Tesi di Dottorato

Orbits of real forms in complex flag manifolds

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Contents

Introduction

A complex flag manifold is a compact complex manifold $\mathfrak{M}$ that is homogeneous for the action of a semisimple complex Lie group $\hat{\mathbf{G}}$; equivalently $\mathfrak{M}$ is of the form $\mathfrak{M} = \hat{\mathbf{G}}/\mathbf{Q}$, with $\mathbf{Q}$ a complex parabolic subgroup of $\hat{\mathbf{G}}$. The orbits M in $\mathfrak{M}$ of a real form $\mathbf{G}$ of $\hat{\mathbf{G}}$ inherit from the complex structure of $\mathfrak{M}$ a $\mathbf{G}$-homogeneous CR structure. In this way we obtain a large class of CR manifolds, that we call *parabolic CR manifolds.* They are homogeneous for the CR action of a real semisimple Lie group. Special examples are the compact standard homogeneous CR manifolds, corresponding to Levi-Tanaka algebras (see e.g. [MN97], [MN98], [MN01]), and the symmetric CR manifolds in [LN05].

The orbits of $\mathbf{G}$ in $\mathfrak{M}$ were already considered in [Wol69]. Here it is proved that there is a unique compact $\mathbf{G}$-orbit in $\mathfrak{M}$, that we call *compact parabolic CR manifold.*

Among the several recent contributions to the study of this subject, we cite [Kas93] in the context of infinite dimensional representation theory, [GM03], [HW03] and [KZ03] for applications to the geometry of symmetric spaces.

In this work we stress the point of view of CR geometry. The main tool we use are *parabolic CR algebras*, that is CR algebras of the form $(\mathfrak{g}, \mathfrak{q})$, where $\mathfrak{g}$ is a real semisimple Lie algebra and $\mathfrak{q}$ is a parabolic subalgebra of the complexification $\hat{\mathfrak{g}} = \mathbb{C} \otimes \mathfrak{g}$ of $\mathfrak{g}$. These algebras, first introduced in [MN05], provide an algebraic description of the local CR structure of homogeneous CR manifolds.

It is possible to find Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{q}$. Several CR and topological invariants of M can thus be described in terms of carefully chosen bases of the root system $\mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}} = \mathbb{C} \otimes \mathfrak{h}$.

The open orbits are complex manifolds and have been extensively studied (see e.g. [FHW06]). In particular, they are all simply connected (see [Wol69]). Also the topology of the real flag manifolds has been thoroughly investigated (see e.g. [CS99], [DKV83], [Wig98]). In this work we show that every parabolic CR manifold M is the total space of a canonical fibration over a real flag manifold. The fiber may be disconnected and each connected component is a simply connected complex manifold, which can be retracted onto an open orbit. This essentially reduces the computation of the fundamental group of M to counting the connected components of the fiber.

The thesis is organized as follows.

The first part deals with general parabolic CR manifolds and comprises Chapters 1–4.

In Chapter 1 we review the notions of CR algebras and homogeneous CR manifolds from [MN05], that was also recently utilized in [Fel06] and [FK06]. We collect here the main results and fix the notation that will be employed in the following chapters.

In Chapter 2 we quickly rehearse parabolic complex Lie subalgebras and complex flag manifolds and begin the study of the CR algebras that are associated to the real orbits M in the complex flag manifolds $\mathfrak{M}$, also investigating the canonical $\mathbf{G}$-equivariant maps of [MN05] in this special situation.

Chapter 3 is the core of our investigation of the CR properties of M. Through the introduction of adapted Cartan subalgebras and S- and V-fit Weyl chambers, we associate to M special systems of simple roots. Weak (i.e. holomorphic according to [BER99]) nondegeneracy and fundamentality (i.e. finite type according to [BG77]) are proved to be equivalent to properties of these systems of simple roots. These can be checked from the pattern of some *cross marked diagrams* associated to M, that generalize those of [MN98], [LN05].

In Chapter 4 we turn to the construction of homogeneous CR manifolds that fiber over our orbit M and that are useful both for finding the S- and V-fit Weyl chambers and for investigating the topological properties of M in the following sections. In particular, we construct the *weakest CR model* of M, that is a step to build a chain of fibrations, with simply connected fibers, that in some instances coincides with, and in general can be considered as a substitute of, the *holomorphic arc components* of [Wol69].

In the second part, that comprises Chapters 5–7, the special case of compact parabolic CR manifolds is studied in detail.

In Chapter 5 we characterize those parabolic CR algebras that correspond to compact CR manifolds and associate to them a special subclass of the diagrams introduced in Chapter 3. Then we study $\mathbf{G}$-equivariant fibrations of compact parabolic CR manifolds and classify totally real and totally complex ones.

In Chapter 6 we investigate several nondegeneracy conditions for compact parabolic CR manifolds, sharpening the results of Chapter 3.

In Chapter 7 we recall the definition of essential pseudoconcavity, a notion that generalizes that of pseudoconcavity, and characterize compact parabolic CR manifolds that are essentially pseudoconcave.

The third part of the thesis, that includes Chapters 8–10, presents some applications of the theory developed in the previous chapters.

In Chapter 8 we investigate the connectivity of the isotropy subgroup of M. This is needed to study the connectivity of the fibers of a fibration of M over a real flag manifold M', that we utilize to compute the fundamental group of M. This is a somehow delicate point: the simply connected fibers of our construction may be not connected. We use classical results from [BT72], [BT65] to characterize Cartan subgroups and isotropy subgroups of connected semisimple real linear groups in terms of characters. Then we discuss the fundamental group of M.

In Chapter 9 we provide several examples which show how effective our results are for the study of CR and topological properties of the orbits.

Finally, in Chapter 10, we describe the space of global CR functions on parabolic CR manifolds

All the results contained in this thesis were first presented in [AM06], [AMN06a], [AMN06b], [Alt07].

Part 1

Orbits in flag manifolds

CHAPTER 1

Homogeneous CR manifolds and CR algebras

In this chapter we review some aspects of the theory of homogeneous CR manifolds. First we recall the basic definitions and results about CR manifolds and CR maps (a general reference for this topic, covering much more than we need here, is [BER99]). Then we introduce homogeneous CR manifolds and review the relation between (germs of) homogeneous CR manifolds and CR algebras, along the lines of [MN05].

1.1 CR manifolds

A CR manifold of type (n, k) is a triple (M, HM, J), consisting of:

(1) a smooth paracompact manifold M of real dimension $(2n + k)$,

(2) a smooth real vector subbundle HM of rank $2n$ of its real tangent bundle TM,

(3) a smooth complex structure $J\colon HM \to HM$ on the fibers of HM.

The integers n and k are the CR *dimension* and CR *codimension* of M. It is also required that J satisfies the *formal integrability conditions*:

$$\big[\mathcal{C}^\infty(M, T^{0,1}M), \mathcal{C}^\infty(M, T^{0,1}M)\big] \subset \mathcal{C}^\infty(M, T^{0,1}M). \tag{1.1}$$

where $T^{0,1}M = \{X + iJX \mid X \in HM\}$ is the complex subbundle of the complexification $\mathbb{C}HM$ of HM corresponding to the eigenvalue $-i$ of J; with $T^{1,0}M = \overline{T^{0,1}M}$ we have $T^{1,0}M \cap T^{0,1}M = 0$ and $T^{1,0}M \oplus T^{0,1}M = \mathbb{C}HM$. Any real smooth manifold is a CR manifold, with $n = 0$. When $k = 0$ instead, we recover the abstract definition of a complex manifold, via the Newlander-Nirenberg theorem.

If (M, HM, J) and (M', HM', J') are CR manifolds, a smooth $f\colon M \to M'$ is a CR *map* if:

(1) $df(HM) \subset HM'$,

(2) $df \circ J = J' \circ df$ on HM.

Assume that (M', HM', J') is a CR manifold and $f\colon M \to M'$ a smooth immersion. For $x \in M$ we define H_xM and J_xv for $v \in H_xM$ by setting:

$$\begin{cases} H_xM = [df(x)]^{-1}\big([df(T_xM) \cap H_{f(x)}M'] \cap [J'(df(T_xM) \cap H_{f(x)}M')]\big) \\ J_x(v) = [df(x)]^{-1}\left(J'([df(x)](v))\right). \end{cases} \tag{1.2}$$

If the dimension H_xM is a constant integer, independent of $x \in M$, then the disjoint union HM of the H_xM's, and the map $J : HM \to HM$, equal to J_x on the fiber H_xM, define a CR manifold (M, HM, J). This is the CR structure on M with the maximal CR dimension among those for which f is a CR map. In this case the map $f : M \to M'$ is called a CR *immersion.* If (M', HM', J') is of type (n', k') and (M, HM, J) of type (n, k), we always have $n + k \leq n' + k'$. The immersion is *generic* when the equality $n+k=n'+k'$ holds. When $f:M\longrightarrow M'$ is also an embedding, we say that f is a *CR embedding* or a *generic CR embedding*, respectively.

A CR map $f\colon(M,HM,J)\longrightarrow(M',HM',J')$ is a *CR submersion* if $f:M\longrightarrow M'$ is a smooth submersion and moreover $df(x)(H_xM)=H_{f(x)}M'$ for all $x\in M$. If (M,HM,J) is of type (n,k) and (M',HM',J') of type (n',k'), the existence of a CR submersion implies that $n\geq n'$ and $k\geq k'$.

When $f\colon M\longrightarrow M'$ is a CR submersion and a smooth fiber bundle, we say that $f\colon(M,HM,J)\longrightarrow(M',HM',J')$ is a *CR fibration*. The fibers are embedded CR submanifolds of M' of type $(n-n',k-k')$.

A *CR diffeomorphism* of (M,HM,J) onto (M',HM',J') is a diffeomorphism $f\colon M\longrightarrow M'$ such that both f and f^{-1} are smooth CR maps. The set of all CR diffeomorphisms of (M,HM,J) onto itself (*CR automorphisms*) is a group with the composition operation.

We say that (M,HM,J) is a *homogeneous CR manifold* if there is a Lie group of CR automorphisms that acts transitively on M.

Let (M,HM,J) be a CR manifold. A vector field $X\in\mathcal{C}^\infty(U,TM)$, defined on an open subset U of M, is an *infinitesimal CR automorphism* if the maps $\varphi_X(t)$ of the local 1-parameter group of local transformations generated by X are CR. This is equivalent to the fact that $[X,\mathcal{C}^\infty(U,T^{0,1}M)]\subset\mathcal{C}^\infty(U,T^{0,1}M)$. We say that $(M,HM,J;\mathbf{o})$ is a *locally homogeneous CR manifolds* at a point $\mathbf{o}\in M$ if, for each $v\in T_{\mathbf{o}}M$, there is an infinitesimal CR automorphism X, defined in an open neighborhood U of $\mathbf{o}$ in M, with $v=X(\mathbf{o})$.

Homogeneous CR manifolds are locally CR homogeneous: a homogeneous CR manifold (M,HM,J) has a real analytic CR structure and therefore (see e.g. [AF79]) admits a generic embedding $M\hookrightarrow\hat{M}$ into a complex manifold $\hat{M}$. Then the Lie algebra $\mathfrak{g}$ of the Lie group $\mathbf{G}$ that acts transitively by CR automorphisms on M can be identified with a Lie algebra of infinitesimal analytic CR automorphisms defined on $U=M$. Each $X^*\in\mathcal{C}^\infty(M,TM)$, corresponding to an $X\in\mathfrak{g}$, is the restriction of the real part of a holomorphic vector field Z^*, defined on an open complex neighborhood $\hat{U}$ of M in $\hat{M}$ (i.e. $X^*=[\operatorname{Re}Z^*]\,|_M$; see e.g. [BER99, §12.4]).

The germs of infinitesimal CR automorphisms of (M,HM,J) at a point $\mathbf{o}\in M$, with the Lie bracket, form a real Lie algebra $\mathfrak{G}=\mathfrak{G}(M,HM,J;\mathbf{o})$. We consider its complexification $\hat{\mathfrak{G}}=\mathbb{C}\otimes\mathfrak{G}$ and denote by $\mathfrak{Q}=\mathfrak{Q}(M,HM,J;\mathbf{o})$ the complex Lie subalgebra of $\hat{\mathfrak{G}}$ consisting of all $Z\in\hat{\mathfrak{G}}$ with $Z(\mathbf{o})\in T^{0,1}_{\mathbf{o}}M$. The fact that $\mathfrak{Q}$ is actually a complex Lie subalgebra of $\hat{\mathfrak{G}}$ is a consequence of the formal integrability of the partial complex structure J.

When (M,HM,J) is locally CR homogeneous at $\mathbf{o}\in M$, the pair $(\mathfrak{G},\mathfrak{Q})=(\mathfrak{G}(M,HM,J;\mathbf{o}),\mathfrak{Q}(M,HM,J;\mathbf{o}))$ completely determines the germ of the CR manifold (M,HM,J) at $\mathbf{o}$. Vice versa, if $\mathfrak{g}$ is a finite dimensional real Lie algebra and $\mathfrak{q}$ a complex Lie subalgebra of its complexification $\hat{\mathfrak{g}}$, the general construction[1] of a germ $(M,\mathbf{o})$ of homogeneous manifold associated to the Lie algebra $\mathfrak{g}$ and to its

[1] If $\mathfrak{g}$ is a finite dimensional real Lie algebra and $\mathfrak{g}_+$ a real Lie subalgebra of $\mathfrak{g}$, we can find a germ $(M,\mathbf{o})$ of analytic real manifold, unique modulo germs of analytic diffeomorphisms, for which there is a real Lie algebras homomorphism $\imath:\mathfrak{g}\longrightarrow\mathcal{C}^\infty_{(\mathbf{o})}(M,TM)$ with $\{\imath(X)(\mathbf{o})\mid X\in\mathfrak{g}\}=T_{\mathbf{o}}M$ and $\mathfrak{g}_+=\{X\in\mathfrak{g}\mid\imath(X)(\mathbf{o})=0\}$.

Lie subalgebra $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ (cf. e.g. [Pos86, Ch.VII]) yields a unique, modulo local CR diffeomorphisms, germ of locally homogeneous CR manifold $(M, HM, J; \mathbf{o})$ at $\mathbf{o} \in M$, such that the complexification $\hat{\imath}$ of the correspondence $\imath : \mathfrak{g} \to \mathcal{C}^\infty_{(\mathbf{o})}(M, TM)$ yields a homomorphism of complex Lie algebras: $\hat{\imath} : \hat{\mathfrak{g}} \to \hat{\mathfrak{G}}(M, HM, J; \mathbf{o})$ with

$$\imath(\mathfrak{g}) \subset \mathfrak{G}(M, HM, J; \mathbf{o})\,, \quad \hat{\imath}(\mathfrak{q}) \subset \mathfrak{Q}(M, HM, J; \mathbf{o}) \tag{1.3}$$

and for which the induced map on the quotients

$$\mathfrak{g}/(\mathfrak{g} \cap \mathfrak{q}) \longrightarrow \mathfrak{G}(M, HM, J; \mathbf{o})/\left(\mathfrak{G}(M, HM, J; \mathbf{o}) \cap \mathfrak{Q}(M, HM, J; \mathbf{o})\right)$$

is an isomorphism. In this case we say that the germ of (M, HM, J) at $\mathbf{o}$ is *associated to the pair* $(\mathfrak{g}, \mathfrak{q})$. (We shall consistently use "*hat*" to indicate complexification: e.g. $\hat{\varphi} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}'$ is the complexification of $\varphi : \mathfrak{g} \to \mathfrak{g}'$).

These remarks led to the introduction in [MN05] of the abstract notion of a CR algebra. A *CR algebra* $(\mathfrak{g}, \mathfrak{q})$ is the pair consisting of a real Lie algebra $\mathfrak{g}$ and of a complex Lie subalgebra $\mathfrak{q}$ of its complexification $\hat{\mathfrak{g}} = \mathbb{C} \otimes \mathfrak{g}$, such that the quotient $\mathfrak{g}/(\mathfrak{g} \cap \mathfrak{q})$ is a finite dimensional real linear space. Note that we do not require that $\mathfrak{g}$ is finite dimensional. The intersection $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ is the *isotropy* of $(\mathfrak{g}, \mathfrak{q})$. Let $\mathcal{H}_+ = \{\mathrm{Re}\, Z \mid Z \in \mathfrak{q}\}$ and denote by $\bar{Z}$ the conjugate of $Z \in \hat{\mathfrak{g}}$ with respect to the real form $\mathfrak{g}$.

A CR algebra $(\mathfrak{g}, \mathfrak{q})$ is:

- *totally real* if $\mathcal{H}_+ = \mathfrak{g}_+$,
- *totally complex* if $\mathcal{H}_+ = \mathfrak{g}$,
- *fundamental* if $\mathcal{H}_+$ generates $\mathfrak{g}$ as a real Lie algebra,
- *transitive*, or *effective* if $\mathfrak{g}_+$ does not contain any nonzero ideal of $\mathfrak{g}$,
- *ideal nondegenerate* if all ideals of $\mathfrak{g}$ contained in $\mathcal{H}_+$ are contained in $\mathfrak{g}_+$,
- *weakly nondegenerate* if there is no complex Lie subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with:
$$\mathfrak{q} \subsetneq \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}},$$
- *strictly nondegenerate* if $\mathfrak{g}_+ = \{X \in \mathcal{H}_+ \mid [X, \mathcal{H}_+] \subset \mathcal{H}_+\}$.

Clearly:

strictly nondegenerate $\Longrightarrow$ weakly nondegenerate $\Longrightarrow$ ideal nondegenerate.

Fundamentality of $(\mathfrak{g}, \mathfrak{q})$ is equivalent to the fact that the associated germ of homogeneous CR manifold $(M, HM, J; \mathbf{o})$ is of *finite type in the sense of* [BG77], i.e. that the smallest involutive distribution of tangent vectors containing HM also contains $T_{\mathbf{o}}M$.

Strict and weak nondegeneracy hold, or do not hold, for all CR algebras that are associated to the same germ $(M, HM, J; \mathbf{o})$ of locally homogeneous CR manifold. They correspond indeed to the nondegeneracy of the (vector valued) Levi form and of its higher order analog, respectively (see e.g. [MN05, §13]). In particular, *weak nondegeneracy* at a point $\mathbf{o} \in M$ of a (germ of) CR manifold (M, HM, J) means that, for every $L \in \mathcal{C}^\infty(M, T^{1,0}M)$ with $L(\mathbf{o}) \neq 0$, there exist finitely many vector fields $\bar{Z}_1, \ldots, \bar{Z}_m \in \mathcal{C}^\infty(M, T^{0,1}M)$ such that $[L, \bar{Z}_1, \ldots, \bar{Z}_m](\mathbf{o}) \notin T^{1,0}_{\mathbf{o}}M \oplus T^{0,1}_{\mathbf{o}}M$.

When $(\mathfrak{g}, \mathfrak{q})$ defines at $\mathbf{o}$ a germ of homogeneous CR manifold, the two notions of weak nondegeneracy, the one for CR algebras and the one above for CR manifolds, coincide and also coincide with the holomorphic nondegeneracy of [BER99] and the finite nondegeneracy of [Fel06].

We have the following:

PROPOSITION 1.1. *Let (M, HM, J) and (M', HM', J') be CR manifolds. Assume that M' is locally embeddable and that there exists a CR fibration $\pi : (M, HM, J) \longrightarrow (M', HM', J')$, with totally complex fibers of positive dimension. Then M is weakly degenerate.*

PROOF. Let f be any smooth CR function defined on a neighborhood U' of $p' \in M'$. Then $\pi^* f$ is a CR function in $U = \pi^{-1}(U')$, that is constant along the fibers of π. Hence, if $L \in \mathcal{C}^\infty(M, T^{1,0}M)$ is tangent to the fibers of π in U, we obtain that $[\bar{Z}_1, \ldots, \bar{Z}_m, L](\pi^* f) = 0$ for every choice of $\bar{Z}_1, \ldots, \bar{Z}_m \in \mathcal{C}^\infty(M, T^{0,1}M)$. Assume by contradiction that M is weakly nondegenerate at some p with $\pi(p) = p'$. Then for some choice of $\bar{Z}_1, \ldots, \bar{Z}_m \in \mathcal{C}^\infty(M, T^{0,1}M)$ we would have $v_p = [\bar{Z}_1, \ldots, \bar{Z}_m, L] \notin T^{1,0}_p M \oplus T^{0,1}_p M$. Since the fibers of π are totally complex, $\pi_*(v_p) \neq 0$. By the assumption that M' is locally embeddable at p, the real parts of the (locally defined) CR functions give local coordinates in M' and therefore there is a CR function f defined on a neighborhood U' of p' with $v_p(\pi^* f) = \pi_*(v_p)(f) \neq 0$. This gives a contradiction, proving our statement. $\square$

Differently, both ideal degenerate and ideal nondegenerate CR algebras may correspond to the same (weakly degenerate) germ of locally homogeneous CR manifold.

From [MN05, Theorem 9.1] we know that if $(\mathfrak{g}, \mathfrak{q})$ is fundamental, effective, and ideal nondegenerate, then $\mathfrak{g}$ is finite dimensional.

From this result we deduce the following:

THEOREM 1.2. *Let $(\mathfrak{g}, \mathfrak{q})$ be a fundamental effective CR algebra. Then there exist a germ of homogeneous complex manifold $(\hat{M}, \mathbf{o})$ at a point $\mathbf{o}$, and a germ of homogeneous generic CR submanifold $(M, HM, J; \mathbf{o})$ of $(\hat{M}, \mathbf{o})$ at $\mathbf{o}$, with associated CR algebra $(\mathfrak{g}, \mathfrak{q})$.*

PROOF. First we note that the statement holds true when $\mathfrak{g}$ is finite dimensional: by [Pos86, Ch.VII] there is a germ of homogeneous complex manifold $(\hat{M}, \mathbf{o})$ at $\mathbf{o}$ corresponding to the complex Lie algebra $\hat{\mathfrak{g}}$ and to its complex Lie subalgebra $\mathfrak{q}$; the inclusion $\mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}$ yields the embedding into $(\hat{M}, \mathbf{o})$ of a germ of homogeneous CR manifold $(M, HM, J; \mathbf{o})$ at $\mathbf{o}$, corresponding to the pair $(\mathfrak{g}, \mathfrak{q})$.

Consider now the general case. We keep the notation introduced above. By [MN05, Lemma 7.2] there is a largest ideal $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathcal{H}_+$. By [MN05, Theorem 9.1], $\mathfrak{g}/\mathfrak{a}$ is finite dimensional and by the first part of the proof there is a germ of complex homogeneous manifold $(\hat{N}, \mathbf{o})$ at $\mathbf{o}$, and a germ of generic CR submanifold $(N, HN, J_N; \mathbf{o})$ of $\hat{N}$ at $\mathbf{o}$, associated to the pair $(\mathfrak{g}/\mathfrak{a}, \mathfrak{q}/(\mathfrak{q} \cap \hat{\mathfrak{a}}))$ (here $\hat{\mathfrak{a}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}$ is the complexification of $\mathfrak{a}$ in $\hat{\mathfrak{g}}$). If $2d$ is the real dimension of $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{g}_+)$, we can take $\hat{M} = \hat{N} \times \mathbb{C}^d$ and, likewise, $M = N \times \mathbb{C}^d$, with $HM = HN \times T(\mathbb{C}^d)$ and $J = J_N \times J_{\mathbb{C}^d}$. Then $(M, HM, J; (\mathbf{o}, 0))$ is associated to $(\mathfrak{g}, \mathfrak{q})$. $\square$

Note that the ideal nondegeneracy of $(\mathfrak{g}, \mathfrak{q})$ implies that all ideals $\mathfrak{r}$ of the complex Lie algebra $\hat{\mathfrak{g}}$ that are contained in $\mathfrak{q}$ are contained in $\mathfrak{q} \cap \bar{\mathfrak{q}}$. Indeed, if $\mathfrak{r}$ is a (complex) ideal of $\hat{\mathfrak{g}}$ contained in $\mathfrak{q}$, then $\mathfrak{a} = (\mathfrak{r} + \bar{\mathfrak{r}}) \cap \mathfrak{g}$ is an ideal of $\mathfrak{g}$ contained in $\mathcal{H}_+ = (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}$, and $\mathfrak{a} \subset \mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ implies that $\mathfrak{r} + \bar{\mathfrak{r}} \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$.

Let $(\mathfrak{g}, \mathfrak{q})$ be a CR algebra with a finite dimensional $\mathfrak{g}$. We denote by $\hat{\mathbf{G}}$ the connected and simply connected complex Lie group with Lie algebra $\hat{\mathfrak{g}}$ and by $\mathbf{Q}$ its analytic Lie subgroup, generated by $\mathfrak{q}$. Let $\mathbf{G}$ be the analytic subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{g}$ and set $\mathbf{G}_+ = \mathbf{G}_+(\mathfrak{g},\mathfrak{q}) := \mathbf{Q}\cap\mathbf{G}$. This is a Lie subgroup of $\mathbf{G}$ with Lie subalgebra $\mathfrak{g}_+$. Denote by $\tilde{\mathbf{G}}$ a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and by $\tilde{\mathbf{G}}_+ = \tilde{\mathbf{G}}_+(\mathfrak{g},\mathfrak{q})$ its analytic Lie subgroup with Lie subalgebra $\mathfrak{g}_+$. Since $\hat{\mathbf{G}}$ is simply connected, the conjugation σ of $\hat{\mathfrak{g}}$ with respect its real form $\mathfrak{g}$ defines an antiholomorphic involution, still denoted by σ, in $\hat{\mathbf{G}}$. Thus $\mathbf{G}$, being the connected component of the identity in the set of fixed points of σ, is closed in $\hat{\mathbf{G}}$. We have the implications:

$$\mathbf{Q} \text{ closed in } \hat{\mathbf{G}} \implies \mathbf{G}_+ \text{ closed in } \mathbf{G}\,, \quad \mathbf{G}_+ \text{ closed in } \mathbf{G} \implies \tilde{\mathbf{G}}_+ \text{ closed in } \tilde{\mathbf{G}}\,.$$

When $\tilde{\mathbf{G}}_+$ is closed in $\tilde{\mathbf{G}}$, we can uniquely define a $\tilde{\mathbf{G}}$-homogeneous CR manifold $\tilde{M}(\mathfrak{g},\mathfrak{q}) = (\tilde{M}, H\tilde{M}, \tilde{J})$, where the underlying smooth manifold $\tilde{M}$ is the $\tilde{\mathbf{G}}$-homogeneous space $\tilde{\mathbf{G}}/\tilde{\mathbf{G}}_+$, and $(\mathfrak{g},\mathfrak{q})$ is associated to the germ $(\tilde{M}, H\tilde{M}, \tilde{J}; \mathbf{o})$ at the base point $\mathbf{o} = e\,\tilde{\mathbf{G}}_+$.

Likewise, for a closed $\mathbf{G}_+ \subset \mathbf{G}$, we define the $\mathbf{G}$-homogeneous CR manifold $M(\mathfrak{g},\mathfrak{q}) = (M, HM, J)$ with $M = \mathbf{G}/\mathbf{G}_+$ and $(\mathfrak{g},\mathfrak{q})$ associated to $(M, HM, J; \mathbf{o})$ for $\mathbf{o} = e\,\mathbf{G}_+$.

If $\mathbf{Q}$ is closed, $\hat{M} = \hat{M}(\hat{\mathfrak{g}},\mathfrak{q}) := \hat{\mathbf{G}}/\mathbf{Q}$ is a $\hat{\mathbf{G}}$-homogeneous complex manifold and $M(\mathfrak{g},\mathfrak{q})$ can be identified, its partial complex structure being that of a generic CR submanifold of $\hat{M}$, to the orbit of $\mathbf{G}$ through the base point $\mathbf{o} = e\,\mathbf{Q}$ of $\hat{M}$.

Our canonical choice of $M(\mathfrak{g},\mathfrak{q})$ aims to obtain a homogeneous CR manifold with a generic CR embedding into a homogeneous complex manifold $\hat{M} = \hat{M}(\hat{\mathfrak{g}},\mathfrak{q})$, that is "good" in some suitable sense.

A morphism of CR algebras $(\mathfrak{g},\mathfrak{q}) \xrightarrow{\varphi} (\mathfrak{g}',\mathfrak{q}')$ is a homomorphism of real Lie algebras $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}'$, with $\hat{\varphi}(\mathfrak{q}) \subset \mathfrak{q}'$.

It is called:

- *a CR immersion* if the quotient map $[\varphi] : \mathfrak{g}/\mathfrak{g}_+ \longrightarrow \mathfrak{g}'/\mathfrak{g}'_+$ is injective and $\hat{\varphi}^{-1}(\mathfrak{q}') = \mathfrak{q}$;
- *a CR submersion* if both $[\varphi] : \mathfrak{g}/\mathfrak{g}_+ \longrightarrow \mathfrak{g}'/\mathfrak{g}'_+$ and $[\hat{\varphi}] : \mathfrak{q}/\hat{\mathfrak{g}}_+ \longrightarrow \mathfrak{q}'/\hat{\mathfrak{g}}'_+$ are onto;
- *a local CR isomorphism* if both $[\varphi] : \mathfrak{g}/\mathfrak{g}_+ \longrightarrow \mathfrak{g}'/\mathfrak{g}'_+$ and $[\hat{\varphi}] : \mathfrak{q}/\hat{\mathfrak{g}}_+ \longrightarrow \mathfrak{q}'/\hat{\mathfrak{g}}'_+$ are isomorphisms;
- *a CR isomorphism* if φ is an isomorphism of real Lie algebras with $\hat{\varphi}(\mathfrak{q}) = \mathfrak{q}'$.

We quote from [MN05]:

PROPOSITION 1.3. *Let $(\mathfrak{g},\mathfrak{q}) \xrightarrow{\varphi} (\mathfrak{g}',\mathfrak{q}')$ be a morphism of CR algebras, with $\mathfrak{g}$ and $\mathfrak{g}'$ finite dimensional. Let $(M, HM, J; \mathbf{o})$ and $(M', HM', J'; \mathbf{o}')$ be the germs of homogeneous CR manifolds at $\mathbf{o} \in M$, $\mathbf{o}' \in M'$, associated to $(\mathfrak{g},\mathfrak{q})$, $(\mathfrak{g}',\mathfrak{q}')$, respectively. Then there is a unique germ of smooth CR map $\Phi : (M, HM, J; \mathbf{o}) \longrightarrow (M', HM', J'; \mathbf{o}')$ with $\Phi(\mathbf{o}) = \mathbf{o}'$ such that $d\Phi_{\mathbf{o}}(\imath(X)) = \imath'(\varphi(X))$. Here $\imath$, $\imath'$ are the homomorphisms of Lie algebras $\imath : \mathfrak{g} \longrightarrow \mathfrak{G}(M, HM, J; \mathbf{o})$ and $\imath' : \mathfrak{g} :\longrightarrow \mathfrak{G}(M', HM', J'; \mathbf{o})$ of* (1.3).

The germ Φ of smooth CR map is a CR immersion, submersion, diffeomorphism if and only if the corresponding morphism φ of CR algebras is a CR immersion, a CR submersion, a local CR isomorphism, respectively.

