]$;

(ii) $[i, j] \cap [h, k] \neq \emptyset$ and $[h, k]$ doesn't contain $[i, j]$;

(iii) $[i, j]$ and $[h, k]$ are linked by one edge of Δ (respectively Δ' or Δ'').

Then $Ext^1_Q(V_1, V_2) = 0$.

Proof. We restrict to decomposition $\bar{d}_j = \sum_{i \in I_+} p_i^j (e_i + \delta e_i) + \sum_{i \in I_\delta} p_i^j e_i$, for $j = 1, 2$. We have nine cases:

(1) $V_1 = E_{i,\sigma_I(i)}$, $V_2 = E_{j,\sigma_I(j)}$ and $V_1 = E_{\sigma_I(j),j}$, $V_2 = E_{\sigma_I(i),i}$ with $i, j \in I_+ \sqcup I_\delta$.

(2) $V_1 = E_{i,\sigma_I(i)}$, $V_2 = E_{\sigma_I(j),j}$ and $V_1 = E_{\sigma_I(j),j}$, $V_2 = E_{i,\sigma_I(i)}$ with $i, j \in I_+ \sqcup I_\delta$ such that $j > i + 1$.

(3) $V_1 = E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}$, $V_2 = E_{k,\sigma_I(k)}$ and $V_1 = E_{k,\sigma_I(k)}$, $V_2 = E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}$ with $i, j, k \in I_+ \sqcup I_\delta$ such that either $j > k + 1$ or $k \geq i$.

(4) $V_1 = E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}$, $V_2 = E_{\sigma_I(k),k}$ or $V_1 = E_{\sigma_I(k),k}$, $V_2 = E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}$ with $i, j, k \in I_+ \sqcup I_\delta$ such that either $j \geq k$ or $k > i + 1$.

(5) $V_1 = E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}$ and $V_2 = E_{h,k} \oplus E_{\sigma_I(k),\sigma_I(h)}$ with $i,j,k,h \in I_+ \sqcup I_\delta$ such that either $k \leq j$ and $i \leq h$ or $k \geq j$ and $i \geq h$.

(6) $V_1 = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$, $V_2 = E_{h,\sigma_I(k)} \oplus E_{k,\sigma_I(h)}$ and $V_1 = E_{\sigma_I(j),i} \oplus E_{\sigma_I(i),j}$ and $V_2 = E_{\sigma_I(k),h} \oplus E_{\sigma_I(h),k}$) with $i,j,k \in I_+ \sqcup I_\delta$.

(7) $V_1 = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$, $V_2 = E_{\sigma_I(k),k}$ (resp. $V_1 = E_{\sigma_I(k),k}$, $V_2 = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$) with $i,j,k \in I_+ \sqcup I_\delta$ such that $k > i+1$ and $i > j$ and $V_1 = E_{\sigma_I(j),i} \oplus E_{\sigma_I(i),j}$, $V_2 = E_{k,\sigma_I(k)}$ (resp. $V_1 = E_{k,\sigma_I(k)}$, $V_2 = E_{\sigma_I(j),i} \oplus E_{\sigma_I(i),j}$) with $i,j,k \in I_+ \sqcup I_\delta$ such that $i > k+1$ and $i < j$.

(8) $V_1 = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$, $V_2 = E_{h,k} \oplus E_{\sigma_I(k),\sigma_I(h)}$ (resp. $V_1 = E_{h,k} \oplus E_{\sigma_I(k),\sigma_I(h)}$, $V_2 = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$) with $i,j,k \in I_+ \sqcup I_\delta$ such that $i > j$ and either $k > i+1$ or $i \geq h$ and $V_1 = E_{\sigma_I(j),i} \oplus E_{\sigma_I(i),j}$, $V_2 = E_{h,k} \oplus E_{\sigma_I(k),\sigma_I(h)}$ (resp. $V_1 = E_{h,k} \oplus E_{\sigma_I(k),\sigma_I(h)}$, $V_2 = E_{\sigma_I(j),i} \oplus E_{\sigma_I(i),j}$) with $i,j,k \in I_+ \sqcup I_\delta$ such that $i < j$ and either $k \geq i$ or $i > h+1$.

(9) $V_1 = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$ and $V_2 = E_{\sigma_I(k),h} \oplus E_{\sigma_I(h),k}$ (resp. $V_1 = E_{\sigma_I(k),h} \oplus E_{\sigma_I(h),k}$ and $V_2 = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$) with $i,j,k \in I_+ \sqcup I_\delta$ such that $h > i+1$, $i > j$ and $h < k$.

We consider (1). By [D, lemma 4.1],

$$Hom_Q(E_{i,\sigma_I(i)}, E_{j,\sigma_I(j)}) = 0 = Hom_Q(E_{\sigma_I(j),j}, E_{\sigma_I(i),i})$$

and by lemma B.2.9,

$$\langle \underline{dim}(E_{i,\sigma_I(i)}), \underline{dim}(E_{j,\sigma_I(j)})\rangle = 0 = \langle \underline{dim}(E_{\sigma_I(j),j}), \underline{dim}(E_{\sigma_I(i),i})\rangle.$$

So we get

$$Ext^1_Q(E_{i,\sigma_I(i)}, E_{j,\sigma_I(j)}) = 0 = Ext^1_Q(E_{\sigma_I(j),j}, E_{\sigma_I(i),i}).$$

Similarly for (2), by [D, lemma 4.1] and by lemma B.2.9, we get $Ext^1_Q(V_1, V_2) = 0$.
We consider (3). We suppose $j > k+1$. By [D, lemma 4.1], we have

$$Hom_Q(E_{i,j}, E_{k,\sigma_I(k)}) = 0 = Hom_Q(E_{\sigma_I(j),\sigma_I(i)}, E_{k,\sigma_I(k)})$$

and so

$$\begin{aligned} &Hom_Q(E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}, E_{k,\sigma_I(k)}) \\ &= Hom_Q(E_{i,j}, E_{k,\sigma_I(k)}) \oplus Hom_Q(E_{\sigma_I(j),\sigma_I(i)}, E_{k,\sigma_I(k)}) = 0. \end{aligned}$$

Moreover, by lemma B.2.9

$$\langle \underline{dim}(E_{i,j}), \underline{dim}(E_{k,\sigma_I(k)})\rangle = 0 = \langle \underline{dim}(E_{\sigma_I(j),\sigma_I(i)}), \underline{dim}(E_{k,\sigma_I(k)})\rangle$$

and hence

$$\langle \underline{dim}(E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}), \underline{dim}(E_{k,\sigma_I(k)})\rangle =$$

$$\langle \underline{dim}(E_{i,j}), \underline{dim}(E_{k,\sigma_I(k)})\rangle + \langle \underline{dim}(E_{\sigma_I(j),\sigma_I(i)}), \underline{dim}(E_{k,\sigma_I(k)})\rangle = 0.$$

So we have

$$Ext^1_Q(E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}, E_{k,\sigma_I(k)}) = 0.$$

Similarly to (3), one proceeds for the other cases. □

Proposition 3.2.33. *Let (Q, σ) be a symmetric quiver of tame tape. Let V be a regular indecomposable symplectic (respectively orthogonal) representation of (Q, σ) such that $\underline{dim}(V) = h$ or $\bar{d}$. Moreover we suppose $V \neq E_{i,j} \oplus E_{\sigma_I(j),\sigma_I(i)}$ with $i, j \in I_+$ such that $e_i(ta) \neq 0$ or $e_j(ta) \neq 0$ for $a \in Q_1^\sigma$. Then, for every non-trivial short exact sequence*

$$0 \to V \to W \to V \to 0,$$

W is not symplectic (respectively it is not orthogonal).

Proof. We give a proof for $(Q = \widetilde{A}^{2,0,1}_{k,l}, \sigma)$ for the symplectic case, one proves similarly the other cases.
(i) Let $\underline{dim}(V) = h$. By lemma 3.2.4, the regular indecomposable symplectic representation of dimension h is $E_{i,\sigma_I(i)}$ containing $E_{\frac{l}{2}+1}$, i.e. the representation V defined by $V(x) = \mathbb{K}$ for every $x \in Q_0$ and

$$V(c) = \begin{cases} 0 & \text{if } c = a \\ Id & \text{otherwise,} \end{cases}$$

for $c \in Q_1$.
By [D, lemma 4.1], $Hom_Q(V, V) = \mathbb{K}$ and since $\langle h, h\rangle = 0$, then $Ext^1_Q(V, V) = \mathbb{K}$. One non-trivial auto-extension W of V is defined by $W(x) = \mathbb{K}^2$ for every $x \in Q_0$, and

$$W(c) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } c = a \\ \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

for $c \in Q_1$. Finally we note that W is not symplectic, because $W(a)$ is not symmetric. Since $Ext^1_Q(V, V) = \mathbb{K}$, the non-trivial auto-extensions of V is not symplectic.
(ii) Let $\underline{dim}(V) = \bar{d}$. The only regular indecomposable symplectic representations which we have to consider are $E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$ and $E_{\sigma_I(j),i} \oplus E_{\sigma_I(i),j}$

with $i, j \in I_+ \sqcup I_\delta$.
Let $V = E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}$, with $j < i$.

$$V(x) = V(\sigma(x)) \begin{cases} \mathbb{K} & \text{if } x \in \{x_r |\ e_m(x_r) \neq 0,\ m \in \{j+1, \dots, i\}\} \\ 0 & \text{if } x \in \{x_r |\ e_m(x_r) = 0, m \in I_+\} \\ \mathbb{K}^2 & \text{otherwise} \end{cases}$$

for $x \in Q_0$ and

$$V(c) = -V(\sigma(c))^t = \begin{cases} 1 & \text{if } c \in \{v_r |\ e_m(tv_r) \neq 0,\ m \in \{j+1, \dots, i\}\} \\ (1,1) & \text{if } c = v_r \text{ s.t. } e_j(tv_r) \neq 0 \\ 0 & \text{if } c \in \{v_r |\ e_m(tv_r) = 0, m \in I_+\} \cup \{a\} \\ Id_{2\times 2} & \text{otherwise} \end{cases}$$

for $c \in Q_1^+$ and $V(b) = Id_{2\times 2}$.
By [D, lemma 4.1],

$$dim_{\mathbb{K}}(Hom_Q(E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}, E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)})) = 3$$

and by lemma B.2.9,

$$\langle \underline{dim}(E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}), \underline{dim}(E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}) \rangle = 2.$$

So we have

$$Ext^1_Q(E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}, E_{i,\sigma_I(j)} \oplus E_{j,\sigma_I(i)}) = \mathbb{K}.$$

Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One non-trivial auto-extension W of V is defined by

$$W(x) = W(\sigma(x)) \begin{cases} \mathbb{K}^2 & \text{if } x \in \{x_r |\ e_m(x_r) \neq 0,\ m \in \{j+1, \dots, i\}\} \\ 0 & \text{if } x \in \{x_r |\ e_m(x_r) = 0, m \in I_+\} \\ \mathbb{K}^4 & \text{otherwise} \end{cases}$$

for $x \in Q_0$ and

$$W(c) = -W(\sigma(c))^t = \begin{cases} Id_{2\times 2} & \text{if } c \in \{v_r |\ e_m(tv_r) \neq 0,\ m \in \{j+1, \dots, i\}\} \\ A & \text{if } c = v_r \text{ s.t. } e_j(tv_r) \neq 0 \\ 0 & \text{if } c \in \{v_r |\ e_m(tv_r) = 0, m \in I_+\} \cup \{a\} \\ Id_{4\times 4} & \text{otherwise,} \end{cases}$$

for $c \in Q_1^+$ and $W(b) = B$. Finally we note that W is not symplectic because $W(b)$ is not symmetric. Since $Ext^1_Q(V, V) = \mathbb{K}$, this concludes the proof for

$(\widetilde{A}_{k,l}^{2,0,1}, \sigma)$. □

Proof of 3.2.31. *(1)* Let d be a symplectic regular dimension vector with decomposition (3.34). First we note that the symmetric dimension vectors appearing in decomposition (3.2.31) are not dimension vectors of the regular indecomposable symplectic representations which are exceptions of proposition 3.2.32 and 3.2.33. Let $\mathcal{O}(d)$ be the open orbit of the regular symplectic representations of dimension d. By [Bo1] and [Z], we obtain each representation V in $\mathcal{O}(d)$ as follows.
There are representations M_i, U_i, V_i and short exact sequences

$$0 \to U_i \to M_i \to V_i \to 0$$

such that $M_{i+1} = U_i \oplus V_i$ and $V = U_{n+1} \oplus V_{n+1}$, with $1 \leq i \leq n$ for some $n \in \mathbb{N}$.
By propositions 3.2.32 and 3.2.33, we have

(i) If $U_i \neq V_i$, then $Ext_Q^1(V_i, U_i) = 0$.

(ii) If $U_i = V_i$, then either $Ext_Q^1(U_i, U_i) = 0$ or no one non-trivial autoextension of U_i is symplectic. So, if $Ext_Q^1(U_i, U_i) \neq 0$ then U_i doesn't appear in decomposition of a symplectic representation.

Hence V decomposes in regular indecomposable symplectic representations of dimension β_i, where β_i are regular symmetric dimension vectors appearing in decomposition (3.34) of d.
(2) One proves similarly to *(1)*. □

Let d be a regular symmetric vector with a decomposition (3.34) or (3.35). We note that if $d = d_1 + d_2$ with d_1 and d_2 summands of this generic decomposition, we have canonical embeddings

$$SpSI(Q,d) \stackrel{\Phi_d}{\to} \bigoplus_{\chi \in char(Sp(Q,d))} SpSI(Q,d_1)_{\chi|_{d_1}} \otimes SpSI(Q,d_2)_{\chi_{d_2}} \qquad (3.36)$$

and

$$OSI(Q,d) \stackrel{\Psi_d}{\to} \bigoplus_{\chi \in char(O(Q,d))} OSI(Q,d_1)_{\chi|_{d_1}} \otimes OSI(Q,d_2)_{\chi_{d_2}}, \qquad (3.37)$$

induced by the restriction homomorphism. We prove theorem 3.2.9 by induction on the number of the summands $e_{[i,j]} + \delta e_{[i,j]}$, $e_{[i,\sigma_I(i)]}$, $e_{[i_{2k},\sigma_I(i_{2k-1})]} + e_{[i_{2k-1},\sigma_I(i_{2k})]}$ and respective summands corresponding to the admissible arcs in $\mathcal{A}'(d)$ and in $\mathcal{A}''(d)$. If this number is 0, then $d = ph$ and it was already proved. We suppose that the generic decomposition of d contains one of those summands and, without loss of generality, we can assume that this summand is one of those corresponding to the arcs in $\mathcal{A}(d)$. In particular we suppose that this summand is $e_{[s,\sigma_I(s)]}$ (one proceeds similarly for the other types), with $s \in I_+ \sqcup I_\delta$, and we can assume $ind[s, \sigma_I(s)] = r = max\{p_k\}$. We call $d_2 = e_{[s,\sigma_I(s)]}$ and so $d_1 = d - e_{[s,\sigma_I(s)]}$. Now we compare the generators of the algebras $SpSI(Q, d)$ and $SpSI(Q, d_1)$ (respectively $OSI(Q, d)$ and $OSI(Q, d_1)$). By induction the generators of $SpSI(Q, d_1)$ (respectively of $OSI(Q, d_1)$) are described by theorem 3.2.9. Since $\Delta'(d) = \Delta'(d_1)$ and $\Delta''(d) = \Delta''(d_1)$, the generators $c_0, \dots, c_t$ (with $t = \frac{p}{2}$, $\frac{p-1}{2}$ or p), those corresponding to the arcs from $\mathcal{A}'(d)$ and those corresponding to the arcs from $\mathcal{A}''(d)$ occur. So it's enough to study the behavior of the semi-invariants corresponding to the arcs from $\mathcal{A}(d)$. We describe the link between the admissible arcs of the polygons $\Delta(d)$ and $\Delta(d_1)$. We have

$$d_1 = ph + \sum_{i \in I_+ \setminus (I_+ \cap [s,\sigma_I(s)])} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta \setminus (I_\delta \cap [s,\sigma_I(s)])} p_i e_i +$$
$$\sum_{i \in I_+ \cap [s,\sigma_I(s)]} p_i(e_i + \delta e_i) + \sum_{i \in I_\delta \cap [s,\sigma_I(s)]} p_i e_i +$$
$$\sum_{i \in I'_+} p'_i(e'_i + \delta e'_i) + \sum_{i \in I'_\delta} p'_i e'_i + \sum_{i \in I''_+} p''_i(e''_i + \delta e''_i).$$

We have two cases

(1) $p_{s-1} = p_{\sigma_I(s)+1} < r - 1$ with $s - 1 \in I_+$,

(2) $p_{s-1} = p_{\sigma_I(s)+1} = r - 1$ with $s - 1 \in I_+$.

in the case (1) the only difference between the structure of $\mathcal{A}(d)$ and $\mathcal{A}(d_1)$ is that the admissible arcs $[s, s+1], [s+1, s+2], \dots, [\sigma_I(s) - 1, \sigma(s)]$ are of index r in $\mathcal{A}(d)$ and of index $r - 1$ in $\mathcal{A}(d_1)$. In the case (2) we have the admissible arc $[s - 1, \sigma_I(s) + 1]$ of index $r - 1$. The admissible arcs $[s, s+1], [s+1, s+2], \dots, [\sigma_I(s) - 1, \sigma_I(s)]$ are of index s in $\mathcal{A}(d)$ and the admissible arcs $[s-1, s], [s, s+1], \dots, [\sigma_I(s) - 1, \sigma_I(s)], [\sigma_I(s), \sigma_I(s) + 1]$ are of index $r - 1$ in $\mathcal{A}(d_1)$.
Now we prove that the embeddings Φ_d and Ψ_d are isomorphisms and this will be done in two steps. The first step is to show case by case that the semi-invariants corresponding to the admissible arcs $[i, j]$ are non zero c^V for some $V \in Rep(Q)$ and, if V satisfy property *(Spp)* or *(Op)*, they are non zero pf^V. The second step is to give an explicit description of the generators of the algebras on the right hand side of Φ_d and Ψ_d. This is based on the knowledge, given by inductive hypothesis, of the algebra $SpSI(Q, d_1)$ (respectively $OSI(Q, d_1)$). We can describe explicitly the generators of the algebra $SpSI(Q, d_2)$ (respectively $OSI(Q, d_2)$) and we can note that they are determinants or pfaffians, knowing that the group

$Sp(Q,d_2)$ (respectively $O(Q,d_2)$) has an open orbit in $SpRep(Q,d_2)$ (respectively $ORep(Q,d_2)$) and hence that $SpSI(Q,d_2)$ (respectively $OSI(Q,d_2)$) is a polynomial ring (lemma A.2.5). At this point we know the generators of the algebras on the right hand side of Φ_d and Ψ_d. Now, using the fact that these are determinants or pfaffians, we prove that they actually are in $SpSI(Q,d)$ (respectively in $OSI(Q,d)$) and that the embeddings Φ_d and Ψ_d are isomorphisms.
We will consider case by case the semi-invariants corresponding to each admissible arc $[i,j]$. To simplify the notation we shall call a both the arrow $a \in Q_1$ and the linear map $V(a)$ defined on a, where V is a representation of Q.