Let $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{G}}'$ be the connected and simply connected real Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. If the analytic subgroup $\tilde{\mathbf{G}}_+$ of $\tilde{\mathbf{G}}$ with Lie algebra

$\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ *and the analytic subgroup* $\tilde{\mathbf{G}}'_+$ *of* $\tilde{\mathbf{G}}'$ *with Lie algebra* $\mathfrak{g}'_+ = \mathfrak{q}' \cap \mathfrak{g}'$ *are both closed, then there is a unique smooth* CR *map* $\tilde{\Phi} : \tilde{M}(\mathfrak{g}, \mathfrak{q}) \longrightarrow \tilde{M}(\mathfrak{g}', \mathfrak{q}')$ *that makes the following diagram commute:*

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\exp} & \tilde{\mathbf{G}} & \longrightarrow & \tilde{M}(\mathfrak{g}, \mathfrak{q}) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\tilde{\Phi}} \\ \mathfrak{g}' & \xrightarrow{\exp} & \tilde{\mathbf{G}}' & \longrightarrow & \tilde{M}(\mathfrak{g}', \mathfrak{q}') \end{array}$$

The map $\tilde{\Phi}$ *is a* CR *immersion, a* CR *submersion or a local* CR *diffeomorphism if and only if the corresponding* CR *morphism of* CR *algebras* φ *is a* CR *immersion, a* CR *submersion or a local* CR *isomorphism, respectively.*

Let $\hat{\mathbf{G}}$ *and* $\hat{\mathbf{G}}'$ *be the connected and simply connected complex Lie groups with Lie algebras* $\hat{\mathfrak{g}}$ *and* $\hat{\mathfrak{g}}'$*, respectively. Let* $\mathbf{G}, \mathbf{Q} \subset \hat{\mathbf{G}}$ *and* $\mathbf{G}', \mathbf{Q}' \subset \hat{\mathbf{G}}'$ *be the analytic subgroups with Lie algebras* $\mathfrak{g}, \mathfrak{q}$ *and* $\mathfrak{g}', \mathfrak{q}'$*, respectively. If* $\mathbf{G}_+ = \mathbf{Q} \cap \mathbf{G}$ *and* $\mathbf{G}' = \mathbf{Q}' \cap \mathbf{G}'$ *are closed, then there is a unique smooth* CR *map* $\Phi : M(\mathfrak{g}, \mathfrak{q}) \longrightarrow M(\mathfrak{g}', \mathfrak{q}')$ *such that the diagram:*

$$\begin{array}{ccc} \tilde{M}(\mathfrak{g}, \mathfrak{q}) & \xrightarrow{\tilde{\Phi}} & \tilde{M}(\mathfrak{g}', \mathfrak{q}') \\ \big\downarrow & & \big\downarrow \\ M(\mathfrak{g}, \mathfrak{q}) & \xrightarrow{\Phi} & M(\mathfrak{g}', \mathfrak{q}') \end{array}$$

where the vertical arrows are the natural projections from the universal coverings, commutes.

The map Φ *is a* CR *immersion, a* CR *submersion or a local* CR *diffeomorphism if and only if the corresponding morphism of* CR *algebras* φ *is a* CR *immersion, a* CR *submersion or a local* CR *isomorphism, respectively.*

If $\mathbf{Q} \subset \hat{\mathbf{G}}$ *and* $\mathbf{Q}' \subset \hat{\mathbf{G}}'$ *are closed, the map* $M(\mathfrak{g}, \mathfrak{q}) \xrightarrow{\Phi} M(\mathfrak{g}', \mathfrak{q}')$ *is the restriction of the holomorphic map* $\hat{\Phi} : \hat{M} = \hat{\mathbf{G}}/\mathbf{Q} \to \hat{M}' = \hat{\mathbf{G}}'/\mathbf{Q}'$ *defined by the commutative diagram:*

$$\begin{array}{ccccc} \hat{\mathfrak{g}} & \xrightarrow{\exp} & \hat{\mathbf{G}} & \longrightarrow & \hat{M} \\ {\scriptstyle\hat{\varphi}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\hat{\Phi}} \\ \hat{\mathfrak{g}}' & \xrightarrow{\exp} & \hat{\mathbf{G}}' & \longrightarrow & \hat{M}' \end{array}$$

where the central vertical arrow is the homomorphism of complex connected simply connected Lie algebras defined by the homomorphism $\hat{\varphi} : \hat{\mathfrak{g}} \longrightarrow \hat{\mathfrak{g}}'$ *of their Lie algebras.* □

To discuss, later on, the structure of the fibers of some CR fibrations, we need to introduce the notion of *semidirect sum* of CR algebras.

Let $(\mathfrak{g}_1, \mathfrak{q}_1)$, $(\mathfrak{g}_2, \mathfrak{q}_2)$ be CR algebras, and assume that $\mathfrak{g}_2$ has a $\mathfrak{g}_1$-module structure and that $\mathfrak{q}_2$ is a $\mathfrak{q}_1$-module for the restriction of the complexification of the action of $\mathfrak{g}_1$ on $\mathfrak{g}_2$. Then $\mathfrak{q} = \mathfrak{q}_1 \rtimes \mathfrak{q}_2$ (semidirect sum) is a complex Lie subalgebra of the complexification of the semidirect sum $\mathfrak{g} = \mathfrak{g}_1 \rtimes \mathfrak{g}_2$, and the CR algebra $(\mathfrak{g}, \mathfrak{q}) = (\mathfrak{g}_1 \rtimes \mathfrak{g}_2, \mathfrak{q}_1 \rtimes \mathfrak{q}_2)$ is called the *semidirect sum* of the CR algebras $(\mathfrak{g}_1, \mathfrak{q}_1)$ and $(\mathfrak{g}_2, \mathfrak{q}_2)$:

$$(\mathfrak{g}, \mathfrak{q}) = (\mathfrak{g}_1 \rtimes \mathfrak{g}_2, \mathfrak{q}_1 \rtimes \mathfrak{q}_2) = (\mathfrak{g}_1, \mathfrak{q}_1) \rtimes (\mathfrak{g}_2, \mathfrak{q}_2) \quad \text{(semidirect sum)}\,. \tag{1.4}$$

We shall assume that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are finite dimensional. Denote by:

$\hat{\mathbf{G}}$, $\hat{\mathbf{G}}_1$, $\hat{\mathbf{G}}_2$ the connected and simply connected complex Lie groups with Lie algebras $\hat{\mathfrak{g}}$, $\hat{\mathfrak{g}}_1$, $\hat{\mathfrak{g}}_2$, respectively;

$\mathbf{G}$, $\mathbf{G}_1$, $\mathbf{G}_2$ the analytic real subgroups of the corresponding complex connected Lie groups $\hat{\mathbf{G}}$, $\hat{\mathbf{G}}_1$, $\hat{\mathbf{G}}_2$, with Lie algebras $\mathfrak{g}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$, respectively;

$\mathbf{Q} \subset \hat{\mathbf{G}}$, $\mathbf{Q}_1 \subset \hat{\mathbf{G}}_1$, $\mathbf{Q}_2 \subset \hat{\mathbf{G}}_2$ the Lie subgroups corresponding to the Lie subalgebras $\mathfrak{q} \subset \hat{\mathfrak{g}}$, $\mathfrak{q}_1 \subset \mathfrak{g}_1$, $\mathfrak{q}_2 \subset \mathfrak{g}_2$, respectively;

$\tilde{\mathbf{G}}$, $\tilde{\mathbf{G}}_1$, $\tilde{\mathbf{G}}_2$ connected and simply connected real Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$, respectively;

$\mathbf{G}_+ = \mathbf{Q} \cap \mathbf{G}$, $\mathbf{G}_{1\,+} = \mathbf{Q} \cap \mathbf{G}_1$, $\mathbf{G}_{2\,+} = \mathbf{Q} \cap \mathbf{G}_2$;

$\tilde{\mathbf{G}}_+ \subset \tilde{\mathbf{G}}$, $\tilde{\mathbf{G}_1}_+ \subset \tilde{\mathbf{G}}_1$, $\tilde{\mathbf{G}}_{2\,+} \subset \tilde{\mathbf{G}}_2$ the analytic subgroups corresponding to the Lie subalgebras $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$, $\mathfrak{g}_{1+} = \mathfrak{q}_1 \cap \mathfrak{g}_1$, $\mathfrak{g}_{2+} = \mathfrak{q}_2 \cap \mathfrak{g}_2$, respectively.

Let $(M, HM, J; \mathbf{o})$, $(M_1, HM_1, J_1; \mathbf{o}_1)$,$(M_2, HM_2, J_2; \mathbf{o}_2)$ be the germs of locally homogeneous CR manifolds associated to the CR algebras $(\mathfrak{g}, \mathfrak{q})$, $(\mathfrak{g}_1, \mathfrak{q}_1)$, $(\mathfrak{g}_2, \mathfrak{q}_2)$, respectively.

We obtain:

Theorem 1.4. *The diffeomorphism* $\mathbf{G}_1 \times \mathbf{G}_2 \ni (g_1, g_2) \longrightarrow g_1 g_2 \in \mathbf{G}_1 \rtimes \mathbf{G}_2$ *defines a germ of CR diffeomorphism:*

$$(M_1, HM_1, J_1; \mathbf{o}_1) \times (M_2, HM_2, J_2; \mathbf{o}_2) \longrightarrow (M, HM, J; \mathbf{o})\,.$$

If $\tilde{\mathbf{G}}_{1\,+}$ and $\tilde{\mathbf{G}}_{2\,+}$ are closed, then $\tilde{\mathbf{G}}_+$ is closed and we obtain a global CR diffeomorphism:

$$\tilde{M}(\mathfrak{g}_1, \mathfrak{q}_1) \times \tilde{M}(\mathfrak{g}_2, \mathfrak{q}_2) \longrightarrow \tilde{M}(\mathfrak{g}, \mathfrak{q})\,.$$

If $\mathbf{G}_{1\,+}$ and $\mathbf{G}_{2\,+}$ are closed, then $\mathbf{G}_+$ is closed and we obtain a global CR diffeomorphism:

$$M(\mathfrak{g}_1, \mathfrak{q}_1) \times M(\mathfrak{g}_2, \mathfrak{q}_2) \longrightarrow M(\mathfrak{g}, \mathfrak{q})\,. \tag{1.5}$$

When $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are closed, also $\mathbf{Q}$ is closed and the map (1.5) *is the restriction of a biholomorphic map*

$$\left(\hat{\mathbf{G}}_1/\mathbf{Q}_1\right) \times \left(\hat{\mathbf{G}}_2/\mathbf{Q}_2\right) \longrightarrow \hat{\mathbf{G}}/\mathbf{Q}\,. \qquad \square$$

1.2 $\mathfrak{g}$-equivariant fibrations

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{q}, \mathfrak{q}'$ complex subalgebras of its complexification $\hat{\mathfrak{g}}$, with $\mathfrak{q} \subset \mathfrak{q}'$. Then the identity map in $\mathfrak{g}$ and the inclusion $\mathfrak{q} \hookrightarrow \mathfrak{q}'$ define a $\mathfrak{g}$-equivariant morphism of CR algebras

$$(\mathfrak{g}, \mathfrak{q}) \longrightarrow (\mathfrak{g}, \mathfrak{q}')\,. \tag{1.6}$$

If $(M, HM, J; \mathbf{o})$ and $(M', HM', J'; \mathbf{o}')$ are germs of locally homogeneous CR manifolds with associated CR algebras $(\mathfrak{g}, \mathfrak{q})$ and $(\mathfrak{g}, \mathfrak{q}')$, respectively, then the identity map $\mathfrak{g} \longrightarrow \mathfrak{g}$ defines, by passing to the quotients, the differential of a CR map $\pi_{(\mathbf{o})} : (M, HM, J; \mathbf{o}) \longrightarrow (M', HM', J'; \mathbf{o}')$ that is locally $\mathbf{G}$-equivariant for a (connected) real Lie group with Lie algebra $\mathfrak{g}$.

Let $\mathbf{G}$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and assume that there are two $\mathbf{G}$-homogeneous CR manifolds (M, HM, J), (M', HM', J') that are associated to $(\mathfrak{g}, \mathfrak{q})$ at some $\mathbf{o} \in M$ and to $(\mathfrak{g}, \mathfrak{q}')$ at some $\mathbf{o}' \in M'$, respectively. Then there is a unique $\mathbf{G}$-equivariant CR map $\pi : (M, HM, J) \longrightarrow (M', HM', J')$ with $\pi(\mathbf{o}) = \mathbf{o}'$.

In general $\pi_{(\mathbf{o})}$ (and π, when defined) are smooth, but not CR, $\mathbf{G}$-equivariant fibrations: a necessary and sufficient condition for (1.6) to be a $\mathfrak{g}$-equivariant CR fibration, and hence for $\pi_{(\mathbf{o})}$ (and π, when defined) to be $\mathbf{G}$-equivariant local (resp. global) CR fibrations is that (see [MN05, Lemma 5.1])

$$\mathfrak{q}' = \mathfrak{q} + \hat{\mathfrak{g}}'_+ \,. \tag{1.7}$$

We call $(\mathfrak{g}, \mathfrak{q}')$ the *basis* of the fibration (1.6). The *fiber* of (1.6) is the CR algebra $(\mathfrak{g}'_+, \mathfrak{q}'')$ where $\mathfrak{q}'' = \hat{\mathfrak{g}}'_+ \cap \mathfrak{q}$. It is a CR algebra associated to the germ $(F, HF, J\,|_{HF}\,; \mathbf{o})$, where $(F; \mathbf{o}) = \left(\pi^{-1}_{(\mathbf{o})}(\mathbf{o}'); \mathbf{o}\right)$, and the germ of partial complex structure $(HF, J\,|_{HF}\,; \mathbf{o})$ that is characterized by requiring that the smooth embedding $(\pi^{-1}(\mathbf{o}'); \mathbf{o}) \hookrightarrow (M; \mathbf{o})$ is a CR immersion.

We know that (1.6) is always a CR fibration, with a totally complex fiber, when $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \overline{\mathfrak{q}}$: indeed in this case $\mathfrak{q}' = \mathfrak{q} + \hat{\mathfrak{g}}'_+$.

From [MN05, §5] we have:

PROPOSITION 1.5. *Let $(\mathfrak{g}, \mathfrak{q})$ be a CR algebra. Then there exist:*

- *a largest ideal $\mathfrak{i}$ of $\mathfrak{g}$ with $\mathfrak{i} \subset \mathfrak{g}_+$;*
- *a largest ideal $\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathcal{H}_+$;*
- *a largest complex Lie subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$;*
- *a smallest complex Lie subalgebra $\mathfrak{q}''$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q} + \bar{\mathfrak{q}} \subset \mathfrak{q}''$.*

We have $\mathfrak{i} \subset \mathfrak{a} \subset \mathfrak{q}' \subset \mathfrak{q}''$ and $\mathfrak{q}'' = \bar{\mathfrak{q}}\,'' = \hat{\mathfrak{g}}\,''$ for a real Lie subalgebra $\mathfrak{g}''$ of $\mathfrak{g}$.

The identity in $\mathfrak{g}$ defines $\mathfrak{g}$-equivariant CR fibrations $(\mathfrak{g}, \mathfrak{q}) \longrightarrow (\mathfrak{g}, \mathfrak{q}') \longrightarrow (\mathfrak{g}, \mathfrak{q}\,'')$, where $(\mathfrak{g}, \mathfrak{q}')$ is weakly nondegenerate and $(\mathfrak{g}, \mathfrak{q}\,'')$ is totally real. For all complex Lie subalgebras $\mathfrak{f}$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q} \subset \mathfrak{f} \subsetneq \mathfrak{q}'$, the CR algebra $(\mathfrak{g}, \mathfrak{f})$ is weakly degenerate. For all complex Lie subalgebras $\mathfrak{f}$ with $\mathfrak{q} \subset \mathfrak{f} \subset \mathfrak{q}'$, the $\mathfrak{g}$-equivariant map $(\mathfrak{g}, \mathfrak{q}) \longrightarrow (\mathfrak{g}, \mathfrak{f})$ is a CR fibration with a totally complex fiber.

The CR algebra $(\mathfrak{g}'', \mathfrak{q})$ is fundamental and, for all real Lie subalgebras $\mathfrak{l}$ of $\mathfrak{g}$ with $\mathfrak{g}'' \subsetneq \mathfrak{l} \subset \mathfrak{g}$ the CR algebra $(\mathfrak{l}, \mathfrak{q})$ is not fundamental. □

CHAPTER 2

Parabolic CR algebras and parabolic CR manifolds

In the first section of this chapter we collect the notions on complex parabolic subalgebras and fix the notation that will then be utilized throughout this work. This is mostly a review of classical results, for which general references are [Bou02, Ch.IV §2.6, Ch.VI §1], [Bou05, Ch.VIII §3], [Kna02, Ch.VII], [War72, Ch.1], [Wol69].

In the second section we introduce the main object of our study, namely parabolic CR algebras and parabolic CR manifolds, and begin to study some of their properties.

2.1 Parabolic subalgebras and complex flag manifolds

Let $\hat{\mathfrak{g}}$ be a complex Lie algebra. A maximal solvable complex Lie subalgebra $\mathfrak{b}$ of $\hat{\mathfrak{g}}$ is called a *Borel*, or *minimal parabolic* complex Lie subalgebra of $\hat{\mathfrak{g}}$. A complex Lie subalgebra $\mathfrak{q}$ of $\hat{\mathfrak{g}}$ is *parabolic* if it contains a complex Borel subalgebra $\mathfrak{b}$ of $\hat{\mathfrak{g}}$.

For our purposes, it will be sufficient to consider the case of a semisimple $\hat{\mathfrak{g}}$. Thus from now on we shall assume that $\hat{\mathfrak{g}}$ is a semisimple complex Lie algebra.

A parabolic subalgebra $\mathfrak{q}$ of $\hat{\mathfrak{g}}$ contains a complex Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$. Let $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ be the root system of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$. We denote by $\mathfrak{h}_{\mathbb{R}}$ the real form of $\hat{\mathfrak{h}}$ on which all roots are real valued. Thus $\mathcal{R}$ is a subset of the real dual space $\mathfrak{h}_{\mathbb{R}}^*$. The Killing form $\kappa_{\hat{\mathfrak{g}}}$ of $\hat{\mathfrak{g}}$ restricts to a real positive scalar product in $\mathfrak{h}_{\mathbb{R}}$. We shall write $(A|B) = \kappa_{\hat{\mathfrak{g}}}(A, B)$ for $A, B \in \mathfrak{h}_{\mathbb{R}}$. We set also $(\xi|\eta) = (T_\xi|T_\eta)$ for $\xi, \eta \in \mathfrak{h}_{\mathbb{R}}^*$ and $(T_\xi|A) = \xi(A)$, $(T_\eta|A) = \eta(A)$ for all $A \in \mathfrak{h}_{\mathbb{R}}$ (dual scalar product in $\mathfrak{h}_{\mathbb{R}}^*$). Roots $\alpha, \beta \in \mathcal{R}$ for which $\alpha \pm \beta \notin \mathcal{R}$ are called *strongly orthogonal.* Note that strongly orthogonal roots are also orthogonal for the scalar product in $\mathfrak{h}_{\mathbb{R}}^*$.

An element $H \in \hat{\mathfrak{h}}$ is *regular* if $\alpha(H) \neq 0$ for all $\alpha \in \mathcal{R}$. Denote by $\mathfrak{C}(\mathcal{R})$ the set of the Weyl chambers of $\mathcal{R}$. They are the connected components of the set of regular elements of $\mathfrak{h}_{\mathbb{R}}$. For $C \in \mathfrak{C}(\mathcal{R})$, and $H \in C$, the set $\mathcal{R}^+(C) = \{\alpha \in \mathcal{R} \mid \alpha(H) > 0\}$ is independent of the choice of $H \in C$: it is called the set of *positive roots* with respect to C. The set $\mathcal{R}^-(C) = \mathcal{R}^+(C^{\mathrm{opp}})$, for $C^{\mathrm{opp}} = \{-H \mid H \in C\}$, is the complement of $\mathcal{R}^+(C)$ in $\mathcal{R}$ and is called the set of *negative roots* with respect to C. A Weyl chamber C also defines a partial order relation "$\prec_C$" in $\mathfrak{h}_{\mathbb{R}}^*$, by:

$$\eta \prec_C \xi \text{ if } \eta(A) < \xi(A) \text{ for all } A \in C. \tag{2.1}$$

In particular $\mathcal{R}^+(C) = \{\alpha \in \mathcal{R} \mid \alpha \succ_C 0\}$.

With $\hat{\mathfrak{g}}^\alpha = \{X \in \hat{\mathfrak{g}} \mid [H, X] = \alpha(H)\, X \ \forall H \in \hat{\mathfrak{h}}\}$, we set

$$\mathcal{Q} = \{\alpha \in \mathcal{R} \mid \hat{\mathfrak{g}}^\alpha \subset \mathfrak{q}\}\,. \tag{2.2}$$

This is the *parabolic set* associated to $\mathfrak{q}$ and $\hat{\mathfrak{h}}$. Parabolic sets of roots are abstractly defined by the two conditions:

$$\text{(2.4 i)} \qquad \alpha, \beta \in \mathcal{Q}\,,\ \alpha + \beta \in \mathcal{R} \implies \alpha + \beta \in \mathcal{Q} \quad \text{(closedness)}$$

$$\text{(2.4 ii)} \qquad \mathcal{Q} \cup \mathcal{Q}^{\mathrm{opp}} = \mathcal{R} \quad \text{where} \quad \mathcal{Q}^{\mathrm{opp}} = \{-\alpha \mid \alpha \in \mathcal{Q}\}\,.$$

Given (2.4 i), condition (2.4 ii) is equivalent to the fact that $\mathcal{Q} \supset \mathcal{R}^+(C)$ for some Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$. We have

$$\mathfrak{q} = \mathfrak{q}_{\mathcal{Q}} = \hat{\mathfrak{h}} \oplus \sum_{\alpha \in \mathcal{Q}} \hat{\mathfrak{g}}^{\,\alpha} \tag{2.5}$$

and the correspondence $\mathcal{Q} \longleftrightarrow \mathfrak{q}_{\mathcal{Q}}$ is one-to-one between parabolic subsets of $\mathcal{R}$ and parabolic subalgebras of $\hat{\mathfrak{g}}$ containing $\hat{\mathfrak{h}}$.

Given a parabolic set $\mathcal{Q} \subset \mathcal{R}$ we set:

$$\mathcal{Q}^r = \mathcal{Q} \cap \mathcal{Q}^{\mathrm{opp}}\{\alpha \in \mathcal{Q} \mid -\alpha \in \mathcal{Q}\} \quad \text{and} \quad \mathcal{Q}^n = \mathcal{Q} \setminus \mathcal{Q}^r = \{\alpha \in \mathcal{Q} \mid -\alpha \notin \mathcal{Q}\}\,. \tag{2.6}$$

Then

$$\mathfrak{q}^n = \sum_{\alpha \in \mathcal{Q}^n} \hat{\mathfrak{g}}^{\alpha} \tag{2.7}$$

is the *nilradical* of $\mathfrak{q}$, i.e. the set of the elements Z of its radical $\mathfrak{r}(\mathfrak{q})$ for which $\mathrm{ad}_{\hat{\mathfrak{g}}}(Z)$ is nilpotent, and

$$\mathfrak{q}^r = \hat{\mathfrak{h}} \oplus \sum_{\alpha \in \mathcal{Q}^r} \hat{\mathfrak{g}}^{\alpha} \tag{2.8}$$

a reductive complement of $\mathfrak{q}^n$ in $\mathfrak{q}$. The complex parabolic Lie subalgebra $\mathfrak{q}$ of $\hat{\mathfrak{g}}$ is its own normalizer and the normalizer of its nilradical $\mathfrak{q}^n$:

$$\mathfrak{q} = \{Z \in \hat{\mathfrak{g}} \mid [Z, \mathfrak{q}] \subset \mathfrak{q}\} = \{Z \in \hat{\mathfrak{g}} \mid [Z, \mathfrak{q}^n] \subset \mathfrak{q}^n\}\,. \tag{2.9}$$

If $A \in \mathfrak{h}_{\mathbb{R}}$, then the set

$$\mathcal{Q}_A = \{\alpha \in \mathcal{R} \mid \alpha(A) \geq 0\} \tag{2.10}$$

is parabolic, with $\mathcal{Q}^r_A = \{\alpha \in \mathcal{R} \mid \alpha(A) = 0\}$ and $\mathcal{Q}^n_A = \{\alpha \in \mathcal{R} \mid \alpha(A) > 0\}$. Vice versa, if $\mathcal{Q}$ is parabolic, set $\delta = \sum\{\alpha \mid \alpha \in \mathcal{Q}^n\}$, and define $T_\delta \in \mathfrak{h}_{\mathbb{R}}$ by $(T_\delta | A) = \delta(A)$ for all $A \in \mathfrak{h}_{\mathbb{R}}$. Then $\mathcal{Q} = \mathcal{Q}_{T_\delta} = \{\alpha \in \mathcal{R} \mid (\alpha|\delta) \geq 0\}$. The set of $A \in \mathfrak{h}_{\mathbb{R}}$ for which $\mathcal{Q} = \mathcal{Q}_A$ is in fact a relatively open convex cone in $\mathfrak{h}_{\mathbb{R}}$.

When $\mathcal{Q} = \mathcal{Q}_A$ for some $A \in \mathfrak{h}_{\mathbb{R}}$, we shall also write $\mathfrak{q}_A$ for $\mathfrak{q}_{\mathcal{Q}_A}$.

The sets $\mathcal{Q}^n$ associated to parabolic $\mathcal{Q}$'s are called *horocyclic* (see [War72, §1.1]). The correspondence $\mathcal{Q}^n \longleftrightarrow \mathfrak{q}^n = \sum_{\alpha \in \mathcal{Q}^n} \hat{\mathfrak{g}}^{\,\alpha}$ is one-to-one between horocyclic sets of roots in $\mathcal{R}$ and nilradicals of complex parabolic Lie subalgebras containing $\hat{\mathfrak{h}}$.

Given a parabolic subset $\mathcal{Q} \subset \mathcal{R}$, we use the notation $\mathcal{Q}^{-n}$ for its *opposite horocyclic set* $[\mathcal{Q}^n]^{\mathrm{opp}} = \mathcal{R} \setminus \mathcal{Q}$: the corresponding nilpotent algebra $\mathfrak{q}^{-n} = \sum_{\alpha \in \mathcal{Q}^{-n}} \hat{\mathfrak{g}}^{\alpha}$ is a complement of $\mathfrak{q}$ in $\hat{\mathfrak{g}}$.

To a parabolic $\mathcal{Q} \subset \mathcal{R}$ we associate the set of Weyl chambers:

$$\mathfrak{C}(\mathcal{R}, \mathcal{Q}) = \{C \in \mathfrak{C}(\mathcal{R}) \mid \mathcal{R}^+(C) \subset \mathcal{Q}\} = \{C \in \mathfrak{C}(\mathcal{R}) \mid \mathcal{R}^+(C) \supset \mathcal{Q}^n\}. \tag{2.11}$$

We denote by $\mathcal{B}(C)$ the simple roots of $\mathcal{R}^+(C)$, for $C \in \mathfrak{C}(\mathcal{R})$. Every $\alpha \in \mathcal{R}$ can be written in a unique way as a linear combination, with integral coefficients (either all ≥ 0 or all ≤ 0), of the simple roots in $\mathcal{B}(C)$:

$$\alpha = \sum_{\beta \in \mathcal{B}(C)} k_\alpha^\beta(C)\beta. \tag{2.12}$$

We define the *support* of a root α as:

$$\operatorname{supp}_C(\alpha) = \{\beta \in \mathcal{B}(C) \mid k_\alpha^\beta(C) \neq 0\}. \tag{2.13}$$

If $\mathcal{Q}$ is parabolic, $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, and $\Phi_C(\mathcal{Q}) = \mathcal{B}(C) \cap \mathcal{Q}^n$, then

$$\mathcal{Q}^n = \{\alpha \in \mathcal{R}^+(C) \mid \operatorname{supp}_C(\alpha) \cap \Phi_C(\mathcal{Q}) \neq \emptyset\}. \tag{2.14}$$

The correspondence

$$\mathcal{B}(C) \supset \Phi_C \longleftrightarrow \mathfrak{q} = \hat{\mathfrak{h}} \oplus \sum_{\alpha \in \mathcal{R}^+(C)} \hat{\mathfrak{g}}^\alpha \oplus \sum_{\substack{\alpha \in \mathcal{R}^-(C) \\ \operatorname{supp}_C(\alpha) \cap \Phi_C = \emptyset}} \hat{\mathfrak{g}}^\alpha \tag{2.15}$$

is one-to-one between subsets Φ_C of $\mathcal{B}(C)$ and complex parabolic Lie subalgebra of $\hat{\mathfrak{g}}$ that contain $\hat{\mathfrak{h}}$ and have an associated parabolic set $\mathcal{Q}$ with $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$.

Having fixed a Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$ and $\Phi_C \subset \mathcal{B}(C)$, we shall denote by $\mathfrak{q}_{\Phi_C}$ the complex parabolic Lie subalgebra of $\hat{\mathfrak{g}}$ defined by the right hand side of (2.15) and by $\mathcal{Q}_{\Phi_C}$ the corresponding parabolic set.

We denote by $\mathbf{W}(\mathcal{R})$ the Weyl group of $\mathcal{R}$, (i.e. the group of isometries of $\mathfrak{h}_\mathbb{R}^*$ generated by the symmetries $\xi \longrightarrow s_\alpha(\xi) = \xi - 2[(\xi|\alpha)/\|\alpha\|^2]\,\alpha$ for $\alpha \in \mathcal{R}$) and by $\mathbf{A}(\mathcal{R})$ the group of all isometries of $\mathfrak{h}_\mathbb{R}^*$ (with respect to the scalar product defined above) that transform $\mathcal{R}$ into itself. For $C \in \mathfrak{C}(\mathcal{R})$ we denote by $\mathbf{A}_C(\mathcal{R})$ the subgroup of $\mathbf{A}(\mathcal{R})$ consisting of the elements $w \in \mathbf{A}(\mathcal{R})$ for which $w(\mathcal{R}^+(C)) = \mathcal{R}^+(C)$. Then $\mathbf{A}(\mathcal{R}) = \mathbf{A}_C(\mathcal{R}) \rtimes \mathbf{W}(\mathcal{R})$.

We define $\mathbf{W}(\mathcal{R}, \mathcal{Q})$ and $\mathbf{A}(\mathcal{R}, \mathcal{Q})$ as the subgroups of $\mathbf{W}(\mathcal{R})$ and $\mathbf{A}(\mathcal{R})$, respectively, that transform $\mathcal{Q}$ into itself. Then we have Chevalley's Lemma (see e.g. [War72, Theorem 1.1.2.8]):

LEMMA 2.1. *The group $\mathbf{W}(\mathcal{R}, \mathcal{Q})$ is generated by the symmetries s_α with $\alpha \in \mathcal{Q}^r$. If $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, then the symmetries s_α with $\alpha \in \mathcal{B}(C) \setminus \Phi_C(\mathcal{Q})$ generate $\mathbf{W}(\mathcal{R}, \mathcal{Q})$ and $\mathbf{A}(\mathcal{R}, \mathcal{Q})$ is a semidirect product $\mathbf{A}(\mathcal{R}, \mathcal{Q}) = \mathbf{A}_C(\mathcal{R}, \mathcal{Q}) \rtimes \mathbf{W}(\mathcal{R}, \mathcal{Q})$, with $\mathbf{A}_C(\mathcal{R}, \mathcal{Q}) = \mathbf{A}_C(\mathcal{R}) \cap \mathbf{A}(\mathcal{R}, \mathcal{Q})$.* □

Let $\hat{\mathbf{G}}$ be a connected complex Lie group with Lie algebra $\hat{\mathfrak{g}}$. If $\mathbf{Q}$ is any Lie subgroup of $\hat{\mathbf{G}}$ with complex Lie subalgebra $\mathfrak{q}$ that is parabolic in $\hat{\mathfrak{g}}$, then $\mathbf{Q}$ is closed, connected and coincides with its normalizer in $\hat{\mathbf{G}}$ and is the normalizer of its Lie algebra for the adjoint representation:

$$\mathbf{Q} = \{g \in \hat{\mathbf{G}} \mid g\,\mathbf{Q}\,g^{-1} = \mathbf{Q}\} = \{g \in \hat{\mathbf{G}} \mid \operatorname{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}) = \mathfrak{q}\} \tag{2.16}$$

The homogeneous space $\mathfrak{M} = \hat{\mathbf{G}}/\mathbf{Q}$ is compact and simply connected. Since the center $\mathbf{Z}(\hat{\mathbf{G}})$ of $\hat{\mathbf{G}}$ is contained in all its parabolic subgroups, the choice of different connected complex Lie groups $\hat{\mathbf{G}}$ yields the same $\mathfrak{M}$. Hence we can consider the complex flag manifold $\mathfrak{M} = \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q})$ as an object associated simply to the pair $(\hat{\mathfrak{g}}, \mathfrak{q})$. We also recall (see [Wol69, §2.7]) that the integral cohomology $H^*(\mathfrak{M}, \mathbb{Z})$ is torsion free and 0 in odd degrees.

If $\hat{\mathfrak{h}}$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$ contained in $\mathfrak{q}$ and $\mathcal{Q} \subset \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ the parabolic set of roots associated to $\mathfrak{q}$, the complex dimension of $\mathfrak{M} = \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q})$ equals the number of roots in $\mathcal{Q}^n$.

2.2 Parabolic CR algebras and CR manifolds

A CR algebra $(\mathfrak{g}, \mathfrak{q})$ is called *parabolic* if $\mathfrak{g}$ is finite dimensional and $\mathfrak{q}$ is a parabolic subalgebra of its complexification $\hat{\mathfrak{g}}$.

By the results stated above, if $(\mathfrak{g}, \mathfrak{q})$ is a parabolic CR algebra, then all the homogeneous spaces $\tilde{M} = \tilde{M}(\mathfrak{g}, \mathfrak{q})$, $M = M(\mathfrak{g}, \mathfrak{q})$, and $\mathfrak{M} = \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q})$ are well defined. We recall that $\hat{\mathbf{G}}$ is the complex connected and simply connected Lie group with Lie algebra $\hat{\mathfrak{g}}$, the groups $\mathbf{G}$ and $\mathbf{Q}$ are the analytic subgroups of $\hat{\mathbf{G}}$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{q}$, respectively. Then $\mathfrak{M} = \hat{\mathbf{G}}/\mathbf{Q}$ is a complex flag manifold and M is an orbit in $\mathfrak{M}$ of the real form $\mathbf{G}$ of $\hat{\mathbf{G}}$.

We say that M is a *parabolic CR manifold*.

Vice versa, if $\mathbf{G}$ is a connected real form of the complex semisimple Lie group $\hat{\mathbf{G}}$, then all $\mathbf{G}$-orbits in the complex flag manifolds $\mathfrak{M} = \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q})$ are homogeneous CR manifolds of the form $M = M(\mathfrak{g}, \mathfrak{q}')$, for some parabolic complex Lie subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$, conjugated to $\mathfrak{q}$ by an inner automorphism.

It is worth noticing that, in the definition of the homogeneous CR manifold $M(\mathfrak{g}, \mathfrak{q}) = \mathbf{G}/\mathbf{G}_+$, we can define the isotropy $\mathbf{G}_+ = \mathbf{G}_+(\mathfrak{g}, \mathfrak{q})$ by

$$\mathbf{G}_+ = \{g \in \mathbf{G} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}) = \mathfrak{q}\}. \tag{2.17}$$

Since the center of $\mathbf{G}$ is always contained in $\mathbf{G}_+$, we obtain an equivalent definition of $M(\mathfrak{g}, \mathfrak{q})$ if we substitute to $\hat{\mathbf{G}}$ any connected complex Lie group $\hat{\mathbf{G}}'$ with the same Lie algebra $\hat{\mathfrak{g}}$ and to $\mathbf{G}$ the analytic subgroup $\mathbf{G}'$ of $\hat{\mathbf{G}}'$ with Lie algebra $\mathfrak{g}$. However, it is more convenient to fix a simply connected $\hat{\mathbf{G}}$, since in this case, by [BT72, Corollaire 4.7], we have:

$$\mathbf{G} = \hat{\mathbf{G}}^{\sigma} = \{g \in \hat{\mathbf{G}} \mid \sigma(g) = g\}, \tag{2.18}$$

where $\sigma : \hat{\mathbf{G}} \to \hat{\mathbf{G}}$ is the anti-holomorphic involution of $\hat{\mathbf{G}}$ corresponding to the conjugation σ of $\hat{\mathfrak{g}}$ defined by the real form $\mathfrak{g}$.

We begin by proving some general facts about parabolic CR algebras, and their associated CR manifolds.

Proposition 2.2. *A parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ is effective if and only if: (i) $\mathfrak{g}$ is semisimple, (ii) no simple ideal of $\hat{\mathfrak{g}}$ is contained in $\mathfrak{q} \cap \bar{\mathfrak{q}}$.*

An effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ with $\mathfrak{g}$ simple is either totally complex or ideal nondegenerate.

PROOF. The statement follows by observing that: (a) for a parabolic $(\mathfrak{g},\mathfrak{q})$ the radical $\mathfrak{r}$ of $\mathfrak{g}$ is contained in $\mathfrak{g}_+$; (b) if an ideal $\mathfrak{a}$ of $\hat{\mathfrak{g}}$ is contained in $\mathfrak{q}\cap\bar{\mathfrak{q}}$, then $\mathfrak{a}+\bar{\mathfrak{a}}$ is the complexification of an ideal $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_+$. ☐

PROPOSITION 2.3. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic CR algebra and let $\mathfrak{g} = \mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_\ell$ be the decomposition of $\mathfrak{g}$ into the direct sum of its simple ideals. Then:*

- (i) *$\mathfrak{q}=\mathfrak{q}_1\oplus\cdots\oplus\mathfrak{q}_\ell$ where $\mathfrak{q}_j=\mathfrak{q}\cap\hat{\mathfrak{g}}_j$ for $j=1,\dots,\ell$;*
- (ii) *for each $j=1,\dots,\ell$, $(\mathfrak{g}_j,\mathfrak{q}_j)$ is an effective parabolic CR algebra;*
- (iii) *$(\mathfrak{g},\mathfrak{q})$ is ideal (resp. weakly, strictly) nondegenerate if and only if for each $j=1,\dots,\ell$, the CR algebra $(\mathfrak{g}_j,\mathfrak{q}_j)$ is ideal (resp. weakly, strictly) nondegenerate;*
- (iv) *$(\mathfrak{g},\mathfrak{q})$ is fundamental if and only if for each $j=1,\dots,\ell$, the CR algebra $(\mathfrak{g}_j,\mathfrak{q}_j)$ is fundamental.*
- (v) *We have ($\cong$ meaning biholomorphic or CR equivalence):*

$$\hat{M}(\hat{\mathfrak{g}},\mathfrak{q})\cong\hat{M}(\hat{\mathfrak{g}}_1,\mathfrak{q}_1)\times\cdots\times\hat{M}(\hat{\mathfrak{g}}_\ell,\mathfrak{q}_\ell)\,,$$
$$\tilde{M}(\mathfrak{g},\mathfrak{q})\cong\tilde{M}(\mathfrak{g}_1,\mathfrak{q}_1)\times\cdots\times\tilde{M}(\mathfrak{g}_\ell,\mathfrak{q}_\ell)\,,$$
$$M(\mathfrak{g},\mathfrak{q})\cong M(\mathfrak{g}_1,\mathfrak{q}_1)\times\cdots\times M(\mathfrak{g}_\ell,\mathfrak{q}_\ell)\,.$$

PROOF. In fact $\hat{\mathfrak{g}}=\bigoplus_{j=1}^{\ell}\hat{\mathfrak{g}}_j$ is a decomposition of $\hat{\mathfrak{g}}$ into a direct sum of ideals. The decomposition (i) of $\mathfrak{q}$ follows then from the decomposition $\hat{\mathfrak{h}}=\bigoplus_{j=1}^{\ell}\left(\hat{\mathfrak{h}}\cap\hat{\mathfrak{g}}_j\right)$ of any Cartan subalgebra of $\hat{\mathfrak{g}}$ contained in $\mathfrak{q}$ (see [Bou05, Ch.VII,§2,Prop.2]).

The proof of the other statements is straightforward. ☐

2.3 Adapted Cartan subalgebras and Cartan involutions

When $\mathfrak{q}$ is parabolic in $\hat{\mathfrak{g}}$, its conjugate $\bar{\mathfrak{q}}$ with respect to the real form $\mathfrak{g}$ is also parabolic in $\hat{\mathfrak{g}}$. Therefore the intersection $\mathfrak{q}\cap\bar{\mathfrak{q}}$ contains a Cartan subalgebra $\hat{\mathfrak{h}}$ that is invariant under conjugation. The intersection $\mathfrak{h}=\hat{\mathfrak{h}}\cap\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{g}_+=\mathfrak{q}\cap\mathfrak{g}$.

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_+=\mathfrak{q}\cap\mathfrak{g}$ is said to be *adapted* to $(\mathfrak{g},\mathfrak{q})$. We also have:

PROPOSITION 2.4. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic CR algebra, with isotropy subalgebra $\mathfrak{g}_+=\mathfrak{q}\cap\mathfrak{g}$. The elements A of the radical $\mathfrak{r}(\mathfrak{g}_+)$ of $\mathfrak{g}_+$ for which $\mathrm{ad}_{\mathfrak{g}}(A):\mathfrak{g}\to\mathfrak{g}$ is nilpotent, form a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}_+$. It admits a reductive complement $\mathfrak{g}_0$ in $\mathfrak{g}_+$:*

$$\mathfrak{g}_+=\mathfrak{n}\oplus\mathfrak{g}_0\,. \tag{2.19}$$

The reductive subalgebra $\mathfrak{g}_0$ is uniquely determined modulo inner automorphisms of $\mathfrak{g}_+$ from the subgroup generated by those of the form $\exp\left(\mathrm{ad}_{\mathfrak{g}_+}(X)\right)$ with $X\in\mathfrak{n}$.

PROOF. Indeed $\mathfrak{q}$, being parabolic, contains the semisimple and nilpotent parts of its elements. If $X\in\mathfrak{q}$ belongs to the real form $\mathfrak{g}$, then also its semisimple and nilpotent parts belong to $\mathfrak{g}$. Therefore $\mathfrak{g}_+$ is splittable, i.e. contains the semisimple and nilpotent part of its elements and we can apply [Bou05, Prop.7, §5, Ch.VII] to obtain our statement. $\square$

Let $\mathfrak{z}_0$ be the center and $\mathfrak{s}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ the semisimple ideal of $\mathfrak{g}_0$. Then

$$\mathfrak{g}_0 = \mathfrak{z}_0 \oplus \mathfrak{s}_0 . \tag{2.20}$$

Thus, a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_+$ of $\mathfrak{g}$ can be taken as the direct sum

$$\mathfrak{h} = \mathfrak{z}_0 \oplus \mathfrak{h}_0 \tag{2.21}$$

of the center $\mathfrak{z}_0$ of $\mathfrak{g}_0$ and a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{s}_0$. Vice versa, every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ adapted to $(\mathfrak{g}, \mathfrak{q})$ has the form (2.21) for some reductive subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_+$.

It is also convenient to consider a Cartan decomposition (see e.g. [Bou05]) :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2.22}$$

of $\mathfrak{g}$, corresponding to a Cartan involution ϑ . The set $\mathfrak{k} = \{X \in \mathfrak{g} \mid \vartheta(X) = X\}$ of fixed points of ϑ is a maximal compact Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \vartheta(X) = -X\}$ its orthogonal for the Killing form $\kappa_{\mathfrak{g}}$ of $\mathfrak{g}$. Any ϑ-invariant Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ of its *compact* (or toroidal) part $\mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{k} \subset \mathfrak{k}$ and its *noncompact* (or vector part) $\mathfrak{h}^- = \mathfrak{h} \cap \mathfrak{p} \subset \mathfrak{p}$.

We say that the Cartan decomposition (2.22) is *adapted* to the effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ if $\mathfrak{k}$ contains a maximal compact Lie subalgebra of $\mathfrak{g}_+$. Then :

LEMMA 2.5. *If a Cartan decomposition* (2.22) *is adapted to the parabolic CR algebra* $(\mathfrak{g}, \mathfrak{q})$, *then every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is adapted to* $(\mathfrak{g}, \mathfrak{q})$ *is conjugate, modulo an inner automorphism of $\mathfrak{g}_+$, to a ϑ-invariant Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}$ that is adapted to* $(\mathfrak{g}, \mathfrak{q})$.

Vice versa, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ adapted to $(\mathfrak{g}, \mathfrak{q})$, *then there exists a Cartan decomposition* (2.22), *adapted to* $(\mathfrak{g}, \mathfrak{q})$, *such that* $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$.

In particular, if $\{\mathfrak{q}_i \mid i \in I\}$ is a family of complex parabolic Lie subalgebras of $\hat{\mathfrak{g}}$ such that $\bigcap_{i\in I} \mathfrak{q}_i$ is parabolic in $\hat{\mathfrak{g}}$, then there exist both a Cartan decomposition (2.22) *and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, compatible with* (2.22), *that are adapted to all the $(\mathfrak{g}, \mathfrak{q}_i)$'s.* $\square$

We say that $(\vartheta, \mathfrak{h})$ is an *adapted Cartan pair* for $(\mathfrak{g}, \mathfrak{q})$ if :
(i) ϑ is the Cartan involution of a Cartan decomposition (2.22) adapted to $(\mathfrak{g}, \mathfrak{q})$;
(ii) $\mathfrak{h}$ is a ϑ-invariant Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$.

Being $\sigma : \hat{\mathfrak{g}} \ni X \longrightarrow \bar{X} \in \hat{\mathfrak{g}}$ the conjugation in $\hat{\mathfrak{g}}$ associated to the real form $\mathfrak{g}$, and having fixed (2.22), we also consider the conjugation $\tau : \hat{\mathfrak{g}} \longrightarrow \hat{\mathfrak{g}}$ of $\hat{\mathfrak{g}}$ with respect to its compact real form $\mathfrak{u} = \mathfrak{k} \oplus i\,\mathfrak{p}$ and use the same symbol ϑ to denote the $\mathbb{C}$-linear extension to $\hat{\mathfrak{g}}$ of the Cartan involution ϑ of $\mathfrak{g}$. We obtain in this way three commuting involutions σ, τ, ϑ of $\hat{\mathfrak{g}}$, each being the composition product of the other two :

$$\tau = \vartheta \circ \sigma = \sigma \circ \vartheta\,, \quad \sigma = \vartheta \circ \tau = \tau \circ \vartheta\,, \quad \vartheta = \sigma \circ \tau = \tau \circ \sigma\,. \tag{2.23}$$

In particular $\mathfrak{u}$ is invariant under σ: $\sigma(\mathfrak{u}) = \mathfrak{u}$.

Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to $(\mathfrak{g}, \mathfrak{q})$. Then $\mathfrak{h}_{\mathbb{R}}$ decomposes as: $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}^- \oplus i\,\mathfrak{h}^+$ and $\mathfrak{h}_{\mathbb{R}}$ is a Cartan subalgebra of a split real form $\mathfrak{g}_{\mathbb{R}}$ of $\hat{\mathfrak{g}}$. The involutions σ, τ and ϑ transform $\mathfrak{h}_{\mathbb{R}}$ into itself. Hence, by transposition, they define involutions on $\mathfrak{h}^*_{\mathbb{R}}$, that we still denote by the same symbols σ, τ and ϑ, and that transform the set of roots $\mathcal{R}$ into itself. We set $\bar{\alpha} = \sigma(\alpha)$ for all $\alpha \in \mathfrak{h}^*_{\mathbb{R}}$. We have:

$$\tau(\alpha) = -\alpha\,, \quad \vartheta(\alpha) = -\bar{\alpha} \quad \forall \alpha \in \mathfrak{h}^*_{\mathbb{R}}\,. \tag{2.24}$$

The reductive complement $\mathfrak{q}^r$ of $\mathfrak{q}^n$ in $\mathfrak{q}$ of (2.8) is $\mathfrak{q} \cap \tau(\mathfrak{q})$, while the reductive complement $\mathfrak{g}_0$ of $\mathfrak{n}$ in $\mathfrak{g}_+$ of (2.19) can be taken equal to $\mathfrak{g}_+ \cap \vartheta(\mathfrak{g}_+)$.

2.4 The fundamental and weakly nondegenerate reductions

We consider the CR fibrations of Proposition 1.5 in the special case of a parabolic CR algebra.

THEOREM 2.6. *Every effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ admits a unique $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \hat{\mathfrak{g}}')$, where $\hat{\mathfrak{g}}' \supset \mathfrak{q}$ is the complexification of a real parabolic subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$, and the fiber is fundamental. The basis $(\mathfrak{g}, \hat{\mathfrak{g}}')$ is a totally real parabolic CR algebra and also the fiber $(\mathfrak{g}', \mathfrak{q})$ is parabolic.*

*This yields a **G**-equivariant CR fibration $\pi : M(\mathfrak{g}, \mathfrak{q}) \to M(\mathfrak{g}, \hat{\mathfrak{g}}')$ with compact basis. Each connected component of the fiber is CR diffeomorphic to $M(\mathfrak{g}', \mathfrak{q})$, hence of finite type.*

PROOF. Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra. The complex subalgebra $\mathfrak{q}''$ generated by $\mathfrak{q} + \bar{\mathfrak{q}}$ is parabolic in $\hat{\mathfrak{g}}$ because contains $\mathfrak{q}$, and is the complexification of a real parabolic subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ because $\bar{\mathfrak{q}}\,'' = \mathfrak{q}''$. Then (1.6) yields a $\mathfrak{g}$-equivariant CR fibration with a totally real basis. The fiber is $(\mathfrak{g}', \mathfrak{q})$. This is parabolic because $\mathfrak{q}$, being parabolic in $\hat{\mathfrak{g}}$, is also parabolic in $\hat{\mathfrak{g}}' \subset \hat{\mathfrak{g}}$.

The final statement follows from the commutative diagram:

$$\begin{tikzcd}
M(\mathfrak{g}, \mathfrak{q}) \arrow[r] \arrow[d, "\pi"'] & \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q}) \arrow[d, "\hat{\pi}"'] \\
M(\mathfrak{g}, \hat{\mathfrak{g}}') \arrow[r] & \hat{M}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}')
\end{tikzcd}$$

that yields an embedding of each fiber of $\pi : M(\mathfrak{g}, \mathfrak{q}) \to M(\mathfrak{g}, \hat{\mathfrak{g}}')$ into a fiber of $\hat{\pi} : \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q}) \to \hat{M}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}')$. The basis $M(\mathfrak{g}, \hat{\mathfrak{g}}\,')$ is compact because $\mathfrak{g} \cap \hat{\mathfrak{g}}'$ is parabolic in $\mathfrak{g}$. □

THEOREM 2.7. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra. Then there is a unique $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{q}')$ with a weakly nondegenerate basis $(\mathfrak{g}, \mathfrak{q}')$ and a totally complex fiber. The basis $(\mathfrak{g}, \mathfrak{q}')$ is a parabolic CR algebra.*

PROOF. We recall from [MN05, §5], that $\mathfrak{q}'$ is the unique maximal subalgebra of $\hat{\mathfrak{g}}$ that contains $\mathfrak{q}$ and is contained in $\mathfrak{q} + \bar{\mathfrak{q}}$. Clearly $\mathfrak{q}'$ is parabolic because it contains the parabolic subalgebra $\mathfrak{q}$. □

2.5 The fiber of a G-equivariant fibration

Next we investigate the general structure of the fiber of a **G**-equivariant fibration $M(\mathfrak{g}, \mathfrak{q}) \to M(\mathfrak{g}, \mathfrak{q}')$ for a pair of complex parabolic subalgebras $\mathfrak{q} \subset \mathfrak{q}'$ of $\hat{\mathfrak{g}}$.

THEOREM 2.8. *Let $\mathfrak{q} \subset \mathfrak{q}'$ be complex parabolic Lie subalgebras of $\hat{\mathfrak{g}}$. With $\mathfrak{g}' = \mathfrak{g} \cap \mathfrak{q}'$, the CR algebra $(\mathfrak{g}', \bar{\mathfrak{q}}' \cap \mathfrak{q})$ is the fiber of the $\mathfrak{g}$-equivariant fibration $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{q}')$ (see Chapter 1).*

The nilradical $\mathfrak{n}'$ of $\mathfrak{g}'$, consisting of the $\mathrm{ad}_{\mathfrak{g}}$-nilpotent elements of the radical $\mathfrak{r}(\mathfrak{g}')$ of $\mathfrak{g}'$, has a reductive complement $\mathfrak{g}'_0$ in $\mathfrak{g}'$ such that:

(*i*) *The CR algebra $(\mathfrak{g}'_0, \hat{\mathfrak{g}}'_0 \cap \mathfrak{q})$ is parabolic.*
(*ii*) *The fiber $(\mathfrak{g}', \hat{\mathfrak{g}}' \cap \mathfrak{q})$ is the semidirect sum of the parabolic CR algebra $(\mathfrak{g}'_0, \hat{\mathfrak{g}}'_0 \cap \mathfrak{q})$ and of the nilpotent CR algebra $(\mathfrak{n}', \hat{\mathfrak{n}}' \cap \mathfrak{q})$.*
(*iii*) *The nilpotent CR algebra $(\mathfrak{n}', \hat{\mathfrak{n}}' \cap \mathfrak{q})$ is totally complex.*
(*iv*) *The connected components of the fibers of the $\mathbf{G}$-equivariant fibration $\pi : M(\mathfrak{g}, \mathfrak{q}) \to M(\mathfrak{g}, \mathfrak{q}')$ are CR diffeomorphic to the Cartesian product of a parabolic CR manifold $M(\mathfrak{g}'_0, \mathfrak{q} \cap \hat{\mathfrak{g}}'_0)$ and of a Euclidean complex manifold ($\cong \mathbb{C}^\ell$).*

PROOF. Fix a Cartan pair $(\vartheta, \mathfrak{h})$, that is adapted for both $(\mathfrak{g}, \mathfrak{q})$ and $(\mathfrak{g}, \mathfrak{q}')$. Since $\mathfrak{q} \subset \mathfrak{q}'$ and $\mathfrak{q}$ is parabolic, we have the inclusions: $\mathfrak{q}^r \subset \mathfrak{q}'^{\,r}$ and $\mathfrak{q}^n \supset \mathfrak{q}'^{\,n}$. The complexification of the fiber $\mathfrak{g}'$ is:

$$\hat{\mathfrak{g}}' = \mathfrak{q}' \cap \bar{\mathfrak{q}}' = \hat{\mathfrak{g}}'_0 \rtimes \hat{\mathfrak{n}}' ,$$

where:

$$\begin{cases} \hat{\mathfrak{g}}'_0 = \mathfrak{q}'^{\,r} \cap \bar{\mathfrak{q}}'^{\,r}, \\ \hat{\mathfrak{n}}' = (\mathfrak{q}'^{\,r} \cap \bar{\mathfrak{q}}'^{\,n}) \oplus (\mathfrak{q}'^{\,n} \cap \bar{\mathfrak{q}}'^{\,r}) \oplus (\mathfrak{q}'^{\,n} \cap \bar{\mathfrak{q}}'^{\,n}) = (\mathfrak{q}' \cap \bar{\mathfrak{q}}'^{\,n}) + (\mathfrak{q}'^{\,n} \cap \bar{\mathfrak{q}}') . \end{cases}$$

Thus $\mathfrak{g}' = \mathfrak{g}'_0 \rtimes \mathfrak{n}'$, where $\mathfrak{n}' = \hat{\mathfrak{n}}' \cap \mathfrak{g}$ is a real form of the nilradical $\hat{\mathfrak{n}}'$ of $\mathfrak{q}' \cap \bar{\mathfrak{q}}'$ and $\mathfrak{g}'_0 := \hat{\mathfrak{g}}'_0 \cap \mathfrak{g}$ a reductive complement of $\mathfrak{n}'$ in $\mathfrak{g}'$.

We have:

$$\hat{\mathfrak{g}}' \cap \mathfrak{q} = (\hat{\mathfrak{g}}'_0 \rtimes \hat{\mathfrak{n}}') \cap \mathfrak{q} = (\hat{\mathfrak{g}}'_0 \cap \mathfrak{q}) \rtimes (\hat{\mathfrak{n}}' \cap \mathfrak{q}) ,$$

so that:

$$(\mathfrak{g}', \hat{\mathfrak{g}}' \cap \mathfrak{q}) = (\mathfrak{g}'_0, \hat{\mathfrak{g}}'_0 \cap \mathfrak{q}) \rtimes (\mathfrak{n}', \hat{\mathfrak{n}}' \cap \mathfrak{q}) .$$

The complex Lie subalgebra $\hat{\mathfrak{g}}'_0 \cap \mathfrak{q}$ is parabolic in $\hat{\mathfrak{g}}'_0$, because $\hat{\mathfrak{g}}'_0$ is reductive, $\mathfrak{q}$ is parabolic in $\hat{\mathfrak{g}}$, and $\hat{\mathfrak{g}}'_0 \cap \mathfrak{q}$ contains a Cartan subalgebra of $\hat{\mathfrak{g}}'_0$ and of $\hat{\mathfrak{g}}$.

Note that $\hat{\mathfrak{n}}' \cap \mathfrak{q}$ is contained in, but in general not equal to, the nilradical $\hat{\mathfrak{n}}$ of $\mathfrak{q} \cap \bar{\mathfrak{q}}$. We have:

$$\hat{\mathfrak{n}}' \cap \mathfrak{q} \supset \mathfrak{q}'^{\,n} \cap \bar{\mathfrak{q}}' ,$$

so that:

$$\hat{\mathfrak{n}}' \cap \mathfrak{q} + \overline{\hat{\mathfrak{n}}' \cap \mathfrak{q}} \supset \mathfrak{q}'^{\,n} \cap \bar{\mathfrak{q}}' + \mathfrak{q}' \cap \bar{\mathfrak{q}}'^{\,n} = \hat{\mathfrak{n}}'$$

shows that actually :

$$\hat{\mathfrak{n}}' \cap \mathfrak{q} + \overline{\hat{\mathfrak{n}}' \cap \mathfrak{q}} = \hat{\mathfrak{n}}'$$

and the nilpotent CR algebra $(\mathfrak{n}', \hat{\mathfrak{n}}' \cap \mathfrak{q})$ is totally complex.

The $\mathbf{G}$-equivariant fibration $\pi : M(\mathfrak{g}, \mathfrak{q}) \to M(\mathfrak{g}, \mathfrak{q}')$ is the restriction of the $\hat{\mathbf{G}}$ equivariant fibration $\hat{\pi} : \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q}) \to \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q}')$. The typical fiber $\mathfrak{F}$ of $\hat{\pi}$ is $\mathbf{Q}'/\mathbf{Q}$. Since $\mathfrak{q}$ is a parabolic complex Lie subalgebra of $\mathfrak{q}'$, the fiber $\mathfrak{F}$ is a complex flag manifold and, in particular, is compact, connected and simply connected. Thus the typical fiber F of $\pi : M(\mathfrak{g}, \mathfrak{q}) \to M(\mathfrak{g}, \mathfrak{q}')$ is a submanifold of a complex flag manifold.

Denote still by $\tau : \hat{\mathbf{G}} \longrightarrow \hat{\mathbf{G}}$ the involution of $\hat{\mathbf{G}}$ associated to the conjugation $\tau : \hat{\mathfrak{g}} \longrightarrow \hat{\mathfrak{g}}$ with respect to the compact real form $\mathfrak{u} = \mathfrak{k} \oplus i\,\mathfrak{p}$. Since $\mathfrak{q}'^{\,n} \subset \mathfrak{q}^n$, the fiber $\mathfrak{F}$ can be viewed also as a flag manifold of the reductive complex closed connected Lie subgroup $\mathbf{Q}'^{\,r} = \mathbf{Q}' \cap \tau(\mathbf{Q}')$ of $\hat{\mathbf{G}}$. The fiber F is contained in the orbit $\mathfrak{F}_0 \subset \mathfrak{F}$ of the closed complex Lie subgroup $\hat{\mathbf{G}}' := \mathbf{Q}' \cap \sigma(\mathbf{Q}')$ of $\mathbf{Q}'$. The group $\hat{\mathbf{G}}'$ is connected because it contains a Cartan subgroup of $\hat{\mathbf{G}}$, and decomposes into the semidirect product

$$\hat{\mathbf{G}}' = \hat{\mathbf{G}}'_0 \rtimes \hat{\mathbf{N}}' \,, \tag{2.25}$$

where $\hat{\mathbf{G}}'_0$ and $\hat{\mathbf{N}}'$ are the analytic complex Lie subgroups of $\hat{\mathbf{G}}$ generated by the Lie subalgebras $\hat{\mathfrak{g}}'_0$ and $\hat{\mathfrak{n}}'$, respectively. We have $\hat{\mathbf{G}}'_0 = \mathbf{Q}'^{\,r} \cap \sigma(\mathbf{Q}'^{\,r})$, so that $\hat{\mathbf{G}}'_0$ is closed in $\hat{\mathbf{G}}$. Moreover, since $\mathrm{ad}_{\hat{\mathfrak{g}}}(Z)$ is nilpotent for all $Z \in \hat{\mathfrak{n}}$, by Engel's theorem and the semisimplicity of $\hat{\mathfrak{g}}$, we obtain that $\exp : \hat{\mathfrak{n}}' \longrightarrow \hat{\mathbf{N}}'$ is an analytic diffeomorphism, and $\hat{\mathbf{N}}'$ is Euclidean.

The validity of (iv) is then a consequence of the next Proposition. □

Proposition 2.9. *Let $\hat{\mathbf{N}}'$ be a connected nilpotent complex Lie group with complex Lie algebra $\hat{\mathfrak{n}}'$ and $\mathfrak{n}'$ a real form of $\hat{\mathfrak{n}}'$. Let $\mathbf{N}'$ be the real analytic Lie subgroup of $\hat{\mathbf{N}}'$ with Lie algebra $\mathfrak{n}'$, and $\mathbf{Q}_0$ a closed connected complex Lie subgroup of $\hat{\mathbf{N}}'$, with Lie algebra $\mathfrak{q}_0 \subset \hat{\mathfrak{n}}'$, and set $\mathbf{N} = \mathbf{Q}_0 \cap \mathbf{N}'$. Assume that the CR algebra $(\mathfrak{n}', \mathfrak{q}_0)$ is totally complex. With $E = \mathbf{N}'/\mathbf{N}$ and $\hat{E} = \hat{\mathbf{N}}'/\mathbf{Q}_0$, the natural map $E \longrightarrow \hat{E}$ obtained from the inclusion $\mathbf{N}' \hookrightarrow \hat{\mathbf{N}}'$ is a diffeomorphism.*

Proof. The condition that $(\mathfrak{n}', \mathfrak{q}_0)$ is totally complex is equivalent to the equality $\mathfrak{n}' + \mathfrak{q}_0 = \hat{\mathfrak{n}}'$. Since $\hat{\mathfrak{n}}'$ is nilpotent, this equality implies (see the proof below) that the map $\mathbf{N}' \times \mathbf{Q}_0 \ni (n, q) \longrightarrow n \cdot q \in \hat{\mathbf{N}}'$ is onto, and hence the inclusion $\mathbf{N}' \hookrightarrow \hat{\mathbf{N}}'$ yields, by passing to the quotients, a smooth one-to-one map $f : E \longrightarrow \hat{E}$. We note that $E = \mathbf{N}'/\mathbf{N}$ is a complex manifold with the homogeneous CR structure defined by the CR algebra $(\mathfrak{n}', \mathfrak{q}_0)$. With this complex structure on E, and with the complex structure that $\hat{E}$ inherits from $\hat{\mathbf{N}}'$, the map f is holomorphic. Being one-to-one, f is a biholomorphism.

We give here a simple argument to prove that $\hat{\mathbf{N}}' = \mathbf{N}'\mathbf{Q}_0$.

Consider the lower central series

$$\begin{aligned}\hat{\mathfrak{n}} = \mathrm{C}^{(0)}(\hat{\mathfrak{n}}') \supset \mathrm{C}^{(1)}(\hat{\mathfrak{n}}') = [\hat{\mathfrak{n}}', \hat{\mathfrak{n}}'] \supset \cdots \supset \mathrm{C}^{(h)}(\hat{\mathfrak{n}}') = \left[\mathrm{C}^{(h-1)}(\hat{\mathfrak{n}}'), \hat{\mathfrak{n}}'\right] \supset \cdots \\ \cdots \supset \mathrm{C}^{(m)}(\hat{\mathfrak{n}}') = \left[\mathrm{C}^{(m-1)}(\hat{\mathfrak{n}}'), \hat{\mathfrak{n}}'\right] = \{0\}\,.\end{aligned}$$

Since $\hat{\mathfrak{n}}'$ is nilpotent and $\hat{\mathbf{N}}'$ is connected, the exponential map $\exp : \hat{\mathfrak{n}}' \longrightarrow \hat{\mathbf{N}}'$ is surjective. Let $Z \in \hat{\mathfrak{n}}'$. We want to prove that there is $g \in \mathbf{N}'$ such that $g^{-1} \cdot \exp(Z) \in \mathbf{Q}_0$. To this aim, let $X \in \mathfrak{n}'$ and $W \in \mathfrak{q}_0$ be such that $Z = X + W$. Let $Z_1 \in \hat{\mathfrak{n}}'$ be such that $\exp(Z_1) = \exp(-X)\exp(Z)\exp(-W)$. We claim that, if $Z \in \mathrm{C}^h(\hat{\mathfrak{n}}')$, then $Z_1 \in \mathrm{C}^{h+1}(\hat{\mathfrak{n}}')$.