3.2.1.1 $\widetilde{A}^{2,0,1}_{k,l}$

We have at most two τ^+-orbits Δ and Δ' of the dimension vectors of nonhomogeneous simple regular representation. We assume $n \geq 2$ and we consider the τ-orbit $\{e_1 = \delta e_1, e_2, \ldots, e_{[\frac{l}{2}]+1}, \delta e_{[\frac{l}{2}]+1}, \ldots, \delta e_2\}$. Let $[i,j] \in \mathcal{A}(d)$. If we consider the arc $[1,1]$ of index 0, i.e. $p_1 = 0, p_2 \neq 0, \ldots, p_{[\frac{l}{2}]+1} \neq 0$, we have the minimal projective resolution of $V_{(0,1)}$

$$0 \longrightarrow P_{\sigma(a_0)} \xrightarrow{d^{V_{(0,1)}}_{min}} P_{a_0} \longrightarrow V_{(0,1)} \longrightarrow 0$$

where $d^{V_{(0,1)}}_{min} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1$ and so

$$c^{V_{(0,1)}} = det(Hom_Q(d^{V_{(0,1)}}_{min}, \cdot)) = det(\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1)$$

in the symplectic case and $pf^{V_{(0,1)}} = pf(\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1)$ in the orthogonal case, since in this case a is skew-symmetric and $\sigma(v_i) = -(v_i)^t$. If we consider the arc $[\sigma_I(2), 2] = [0,2]$ of index 0, i.e. $p_{\sigma_I(2)} = 0 = p_2, p_1 \neq 0$, we have the minimal projective resolution of $V_{(1,0)}$

$$0 \longrightarrow P_{\sigma(a_0)} \xrightarrow{d^{V_{(1,0)}}_{min}} P_{a_0} \longrightarrow V_{(1,0)} \longrightarrow 0$$

where $d^{V_{(1,0)}}_{min} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1$ and so

$$c^{V_{(1,0)}} = det(Hom_Q(d^{V_{(1,0)}}_{min}, \cdot)) = det(\sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1)$$

in the symplectic case and $pf^{V_{(1,0)}} = pf(\sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1)$ in the orthogonal case, since in this case b is skew-symmetric and $\sigma(u_i) = -(u_i)^t$.
We note that for $l = 2$ we have only the admissible arcs $[1,1]$ an $[\sigma_I(2), 2]$.
We assume now that $l \geq 4$ (l is even) and $[i,j]$ is not an admissible arc

considered above. If $1 \leq i < j \leq \frac{l}{2} + 1$, then we identify $[i, j]$ with the path $v_{j-1} \cdots v_i$ in Q and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{x_{j-1}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{x_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = v_{j-1} \cdots v_i$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det(v_{j-1} \cdots v_i).$$

We note that

$$c^{\tau^- \nabla E_{i,j-1}} = c^{E_{\sigma_I(j), \sigma_I(i)-1}} = det(\sigma(v_i) \cdots \sigma(v_{j-1})) = det(v_{j-1} \cdots v_i) = c^{E_{i,j-1}}.$$

If $j = \sigma_I(i)$ then in the symplectic case we get $c^{E_{i,\sigma_I(i)-1}} = det(\sigma(v_i) \cdots a \cdots v_i)$ and in the orthogonal case, we get $pf^{E_{i,\sigma_I(i)-1}} = pf(\sigma(v_i) \cdots a \cdots v_i)$, since $\sigma(v_i) \cdots a \cdots v_i$ is skew-symmetric. Now we consider the arcs $[i, j]$ which have e_1 as internal vertex. For these arcs, $2 \leq j < i - 1 < l$ and $[i, j]$ can be identify with the path in Q consisting of the path $v_l \cdots v_{i-1} = \sigma(v_1) \cdots v_{i-1}$, then coming back by $\sigma(u_1) \cdots b \cdots u_1$ and at last passing for $v_{j-1} \cdots v_1$. We have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{\sigma(a_0)} \oplus P_{x_{j-1}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{a_0} \oplus P_{x_{i-2}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = \begin{pmatrix} \sigma(u_1) \cdots b \cdots u_1 & v_{j-1} \cdots v_1 \\ \sigma(v_1) \cdots v_{i-1} & 0 \end{pmatrix}$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det \begin{pmatrix} \sigma(u_1) \cdots b \cdots u_1 & \sigma(v_1) \cdots v_{i-1} \\ v_{j-1} \cdots v_1 & 0 \end{pmatrix}.$$

In particular we note that if $i = \sigma_I(j)$, in the orthogonal case, we get

$$pf^{E_{\sigma_I(j),j-1}} = pf \begin{pmatrix} \sigma(u_1) \cdots b \cdots u_1 & \sigma(v_1) \cdots \sigma(v_{j-1}) \\ v_{j-1} \cdots v_1 & 0 \end{pmatrix},$$

since b is skew-symmetric and $\sigma(v_i) = -(v_i)^t$. Finally we note that $V_{(0,1)}$, $V_{(1,0)}$, $E_{i,\sigma_I(i)-1}$ and $E_{\sigma_I(j),j-1}$ satisfy property *(Spp)*. Similarly we define the semi-invariants for the admissible arcs $[i, j]$ in $\mathcal{A}'(d)$, exchanging the upper paths of $\widetilde{A}_{k,l}^{2,0,1}$ with the lower ones.

3.2.1.2 $\widetilde{A}_{k,l}^{2,0,2}$

We have at most two τ^+-orbits Δ and Δ' of the dimension vectors of non-homogeneous simple regular representation. We assume $n \geq 2$ and we consider the τ-orbit

$$\{e_2, \ldots, e_{[\frac{l}{2}]+2}, \delta e_{[\frac{l}{2}]+2}, \ldots, \delta e_2 = e_1\}.$$

Let $[i,j] \in \mathcal{A}(d)$. If we consider the arc $[\sigma_I(2),2] = [1,2]$ of index 0, i.e. $p_2 = 0, p_3 \neq 0, \ldots, p_{[\frac{l}{2}]+2} \neq 0$, we have the minimal projective resolution of $V_{(1,0)}$

$$0 \longrightarrow P_{\sigma(a_0)} \overset{d_{min}^{V_{(1,0)}}}{\longrightarrow} P_{a_0} \longrightarrow V_{(1,0)} \longrightarrow 0$$

where $d_{min}^{V_{(1,0)}} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1$ and so

$$c^{V_{(1,0)}} = det(Hom_Q(d_{min}^{V_{(1,0)}}, \cdot)) = det(\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1)$$

in the symplectic case and $pf^{V_{(1,0)}} = pf(\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})av_{\frac{l}{2}}\cdots v_1)$ in the orthogonal case, since in this case a is skew-symmetric and $\sigma(v_i) = -(v_i)^t$. If we consider the arc $[\sigma_I(3),3] = [0,3]$ of index 0, i.e. $p_3 = 0, p_2 \neq 0$, we have the minimal projective resolution of $V_{(0,1)}$

$$0 \longrightarrow P_{y_{\frac{k}{2}}} \oplus P_{\sigma(a_0)} \overset{d_{min}^{V_{(0,1)}}}{\longrightarrow} P_{\sigma(y_{\frac{k}{2}})} \oplus P_{a_0} \longrightarrow V_{(0,1)} \longrightarrow 0$$

where $d_{min}^{V_{(0,1)}} = \begin{pmatrix} b & \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}}) \\ u_{\frac{k}{2}}\cdots u_1 & 0 \end{pmatrix}$ and so

$$c^{V_{(0,1)}} = det(Hom_Q(d_{min}^{V_{(0,1)}}, \cdot)) = det\begin{pmatrix} b & u_{\frac{k}{2}}\cdots u_1 \\ \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}}) & 0 \end{pmatrix}$$

in the symplectic case and

$$pf^{V_{(0,1)}} = pf\begin{pmatrix} b & u_{\frac{k}{2}}\cdots u_1 \\ \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}}) & 0 \end{pmatrix}$$

in the orthogonal case, since b is skew-symmetric and $\sigma(u_i) = -(u_i)^t$. We note that for $l = 2$ we have only the admissible arcs $[\sigma_I(2),2]$ an $[\sigma_I(3),3]$. We assume now that $l \geq 4$ and $[i,j]$ is not an admissible arc considered above. If $2 \leq i < j \in I \leq \frac{l}{2}+2$, then we identify $[i,j]$ with the path $v_{j-2}\cdots v_{i-1}$ in Q and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{x_{j-2}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{x_{i-2}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = v_{j-2}\cdots v_{i-1}$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det(v_{j-2}\cdots v_{i-1}).$$

We note that

$$c^{\tau^-\nabla E_{i,j-1}} = c^{E_{\sigma_I(j),\sigma_I(i)-1}} = det(\sigma(v_{i-1})\cdots\sigma(v_{j-2})) = det(v_{j-2}\cdots v_{i-1}) = c^{E_{i,j-1}}.$$

Moreover, if $j = \sigma_I(i)$ then, only in the orthogonal case, we get $pf^{E_{i,\sigma_I(i)-1}} = pf(\sigma(v_{i-1}) \cdots a \cdots v_{i-1})$ since $\sigma(v_{i-1}) \cdots a \cdots v_{i-1}$ is skew-symmetric. Now we consider the arcs $[i, j]$ which have $\delta e_2 = e_1$ and e_2 as internal vertex. For these arcs, $3 \le j < i - 1 < l + 1$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{y_{\frac{k}{2}}} \oplus P_{\sigma(a_0)} \oplus P_{x_{j-2}} \xrightarrow{d_{min}^{E_{i,j-1}}} P_{\sigma(y_{\frac{k}{2}})} \oplus P_{a_0} \oplus P_{x_{i-3}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = \begin{pmatrix} b & \sigma(u_1) \cdots \sigma(u_{\frac{k}{2}}) & 0 \\ u_{\frac{k}{2}} \cdots u_1 & 0 & v_{j-2} \cdots v_1 \\ 0 & \sigma(v_1) \cdots v_{i-2} & 0 \end{pmatrix}$ and so

$$c^{E_{i,j-1}} = det \begin{pmatrix} b & u_{\frac{k}{2}} \cdots u_1 & 0 \\ \sigma(u_1) \cdots \sigma(u_{\frac{k}{2}}) & 0 & \sigma(v_1) \cdots v_{i-2} \\ 0 & v_{j-2} \cdots v_1 & 0 \end{pmatrix}.$$

In particular we note that if $i = \sigma_I(j)$, in the orthogonal case, we get

$$pf^{E_{\sigma_I(j),j-1}} = pf \begin{pmatrix} b & u_{\frac{k}{2}} \cdots u_1 & 0 \\ \sigma(u_1) \cdots \sigma(u_{\frac{k}{2}}) & 0 & \sigma(v_1) \cdots \sigma(v_{j-2}) \\ 0 & v_{j-2} \cdots v_1 & 0 \end{pmatrix},$$

since b is skew-symmetric, $\sigma(v_i) = -(v_i)^t$ and $\sigma(u_i) = -(u_i)^t$. Finally we note that $V_{(0,1)}$, $V_{(1,0)}$, $E_{i,\sigma_I(i)-1}$ and $E_{\sigma_I(j),j-1}$ satisfy property *(Spp)*. Similarly we define the semi-invariants for the admissible arcs $[i, j]$ in $\mathcal{A}'(d)$, exchanging the upper paths of $\widetilde{A}_{k,l}^{2,0,2}$ with the lower ones.

3.2.1.3 $\widetilde{A}_{k,l}^{0,2}$

We have at most two τ^+-orbits Δ and Δ' of the dimension vectors of non-homogeneous simple regular representation. We assume $n \ge 2$ and we consider the τ-orbit

$$\{e_1 = \delta e_1, e_2, \ldots, e_{[\frac{l-1}{2}]+1}, e_{[\frac{l-1}{2}]+2} = \delta e_{[\frac{l-1}{2}]+2}, \delta e_{[\frac{l-1}{2}]+1}, \ldots, \delta e_2\}.$$

Let $[i, j] \in \mathcal{A}(d)$. If we consider the arc $[1, 1]$ of index 0, i.e. $p_1 = 0, p_2 \neq 0, \ldots, p_{[\frac{l-1}{2}]+2} \neq 0$, we have the minimal projective resolution of $V_{(0,1)}$

$$0 \longrightarrow P_{\sigma(a_0)} \xrightarrow{d_{min}^{V_{(0,1)}}} P_{a_0} \longrightarrow V_{(0,1)} \longrightarrow 0$$

where $d_{min}^{V_{(0,1)}} = \sigma(v_1) \cdots \sigma(v_{\frac{l}{2}}) v_{\frac{l}{2}} \cdots v_1$ and so

$$c^{V_{(0,1)}} = det(Hom_Q(d_{min}^{V_{(0,1)}}, \cdot)) = det(\sigma(v_1) \cdots \sigma(v_{\frac{l}{2}}) v_{\frac{l}{2}} \cdots v_1)$$

in the orthogonal case and $pf^{V_{(0,1)}} = pf(\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1)$ in the symplectic case, since by definition of symplectic representation $\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1$ is skew-symmetric. If we consider the arc $[\sigma_I(2), 2] = [0, 2]$ of index 0, i.e. $p_{\sigma_I(2)} = 0 = p_2, p_1 \neq 0$, we have the minimal projective resolution of $V_{(1,0)}$

$$0 \longrightarrow P_{\sigma(a_0)} \overset{d_{min}^{V_{(1,0)}}}{\longrightarrow} P_{a_0} \longrightarrow V_{(1,0)} \longrightarrow 0$$

where $d_{min}^{V_{(1,0)}} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})u_{\frac{k}{2}}\cdots u_1$ and so

$$c^{V_{(1,0)}} = det(Hom_Q(d_{min}^{V_{(1,0)}}, \cdot)) = det(\sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})u_{\frac{k}{2}}\cdots u_1)$$

in the orthogonal case and $pf^{V_{(1,0)}} = pf(\sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})u_{\frac{k}{2}}\cdots u_1)$ in the symplectic case, since by definition of symplectic representation $\sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})u_{\frac{k}{2}}\cdots u_1$ is skew-symmetric. We note that for $l = 2$ we have only the admissible arcs $[1, 1]$ an $[\sigma_I(2), 2]$. We assume now that $l \geq 4$ (l is even) and $[i, j]$ is not an admissible arc considered above. If $1 \leq i < j \leq l + 1$, then we identify $[i, j]$ with the path $v_{j-1}\cdots v_i$ in Q and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{x_{j-1}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{x_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = v_{j-1}\cdots v_i$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det(v_{j-1}\cdots v_i).$$

We note that

$$c^{\tau^-\nabla E_{i,j-1}} = c^{E_{\sigma_I(j),\sigma_I(i)-1}} = det(\sigma(v_i)\cdots\sigma(v_{j-1})) = det(v_{j-1}\cdots v_i) = c^{E_{i,j-1}}.$$

Moreover, if $j = \sigma_I(i)$ then, only in the symplectic case, we get $pf^{E_{i,\sigma_I(i)-1}} = pf(\sigma(v_i)\cdots v_i)$, since $\sigma(v_i)\cdots v_i$ is skew-symmetric. Now we consider the arcs $[i, j]$ which have e_1 as internal vertex. For these arcs, $2 \leq j < i - 1 < l$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{\sigma(a_0)} \oplus P_{x_{j-1}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{a_0} \oplus P_{x_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = \begin{pmatrix} \sigma(u_1)\cdots u_1 & v_{j-1}\cdots v_1 \\ \sigma(v_1)\cdots v_i & 0 \end{pmatrix}$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det\begin{pmatrix} \sigma(u_1)\cdots u_1 & \sigma(v_1)\cdots v_i \\ v_{j-1}\cdots v_1 & 0 \end{pmatrix}.$$

In particular we note that if $i = \sigma_I(j)$, in the symplectic case, we get

$$pf^{E_{\sigma_I(j),j-1}} = pf \begin{pmatrix} \sigma(u_1)\cdots u_1 & \sigma(v_1)\cdots\sigma(v_{j-1}) \\ v_{j-1}\cdots v_1 & 0 \end{pmatrix},$$

since $\sigma(u_1)\cdots u_1$ and $\sigma(v_i) = -(v_i)^t$. Finally we note that $V_{(0,1)}$, $V_{(1,0)}$, $E_{i,\sigma_I(i)-1}$ and $E_{\sigma_I(j),j-1}$ satisfy *(Op)*. Similarly we define the semi-invariants for the admissible arcs $[i,j]$ in $\mathcal{A}'(d)$, exchanging the upper paths of $\widetilde{A}^{0,2}_{k,l}$ with the lower ones.

3.2.1.4 $\widetilde{A}^{1,1}_{k,l}$

We have at most two τ^+-orbits Δ and Δ' of the dimension vectors of non-homogeneous simple regular representation. We assume $n \geq 2$ and we consider the τ-orbit

$$\{e_1 = \delta e_1, e_2, \ldots, e_{[\frac{l-1}{2}]+1}, e_{[\frac{l-1}{2}]+2} = \delta e_{[\frac{l-1}{2}]+2}, \delta e_{[\frac{l-1}{2}]+1}, \ldots, \delta e_2\}.$$

Let $[i,j] \in \mathcal{A}(d)$. If we consider the arc $[1,1]$ of index 0, i.e. $p_1 = 0, p_2 \neq 0, \ldots, p_{[\frac{l-1}{2}]+2} \neq 0$, we have the minimal projective resolution of $V_{(0,1)}$

$$0 \longrightarrow P_{\sigma(a_0)} \xrightarrow{d^{V_{(0,1)}}_{min}} P_{a_0} \longrightarrow V_{(0,1)} \longrightarrow 0$$

where $d^{V_{(0,1)}}_{min} = \sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1$ and so

$$c^{V_{(0,1)}} = det(Hom_Q(d^{V_{(0,1)}}_{min}, \cdot)) = det(\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1)$$

in the orthogonal case and $pf^{V_{(0,1)}} = pf(\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1)$ in the symplectic case, since by definition of symplectic representation $\sigma(v_1)\cdots\sigma(v_{\frac{l}{2}})v_{\frac{l}{2}}\cdots v_1$ is skew-symmetric . If we consider the arc $[\sigma_I(2), 2] = [0,2]$ of index 0, i.e. $p_{\sigma_I(2)} = 0 = p_2, p_1 \neq 0$, then we have the minimal projective resolution of $V_{(1,0)}$

$$0 \longrightarrow P_{\sigma(a_0)} \xrightarrow{d^{V_{(1,0)}}_{min}} P_{a_0} \longrightarrow V_{(1,0)} \longrightarrow 0$$

where $d^{V_{(1,0)}}_{min} = \sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1$ and so

$$c^{V_{(1,0)}} = det(Hom_Q(d^{V_{(1,0)}}_{min}, \cdot)) = det(\sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1)$$

in the symplectic case and $pf^{V_{(1,0)}} = pf(\sigma(u_1)\cdots\sigma(u_{\frac{k}{2}})bu_{\frac{k}{2}}\cdots u_1)$ in the orthogonal case, since b is skew-symmetric and $\sigma(u_i) = -(u_i)^t$. We note that for $l = 2$ we have only the admissible arcs $[1,1]$ an $[\sigma_I(2), 2]$. We assume now that $l \geq 4$ (l is even) and $[i,j]$ is not an admissible arc considered above. If $1 \leq i < j \leq l+1$, then we identify $[i,j]$ with the path $v_{j-1}\cdots v_i$ in Q and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{x_{j-1}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{x_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = v_{j-1}\cdots v_i$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det(v_{j-1}\cdots v_i).$$

We note that

$$c^{\tau^-\nabla E_{i,j-1}} = c^{E_{\sigma_I(j),\sigma_I(i)-1}} = det(\sigma(v_i)\cdots\sigma(v_{j-1})) = det(v_{j-1}\cdots v_i) = c^{E_{i,j-1}}.$$

Moreover, if $j = \sigma_I(i)$ then, only in the symplectic case, we get $pf(\sigma(v_i)\cdots v_i) = pf^{E_{i,\sigma_I(i)-1}}$ since $\sigma(v_i)\cdots v_i$ is skew-symmetric. Now we consider the arcs $[i,j]$ which have e_1 as internal vertex. For these arcs, $2 \leq j < i-1 < l$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{\sigma(a_0)} \oplus P_{x_{j-1}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{a_0} \oplus P_{x_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = \begin{pmatrix} \sigma(u_1)\cdots b\cdots u_1 & \sigma(v_1)\cdots v_i \\ v_{j-1}\cdots v_1 & 0 \end{pmatrix}$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det\begin{pmatrix} \sigma(u_1)\cdots b\cdots u_1 & \sigma(v_1)\cdots v_i \\ v_{j-1}\cdots v_1 & 0 \end{pmatrix}.$$

In particular we note that if $i = \sigma_I(j)$, in the orthogonal case, we get

$$pf^{E_{\sigma_I(j),j-1}} = pf\begin{pmatrix} \sigma(u_1)\cdots b\cdots u_1 & \sigma(v_1)\cdots\sigma(v_{j-1}) \\ v_{j-1}\cdots v_1 & 0 \end{pmatrix},$$

since b is skew-symmetric, $\sigma(v_i) = -(v_i)^t$ and $\sigma(u_i) = -(u_i)^t$. Finally we note that $V_{(0,1)}$, $E_{i,\sigma_I(i)-1}$ satisfy *(Op)* and $V_{(1,0)}$, $E_{\sigma_I(j),j-1}$ satisfy property *(Spp)*. Similarly we define the semi-invariants for the admissible arcs $[i,j]$ in $\mathcal{A}'(d)$, exchanging the upper paths of $\widetilde{A}_{k,l}^{1,1}$ with the lower ones and tracing out the procedure done for $\widetilde{A}_{k,l}^{2,0,1}$.

3.2.1.5 $\widetilde{A}_{k,k}^{0,0}$

We have at most two τ^+-orbits Δ and Δ' of the dimension vectors of non-homogeneous simple regular representation but in this case $\Delta = \delta\Delta'$ so it's enough to study the semi-invariants associated to the arcs in $\mathcal{A}(d)$, because these are equal to those ones associated to the arcs in $\mathcal{A}'(d)$. We assume

$k \geq 2$ and we consider the τ-orbit $\{e_0, e_1, e_2, \dots, e_{k-1}\}$. Let $[i,j] \in \mathcal{A}(d)$. If we consider the arc $[1,1]$ of index 0, i.e. $p_1 = 0, p_2 \neq 0, \dots, p_{k-1} \neq 0$, we have the minimal projective resolution of $V_{(0,1)}$

$$0 \longrightarrow P_{\sigma(a_0)} \stackrel{d_{min}^{V_{(0,1)}}}{\longrightarrow} P_{a_0} \longrightarrow V_{(0,1)} \longrightarrow 0$$

where $d_{min}^{V_{(0,1)}} = v_k \cdots v_1$ and so

$$c^{V_{(0,1)}} = det(Hom_Q(d_{min}^{V_{(0,1)}}, \cdot)) = det(v_k \cdots v_1).$$

If we consider the arc $[0,2]$ of index 0, i.e. $p_0 = 0 = p_2, p_1 \neq 0$, then we have the minimal projective resolution of $V_{(1,0)}$

$$0 \longrightarrow P_{\sigma(a_0)} \stackrel{d_{min}^{V_{(1,0)}}}{\longrightarrow} P_{a_0} \longrightarrow V_{(1,0)} \longrightarrow 0$$

where $d_{min}^{V_{(0,1)}} = u_k \cdots u_1$ and so

$$c^{V_{(1,0)}} = det(Hom_Q(d_{min}^{V_{(1,0)}}, \cdot)) = det(u_k \cdots u_1).$$

We note that for $k = 2$ we have only the admissible arcs $[1,1]$ an $[0,2]$. We assume now that $k \geq 3$ and $[i,j]$ is not an admissible arc considered above. If $1 \leq i < j \leq k$, then we identify $[i,j]$ with the path $v_{j-1} \cdots v_i$ in Q and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{x_{j-1}} \stackrel{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{x_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = v_{j-1} \cdots v_i$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det(v_{j-1} \cdots v_i).$$

Now we consider the arcs $[i,j]$ which have e_1 as internal vertex. For these arcs, $2 \leq j < i-1 < k-1$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{\sigma(a_0)} \oplus P_{x_{j-1}} \stackrel{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{a_0} \oplus P_{x_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = \begin{pmatrix} u_k \cdots u_1 & v_{j-1} \cdots v_1 \\ v_k \cdots v_i & 0 \end{pmatrix}$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det \begin{pmatrix} u_k \cdots u_1 & v_k \cdots v_i \\ v_{j-1} \cdots v_1 & 0 \end{pmatrix}.$$

3.2.1.6 $\widetilde{D}_n^{1,0}$

In this case there are three τ-orbit $\Delta = \{e_1 = \delta e_1, e_2, \dots, e_{n-1}, \delta e_{n-1}, \dots, \delta e_2\}$, $\Delta' = \{e'_0 = \delta e'_0, e'_1 = \delta e'_1\}$ and $\Delta'' = \{e''_0 = \delta e''_1\}$. The only admissible arcs in $\Delta'(d)$ and $\Delta''(d)$ are $[0,0]$ and $[1,1]$, recalling that $e'_0 + e'_1 = h = e''_0 + e''_1$. For such arcs in Δ' we have the minimal projective resolution of $E'_{0,1}$

$$0 \longrightarrow P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \overset{d_{min}^{E'_{0,1}}}{\longrightarrow} P_{t_1} \oplus P_{t_2} \longrightarrow E'_{0,1} \longrightarrow 0$$

where $d_{min}^{E'_{0,1}} = \begin{pmatrix} \sigma(a)\bar{c}a & 0 \\ \sigma(a)\bar{c}b & \sigma(b)\bar{c}b \end{pmatrix}$, similarly for $E'_{1,0}$ and so

$$c^{E'_{1,0}} = c^{E'_{0,1}} = det(Hom_Q(d_{min}^{E'_{0,1}}, \cdot)) = det \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(a)\bar{c}b \\ 0 & \sigma(b)\bar{c}b \end{pmatrix}.$$

We note that the matrices $\sigma(a)\bar{c}b$, $\sigma(a)\bar{c}a$ and $\sigma(b)\bar{c}b)$ have different size for $[0,0]$ and for $[1,1]$. Whereas in Δ'' we have we have the minimal projective resolution of $c^{E''_{0,1}} = c^{E''_{1,0}}$

$$0 \longrightarrow P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \overset{d_{min}^{E''_{0,1}}}{\longrightarrow} P_{t_1} \oplus P_{t_2} \longrightarrow E''_{0,1} \longrightarrow 0$$

where $d_{min}^{E''_{0,1}} = \begin{pmatrix} 0 & \sigma(b)\bar{c}a \\ \sigma(a)\bar{c}b & \sigma(b)\bar{c}b \end{pmatrix}$ and so

$$c^{E''_{1,0}} = c^{E''_{0,1}} = det(Hom_Q(d_{min}^{E''_{0,1}}, \cdot)) = det \begin{pmatrix} 0 & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the symplectic case and

$$pf^{E''_{0,1}} = pf^{E''_{1,0}} = pf \begin{pmatrix} 0 & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the orthogonal case, since $\bar{c}$ is skew-symmetric, $\sigma(b) = -b^t$ and $\sigma(a) = -a^t$. We assume $n \geq 3$ and we take $[i,j] \in \mathcal{A}(d)$. If we consider the arc $[1,1]$, we have the minimal projective resolution $V_{(1,1)}$

$$0 \longrightarrow P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \overset{d_{min}^{V_{(1,1)}}}{\longrightarrow} P_{t_1} \oplus P_{t_2} \longrightarrow V_{(1,1)} \longrightarrow 0$$

where $d_{min}^{V_{(1,1)}} = \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(b)\bar{c}a \\ \sigma(a)\bar{c}b & \sigma(b)\bar{c}b \end{pmatrix}$ and so

$$c^{V_{(1,1)}} = det(Hom_Q(d_{min}^{V_{(1,1)}}, \cdot)) = det \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the symplectic case and

$$pf^{V_{(1,1)}} = pf \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the orthogonal case. If $[i,j]$ doesn't contain e_1 as an internal vertex, then we have $1 \leq i < j \leq 2n$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{z_{j-1}} \stackrel{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{z_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = c_{j-2} \cdots c_{i-1}$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det(c_{j-2} \cdots c_{i-1}),$$

where $c_0 = (a,b)$ and $c_{2n-1} = \sigma(c_0)$. In particular in the orthogonal case if $j = \sigma_I(i)$ then $pf^{E_{i,\sigma_I(i)-1}} = pf(\sigma(c_{i-1}) \cdots c_{i-1})$, since in this case $\sigma(c_i) = -(c_i)^t$ and c_{n-2} is skew-symmetric. If $[i,j]$ contains e_1 as an internal vertex, i.e. $2 \leq j < i \leq 2n-1$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{z_{j-1}} \oplus P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \stackrel{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{z_{i-1}} \oplus P_{t_1} \oplus P_{t_2} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = \begin{pmatrix} 0 & \sigma(a)c_{2n-3,i-1} & \sigma(b)c_{2n-3,i-1} \\ c_{j-2,1}a & \sigma(a)c_{2n-3,1}a & \sigma(b)c_{2n-3,1}a \\ c_{j-2,1}b & \sigma(a)c_{2n-3,1}b & \sigma(b)c_{2n-3,1}b \end{pmatrix}$ and so

$$c^{E_{i,j-1}} = \begin{pmatrix} 0 & c_{j-2,1}a & c_{j-2,1}b \\ \sigma(a)c_{2n-3,i-1} & \sigma(a)c_{2n-3,1}a & \sigma(a)c_{2n-3,1}b \\ \sigma(b)c_{2n-3,i-1} & \sigma(b)c_{2n-3,1}a & \sigma(b)c_{2n-3,1}b \end{pmatrix}$$

where $c_{k,l} = c_k \cdots c_l$ and $c_{0,1} = id$. If $\sigma_I(i) = j$ then, only in the orthogonal case, we have

$$pf^{E_{\sigma_I(j),j-1}} = pf \begin{pmatrix} 0 & c_{j-2,1}a & c_{j-2,1}b \\ \sigma(a)\sigma(c_{j-2,1}) & \sigma(a)c_{2n-3,1}a & \sigma(a)c_{2n-3,1}b \\ \sigma(b)\sigma(c_{j-2,1}) & \sigma(b)c_{2n-3,1}a & \sigma(b)c_{2n-3,1}b \end{pmatrix},$$

since $\sigma(c_{j-2,1}) = -(c_{j-2,1})^t$, $\sigma(a) = -a^t$, $\sigma(b) = -b^t$ and $c_{2n-3,1}$ is skew-symmetric.
Finally we note that $E''_{1,0}$, $V_{(1,1)}$, $E_{i,\sigma_I(i)-1}$ and $E_{\sigma_I(j),j-1}$ satisfy property *(Spp)*.

3.2.1.7 $\widetilde{D}_n^{0,1}$

There are again three τ-orbit $\Delta = \{e_1 = \delta e_1, e_2, \dots, e_{n-1} = \delta e_{n-1}, \dots, \delta e_2\}$, $\Delta' = \{e'_0 = \delta e'_0, e'_1 = \delta e'_1\}$ and $\Delta'' = \{e''_0 = \delta e''_1\}$. The only admissible arcs in $\Delta'(d)$ and $\Delta''(d)$ are $[0,0]$ and $[1,1]$, recalling that $e'_0 + e'_1 = h = e''_0 + e''_1$. For such arcs in Δ' we have the minimal projective resolution of $E'_{0,1}$

$$0 \longrightarrow P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \stackrel{d_{min}^{E'_{0,1}}}{\longrightarrow} P_{t_1} \oplus P_{t_2} \longrightarrow E'_{0,1} \longrightarrow 0$$

where $d_{min}^{E'_{0,1}} = \begin{pmatrix} \sigma(a)\bar{c}a & 0 \\ \sigma(a)\bar{c}b & \sigma(b)\bar{c}b \end{pmatrix}$, similarly for $E'_{1,0}$ and so

$$c^{E'_{1,0}} = c^{E'_{0,1}} = det(Hom_Q(d_{min}^{E'_{0,1}}, \cdot)) = det \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(a)\bar{c}b \\ 0 & \sigma(b)\bar{c}b \end{pmatrix}.$$

We note that the matrices $\sigma(a)\bar{c}b$, $\sigma(a)\bar{c}a$ and $\sigma(b)\bar{c}b)$ have different size for $[0,0]$ and for $[1,1]$. Whereas in Δ'' we have the minimal projective resolution of $c^{E''_{0,1}} = c^{E''_{1,0}}$

$$0 \longrightarrow P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \stackrel{d_{min}^{E''_{0,1}}}{\longrightarrow} P_{t_1} \oplus P_{t_2} \longrightarrow E''_{0,1} \longrightarrow 0$$

where $d_{min}^{E''_{0,1}} = \begin{pmatrix} 0 & \sigma(b)\bar{c}a \\ \sigma(a)\bar{c}b & \sigma(b)\bar{c}b \end{pmatrix}$ and so

$$c^{E''_{1,0}} = c^{E''_{0,1}} = det(Hom_Q(d_{min}^{E''_{0,1}}, \cdot)) = det \begin{pmatrix} 0 & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the orthogonal case and

$$pf^{E''_{0,1}} = pf^{E''_{1,0}} = pf \begin{pmatrix} 0 & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the symplectic case, since $\bar{c}$ is skew-symmetric, $\sigma(b) = -b^t$ and $\sigma(a) = -a^t$. We assume $n \geq 3$ and we take $[i,j] \in \mathcal{A}(d)$. If we consider the arc $[1,1]$, we have the minimal projective resolution $V_{(1,1)}$

$$0 \longrightarrow P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \stackrel{d_{min}^{V_{(1,1)}}}{\longrightarrow} P_{t_1} \oplus P_{t_2} \longrightarrow V_{(1,1)} \longrightarrow 0$$

where $d_{min}^{V_{(1,1)}} = \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(b)\bar{c}a \\ \sigma(a)\bar{c}b & \sigma(b)\bar{c}b \end{pmatrix}$ and so

$$c^{V_{(1,1)}} = det(Hom_Q(d_{min}^{V_{(1,1)}}, \cdot)) = det \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the orthogonal case and

$$pf^{V_{(1,1)}} = pf \begin{pmatrix} \sigma(a)\bar{c}a & \sigma(a)\bar{c}b \\ \sigma(b)\bar{c}a & \sigma(b)\bar{c}b \end{pmatrix}$$

in the symplectic case. If $[i,j]$ doesn't contain e_1 as an internal vertex, then we have $1 \leq i < j \leq 2n-3$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{z_{j-1}} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{z_{i-1}} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = c_{j-2} \cdots c_{i-1}$ and so

$$c^{E_{i,j-1}} = det(Hom_Q(d_{min}^{E_{i,j-1}}, \cdot)) = det(c_{j-2} \cdots c_{i-1}),$$

where $c_0 = (a,b)$ and $c_{2n-4} = \sigma(c_0)$. In particular in the symplectic case if $j = \sigma_I(i)$ then $pf^{E_{i,\sigma_I(i)-1}} = pf(\sigma(c_{i-1}) \cdots c_{i-1})$. If $[i,j]$ contains e_1 as an internal vertex, i.e. $2 \leq j < i \leq 2n-4$ and we have the minimal projective resolution of $E_{i,j-1}$

$$0 \longrightarrow P_{z_{j-1}} \oplus P_{\sigma(t_1)} \oplus P_{\sigma(t_2)} \overset{d_{min}^{E_{i,j-1}}}{\longrightarrow} P_{z_{i-1}} \oplus P_{t_1} \oplus P_{t_2} \longrightarrow E_{i,j-1} \longrightarrow 0$$

where $d_{min}^{E_{i,j-1}} = \begin{pmatrix} 0 & \sigma(a)c_{2n-6,i-1} & \sigma(b)c_{2n-6,i-1} \\ c_{j-2,1}a & \sigma(a)c_{2n-6,1}a & \sigma(b)c_{2n-6,1}a \\ c_{j-2,1}b & \sigma(a)c_{2n-6,1}b & \sigma(b)c_{2n-6,1}b \end{pmatrix}$ and so

$$c^{E_{i,j-1}} = \begin{pmatrix} 0 & c_{j-2,1}a & c_{j-2,1}b \\ \sigma(a)c_{2n-6,i-1} & \sigma(a)c_{2n-6,1}a & \sigma(a)c_{2n-6,1}b \\ \sigma(b)c_{2n-6,i-1} & \sigma(b)c_{2n-6,1}a & \sigma(b)c_{2n-6,1}b \end{pmatrix}.$$

If $\sigma_I(i) = j$ then, only in the symplectic case, we have

$$pf^{E_{\sigma_I(j),j-1}} = pf \begin{pmatrix} 0 & c_{j-2,1}a & c_{j-2,1}b \\ \sigma(a)\sigma(c_{j-2,1}) & \sigma(a)c_{2n-6,1}a & \sigma(a)c_{2n-6,1}b \\ \sigma(b)\sigma(c_{j-2,1}) & \sigma(b)c_{2n-6,1}a & \sigma(b)c_{2n-6,1}b \end{pmatrix},$$

since $\sigma(c_{j-2,1}) = -(c_{j-2,1})^t$, $\sigma(a) = -a^t$, $\sigma(b) = -b^t$ and $c_{2n-6,1}$ is skew-symmetric.
Finally we note that $E''_{1,0}$, $V_{(1,1)}$, $E_{i,\sigma_I(i)-1}$ and $E_{\sigma_I(j),j-1}$ satisfy property *(Op)*.

3.2.1.8 End of proof of theorem 3.2.9, theorem 3.2.6 and proposition 3.2.8

We prove the second step of proof of theorem 3.2.9. By the analysis case by case we note that if $[i,j]$ is admissible then the semi-invariants associated to $[i,j]$ define a nonzero element of $SpSI(Q,d)$ (respectively of $OSI(Q,d)$). For a symmetric dimension vector d we denote

$$Sp\Gamma(Q,d) = \{\chi \in \mathbb{Z}^{Q_0} \cup \frac{1}{2}\mathbb{Z}^{Q_0} |\, SpSI(Q,d)_\chi \neq 0\} \tag{3.38}$$

and

$$O\Gamma(Q,d) = \{\chi \in \mathbb{Z}^{Q_0} \cup \frac{1}{2}\mathbb{Z}^{Q_0} |\, OSI(Q,d)_\chi \neq 0\} \tag{3.39}$$

the semigroup of weights of symplectic (respectively orthogonal) semi-invariants. We note that (3.38) and (3.39) involve also $\frac{1}{2}\mathbb{Z}^{Q_0}$ because in $SpSI(Q,d)$ and in $OSI(Q,d)$ also pfaffians can appear. To simplify the notation, we shall call $\chi_{[i,j]}$, $\chi'_{[i,j]}$ and $\chi''_{[i,j]}$ be respectively the weights of the semi-invariants associated to admissible arcs $[i,j]$ respectively from $\mathcal{A}(d)$, $\mathcal{A}'(d)$ and $\mathcal{A}''(d)$. In the next the following proposition will be useful. We will state it only for Δ, because for Δ' and Δ'' the statements are similar. Let d be a regular symmetric dimension vector with canonical decomposition $d = ph + d'$ with $p \geq 1$.