While proving this claim, we can assume that $\hat{\mathbf{N}}'$ is also simply connected, so that all Lie subgroups $\hat{\mathbf{N}}'_h = \exp\left(\mathrm{C}^{(h)}(\hat{\mathfrak{n}}')\right)$ are closed and simply connected. For each integer $h \geq 0$ we have a commutative diagram:

$$\begin{array}{ccc} \hat{\mathfrak{n}}' & \xrightarrow{\exp} & \hat{\mathbf{N}}' \\ {\scriptstyle \pi_h}\Big\downarrow & & \Big\downarrow{\scriptstyle p_h} \\ \hat{\mathfrak{n}}'/\mathrm{C}^{h+1}(\hat{\mathfrak{n}}') & \xrightarrow{[\exp]} & \hat{\mathbf{N}}'/\hat{\mathbf{N}}'_{h+1} \end{array}$$

where [exp] denotes the exponential map on the quotient. If $Z \in \mathrm{C}^h(\hat{\mathfrak{n}}')$, then $\pi_h(Z)$ belongs to the center of the quotient Lie algebra $\hat{\mathfrak{n}}'/\mathrm{C}^{h+1}(\hat{\mathfrak{n}}')$. Hence we obtain:

$$\begin{aligned} [\exp]\,(\pi_h(Z_1)) &= [\exp](-\pi_h(X)) \cdot [\exp](\pi_h(Z)) \cdot [\exp](\pi_h(X-Z)) \\ &= [\exp](-\pi_h(X))) \cdot [\exp]\,(\pi_h(Z) + \pi_h(X-Z)) \\ &= [\exp](-\pi_h(X))) \cdot [\exp](\pi_h(X))) = \mathbf{1}_{\hat{\mathbf{N}}'/\hat{\mathbf{N}}'_{h+1}} . \end{aligned}$$

Since [exp] is a diffeomorphism, we obtain that $Z \in \mathrm{C}^{h+1}(\hat{\mathfrak{n}}')$.

We show by recurrence that for every $Z \in \mathrm{C}^{(m-i)}(\hat{\mathfrak{n}}')$ there is some $X \in \mathfrak{n}$ such that $\exp(-X) \cdot \exp(Z) \in \mathbf{Q}_0$. This is trivially true when $m = 0$, as $Z = 0$ in this case. If $Z \in \mathrm{C}^{(m-i)}(\hat{\mathfrak{n}}')$ for some $i > 0$, and $X \in \mathfrak{n}'$ is such that $X - Z \in \mathfrak{q}_0$, then $\exp(-X) \cdot \exp(Z) \cdot \exp(X - Z) = \exp(Z_1)$ for some $Z_1 \in \mathrm{C}^{(m-i+1)}(\hat{\mathfrak{n}}')$. By the recursive assumption, there is $X_1 \in \mathfrak{n}'$ such that $\exp(-X_1)\exp(Z_1) \in \mathbf{Q}_0$. Then $g = \exp(X_1) \cdot \exp(X) \in \mathbf{N}'$ and $g^{-1} \cdot \exp(Z) \in \mathbf{Q}_0$. For $i = m$ we obtain our contention. ☐

From Theorem 2.8 we obtain:

THEOREM 2.10. *Let $M = M(\mathfrak{g}, \mathfrak{q})$ and $M' = M(\mathfrak{g}, \mathfrak{q}')$ be parabolic CR manifolds. If $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$, then the $\mathbf{G}$-equivariant fibration $M \longrightarrow M'$ is a CR fibration and has a totally complex simply connected fiber.*

PROOF. We already noted in Chapter 1 that the CR algebra associated to the fiber F of the fibration $M \longrightarrow M'$, and hence F itself, is totally complex when $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$. By Theorem 2.8, the connected components of the fiber are the product of a Euclidean complex nilmanifold and of a manifold $M(\mathfrak{g}'_0, \hat{\mathfrak{g}}'_0 \cap \mathfrak{q})$, for a totally complex parabolic CR algebra $(\mathfrak{g}'_0, \hat{\mathfrak{g}}'_0 \cap \mathfrak{q})$. This $M(\mathfrak{g}'_0, \hat{\mathfrak{g}}'_0 \cap \mathfrak{q})$ is an open orbit of a connected real form $\mathbf{G}'_0$ of a connected complex Lie group $\hat{\mathbf{G}}'_0$ with Lie algebra $\hat{\mathfrak{g}}'_0$, and thus is simply connected by [Wol69, Theorem 5.4]. ☐

COROLLARY 2.11. *Let $(\mathfrak{g}, \mathfrak{q}')$ be the weakly nondegenerate reduction of the effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$. Then*

$$f : M = M(\mathfrak{g}, \mathfrak{q}) \longrightarrow M' = M(\mathfrak{g}, \mathfrak{q}') \tag{2.26}$$

is a $\mathbf{G}$-equivariant CR fibration with complex simply connected fibers. ☐

We give here a simple general criterion that ensures the existence and the connectedness of the fiber of some $\mathbf{G}$-equivariant fibrations.

PROPOSITION 2.12. *Keep the notation introduced above. The isotropy subgroup $\mathbf{G}_+$ is the closed real semi-algebraic subgroup of $\mathbf{G}$:*

$$\mathbf{G}_+ = \mathbf{N}_{\mathbf{G}}(\mathfrak{q}_\Phi) = \{g \in \mathbf{G} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}_\Phi) = \mathfrak{q}_\Phi\} .$$

The isotropy subgroup $\mathbf{G}_+$ *admits a Chevalley decomposition*

$$\mathbf{G}_+ = \mathbf{G}_0 \rtimes \mathbf{N}$$

where:

(i) $\mathbf{N}$ *is a unipotent, closed, connected, and simply connected subgroup with Lie algebra* $\mathfrak{n}$*;*

(ii) $\mathbf{G}_0$ *is a reductive Lie subgroup, with Lie algebra* $\mathfrak{g}_0$*, and is the centralizer of its center* $\mathfrak{z} = \mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{g}_0)$ *in* $\mathbf{G}$*:*

$$\mathbf{G}_0 = \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}) = \{g \in \mathbf{G} \mid \mathrm{Ad}_{\mathfrak{g}}(g)(H) = H \quad \forall H \in \mathfrak{z}\}\,.$$

Proof. Let $g \in \mathbf{G}_+$. Then $\mathrm{Ad}_{\mathfrak{g}}(g)(\mathfrak{g}_0)$ is a reductive complement of $\mathfrak{n}$ in $\mathfrak{g}_+$. Since all reductive complements of $\mathfrak{n}$ are conjugated by an inner automorphism from $\mathrm{Ad}_{\mathfrak{g}_+}(\mathbf{N})$, we can find a $g_n \in \mathbf{N}$ such that $\mathrm{Ad}_{\mathfrak{g}_+}(g_n^{-1}g)(\mathfrak{g}_0) = \mathfrak{g}_0$. Consider the element $g_r = g_n^{-1}g$. We have:

$$\begin{array}{llll} \mathrm{Ad}_{\mathfrak{g}}(g_r)(\mathfrak{g}_0) & = \mathfrak{g}_0, & \mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}_\Phi) & = \mathfrak{q}_\Phi, \\ \mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}_\Phi^n) & = \mathfrak{q}_\Phi^n, & \mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\bar{\mathfrak{q}}_\Phi) & = \bar{\mathfrak{q}}_\Phi, \end{array}$$

because $g_r \in \mathbf{Q} \cap \bar{\mathbf{Q}}$. We consider the parabolic subalgebra of $\hat{\mathfrak{g}}$ defined by:

$$\mathfrak{q}_{\Phi'} = \mathfrak{q}_\Phi^n \oplus (\mathfrak{q}_\Phi^r \cap \bar{\mathfrak{q}}_\Phi) = \mathfrak{q}_\Phi^n + (\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi)\ .$$

It has the property that $\mathfrak{q}_{\Phi'}^r = \bar{\mathfrak{q}}_{\Phi'}^r$ is the complexification of $\mathfrak{g}_0$. Clearly $\mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}_{\Phi'}) = \mathfrak{q}_{\Phi'}$ and $\mathrm{Ad}_{\hat{\mathfrak{g}}}(g_r)(\mathfrak{q}_{\Phi'}^r) = \mathfrak{q}_{\Phi'}^r$. Hence $g_r \in \mathbf{Q}_{\Phi'}^r$ and the statement follows because $\mathbf{Q}_{\Phi'}^r = \mathbf{Z}_{\hat{\mathbf{G}}}(\mathfrak{z}_{\mathfrak{q}_{\Phi'}^r}(\mathfrak{q}_{\Phi'}^r))$ is the centralizer of the center of its Lie algebra and $\mathfrak{z}$ is a real form of $\mathfrak{z}_{\mathfrak{q}_{\Phi'}^r}(\mathfrak{q}_{\Phi'}^r)$. □

Proposition 2.13. *Let* $(\mathfrak{g}, \mathfrak{q})$*,* $(\mathfrak{g}, \mathfrak{q}')$ *be two effective parabolic CR algebras. Assume that* $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g} \subset \mathfrak{g}'_+ = \mathfrak{q}' \cap \mathfrak{g}$ *and that* $\mathfrak{g}_+$ *contains a Cartan subalgebra* $\mathfrak{h}$ *that is maximally noncompact among the Cartan subalgebras of* $\mathfrak{g}$ *that are contained in* $\mathfrak{g}'_+$*. Then the germ of local* $\mathbf{G}$*-equivariant submersion* $(M(\mathfrak{g}, \mathfrak{q}), \mathbf{o}) \longrightarrow (M(\mathfrak{g}, \mathfrak{q}'), \mathbf{o}')$*, defined by the projection* $\mathfrak{g}/\mathfrak{g}_+ \longrightarrow \mathfrak{g}/\mathfrak{g}'$*, extends to a* $\mathbf{G}$*-equivariant fibration* $\pi : M(\mathfrak{g}, \mathfrak{q}) \longrightarrow M(\mathfrak{g}, \mathfrak{q}')$ *with connected fibers.*

Proof. Decompose $\mathbf{G}_+ = \mathbf{G}_0 \rtimes \mathbf{N}$ and $\mathbf{G}'_+ = \mathbf{G}'_0 \rtimes \mathbf{N}'$. Let $\mathbf{H} = \mathbf{Z}_{\mathbf{G}}(\mathfrak{h}) = \{h \in \mathbf{G} \mid \mathrm{Ad}_{\mathfrak{g}}(h)(H) = H\,,\ \forall H \in \mathfrak{h}\}$ be the Cartan subgroup of $\mathbf{G}$ corresponding to $\mathfrak{h}$. We have $\mathrm{Ad}_{\hat{\mathfrak{g}}}(h)(\mathfrak{q}) = \mathfrak{q}$ and $\mathrm{Ad}_{\hat{\mathfrak{g}}}(h)(\mathfrak{q}') = \mathfrak{q}'$ for all $h \in \mathbf{H}$. Hence $\mathbf{H} \subset \mathbf{G}_0 \cap \mathbf{G}'_0$. Since $\mathfrak{h}$ is maximally noncompact in $\mathfrak{g}'_0$ and a fortiori in $\mathfrak{g}_0$, by [Kna02, Prop.7.90], all connected components of $\mathbf{G}'_0$ and $\mathbf{G}_0$, and also of $\mathbf{G}'_+$ and $\mathbf{G}_+$, intersect $\mathbf{H}$. The connected component of the identity $\mathbf{G}_+^0$ of $\mathbf{G}_+$ is contained in the connected component $[\mathbf{G}'_+]^0$ of the identity in $\mathbf{G}'_+$. Since $\mathbf{G}_+$ is generated by $\mathbf{G}_+^0$ and $\mathbf{H}$, and likewise $\mathbf{G}'_+$ is generated by $[\mathbf{G}'_+]^0$ and $\mathbf{H}$, we obtain at the same time that $\mathbf{G}_+ \subset \mathbf{G}'_+$ and that the fiber $\mathbf{G}'_+/\mathbf{G}_+$ is connected. □

Using Proposition 2.13, we can prove:

THEOREM 2.14. *Let* $(\mathfrak{g}, \mathfrak{q})$ *and* $(\mathfrak{g}, \mathfrak{q}')$ *be two effective parabolic* CR *algebras such that:*

$$\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{q}' \cap \bar{\mathfrak{q}}'\,.$$

Then the CR *manifolds* $M = M(\mathfrak{g}, \mathfrak{q})$ *and* $M' = M(\mathfrak{g}, \mathfrak{q}')$ *are diffeomorphic, by a* **G***-equivariant diffeomorphism.*

PROOF. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g} = \mathfrak{g}'_+ = \mathfrak{q}' \cap \mathfrak{g}$ and maximally noncompact as a Cartan subalgebra of $\mathfrak{g}_+$.

Let $A, A' \in \mathfrak{h}_{\mathbb{R}}$ be such that $\mathfrak{q} = \mathfrak{q}_A$, $\mathfrak{q}' = \mathfrak{q}_{A'}$. We can assume that $\mathfrak{q} \neq \mathfrak{q}'$, so that A and A' are linearly independent. Then we set $A_t = A + t(A' - A)$, for $0 \leq t \leq 1$, so that $A_0 = A$ and $A_1 = A'$. Let us take a partition $t_0 = 0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that the rank of $\mathrm{ad}_{\hat{\mathfrak{g}}}(A_t)$ is constant for t in the open intervals $t_{j-1} < t < t_j$, $1 \leq j \leq m$, so that $\mathfrak{q}_{A_t} = \mathfrak{q}_{A_{t'}}$ for $t_{j-1} < t, t' < t_j$. Let $M_j = M(\mathfrak{g}, \mathfrak{q}_{A_{t_j}})$, for $0 \leq j \leq m$, and $N_j = M(\mathfrak{g}, \mathfrak{q}_{A_{(t_{j-1}+t_j)/2}})$, for $1 \leq j \leq m$. Since: $\mathfrak{q}_{A_{(t_{j-1}+t_j)/2}} \subset \mathfrak{q}_{A_{t_{j-1}}} \cap \mathfrak{q}_{A_{t_j}}$, there are **G**-equivariant maps:

$$M_{j-1} \xleftarrow{\;f_j\;} N_j \xrightarrow{\;F_j\;} M_j$$

for all $1 \leq j \leq m$. By Proposition 2.13, all these maps, being covering maps with connected fibers, are diffeomorphisms. Thus:

$$(F_m \circ f_m^{-1}) \circ (F_{m-1} \circ f_{m-1}^{-1}) \circ \cdots \circ (F_1 \circ f_1^{-1}) : M \longrightarrow M'$$

is a **G**-equivariant diffeomorphism. □

CHAPTER 3

Fit Weyl chambers and CR geometry of $M(\mathfrak{g},\mathfrak{q})$

Let $M(\mathfrak{g},\mathfrak{q})$ be a parabolic CR manifold and $(\vartheta,\mathfrak{h})$ a Cartan pair adapted to $(\mathfrak{g},\mathfrak{q})$. In this Chapter we introduce some special Weyl chambers, that we call S-fit and V-fit, and describe some geometric properties of M, namely fundamentality and weak nondegeneracy, in terms of properties of the simple roots associated to these special Weyl chambers.

We keep the notation of the preceding chapters, for roots, parabolic sets, Cartan decomposition, etc. In particular, we denote by $\sigma : \mathfrak{h}^*_{\mathbb{R}} \ni \alpha \to \bar{\alpha} \in \mathfrak{h}^*_{\mathbb{R}}$ the adjoint map of the restriction to $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}^- \oplus i\,\mathfrak{h}^+$ of the conjugation in $\hat{\mathfrak{g}}$ defined by the real form $\mathfrak{g}$. We say that a root α is *real* if $\bar{\alpha} = \alpha$, *imaginary* if $\bar{\alpha} = -\alpha$, *complex* if $\bar{\alpha} \neq \pm\alpha$ and denote by $\mathcal{R}_{\mathrm{re}}$, $\mathcal{R}_{\mathrm{im}}$ and $\mathcal{R}_{\mathrm{cp}}$ the sets of real, imaginary and complex roots, respectively. When α is imaginary, the eigenspace $\hat{\mathfrak{g}}^{\alpha}$ is contained either in $\hat{\mathfrak{k}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}$, or in $\hat{\mathfrak{p}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{p}$. In the first case we say that α is *compact*, in the second that α is *noncompact.* Thus $\mathcal{R}_{\mathrm{im}}$ is the disjoint union of the set $\mathcal{R}_{\bullet}$ of the compact and of the set $\mathcal{R}_*$ of the noncompact imaginary roots: $\mathcal{R}_{\mathrm{im}} = \mathcal{R}_{\bullet} \cup \mathcal{R}_*$.

3.1 S-fit and V-fit Weyl chambers

The conjugation σ defines an involution in $\mathfrak{h}^*_{\mathbb{R}}$ that belongs to the group $\mathbf{A}(\mathcal{R})$ of isometries of the root system $\mathcal{R}$. Vice versa, every involution σ in $\mathbf{A}(\mathcal{R})$ can be obtained from a conjugation with respect to a real form $\mathfrak{g}$ of $\hat{\mathfrak{g}}$. Note that, in general, σ does not uniquely determine the isomorphism class of $\mathfrak{g}$. Let us describe the structure of an arbitrary involution σ in $\mathbf{A}(\mathcal{R})$:

Theorem 3.1. *Let σ be an involution in $\mathbf{A}(\mathcal{R})$. Then there exist: a set of pairwise strongly orthogonal roots $\alpha_1, \ldots, \alpha_m$ in $\mathcal{R}$, with $\sigma(\alpha_j) = -\alpha_j$ for $j = 1,\ldots,m$, a Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$, and an involution $\jmath \in \mathbf{A}_C(\mathcal{R})$, with $\jmath(\alpha_i) = \alpha_i$, and hence commuting with s_{α_i}, for all $i = 1,\ldots,m$, such that:*

$$\sigma = \jmath \circ s_{\alpha_1} \circ \cdots \circ s_{\alpha_m}\,; \tag{3.1}$$

$$\alpha \in \mathcal{R}^+(C) \Longrightarrow \begin{cases} \textit{either } \sigma(\alpha) = -\alpha \\ \textit{or } \sigma(\alpha) \in \mathcal{R}^+(C) \end{cases} \tag{3.2}$$

Recall that two roots $\alpha,\beta \in \mathcal{R}$ are strongly orthogonal if $\alpha \pm \beta \notin \mathcal{R}$.

Proof. Let $F^-(\sigma) = \{\alpha \in \mathfrak{h}^*_{\mathbb{R}} \mid \sigma(\alpha) = -\alpha\}$, take a maximal subset $\alpha_1,\ldots,\alpha_m$ of pairwise orthogonal roots in $F^-(\sigma) \cap \mathcal{R}$ and consider $\jmath = \sigma \circ s_{\alpha_1} \circ \cdots \circ s_{\alpha_m}$. We have $\jmath(\alpha_i) = \sigma(-\alpha_i) = \alpha_i$ for all $i = 1,\ldots,m$. We claim that $\jmath(\alpha) \neq -\alpha$ for all $\alpha \in \mathcal{R}$. Indeed, if there was $\alpha \in \mathcal{R}$ with $\jmath(\alpha) = -\alpha$, from $(\alpha|\alpha_i) = (\jmath(\alpha)|\jmath(\alpha_i)) = -(\alpha|\alpha_i)$ we obtain that $(\alpha|\alpha_i) = 0$ for all $i = 1,\ldots,m$. Hence $s_{\alpha_i}(\alpha) = \alpha$ for all α and therefore $\sigma(\alpha) = \jmath(\alpha) = -\alpha$, contradicting the fact that $\alpha_1,\dots,\alpha_m$ was a maximal system of pairwise orthogonal roots in $\mathcal{R}\cap F^-(\sigma)$.

To obtain that $\alpha_1,\dots,\alpha_m$ are strongly orthogonal it suffices to choose the sequence $\alpha_1,\dots,\alpha_m$ with a maximal sum $\sum_{i=1}^m \|\alpha_i\|^2$. Indeed, if α_j and α_h are orthogonal, but not strongly orthogonal, then both $\alpha_j+\alpha_h$ and $\alpha_j-\alpha_h$ are roots. Setting $\alpha'_i=\alpha_i$ for $i\neq j,h$, and $\alpha'_j=\alpha_j+\alpha_h$, $\alpha'_h=\alpha_j-\alpha_h$, we obtain a new sequence $\alpha'_1,\dots,\alpha'_m$ of pairwise orthogonal roots in $F^-(\sigma)\cap\mathcal{R}$. It is contained in a maximal one and the inequality $\sum_{i=1}^m \|\alpha'_i\|^2 = \left(\sum_{i=1}^m\|\alpha_i\|^2\right)+\|\alpha_j\|^2+\|\alpha_h\|^2 > \sum_{i=1}^m\|\alpha_i\|^2$, contradicts the maximality of $\sum_{i=1}^m\|\alpha_i\|^2$.

We claim that there exists a Weyl chamber C such that:

$$(*) \qquad \jmath(\mathcal{R}^+(C)) = \mathcal{R}^+(C)\,.$$

Indeed $(*)$ is equivalent to $\jmath(\mathcal{B}(C))\subset\mathcal{R}^+(C)$. For a Weyl chamber C, denote by n_C the number of the elements in $\mathcal{R}^+(C)\cap\jmath(\mathcal{R}^+(C))$. Fix C with n_C maximum. If C does not satisfy $(*)$, take $\alpha\in\mathcal{B}(C)$ with $\jmath(\alpha)\notin\mathcal{R}^+(C)$ and consider the chamber $C'=s_\alpha(C)$. From $\mathcal{R}^+(C')=(\mathcal{R}^+(C)\setminus\{\alpha\})\cup\{-\alpha\}$ and $\jmath(-\alpha)\in\mathcal{R}^+(C)\setminus\{\alpha\}\subset\mathcal{R}^+(C')$, we obtain $n_{C'}=n_C+1$, contradicting our choice of C. Hence C satisfies $(*)$ and therefore also (3.2). This completes the proof. □

Using Theorem 3.1, we obtain the formula:

$$(3.3)\qquad \sigma(\beta)=\jmath(\beta)-\sum_{j=1}^m\left(\beta|\alpha_j^\vee\right)\alpha_j\,,\quad\text{with}\quad \alpha_j^\vee=2\alpha_j/\|\alpha_j\|^2\,,\quad\forall\beta\in\mathfrak{h}^*_{\mathbb{R}}\,.$$

Likewise, we have the following:

THEOREM 3.2. *Let σ be an involution in $\mathbf{A}(\mathcal{R})$. Then there exists a set of pairwise strongly orthogonal roots $\delta_1,\dots,\delta_m\in\mathcal{R}$, with $\sigma(\delta_j)=\delta_j$ for $j=1,\dots,m$, a Weyl chamber $C\in\mathfrak{C}(\mathcal{R})$ and an involution ϖ, that commutes with s_{δ_j}, satisfies $\varpi(\delta_j)=-\delta_j$ for all $j=1,\dots,m$, and transforms C into C^{opp}, such that:*

$$(3.4)\qquad \sigma=\varpi\circ s_{\delta_1}\circ\cdots\circ s_{\delta_m}\,,$$

$$(3.5)\qquad \alpha\in\mathcal{R}^+(C)\Longrightarrow\begin{cases}\text{either } \sigma(\alpha)=\alpha\\ \text{or } \sigma(\alpha)\in\mathcal{R}^-(C)\,.\end{cases}$$

PROOF. We take $\sigma'=s_0\circ\sigma$, where s_0 is the symmetry with respect to the origin of $\mathfrak{h}^*_{\mathbb{R}}$. By the preceding Theorem, $\sigma'=\jmath\circ s_{\delta_1}\circ\cdots\circ s_{\delta_m}$, where $\jmath\in\mathbf{A}_C(\mathcal{R})$ for some $C\in\mathfrak{C}(\mathcal{R})$, and $\delta_1,\dots,\delta_m$ is a maximal system of strongly orthogonal roots in $F^-(\sigma')\cap\mathcal{R}=\{\alpha\in\mathcal{R}\mid\sigma(\alpha)=\alpha\}$, with $\jmath(\delta_j)=\delta_j$. The statement follows by taking $\varpi=s_0\circ\jmath$. □

With ϖ and $\delta_1,\dots,\delta_m$ as in Theorem 3.2, we obtain the formula:

$$(3.6)\qquad \sigma(\beta)=\varpi(\beta)+\sum_{j=1}^m\left(\beta|\delta_j^\vee\right)\delta_j\,,\quad\text{with}\quad \delta_j^\vee=2\,\delta_j/\|\delta_j\|^2\,,\quad\forall\beta\in\mathfrak{h}^*_{\mathbb{R}}\,.$$

A Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$ that satisfies condition (3.2) (resp. (3.5)) is said to be[2] *S-adapted* (resp. *V-adapted*) to the conjugation σ.

For general CR algebras $(\mathfrak{g}, \mathfrak{q})$, there could be no adapted Cartan subalgebras $\mathfrak{h}$ that are either maximally compact or maximally noncompact in $\mathfrak{g}$. This is a major drawback in the classification of the orbits of $\mathbf{G}$ in $\mathfrak{M}$ (see e.g. the references in [BL02]), but, while discussing fundamentality, weak nondegeneracy and some topological properties, it turns out that the choice of $\mathfrak{h}$ is not as crucial as that of special Weyl chambers C in $\mathfrak{C}(\mathcal{R}, \mathcal{Q})$. In general $\mathfrak{C}(\mathcal{R}, \mathcal{Q})$ may not contain any Weyl chamber that is either S- or V-adapted to σ. In the following lemmas we describe chambers in $\mathfrak{C}(\mathcal{R}, \mathcal{Q})$ that are as close as possible to being S- or V-adapted.

LEMMA 3.3. *Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to $(\mathfrak{g}, \mathfrak{q})$. Then there exists a Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ that satisfies the equivalent conditions:*

(*i*) *If $\alpha \notin \mathcal{R}_{\mathrm{im}}$, $\alpha \succ_C 0$, and $\bar{\alpha} \prec_C 0$, then both α and $-\bar{\alpha}$ belong to $\mathcal{Q}^n$.*

(*ii*) *$\bar{\alpha} \succ_C 0$ for all $\alpha \in \mathcal{B}(C) \setminus (\Phi_C \cup \mathcal{R}_{\mathrm{im}})$.*

Assume that C satisfies the equivalent conditions (i) and (ii). Then:

(*iii*) *If moreover $\mathfrak{h}$ is maximally noncompact among the Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$, then $\mathcal{B}(C) \cap \mathcal{R}_* \subset \Phi_C$.*

PROOF. Choose $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ with a maximal $\mathcal{R}^+(C) \cap \sigma(\mathcal{R}^+(C))$. Then (*ii*) is satisfied. Indeed, if there was $\alpha \in \mathcal{B}(C) \setminus (\Phi_C \cup \mathcal{R}_{\mathrm{im}})$ with $\bar{\alpha} \prec_C 0$, we would take $C' = s_\alpha(C)$. Then $\mathcal{R}^+(C') = (\mathcal{R}^+(C) \setminus \{\alpha\}) \cup \{-\alpha\} \subset \mathcal{Q}$, so that $C' \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, and $\mathcal{R}^+(C') \cap \sigma(\mathcal{R}^+(C')) \supsetneqq \mathcal{R}^+(C) \cap \sigma(\mathcal{R}^+(C))$, yielding a contradiction. Clearly (*i*) $\Rightarrow$ (*ii*). Vice versa, if $\alpha = \sum_{\beta \in \mathcal{B}(C)} k_\alpha^\beta \beta \in \mathcal{R}^+(C)$ and $\bar{\alpha} \prec_C 0$, then either $\alpha \in \mathcal{R}_{\mathrm{im}}$, or else there is some $\beta \in \mathrm{supp}_C(\alpha) \cap \mathcal{R}_{\mathrm{cp}}$ with $\bar{\beta} \prec_C 0$; by (*ii*), we have $\beta \in \Phi_C$ and hence $\alpha \in \mathcal{Q}^n$. The same argument, applied to $-\bar{\alpha}$, shows that also $-\bar{\alpha} \in \mathcal{Q}^n$. This completes the proof of the equivalence (*i*) $\Leftrightarrow$ (*ii*).

Finally, if $\alpha \in (\mathcal{B}(C) \cap \mathcal{R}_*) \setminus \Phi_C$, both α and $\bar{\alpha} = -\alpha$ belong to $\mathcal{Q}$. Let $\Gamma = \{(X_\alpha, H_\alpha)_{\alpha \in \mathcal{R}}\}$ be a Chevalley system, as in [Bou05]. Then $T_\alpha = X_\alpha - X_{-\alpha} \in \mathfrak{p} \cap \mathfrak{g}_+$ (for this construction cf. [Sug59]) is a semisimple element of $\mathfrak{g}$ that commutes with all elements of $\mathfrak{h}^-$ and of $\mathfrak{j}^+ = \{H \in \mathfrak{h}^+ \mid \alpha(H) = 0\}$. Hence $\mathfrak{j} = \mathfrak{h}^- \oplus \mathbb{R}\, T_\alpha \oplus \mathfrak{j}^+$ is a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{g}^+$, with $\mathfrak{j}^- = \mathfrak{h}^- \oplus \mathbb{R}\, T_\alpha \supsetneqq \mathfrak{h}^-$. Thus, if $\mathfrak{h}^-$ is maximal, we have $\mathcal{B}(C_0) \cap \mathcal{R}_* \subset \Phi_C$. $\square$

An alternative construction of a Weyl chamber C satisfying (*i*) and (*ii*) of Lemma 3.3 is the following (which is a particular case of a general construction that will be described in Chapter 4). Fix a Weyl chamber $C_0 \in \mathfrak{C}(\mathcal{R})$ that is S-adapted to σ (recall that this means $\sigma(\mathcal{R}^+(C_0) \setminus \mathcal{R}_{\mathrm{im}}) \subset \mathcal{R}^+(C_0)$), and consider the Borel subalgebra:

$$\mathfrak{b}_{C_0} = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \mathcal{R}^+(C_0)} \hat{\mathfrak{g}}^\alpha \subset \hat{\mathfrak{g}}\,.$$

Then $\mathfrak{b} = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \mathfrak{b}_{C_0})$ is a Borel subalgebra of $\hat{\mathfrak{g}}$, corresponding to a Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ that satisfies (*i*) and (*ii*) of Lemma 3.3.

[2] If we choose $\mathfrak{h}$ maximally noncompact, then in an S-adapted Weyl chamber C the conjugation can be described by a Satake diagram; if instead we take $\mathfrak{h}$ with a maximal compact part, in a V-adapted Weyl chamber the conjugation is described by a Vogan diagram (see e.g. [Ara62], [Kna02].

A Weyl chamber $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ that satisfies the equivalent conditions (i) and (ii) of Lemma 3.3 is called *S-fit* to $(\mathfrak{g},\mathfrak{q})$.

With arguments similar to those employed to prove Lemma 3.3, we obtain:

LEMMA 3.4. *Let $(\vartheta,\mathfrak{h})$ be a Cartan pair adapted to $(\mathfrak{g},\mathfrak{q})$. Then there exists a Weyl chamber $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ that satisfies the equivalent conditions:*

(iv) *If $\alpha \notin \mathcal{R}_{\mathrm{re}}$, $\alpha \succ_C 0$, and $\bar{\alpha} \succ_C 0$, then both α and $\bar{\alpha}$ belong to $\mathcal{Q}^n$.*

(v) *$\bar{\alpha} \prec_C 0$ for all $\alpha \in \mathcal{B}(C) \setminus (\Phi_C \cup \mathcal{R}_{\mathrm{re}})$.*

Assume that C satisfies the equivalent conditions (iv) and (v). Then:

(vi) *If moreover $\mathfrak{h}$ is maximally compact among the Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$, then $\mathcal{B}(C) \cap \mathcal{R}_{\mathrm{re}} \subset \Phi_C$.* $\square$

A Weyl chamber $C \in \mathfrak{C}(\mathcal{Q},\mathcal{R})$ that satisfies the equivalent conditions (iv) and (v) of Lemma 3.4 is called *V-fit* to $(\mathfrak{g},\mathfrak{q})$.

V-fit Weyl chambers can also be obtained as follows: if $\mathfrak{b}_{C_0} = \hat{\mathfrak{h}} \oplus \sum_{\alpha\in\mathcal{R}^+(C_0)} \hat{\mathfrak{g}}^\alpha$ is the Borel subalgebra associated to a Weyl chamber C_0 that is V-adapted to the conjugation σ, then $\mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \mathfrak{b}_{C_0})$ is the Borel subalgebra associated to a Weyl chamber $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ that is V-fit to $(\mathfrak{g},\mathfrak{q})$.