Proposition 3.2.34. *Let (Q,σ) be a symmetric quiver of tame type. Let d_2 be of type $e_{[s,\sigma_I(s)]}$, $e_{[s,t]} + \delta e_{[s,t]}$ or $e_{[i_{2k},\sigma_I(i_{2k-1})]} + e_{[i_{2k-1},\sigma_I(i_{2k})]}$.*
(i) If $d_2 = e_{[s,\sigma_I(s)]}$, then

- *(a) For every arc $[i,j]$ of Δ' and Δ'' we have $\chi'_{[i,j]}|_{supp(d_2)}, \chi''_{[i,j]}|_{supp(d_2)} \in Sp\Gamma(Q,d_2)$ (respectively in $O\Gamma(Q,d_2)$).*
- *(b) For every arc $[i,j]$ of Δ that doesn't intersect $[s,\sigma_I(s)]$ or contains $[s-1,\sigma_I(s)+1]$ we have $\chi_{[i,j]}|_{supp(d_2)} \in Sp\Gamma(Q,d_2)$ (respectively in $O\Gamma(Q,d_2)$).*
- *(c) Let $\rho_1,\ldots,\rho_r$ be the weights of generators of the polynomial algebra $SpSI(Q,d_2)$ (respectively $OSI(Q,d_2)$). Then $r \geq n'-s$, where $n' \in I_+ \sqcup I_\delta$ is either a σ_I-fixed vertex or the extremal vertex of a σ_I-fixed edge, and $\rho_1,\ldots,\rho_r$ can be reordered such that $\rho_1 = \chi_{[s,s+1]},\ldots,\rho_{n'-s} = \chi_{[n'-1,n']}$ and for every $m > n'-s$ we have $\langle \rho_m, e_n\rangle = 0$ for $n = s,\ldots,n'$.*

(ii) Let $d_2 = e_{[s,t]} + \delta e_{[s,t]}$, then

- *(a) For every arc $[i,j]$ of Δ' and Δ'' we have $\chi'_{[i,j]}|_{supp(d_2)}, \chi''_{[i,j]}|_{supp(d_2)} \in Sp\Gamma(Q,d_2)$ (respectively in $O\Gamma(Q,d_2)$).*
- *(b) For every symmetric arc $[i,j]$ of Δ that doesn't intersect $[s,t]\cup[\sigma_I(t),\sigma_I(s)]$ or contains $[s-1,\sigma_I(s-1)]$ or $[\sigma_I(t+1),t+1]$, we have $\chi_{[i,j]}|_{supp(d_2)} \in Sp\Gamma(Q,d_2)$ (respectively in $O\Gamma(Q,d_2)$).*
- *(c) For every arc $[i,j] \subset I_+$ (respectively $[i,j] \subset I_-$) that doesn't intersect $[s,t]$ (respectively $[\sigma_I(t),\sigma_I(s)]$ or contains $[s-1,t+1]$ we have $\chi_{[i,j]}|_{supp(d_2)} \in Sp\Gamma(Q,d_2)$ (respectively in $O\Gamma(Q,d_2)$)..*

(d) *Let $\rho_1, \ldots, \rho_r$ be the weights of generators of the polynomial algebra $SpSI(Q, d_2)$ (respectively $OSI(Q, d_2)$). Then $r \geq t - s$ and $\rho_1, \ldots, \rho_r$ can be reordered such that $\rho_1 = \chi_{[s,s+1]}, \ldots, \rho_{t-s} = \chi_{[t-1,t]}$ and for every $m > t - s$ we have $\langle \rho_m, e_n \rangle = 0$ for $n = s, \ldots, t$.*

(iii) *Let $d_2 = e_{[i_{2k}, i_{\sigma_I(i_{2k-1})}]} + e_{[i_{2k-1}, i_{\sigma_I(i_{2k})}]}$, then*

(a) *For every arc $[i, j]$ of Δ' and Δ'' we have $\chi'_{[i,j]}|_{supp(d_2)}, \chi''_{[i,j]}|_{supp(d_2)} \in Sp\Gamma(Q, d_2)$ (respectively in $O\Gamma(Q, d_2)$).*

(b) *For every arc $[i, j]$ of Δ that doesn't intersect $[i_{2k-1}, \sigma_I(i_{2k-1})]$ or contains $[i_{2k-1} - 1, \sigma_I(i_{2k-1}) + 1]$ we have $\chi_{[i,j]}|_{supp(d_2)} \in Sp\Gamma(Q, d_2)$ (respectively in $O\Gamma(Q, d_2)$).*

(c) *Let $\rho_1, \ldots, \rho_r$ be the weights of generators of the polynomial algebra $SpSI(Q, d_2)$ (respectively $OSI(Q, d_2)$). Then $r \geq n' - s$, where $n' \in I_+ \sqcup I_\delta$ is either a σ_I-fixed vertex or the extremal vertex of a σ_I-fixed edge, and $\rho_1, \ldots, \rho_r$ can be reordered such that $\rho_1 = \chi_{[s,s+1]}, \ldots, \rho_{n'-s} = \chi_{[n'-1,n']}$ and for every $m > n' - s$ we have $\langle \rho_m, e_n \rangle = 0$ for $n = s, \ldots, n'$.*

Proof. It proceeds type by type analysis, considering the description of the weights of symplectic and orthogonal semi-invariants done above. We recall that $\gamma\chi_{[i,j]} = \chi_{[\sigma_I(j), \sigma_I(i)]}$ and we observe that if x is a σ-fixed vertex and χ is a weight, then $\chi(x) = 0$. We prove only the symplectic case for $Q = \widetilde{A}^{1,1}_{k,l}$ and for $d_2 = e_{[s,\sigma_I(s)]}$, because the procedure to prove all other cases is similar. We order the vertices of $\widetilde{A}^{1,1}_{k,l}$ such that the only source is 1 (so the only sink is $\sigma(1)$), $hv_{i-1} = i$ for every $i \in \{2, \ldots, \frac{l}{2}+1\}$, $hu_i = \frac{l}{2}+i+1$ for every $i \in \{1, \ldots, \frac{k}{2}\}$ and then the respective conjugates by σ of these. We shall call $w_{(t^1)_{i_1}, \ldots, (t^f)_{i_f}}$, where $t^1, \ldots, t^f \in \mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$ and $\{i_1, \ldots, i_f\}$ is an ordered subset of $\{1, \ldots, \frac{l}{2} + \frac{k}{2} + 1, \sigma(\frac{l}{2} + \frac{k}{2} + 1), \ldots, \sigma(1)\}$, the vector such that

$$w_{(t^1)_{i_1}, \ldots, (t^f)_{i_f}}(y) = \begin{cases} (t^j)_{i_j} & y = i_j, \forall j = 1, \ldots, f \\ 0 & \text{otherwise.} \end{cases}$$

Moreover we can associate in bijective way the vertex $i \in \{2, \ldots, \frac{l}{2}\} \subset (\widetilde{A}^{1,1}_{k,l})^+_0$ to $i \in I_+$, the vertex $\frac{l}{2} + i + 1$ of $\widetilde{A}^{1,1}_{k,l}$ to $i + 1 \in I'_+$ and the vertex $\frac{l}{2}$ to $[\frac{l-1}{2}] + 2 \in I_\delta$.
(a) By section 3.2.1.4 we have

$$\chi'_{[i,j]} = w_{(1)_{\frac{l}{2}+i+1}, (-1)_{\frac{l}{2}+j+1}} \quad \text{for} \quad 1 \leq i < j \leq \frac{k}{2} + 1,$$

if $[i, j]$ has not e_1 as internal vertex;

$$\chi'_{[i,j]} = w_{(1)_1, (-1)_{\frac{l}{2}+j+1}, (1)_{\frac{l}{2}+i+1}, (-1)_{\sigma(1)}} \quad \text{for} \quad j < i - 1$$

if $[i,j]$ has e_1 as internal vertex and in particular if $j=\sigma_I(i)$ we have

$$\chi'_{[i,j]} = w_{(\frac{1}{2})_1,(-\frac{1}{2})_{\frac{l}{2}+i+1},(\frac{1}{2})_{\sigma(\frac{l}{2}+i+1)},(-\frac{1}{2})_{\sigma(1)}}.$$

Now if $\langle \chi'_{[i,j]}, e_{[s,\sigma_I(s)]}\rangle \neq 0$ then $\chi'_{[i,j]} \notin SpSI(Q,d_2)$, but we note that $\langle \chi'_{[i,j]}, e_{[s,\sigma_I(s)]}\rangle = 0$ for every i and j, so we have *(a)*.
(b) By section 3.2.1.4 we have

$$\chi_{[i,j]} = w_{(1)_i,(-1)_j} \quad \text{for} \quad 1 \leq i < j \leq \frac{l}{2} \quad \text{and} \quad \chi_{[\frac{l}{2}+1,\sigma(\frac{l}{2}+1)]} = w_{(\frac{1}{2})_{\frac{l}{2}+1},(-\frac{1}{2})_{\sigma(\frac{l}{2}+1)}}$$

if $[i,j]$ has not e_1 as internal vertex;

$$\chi_{[i,j]} = w_{(1)_1,(-1)_j,(1)_i,(-1)_{\sigma(1)}} \quad \text{for} \quad j < i-1.$$

if $[i,j]$ has e_1 as internal vertex.
Now we note that $\langle \chi_{[i,j]}, e_{[s,\sigma_I(s)]}\rangle \neq 0$ if $[i,j] \cap [s,t] \neq \emptyset$ and $[i,j] \not\supseteq [s-1, \sigma(s-1) = \sigma(s)+1]$, so we have *(b)*.
(c) First we note that we can choose symmetric arcs of each length from a fixed vertex of Δ, because the result of theorem 3.2.9 is invariant respect to the Coxeter transformation τ^+. We note that $[s,\sigma_I(s)]$ has e_1 as internal vector. The generators of $SpSI(Q,d_2)$ associated to $\Delta(d_2)$ are $c^{E_i} = det(v_i)$ of weight $\chi_{[i,i+1]} = w_{(1)_i,(-1)_{i+1}}$ for every $i \in \{1,\ldots,s-1\}$ and $c^{E_{s,\sigma_I(s)-1}} = det\begin{pmatrix} \sigma(u_1)\cdots b \cdots u_1 & \sigma(v_1)\cdots\sigma(v_s) \\ v_{s-1}\cdots v_1 & 0 \end{pmatrix}$ of weight $\chi_{[s,\sigma_I(s)]} = w_{(1)_1,(-1)_s,(1)_{\sigma(s)},(-1)_{\sigma(1)}}$. So we call $\rho_i = \chi_{[i,i+1]}$ for every $i \in \{1,\ldots,s-1\}$ and $\rho_{n'-s} = \chi_{[s,\sigma_I(s)]}$, where in this case $n' = \frac{[l-1]}{2}+2$. The other generators are associated to $\Delta'(d_2)$ and so, as done in the part *(a)* of this proposition, their weight ρ_m, for $m \in \{n'-s+1,\ldots,r\}$, are such that $\langle \rho_m, e_n\rangle = 0$ for $n \in \{s,\ldots,n'\}$. □

We assume now that $d = d_1 + d_2$ where $d_1 = ph + d'_1$ with $p \geq 1$ and $d_2 = e_{[s,\sigma_I(s)]}$, $e_{[s,t]} + \delta e_{s,t]}$ or $e_{[i_{2k},i_{\sigma_I(i_{2k-1})}]} + e_{[i_{2k-1},i_{\sigma_I(i_{2k})}]}$.. So we take the corresponding arc in a chosen position (for which we proved proposition 3.2.34).

Proposition 3.2.35. *Let d, d_1, d_2 be as above. We suppose that the semigroup $Sp\Gamma(Q,d_1)$ (respectively $O\Gamma(Q,d_1)$) is generated by the weights $\chi_{[i,j]}$, $\chi'_{[i,j]}$, $\chi''_{[i,j]}$ for admissible arcs $[i,j]$ of the labelled polygons $\Delta(d_1)$, $\Delta'(d_1)$, $\Delta''(d_1)$. Then $Sp\Gamma(Q,d_1) \cap Sp\Gamma(Q,d_2)$ (respectively $O\Gamma(Q,d_1) \cap O\Gamma(Q,d_2)$) is generated by the weights $\chi_{[i,j]}$, $\chi'_{[i,j]}$, $\chi''_{[i,j]}$ for admissible arcs $[i,j]$ of the labelled polygons $\Delta(d)$, $\Delta'(d)$, $\Delta''(d)$.*

Proof. We prove it only for the othogonal case and for $d_2 = e_{[s,\sigma_I(s)]}$, because the symplectic case is similar.

We are two cases.
(1) Assume $p_{s-1} = p_{\sigma_I(s)+1} < r-1$. The admissible arcs of $\Delta(d_1)$, $\Delta'(d_1)$, $\Delta''(d_1)$ and $\Delta(d)$, $\Delta'(d)$, $\Delta''(d)$ are the same. By proposition 3.2.34 $O\Gamma(Q, d_2)$ contains $\chi_{[s,s+1]}, \ldots, \chi_{[\sigma_I(s)-1,\sigma_I(s)]}$ and all the other weights corresponding to the admissible arcs of $\Delta(d)$, $\Delta'(d)$ and $\Delta''(d)$.
(2) Assume $p_{s-1} = p_{\sigma_I(s)+1} = r-1$. We prove that $O\Gamma(Q, d_1) \cap O\Gamma(Q, d_2)$ is generated by $\chi'_{[i,j]}$ for every admissible arc $[i, j]$ of $\Delta'(d_1) = \Delta'(d)$, $\chi''_{[i,j]}$ for every admissible arc $[i, j]$ of $\Delta''(d_1) = \Delta''(d)$ and $\chi_{[i,j]}$ for every admissible arc $[i, j]$ of $\Delta(d_1)$ of index smaller than $r-1$ or not intersecting $[s, \sigma_I(s)]$, i.e. $\chi_{[s,s+1]}, \ldots, \chi_{[\sigma_I(s)-1,\sigma_I(s)]}$ and $\chi_{[s-1,\sigma_I(s)+1]} = \chi_{[s-1,s]} + \cdots + \chi_{[\sigma_I(s),\sigma_I(s)+1]}$.
Let

$$\chi = \sum_{[i,j]\in\mathcal{A}(d_1)} n_{i,j}\chi_{[i,j]} + \sum_{[i,j]\in\mathcal{A}'(d_1)} n'_{i,j}\chi'_{[i,j]} + \sum_{[i,j]\in\mathcal{A}''(d_1)} n''_{i,j}\chi''_{[i,j]},$$

with $n_{i,j}, n'_{i,j}, n''_{i,j} \geq 0$, be an element of $O\Gamma(Q, d_1)$. We assume that χ is also in $O\Gamma(Q, d_2)$. By proposition 3.2.34, we note that all the generators of $O\Gamma(Q, d_1)$ except of $\chi_{[s-1,s]}$ and $\chi_{[\sigma_I(s),\sigma_I(s)+1]}$ are also in $O\Gamma(Q, d_2)$. Hence, if χ contains neither $\chi_{[s-1,s]}$ nor $\chi_{[\sigma_I(s),\sigma_I(s)+1]}$, then χ is a linear combination of desired generators. So we have to prove that if χ contains $\chi_{[s-1,s]}$ (resp. $\chi_{[\sigma_I(s),\sigma_I(s)+1]}$) with positive coefficient, then it contains $\chi_{[s,s+1]}, \ldots, \chi_{[\sigma_I(s),\sigma_I(s)+1]}$ (resp. $\chi_{[s-1,s]}, \ldots, \chi_{[\sigma_I(s)-1,\sigma_I(s)]}$). Thus we can subtract $\chi_{[s-1,\sigma_I(s)+1]}$ from χ.
We assume that χ contains $\chi_{[s-1,s]}$ with positive coefficient (the proof is similar for $\chi_{[\sigma_I(s),\sigma_I(s)+1]}$). We note that $\langle \chi_{[s-1,s]}, e_s\rangle = -1$ and, by proposition 3.2.34, the other generators of $O\Gamma(Q, d_1)$, except $\chi_{[s,s+1]}$, have zero product scalar with e_s. Moreover, $\chi \in O\Gamma(Q, d_2)$ and so, by proposition 3.2.34, $\langle \chi, e_s\rangle \geq 0$. Hence χ contains $\chi_{[s,s+1]}$ with positive coefficient. By proposition 3.2.34, it follows that $\langle \chi, e_s + e_{s+1}\rangle \geq 0$. But $\langle \chi_{[s-1,s]} + \chi_{[s,s+1]}, e_s + e_{s+1}\rangle = -1$ and $\chi_{[s+1,s+2]}$ is the only generator of $O\Gamma(Q, d_1)$ with positive scalar product with $e_s + e_{s+1}$. Continuing in this way, we check that χ contains $\chi_{[s-1,s]}, \chi_{[s,s+1]}, \ldots, \chi_{[\sigma_I(s)-1,\sigma_I(s)]}, \chi_{[\sigma_I(s),\sigma_I(s)+1]}$ with positive coefficients. So we can subtract $\chi_{[s-1.\sigma_I(s)+1]}$ from χ and continue. In this way we complete the proof. □

Now we can finish the proof of theorem 3.2.9. Since theorem 3.2.9 is equivalent to conjectures 1.2.1 and 1.2.2 for tame type and regular dimension vectors, then, in this way, we finish also the proof of conjectures 1.2.1 and 1.2.2.
Again we consider the embeddings

$$SpSI(Q, d) \overset{\Phi_d}{\to} \bigoplus_{\chi\in char(Sp(Q,d))} SpSI(Q, d_1)_{\chi|_{d_1}} \otimes SpSI(Q, d_2)_{\chi|_{d_2}} \qquad (3.40)$$

and

$$OSI(Q,d) \stackrel{\Psi_d}{\to} \bigoplus_{\chi \in char(O(Q,d))} OSI(Q,d_1)_{\chi|_{d_1}} \otimes OSI(Q,d_2)_{\chi|_{d_2}} \tag{3.41}$$

where Q, d, d_1 and d_2 are as above. The semigroup of weights of the right hand side of Φ_d and Ψ_d are respectively $Sp\Gamma(Q,d_1) \cap Sp\Gamma(Q,d_2)$ and $O\Gamma(Q,d_1) \cap O\Gamma(Q,d_2)$. These are generated by $\chi_{[i,j]}$, $\chi'_{[i,j]}$, $\chi''_{[i,j]}$ for admissible arcs $[i,j]$ of the labelled polygons $\Delta(d)$, $\Delta'(d)$, $\Delta''(d)$, by proposition 3.2.35. So the algebras on the right hand side of Φ_d and Ψ_d are generated by the semi-invariants of weights $\chi_{[i,j]}$, $\chi'_{[i,j]}$, $\chi''_{[i,j]}$ and by the semi-invariants of weights $\langle h, \cdot \rangle$ (or $\frac{1}{2}\langle h, \cdot \rangle$).
Finally, we note that the embeddings Φ_d and Ψ_d are isomorphisms because they are also isomorphisms in the weight $\langle h, \cdot \rangle$ (or $\frac{1}{2}\langle h, \cdot \rangle$) and so we completed the proof of theorem 3.2.9. Moreover, in that way, we also proved proposition 3.2.8, expliciting the semi-invariants of type c^V for every admissible arc $[i,j]$, and theorem 3.2.6, by isomorphisms Φ_d and Ψ_d considering $d_1 = ph$ and $d_2 = d'$.

Appendix A

Representations of GL and invariant theory

A.1 Highest weight theory for GL and Schur modules

We recall the basics of representation theory of general linear group.
We fix an algebraically closed field $\mathbb{K}$.

Definition A.1.1. *Let G be an algebraic group.* (V, ρ) *is a rational representation if* V *is a vector space of dimension* m*,* $\rho : G \times V \longrightarrow V$ *such that* $\rho(g, v) = g \cdot v$ *is a rational action, i.e.*

a) $g \cdot (h \cdot v) = (gh) \cdot v$ *for every* $g, h \in G$ *and* $v \in V$*,*

b) $e \cdot v = v$ *for every* $v \in V$ *where* e *is identity in G,*

c) ρ *is a morphism of varieties.*

Definition A.1.2. *G is linearly reductive if and only if every rational linear representation of G is semisimple.*

Let G be a linearly reductive group and let $\rho : G \to GL(V)$ be a finite dimensional rational representation of G. Let H be a *maximal torus of G*, i.e. a maximal subgroup of G isomorphic to $(\mathbb{K}^*)^h$ for some $h \in \mathbb{N}$, restricting ρ to H we obtain a rational representation of H. So we can decompose V into the direct sum of eigenspaces

$$V = \bigoplus_{\chi \in char(H)} V_\chi$$

where $char(H) = \{$homomorphisms of algebraic groups $\chi : H \to \mathbb{K}^*\}$ is the set of characters of H and $V_\chi = \{v \in V | \rho(t)(v) = \chi(t)v, \ \forall t \in H\}$. The elements $\chi \in char(H)$ such that $V_\chi \neq 0$ are called weights of ρ, V_χ is called *weight space of weight* χ and $dim\, V_\chi$ is called *multiplicity of the weight*

χ. The set of weights $char(G)$ forms a free abelian group $\mathcal{X} = char(G)$. Let $\Phi = \Phi(G, H)$ be the set of roots of G relative to H. Φ is an abstract root system in a real vector space E. Let Δ be a base of Φ. So $\mathcal{X}$ has a dual base by the inner product on E defined by Cartan matrix of Φ (see [Hu, Appendix]. A weight is called *dominant weight* if it is a linear combination of elements of a such base of $\mathcal{X}$ with integer non-negative coefficients.