3.2 Fundamental parabolic CR algebras

Fundamental parabolic CR algebras can be more easily characterized when described in terms of a S-fit Weyl chamber. Indeed we have:

THEOREM 3.5. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic CR algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ adapted to $(\mathfrak{g},\mathfrak{q})$ and fix an S-fit Weyl chamber $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ for $(\mathfrak{g},\mathfrak{q})$. Then $(\mathfrak{g},\mathfrak{q})$ is fundamental if and only if:*

$$\forall \alpha_0 \in \Phi_C \cap \bar{\mathcal{Q}}^n \begin{cases} \text{either } \exists \beta \in \mathcal{B}(C)\setminus\Phi_C \text{ with } \alpha_0 \in \operatorname{supp}_C(\bar{\beta}), \\ \text{or } \exists\beta\in\Phi_C \text{ with } \bar{\beta}\in\mathcal{R}^-(C) \text{ and } \alpha_0\in\operatorname{supp}_C(\bar{\beta}). \end{cases} \tag{3.7}$$

PROOF. The parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$ is fundamental if, and only if, there is no complex parabolic subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q}+\bar{\mathfrak{q}} \subset \mathfrak{q}' \subsetneq \hat{\mathfrak{g}}$. All complex parabolic $\mathfrak{q}'$ that contain $\mathfrak{q}$ are of the form $\mathfrak{q}' = \mathfrak{q}_{\Psi_C}$ for some set of simple roots $\Psi_C \subset \Phi_C$. We can limit ourselves to consider the cases where $\Psi_C = \{\alpha_0\}$ for some $\alpha_0 \in \Phi_C$. Thus we obtain:

$$(\mathfrak{g},\mathfrak{q}_{\Phi_C}) \text{ is not fundamental} \iff \begin{cases} \exists \alpha_0\in\Phi_C \text{ such that} \\ \{\beta\in\mathcal{R} \mid \beta \succeq_C \alpha_0\} \subset \bar{\mathcal{Q}}^n. \end{cases} \tag{3.8}$$

Indeed, $\mathfrak{q}+\bar{\mathfrak{q}} \subset \mathfrak{q}'$ if and only if $\mathfrak{q}'^n \subset \mathfrak{q}^n \cap \bar{\mathfrak{q}}^n$. Thus it suffices to further restrict our consideration to simple roots $\alpha_0 \in \Phi_C \cap \bar{\mathcal{Q}}^n$ and check whether $\mathcal{F} = \{\beta\in\mathcal{R} \mid \beta\succeq_C \alpha_0\}$ is contained or not in $\bar{\mathcal{Q}}^n$.

First we show that condition (3.7) is sufficient. Let $\alpha_0 \in \Phi_C \cap \bar{\mathcal{Q}}^n$. If $\beta \in \mathcal{B}(C)\setminus\Phi_C$ and $\alpha_0 \in \operatorname{supp}_C(\bar{\beta})$, then $\bar{\beta} \succeq_C \alpha_0$ by the assumption that C is S-fit to $(\mathfrak{g},\mathfrak{q})$; hence $\bar{\beta} \in \mathcal{F}\setminus\bar{\mathcal{Q}}^n$. Likewise, if $\beta\in\Phi_C$, $\bar{\beta}\prec_C 0$ and $\alpha_0\in\operatorname{supp}_C(\bar{\beta})$, then $-\bar{\beta}\in\mathcal{F}\setminus\bar{\mathcal{Q}}^n$. This completes the proof of sufficiency.

To prove that (3.7) is also necessary, fix again $\alpha_0 \in \Phi_C\cap\bar{\mathcal{Q}}^n$. If $(\mathfrak{g},\mathfrak{q})$ is fundamental, then $\mathcal{F}\not\subset\bar{\mathcal{Q}}^n$, and hence there is a root α with $\alpha\succeq_C\alpha_0$ and $\bar{\alpha}\notin\mathcal{Q}^n$. If $\bar{\alpha}\succ_C 0$, then $\operatorname{supp}_C(\bar{\alpha})\cap\Phi_C=\emptyset$. Since $\alpha_0\in\operatorname{supp}_C(\alpha)\subset\bigcup_{\beta\in\operatorname{supp}_C(\bar{\alpha})}\operatorname{supp}_C(\bar{\beta})$, there is at least a $\beta \in \mathcal{B}(C) \setminus \Phi_C$ with $\alpha_0 \in \operatorname{supp}_C(\bar{\beta})$. Assume now that α_0 does not belong to $\operatorname{supp}_C(\bar{\beta})$ for any $\beta \in \mathcal{B} \setminus \Phi_C$. Then, for a root $\alpha \succ_C \alpha_0$ that does not belong to $\bar{\mathcal{Q}}^n$, we have $\bar{\alpha} \prec_C 0$. From $\alpha_0 \in \bigcup_{\beta \in \operatorname{supp}_C(\bar{\alpha})} \operatorname{supp}_C(\bar{\beta})$ we obtain that $\alpha_0 \in \operatorname{supp}_C(\bar{\beta})$ for some $\beta \in \mathcal{B}(C)$ with $\bar{\beta} \prec_C 0$, and hence in Φ_C. $\square$

Theorem 3.5 provides a criterion, only involving the conjugation of the simple roots of an S-fit Weyl chamber, for an effective parabolic CR algebra to be totally real. We found convenient to formulate the criterion for an arbitrary Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$.

Proposition 3.6. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra, $\mathfrak{h}$ an adapted Cartan subalgebra, and $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$. A necessary and sufficient condition in order that $(\mathfrak{g}, \mathfrak{q})$ be totally real is that:*

$$\begin{cases} \alpha \in \mathcal{B}(C) \setminus \Phi_C & \Longrightarrow \quad \operatorname{supp}_C(\bar{\alpha}) \cap \Phi_C = \emptyset \\ \quad \alpha \in \Phi_C & \Longrightarrow \quad \bar{\alpha} \in \mathcal{R}^+(C)\,. \end{cases} \tag{3.9}$$

Proof. The case where $\mathcal{Q} = \mathcal{R}$ is trivial. Assume that $\mathcal{Q} \neq \mathcal{R}$. The first condition in (3.9) implies that $\mathcal{B}(C) \setminus \Phi_C \subset \bar{\mathcal{Q}}^r$. Hence $\mathcal{Q}^r = \bar{\mathcal{Q}}^r$. In particular, if $\alpha \in \mathcal{Q}^n$, then $\bar{\alpha} \in \mathcal{R} \setminus \mathcal{Q}^r$. Then, since $\mathcal{Q}^n = \mathcal{R}^+(C) \setminus \mathcal{Q}^r$, the second condition implies that $\bar{\alpha} \in \mathcal{Q}^n$ for all $\alpha \in \Phi_C$. Hence $\Phi_C \subset \bar{\mathcal{Q}}^n$. Therefore $\mathcal{B}(C) \subset \overline{\mathcal{Q}}$, so that we also have $\overline{\mathcal{Q}}^n = \mathcal{Q}^n$ and hence $\mathcal{Q} = \overline{\mathcal{Q}}$. The condition is obviously also necessary. $\square$

Proposition 3.6 prompts a recursive method to construct the totally real basis of the canonical $\mathfrak{g}$-equivariant fibration $(\mathfrak{g}, \mathfrak{q}) \longrightarrow (\mathfrak{g}, \mathfrak{q}')$, with totally real basis and fundamental fiber.

After taking any $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, we define recursively:

$$\begin{cases} \Upsilon_C^{(0)} & = \{\alpha \in \Phi_C \mid \bar{\alpha} \in \mathcal{R}^+(C)\} \\ \Upsilon_C^{(1)} & = \Upsilon_C^{(0)} \setminus \bigcup_{\alpha \in \mathcal{B}(C) \setminus \Upsilon_C^{(0)}} \operatorname{supp}_C(\bar{\alpha}) \\ \Upsilon_C^{(h+1)} & = \Upsilon_C^{(h)} \setminus \bigcup_{\alpha \in \mathcal{B}(C) \setminus \Upsilon_C^{(h)}} \operatorname{supp}_C(\bar{\alpha}) \quad \text{for} \quad h \geq 1 \\ \Upsilon_C & = \bigcap_{h \geq 0} \Upsilon_C^{(h)} \quad (\text{finite intersection})\,. \end{cases} \tag{3.10}$$

One easily verifies, using the previous results, that:

Proposition 3.7. *The natural $\mathfrak{g}$-equivariant fibration $(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Upsilon_C})$ is the fundamental reduction of $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$. In particular, $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is fundamental if and only if $\Upsilon_C = \emptyset$.* $\square$

3.3 Weakly nondegenerate parabolic CR algebras

We turn now to weak nondegeneracy for an effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$. We shall see that this property can be better examined in terms of V-fit Weyl chambers. We start with a Lemma:

LEMMA 3.8. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic CR algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ adapted to $(\mathfrak{g},\mathfrak{q})$, and $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ a V-fit Weyl chamber for $(\mathfrak{g},\mathfrak{q})$. Let $\Phi_C = \mathcal{B}(C) \cap \mathcal{Q}^n$ and $\mathfrak{q}' = \mathfrak{q}_{\Psi_C} \supset \mathfrak{q}$ (cf. (2.15) for the notation), with $\Psi_C \subset \Phi_C$. Then the $\mathfrak{g}$-equivariant fibration of CR algebras $(\mathfrak{g},\mathfrak{q}) \to (\mathfrak{g},\mathfrak{q}')$ is a CR fibration with a totally complex fiber if and only if*

$$\bar{\alpha} \prec_C 0 \quad \forall \alpha \in \Phi_C \setminus \Psi_C\,. \tag{3.11}$$

PROOF. The $\mathfrak{g}$-equivariant CR homomorphism $(\mathfrak{g},\mathfrak{q}) \longrightarrow (\mathfrak{g},\mathfrak{q}')$ is a CR fibration with a totally complex fiber if, and only if, $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$. Thus, we need to show that these inclusions are equivalent to (3.11) when $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ satisfies (iv) and (v) of Lemma 3.4.

Clearly, it suffices to consider the case where the difference $\Phi_C \setminus \Psi_C$ consists of a single simple root α_0. We shall assume in the following that $\Phi_C \setminus \Psi_C = \{\alpha_0\}$.

First we prove that, if $\bar{\alpha}_0 \prec_C 0$, then $\mathcal{Q}' = \mathcal{Q}_{\Psi_C} \subset \mathcal{Q} \cup \bar{\mathcal{Q}}$. Assume by contradiction that there is $\beta \in \mathcal{Q}' \setminus \big(\mathcal{Q} \cup \bar{\mathcal{Q}}\big)$. Then $-\beta, -\bar{\beta} \in \mathcal{Q}^n \subset \mathcal{R}^+(C)$. Moreover $\beta \in \mathcal{Q}'^r$, because $\mathcal{Q}'^n \subset \mathcal{Q}^n \subset \big(\mathcal{Q} \cup \bar{\mathcal{Q}}\big)$. Thus $\operatorname{supp}_C(\beta) \cap \Psi_C = \emptyset$, and hence $\operatorname{supp}_C(\beta) \cap \Phi_C = \{\alpha_0\}$, because $-\beta \in \mathcal{Q}^n$. As C is V-fit, $\bar{\alpha} \prec_C 0$ for all $\alpha \in \operatorname{supp}_C(\beta) \setminus \mathcal{R}_{\mathrm{re}}$. Since β is not real, this implies that $\bar{\beta} \succ_C 0$, giving a contradiction. Hence $\mathcal{Q}' \subset \big(\mathcal{Q} \cup \overline{\mathcal{Q}}\big)$, and this proves that $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$ when $\bar{\alpha}_0 \in \mathcal{R}^-(C)$.

Vice versa, assume that $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$. In particular, $-\alpha_0 \in \bar{\mathcal{Q}}$. If $\bar{\alpha}_0 \in \mathcal{Q}^r$, then $\bar{\alpha}_0$ and $\alpha_0 = \bar{\bar{\alpha}}_0$ belong to opposite cones $\mathcal{R}^{\pm}(C)$ and thus $\bar{\alpha}_0 \prec_C 0$. When $\bar{\alpha}_0 \notin \mathcal{Q}^r$, we have $-\bar{\alpha}_0 \in \mathcal{Q}^n \subset \mathcal{R}^+(C)$, and thus $\bar{\alpha}_0 \prec_C 0$. The proof is complete.□

Using Lemma 3.8, we obtain a characterization of weakly nondegenerate CR algebras:

THEOREM 3.9. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic CR algebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ adapted to $(\mathfrak{g},\mathfrak{q})$. Let $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ be V-fit to $(\mathfrak{g},\mathfrak{q})$. Then $(\mathfrak{g},\mathfrak{q})$ is weakly nondegenerate if and only if:*

$$\bar{\alpha} \succ_C 0 \qquad \forall \alpha \in \Phi_C\,. \tag{3.12}$$

PROOF. Fix a Weyl chamber $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ for which (iv) and (v) of Lemma 3.4 are valid. By Lemma 3.8, the necessary and sufficient condition for $(\mathfrak{g},\mathfrak{q})$ to be weakly degenerate is that there exists $\alpha_0 \in \Phi_C$ contradicting (3.12). □

COROLLARY 3.10. *Let $(\mathfrak{g},\mathfrak{q})$ be a weakly nondegenerate parabolic CR algebra. Let $\mathcal{Q}$ be the parabolic set associated to $\mathfrak{q}$ in $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}},\hat{\mathfrak{h}})$, for an admissible Cartan subalgebra $\mathfrak{h}$ of $(\mathfrak{g},\mathfrak{q})$. Then $(\mathfrak{g},\mathfrak{q})$ is totally real if and only if $\bar{\mathcal{Q}}^n \cap \mathcal{Q}^r = \emptyset$.*

PROOF. Choose a V-fit Weyl chamber $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ for $(\mathfrak{g},\mathfrak{q})$. The second condition in (3.9) being automatically satisfied because C is V-fit and $(\mathfrak{g},\mathfrak{q})$ weakly nondegenerate, we observe that the first line in (3.9) is equivalent to the condition that $\bar{\mathcal{Q}}^n \cap \mathcal{Q}^r = \emptyset$. □

3.4 Cross-marked diagrams and examples

In the examples that will follow, here and in the next chapters, to describe specific parabolic CR algebras, we shall utilize *cross-marked diagrams.* They are Dynkin diagrams, where the simple roots in $\mathcal{B}(C)$, for a Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, are indicated by:

- $\circ$ if the root is real;
- $\bullet$ if the root is compact imaginary;
- $\circledast$ if the root is noncompact imaginary;
- $\oplus$ if the root is complex and its conjugate belongs to $\mathcal{R}^+(C)$;
- $\ominus$ if the root is complex and its conjugate belongs to $\mathcal{R}^-(C)$

and we cross-mark the roots in Φ_C. Some extra information about the action of σ on the simple roots in $\mathcal{B}(C)$ is provided by some arrows and dotted arrows joining pairs of simple roots that have the same, or opposite, restriction to $\mathfrak{h}^- \subset \mathfrak{h}_{\mathbb{R}}$, or that are the edges of segments whose nodes are the support of real or imaginary roots. However, as we shall see, the most important information is carried by the colors of the nodes.

EXAMPLE 3.1. Consider the CR manifold M consisting of 3-planes ℓ_3 of $\mathbb{C}^6$ with $\dim_{\mathbb{C}}(\ell_3 \cap \bar{\ell}_3) = 1$. This is an orbit of $\mathbf{SL}(6, \mathbb{R})$ in the Grassmannian of 3-planes of $\mathbb{C}^6$. Let $\varepsilon_1, \ldots, \varepsilon_6$ be the canonical basis of $\mathbb{C}^6$. It is convenient to represent the Lie algebra $\mathfrak{sl}(6, \mathbb{R})$ in the basis

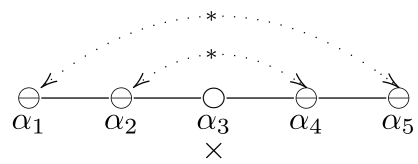
$$e_1 = \varepsilon_1 + i\varepsilon_6\,,\ e_2 = \varepsilon_2 + i\varepsilon_5\,,\ e_3 = \varepsilon_3\,,\ e_4 = \varepsilon_4\,,\ e_5 = \varepsilon_2 - i\varepsilon_5\,,\ e_6 = \varepsilon_1 - i\varepsilon_6\,.$$

Then M is the orbit of the 3 plane generated by e_1, e_2, e_3.

We can take $\mathfrak{h}_{\mathbb{R}}$ to be the set of real diagonal matrices. The parabolic $\mathfrak{q}$ is $\mathfrak{q}_A$ for $A = \mathrm{diag}(1, 1, 1, -1, -1, -1)$. The Weyl chamber C corresponding to the canonical basis $\alpha_i = e_i - e_{i+1}$ $(i = 1, \ldots, 5)$ belongs to $\mathfrak{C}(\mathcal{R}, \mathcal{Q})$ and is V-fit. We have indeed

$$\begin{cases} \bar{\alpha}_1 = e_6 - e_5 = -\alpha_5 \\ \bar{\alpha}_2 = e_5 - e_3 = -(\alpha_3 + \alpha_4) \\ \bar{\alpha}_3 = e_3 - e_4 = \alpha_3 \\ \bar{\alpha}_4 = e_4 - e_2 = -(\alpha_2 + \alpha_3) \\ \bar{\alpha}_5 = e_2 - e_1 = -\alpha_1 \end{cases}$$

so that the associated diagram is:

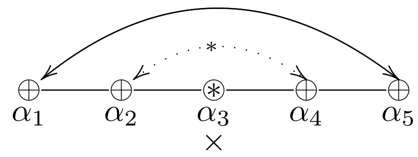


We have $\Phi_C = \{\alpha_3\}$, $\sigma\left([\mathcal{R}^+(C) \cap \mathcal{Q}^r] \setminus \mathcal{R}_{\mathrm{re}}\right) \subset \mathcal{R}^-(C)$, $\bar{\Phi}_C = \Phi_C \subset \mathcal{R}^+(C)$. Hence, by Lemma 3.4, $(\mathfrak{g}, \mathfrak{q})$ is weakly nondegenerate. Our M is a CR manifold of hypersurface type $(8, 1)$. Its Levi form has two positive, two negative and four zero eigenvalues. Hence M is fundamental and weakly, but not strictly nondegenerate.

We obtain an S-fit chamber by describing $\mathfrak{sl}(6, \mathbb{R})$ in the basis:

$$e_1 = \varepsilon_1\,,\ e_2 = \varepsilon_2 + i\varepsilon_5\,,\ e_3 = \varepsilon_3 + i\varepsilon_4\,,\ e_4 = \varepsilon_3 - i\varepsilon_4\,,\ e_5 = \varepsilon_2 - i\varepsilon_5\,,\ e_6 = \varepsilon_6\,.$$

One verifies that, with the basis of simple roots $\alpha_i = e_i - e_{i+1}$ $(1 \leq i \leq 5)$ the corresponding diagram is:



We have $\bar{\alpha}_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \succ_C \alpha_3$, in accordance with the fact that $(\mathfrak{g},\mathfrak{q})$ is fundamental.

Example 3.2. Consider the CR manifold M^* consisting of all flags $\ell_1 \subset \ell_2 \subset \ell_4 \subset \ell_5 \subset \mathbb{C}^6$ (the subscript is the dimension of the linear subspace), with

$$\dim_{\mathbb{C}}(\ell_1 \cap \bar{\ell}_1) = 0\,, \quad \ell_2 = \ell_1 + \bar{\ell}_1\,, \quad \dim_{\mathbb{C}}(\ell_4 \cap \bar{\ell}_4) = 3\,, \quad \dim_{\mathbb{C}}(\ell_5 \cap \bar{\ell}_5) = 3\,.$$

We note that M^* is not connected. Thus an orbit of $\mathbf{SL}(6,\mathbb{R})$ in M^* will be a connected component M of M^*. We can better describe such an M in terms of the choice of a suitable basis of $\mathbb{C}^6$. Denoting by $\varepsilon_1,\dots,\varepsilon_6$ the canonical basis of $\mathbb{C}^6$, we introduce the basis:

$$e_1 = \varepsilon_1 + i\varepsilon_2,\ e_2 = \varepsilon_1 - i\varepsilon_2,\ e_3 = \varepsilon_3,\ e_4 = \varepsilon_4 + i\varepsilon_6,\ e_5 = \varepsilon_5,\ e_6 = \varepsilon_4 - i\varepsilon_6\,.$$

Our M is the orbit, under the action of $\mathbf{SL}(6,\mathbb{R})$, of the flag

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1, e_2, e_3, e_4\rangle \subset \langle e_1, e_2, e_3, e_4, e_5\rangle\,.$$

We can take the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(6,\mathbb{R})$ that is described, in the basis $e_1,\dots,e_6$, by:

$$\mathfrak{h} = \{\mathrm{diag}(\lambda,\bar{\lambda},a,\mu,b,\bar{\mu}) \mid \lambda,\mu \in \mathbb{C},\ a,b \in \mathbb{R}\,,\ 2\mathrm{Re}(\lambda+\mu)+a+b=0\}$$

and consider the corresponding root system $\mathcal{R}$ of $\mathfrak{sl}(6,\mathbb{C})$ with respect to $\hat{\mathfrak{h}}$. We identify $\mathfrak{h}_{\mathbb{R}}$ with the space of real diagonal matrices in $\mathfrak{sl}(6,\mathbb{C})$ and the e_h's to the evaluation of the h-th diagonal entry of $H \in \mathfrak{h}_{\mathbb{R}}$. Let $(\mathfrak{g},\mathfrak{q})$ be the effective parabolic CR algebra associated to M and $\mathcal{Q}$ the parabolic set of $\mathfrak{q}$. Then the Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$ with basis $\mathcal{B}(C) = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq 5\}$ belongs to $\mathfrak{C}(\mathcal{R},\mathcal{Q})$ and $\Phi_C = \{\alpha_1,\alpha_2,\alpha_4,\alpha_5\}$. We have:

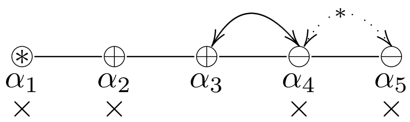
$$\begin{cases} \bar{\alpha}_1 = -\alpha_1 \\ \bar{\alpha}_2 = \alpha_1+\alpha_2 \\ \bar{\alpha}_3 = \alpha_3+\alpha_4+\alpha_5 \\ \bar{\alpha}_4 = -\alpha_5 \\ \bar{\alpha}_5 = -\alpha_4\,. \end{cases}$$

Thus (3.10) yields in this case:

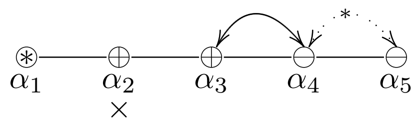
$$\begin{cases} \Upsilon_C^{(0)} = \{\alpha_2\} \\ \Upsilon_C^{(1)} = \Upsilon_C^{(0)} = \Upsilon_C\,. \end{cases}$$

Thus the basis of the fundamental reduction is the Grassmannian of 2-planes ℓ_2 in $\mathbb{C}^6$ with $\bar{\ell}_2 = \ell_2$. We give below the diagrams for $(\mathfrak{g},\mathfrak{q})$, for its fundamental reduction $(\mathfrak{g}, \mathfrak{q}')$, and for the fiber $(\mathfrak{g}, \mathfrak{q}'')$, (recall that the basis of the fundamental reduction of a parabolic is still parabolic).

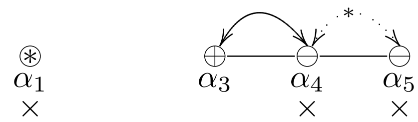
$(\mathfrak{g}, \mathfrak{q})$



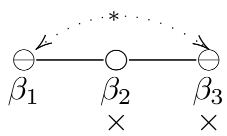
$(\mathfrak{g}, \mathfrak{q}')$



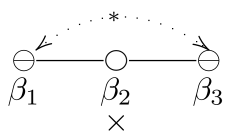
$(\mathfrak{g}, \mathfrak{q}'')$



The fiber M'' is the product of a complex disk (a connected component of the set of $\mathbb{CP}^1 \setminus \mathbb{RP}^1 \simeq S^2 \setminus S^1$) and a connected CR manifold N, consisting of flags in $\mathbb{C}^6/\mathbb{C}^2$: these can be identified to pairs $\ell_2 \subset \ell_3 \subset \mathbb{C}^4$ such that $\dim_{\mathbb{C}}(\ell_2 \cap \bar{\ell}_2) = 1$, $\ell_3 \not\supset \ell_2 + \bar{\ell}_2$ and $\dim_{\mathbb{C}}(\ell_3 \cap \bar{\ell}_3) = 2$. It is convenient to utilize the basis e_4, e_3, e_5, e_6 of $\mathbb{C}^4 \simeq \langle e_3, e_4, e_5, e_6 \rangle \subset \mathbb{C}^6$. Then $\beta_1 = e_4 - e_3, \beta_2 = e_3 - e_5, \beta_3 = e_5 - e_6$ is the basis related to a Weyl chamber C_N in which the diagram associated to the parabolic CR algebra of N is:



Since C_N is V-fit, we see from this diagram that N is weakly degenerate. The basis N' of its weakly nondegenerate reduction, consists of planes ℓ_2 of $\mathbb{C}^4$ with $\dim_{\mathbb{C}}(\ell_2 \cap \bar{\ell}_2) = 1$, and corresponds to the diagram:



The parabolic CR manifold N' is of hypersurface type $(3, 1)$, with a degenerate Levi form of signature $(1, -1, 0)$. Thus it is 1-pseudoconcave (see e.g. [HN96]) and weakly, but not strictly, nondegenerate. The fiber F of the $\mathbf{SL}(4, \mathbb{R})$-equivariant weakly nondegenerate reduction $N \longrightarrow N'$, that lies above a given 2-plane ℓ_2 with $\dim_{\mathbb{C}}(\ell_2 + \bar{\ell}_2) = 3$, is isomorphic to the pencil of 3-planes in $\mathbb{C}^4$, that contain ℓ_2 and are distinct from $(\ell_2 + \bar{\ell}_2)$. Thus $F \simeq \mathbb{CP}^1 \setminus \{\text{a point}\} \simeq \mathbb{C}$. Note that the CR algebra that is naturally associated to the fiber fails in this case to be parabolic. In fact, the CR algebra $(\mathfrak{g}^\sharp, \mathfrak{q}^\sharp)$ of the fiber is given by:

$$\mathfrak{g}^\sharp = \left\{ \begin{pmatrix} \lambda & 0 & \bar{\zeta} & 0 \\ z & h & s & \bar{z} \\ 0 & 0 & k & 0 \\ 0 & 0 & \zeta & \bar{\lambda} \end{pmatrix} \middle| \begin{array}{l} \lambda, z, \zeta \in \mathbb{C} \\ h, k, s \in \mathbb{R} \\ h + k + 2\mathrm{Re}\lambda = 0 \end{array} \right\},$$

$$\mathfrak{q}^\sharp = \left\{ \begin{pmatrix} \lambda_1 & 0 & \zeta_2 & 0 \\ 0 & \lambda_2 & \theta & z_2 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & \zeta_1 & \lambda_4 \end{pmatrix} \middle| \begin{array}{l} \lambda_i, z_i, \zeta_i, \theta \in \mathbb{C} \\ \sum_{i=1}^4 \lambda_i = 0 \end{array} \right\},$$

where both $\mathfrak{g}^\sharp$ and its complexification $\hat{\mathfrak{g}}^\sharp$ are nilpotent.

CHAPTER 4

Canonical fibrations over a parabolic CR manifold

Given an effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$, we constructed in the previous chapters new parabolic complex subalgebras $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q} \subset \mathfrak{q}'$, to obtain smooth fibrations $M(\mathfrak{g}, \mathfrak{q}) \longrightarrow M(\mathfrak{g}, \mathfrak{q}')$, namely with a weakly nondegenerate basis and totally complex fibers, and with totally real basis and fundamental fibers. Here, we consider smooth fibrations $M(\mathfrak{g}, \mathfrak{q}') \longrightarrow M(\mathfrak{g}, \mathfrak{q})$, obtained by choosing special parabolic $\mathfrak{q}' \subset \hat{\mathfrak{g}}$ with $\mathfrak{q}' \subset \mathfrak{q}$, and that will be useful to find suitable Weyl chambers in $\mathfrak{C}(\mathcal{R}, \mathcal{Q})$ and to investigate the topology of the general $M(\mathfrak{g}, \mathfrak{q})$.

We keep the notation of the preceding chapters. In particular, we fix a Cartan pair $(\vartheta, \mathfrak{h})$, assuming that it is adapted to all the parabolic CR algebras that we shall consider.

We have:

Proposition 4.1. *Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebras $(\mathfrak{g}, \mathfrak{q})$ and $(\mathfrak{g}, \mathfrak{e})$. Then:*

$$\mathfrak{l} = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \mathfrak{e}) \tag{4.1}$$

is a parabolic complex Lie subalgebra of $\hat{\mathfrak{g}}$ with:

$$\begin{cases} \mathfrak{h} \subset \mathfrak{l} \\ \mathfrak{l}^n = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \mathfrak{e}^n) \supset \mathfrak{q}^n \\ \mathfrak{l}^r = \mathfrak{q}^r \cap \mathfrak{e}^r \subset \mathfrak{q}^r \\ \mathfrak{l} = \mathfrak{l}^n \oplus \mathfrak{l}^r \subset \mathfrak{q}\,. \end{cases} \tag{4.2}$$

Proof. Let $\mathcal{Q}$, $\mathcal{E}$ be the parabolic sets in $\mathcal{R}$ corresponding to the complex parabolic Lie subalgebras $\mathfrak{q}$, $\mathfrak{e}$, respectively. To prove that $\mathfrak{l}$ is parabolic, we need to prove that

$$\mathcal{L} = \mathcal{Q}^n \cup (\mathcal{Q}^r \cap \mathcal{E}) = \mathcal{Q}^n \cup (\mathcal{Q}^r \cap \mathcal{E}^r) \cup (\mathcal{Q}^r \cap \mathcal{E}^n) \tag{4.3}$$

is a parabolic subset of $\mathcal{R}$.

Let $A, B \in \mathfrak{h}_{\mathbb{R}}$ be such that $\mathcal{Q} = \mathcal{Q}_A$, $\mathcal{E} = \mathcal{Q}_B$ and fix $\varepsilon > 0$ so small that $\varepsilon\,|\alpha(B)| < \alpha(A)$ for all $\alpha \in \mathcal{Q}^n$. Then we claim that

$$\mathcal{L} = \{\alpha \in \mathcal{R} \,|\, \alpha(A + \varepsilon B) \geq 0\}\,.$$

Indeed, when $\alpha \in \mathcal{Q}^n$, then $\alpha(A + \varepsilon B) \geq \alpha(A) - \varepsilon\,|\alpha(B)| > 0$; when $\alpha \notin \mathcal{Q}$, then $-\alpha \in \mathcal{Q}^n$ and hence $\alpha(A+\varepsilon B) < 0$; finally for $\alpha \in \mathcal{Q}^r$, we have $\alpha(A+\varepsilon B) = \varepsilon\,\alpha(B)$ and hence $\alpha \in \mathcal{L}$ if and only if $\alpha \in \mathcal{E}$.

The proof of (4.2) is straightforward. □

Vice versa, when $\mathfrak{l}$ is a complex parabolic subalgebra of $\hat{\mathfrak{g}}$ with $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{q}$, then $\mathfrak{l} = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \mathfrak{l})$, so that (4.1) gives a way to construct all complex parabolic subalgebras of $\hat{\mathfrak{g}}$ with $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{q}$.

4.1 The canonical CR lift

We give a first application of the above construction.