Theorem A.1.3. *Let B be a Borel subgroup of G, i.e. a closed, connected and solvable subgroup of G which is maximal for these properties, containing H.*

(a) For every irreducible rational representation V of G there exists a unique B-stable 1-dimensional subspace which is a weight space V_μ, for some dominant weight μ of multiplicity 1 (μ is called the highest weight of V and any generator of V_μ is called highest weight vector).

(b) For every dominant weight $\mu \in char(H)$ there exists an irreducible rational representation V of G with highest weight μ (called the highest weight representation of G) which is unique up to isomorphism, i.e. if V' is another irreducible rational representation of G with highest weight μ' then V is isomorphic to V' if and only if μ equals μ'.

Proof. See [Hu, theorem 31.3]. □

The groups $GL(n)$ and $SL(n)$ are linearly reductive (see [GW, theorem 2.4.5]. Hence for $GL(n) = GL(E)$, where $E = \mathbb{K}^n$ with $\mathbb{K}$ an algebraically closed field of characteristic 0, it's enough to classify irreducible rational representations.
If V is a vector space of dimension m, a rational representation $\rho : GL(E) \to GL(V)$ is called polynomial if and only if the entries $\rho_{ij}(g)$ of ρ (for $1 \leq i, j \leq m$) are polynomials in $\{g_{ij}\}_{1 \leq i,j \leq n}$, where $g = (g_{ij})_{1 \leq i,j \leq n} \in GL(E)$. A polynomial representation $\rho : GL(E) \to GL(V)$ is homogeneous of degree d if and only if the entries $\rho_{ij}(g)$ of ρ (for $1 \leq i, j \leq m$) are homogeneous of degree d in $\{g_{ij}\}_{1 \leq i,j \leq n}$.

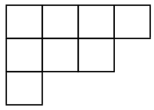
Proposition A.1.4. *a) Every rational representation V of $GL(E)$ is of the form $V = V' \otimes (\bigwedge^n E)^{\otimes t}$ for some t, where V' is a polynomial representation and $\bigwedge^n E$ is the n-th exterior power of E.*

b) Every polynomial representation of $GL(E)$ is a direct sum of homogeneous representations.

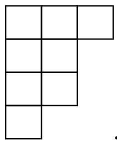
Proof. See [FH, sec. 15.5]. □

Hence it's enough to classify irreducible homogeneous representations of degree d.
Let λ be a partition of d, i.e. $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda = \lambda_1 \geq \ldots \geq \lambda_k \geq 0$

and $\lambda_1 + \ldots + \lambda_k = d$. We identify partitions $(\lambda_1, \ldots, \lambda_k, 0)$ with $(\lambda_1, \ldots, \lambda_k)$. We shall denote $d = |\lambda|$ and we shall call the *height of* λ, denoted by $ht(\lambda)$, the number k of nonzero components of λ. Graphically we represent λ as a set of boxes with λ_i boxes in the i-th row (called *Young diagram of* λ), so $|\lambda|$ and $ht(\lambda)$ are, respectively, the number of boxes and the number of rows of the diagram of λ. For example, if $\lambda = (4, 3, 1)$, then the Young diagram of λ is:



For a partition λ we denote its conjugate (or transpose) partition $\lambda' = (\lambda'_1, \ldots, \lambda'_t)$, where λ'_j is the number of boxes in the j-th column of the Young diagram of λ. For example, if $\lambda = (4, 3, 1)$ then $\lambda' = (3, 2, 2, 1)$ and the Young diagram of λ' is:



.

Let T be a *tableau* of shape λ, i.e. a filling of the Young diagram of λ with numbers $1, \ldots, d$. We define the *Young idempotent* e_T to be an element of the group ring $\mathbb{K}[S_d]$. In the symmetric group S_d we define the subgroups R_T and C_T to be the sets of permutations in S_d preserving respectively the rows and the columns of T. We define

$$e_T = \sum_{\sigma \in R_T, \tau \in C_T} sgn(\tau)\sigma\tau.$$

Finally we define the *Schur module*

$$S_\lambda V := e_T V^{\otimes d},$$

where V is a finite dimensional vector space, $dimV = n$. If T and T' are two tableaux of the same partition λ, then $e_T V^{\otimes d}$ and $e'_T V^{\otimes d}$ are isomorphic as $GL(V)$-modules [W, lemma 2.2.13]; thus $S_\lambda V = e_T V^{\otimes d}$ depends on the partition λ and not on the tableau T. The representations $S_\lambda V$ give all irreducible representations of $GL(V)$ homogeneous of degree d [P, chap. 9 sec. 8.1].
For the Schur modules sometimes we shall use the notation $S_\lambda V$ and sometimes the notation $S_{(\lambda_1, \ldots, \lambda_k)} V$, it depends if we want to consider or not the components of λ.
Now we give two examples of Schur modules. If V is finite dimensional vector space we shall call $S_n(V)$ the n-th symmetric power of V, so the symmetric algebra of V is $S(V) = \bigoplus_{n \geq 0} S_n(V)$, and $\bigwedge^n(V)$ the n-th exterior power of V, so the exterior algebra of V is $\bigwedge(V) = \bigoplus_{n \geq 0} \bigwedge^n(V)$.

Example A.1.5. *Let V be an n-dimensional vector space*

(a) If $\lambda = (d, \overbrace{0, \ldots, 0}^{n-1}) = (d, 0^{n-1})$ then $S_{(d,0^{n-1})}V$ is just the d-th symmetric power $S_d(V)$.

(b) If $\lambda = (\overbrace{1, \ldots, 1}^{d}, \overbrace{0, \ldots, 0}^{n-d}) = (1^d, 0^{n-d})$ then $S_{(1^d,0^{n-d})}V$ is just the d-th exterior power $\bigwedge^d(V)$; in particular if $d = dim\, V$, $S_{(1^{dim\, V})}V = \bigwedge^{dim\, V}(V) := D$ is called a determinant representation of G.

c) If $k > n$ and $\lambda_k > 0$, we have $S_{(\lambda_1,\ldots,\lambda_k)}V = 0$.

Introducing the convention $\bigwedge^n(V^*) = S_{\underbrace{(-1, \ldots, -1)}_{n}}V$ and $S_{(\lambda_1,\ldots,\lambda_n)}V \otimes \bigwedge^n(V^*) = S_{(\lambda_1-1,\ldots,\lambda_n-1)}V$, we see that there is a bijective correspondence between rational irreducible representations of $GL(n)$ and vectors $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \cdots \geq \lambda_n$.

We give an alternative description of Schur modules equivalent to that already given [W, lemma 2.2.13]. Let V be an n-dimensional vector space. Let

$$m : \bigwedge^r V \otimes \bigwedge^s V \to \bigwedge^{r+s} V,$$

such that

$$m(u_1 \wedge \ldots \wedge u_r \otimes v_1 \wedge \ldots \wedge v_s) = u_1 \wedge \ldots \wedge u_r \wedge v_1 \wedge \ldots \wedge v_s,$$

be the multiplication in the exterior algebra $\bigwedge V$ and let

$$\Delta : \bigwedge^{r+s} V \to \bigwedge^r V \otimes \bigwedge^s V,$$

such that

$$\Delta(u_1 \wedge \ldots \wedge u_{r+s}) = \sum_{\sigma \in S^{r,s}_{r+s}} (-1)^{sgn(\sigma)} u_{\sigma(1)} \wedge \ldots \wedge u_{\sigma(r)} \otimes u_{\sigma(r+1)} \wedge \ldots \wedge u_{\sigma(r+s)}$$

where $S^{r,s}_{r+s} = \{\sigma \in S_{r+s} | \sigma(1) < \cdots < \sigma(r); \sigma(r+1) < \cdots < \sigma(r+s)\}$, be the comultiplication in the exterior algebra $\bigwedge V$. We consider $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of d. We can define the Schur module as

$$S_\lambda V := \bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda_k} V / R(\lambda, V),$$

where

$$R(\lambda, V) = \sum_{1 \leq a \leq k-1} \bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda_{a-1}} V \otimes R_{a,a+1}(V) \otimes \bigwedge^{\lambda_{a+2}} V \otimes \cdots \otimes \bigwedge^{\lambda_k} V$$

where $R_{a,a+1}(V)$ is the submodule spanned by the images of the following maps $\theta(\lambda, a, u, v; V)$ with $u + v < \lambda_{a+1}$:

$$\begin{array}{c} \bigwedge^u V \otimes \bigwedge^{\lambda_a - u + \lambda_{a+1} - v} V \otimes \bigwedge^v V \\ \downarrow 1 \otimes \Delta \otimes 1 \\ \bigwedge^u V \otimes \bigwedge^{\lambda_a - u} \otimes \bigwedge^{\lambda_{a+1} - v} V \otimes \bigwedge^v V \\ \downarrow m_{12} \otimes m_{34} \\ \bigwedge^{\lambda_a} V \otimes \bigwedge^{\lambda_{a+1}} V. \end{array}$$

Let us choose an ordered basis $\{e_1, \ldots, e_n\}$ of V. If T is a tableau of shape λ with entries in $\{1, \ldots, n\}$, we associate to T the element in $S_\lambda V$

$$e_{T(1,1)} \wedge \ldots \wedge e_{T(1,\lambda_1)} \otimes \ldots \otimes e_{T(k,1)} \wedge \ldots \wedge e_{T(k,\lambda_k)} + R(\lambda, V),$$

where $T(i, j)$ is the entry of T in the i-th row and j-th column of the Young diagram of λ.
We recall some properties and some known results about Schur modules.
A filling of the Young diagram of a partition λ with the numbers $1, \ldots, n$ weakly increasing along each row and strictly increasing along each column is called *column standard tableau corresponding to the basis* $\{e_1, \ldots, e_n\}$.

Theorem A.1.6. *Let $\{e_1, \ldots, e_n\}$ be a basis of V. The column standard tableaux corresponding to this basis form a basis of $S_\lambda V$*

Proof. See [W, prop. 2.1.4]. □

If V is an n-dimensional vector space, a Borel subgroup of $GL(V) = GL(n)$ is the subgroup of all upper triangular matrices, the maximal torus H of $GL(n)$ is the subgroup of diagonal matrices and the sequences $(\lambda_1, \ldots, \lambda_n)$, with $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \ldots \geq \lambda_n$, are the *dominant integral weights for* $GL(n)$; we shall write $x = diag(x_1, \ldots, x_n)$ in H for the diagonal matrix with these entries. The decomposition of V into direct sum of weight spaces is

$$\bigoplus_{a=(a_1,\ldots,a_n)\in\mathbb{Z}^n} V_a = \{v \in V | x \cdot v = \prod_{i=1}^n x_i^{a_i} v \ \forall x \in H\},$$

see [B, chap. 3 sec. 8].

Theorem A.1.7. *Let V be an n-dimensional vector space.*

1) *If λ is a partition with at most n components then the representation $S_\lambda V$ of $GL(n)$ is an irreducible representation of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$.*

2) *For any $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \geq \cdots \geq \mu_n$ integers, there is a unique irreducible representation of $GL(n)$ with highest weight μ, which can be realized as $S_\lambda V \otimes D^{\otimes k}$, for any $k \in \mathbb{Z}$ and where $\lambda_i = \mu_i - k \geq 0$ for every $i \in \{1, \ldots, n\}$.*

Proof. See [F, sec. 8.2 theorem 2]. □

By theorem A.1.3 and by the previous one, every irreducible rational representation is a Schur module tensored with a power of a determinant representation.

Theorem A.1.8 (Properties of Schur modules)**.** *Let V be vector space of dimension n and λ be the highest weight for $GL(n)$.*

(i) $S_\lambda V = 0 \Leftrightarrow ht(\lambda) > n$.

(ii) $dim\ S_\lambda V = 1 \Leftrightarrow \lambda = (\overbrace{k, \ldots, k}^{n}) = (k^n)$ *for some* $k \in \mathbb{Z}$.

(iii) $\left(S_{(\lambda_1,\ldots,\lambda_n)}V\right)^* \cong S_{(\lambda_1,\ldots,\lambda_n)}V^* \cong S_{(-\lambda_n,\ldots,-\lambda_1)}V$.

Proof. See [FH, theorem 6.3].

Theorem A.1.9 (Cauchy formulas)**.** *Let V and W be two finite dimensional vector spaces.*

a) As a representation of $GL(V) \times GL(W)$, $S_d(V \otimes W)$ decomposes as

$$S_d(V \otimes W) = \bigoplus_{|\lambda|=d} S_\lambda V \otimes S_\lambda W;$$

b) As a representation of $GL(V) \times GL(W)$, $S_d(V \otimes W)$ decomposes as

$$\bigwedge^d(V \otimes W) = \bigoplus_{|\lambda|=d} S_\lambda V \otimes S_{\lambda'} W;$$

c) As a representation of $GL(V)$, $S_d(S_2(V))$ decomposes as

$$S_d(S_2(V)) = \bigoplus_{|\lambda|=d} S_{2\lambda} V,$$

where $2\lambda = (2\lambda_1, \ldots, 2\lambda_k)$ if $\lambda = (\lambda_1, \ldots, \lambda_k)$;

d) As a representation of $GL(V)$ the ring $S_d(\bigwedge^2(V))$ decomposes as

$$S_d(\bigwedge^2(V)) = \bigoplus_{|\lambda|=d} S_{2\lambda'} V.$$

Proof. See [P] chap. 9 sec. 6.3 and sec. 8.4 , chap. 11 sec. 4.5.

Finally we consider the tensor product of Schur modules

Lemma A.1.10.
$$S_\lambda V \otimes S_\mu V = \bigoplus_\nu c^\nu_{\lambda\mu} S_\nu V,$$
where $c^\nu_{\lambda\mu}$'s are called Littlewood-Richardson coefficients.

There is a combinatorial formula to calculate $c^\nu_{\lambda\mu}$.
Let
$$D_\lambda = \{(i,j)|\, 1 \leq i \leq k\, 1 \leq j \leq \lambda_i\}$$
be the Young diagram of λ and let $f : D_{\nu/\lambda} \to \{1, \dots, n\}$ be a column standard tableau. We denote $CST(\nu/\lambda, \{1, \dots, n\})$ the set of column standard tableaux of shape ν/λ with values in $\{1, \dots, n\}$. We define $cont(f)$, the *content of f*, to be the sequence $\{|f^{-1}(1)|, \dots, |f^{-1}(n)|\}$. We define $w(f)$ to be the word we get from f when we read it by rows, starting with the first row, from right to left in each row. A word $w = (w_1, \dots, w_m)$ on the alphabet $\{1, \dots, n\}$ is a *lattice permutation* if for each $1 \leq u \leq m$ and for each $1 \leq i \leq n-1$ we have
$$|\{1 \leq j \leq u|\, w_j = i\}| \geq |\{1 \leq j \leq u|\, w_j = i+1\}|.$$
Finally we define the set
$$LR^\nu_{\lambda,\mu} = \{f \in CST(\nu/\lambda, \{1, \dots, n\})|\, cont(f) = (\mu_1, \dots, \mu_n),\ w(f) \text{ is a lattice permutation}\}.$$

Theorem A.1.11 (Littlewood-Richardson rule). *Let λ, μ, ν be partitions, then*
$$c^\nu_{\lambda\mu} = |LR^\nu_{\lambda,\mu}|.$$

Proof. See [P, chap. 12 sec. 5.3]. □

Corollary A.1.12. *If $\lambda = (l^s)$ and $\mu = (m^t)$, then $S_\lambda V \otimes S_\mu V$ is multiplicity free, i.e. $S_\lambda V \otimes S_\mu V = \bigoplus_\nu S_\nu V$. Moreover if $s \geq t$ then $\nu = (\nu_1, \dots, \nu_{s+t})$ with $\nu_i = l + c_i$ for $1 \leq i \leq t$, $\nu_i = l$ for $t < i \leq s$ and $\nu_{s+i} = m - c_{t-i+1}$ for $1 \leq i \leq t$, where $m \geq c_1 \geq \dots \geq c_t \geq 0$ and $l + c_t \geq m$.*

Proof. We note that we can suppose in the statement $s \geq t$, since the tensor product is commutative. The proof is a consequence of Littlewood-Richardson rule. □

A.2 Invariant theory

In this section we recall definitions and fundamental results of invariant theory.
If G is a group which acts on a finite dimensional vector space V, we shall call $V^G = \{v \in V | g \cdot v = v\ \forall g \in G\}$ the space of invariants of V and we have a general lemma

Lemma A.2.1. *Let G be a group which acts on two finite dimensional vector space V and W. If G acts trivially on V, then $(V \otimes W)^G = V \otimes W^G$.*

If G is an algebraic group and V is a rational representation of G, then G acts on the coordinate ring of V $\mathbb{K}[V]$ as follows: if $f \in \mathbb{K}[V]$ and $g \in G$,

$$(g \cdot f)(v) = f(g^{-1} \cdot v).$$

The ring of G-invariants in $\mathbb{K}[V]$ is

$$\mathbb{K}[V]^G = \{f \in \mathbb{K}[V] | g \cdot f = f \ \forall g \in G\}.$$

Theorem A.2.2 (Hilbert)**.** *If G is linearly reductive and acts rationally on a finite dimensional vector space V then $\mathbb{K}[V]^G$ is finitely generated.*

Proof. See [P, chap. 14 sec. 1.1].

Now we formulate the first fundamental theorem for the linear group.

Theorem A.2.3 (FFT for GL)**.** *Let V be a finite dimensional vector space. We take the space $(V^*)^p \times V^q = \big\{(\alpha_1, \dots, \alpha_p, v_1, \dots, v_q) | \alpha_j \in V^*, v_i \in V \ \forall j \in \{1, \dots, p\}$ and $\forall i \in \{1, \dots, q\}\big\}$ as a representation of $GL(V)$. On this space we consider the pq polynomial functions $u_{ij}(\alpha_1, \dots, \alpha_p, v_1, \dots, v_q) = \alpha_j(v_i)$ which are $GL(V)$-invariant. Then*

$$\mathbb{K}[(V^*)^p \times V^q]^{GL(V)} = \mathbb{K}[u_{ij}]_{\substack{1 \le i \le q \\ 1 \le j \le p}}$$

Proof. See [P, chap. 9 sec.1.4].

Now we give the definition of semi-invariant and of character of an algebraic group.

Definition A.2.4. *Let G be an algebraic group and let V be a rational representation of G.*

(i) *$\chi : G \to \mathbb{K}^*$ is a character of G if it is a homomorphism of algebraic groups;*

(ii) *$f \in \mathbb{K}[V]$ is a semi-invariant of weight χ of the action of G on V if for every $g \in G$, $g \cdot f = \chi(g) f$ where χ is a character of G.*

If $char(G)$ is the set of characters of G, then the ring of semi-invariants of the action of G on V is

$$SI(G, V) = \bigoplus_{\chi \in char(G)} SI(G, V)_\chi$$

where $SI(G, V)_\chi = \{f \in \mathbb{K}[V] | \forall g \in G, g \cdot f = \chi(g) f\}$ is called weight space. In general we have the following lemma proved in [SK].

Lemma A.2.5 (Sato-Kimura)**.** *Let G be a linear algebraic group acting rationally on the vector space V. If there is a Zariski open G-orbit in V then the ring $SI(G,V)$ spanned by the semi-invariants is a polynomial ring:*

$$SI(G,V) = k[f_1,\ldots,f_s]$$

for some collection of algebraically independent and irreducible semi-invariants $f_1,\ldots,fs$. Moreover if $f_i \in SI(G,V)_{\chi_i}$ then the χ_i are linearly independent over $\mathbb{Z}$ in the space of characters of G.

Corollary A.2.6. *Under the assumptions of the lemma A.2.5, the set of characters χ such that $SI(G,V)_\chi \neq 0$ forms a free abelian semigroup, isomorphic to $\mathbb{N}^s$. In particular, if f is any semi-invariant of weight χ, then $f = uf_1^{a_1}\cdots f_s^{a_s}$, where u is a unit in $\mathbb{K}$ and the $a_i \geq 0$ are the unique integers such that $\chi = \sum_{i=1}^s a_i\chi_i$ in the space of characters of G. Thus $SI(G,V)$ is a polynomial ring.*

If $G = GL(n)$, there exists an isomorphism $\mathbb{Z} \cong char(GL(n))$ which sends an element a of $\mathbb{Z}$ in $(det)^a$ (where det associates to $g \in GL(n)$ its determinant). So we have

$$SI(G,V) = \mathbb{K}[V]^{SL(V)}.$$

Finally other two results on Schur modules and invariant theory.

Proposition A.2.7. *Let V be a finite dimensional vector space of dimension n.*

$$(S_\lambda V)^{SL(V)} \neq 0 \Longleftrightarrow \lambda = (k^n)$$

for some k and in this case $S_\lambda V$, and so also $(S_\lambda V)^{SL(V)}$, have dimension one.

Proposition A.2.8. *Let V be a finite dimensional vector space of dimension n and let λ and μ be two dominant integral weights. Then*

$$S_\lambda V \otimes S_\mu V \text{ contains a semi-invariant}$$

$$\Longleftrightarrow$$

$$\begin{array}{rcl} \lambda_1 - \lambda_2 & = & \mu_{n-1} - \mu_n \\ \lambda_2 - \lambda_3 & = & \mu_{n-2} - \mu_{n-1} \\ & \vdots & \\ \lambda_{n-1} - \lambda_n & = & \mu_1 - \mu_2 \end{array}$$

and in this case the semi-invariant is unique (up to a non zero scalar) and has weight $\lambda_1 + \mu_n = \lambda_2 + \mu_{n-1} = \cdots = \lambda_n + \mu_1$.

Proof. It is a corollary of (5.6) in [M, I.5]. □

Let $Sp(2n) = \{A \in GL(2n) | AJA = J\}$ be the simplectic group, let $O(n) = \{A \in GL(n) | A^t A = I\}$ be the orthogonal group and $SO(n) = \{A \in O(n) | det\, A = 1\}$ be the special orthogonal group, where I is the identity matrix and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Proposition A.2.9. *Let V be an orthogonal space of dimension n and let W be a symplectic space of dimension $2n$.*

(a) $dim\,(S_\lambda V)^{O(V)} = \begin{cases} 1 & if \quad \lambda = 2\mu \\ 0 & otherwise \end{cases}$,

(b) $dim\,(S_\lambda V)^{SO(V)} = \begin{cases} 1 & if \quad \lambda = 2\mu + (k^n) \\ 0 & otherwise \end{cases}$,

(c) $dim\,(S_\lambda W)^{Sp(W)} = \begin{cases} 1 & if \quad \lambda = 2\mu' \\ 0 & otherwise \end{cases}$

for some partition μ and for some $k \in \mathbb{Z}_{\geq 0}$.

Proof. See [P] chap. 11 cor. 5.2.1 and 5.2.2. □

We end this section recalling definition and properties of the *Pfaffian* of a skew-symmetric matrix.
Let $A = (a_{ij})_{1 \leq i,j \leq 2n}$ be a skew-symmetric $2n \times 2n$ matrix. Given $2n$ vectors $x_1, \ldots, x_{2n}$ in $\mathbb{K}^{2n}$, with $\mathbb{K}$ an algebraically closed field with characteristic 0, we define

$$F_A(x_1, \ldots, x_{2n}) = \frac{1}{n!2^n} \sum_{s \in S_{2n}} sgn(s) \prod_{i=1}^{n} (x_{s(2i-1)}, x_{s(2i)}),$$

where S_{2n} is the symmetric group on $2n$ elements, $sgn(s)$ is the sign of permutation s and $(\cdot, \cdot)$ is the skew-symmetric bilinear form associated to A. So F_A is a skew-symmetric multilinear function of $x_1, \ldots, x_{2n}$. Since, up to a scalar, the only one skew-symmetric multilinear function of $2n$ vectors in $\mathbb{K}^{2n}$ is the determinant, there is a complex number $Pf(A)$, called *Pfaffian of A*, such that

$$F_A(x_1, \ldots, x_{2n}) = Pf(A) det[x_1, \ldots, x_{2n}]$$

where $[x_1, \ldots, x_{2n}]$ is the matrix which has the vector x_i for i-th column. In particular one proves that

$$Pf(A) = \frac{1}{n!2^n} \sum_{s \in S_{2n} \setminus B_n} sgn(s) \prod_{i=1}^{n} a_{s(2i-1)s(2i)}$$

where B_n is a subgroup of S_{2n} isomorphic to the semidirect product $S_n \ltimes (\mathbb{Z}_2)^n$. We can write the Pfaffian of A avoiding to sum on all possible permutations,

$$Pf(A) = \sum_{\substack{i_1<j_1,\ldots,i_n<j_n \\ i_1<\ldots<i_n}} sgn(s) a_{1_1 j_1} \cdots a_{i_n j_n}$$

where s is the permutation $\begin{bmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{bmatrix}$.

Proposition A.2.10. *Let A be a skew-symmetric $2n \times 2n$ matrix.*

(i) For every invertible $2n \times 2n$ matrix B,

$$Pf(BAB^t) = det(B)Pf(A);$$

(ii) $det(A) = Pf(A)^2$.

Proof. See [P, chap. 5 sec. 3.6]. □

Appendix B

Quiver representations and semi-invariants

B.1 Auslander-Reiten theory

A quiver Q is a pair (Q_0, Q_1) where Q_0 is the set of vertices and Q_1 is the set of arrows. Let

$$a : ta \longrightarrow ha, \quad ta, ha \in Q_0$$

be an arrow in Q_1. We shall call ta the tail of the arrow a and ha the head of the arrow a. A path p in Q is a sequence of arrows $p = a_1 \cdots a_n$ such that $ha_i = ta_{i+1}$, $(1 \leq i \leq n-1)$. For every $x \in Q_0$ we also have a trivial path e_x such that $he_x = te_x = x$. We say that Q *has no oriented cycles* if there are no paths $p = a_1 \cdots a_n$ such that $ta_1 = ha_n$.

We fix an algebraically closed field $\mathbb{K}$. A representation V of Q is a family of finite dimensional vector spaces $\{V(x) | x \in Q_0\}$ and of linear maps $\{V(a) : V(ta) \to V(ha)\}_{a \in Q_1}$. The dimension vector of V is a function $\underline{dim}(V) : Q_0 \to \mathbb{Z}_{\geq 0}$ defined by $\underline{dim}(V)(x) := dim V(x)$.

A morphism $f : V \to W$ of two representations is a family of linear maps $\{f(x) : V(x) \to W(x) | f(ha)V(a) = W(a)f(ta) \forall a \in Q_1\}_{x \in Q_0}$. We denote the space of morphisms from V to W by $Hom_Q(V, W)$ and the space of extensions of V by W by $Ext^1_Q(V, W)$.

Definition B.1.1. *The non symmetric bilinear form on the space $\mathbb{Z}^{Q_0}$ of dimension vectors given by*

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$$

is the Euler form of Q, where $\alpha, \beta \in \mathbb{Z}^{Q_0}$.

If $\underline{dim}\, V = \alpha$ and $\underline{dim}\, W = \beta$, we have

$$\langle \alpha, \beta \rangle = dim\, Hom_Q(V, W) - dim\, Ext^1_Q(V, W)$$

We shall call $Rep(Q, \alpha)$ the variety of representations of Q of dimension vector α.

Definition B.1.2. *Let Q be a quiver and let α be a dimension vector. A general representation of Q is a representation from some nonempty Zariski open set in $Rep(Q, \alpha)$.*

We recall the definitions of simple, projective and injective representation of a quiver $Q = (Q_0, Q_1)$. For each vertex x, a simple representation S_x is the representation for which $S_x(x) = \mathbb{K}$, $S_x(y) = 0$ for every $y \in Q_0 \setminus \{x\}$ and $S_x(a)$ is the zero map for every $a \in Q_1$. For every $x \in Q_0$ we define an indecomposable projective representation P_x as follows:

$$P_x(y) = [x, y] \text{ and } P_x(a) := a\circ : [x, ta] \to [x, ha]$$

with $x, y \in Q_0$ and $a \in Q_1$, where $[x, y]$ is a vector space over $\mathbb{K}$ with a basis labelled by all paths from x to y in Q and $a\circ$ is the map which sends the path p to the path $a \circ p$. Every indecomposable projective representation of Q is isomorphic to P_x for some $x \in Q_0$ and moreover we have $Hom_Q(P_x, V) \cong V(x)$ for every representation V of Q, see [ARS, sec III.1]. Similarly every indecomposable injective representation of Q is isomorphic to I_x, where I_x is defined as follows:

$$I_x(y) = [y, x]^* \text{ and } I_x(a) := (\circ a)^* : [ta, x]^* \to [ha, x]^*$$

with $x, y \in Q_0$ and $a \in Q_1$, where $[y, x]^*$ is the dual space of $[y, x]$ and $\circ a : [ha, x] \to [ta, x]$ is the map which sends p to $p \circ a$. In this case we have $Hom_Q(V, I_x) \cong V(x)^*$ for every representation V of Q, where $V(x)^*$ is the dual space of $V(x)$.
Now we recall some definitions and results of Auslander-Reiten Theory, for deepening see [ARS] and [ASS].
We define *the path algebra* $\mathbb{K}Q$ of a quiver Q, the $\mathbb{K}$-algebra which has the paths of Q as basis. The multiplication in $\mathbb{K}Q$ is defined by

$$p \cdot q = \begin{cases} pq & \text{if } tp = hq \\ 0 & \text{otherwise.} \end{cases}$$

Proposition B.1.3. *1) $\mathbb{K}Q$ is a finite-dimensional $\mathbb{K}$-algebra if and only if Q has no oriented cycles.*

2) The categories $Rep(Q)$ of representations of Q and $\mathbb{K}Q - mod$ of left $\mathbb{K}Q$-modules are equivalent.

Proof. See [ARS, sec. 3.1 prop. 1.1 and prop. 1.3] and [ASS, sec. II.1 lemma 1.4(c) and sec. III.1 cor. 1.7]. □

Let A be a finite-dimensional $\mathbb{K}$-algebra, a morphism $f : V \to W$ in the category of left A-modules $A - mod$ is called a *retraction* if there exists $g : W \to V$ such that $fg = id_W$ and it is called a *section* if there exists $g : W \to V$ such that $gf = id_V$.

Definition B.1.4. *Let $f : V \to W$ be a morphism in $A - mod$.*

(a) f is called minimal right almost split if

(i) every endomorphism $h : V \to V$ such that $fh = f$, is an isomorphism (right minimal morphism),

(ii) f is not a retraction,

(iii) for every $g : V' \to W$ which is not a retraction there exists $g' : V' \to V$ such that $fg' = g$.

(b) f is called irreducible if it is neither a section nor a retraction and if $f = ts$, for some $s : V \to X$ and $t : X \to W$, then s is a section or t is a retraction.

Now we are able to define the Auslander-Reiten quiver and the almost split sequences.

Definition B.1.5. *Let Q be a quiver and $\mathbb{K}Q$ be the path algebra of Q. The quiver $AR(Q) = (AR(Q)_0, AR(Q)_1)$, where the set of vertices $AR(Q)_0$ is the set of indecomposables of $\mathbb{K}Q$ and the set of arrows $AR(Q)_1$ is the set of the irreducible morphisms not zero between indecomposables, is called Auslander-Reiten quiver of Q.*

Theorem B.1.6. *If W is an indecomposable non-projective A-module (respectively V is an indecomposable non-injective A-module) then there exists an exact sequence $0 \to V \xrightarrow{f} Z \xrightarrow{g} W \to 0$ such that f and g are both irreducible, called almost split sequence.*

Proof. See [ARS, sec. 5.1 theorem 1.15]. □

If V is an A-module, a right minimal morphism $p : P \to V$, with P projective, is called a *projective cover of V*. One can prove that every A-module V has a minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} V \to 0$, i.e. an exact sequence where p_0 is a projective cover of V and p_1 is a projective cover of $Ker\, p_0$ ([ARS, sec. 1.4 theorem 4.2] and [ASS, sec. I.5 theor. 5.8]).
Let $V \in A - mod$, we assume that V has no projective summands and let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} V \to 0$ be a minimal presentation of V. Applying the functor $Hom_A(\cdot, A)$ on it, we obtain a minimal presentation

$$Hom(P_0, A) \xrightarrow{Hom(p_1,A)} Hom(P_1, A) \longrightarrow Coker(Hom(p_1, A)) \longrightarrow 0.$$

We define $coKer(Hom(p_1, A)) := Tr(V)$, the *transpose of V*. Thus the transpose is a contravariant functor $Tr : A - mod \to mod - A$ ($mod - A$ is the category of right A-modules) which equals zero on projective modules. We can define also $Tr : mod - A \to A - mod$ considering

$$mod - A \cong A^{op} - mod \xrightarrow{Tr} mod - A^{op} \cong A - mod.$$

Proposition B.1.7. *If $A = \mathbb{K}Q$ and V is a representation of Q without projective direct summands, then $Tr(V) = Ext^1_A(V, A)$.*

Proof. See [ARS, sec. 4.1 corollary 1.14]. □

Definition B.1.8. *The functor*

$$\tau^+ := \nabla \circ Tr : A - mod \xrightarrow{Tr} mod - A \cong A^{op} - mod \xrightarrow{\nabla} A - mod,$$

where ∇ is the duality functor sending the representation V to V^*, is called Auslander-Reiten translation (AR-translation). Similarly we can define the functor $\tau^- := Tr \circ \nabla$.

We note that, by definition, $\nabla\tau^- = \tau^+\nabla$ and $\nabla\tau^+ = \tau^-\nabla$.
The following theorem records an important property of the AR-translation.

Theorem B.1.9 (Auslander-Reiten duality)**.** *Let $A = \mathbb{K}Q$ and let V and W be two A-modules.*

(a) If V has no projective summands, then there exist isomorphisms of vector spaces

$$Hom_Q(W, \tau^+V) \cong Ext^1_Q(V, W)^* \text{ and } Ext^1_Q(W, \tau^+V) \cong Hom_Q(V, W)^*.$$

(b) If V has no injective summands, then there exist isomorphisms of vector spaces

$$Hom_Q(\tau^-V, W) \cong Ext^1_Q(W, V)^* \text{ and } Ext^1_Q(\tau^-V, W) \cong Hom_Q(W, V)^*.$$

Proof See [ASS, sec. IV.2 cor. 2.14]. □

Corollary B.1.10. *Let $A = \mathbb{K}Q$ and let V and W be two A-modules.*

(a) If V and W have no projective summands, then there exist isomorphisms of vector spaces

$$Hom_Q(\tau^+V, \tau^+W) \cong Hom_Q(V, W)$$

and

$$Ext^1_Q(\tau^+V, \tau^+W) \cong Ext^1_Q(V, W).$$

(b) If V and has no injective summands, then there exist isomorphisms of vector spaces

$$Hom_Q(\tau^-V, \tau^-W) \cong Hom_Q(V, W)$$

and

$$Ext^1_Q(\tau^-V, \tau^-W) \cong Ext^1_Q(V, W).$$

Proof. It is an immediate consequence of theorem 1.9. □

By AR-duality, if we consider τ^+ and τ^- as linear transformations on the space of dimension vectors, i.e. if V is a representation of a quiver with dimension α then $\tau^{\pm}\alpha := \underline{dim}\,\tau^{\pm}V$, we have, for every α and β dimension vectors, then

(i) $\langle\alpha, \beta\rangle = -\langle\tau^-\beta, \alpha\rangle$

(ii) $\langle\alpha, \beta\rangle = -\langle\beta, \tau^+\alpha\rangle$

(iii) $\langle\alpha, \beta\rangle = \langle\tau^{\pm}\alpha, \tau^{\pm}\beta\rangle$.

At last another result about the existence of the almost split sequences.

Theorem B.1.11 (Auslander-Reiten 1975)**.** *1) For every finitely generated indecomposable non-projective module V there is an almost split sequence $0 \to \tau^+V \to X \to V \to 0$ in $A - mod$ with finitely generated modules.*

2) For every finitely generated indecomposable non-injective module V there is an almost split sequence $0 \to V \to Z \to \tau^-V \to 0$ in $A - mod$ with finitely generated modules.

Proof. It is a direct consequence of the theorem 1.8, see also [ASS, sec. IV.3 theor. 3.1]. □

B.2 Quivers of tame type

Definition B.2.1. *A quiver Q is called of tame type if the underlying graph of Q is of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$.*

For all of the next results we refer to [DR].

Proposition B.2.2. *Let Q be a quiver of tame type, then the quadratic form $q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ defined by*

$$q_Q(\alpha) := \sum_{x\in Q_0} \alpha(x)^2 - \sum_{a\in Q_1} \alpha(ta)\alpha(ha)$$

is positive semi-definite and there exists a unique vector $h \in \mathbb{N}^{Q_0}$ such that $\mathbb{Z}h$ is the radical of q_Q or, equivalently, such that $\tau^+h = h$ and $|h| := \sum_{x\in Q_0} h(x)$ is minimum in $\mathbb{N}^{Q_0}$. For quivers of type $\widetilde{A}$ and $\widetilde{D}$ the vector h has the following form

$$\widetilde{A}: \quad \begin{matrix} & 1 & \cdots & 1 & \\ 1 & & & & 1 \\ & 1 & \cdots & 1 & \end{matrix} \tag{B.1}$$

$$\widetilde{D}: \quad \begin{array}{ccccc} 1 & & & & 1 \\ & 2 & \cdots & 2 & \\ 1 & & & & 1 \end{array} \tag{B.2}$$

Definition B.2.3. *Let V be an indecomposable representation of Q.*

(i) V is preprojective if and only if $(\tau^+)^i V = 0$ for $i >> 0$.

(ii) V is preinjective if and only if $(\tau^-)^i V = 0$ for $i >> 0$.

(iii) V is regular if and only if $(\tau^+)^i V \neq 0$ for every $i \in \mathbb{Z}$.

Definition B.2.4. *Let V be a representation of Q. The linear map*

$$\partial : \mathbb{N}^{Q_0} \longrightarrow \mathbb{Z}$$

defined by $\partial(\underline{dim}\, V) := \langle h, \underline{dim}\, V\rangle$ is called defect of V.

Lemma B.2.5. *Let V an indecomposable representation of Q. V is preprojective, preinjective or regular if and only if the defect of V is respectively negative, positive or zero.*

The regular representations of Q form an Abelian category $Reg_{\mathbb{K}}(Q)$. Moreover $Reg_{\mathbb{K}}(Q)$ is serial, i.e. every indecomposable regular representation has only one regular composition series and so it is only determined by its regular socle and by its regular length.

Definition B.2.6. *A simple regular module E is called homogeneous if and only if $\underline{dim}\, E = h$.*

Proposition B.2.7. *Let Q be a quiver of tame type. Then there exist at most three τ^+-orbits $\Delta = \{e_i | \, i \in I = \{0, \dots, u\}\}$, $\Delta' = \{e'_i | \, i \in I' = \{0, \dots, v\}\}$, $\Delta'' = \{e''_i | \, i \in I'' = \{0, \dots, w\}\}$, of dimension vectors of non-homogeneous simple regular representations of Q (I, I', I'' could be empty). We can assume that $\tau^+(e_i) = e_{i+1}$ for $i \in I$ ($e_{u+1} = e_0$), $\tau^+(e'_i) = e'_{i+1}$ for $i \in I'$ ($e'_{v+1} = e'_0$) and $\tau^+(e''_i) = e''_{i+1}$ for $i \in I''$ ($e''_{w+1} = e_0$).*

We denote the set of all regular representations of Q with $\mathcal{D}_r$. Every vector $d \in \mathcal{D}_r$ can be decomposed as

$$d = ph + \sum_{i \in I} p_i e_i + \sum_{i \in I'} p'_i e'_i + \sum_{i \in I''} p''_i e''_i \tag{B.3}$$

for some $p, p_i, p'_i, p''_i \in \mathbb{N}$ such that at least one of coefficients in each family $\{p_i | \, i \in I\}$, $\{p'_i | \, i \in I'\}$, $\{p''_i | \, i \in I''\}$ is zero. The decomposition (B.3) is called *canonical decomposition of d.* It is unique because the only linear relations between h, e_i, e'_i and e''_i are

$$h = \sum_{i \in I} e_i = \sum_{i \in I'} e'_i = \sum_{i \in I''} e''_i.$$

We observe that the category $Reg_{\mathbb{K}}(Q)$ can be decomposed as direct sum of categories $\mathcal{R}_t$, with $t = (\varphi, \psi) \in \mathbb{P}_1(\mathbb{K})$. In all categories $\mathcal{R}_t$, but at most three of these, there is only one simple object V_t which is necessarily homogeneous.