PROPOSITION 4.2. *Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$. Consider the complex parabolic Lie subalgebra*

$$\mathfrak{w} = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \vartheta(\mathfrak{q})) \tag{4.4}$$

of $\hat{\mathfrak{g}}$. The σ-invariant reductive subalgebra $\mathfrak{q}^r \cap \bar{\mathfrak{q}}^r$ is a complement in $\mathfrak{w}$ of its nilradical $\mathfrak{w}^n$. We have:

$$\begin{cases} \mathfrak{w}^r = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^r, \quad \mathfrak{w}^n = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \vartheta(\mathfrak{q}^n)) \supset \mathfrak{q}^n, \quad \mathfrak{w} = \mathfrak{w}^n \oplus \mathfrak{w}^r \subset \mathfrak{q}, \\ \mathfrak{w}^n \cap \bar{\mathfrak{w}}^n = \mathfrak{q}^n \cap \bar{\mathfrak{q}}^n, \quad \mathfrak{w}^r = \bar{\mathfrak{w}}^r, \quad \mathfrak{w} \cap \bar{\mathfrak{w}} = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^r \oplus \mathfrak{q}^n \cap \bar{\mathfrak{q}}^n, \end{cases} \tag{4.5}$$

and $\mathfrak{w}$ is the smallest parabolic subalgebra of $\hat{\mathfrak{g}}$ that satisfies the conditions:

$$\mathfrak{q}^r \cap \bar{\mathfrak{q}}^r \subset \mathfrak{w} \subset \mathfrak{q} \quad \textit{and} \quad \mathfrak{w} + \bar{\mathfrak{w}} = \mathfrak{q} + \bar{\mathfrak{q}}. \tag{4.6}$$

PROOF. By Proposition 4.1, $\mathfrak{w}$ is complex parabolic in $\hat{\mathfrak{g}}$. Indeed $\vartheta(\mathfrak{q})$ is complex parabolic in $\hat{\mathfrak{g}}$ and contains $\mathfrak{h}$. The parabolic set associated to $\vartheta(\mathfrak{q})$ is $\vartheta(\mathcal{Q}) = \{\alpha \mid -\bar{\alpha} \in \mathcal{Q}\} = \bar{\mathcal{Q}}^r \cup \bar{\mathcal{Q}}^{-n}$. Hence the parabolic set corresponding to $\mathfrak{w}$ is:

$$\mathcal{W} = \mathcal{Q}^n \cup (\mathcal{Q}^r \cap \vartheta(\mathcal{Q})) = \left(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^r\right) \cup \mathcal{Q}^n \cup \left(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^{-n}\right). \tag{4.7}$$

We obtain (4.5) by using Proposition 4.1.

We have $\mathcal{W}^r = \bar{\mathcal{W}}^r = \mathcal{Q}^r \cap \bar{\mathcal{Q}}^r$, and $\mathcal{W} \cup \bar{\mathcal{W}} = \mathcal{W}^r \cup \mathcal{W}^n \cup \bar{\mathcal{W}}^n = \mathcal{Q} \cup \bar{\mathcal{Q}}$. The right hand side of this equality can be written as a disjoint union:

$$\mathcal{Q} \cup \bar{\mathcal{Q}} = \left(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^r\right) \cup \left(\mathcal{Q}^n \cup \bar{\mathcal{Q}}^n\right) \cup \left(\mathcal{Q}^r \setminus \bar{\mathcal{Q}}\right) \cup \left(\bar{\mathcal{Q}}^r \setminus \mathcal{Q}\right).$$

In particular, $\mathcal{Q}^r \setminus \bar{\mathcal{Q}} \subset \mathcal{W}$ for the parabolic set $\mathcal{W}$ of any complex parabolic $\mathfrak{w}$ that satisfies (4.6), and this shows that the $\mathfrak{w}$ we constructed is the smallest complex parabolic subalgebra of $\hat{\mathfrak{g}}$ that satisfies (4.6). $\square$

Since $\mathfrak{w} \subset \mathfrak{q}$, we have a $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{w}) \longrightarrow (\mathfrak{g}, \mathfrak{q})$. We call $(\mathfrak{g}, \mathfrak{w})$ the *canonical CR-lift* of $(\mathfrak{g}, \mathfrak{q})$.

THEOREM 4.3. *The canonical CR lift $(\mathfrak{g}, \mathfrak{w}) \to (\mathfrak{g}, \mathfrak{q})$ is a $\mathfrak{g}$-equivariant CR fibration (in particular a CR submersion) with totally complex fibers. When $(\mathfrak{g}, \mathfrak{q})$ is weakly nondegenerate, then $(\mathfrak{g}, \mathfrak{w}) \to (\mathfrak{g}, \mathfrak{q})$ is the weakly nondegenerate reduction of $(\mathfrak{g}, \mathfrak{w})$.*

PROOF. The statement is an immediate consequence of Lemma 3.8 and of the next lemma. $\square$

LEMMA 4.4. *Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$. Let $(\mathfrak{g}, \mathfrak{w})$ be its canonical CR-lift. Then a Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ is V-fit for $(\mathfrak{g}, \mathfrak{q})$ if and only if $C \in \mathfrak{C}(\mathcal{R}, \mathcal{W})$ and is V-fit for $(\mathfrak{g}, \mathfrak{w})$.*

PROOF. Assume that $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ is V-fit for $(\mathfrak{g},\mathfrak{q})$. We want to show that $\mathcal{R}^+(C) \subset \mathcal{W}$. Since $\mathcal{Q}^n \cup \big(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^r\big) \subset \mathcal{W}$, it suffices to prove that $\alpha \in \mathcal{W}$ for $\alpha \in (\mathcal{R}^+(C) \cap \mathcal{Q}^r) \setminus \bar{\mathcal{Q}}^r$. Since C is V-fit for $(\mathfrak{g},\mathfrak{q})$, for such a root α we have $\bar{\alpha} \prec_C 0$. Hence $\vartheta(\alpha) = -\bar{\alpha} \in \mathcal{R}^+(C) \subset \mathcal{Q}$, i.e. $\alpha \in \mathcal{Q}^r \cap \vartheta(\mathcal{Q}) \subset \mathcal{W}$. A chamber $C \in \mathfrak{C}(\mathcal{R},\mathcal{W})$ that is V-fit for $(\mathfrak{g},\mathfrak{q})$ is also V-fit for $(\mathfrak{g},\mathfrak{w})$, because $\mathfrak{w} \subset \mathfrak{q}$.

Let now $C \in \mathfrak{C}(\mathcal{R},\mathcal{W}) \subset \mathfrak{C}(\mathcal{R},\mathcal{Q})$ be V-fit for $(\mathfrak{g},\mathfrak{w})$. Note that $\mathcal{R}^+(C) \cap \mathcal{Q}^r = (\mathcal{R}^+(C) \cap \mathcal{W}^r) \cup (\mathcal{W}^n \setminus \mathcal{Q}^n)$, and $\mathcal{W}^n \setminus \mathcal{Q}^n = \mathcal{Q}^r \cap \bar{\mathcal{Q}}^{-n}$. Since $\bar{\alpha} \prec_C 0$ for all $\alpha \in \bar{\mathcal{Q}}^{-n}$, we obtain that $\sigma(\mathcal{R}^+(C) \cap \mathcal{Q}^r \setminus \mathcal{R}_{\mathrm{re}}) \subset \mathcal{R}^-(C)$ and therefore C is also V-fit for $(\mathfrak{g},\mathfrak{q})$. $\square$

In terms of a base of the root system we have:

PROPOSITION 4.5. *Let $(\vartheta,\mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$, and let $(\mathfrak{g},\mathfrak{w})$ be its canonical lift. Let $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ be V-fit for $(\mathfrak{g},\mathfrak{q})$, and $\Phi_C = \mathcal{B}(C) \cap \mathcal{Q}^n$, $\tilde{\Phi}_C = \mathcal{B}(C) \cap \mathcal{W}^n$. Then:*

$$\tilde{\Phi}_C = \Phi_C \cup \{\alpha \in \mathcal{B}_C \,|\, \mathrm{supp}_C(\bar{\alpha}) \cap \Phi_C \neq \emptyset\}\,. \tag{4.8}$$

PROOF. By Lemma 4.4, $\mathfrak{w} = \mathfrak{q}_{\tilde{\Phi}_C}$ for some set of simple roots $\tilde{\Phi}_C$ with $\Phi_C \subset \tilde{\Phi}_C \subset \mathcal{B}(C)$. Let $\mathcal{Q} = \mathcal{Q}_A = \{\alpha \in \mathcal{R} \,|\, \alpha(A) \geq 0\}$, with $A \in \mathfrak{h}_{\mathbb{R}}$. Fix a real $\varepsilon > 0$ with $\varepsilon\,|\bar{\alpha}(A)| < \alpha(A)$ for all $\alpha \in \mathcal{Q}^n$. Then:

$$\mathcal{W} = \{\alpha \in \mathcal{R} \,|\, \alpha(A - \varepsilon \bar{A}) \geq 0\,\}\,. \tag{4.9}$$

Indeed, $\mathcal{W} \subset \mathcal{Q}$ because $\alpha(A - \varepsilon\bar{A}) < 0$ when $\alpha \in \mathcal{R}$ and $\alpha(A) < 0$; moreover $\mathcal{Q}^n \subset \mathcal{W}^n$, and a root $\alpha \in \mathcal{Q}^r$ belongs to $\mathcal{W}$ if and only if $\bar{\alpha}(A) \leq 0$, i.e. if and only if $\vartheta(\alpha) \in \mathcal{Q}$.

Thus we have:

$$\begin{aligned}\tilde{\Phi}_C &= \mathcal{B}(C) \cap \mathcal{W}^n = \{\alpha \in \mathcal{B}(C) \,|\, \alpha(A - \varepsilon\bar{A}) > 0\} \\ &= \Phi_C \cup \{\alpha \in \mathcal{B}(C) \,|\, \alpha(A) = 0\,,\ \bar{\alpha}(A) < 0\} \\ &= \Phi_C \cup \{\alpha \in \mathcal{B}(C) \,|\, \mathrm{supp}_C(\bar{\alpha}) \cap \Phi_C \neq \emptyset\,\}\,,\end{aligned}$$

because $\tilde{\Phi}_C \setminus \Phi_C \subset \mathcal{R}_{\mathrm{cp}}$ and, for a complex α in $\mathcal{B}(C) \setminus \Phi_C$ we have $\bar{\alpha} \in \mathcal{R}^-(C)$: hence $\bar{\alpha}(A) < 0$ whenever $\bar{\alpha}(A) \neq 0$, i.e. $\mathrm{supp}_C(\bar{\alpha}) \cap \Phi_C \neq \emptyset$. $\square$

We can slightly improve the criterion of weak non-degeneracy of Theorem 3.9, by using Weyl chambers adapted to the canonical CR lift. We have indeed:

PROPOSITION 4.6. *Let $(\vartheta,\mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$, and let $(\mathfrak{g},\mathfrak{w})$ be the canonical CR-lift of $(\mathfrak{g},\mathfrak{q})$. If $C \in \mathfrak{C}(\mathcal{R},\mathcal{W})$, then:*

$$(\mathfrak{g},\mathfrak{q}) \quad \text{is weakly non-degenerate if and only if} \quad \bar{\alpha} \succ_C 0 \quad \forall \alpha \in \Phi_C\,. \tag{4.10}$$

PROOF. ($\Rightarrow$) We argue by contradiction. Assume that $\bar{\alpha}_0 \prec_C 0$ for some $\alpha_0 \in \Phi_C$. We want to prove that $\mathcal{Q}' \cup \bar{\mathcal{Q}}' = \mathcal{Q} \cup \bar{\mathcal{Q}}$ for the parabolic set

$$\mathcal{Q}' = \mathcal{Q}_{\Psi_C} = \mathcal{Q} \cup \{\beta \in \mathcal{R} \,|\, \mathrm{supp}_C(\beta) \cap \Phi_C \subset \{\alpha_0\}\} , \tag{4.11}$$

corresponding to $\Psi_C = \Phi_C \setminus \{\alpha_0\} \subset \mathcal{B}(C)$. It suffices to verify that $\bar{\beta} \in \mathcal{Q}$ if $\beta \prec_C 0$ and $\mathrm{supp}_C(\beta) \cap \Phi_C = \{\alpha_0\}$. Since $C \in \mathfrak{C}(\mathcal{R}, \mathcal{W})$, each $\alpha \in \mathcal{B}(C)$ either belongs to $\mathcal{W}^n = \mathcal{Q}^n \cup \big(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^{-n}\big)$, or to $\mathcal{W}^r = \mathcal{Q}^r \cap \bar{\mathcal{Q}}^r$. Thus $\mathcal{B}(C) \setminus \Phi_C \subset \bar{\mathcal{Q}}^r \cup \bar{\mathcal{Q}}^{-n}$, and, being $\bar{\alpha}_0 \prec_C 0$, we get $-\bar{\alpha} \in \mathcal{Q}$ for all $\alpha \in \mathrm{supp}_C(\beta)$, yielding $\beta \in \bar{\mathcal{Q}}$.
($\Leftarrow$) Assume that $(\mathfrak{g}, \mathfrak{q})$ is weakly degenerate. Then, for some $\alpha_0 \in \Phi_C$, (4.11) defines a parabolic set $\mathcal{Q}'$ with $\mathcal{Q}' \cup \bar{\mathcal{Q}}' = \mathcal{Q} \cup \bar{\mathcal{Q}}$. In particular, $-\alpha_0 \in \bar{\mathcal{Q}}$. If $-\alpha_0 \in \bar{\mathcal{Q}}^n$, then $-\bar{\alpha}_0 \in \mathcal{Q}^n \subset \mathcal{R}^+(C)$ and $\bar{\alpha}_0 \prec_C 0$. Otherwise, $\bar{\alpha}_0 \in \mathcal{Q}^r \cap \bar{\mathcal{Q}}^n$ and, because $\mathcal{Q}^r \cap \bar{\mathcal{Q}}^n \subset \mathcal{W}^{-n}$, the condition that $C \in \mathfrak{C}(\mathcal{R}, \mathcal{W})$ implies that $\bar{\alpha}_0 \prec_C 0$.$\square$

Proposition 4.6 gives a way to construct the weakly non-degenerate reduction of a parabolic CR algebra:

COROLLARY 4.7. *Let $(\mathfrak{g}, \mathfrak{w})$ be the canonical CR-lift of the parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ adapted to $(\mathfrak{g}, \mathfrak{q})$. If $C \in \mathfrak{C}(\mathcal{R}, \mathcal{W}) \subset \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, $\Phi_C = \mathcal{B}(C) \cap \mathcal{Q}^n$, and*

$$\Psi_C = \{\alpha \in \Phi(C) \,|\, \bar{\alpha} \in \mathcal{R}^+(C)\} , \tag{4.12}$$

then the $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ is the weakly nondegenerate reduction of $(\mathfrak{g}, \mathfrak{q})$. $\square$

Furthermore we have:

PROPOSITION 4.8. *Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$, and let $(\mathfrak{g}, \mathfrak{w})$ be the canonical lift of $(\mathfrak{g}, \mathfrak{q})$. Then the natural $\mathbf{G}$-equivariant projection $M(\mathfrak{g}, \mathfrak{w}) \to M(\mathfrak{g}, \mathfrak{q})$ is a CR fibration with totally complex connected fibers.*

PROOF. Our CR manifolds are described as homogeneous spaces by the quotients $M(\mathfrak{g}, \mathfrak{q}) = \mathbf{G}/\mathbf{G}_+$ with $\mathbf{G}_+ = \mathbf{N}_{\mathbf{G}}(\mathfrak{q})$ and $M(\mathfrak{g}, \mathfrak{w}) = \mathbf{G}/\mathbf{W}_+$, where $\mathbf{W}_+ = \mathbf{N}_{\mathbf{G}}(\mathfrak{w})$.

We want to prove that every connected component of $\mathbf{G}_+$ contains an element of $\mathbf{W}_+$.

Let $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ be the Lie algebra of $\mathbf{G}_+$ and consider its decomposition in (2.19) of Proposition 2.4: we have $\mathfrak{g}_+ = \mathfrak{n} \oplus \mathfrak{g}_0$, where $\mathfrak{n}$ is the ideal of the nilpotent elements of the radical of $\mathfrak{g}_+$ and $\mathfrak{g}_0 = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^r \cap \mathfrak{g}$ is a ϑ-invariant reductive complement of $\mathfrak{n}$ in $\mathfrak{g}_+$.

Being algebraic, $\mathbf{G}_+$ has a *Chevalley decomposition* (see [Che55, Chap.5, Sect.4]) into the semidirect product $\mathbf{N} \rtimes \mathbf{G}_0$ of the analytic subgroup with Lie algebra $\mathfrak{n}$ and of a closed Lie subgroup $\mathbf{G}_0$ with Lie algebra $\mathfrak{g}_0$.

Let $g \in \mathbf{G}_+$ and denote by Γ_g the connected component of g in $\mathbf{G}_+$. Since $\mathbf{N}$ is connected, we can as well take from the start g in $\mathbf{G}_0$, so that in particular $\mathrm{Ad}_{\mathfrak{g}}(g)(\mathfrak{g}_0) = \mathfrak{g}_0$.

Since $\mathfrak{g}_0$ is a real form of $\mathfrak{q}^r \cap \bar{\mathfrak{q}}^r$, by complexification we obtain that $\mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}^r \cap \bar{\mathfrak{q}}^r) = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^r$. Thus $\mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{w})$ is a parabolic complex subalgebra of $\hat{\mathfrak{g}}$ with $\mathfrak{q}^r \cap \bar{\mathfrak{q}}^r \subset \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{w}) \subset \mathfrak{q}$. Since $\dim_{\mathbb{C}}(\mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{w})) = \dim_{\mathbb{C}}(\mathfrak{w})$, it follows from the characterization of $\mathfrak{w}$ in Proposition 4.2, that $\mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{w}) = \mathfrak{w}$, and hence $g \in \mathbf{W}_+$. $\square$

4.2 The weakest CR model

Next we describe a construction that is similar to the one discussed above. We keep the notation introduced therein.

PROPOSITION 4.9. *Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$, and let:*

$$\mathfrak{v} = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \bar{\mathfrak{q}}) \ . \tag{4.13}$$

Then $\mathfrak{v}$ is a parabolic subalgebra of $\hat{\mathfrak{g}}$, such that:

$$\begin{cases} \mathfrak{v}^r = \bar{\mathfrak{v}}^{\,r} = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^{\,r}\,, \quad \mathfrak{v}^n = \mathfrak{q}^n \oplus (\mathfrak{q}^r \cap \bar{\mathfrak{q}}^{\,n}) \supset \mathfrak{q}^n\,, \quad \mathfrak{v} = \mathfrak{v}^r \oplus \mathfrak{v}^n \subset \mathfrak{q} \\ \mathfrak{v}^n \cap \bar{\mathfrak{v}}^n = \mathfrak{q}^n \cap \bar{\mathfrak{q}}^{\,n}\,, \quad \bar{\mathfrak{v}}^{\,r} = \mathfrak{v}^r\,, \quad \mathfrak{v} \cap \bar{\mathfrak{v}} = \mathfrak{q} \cap \bar{\mathfrak{q}}\,. \end{cases} \tag{4.14}$$

It is uniquely determined by the condition of being the smallest complex parabolic subalgebras $\mathfrak{q}' \subset \hat{\mathfrak{g}}$ with:

$$\mathfrak{q} \cap \bar{\mathfrak{q}} \subset \mathfrak{q}' \subset \mathfrak{q}\,. \tag{4.15}$$

We note that the latter characterization of $\mathfrak{v}$ is independent from the choice of the Cartan pair $(\vartheta, \mathfrak{h})$.

PROOF. All Cartan subalgebras $\mathfrak{h}$ adapted to $(\mathfrak{g}, \mathfrak{q})$ are also contained in $\bar{\mathfrak{q}}$. We apply Proposition 4.1. The parabolic set corresponding to $\mathfrak{v}$ is:

$$\mathcal{V} = \mathcal{Q}^n \cup \left(\mathcal{Q}^r \cap \bar{\mathcal{Q}}\right) = \left(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^{\,r}\right) \cup \mathcal{Q}^n \cup \left(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^{\,n}\right)\,. \tag{4.16}$$

Then it is easy to verify (4.14) by using Proposition 4.1.

All complex parabolic $\mathfrak{q}' \subset \hat{\mathfrak{g}}$ that satisfy (4.15), also satisfy $\mathfrak{q}'^{\,n} \supset \mathfrak{q}^n$, because $\mathfrak{q}' \subset \mathfrak{q}$, and $\mathfrak{q}^r \cap \bar{\mathfrak{q}} \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$; hence $\mathfrak{v} \subset \mathfrak{q}'$ if $\mathfrak{q}'$ satisfies (4.15). □

The parabolic subalgebra $(\mathfrak{g}, \mathfrak{v})$ defined in Proposition 4.9 is called *the weakest CR model* of $(\mathfrak{g}, \mathfrak{q})$.

By Theorem 2.14, since $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{v} \cap \bar{\mathfrak{v}}$, we have:

THEOREM 4.10. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra and $(\mathfrak{q}, \mathfrak{v})$ its weakest CR model. Then the holomorphic projection $\hat{M}(\hat{\mathfrak{g}}, \mathfrak{v}) \longrightarrow \hat{M}(\hat{\mathfrak{g}}, \mathfrak{q})$ restricts to a smooth diffeomorphism $M(\mathfrak{g}, \mathfrak{v}) \longrightarrow M(\mathfrak{g}, \mathfrak{q})$.* □

An alternative construction of the weakest CR model is given by the following Lemma.

LEMMA 4.11. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ adapted to $(\mathfrak{g}, \mathfrak{q})$. Let $\mathcal{Q} = \mathcal{Q}_A = \{\alpha \in \mathcal{R} \mid \alpha(A) \geq 0\}$, with $A \in \mathfrak{h}_{\mathbb{R}}$. Then the parabolic set $\mathcal{V}$ of the parabolic complex $\mathfrak{v} \subset \hat{\mathfrak{g}}$ of the weakest CR model $(\mathfrak{g}, \mathfrak{v})$ of $(\mathfrak{g}, \mathfrak{q})$ is given by:*

$$\mathcal{V} = \{\alpha \in \mathcal{R} \mid \alpha(A + \varepsilon \bar{A}) \geq 0\} \tag{4.17}$$

where ε is any positive real number with $\varepsilon|\bar{\alpha}(A)| < \alpha(A)$ for all $\alpha \in \mathcal{Q}^n$.

PROOF. Since $\alpha(A+\varepsilon\bar{A})<0$ when $\alpha\notin\mathcal{Q}$, we have $\mathcal{Q}^n\subset\mathcal{V}^n$ and hence $\mathcal{V}\subset\mathcal{Q}$. Moreover $\alpha\in\mathcal{Q}^r$ belongs to $\mathcal{V}$ if and only if $\bar{\alpha}(A)\geq 0$, i.e. if and only if $\alpha\in\bar{\mathcal{Q}}$. □

Lemma 4.11 yields:

PROPOSITION 4.12. *Let $(\vartheta,\mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$, and let $(\mathfrak{g},\mathfrak{w})$ be the canonical lift of $(\mathfrak{g},\mathfrak{q})$. Then $(\mathfrak{g},\mathfrak{w})$ coincides with its weakest CR model.*

PROOF. We use the notation of Lemma 4.11. By Proposition 4.5, $\mathfrak{w}=\mathfrak{q}_B$ with $B=A-\varepsilon\bar{A}\in\mathfrak{h}_{\mathbb{R}}$ for $0<\varepsilon<\varepsilon_0$. Then, by Lemma 4.11, the weakest CR model of $(\mathfrak{g},\mathfrak{w})$ is $(\mathfrak{g},\mathfrak{v}')$ where $\mathfrak{v}'=\mathfrak{q}_C$ with $C=A-\varepsilon'\bar{A}\in\mathfrak{h}_{\mathbb{R}}$, with $\varepsilon'=\frac{\varepsilon-\delta}{1-\varepsilon\delta}$ for $0<\delta<\delta_0$ sufficiently small, and hence $\mathfrak{v}'=\mathfrak{w}$. □

We have:

LEMMA 4.13. *Let $(\mathfrak{g},\mathfrak{v})$ be the weakest CR model of $(\mathfrak{g},\mathfrak{q})$ and $\mathfrak{h}$ a CR algebra of $\mathfrak{g}$ adapted to $(\mathfrak{g},\mathfrak{q})$ (and hence also to $(\mathfrak{g},\mathfrak{v})$). If $\mathcal{Q}$, $\mathcal{V}$ are the parabolic sets corresponding to $\mathfrak{q}$ and $\mathfrak{v}$ in $\mathcal{R}=\mathcal{R}(\hat{\mathfrak{g}},\hat{\mathfrak{h}})$, then:*

$$\mathcal{Q}^n\cap\bar{\mathcal{Q}}^{-n}=\mathcal{V}^n\cap\bar{\mathcal{V}}^{-n}\,. \tag{4.18}$$

PROOF. We have: $\mathcal{V}^n=\mathcal{Q}^n\cup\left(\mathcal{Q}^r\cap\bar{\mathcal{Q}}^n\right)$ and hence $\bar{\mathcal{V}}^{-n}=\bar{\mathcal{Q}}^{-n}\cup\left(\bar{\mathcal{Q}}^r\cap\mathcal{Q}^{-n}\right)$. Since:
$\left(\mathcal{Q}^r\cap\bar{\mathcal{Q}}^n\right)\cap\left(\bar{\mathcal{Q}}^r\cap\mathcal{Q}^{-n}\right)=\emptyset,\quad\left(\mathcal{Q}^r\cap\bar{\mathcal{Q}}^n\right)\cap\bar{\mathcal{Q}}^{-n}=\emptyset,\quad\mathcal{Q}^n\cap\left(\bar{\mathcal{Q}}^r\cap\mathcal{Q}^{-n}\right)=\emptyset,$
we obtain (4.18). □

The connection between the weakest CR model and S-fit Weyl chambers is the following:

LEMMA 4.14. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective parabolic CR algebra and $(\mathfrak{g},\mathfrak{v})$ its weakest CR model. Then a Weyl chamber $C\in\mathcal{C}(\mathcal{R},\mathcal{Q})$ is S-fit for $(\mathfrak{g},\mathfrak{q})$ if and only if $C\in\mathcal{C}(\mathcal{R},\mathcal{V})$ and is S-fit for $(\mathfrak{g},\mathfrak{v})$.*

PROOF. We have $\mathfrak{C}(\mathcal{R},\mathcal{V})\subset\mathfrak{C}(\mathcal{R},\mathcal{Q})$ because $\mathcal{V}\subset\mathcal{Q}$. Since $\mathcal{V}^n=\mathcal{Q}^n\cup\left(\mathcal{Q}^r\cap\bar{\mathcal{Q}}^n\right)$, for $C\in\mathfrak{C}(\mathcal{R},\mathcal{V})$ and $\alpha\in\mathcal{V}^n\setminus\mathcal{Q}^n$, we get $\bar{\alpha}\in\mathcal{Q}^n\subset\mathcal{V}^n\subset\mathcal{R}^+(C)$. If moreover $C\in\mathfrak{C}(\mathcal{R},\mathcal{V})$ is S-fit for $(\mathfrak{g},\mathfrak{v})$, then $\bar{\alpha}\succ_C 0$ also for $\alpha\in(\mathcal{V}^r\cap\mathcal{R}^+(C))\setminus\mathcal{R}_{\mathrm{im}}$; since $\mathcal{Q}^r\cap\mathcal{R}^+(C)=(\mathcal{V}^r\cap\mathcal{R}^+(C))\cup(\mathcal{V}^n\setminus\mathcal{Q}^n)$, it follows that C is S-fit also for $(\mathfrak{g},\mathfrak{q})$.

To complete the proof, it suffices to show that an S-fit Weyl chamber for $(\mathfrak{g},\mathfrak{q})$ is admissible for $(\mathfrak{g},\mathfrak{v})$. Let C be S-fit for $(\mathfrak{g},\mathfrak{q})$. The elements α of $\mathcal{V}^n\setminus\mathcal{Q}^n$ belong to $\mathcal{Q}^r\cap\bar{\mathcal{Q}}^n$: this implies that $\bar{\alpha}\succ_C 0$ and therefore that $\alpha\succ_C 0$, by the assumption that C is S-fit for $(\mathfrak{g},\mathfrak{q})$. Hence $\mathcal{V}^n\subset\mathcal{R}^+(C)$ and therefore $C\in\mathfrak{C}(\mathcal{R},\mathcal{V})$ when $C\in\mathfrak{C}(\mathcal{R},\mathcal{Q})$ is S-fit for $(\mathfrak{g},\mathfrak{q})$. □

In terms of a base of the root system we have:

PROPOSITION 4.15. *Let $(\mathfrak{g},\mathfrak{q})$ be a parabolic CR algebra, with $\mathfrak{q}=\mathfrak{q}_{\Phi_C}$ for an S-fit Weyl chamber $C\in\mathfrak{C}(\mathcal{R},\mathcal{Q})$. Then its weakest CR model is $(\mathfrak{g},\mathfrak{q}_{\Phi^\sharp_C})$, for*

$$\Phi^\sharp_C=\Phi_C\cup\{\beta\in\mathcal{B}(C)\cap\mathcal{R}_{\mathrm{cp}}\,|\,\mathrm{supp}_C(\bar\beta)\cap\Phi_C\neq\emptyset\}\,. \tag{4.19}$$

PROOF. Let $C\in\mathfrak{C}(\mathcal{R},\mathcal{Q})$ be S-fit for $(\mathfrak{g},\mathfrak{q})$. If $\mathcal{Q}=\mathcal{Q}_A$ for $A\in\mathfrak{h}_{\mathbb{R}}$, then $\Phi_C=\{\beta\in\mathcal{B}(C)\,|\,\beta(A)>0\}$ and, by Lemma 4.11 and Lemma 4.14, $C\in\mathfrak{C}(\mathcal{Q},\mathcal{V})$ and thus $\mathfrak{v}=\mathfrak{q}_{\Phi^\sharp_C}$ for $\Phi^\sharp_C=\{\beta\in\mathcal{B}(C)\,|\,\beta(A)+\varepsilon\bar\beta(A)>0\}$ with $\varepsilon>0$ and sufficiently small, yielding the characterization in the statement of the Proposition. $\square$

From Corollary 3.10 we obtain:

COROLLARY 4.16. *Let $(\mathfrak{g},\mathfrak{v})$ be the weakest CR model of a parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$. Then:*

(i) $(\mathfrak{g},\mathfrak{v})$ *is either totally real, or weakly degenerate.*
(ii) *If $(\mathfrak{g},\mathfrak{q})$ is weakly non-degenerate, then $\mathfrak{v}=\mathfrak{q}$ if and only if $(\mathfrak{g},\mathfrak{q})$ is totally real.* $\square$

By using Corollary 4.16, starting from a parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$, we can construct a chain of parabolic CR algebras $(\mathfrak{g},\mathfrak{q}_h)$, $(\mathfrak{g},\mathfrak{v}_h)$ and $\mathfrak{g}$-equivariant CR homomorphisms:

$$\begin{tikzcd}
(\mathfrak{g},\mathfrak{q}) \arrow[d] & & & \\
(\mathfrak{g},\mathfrak{q}_1) & (\mathfrak{g},\mathfrak{v}_1) \arrow[l] \arrow[d] & & \\
& (\mathfrak{g},\mathfrak{q}_2) & (\mathfrak{g},\mathfrak{v}_2) \arrow[l] \arrow[d] & \\
& & (\mathfrak{g},\mathfrak{q}_3) & (\mathfrak{g},\mathfrak{v}_3) \arrow[l] \\
& & & \vdots
\end{tikzcd} \tag{4.20}$$

where each vertical arrow is a weakly nondegenerate reduction and each horizontal arrow is a lifting to the weakest CR model.

If we denote by $\mathfrak{P}(\hat{\mathfrak{g}})$ the set of all parabolic complex Lie subalgebras of $\hat{\mathfrak{g}}$, and set $\mathfrak{v}_0=\mathfrak{g}_{-1}=\mathfrak{g}$, we have, for all integers $h>0$:

$$\begin{cases}\mathfrak{q}_h=\text{the largest }\mathfrak{a}\in\mathfrak{P}(\hat{\mathfrak{g}})\text{ such that }\quad\mathfrak{v}_{h-1}\subset\mathfrak{a}\subset\mathfrak{v}_{h-1}+\bar{\mathfrak{v}}_{h-1}\,,\\ \mathfrak{v}_h=\text{the smallest }\mathfrak{a}\in\mathfrak{P}(\hat{\mathfrak{g}})\text{ such that }\quad\mathfrak{q}_h\cap\bar{\mathfrak{q}}_h\subset\mathfrak{a}\subset\mathfrak{q}_h\,.\end{cases} \tag{4.21}$$

This characterization shows that the construction in (4.20) is uniquely determined and independent of the choices of the adapted Cartan pairs in $(\mathfrak{g},\mathfrak{q}_h)$ and $(\mathfrak{g},\mathfrak{v}_h)$.