Definition B.2.8. *(1) We call E_i, E'_i and E''_i the simple non-homogeneous regular representations respectively of dimension e_i, e'_i and e''_i.*
(2) We call $V_{(\varphi,\psi)}$, where $(\varphi, \psi) \in \mathbb{P}_1(\mathbb{K})$, the indecomposable regular representation of dimension h.
(3) We define $E_{i,j}$ to be the indecomposable regular representations with socle E_i and dimension $\sum_{k=i}^{j} e_k$, where e_k are vertices of the arc with clockwise orientation e_i — ⋯ — e_j in Δ, without repetitions of e_k. We denote $E_i := E_{i,i}$ and similarly we define $E'_{i,j}$ and $E''_{i,j}$.

Lemma B.2.9.

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i = j - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Schur's lemma, we have

$$dim_{\mathbb{K}}(Hom_Q(E_i, E_j)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

By [DR, lemma 3.3], we have $dim_{\mathbb{K}}(Ext^1_Q(E_i, E_j)) = 0$ for every $i \neq j - 1$. So by the relation

$$\langle e_i, e_j \rangle = dim_{\mathbb{K}}(Hom_Q(E_i, E_j)) - dim_{\mathbb{K}}(Ext^1_Q(E_i, E_j)),$$

we obtain the thesis. □

B.3 Reflection functors and Coxeter functors

Definition B.3.1. *Let Q be a quiver.*

a) The vertex $x \in Q_0$ is a sink if there are no arrows $a \in Q_1$ such that $ta = x$.

b) The vertex $x \in Q_0$ is a source if there are no arrows $a \in Q_1$ such that $ha = x$.

Let Q be a quiver and let $x \in Q_0$ be a sink (respectively a source). We define the quiver $c_x(Q)$ in which the direction of the arrows connecting to x are reversed.

Definition B.3.2. *Let $\{a_1, \ldots, a_k\}$ be the set of arrows in Q whose head (respectively tail) equals x. We put*

$$c_x(Q)_0 = Q_0$$

$$c_x(Q)_1 = \{c_x(a); a \in Q_1\}$$

where $tc_x(a_i) = ha_i$, $hc_x(a_i) = ta_i$ for every $i \in \{1, \ldots, k\}$ and $tc_x(b) = tb$, $hc_x(b) = hb$ for every $b \in Q_1 \setminus \{a_1, \ldots, a_k\}$.

Now we define the functors C_x^+ and C_x^- from $Rep(Q)$ to $Rep(c_x(Q))$.

Definition B.3.3. *Let Q be a quiver and $x \in Q_0$ be a sink. Let $\{a_1, \ldots, a_k\}$ be the set of arrows in Q whose head equals x. Let $V \in Rep(Q)$. We define the representation $C_x^+(V) := W \in Rep(c_x(Q))$ as follows.*

$$W(y) = \begin{cases} V(y) & if \quad x \neq y \\ Ker\big(\bigoplus_{i=1}^k V(ta_i) \xrightarrow{h} V(x)\big) & otherwise, \end{cases}$$

where $h(v_1, \ldots, v_k) = V(a_1)(v_1) + \cdots + V(a_k)(v_k)$ with $(v_1, \ldots, v_k) \in \bigoplus_{i=1}^k V(ta_i)$.

$$W(c_x(a)) = \begin{cases} V(a) & if \quad ha \neq x \\ W(x) \hookrightarrow \bigoplus_{i=1}^k V(ta_i) \xrightarrow{p_j} V(ta_j) & if \quad a = a_j \end{cases}$$

where p_j denotes the projection on the j-th factor.

Definition B.3.4. *Let Q be a quiver and $x \in Q_0$ be a source. Let $\{b_1, \ldots, b_l\}$ be the set of arrows in Q whose tail equals x. Let $V \in Rep(Q)$. We define the representation $C_x^-(V) := W \in Rep(c_x(Q))$ as follows.*

$$W(y) = \begin{cases} V(y) & if \quad x \neq y \\ Coker\big(V(x) \xrightarrow{\tilde{h}} \bigoplus_{i=1}^l V(hb_i)\big) & otherwise, \end{cases}$$

where $\tilde{h}(v) = (V(b_1)(v), \ldots, V(b_l)(v))$ with $v \in V(x)$.

$$W(c_x(a)) = \begin{cases} V(a) & if \quad ta \neq x \\ V(hb_j) \xrightarrow{i_j} \bigoplus_{i=1}^l V(hb_i) \twoheadrightarrow W(x) & if \quad a = b_j \end{cases}$$

where i_j denotes the immersion of the j-th factor.

Let $f = (f_y)_{y \in Q_0} : V \to W$ be a morphism in $Rep(Q)$.
If x is a sink and $\{a_1, \ldots, a_k\}$ is the set of arrows whose head equals x, we define $C_x^+ f = ((C_x^+ f)_y)_{y \in Q_0} : C_x^+ V \to C_x^+ W$ a morphism in $Rep(c_x Q)$ as

follows. For every $y \neq x$, we have $f_y = (C_x^+ f)_y$, whereas $(C_x^+ f)_x$ is the unique $\mathbb{K}$-linear map which makes the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (C_x^+ V)_x & \longrightarrow & \bigoplus_{i=1}^k V_{ta_i} & \xrightarrow{h} & V_x \\ & & \downarrow (C_x^+ f)_x & & \downarrow \bigoplus_{i=1}^k f_{ta_i} & & \downarrow f_x \\ 0 & \longrightarrow & (C_x^+ W)_x & \longrightarrow & \bigoplus_{i=1}^k W_{ta_i} & \xrightarrow{h'} & W_x \end{array}$$

commutative.
If x is a source and $\{b_1, \dots, b_l\}$ is the set of arrows whose tail equals x, we define $C_x^- f = ((C_x^- f)_y)_{y \in Q_0} : C_x^- V \to C_x^- W$ a morphism in $Rep(c_x Q)$ as follows. For every $y \neq x$, we have $f_y = (C_x^- f)_y$, whereas $(C_x^- f)_x$ is the unique $\mathbb{K}$-linear map which makes the diagram

$$\begin{array}{ccccccc} V_x & \xrightarrow{\tilde{h}} & \bigoplus_{i=1}^l V_{tb_i} & \longrightarrow & (C_x^- V)_x & \longrightarrow & 0 \\ \downarrow f_x & & \downarrow \bigoplus_{i=1}^l f_{tb_i} & & \downarrow (C_x^- f)_x & & \\ W_x & \xrightarrow{\tilde{h'}} & \bigoplus_{i=1}^l W_{tb_i} & \longrightarrow & (C_x^- W)_x & \longrightarrow & 0 \end{array}$$

commutative.
In particular, by definition, we have $Hom(V, W) = 0$ if and only if $Hom(C_x^+ V, C_x^+ W) = 0$, with x a sink and $Hom(V, W) = 0$ if and only if $Hom(C_x^- V, C_x^- W) = 0$, with x a source.
C_x^+, for every x sink, and C_x^-, for every x source, are called *reflection functors*.
We state the main result about reflection functors.

Theorem B.3.5 (Bernstein-Gelfand-Ponomarev). *1) Let $x \in Q_0$ be a sink. Let $V \in Rep(Q)$ be an indecomposable representation of dimension α. Then we have two possibilities*

a) $V = S_x$ and then $C_x^+(V) = 0$,

b) $C_x^+(V)$ is indecomposable and $C_x^- C_x^+(V) \cong V$ and the dimension of $C_x^+(V)$ equals $c_x(\alpha)$ where

$$c_x(\alpha)(y) = \begin{cases} \alpha(y) & if \quad y \neq x \\ \sum_{i=1}^k \alpha(ta_i) - \alpha(x) & otherwise. \end{cases}$$

2) Let $x \in Q_0$ be a source. Let $V \in Rep(Q)$ be an indecomposable representation of dimension α. Then we have two possibilities

a) $V = S_x$ and then $C_x^-(V) = 0$,

b) $C_x^-(V)$ is indecomposable and $C_x^+ C_x^-(V) \cong V$ and the dimension of $C_x^-(V)$ equals $c_x(\alpha)$ where

$$c_x(\alpha)(y) = \begin{cases} \alpha(y) & if \quad y \neq x \\ \sum_{i=1}^l \alpha(hb_i) - \alpha(x) & otherwise. \end{cases}$$

3) *Let* $V_1, V_2 \in Rep(Q)$

$$C_x^{\pm}(V_1 \oplus V_2) = C_x^{\pm}(V_1) \oplus C_x^{\pm}(V_2).$$

Proof. See [BGP, theorem 1.1].

Definition B.3.6. *A sequence* $x_1, \ldots, x_m$ *of vertices of* Q *is an admissible sequence of sinks (respectively of sources) if* x_{i+1} *is a sink (respectively a source) in* $c_{x_i} \cdots c_{x_1}(Q)$ *for* $i = 0, 1, \ldots, m-1$.

Corollary B.3.7. *Let* Q *be a quiver and let* $x_1, \ldots, x_m$ *be an admissible sequence of sinks.*

1) *For every* $i = 1, \ldots, m$, $C_{x_1}^{-} \cdots C_{x_{i-1}}^{-}(S_{x_i})$ *is either 0 or indecomposable (here* $S_{x_i} \in Rep(c_{x_{i-1}} \cdots c_{x_1}(Q))$*).*

2) *Let* $V \in Rep(Q)$ *be an indecomposable. We assume* $C_{x_k} \cdots C_{x_1}(V) = 0$ *for some* k. *Then there exists* $i \in \{0, \ldots, k-1\}$ *such that* $V \cong C_{x_1}^{-} \cdots C_{x_{i-1}}^{-}(S_{x_i})$.

Proof. Follows by induction from theorem 1.7.

Definition B.3.8. *Let* Q *be a quiver with* n *vertices without oriented cycles. We choose the numbering* $(x_1, \ldots, x_n)$ *of vertices such that* $ta > ha$ *for every* $a \in Q_1$. *We define*

$$C^+ := C_{x_n}^{+} \cdots C_{x_1}^{+} \quad \textit{and} \quad C^- := C_{x_1}^{-} \cdots C_{x_n}^{-}.$$

The functors $C^+, C^- : Rep(Q) \to Rep(Q)$ *are called Coxeter functors.*

These functors don't depend on the choice of numbering of vertices because of the following interpretation of the Coxeter functors in terms of the Auslander-Reiten functors.

Lemma B.3.9. *Let* $\mathbb{K}Q$ *be the path algebra of a quiver* Q *without oriented cycles and* $(x_1, \ldots, x_n)$ *be an admissible numbering of vertices.*

(i) *If* V *is an indecomposable nonprojective* $\mathbb{K}Q$*-module, then there are isomorphisms* $C^+V \cong \tau^+V$ *and* $C^-C^+V \cong V$.

(ii) *If* W *is an indecomposable noninjective* $\mathbb{K}Q$*-module, then there are isomorphisms* $C^-W \cong \tau^-W$ *and* $C^+C^-W \cong W$.

Proof. See [ASS, chap. VII lemma 5.8]. □

B.4 Semi-invariants of quivers without oriented cycles

For a dimension vector α we have

$$Rep(Q,\alpha) := \bigoplus_{a\in Q_1} Hom(\mathbb{K}^{\alpha(ta)}, \mathbb{K}^{\alpha(ha)}),$$

the space of α-dimensional representations of Q. Moreover we define the group

$$GL(Q,\alpha) := \prod_{x\in Q_0} GL(\mathbb{K}, \alpha(x))$$

and its subgroup

$$SL(Q,\alpha) := \prod_{x\in Q_0} SL(\mathbb{K}, \alpha(x)).$$

These groups act on $Rep(Q,\alpha)$ as follows: if $V \in Rep(Q,\alpha)$ and $g = (g_x)_{x\in Q_0} \in GL(Q,\alpha)$, then $g \cdot V = \{g_{ha}V(a)g_{ta}^{-1}\}_{a\in Q_1}$. Finally we denote the ring of semi-invariants by

$$SI(Q,\alpha) := \mathbb{K}[Rep(Q,\alpha)]^{SL(Q,\alpha)} = \{f \in Rep(Q,\alpha) | \forall g \in SL(Q,\alpha) g{\cdot}f = f\},$$

where the action of $GL(Q,\alpha)$ on $\mathbb{K}[Rep(Q,\alpha)]$, the coordinate ring of polynomial functions on $Rep(Q,\alpha)$, is induced by the action of $GL(Q,\alpha)$ on $Rep(Q,\alpha)$ by the rule

$$(g \cdot f)(V) := f(g^{-1} \cdot V),$$

with $g \in GL(Q,\alpha)$, $f \in \mathbb{K}[Rep(Q,\alpha)]$ and $V \in Rep(Q,\alpha)$.

Definition B.4.1. *If f is a semi-invariant of a quiver Q, we call $Z(f)$ the vanishing set of f.*

Lemma B.4.2. *Let f and f' be two semi-invariants of a quiver Q such that $Z(f) = Z(f')$ is irreducible. Then $f = \lambda \cdot f'$ for some non zero $\lambda \in \mathbb{K}$.*

Proof. Since $Z(f)$ is irreducible, also f is an irreducible polynomial. From $Z(f) = Z(f')$ it follows that $f'|f$ and so $f = \lambda \cdot f'$ for some non zero $\lambda \in \mathbb{K}$. □.

Remark B.4.3. *Let α be a dimension vector. Any set S of generators of $SI(Q,\alpha)$ contains a subset of irreducible generators. Indeed if $f \in S$ is a reducible polynomial, then it can be expressed as a product of irreducible elements from S.*

Now we define the semi-invariants which appear in the principal theorem.

Lemma B.4.4. *The spaces $Hom_Q(V,W)$ and $Ext^1_Q(V,W)$ are respectively the kernel and the cokernel of the following linear map*

$$d^V_W : \bigoplus_{x \in Q_0} Hom(V(x), W(x)) \longrightarrow \bigoplus_{a \in Q_1} Hom(V(ta), W(ha))$$

where d^V_W is given by

$$\{f(x) | x \in Q_0\} \longmapsto \{f(ha)V(a) - W(a)f(ta) | a \in Q_1\}.$$

Proof. See [R].

If a representation V has dimension vector α, then d^V_W can be seen as the $\mathbb{K}$-linear map which sends $\bigoplus_{x \in Q_0} W(x)^{\alpha(x)}$ to $\bigoplus_{a \in Q_1} W(ha)^{\alpha(ta)}$.
For every representation V of a quiver Q without oriented cycles of dimension α, we can construct a projective resolution, called *Ringel resolution of* V:

$$0 \longrightarrow \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \xrightarrow{d^V} \bigoplus_{x \in Q_0} V(x) \otimes P_x \xrightarrow{p_V} V \longrightarrow 0 \qquad \text{(B.4)}$$

where P_x is the indecomposable projective associated to vertex x for every $x \in Q_0$ (see section B.1 of appendix), d^V restricted to $V(ta) \otimes P_{ha}$ sends $v \otimes e_{ha}$ to $V(a)(v) \otimes e_{ha} - v \otimes a$ and p_V restricted to $V(x) \otimes P_x$ sends v to $v \otimes e_x$, see [R]. Moreover, applying the functor $Hom_Q(\cdot, W)$ to Ringel resolution of V, we have $Hom_Q(d^V, W) = d^V_W$ for every representation W of Q.
Any character τ of $GL(Q,\alpha)$ has the form

$$\tau : \{g_x \in GL(\alpha(x)) | x \in Q_0\} \mapsto \prod_{x \in Q_0} (det g_x)^{\chi(e_x)}$$

with e_x a dimension vector, defined by $e_x(x) = 1$ and $e_x(y) = 0$ if $x = y$, and $\chi(e_x) \in \mathbb{Z}\ \forall x \in Q_0$. A vector $\chi \in \mathbb{Z}^{|Q_0|}$ is called *weight*.
The ring $SI(Q,\alpha)$ decomposes in graded components as

$$SI(Q,\alpha) = \bigoplus_{\tau \in char(GL(Q,\alpha)} SI(Q,\alpha)_\tau$$

where $SI(Q,\alpha)_\tau = \big\{f \in \mathbb{K}[Rep(Q,\alpha)] | g \cdot f = \tau(g) f\ \forall g \in GL(Q,\alpha)\big\}$.

Remark B.4.5. *(1) Each vector $\chi \in \mathbb{Z}^{|Q_0|}$ determines a unique character τ_χ.*

(2) A character τ for some semi-invariant might not uniquely determine the weight of the semi-invariant, e.g. if $\alpha(x) = 0$, then g_x is a 0×0 matrix, in which case $det(g_x) = 1$, therefore for any $\chi(x) \in \mathbb{Z}$, $det(g_x)^{\chi(x)} = det(g_x) = 1$.

If α and β are dimension vectors such that $\langle \alpha, \beta \rangle = 0$, $V \in Rep(Q, \alpha)$ and $W \in Rep(Q, \beta)$, then the matrix of d_W^V is a square matrix.

Definition B.4.6. *We define the semi-invariant $c(V, W) := det\, d_W^V$ of the action of $GL(Q, \alpha) \times GL(Q, \beta)$ on $Rep(Q, \alpha) \times Rep(Q, \beta)$ (see [S]). For a fixed V the restriction of c to $\{V\} \times Rep(Q, \beta)$ defines a semi-invariant $c^V = c(V, \cdot)$ in $SI(Q, \beta)$ of weight $\langle \alpha, \cdot \rangle$ [S, lemma 1.4]. Similarly, for a fixed W the restriction of c to $Rep(Q, \alpha) \times \{W\}$ defines a semi-invariant $c_W = c(\cdot, W)$ in $SI(Q, \alpha)$ of weight $-\langle \cdot, \beta \rangle$ [S, lemma 1.4]. These semi-invariants are called Schofield semi-invariants.*

These semi-invariants have the following properties.

Lemma B.4.7. *Suppose that V', V, V'' and W', W, W'' are representations of Q, that $\langle \underline{dim}(V), \underline{dim}(W) \rangle = 0$ and that there are exact sequences*

$$0 \to V' \to V \to V'' \to 0, \qquad 0 \to W' \to W \to W'' \to 0$$

then

(i) If $\langle \underline{dim}(V'), \underline{dim}(W) \rangle < 0$, then $c^V(W) = 0$

(ii) If $\langle \underline{dim}(V'), \underline{dim}(W) \rangle = 0$, then $c^{V'}(W) = c^{V''}(W) c^V(W)$

(iii) If $\langle \underline{dim}(V), \underline{dim}(W') \rangle > 0$, then $c^V(W) = 0$

(iv) If $\langle \underline{dim}(V), \underline{dim}(W') \rangle = 0$, then $c^V(W) = c^V(W') c^V(W'')$

and similarly for c_W.

Proof. See [DW1, lemma 1]. □

Remark B.4.8. *A consequence of lemma B.4.4 in [S] is that any projective resolution of V (respectively injective coresolution of W) can be used to calculate c^V (respectively c_W).So if P is a projective module and I is an injective module then $c^P = 0$ and $c_I = 0$.*

Now we formulate the result of Derksen and Weyman about the set of generators of the ring of semi-invariants $SI(Q, \alpha)$, defined in section 1.1, where Q is a quiver without oriented cycles and α is a dimension vector. So we assume throughout this section that there are no oriented cycles in Q.

Theorem B.4.9 (Derksen-Weyman)**.** *Let Q be a quiver without oriented cycles and let β be a dimension vector. The ring $SI(Q, \beta)$ is spanned by semi-invariants of the form c^V of weight $\langle \underline{dim}(V), \cdot \rangle$, for which $\langle \underline{dim}(V), \beta \rangle = 0$. It is also spanned by semi-invariants of the form c_W of weight $-\langle \cdot, \underline{dim}(W) \rangle$, for which $\langle \beta, \underline{dim}(W) \rangle = 0$.*

Proof. See [DW1, theorem 1]. □

Remark B.4.10. *If $\langle \underline{dim}(V), \underline{dim}(W)\rangle = 0$ then we have $c(V,W) = c^V(W) = c_W(V) = 0$ if and only if $Hom_Q(V,W) \neq 0$ which is equivalent to $Ext^1_Q(V,W) \neq 0$ by lemma B.4.4.*

Remark B.4.11. *i) If $V, V' \in Rep(Q)$ and $V \cong V'$ then c^V and $c^{V'}$ are equal up to a scalar.*

ii) If $V = V' \oplus V''$ is decomposable then, by lemma B.4.7, we have $c^V = 0$ in $SI(Q,\beta)$ if $\langle \underline{dim}(V'), \beta\rangle \neq 0$, and $c^V = c^{V'}c^{V''}$ in $SI(Q,\beta)$ if $\langle \underline{dim}(V'), \beta\rangle = 0$.

So the algebra $SI(Q,\beta)$ is generated by all c^V where V is indecomposable and $\langle \underline{dim}\, V, \beta\rangle = 0$.

Moreover in [DW1] Derksen and Weyman show the following

Corollary B.4.12 (Reciprocity)**.** *Let α and β be the dimension vectors satisfying $\langle \alpha, \beta\rangle = 0$. Then*

$$dimSI(Q,\beta)_{\langle \alpha, \cdot\rangle} = dimSI(Q,\alpha)_{-\langle \cdot, \beta\rangle}.$$

B.5 c^V, reflection functors and duality functor

The following results show the relation between c^V and C_x^+ (respectively C_x^-).

Lemma B.5.1. *Let V be an indecomposable representation of Q of dimension α such that $Z(c^V)$ is irreducible and let x be a sink of Q. Then*

$$c^V = \lambda \cdot (c^{C_x^+ V} \circ C_x^+)$$

on $Rep(Q,\beta)$ such that $\langle \alpha, \beta\rangle = 0$ and for some non zero $\lambda \in \mathbb{K}$.

Proof. First we note that, by remark B.4.3 and by theorem B.4.9, it's not restrictive to suppose $Z(c^V)$ is irreducible. By remark B.4.10, the vanishing set of c^V is the hypersurface

$$Z(c^V) = \{W \in Rep(Q,\beta) |\, Hom_Q(V,W) \neq 0\}$$

and the vanishing set of $c^{C_x^+ V}$ is the hypersurface

$$Z(c^{C_x^+ V}) = \{C_x^+ W \in Rep(c_x(Q), c_x(\beta)) |\, Hom_Q(C_x^+ V, C_x^+ W) \neq 0\}.$$

By definition of reflection functor, for every $W \in Rep(Q,\beta)$,

$$Hom_Q(V,W) \neq 0 \Leftrightarrow Hom_Q(C_x^+ V, C_x^+ W) \neq 0.$$

Hence $Z(c^V) = Z(c^{C_x^+ V})$.
So, by lemma B.4.2, we conclude that there exist non zero $\lambda \in \mathbb{K}$ such that $c^V = \lambda \cdot (c^{C_x^+ V} \circ C_x^+)$. □

Similarly one proves the following

Lemma B.5.2. *Let V be an indecomposable representation of Q of dimension α such that $Z(c^V)$ is irreducible and let x be a source of Q. Then*

$$c^V = \lambda \cdot (c^{C_x^- V} \circ C_x^-)$$

on $Rep(Q, \beta)$ such that $\langle \alpha, \beta \rangle = 0$ and for some non zero $\lambda \in \mathbb{K}$.

Next we study the relation between c^V and duality functor ∇.

Lemma B.5.3. *Let (Q, σ) be a symmetric quiver. For every representation V of the underlying quiver Q such that $Z(c^V)$ is irreducible, we have*

$$c^V = \lambda \circ (c^{\tau^- \nabla V} \circ \nabla) \tag{B.5}$$

for some non zero $\lambda \in \mathbb{K}$.

Proof. First we note that, by remark B.4.3 and by theorem B.4.9, it's not restrictive to suppose $Z(c^V)$ is irreducible. Let β be a dimension vector such that $\langle \underline{dim}\, V, \beta \rangle = 0$. By equation (1.16) we note that, for every $W \in Rep(Q, \beta)$,

$$Hom_Q(V, W) = 0 \Leftrightarrow Hom_Q(\nabla W, \nabla V) = 0 \Leftrightarrow Hom_Q(\tau^- \nabla V, \nabla W) = 0. \tag{B.6}$$

Thus, by remark B.4.10, the vanishing set of c^V is the hypersurface

$$Z(c^V) = \{W \in Rep(Q, \beta) | \, Hom_Q(V, W) \neq 0\}$$

and the vanishing set of $c^{\tau^- \nabla V}$ is the hypersurface

$$Z(c^{\tau^- \nabla V}) = \{\nabla W \in Rep(Q, \delta\beta) | \, Hom_Q(\nabla W, \nabla V) \neq 0\}.$$

Finally, by equation (B.6), $Z(c^V) = Z(c^{\tau^- \nabla V})$.
So, by lemma B.4.2, we conclude that there exist non zero $\lambda \in \mathbb{K}$ such that $c^V = \lambda \cdot (c^{\tau^- \nabla V} \circ \nabla)$. □

B.6 c^V's, weights and partitions

Lemma B.6.1. *Let Q be a quiver, let x be a sink and let α be a vector dimension.*

(i) If V is indecomposable not projective such that $C_x^+ V$ is not projective and $0 = \langle \underline{dim} V, \alpha \rangle (= \langle c_x \underline{dim} V, c_x \alpha \rangle)$, then $c^V \in SI(Q, \alpha)$ and $c^{C_x^+ V} \in SI(c_x Q, c_x \alpha)$.

(ii) *If $V = S_x$ and $\langle \underline{dim} S_x, c_x\alpha\rangle = 0$, then we have $c^V \in SI(c_xQ, c_x\alpha)$, where S_x is considered as representation of c_xQ, but c^V is zero for Q.*

(iii) *If $V = C^- S_x$ and $\langle \underline{dim} C^- S_x, \alpha\rangle = 0$, then we have $c^V \in SI(Q, \alpha)$ but $c^{C_x^+ V}$ is zero for c_xQ.*

Proof. First of all we observe that if x is a sink and $V \neq S_x$ is projective then $C_x^+ V$ is projective since C^+ doesn't depend on any admissible numbering of vertices. Moreover $\langle \underline{dim} S_x, c_x\alpha\rangle = 0$ and $\langle \underline{dim} C^- S_x, \alpha\rangle = 0$ are not both zero. By theorem B.1.9 and since x is a sink, $0 = \langle \underline{dim} C^- S_x, \alpha\rangle = -\langle \alpha, \underline{dim} S_x\rangle = -\alpha_x + \sum_{a\in Q_1: ha=x} \alpha_{ta}$ and $0 = \langle \underline{dim} S_x, c_x\alpha\rangle = (c_x\alpha)_x - \sum_{a\in c_x(Q)_1} (c_x\alpha)_{ha} = \sum_{a\in Q_1: ha=x} \alpha_{ta} - \alpha_x - \sum_{a\in Q_1: ha=x} \alpha_{ta} = -\alpha_x$ and so $\sum_{a\in Q_1: ha=x} \alpha_{ta} = 0$ which is an absurd unless $\alpha_{ta} = 0$ for every a such that $ha = x$ but in such case $c^{S_x} = 0$ for c_xQ and $c^{C^- S_x} = 0$ for Q.
Proof of (i). Since $\langle \underline{dim} V, \alpha\rangle = 0$, by theorem B.4.9, the c^V's are generators of $SI(Q, \alpha)$ and $c^{C_x^+ V}$'s are generators of $SI(c_xQ, c_x\alpha)$. Moreover we note that the number of generators of $SI(Q, \alpha)$ is equal to the number of generators of $SI(c_xQ, c_x\alpha)$.
Proof of (ii). We can study S_x since if $V \neq S_x$ is projective, by remark above, we have $c^V = 0$ and also $c^{C_x^+ V} = 0$. S_x is projective in Q and so c^{S_x} is zero in $SI(Q, \alpha)$ but S_x, considered as a representation of c_xQ, is injective. So, if $\langle \underline{dim} S_x, c_x\alpha\rangle = 0$ then $c^{S_x} \in SI(c_xQ, c_x\alpha)$.
Proof of (iii). $C^- S_x$ is not projective otherwise $S_x = C^+(C^- S_x) = 0$ which is an absurd. Thus if $\langle \underline{dim} C^- S_x, \alpha\rangle = 0$ then $c^{C^- S_x} \in SI(Q, \alpha)$. Moreover $C^+ C_x^+ C^- S_x = C_x^+ C^+ C^- S_x = C_x^+ S_x = 0$ hence $C_x^+ C^- S_x$ is projective and so $c^{C_x^+ C^- S_x} = 0$ in $SI(c_xQ, c_x\alpha)$. □

We recall that if Q is Dynkin, then $SI(Q, \alpha)$ has a finite number of generators by remark B.4.11.

Corollary B.6.2. *Let Q be a Dynkin quiver and let x be a sink. We call $N(Q, \alpha)$ the number of generators of $SI(Q, \alpha)$ and $N(c_xQ, c_x\alpha)$ the number of generators of $SI(c_xQ, c_x\alpha)$. We have three possibilities.*

(a) $N(Q, \alpha) = N(c_xQ, c_x\alpha)$ *if* $\langle \underline{dim} S_x, c_x\alpha\rangle \neq 0$ *and* $\langle \underline{dim} C^- S_x, \alpha\rangle \neq 0$;

(b) $N(Q, \alpha) + 1 = N(c_xQ, c_x\alpha)$ *if* $\langle \underline{dim} S_x, c_x\alpha\rangle = 0$;

(c) $N(c_xQ, c_x\alpha) + 1 = N(Q, \alpha)$ *if* $\langle \underline{dim} C^- S_x, \alpha\rangle = 0$.

Proof. (a) follows directly from (i) of the previous lemma. (b): the generators of $SI(c_xQ, c_x\alpha)$ are those of $SI(Q, \alpha)$ and c^{S_x}. (c): the generators of $SI(Q, \alpha)$ are those of $SI(c_xQ, c_x\alpha)$ and $c^{C^- S_x}$. □

Now we study weights of a quiver A_n and associated partitions. We denote vertices of A_n with $\{1, \ldots, n\}$ in increasing way from left to right and we call a_i the arrow which has i on the left and $i+1$ on the right. Let $V_{i,j}$ be the indecomposable of A_n with dimension vector

$$(v_{i,j})_h = \begin{cases} 1 & \text{if} \quad i \le h \le j \\ 0 & \text{otherwise.} \end{cases}$$

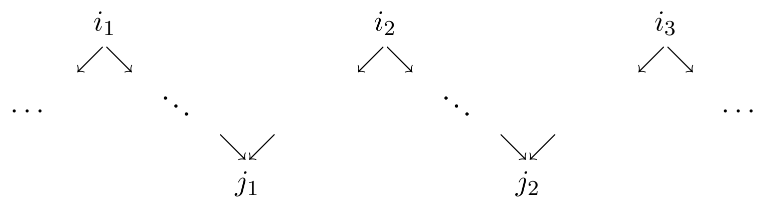
Let $E = (E_{i,j})_{1 \le i,j \le n}$ be the Euler matrix of a quiver Q , i.e the matrix associated to the Euler form $\langle \cdot, \cdot \rangle$. In general we have

$$E_{i,j} = \begin{cases} 1 & \text{if } i = j \\ \sharp\{a \in Q_1 | ta = i, ha = j\} & \text{otherwise.} \end{cases}$$

If $Q = A_n$

$$E_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \to j \\ 0 & \text{otherwise.} \end{cases}$$

Let $\langle v_{i,j}, \cdot \rangle = v_{i,j}E = \chi = (\chi_l)_{1 \le l \le n}$ be the weight of $c^{V_{i,j}}$.
We consider the following notation for A_n, let $s, p \ge 1$ be respectively the number of sources and the number of sinks in A_n (there are at least one source and one sink, which occurs in the equioriented case).



where i_k and j_h in $\{1, \dots, n\}$ with $1 \le k \le s$ and $1 \le h \le p$ are respectively sources and sinks of Q. By the previous picture we note that in A_n sinks and sources alternate.
Let $K = \{k \in \{1, \dots, s\} | i \le i_k \le j\}$ and $H = \{h \in \{1, \dots, p\} | i \le j_h \le j\}$

Lemma B.6.3. *The weight of $c^{V_{i,j}}$ is $\chi = (\chi_l)_{l \in \{1,\dots,n\}}$ such that*

$$\chi_l = \begin{cases} 1 & l = i_k \text{ with } k \in K \text{ or } l = i \text{ and } \quad ta_i = i \text{ or } l = j \text{ and } ta_{j-1} = j \\ -1 & l = j_h \text{ with } h \in H \text{ or } l = i-1 \text{ and } ha_{i-1} = i-1 \text{ or } l = j+1 \text{ and } ha_j = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $v_{i,j}E = \chi = (\chi_l)_{1 \le l \le n}$ is the weight of $c^{V_{ij}}$ then $\chi_l = E_{i,l} + E_{i+i,l} + \dots + E_{j,l}$ for every $l \in \{1, \dots, n\}$. So

$$\chi_l = \begin{cases} E_{l-1,l} + E_{l,l} + E_{l+1,l} & l \in \{i+1, \dots, j-1\} \\ E_{l+1,l} & l = i-1 \\ E_{l-1,l} & l = j+1 \\ E_{l,l} + E_{l+1,l} & l = i \\ E_{l-1,l} + E_{l,l} & l = j \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\chi_l = 0$ for every $l \in \{1,\dots,i-2\} \cup \{j+2,\dots,n\}$,

$$\chi_{i-1} = \begin{cases} -1 & i-1 \leftarrow i \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_{j+1} = \begin{cases} -1 & j \rightarrow j+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_{i} = \begin{cases} 1 & i \rightarrow i+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_{j} = \begin{cases} 1 & j-1 \leftarrow j \\ 0 & \text{otherwise} \end{cases}$$

and for every $l \in \{i+1,\dots,j-1\}$

$$\chi_{l} = \begin{cases} 1 & l-1 \leftarrow l \rightarrow l+1 \\ -1 & l-1 \rightarrow l \leftarrow l+1 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Corollary B.6.4. *Let $Q = A_n$ and let w be the weight of $c^{V_{i,j}}$.*

(i) Let $\chi_l = 1$ for some $l \in \{i,\dots,j\}$ and let $k > l$ in $\{i+1,\dots,j-1\} \cup \{j+1\}$ be the first index such that $\chi_k \neq 0$, then $\chi_k = -1$.

(ii) Let $\chi_l = -1$ for some $l \in \{i+1,\dots,j-1\} \cup \{i-1, j+1\}$ and let $k > l$ in $\{i,\dots,j\}$ be the first index such that $\chi_k \neq 0$, then $\chi_k = 1$. $\square$

Let β be the dimension vector of an indecomposable representation of A_n and let $\chi = \langle \beta, \cdot \rangle$. Let m_1 be the first vertex such that $\chi(m_1) \neq 0$, in particular we suppose $\chi(m_1) = 1$ and m_t the last vertex such that $\chi(m_t) \neq 0$, in particular we suppose $\chi(m_t) = 1$, the other case proves in a similar way. Between m_1 and m_t, -1 and 1 alternate in correspondence respectively to sinks and to sources. In this case we have $[\frac{t}{2}]+1 = s+1$ occurrences of 1 and $s = [\frac{t}{2}]$ occurrences of -1. We call $i_0 = m_1$, $j_s = m_{t-1}$, $i_1,\dots,i_s$ the sources and $j_1,\dots,j_{s-1=p}$ the sinks between i_0 and j_s. Let V be a representation with $\underline{dim}\, V = \alpha$ such that $\langle \beta, \alpha \rangle = 0$ and $SL(V) = SL(V_1) \times \cdots \times SL(V_n)$, so we have, by Cauchy formula

$$\mathbb{K}[Rep(A_n,\alpha)]^{SL(V)} = SI(A_n,\alpha) = \left(\bigoplus_{\lambda: Q_1 \to \Lambda} \bigotimes_{c \in Q_1} S_{\lambda(c)} V_{tc} \otimes S_{\lambda(c)} V^*_{hc} \right)^{SL(V)}$$

where Λ is the set of all partitions.
$\chi(k) = 0$ for every $k < i_0$ so either $\lambda(a_{k-1}) = \lambda(a_k)$ or $\lambda(a_{k-1}) = 0 = \lambda(a_k)$ for every $k < i_0$. Since $\chi(1) = 0$ then $\lambda(a_1) = 0$ and thus $\lambda(a_k) = 0$ for every $k < i_0$. So we have $(S_{\lambda(a_{i_0})} V_{i_0})^{SLV_{i_0}} \neq 0$ if and only if $\lambda(a_{i_0}) =$

$(\overbrace{1,\dots,1}^{\alpha_{a_{i_0}}})$. Now $\chi(k) = 0$ for every $i_0 < k < j_1$ and $\chi(j_1) = -1$ then we have $\lambda(a_{i_0+1}) = \lambda(a_{i_0})$ otherwise $(S_{\lambda(a_{i_0})}V^*_{i_0+1} \otimes S_{\lambda(a_{i_0+1})}V_{i_0+1})^{SLV_{i_0+1}}$ doesn't have weight 0. So $\lambda(a_k) = \lambda(a_{i_0})$ for every $i_0 < k < j_1$. For j_1 we have $\lambda(a_{j_1})$ and $\lambda(a_{i_0})$ are complementary with respect to a column of height α_{j_1} because $-\lambda(a_{j_1})_h - \lambda(a_{i_0})_{\alpha_{j_1}-h+1} = -1$ for every $\in \{1,\dots,\alpha_{j_1}\}$, by proposition A.2.9. We proceed in a similar way with the other vertices until i_s for which $\chi(i_s) = 1$. Since $\chi(k) = 0$ for every $k > i_s$, we have either $\lambda(a_{k-1}) = \lambda(a_k)$ or $\lambda(a_{k-1}) = 0 = \lambda(a_k)$ for every $k > i_s$ but because $\lambda(a_{n-1}) = 0$, $\lambda(a_k) = 0$ for every $k > i_s$. Moreover $\lambda(a_{i_s-1})$ is both a column of height α_{i_s} and the complementary of $\lambda(a_{i_{s-1}-1})$ with respect to a column of height $\alpha_{i_{s-1}}$.
So we proved the following

Lemma B.6.5. *Let Q be a quiver of type A_n, let α be a dimension vector and β be a dimension vector of an indecomposable representation of Q. Let χ be the weight $\langle\beta,\cdot\rangle$ and we suppose it is such that $\chi(i) \neq 0$ for every $i \in I = \{m_j\}_{j\in\{1,\dots,t\}}$, where I is a subset of $\{1,\dots,n\}$. Then the family of partitions associated to χ is $\underline{\lambda} = (\lambda(a_1),\dots,\lambda(a_{n-1}))$ such that $\lambda(a_i) = 0$ for every $i \in \{1,\dots,m_1 - 1\} \cup \{m_t,\dots,n-1\}$, $\lambda(a_{m_1})$ and $\lambda(a_{m_t-1})$ are columns respectively of height α_{m_1} and α_{m_t} and $\lambda(a_i)$ is the complementary of $\lambda(a_{i-1})$ with respect to a column of height α_i for every $i \in \{m_j\}_{j\in\{2,\dots,t-1\}}$. Moreover we have $\alpha_{m_t} = \alpha_{m_{t-1}} - \alpha_{m_{t-2}} + \ldots \pm \alpha_{m_1}$.*

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