We know that the $\mathbf{G}$-equivariant maps $M(\mathfrak{g},\mathfrak{v}_h)\to M(\mathfrak{g},\mathfrak{q}_h)$ (for $h>0$) are smooth diffeomorphisms, while the $\mathbf{G}$-equivariant maps $M(\mathfrak{g},\mathfrak{v}_h)\to M(\mathfrak{g},\mathfrak{q}_{h+1})$ (for $h\geq 0$) are CR fibrations. In particular, there is a smallest integer $m\geq -1$ such that $\dim_{\mathbb{R}}M(\mathfrak{g},\mathfrak{v}_h)>\dim_{\mathbb{R}}M(\mathfrak{g},\mathfrak{v}_{h+1})$ for $h<m+1$, and $M(\mathfrak{g},\mathfrak{v}_h)=M(\mathfrak{g},\mathfrak{v}_{m+1})=M(\mathfrak{g},\mathfrak{q}_m)$ for all $h>m+1$. Hence, by the characterization of the fibers in Theorem 2.8, we have:

PROPOSITION 4.17. *With the notation above: let* $(\mathfrak{g},\mathfrak{q}_h)$, $(\mathfrak{g},\mathfrak{v}_h)$ *be the sequence of weakly nondegenerate parabolic CR algebra and of their weakest CR models defined above. Then there exists a smallest integer $m \geq 0$ such that $\mathfrak{q}_h = \mathfrak{v}_h = \mathfrak{q}_m$ for all $h > m$. Moreover, $(\mathfrak{g},\mathfrak{q}_m)$ is totally real and m is the smallest nonnegative integer for which $(\mathfrak{g},\mathfrak{q}_m)$ is totally real.*

By composition, we obtain a $\mathbf{G}$-equivariant fibration $M(\mathfrak{g},\mathfrak{q}) \to M(\mathfrak{g},\mathfrak{q}_m)$ where the basis $M(\mathfrak{g},\mathfrak{q}_m)$ is a totally real parabolic CR manifold and each connected component of the fiber is a Cartesian product of Euclidean complex nilmanifolds and of simply connected totally complex parabolic CR manifolds. □

A concrete example of this sequence of fibrations is given below.

EXAMPLE 4.1. Let $\varepsilon_1,\dots,\varepsilon_6$ be the canonical basis of $\mathbb{R}^6 \subset \mathbb{C}^6$. Let $\mathbf{G} = \mathbf{SL}(6,\mathbb{R})$ consist of the matrices of $\hat{\mathbf{G}} = \mathbf{SL}(6,\mathbb{C})$ which have real entries in the canonical basis. We consider in $\mathbb{C}^6$ the basis:

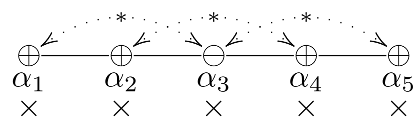
$$e_1 = \varepsilon_1 + i\varepsilon_4,\ e_2 = \varepsilon_2 + i\varepsilon_5,\ e_3 = \varepsilon_3 + i\varepsilon_6,\ e_4 = \varepsilon_1 - i\varepsilon_4,\ e_5 = \varepsilon_2 - i\varepsilon_5,\ e_6 = \varepsilon_3 - i\varepsilon_6$$

and we want to investigate the $\mathbf{G}$-orbit M of the flag

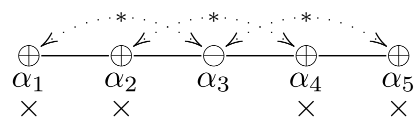
$$\langle e_1\rangle \subset \langle e_1,e_2\rangle \subset \langle e_1,e_2,e_3\rangle \subset \langle e_1,e_2,e_3,e_4\rangle \subset \langle e_1,e_2,e_3,e_4,e_5\rangle\,.$$

Let $(\mathfrak{g},\mathfrak{q})$, with $\mathfrak{g} = \mathfrak{sl}(6,\mathbb{R})$, be the associated parabolic CR algebra. We consider the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(6,\mathbb{R})$ that is represented, in the basis $e_1,\dots,e_6$, by the diagonal matrices $\operatorname{diag}(\lambda_1,\lambda_2,\lambda_3,\bar\lambda_1,\bar\lambda_2,\bar\lambda_3)$ with $\operatorname{Re}(\lambda_1+\lambda_2+\lambda_3)=0$. Then $\mathfrak{h}_{\mathbb{R}}$ consists of the 6×6 traceless real diagonal matrices.

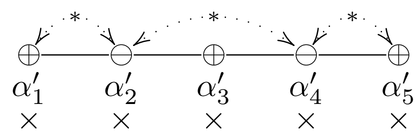
The cross-marked diagram associated to $(\mathfrak{g},\mathfrak{q})$ in the adapted Weyl chamber C with simple roots $\mathcal{B}(C) = \{\alpha_i = e_1 - e_{i+1}\,,\ 1 \leq i \leq 5\}$, is the following:



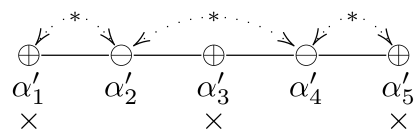
Since $\mathfrak{q}$ is a complex Borel subalgebra of $\hat{\mathfrak{g}}$, the chamber C is S-fit and then $(\mathfrak{g},\mathfrak{q})$ is fundamental by Theorem 3.5, because $\bar\alpha_3 = -(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)$. The chamber C is also V-fit, and we obtain the weakly non-degenerate basis $(\mathfrak{g},\mathfrak{q}_1)$ by dropping the cross under the simple root α_3 with $\bar\alpha_3 \prec_C 0$. The diagram associated to $(\mathfrak{g},\mathfrak{q}_1)$ is:



The parabolic $\mathfrak{q}_1$ is defined by the element $A_1 = \operatorname{diag}(2,1,0,0,-1,-2) \in \mathfrak{h}_{\mathbb{R}}$. To compute its weakest CR model $(\mathfrak{g},\mathfrak{v}_1)$, we observe that $\bar A_1 = \operatorname{diag}(0,-1,-2,2,1,0)$, so that $A_1 + \varepsilon\bar A_1 = \operatorname{diag}(2, 1-\varepsilon, -2\varepsilon, 2\varepsilon, -1+\varepsilon, -2)$. To take a Weyl chamber adapted to $(\mathfrak{g},\mathfrak{v}_1)$, it is convenient to consider the basis obtained from $e_1,\dots,e_6$ by reordering its elements according to the decreasing ordering of the diagonal entries of $A_1 + \varepsilon\bar A_1$. We obtain the new basis: $e_1, e_2, e_4, e_3, e_5, e_6$. With $\alpha'_1 = e_1 - e_2$, $\alpha'_2 = e_2 - e_4$, $\alpha'_3 = e_4 - e_3$, $\alpha'_4 = e_3 - e_5$, $\alpha'_5 = e_5 - e_6$ being the simple roots a Weyl chamber $C' \in \mathfrak{C}(\mathcal{R},\mathcal{V}_1)$, we obtain the diagram:



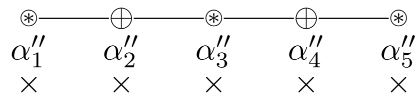
The weakly nondegenerate reduction $(\mathfrak{g}, \mathfrak{q}_2)$ of $(\mathfrak{g}, \mathfrak{v}_1)$ has the diagram:



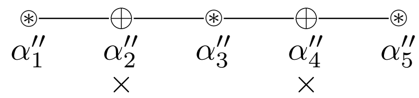
We have $\mathfrak{q}_2 = \mathfrak{q}_{A_2}$, with

$$A_2 = \operatorname{diag}(2, 1, -1, 1, -1, -2), \quad \bar{A}_2 = \operatorname{diag}(1, -1, -2, 2, 1, -1),$$

so that $A_2 + \varepsilon\bar{A}_2 = \operatorname{diag}(2+\varepsilon, 1-\varepsilon, -1-2\varepsilon, 1+2\varepsilon, -1+\varepsilon, -2-\varepsilon)$. To describe $(\mathfrak{g}, \mathfrak{v}_2)$, for $\mathfrak{v}_2 = \mathfrak{q}_{A_2+\varepsilon\bar{A}_2}$, it is convenient to consider the Weyl chamber $C'' \in \mathfrak{C}(\mathcal{R}, \mathcal{V}_2)$ that corresponds to the simple roots related to the ordered basis $e_1, e_4, e_2, e_5, e_3, e_6$ of $\mathbb{C}^6$, for which the entries of $A_2 + \varepsilon\bar{A}_2$ are decreasing: with $\alpha_1'' = e_1 - e_4,\ \alpha_2'' = e_4 - e_2,\ \alpha_3'' = e_2 - e_5,\ \alpha_4'' = e_5 - e_3,\ \alpha_5'' = e_3 - e_6$, we obtain for $(\mathfrak{g}, \mathfrak{v}_2)$ the diagram:



Since $\mathfrak{v}_2$ is Borel, C'' is V-fit for $(\mathfrak{g}, \mathfrak{v}_2)$ and the diagram of the weakly nondegenerate reduction $(\mathfrak{g}, \mathfrak{q}_3)$ of $(\mathfrak{g}, \mathfrak{v}_2)$ is obtained by dropping the crosses under the α_i'''s with $\bar{\alpha}_i'' \prec_{C''} 0$:



The parabolic CR algebra $(\mathfrak{g}, \mathfrak{q}_3)$ is totally real, as $\mathfrak{q}_3 = \mathfrak{q}_{A_3}$ with

$$A_3 = \operatorname{diag}(1, 0, -1, 1, 0, -1) = \bar{A}_3.$$

Hence $(\mathfrak{g}, \mathfrak{q}_2) = (\mathfrak{g}, \mathfrak{v}_h) = (\mathfrak{g}, \mathfrak{q}_h)$ for all $h \geq 3$. The map $M(\mathfrak{g}, \mathfrak{q}) \longrightarrow M(\mathfrak{g}, \mathfrak{q}_3)$ is given by $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \longrightarrow (\ell_1 + \bar{\ell}_1, \ell_2 + \bar{\ell}_2)$.

Part 2

The compact orbit

CHAPTER 5

Compact parabolic CR algebras and manifolds

In this chapter we describe *compact* parabolic CR algebras: they are defined as the parabolic CR algebras $(\mathfrak{g},\mathfrak{q})$ for which the associated parabolic CR manifold $M(\mathfrak{g},\mathfrak{q})$ is compact, and correspond to the unique closed orbit of a real connected semisimple Lie group in a flag manifold of its complexification.

5.1 Satake diagrams

We recall the following result, due to Araki [Ara62] (cf. Theorem 3.1):

Proposition 5.1. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, $(\vartheta,\mathfrak{h})$ a Cartan pair, $\sigma:\mathfrak{h}^*_{\mathbb{R}}\longrightarrow\mathfrak{h}^*_{\mathbb{R}}$ the involution associated to the conjugation of $\hat{\mathfrak{g}}$ induced by $\mathfrak{g}$. Then $\mathfrak{h}$ is maximally noncompact if and only if $\hat{\mathfrak{g}}^\alpha\subset\hat{\mathfrak{k}}$ for all $\alpha\in\mathcal{R}_{\mathrm{im}}$.*

Assume now that $\mathfrak{h}$ is maximally noncompact and let $\mathcal{R}=\mathcal{R}(\hat{\mathfrak{g}},\hat{\mathfrak{h}})$. Then there exists a Weyl chamber $C\in\mathfrak{C}(\mathcal{R})$ such that:

(i) $\bar{\alpha}=\sigma(\alpha)\succ 0$ for all $\alpha\in\mathcal{R}^+(C)\setminus\mathcal{R}_\bullet$, i.e. C is S-adapted to σ;

(ii) there are pairwise strongly orthogonal roots $\beta_1,\dots,\beta_m\in\mathcal{R}_\bullet$ such that $s_{\beta_1}\circ\cdots\circ s_{\beta_m}$ is the element $w_{(C,\bar{C})}$ of the Weyl group that transforms C into $\bar{C}$; in particular $w_{(C,\bar{C})}$ is an involution: $w^2_{(C,\bar{C})}=\mathbf{1}$;

(iii) there is an involution $\varepsilon_C\in\mathbf{A}_{\hat{\mathfrak{h}}}$, such that $\varepsilon_C(C)=C$, that commutes with σ and with $w_{(C,\bar{C})}$, such that:

$$\sigma=\varepsilon_C\circ w_{(C,\bar{C})}\,. \tag{5.1}$$

The Weyl chamber C is uniquely determined modulo the analytic Weyl group $\mathbf{W}_{\mathfrak{h}}=\mathbf{N}_{\mathbf{G}}(\mathfrak{h})/\mathbf{Z}_{\mathbf{G}}(\mathfrak{h})$. □

With the notation of Proposition 5.1, and in particular with C and ε_C satisfying (i), (ii), and (iii), it follows from (3.3) that for all $\alpha\in\mathcal{B}(C)\setminus\mathcal{R}_\bullet$ there are integers $n_{\alpha,\beta}\geq 0$ such that:

$$\bar{\alpha}=\varepsilon_C(\alpha)+\sum_{\beta\in\mathcal{B}\cap\mathcal{R}_\bullet}n_{\alpha,\beta}\beta.$$

To the Weyl chamber C we associate the *Satake diagram* of $\mathfrak{g}$. It is obtained from the Dynkin diagram of $\hat{\mathfrak{g}}$, whose nodes correspond to the roots in $\mathcal{B}(C)$, by painting black those corresponding to imaginary roots and joining by a curved arrow those corresponding to distinct roots $\alpha_1,\alpha_2\in\mathcal{B}(C)\setminus\mathcal{R}_\bullet$ with $\varepsilon_C(\alpha_1)=\alpha_2$.

Satake diagrams coincide with the diagrams defined in § 3.4, with the difference that, in a Satake diagram, real and complex roots are both represented by a white node (i.e. "∘"). However, there is no loss of information. In fact, a root α corresponding to a white node in a Satake diagram is real if and only if there is no arrow issuing from it and it is not connected by a line to a black node.

5.2 Compact parabolic CR algebras

A real Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is *triangular* if all linear maps $\mathrm{ad}_{\mathfrak{g}}(X) \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g})$ with $X \in \mathfrak{t}$ can be simultaneously represented by triangular matrices in a suitable basis of $\mathfrak{g}$. All maximal triangular subalgebras of $\mathfrak{g}$ are conjugate by an inner automorphism (cf. [Mos61, §5.4]). A real Lie subalgebra of $\mathfrak{g}$ containing a maximal triangular subalgebra of $\mathfrak{g}$ is called a *t-subalgebra.*

An effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ will be called *compact* if $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ is a t-subalgebra of $\mathfrak{g}$.

We observe that a maximal triangular subalgebra of $\mathfrak{g}$ contains a maximal Abelian subalgebra of semisimple elements of $\mathfrak{g}$ having real eigenvalues. Hence we have:

Proposition 5.2. *An effective compact parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ admits an adapted Cartan pair $(\vartheta, \mathfrak{h})$ in which $\mathfrak{h}$ is a maximally noncompact Cartan subalgebra of $\mathfrak{g}$.* □

Theorem 5.3. *Let $\mathfrak{g}$ be a semisimple real Lie algebra and $\mathfrak{q}$ a parabolic subalgebra of its complexification $\hat{\mathfrak{g}}$. Then, up to CR isomorphisms, there is a unique compact parabolic effective CR algebra $(\mathfrak{g}', \mathfrak{q}')$ with $\mathfrak{g}'$ isomorphic to $\mathfrak{g}$ and $\mathfrak{q}'$ isomorphic to $\mathfrak{q}$.*

Proof. Fix a maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Its complexification $\hat{\mathfrak{t}}$ is solvable and therefore is contained in a maximal solvable subalgebra, i.e. a Borel subalgebra, $\mathfrak{b}$ of $\hat{\mathfrak{g}}$. Modulo an inner automorphism of $\hat{\mathfrak{g}}$, we can assume that $\mathfrak{b} \subset \mathfrak{q}$. The CR algebra $(\mathfrak{g}, \mathfrak{q})$ is compact parabolic.

Let $\mathfrak{q}$, $\mathfrak{q}'$ be parabolic subalgebras of $\hat{\mathfrak{g}}$ such that $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ and $\mathfrak{g}'_+ = \mathfrak{q}' \cap \mathfrak{g}$ are t-subalgebras of $\mathfrak{g}$. By an inner automorphism of $\mathfrak{g}$, we can assume that $\mathfrak{g}_+$ and $\mathfrak{g}'_+$ contain the same maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and hence a same maximal Abelian subalgebra of $\mathfrak{g}$ of semisimple elements having real eigenvalues. Hence, using another inner automorphism of $\mathfrak{g}$, we can assume that $\mathfrak{q}$ and $\mathfrak{q}'$ contain the same maximally noncompact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

The inner automorphism of $\hat{\mathfrak{g}}$ transforming $\mathfrak{q}$ into $\mathfrak{q}'$ can now be taken to be an element of the analytic Weyl group, leaving the Cartan subalgebra $\mathfrak{h}$ and hence $\mathfrak{g}$ invariant. It defines a CR isomorphism between $(\mathfrak{g}, \mathfrak{q})$ and $(\mathfrak{g}, \mathfrak{q}')$. □

We recall that a CR algebra $(\mathfrak{g}, \mathfrak{q})$ is *totally real* if $\mathfrak{q} = \overline{\mathfrak{q}}$, or, equivalently, if $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g} = \mathcal{H}_+ = \mathfrak{g} \cap (\mathfrak{q} + \overline{\mathfrak{q}})$. This is equivalent to the fact that $M(\mathfrak{g}, \mathfrak{q})$ is totally real, i.e. a CR manifold with CR dimension 0. For a totally real effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ the real subalgebra $\mathfrak{g}_+$ of $\mathfrak{g}$ is parabolic, hence a t-subalgebra of $\mathfrak{g}$. Thus we have:

Proposition 5.4. *A totally real effective parabolic CR algebra is compact.* □

Effective compact parabolic CR algebras correspond to compact orbits. In fact we have:

Theorem 5.5. *The CR manifold $M(\mathfrak{g}, \mathfrak{q})$, associated to an effective parabolic subalgebra $(\mathfrak{g}, \mathfrak{q})$, is compact if and only if $(\mathfrak{g}, \mathfrak{q})$ is compact.*

PROOF. Since $\mathbf{G}$ is a linear group, a $\mathbf{G}$-homogeneous space $\mathbf{G}/\mathbf{G}_+$ is compact if and only if $\mathbf{G}_+$ contains a maximal connected triangular subgroup (see [Oni93, II, Ch.5, §1.1]), i.e. if $\mathfrak{g}_+$ is a t-subalgebra of $\mathfrak{g}$. $\square$

In the following we will use the characterization of compact parabolic CR algebras given above. However, we also give a characterization of effective compact parabolic CR algebras $(\mathfrak{g}, \mathfrak{q})$ in terms of the set of roots $\mathcal{Q}$ associated to $\mathfrak{q}$ by any choice of an adapted Cartan pair $(\vartheta, \mathfrak{h})$.

PROPOSITION 5.6. *A necessary and sufficient condition in order that an effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ be compact is that:*

$$\mathcal{Q}^n \cap \bar{\mathcal{Q}}^{-n} \subset \mathcal{R}_\bullet \,, \tag{5.2}$$

i.e. all roots in $\mathcal{Q}^n \cap \bar{\mathcal{Q}}^{-n}$ are compact imaginary.

PROOF. A necessary and sufficient condition for $(\mathfrak{g}, \mathfrak{q})$ to be compact is that $\mathfrak{g}_+ + \mathfrak{k} = \mathfrak{g}$. By complexification this condition can be rewritten as:

$$\hat{\mathfrak{g}} = \mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}} \,. \tag{5.3}$$

Since $\hat{\mathfrak{h}} \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$, it suffices to show that (5.2) is equivalent to:

$$\hat{\mathfrak{g}}^\alpha \subset \mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}} \quad \text{for all } \alpha \in \mathcal{R} \,. \tag{5.4}$$

To make the equivalence more clear, we first prove:

LEMMA 5.7. *For all $\alpha \notin \left(\mathcal{Q}^n \cap \bar{\mathcal{Q}}^{-n}\right) \cup \left(\mathcal{Q}^{-n} \cap \bar{\mathcal{Q}}^n\right) = \left(\mathcal{Q}^n \setminus \bar{\mathcal{Q}}\right) \cup \left(\bar{\mathcal{Q}}^n \setminus \mathcal{Q}\right)$ we have*

$$\hat{\mathfrak{g}}^\alpha \subset \mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}} \,.$$

PROOF. We get: $\mathcal{R} \setminus \left[\left(\mathcal{Q}^n \setminus \bar{\mathcal{Q}}\right) \cup \left(\bar{\mathcal{Q}}^n \setminus \mathcal{Q}\right)\right] = \mathcal{Q}^r \cup \bar{\mathcal{Q}}^r \cup \left(\mathcal{Q}^n \cap \bar{\mathcal{Q}}\right) \cup \left(\bar{\mathcal{Q}}^n \cap \mathcal{Q}\right)$.

Clearly $\hat{\mathfrak{g}}^\alpha \subset \mathfrak{q} \cap \bar{\mathfrak{q}} \subset \mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}}$ when $\alpha \in \left(\mathcal{Q}^n \cap \bar{\mathcal{Q}}\right) \cup \left(\bar{\mathcal{Q}}^n \cap \mathcal{Q}\right)$.

Since the statement is invariant if we interchange $\mathfrak{q}$ and $\bar{\mathfrak{q}}$, to complete the proof of the lemma it suffices to show that $\hat{\mathfrak{g}}^\alpha \subset \mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}}$ for all $\alpha \in \mathcal{Q}^r$.

Let $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ be V-fit, so that $\sigma\left((\mathcal{R}^+(C) \cap \mathcal{Q}^r) \setminus \mathcal{R}_{\mathrm{re}}\right) \subset \mathcal{R}^-(C)$. We also use the notation $\hat{\mathfrak{k}}^{\alpha,-\bar{\alpha}} = \hat{\mathfrak{k}} \cap (\hat{\mathfrak{g}}^\alpha + \hat{\mathfrak{g}}^{-\bar{\alpha}})$. This is a one-dimensional complex subspace of $\hat{\mathfrak{g}}$ when $\alpha \in \mathcal{R} \setminus \mathcal{R}_*$.

For $\alpha \in \mathcal{Q}^r \cap (\mathcal{R}_{\mathrm{re}} \cup \mathcal{R}_{\mathrm{im}})$, we have $\alpha \in \mathcal{Q}^r \cap \bar{\mathcal{Q}}^r \subset \mathcal{Q} \cap \bar{\mathcal{Q}}$ and hence $\hat{\mathfrak{g}}^\alpha \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$.

If $\alpha \in \mathcal{Q}^r \cap \mathcal{R}^+(C) \cap \mathcal{R}_{\mathrm{cp}}$, then $-\bar{\alpha} \in \mathcal{R}^+(C) \subset \mathcal{Q}$. Hence $-\alpha, -\bar{\alpha} \in \mathcal{Q} \cap \bar{\mathcal{Q}}$, so that: $\hat{\mathfrak{g}}^{-\alpha} \subset \mathfrak{q} \cap \bar{\mathfrak{q}} \subset \mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}}$, $\hat{\mathfrak{g}}^\alpha \subset \hat{\mathfrak{g}}^{-\bar{\alpha}} + \hat{\mathfrak{k}}^{(\alpha,-\bar{\alpha})} \subset \mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}}$. $\square$

We conclude now the proof of Proposition 5.6. If condition (5.2) is satisfied, then $\left(\mathcal{Q}^n \cap \bar{\mathcal{Q}}^{-n}\right) \cup \left(\mathcal{Q}^{-n} \cap \bar{\mathcal{Q}}^n\right) \subset \mathcal{R}_\bullet$, hence $\mathfrak{g}^\alpha \subset \hat{\mathfrak{k}}$ for all $\alpha \in \left(\mathcal{Q}^n \cap \bar{\mathcal{Q}}^{-n}\right) \cup \left(\mathcal{Q}^{-n} \cap \bar{\mathcal{Q}}^n\right)$. In view of Lemma 5.7, we obtain (5.4) and hence (5.3).

Vice versa, assume that there is $\alpha \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^{-n} \setminus \mathcal{R}_\bullet$. If $\alpha \in \mathcal{R}_*$, then $\hat{\mathfrak{g}}^\alpha \in \hat{\mathfrak{p}}$, and cannot be contained in $\mathfrak{q} \cap \bar{\mathfrak{q}} + \hat{\mathfrak{k}}$. Otherwise, $\alpha \in \mathcal{R}_{\mathrm{cp}}$ and $\alpha, -\bar{\alpha} \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^{-n}$. This implies that $\hat{\mathfrak{g}}^\alpha \oplus \hat{\mathfrak{g}}^{\bar{\alpha}} \oplus \hat{\mathfrak{g}}^{-\alpha} \oplus \hat{\mathfrak{g}}^{-\bar{\alpha}} = \hat{\mathfrak{k}}^{(\alpha,-\bar{\alpha})} \oplus \hat{\mathfrak{k}}^{(-\alpha,\bar{\alpha})} \oplus \hat{\mathfrak{p}}^{(\alpha,-\bar{\alpha})} \oplus \hat{\mathfrak{p}}^{(\alpha,-\bar{\alpha})}$ has intersection $\{0\}$ with $\mathfrak{q} \cap \bar{\mathfrak{q}}$ and therefore (5.3) cannot possibly hold true. $\square$

We also have:

PROPOSITION 5.8. *An effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ is compact if and only if its weakest CR model (cf. § 4.2) $(\mathfrak{g}, \mathfrak{v})$ is compact.*

PROOF. The statement follows by (4.18) and the characterization of compact parabolic CR algebras given in Lemma 5.6. $\square$

With respect to an adapted Cartan pair $(\vartheta, \mathfrak{h})$, compact parabolic CR algebras are characterized as those which can be described in a S-adapted Weyl chamber. More precisely, we have:

PROPOSITION 5.9. *If $(\mathfrak{g}, \mathfrak{q})$ is an effective compact parabolic CR algebra and $\mathfrak{h}$ is a maximally noncompact Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$, then there exists an S-fit and S-adapted Weyl chamber C for $(\mathfrak{g}, \mathfrak{h})$.*

PROOF. Let $(\mathfrak{g}, \mathfrak{q})$ be compact. Modulo an inner automorphism of $\hat{\mathfrak{g}}$, we can assume that any given parabolic subalgebra $\mathfrak{q}$ of $\hat{\mathfrak{g}}$ contains a Borel subalgebra $\mathfrak{b}$ of the form:

$$\mathfrak{b} = \hat{\mathfrak{h}} \oplus \sum_{\alpha \in \mathcal{R}^+(C)} \hat{\mathfrak{g}}^{\alpha}$$

for a Weyl chamber $C \in \mathfrak{C}(R)$ that is S-adapted to the conjugation σ defined by the real form $\mathfrak{g}$. Then $\mathfrak{b} \cap \mathfrak{g}$ is contained in $\mathfrak{g}_+$ and contains a maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ (see for instance [Vin94, 4.4, 4.5]). The statement follows from the uniqueness stated in Theorem 5.3. $\square$

We may summarize the above discussion in the following:

THEOREM 5.10. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra and $(\vartheta, \mathfrak{h})$ an adapted Cartan pair with $\mathfrak{h}$ maximally noncompact among the Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{g}_+$. Then $(\mathfrak{g}, \mathfrak{q})$ is compact if and only if:*

(i) $\mathfrak{h}$ is maximally noncompact among all Cartan subalgebras of $\mathfrak{g}$;
(ii) any S-fit Weyl chamber in $\mathfrak{C}(\mathcal{R}, \mathcal{Q})$ is S-adapted.

PROOF. If $(\mathfrak{g}, \mathfrak{q})$ satisfies (i), (ii), then $\mathfrak{g}_+$ contains a maximal triangular subalgebra of $\mathfrak{g}$ (see [Vin94, 4.4, 4.5]), hence $(\mathfrak{g}, \mathfrak{q})$ is compact.

Vice versa, let $\mathfrak{g}, \mathfrak{q}$ be an effective compact parabolic CR algebra, and $(\vartheta, \mathfrak{h})$ an adapted Cartan pair with $\mathfrak{h}$ maximally noncompact in $\mathfrak{g}_+$. By Proposition 5.2, $\mathfrak{h}$ is maximally noncompact in $\mathfrak{g}$, thus (i) is proved. Let $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ be an S-fit Weyl chamber. Assume by contradiction that there exists $\alpha \in \mathcal{B} \setminus \mathcal{R}_{\text{im}}$ such that $\bar{\alpha} \prec 0$. Since C is S-fit, we have that $\alpha \in \mathcal{R}_{\text{cp}} \cap \mathcal{Q}^n$ and $\bar{\alpha} \in \mathcal{R}^-(C) \cap \mathcal{Q}^r$. This implies that all roots $\beta_i \in \text{supp}_C \bar{\alpha}$ belong to $\mathcal{B} \cap \mathcal{Q}^r$, hence for each $\beta \in \text{supp}_C \bar{\alpha}$ either $\beta_i \in \mathcal{R}_{\text{im}}$ or $\bar{\beta}_i \succ_C 0$. If $\bar{\alpha} = \sum_i k_i \beta_i$, with $k_i < 0$, then $\alpha = \sum_i k_i \bar{\beta}_i$, so we should have $\text{supp}_C \bar{\beta}_i \ni \alpha$ for some $\beta_i \in \mathcal{R}_{\text{im}} \cap \in \text{supp}_C \bar{\alpha}$, but this yields a contradiction, because $\alpha \in \Phi_C$ and $\text{supp}_C \bar{\alpha} \subset \mathcal{Q}^r$. $\square$

In view of this characterization, we can describe compact parabolic CR algebras by *cross-marked Satake diagrams*. Let $\boldsymbol{\mathcal{S}}$ be the Satake diagram of the semisimple real Lie algebra $\mathfrak{g}$. The nodes of $\boldsymbol{\mathcal{S}}$ correspond to the simple roots $\mathcal{B}(C)$ of a Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$ adapted to the conjugation σ defined by $\mathfrak{g}$. Fix a subset Φ_C of $\mathcal{B}(C)$ and consider the diagram $(\boldsymbol{\mathcal{S}}, \Phi_C)$ obtained from $\boldsymbol{\mathcal{S}}$ by adding a cross-mark on each node of $\boldsymbol{\mathcal{S}}$ corresponding to a root in Φ_C.

We associate to the pair $(\boldsymbol{\mathcal{S}}, \Phi_C)$ the CR algebra $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$, where $\mathfrak{q}_{\Phi_C}$ is the parabolic subalgebra defined by (2.15).

Two cross-marked Satake diagrams $(\boldsymbol{\mathcal{S}}, \Phi_C)$ and $(\boldsymbol{\mathcal{S}}, \Psi_C)$ are said to be *equivalent* if there exists an $\varepsilon \in \mathbf{Aut}(\boldsymbol{\mathcal{S}})$ such that $\Psi_C = \varepsilon(\Phi_C)$.

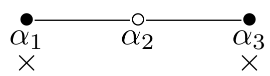
Theorems 5.3 and 5.10 yield:

THEOREM 5.11. *The correspondence:*

$$(\boldsymbol{\mathcal{S}}, \Phi_C) \longleftrightarrow (\mathfrak{g}, \mathfrak{q}_{\Phi_C})$$

is bijective between cross-marked Satake diagrams (modulo equivalence of cross-marked Satake diagrams) and effective compact parabolic CR algebras (modulo CR isomorphisms).

EXAMPLE 5.1. The diagram



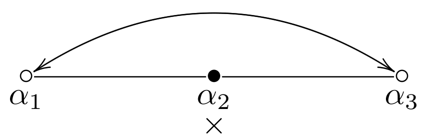
corresponds to

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{H}) \subset \mathfrak{sl}(4, \mathbb{C}),$$
$$\mathfrak{q} = \{Z \in \mathfrak{sl}(4, \mathbb{C}) \mid Z(\langle e_1 \rangle) \subset \langle e_1 \rangle,\ Z(\langle e_1, e_2, e_3 \rangle) \subset \langle e_1, e_2, e_3 \rangle\},$$

where e_1, e_2, e_3, e_4 is the canonical basis of $\mathbb{C}^4$ with $e_1\mathbb{H} = \langle e_1, e_2 \rangle$ and $e_3\mathbb{H} = \langle e_3, e_4 \rangle$.

The associated compact orbit is the CR manifold $M = M^{3,2}$ whose points are the pairs (ℓ_1, ℓ_3) consisting of a complex line ℓ_1 and a complex 3-plane ℓ_3 of $\mathbb{C}^4$ with $\ell_1 \cdot \mathbb{H} \subset \ell_3$. It is strictly nondegenerate, of CR dimension 3 and CR codimension 2; all its nonzero Levi forms have one positive, one negative and one zero eigenvalues (see for instance [HN]).

EXAMPLE 5.2. The diagram:



corresponds to

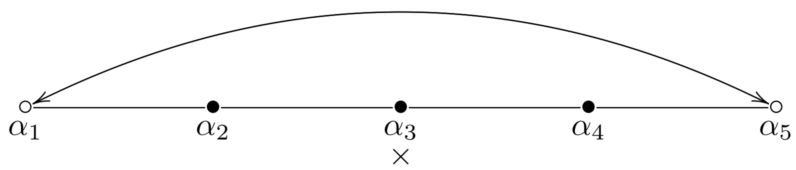
$$\mathfrak{g} = \mathfrak{su}(1,3) \subset \hat{\mathfrak{g}} = \mathfrak{sl}(4, \mathbb{C})$$
$$\mathfrak{q} = \{Z \in \mathfrak{sl}(4, \mathbb{C}) \mid Z(\langle e_1, e_2 \rangle) \subset \langle e_1, e_2 \rangle\}$$

where e_1, e_2, e_3, e_4 is a basis of $\mathbb{C}^4$ such that:

$$\mathfrak{su}(1,3) = \left\{ Z \in \mathfrak{sl}(4, \mathbb{C}) \;\middle|\; \begin{pmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix} Z + Z^* \begin{pmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix} = 0 \right\}$$

The associated compact orbit is a CR manifold $M = M^{3,1}$, of hypersurface type, with a Levi form having one positive, one negative and one zero eigenvalues, and is weakly nondegenerate but not strictly nondegenerate.

Example 5.3. The diagram:



corresponds to

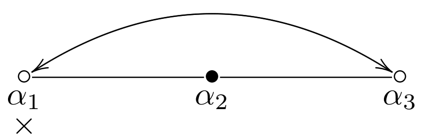
$$\mathfrak{g} = \mathfrak{su}(1,5) \subset \hat{\mathfrak{g}} = \mathfrak{sl}(6,\mathbb{C})$$
$$\mathfrak{q} = \{ Z \in \mathfrak{sl}(6,\mathbb{C}) \mid Z(\langle e_1, e_2, e_3 \rangle) \subset \langle e_1, e_2, e_3 \rangle \}$$

where $e_1, e_2, e_3, e_4, e_5, e_6$ is a basis of $\mathbb{C}^4$ such that

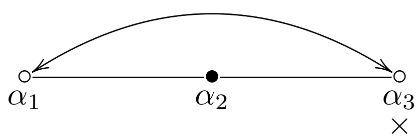
$$\mathfrak{su}(1,5) = \left\{ Z \in \mathfrak{sl}(4,\mathbb{C}) \;\middle|\; \begin{pmatrix} & & 1 \\ & I_3 & \\ 1 & & \end{pmatrix} Z + Z^* \begin{pmatrix} & & 1 \\ & I_3 & \\ 1 & & \end{pmatrix} = 0 \right\}$$

The associated compact orbit is the CR manifold $M = M^{8,1}$, of hypersurface type, with a Levi form having two positive, two negative and four zero eigenvalues, and is weakly nondegenerate but not strictly nondegenerate.

Example 5.4. The two diagrams:



and



are isomorphic. Indeed the map $F(\alpha_i) = \alpha_{4-i}$ for $i = 1, 2, 3$ defines an isomorphism of cross-marked Satake diagrams. The corresponding effective compact parabolic CR-algebras correspond to $\mathfrak{g} = \mathfrak{su}(1,3)$ and $\mathfrak{q} = \mathfrak{q}_{\{\alpha_1\}}$, $\mathfrak{q} = \mathfrak{q}_{\{\alpha_3\}}$, respectively. Let

$$K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and identify $\mathfrak{g}$ with the Lie algebra of 4×4 complex matrices with trace zero that satisfy $X^*K + KX = 0$. The CR isomorphism $(\mathfrak{g}, \mathfrak{q}_{\alpha_1}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\alpha_3})$ is given by the map $\mathfrak{su}(1,3) \ni X \longrightarrow -{}^tX \in \mathfrak{su}(1,3)$.

5.3 $\mathfrak{g}$-equivariant fibrations

In this section we discuss $\mathfrak{g}$-equivariant fibrations of compact parabolic effective CR algebras. Here we focus on the CR algebra aspects, preparing for applications that will be discussed later.

We keep the notation of the previous sections. In particular, $\mathfrak{g}$ is a semisimple real Lie algebra, $(\vartheta, \mathfrak{h})$ an adapted Cartan pair, with $\mathfrak{h}$ a maximally noncompact Cartan subalgebra of $\mathfrak{g}$, $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$, C an S-fit and S-adapted Weyl chamber, $\mathcal{B} = \mathcal{B}(C)$ is the set of simple roots in $\mathcal{R}^+ = \mathcal{R}^+(C)$.

Let $\Psi_C \subset \Phi_C \subset \mathcal{B}$. Then $\mathfrak{q}_{\Phi_C} \subset \mathfrak{q}_{\Psi_C}$ and the identity on $\mathfrak{g}$ defines a natural $\mathfrak{g}$-equivariant morphism of CR algebras:

$$\pi : (\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Psi_C}) \,. \tag{5.5}$$

We prove that the fiber of (5.5) can still be described by a compact parabolic CR algebra. The fiber is:

$$(\mathfrak{g}',\mathfrak{q}'),\quad \text{where}\quad \begin{cases} \mathfrak{g}' = \mathfrak{g}\cap\mathfrak{q}_{\Psi_C} = \mathfrak{g}\cap\bar{\mathfrak{q}}_{\Psi_C} \\ \hat{\mathfrak{g}}' = \mathfrak{q}_{\Psi_C}\cap\bar{\mathfrak{q}}_{\Psi_C} \\ \mathfrak{q}' = \hat{\mathfrak{q}}_{\Phi_C}\cap\hat{\mathfrak{g}}' = \hat{\mathfrak{q}}_{\Phi_C}\cap\mathfrak{q}_{\Psi_C}\cap\bar{\mathfrak{q}}_{\Psi_C} = \mathfrak{q}_{\Phi_C}\cap\bar{\mathfrak{q}}_{\Psi_C}\,. \end{cases} \tag{5.6}$$

Denote by $\mathcal{R}'$ and $\mathcal{Q}'$ the sets of roots $\alpha\in\mathcal{R}$ for which $\hat{\mathfrak{g}}^{\alpha}$ is contained in $\hat{\mathfrak{g}}'$ and $\hat{\mathfrak{q}}'$, respectively:

$$\begin{cases} \mathcal{R}' = \mathcal{Q}_{\Psi_C}\cap\bar{\mathcal{Q}}_{\Psi_C} \\ \mathcal{Q}' = \mathcal{Q}_{\Phi_C}\cap\bar{\mathcal{Q}}_{\Psi_C}\,, \end{cases} \tag{5.7}$$

define:

$$\begin{cases} \mathcal{R}'' = \mathcal{R}'\cap(-\mathcal{R}') \;=\; \mathcal{Q}^r_{\Psi_C}\cap\bar{\mathcal{Q}}^r_{\Psi_C} \\ \mathcal{Q}'' = \mathcal{Q}'\cap\mathcal{R}'' \\ \mathcal{A} = \mathcal{R}'\setminus\mathcal{R}'' \;=\; \left(\mathcal{Q}^n_{\Psi_C}\cap\bar{\mathcal{Q}}_{\Psi_C}\right)\cup\left(\bar{\mathcal{Q}}^n_{\Psi_C}\cap\mathcal{Q}_{\Psi_C}\right) \end{cases} \tag{5.8}$$

and set:

$$\begin{cases} \hat{\mathfrak{g}}'' = \hat{\mathfrak{h}}\oplus\bigoplus_{\alpha\in\mathcal{R}''}\hat{\mathfrak{g}}^{\alpha} \\ \mathfrak{q}'' = \mathfrak{q}'\cap\hat{\mathfrak{g}}'' \\ \hat{\mathfrak{a}} = \bigoplus_{\alpha\in\mathcal{A}}\hat{\mathfrak{g}}^{\alpha}\,. \end{cases} \tag{5.9}$$

Then $\mathcal{R}''$ is σ-invariant, $\hat{\mathfrak{g}}'' = \mathfrak{q}^r_{\Psi_C}\cap\bar{\mathfrak{q}}^r_{\Psi_C}$ is reductive, $\mathfrak{q}''$ is parabolic in $\hat{\mathfrak{g}}''$ and $\hat{\mathfrak{a}} = (\mathfrak{q}^n_{\Psi_C}\cap\bar{\mathfrak{q}}_{\Psi_C}) + (\mathfrak{q}_{\Psi_C}\cap\bar{\mathfrak{q}}^n_{\Psi_C})$ is an ideal in $\hat{\mathfrak{g}}'$, which is invariant with respect to the conjugation defined by the real form $\mathfrak{g}$.

LEMMA 5.12. $\hat{\mathfrak{a}}\subset\mathfrak{q}_{\Phi_C}$.

PROOF. We first show that $\mathcal{Q}^n_{\Psi_C}\cap\bar{\mathcal{Q}}_{\Psi_C}\cap\mathcal{R}_\bullet=\emptyset$. Assume by contradiction that there is $\alpha\in\mathcal{Q}^n_{\Psi_C}\cap\bar{\mathcal{Q}}_{\Psi_C}\cap\mathcal{R}_\bullet$. From $\alpha\in\mathcal{Q}^n_{\Psi_C}$ we obtain that $\bar{\alpha}=-\alpha\in\bar{\mathcal{Q}}^n_{\Psi_C}$, that is $\alpha\notin\bar{\mathcal{Q}}_{\Psi_C}$, which gives a contradiction.

Since $\mathcal{Q}^n_{\Psi_C}$ is contained in $\mathcal{R}^+$ and $\mathcal{Q}^n_{\Psi_C}\cap\bar{\mathcal{Q}}_{\Psi_C}$ does not contain imaginary roots, also its conjugate $\overline{\mathcal{Q}^n_{\Psi_C}\cap\bar{\mathcal{Q}}_{\Psi_C}} = \bar{\mathcal{Q}}^n_{\Psi_C}\cap\mathcal{Q}_{\Psi_C}$ is contained in $\mathcal{R}^+$. Hence $\mathcal{A}\subset\mathcal{R}^+\subset\mathcal{Q}_{\Phi_C}$. □

LEMMA 5.13. *$\mathcal{B}''=\mathcal{B}\cap\mathcal{R}''$ is a basis of $\mathcal{R}''$.*

PROOF. Indeed, assume that $\alpha\in\mathcal{R}''$ is the sum of two positive roots: $\alpha=\beta+\gamma$ with $\beta,\gamma\in\mathcal{R}^+$. Then $\alpha\in\mathcal{Q}^r_{\Psi_C}$ implies that also $\beta,\gamma\in\mathcal{Q}^r_{\Psi_C}$. If $\beta,\gamma\notin\mathcal{R}_\bullet$, then by the same argument applied to $\bar{\alpha}=\bar{\beta}+\bar{\gamma}\in\mathcal{Q}^r_{\Psi_C}$ we obtain that β,γ also belong to $\bar{\mathcal{Q}}^r_{\Psi_C}$ and hence to $\mathcal{R}''$.

Consider now the case where, for instance, $\beta\in\mathcal{R}_\bullet$. Then $\bar{\beta}=-\beta\in\mathcal{Q}^r_{\Psi_C}$ implies that $\beta\in\mathcal{R}''$ and therefore $\gamma=\alpha-\beta\in\mathcal{R}''$, showing that also in this case α is not simple in $[\mathcal{R}'']^+=\mathcal{R}^+\cap\mathcal{R}''$. This shows that $\mathcal{B}''$ is exactly the set of simple roots in $[\mathcal{R}'']^+$, and thus a basis of $\mathcal{R}''$. □

We have obtained:

Proposition 5.14. *The CR algebra $(\mathfrak{g}'', \mathfrak{q}'')$ is compact parabolic. Its cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}'', {\Phi_C}'')$ is the subdiagram of $(\boldsymbol{\mathcal{S}}, \Phi_C)$ consisting of the simple roots α such that:*

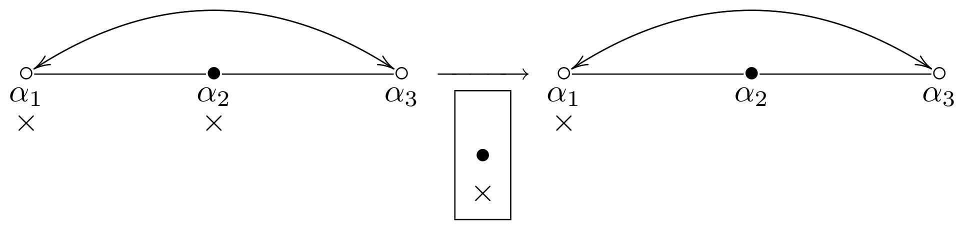
either (i) $\alpha \in \mathcal{R}_\bullet \setminus \Psi_C$, or (ii) $\alpha \notin \mathcal{R}_\bullet$ and $(\{\alpha\} \cup \operatorname{supp}(\bar{\alpha})) \cap \Psi_C = \emptyset$.

The cross-marks are left on the nodes corresponding to roots in $\Phi_C \cap \mathcal{B}''$. $\square$

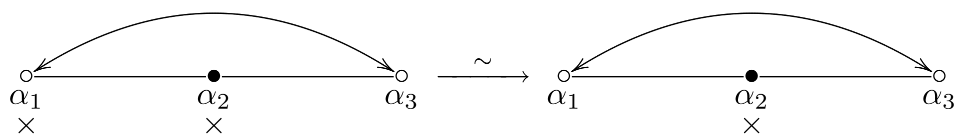
We say that a Satake diagram is *σ-connected* if either is connected or consists of two connected components, joined by curved arrows.

Theorem 5.15. *Let (5.5) be a $\mathfrak{g}$-equivariant fibration. Then the effective quotient of its fiber is the compact parabolic CR algebra whose cross-marked Satake diagram consists of the union of all σ-connected components of the diagram $\boldsymbol{\mathcal{S}}''$ described in Proposition 5.14, containing at least one cross-marked node.* $\square$

Example 5.5. Let $\mathfrak{g} = \mathfrak{su}(1, 3)$ and let $\Phi_C = \{\alpha_1, \alpha_2\}$, $\Psi_C = \{\alpha_1\}$. Then the cross-marked Satake diagrams corresponding to the CR algebra $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$, the basis $(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ and the corresponding effective fiber are given by:



In the case $\Psi_C = \{\alpha_2\}$ we have instead:



The fiber is trivial and the map is a CR morphism, but not a CR isomorphism. The corresponding map $M(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow M(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ is an analytic diffeomorphism and a CR map, but not a CR diffeomorphism.

Recall that $\mathfrak{g}$-equivariant morphism of CR algebras (5.5) is a *CR-fibration* if the quotient map

$$\mathfrak{q}_{\Phi_C} / \left(\mathfrak{q}_{\Phi_C} \cap \bar{\mathfrak{q}}_{\Phi_C}\right) \longrightarrow \mathfrak{q}_{\Psi_C} / \left(\mathfrak{q}_{\Psi_C} \cap \bar{\mathfrak{q}}_{\Psi_C}\right) \tag{5.10}$$

is onto. Set $M_{\Phi_C} = M(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$, $M_{\Psi_C} = M(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$, and $F = M(\mathfrak{g}'', \mathfrak{q}'')$. The condition that (5.5) is a *CR-fibration* is equivalent to the fact that every point of M_{Φ_C} has an open neighborhood which is CR diffeomorphic to the product of an open submanifold of M_{Ψ_C} and F.

The following Proposition provides a criterion to detect if a $\mathfrak{g}$-equivariant fibration is a CR fibration.

Proposition 5.16. *The following conditions are equivalent:*

(*i*) (5.5) *is a CR-fibration;*

(*ii*) $\mathcal{Q}^r_{\Psi_C} \setminus \mathcal{Q}_{\Phi_C} \subset \bar{\mathcal{Q}}_{\Psi_C}$;

(*iii*) $\mathcal{Q}^r_{\Psi_C} \cap \mathcal{Q}^n_{\Phi_C} \subset \bar{\mathcal{Q}}^r_{\Psi_C}$.

PROOF. First we prove the equivalence $(i) \Leftrightarrow (ii)$. A necessary and sufficient condition in order that (5.5) be a CR-fibration is that the sum of the CR dimensions of $(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ and of the fiber $(\mathfrak{g}', \mathfrak{q}')$ equals the CR-dimension of the total space $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$:

$$\begin{aligned}\dim_{\mathbb{C}}\mathfrak{q}_{\Phi_C} - \dim_{\mathbb{C}}\mathfrak{q}_{\Phi_C} \cap \bar{\mathfrak{q}}_{\Phi_C} =& \dim_{\mathbb{C}}\mathfrak{q}_{\Psi_C} - \dim_{\mathbb{C}}\mathfrak{q}_{\Psi_C} \cap \bar{\mathfrak{q}}_{\Psi_C} \\ &+\dim_{\mathbb{C}}\mathfrak{q}_{\Phi_C} \cap \bar{\mathfrak{q}}_{\Psi_C} - \dim_{\mathbb{C}}\mathfrak{q}_{\Phi_C} \cap \bar{\mathfrak{q}}_{\Phi_C} .\end{aligned}$$

Since all subspaces considered in this formula contain $\hat{\mathfrak{h}}$, this is equivalent to:

$$(*)\ \ |\mathcal{Q}_{\Phi_C}| = |\mathcal{Q}_{\Psi_C}| - |\mathcal{Q}_{\Psi_C} \cap \bar{\mathcal{Q}}_{\Psi_C}| + |\mathcal{Q}_{\Phi_C} \cap \bar{\mathcal{Q}}_{\Psi_C}| = |\mathcal{Q}_{\Psi_C} \setminus \bar{\mathcal{Q}}_{\Psi_C}| + |\mathcal{Q}_{\Phi_C} \cap \bar{\mathcal{Q}}_{\Psi_C}|,$$

(where we used $|A|$ for the number of elements of the finite set A). Since $\mathcal{Q}_{\Phi_C} \subset \mathcal{Q}_{\Psi_C}$, we always have:

$$\mathcal{Q}_{\Phi_C} \subset \left(\mathcal{Q}_{\Psi_C} \setminus \bar{\mathcal{Q}}_{\Psi_C}\right) \cup \left(\mathcal{Q}_{\Phi_C} \cap \bar{\mathcal{Q}}_{\Psi_C}\right) .$$

The two sets on the right hand side are disjoint. Hence $(*)$ is equivalent to:

$$\mathcal{Q}_{\Psi_C} \setminus \bar{\mathcal{Q}}_{\Psi_C} \subset \mathcal{Q}_{\Phi_C}.$$

As $\mathcal{Q}^n_{\Psi_C} \subset \mathcal{R}^+ \subset \mathcal{Q}_{\Phi_C}$, this is equivalent to

$$\mathcal{Q}^r_{\Psi_C} \setminus \mathcal{Q}_{\Phi_C} \subset \bar{\mathcal{Q}}_{\Psi_C}.$$

Next we prove that $(ii) \Rightarrow (iii)$. We distinguish several cases.

If $\alpha \in \mathcal{Q}^r_{\Psi_C} \cap \mathcal{R}_\bullet$, then $\bar{\alpha} = -\alpha \in \mathcal{Q}^r_{\Psi_C}$, that is $\alpha \in \bar{\mathcal{Q}}^r_{\Psi_C}$.

If $\alpha \in \mathcal{Q}^r_{\Psi_C} \cap \mathcal{Q}^n_{\Phi_C}$ and $\alpha \notin \mathcal{R}_\bullet$, then $\bar{\alpha} \succ 0$, hence $\alpha \in \bar{\mathcal{Q}}_{\Psi_C}$. On the other hand $-\alpha \in \mathcal{Q}^r_{\Psi_C} \setminus \mathcal{Q}_{\Phi_C}$ and, by (ii), $-\alpha \in \bar{\mathcal{Q}}_{\Psi_C}$, thus $\alpha \in \bar{\mathcal{Q}}^r_{\Psi_C}$.

Finally we prove that $(iii) \Rightarrow (ii)$. Let $\alpha \in \mathcal{Q}^r_{\Psi_C} \setminus \mathcal{Q}_{\Phi_C}$. Then $-\alpha \in \mathcal{Q}^r_{\Psi_C} \cap \mathcal{Q}^n_{\Phi_C}$, and (iii) implies that $-\alpha \in \bar{\mathcal{Q}}^r_{\Psi_C}$, which is equivalent to $\alpha \in \bar{\mathcal{Q}}^r_{\Psi_C}$. $\square$

In particular, we obtain:

PROPOSITION 5.17. *If $\bar{\mathcal{Q}}_{\Psi_C} = \mathcal{Q}_{\Psi_C}$, then* (5.5) *is a CR-fibration.*

PROOF. Indeed condition (iii) of Proposition 5.16 is trivially satisfied if $\bar{\mathcal{Q}}_{\Psi_C} = \mathcal{Q}_{\Psi_C}$. $\square$

We recall (see [MN05]) that a CR algebra $(\mathfrak{g}, \mathfrak{q})$ is *totally complex* if $\mathfrak{q} + \bar{\mathfrak{q}} = \hat{\mathfrak{g}}$. This condition is equivalent to $\mathfrak{g} + \mathfrak{q} = \hat{\mathfrak{g}}$ and to the fact that every homogeneous CR manifold M with associated CR algebra $(\mathfrak{g}, \mathfrak{q})$ is actually a complex manifold.

PROPOSITION 5.18. *If $\bar{\mathcal{Q}}_{\Psi_C} \cup \mathcal{Q}_{\Psi_C} = \bar{\mathcal{Q}}_{\Phi_C} \cup \mathcal{Q}_{\Phi_C}$, then* (5.5) *is a CR-fibration with a totally complex fiber.*

PROOF. Indeed we obtain: $\mathcal{Q}_{\Psi_C} \setminus \mathcal{Q}_{\Phi_C} \subset \bar{\mathcal{Q}}_{\Phi_C} \subset \bar{\mathcal{Q}}_{\Psi_C}$, and hence (ii) of Proposition 5.16 follows because $\mathcal{Q}^r_{\Psi_C} \supset \mathcal{Q}^r_{\Phi_C}$, $\mathcal{Q}^n_{\Psi_C} \subset \mathcal{Q}^n_{\Phi_C}$.

To show that the fiber is totally complex, we need to verify that $\mathfrak{q}_{\Psi_C} \cap \bar{\mathfrak{q}}_{\Psi_C} = \mathfrak{q}_{\Phi_C} \cap \bar{\mathfrak{q}}_{\Psi_C} + \mathfrak{q}_{\Psi_C} \cap \bar{\mathfrak{q}}_{\Phi_C}$. This is obvious because $\mathfrak{q}_{\Phi_C} \subset \mathfrak{q}_{\Psi_C} \subset \mathfrak{q}_{\Phi_C} + \bar{\mathfrak{q}}_{\Phi_C}$. □

Our next aim is to characterize $\mathfrak{g}$-equivariant CR fibrations in terms of cross marked Satake diagrams. For this we introduce some notation.

The *component* $\check{\Psi}_C(\alpha)$ of a root $\alpha \in \mathcal{B}(C)$ is the set of roots $\beta \in \mathcal{B}(C)$ belonging to the connected component of the node corresponding to α in the graph obtained from $\boldsymbol{\mathcal{S}}$ by deleting those nodes that correspond to roots in $\Psi_C \setminus \{\alpha\}$ and the lines and arrows issuing from them.

Given a subset $\mathcal{E}$ of $\mathcal{B}(C)$, its *exterior boundary* $\partial_e\mathcal{E}$ in $\boldsymbol{\mathcal{S}}$ is the set of roots α in $\mathcal{B}(C) \setminus \mathcal{E}$ such that, for some $\beta \in \mathcal{E}$, $\alpha + \beta \in \mathcal{R}$.

It will be convenient in the following to identify the nodes of $\boldsymbol{\mathcal{S}}$ with the corresponding roots in $\mathcal{B}(C)$. In particular, for a connected subset $\mathcal{E}$ of a Satake diagram $\boldsymbol{\mathcal{S}}$, we set $\delta(\mathcal{E}) = \sum_{\alpha\in\mathcal{E}} \alpha \in \mathcal{R}$.

We denote by $\Xi = \mathcal{B}(C) \setminus \mathcal{R}_\bullet$ the set of non imaginary simple roots.

LEMMA 5.19. *If* $\alpha \in \mathcal{R} \setminus \mathcal{R}_\bullet$, *then*

$$\mathrm{supp}_C(\bar{\alpha}) \supset \big(\partial_e(\mathrm{supp}_C(\alpha)) \cap \mathcal{R}_\bullet\big) \cup \varepsilon_C\big(\mathrm{supp}_C(\alpha) \setminus \mathcal{R}_\bullet\big).$$

PROOF. By inspecting the conjugation diagrams in [Ara62], we find that, if $\alpha \in \Xi$:

$$\mathrm{supp}_C(\bar{\alpha}) = \big(\check{\Xi}(\alpha) \setminus \{\alpha\}\big) \cup \{\varepsilon_C(\alpha)\}. \tag{5.11}$$

If $\alpha = \sum k_i\alpha_i \in \mathcal{R} \setminus \mathcal{R}_\bullet$, then

$$\mathrm{supp}_C(\bar{\alpha}) \supset \Big(\bigcup_{\substack{k_i>0\\ \alpha_i\in\Xi}} \mathrm{supp}_C(\bar{\alpha}_i)\Big) \setminus \big(\mathrm{supp}_C(\alpha) \cap \mathcal{R}_\bullet\big),$$

in particular $\mathrm{supp}_C(\bar{\alpha})$ contains $\varepsilon_C\big(\mathrm{supp}_C(\alpha) \setminus \mathcal{R}_\bullet\big)$.

If $\beta \in \partial_e\big(\mathrm{supp}_C(\alpha)\big) \cap \mathcal{R}_\bullet$, then, since $\mathrm{supp}_C(\alpha) \not\subset \mathcal{R}_\bullet$, there exists $\alpha_i \in \mathrm{supp}_C(\alpha) \cap \Xi$ such that $\beta \in \check{\Xi}(\alpha_i)$. This implies that $\mathrm{supp}_C(\bar{\alpha}) \ni \beta$. □

THEOREM 5.20. *A necessary and sufficient condition for* (5.5) *to be a CR $\mathfrak{g}$-equivariant fibration is that for every* $\alpha \in \Phi_C \setminus \Psi_C$ *either one of the following conditions hold:*

(i) $\check{\Psi}_C(\alpha) \subset \mathcal{R}_\bullet$;

(ii) $\check{\Psi}_C(\alpha) \not\subset \mathcal{R}_\bullet$, $\varepsilon_C(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet) \cap \Psi_C = \emptyset$, *and* $\partial_e\check{\Psi}_C(\alpha) \cap \mathcal{R}_\bullet = \emptyset$.

PROOF. Condition (ii) in Proposition 5.16 is equivalent to the fact that, for every root β:

$$\left.\begin{aligned} \mathrm{supp}_C(\beta) \cap \Psi_C = \emptyset \\ \mathrm{supp}_C(\beta) \cap \Phi_C \neq \emptyset \end{aligned}\right\} \implies \mathrm{supp}_C(\bar{\beta}) \cap \Psi_C = \emptyset. \tag{5.12}$$

Fix $\alpha \in \Phi_C \setminus \Psi_C$ and let $\beta = \delta(\check{\Psi}_C(\alpha))$. Then, according to Lemma 5.19, either $\beta \in \mathcal{R}_\bullet$ or $\operatorname{supp}_C(\bar{\beta}) \supset \varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big) \cup \big(\partial_e \check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big)$, showing that either (i) or (ii) must be valid.

Fix again $\alpha \in \Phi_C \setminus \Psi_C$ and let $\alpha_j \in \check{\Psi}_C(\alpha)$. If $\alpha_j \in \mathcal{R}_\bullet$ then $\bar{\alpha}_j = -\alpha_j$ and $\operatorname{supp}_C(\bar{\alpha}_j) \cap \Psi_C = \emptyset$. If $\alpha_j \notin \mathcal{R}_\bullet$, formula (5.11) implies that either $\operatorname{supp}_C(\bar{\alpha}_j) \subset \check{\Psi}_C(\alpha)$ or $\bar{\alpha}_j = \varepsilon_C(\alpha_j)$. In both cases $\operatorname{supp}_C(\bar{\alpha}_j) \cap \Psi_C = \emptyset$. For a generic $\beta \in \mathcal{R} \setminus \mathcal{R}_\bullet$ such that $\operatorname{supp}_C(\beta) \subset \check{\Psi}_C(\alpha)$ we have that:

$$\operatorname{supp}_C(\bar{\beta}) \subset \bigcup_{\alpha_j \in \operatorname{supp}_C(\beta)} \operatorname{supp}_C(\bar{\alpha}_j),$$

hence $\operatorname{supp}_C(\bar{\beta}) \cap \Psi_C = \emptyset$. □

5.4 Totally real and totally complex compact parabolic CR algebras

We already observed (Theorem 5.5) that a totally real parabolic CR algebra is compact, hence can be described by a cross marked Satake diagram. Now we characterize the cross-marked Satake diagrams that correspond to totally real parabolic CR algebras.

THEOREM 5.21. *An effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ with corresponding cross marked Satake diagram $(\boldsymbol{\mathcal{S}}, \Phi_C)$ is totally real if and only if the followig conditions hold true:*

(i) $\Phi_C \cap \mathcal{R}_\bullet = \emptyset$;
(ii) $\varepsilon_C(\Phi_C) = \Phi_C$.

PROOF. The conditions are clearly necessary. Indeed, if $\alpha \in \Phi_C \cap \mathcal{R}_\bullet$ or $\alpha \in \varepsilon_C(\Phi_C) \setminus \Phi_C$, then $\alpha \in \mathcal{Q}$ but $\bar{\alpha} \notin \mathcal{Q}$.

Vice versa, if condition (i) holds true then $\mathcal{R}_\bullet \subset \mathcal{Q} \cap \bar{\mathcal{Q}}$. Furthermore, if $\alpha \in \mathcal{R}_{\mathrm{cp}} \cap \mathcal{R}^-(C)$ then condition (ii) implies that $\alpha \in \mathcal{Q}^{-n}$ if and only if $\alpha \in \bar{\mathcal{Q}}^{-n}$. This shows that $\mathcal{Q} = \bar{\mathcal{Q}}$, thus $\mathfrak{q} = \bar{\mathfrak{q}}$. □

We also characterize the cross-marked Satake diagrams that correspond to totally complex parabolic CR algebras.

THEOREM 5.22. *A simple effective compact parabolic CR algebra $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ with associated cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}, \Phi_C)$ is totally complex if and only if either:*

(i) *$\mathfrak{g}$ is a compact real form, or*
(ii) *$\mathfrak{g}$ is of the complex type and all cross-marked nodes are in the same connected component of $\boldsymbol{\mathcal{S}}$, or*
(iii) *$(\boldsymbol{\mathcal{S}}, \Phi_C)$ is one of the following:*

(A II) $\qquad \begin{cases} \Phi = \{\alpha_1\} \\ \Phi = \{\alpha_\ell\} \end{cases}$

(D II) $\qquad \begin{cases} \Phi = \{\alpha_\ell\} \\ \Phi = \{\alpha_{\ell-1}\} \end{cases}$

PROOF. The CR algebra $(\mathfrak{g}, \mathfrak{q})$ is totally complex if and only if $\mathfrak{g} + \mathfrak{q} = \hat{\mathfrak{g}}$. This is equivalent to the fact that the parabolic CR manifold $\mathbf{G} \cdot o$ is open in the complex flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$. Since it is also closed, it follows that $\mathbf{G}$ is transitive on $\hat{\mathbf{G}}/\mathbf{Q}$. The result then follows from [Wol69, Corollary 1.7]. $\square$

CHAPTER 6

Nondegeneracy condition for compact parabolic CR algebras

In this chapter we find conditions on cross marked Satake diagrams of a compact parabolic CR algebra that are equivalent to geometric CR nondegeneracy conditions for the corresponding compact parabolic CR manifold.

6.1 Fundamental compact parabolic CR algebras

We give a criterion to read off the property of being fundamental from the cross-marked Satake diagram:

THEOREM 6.1. *An effective compact parabolic CR algebra* $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ *is fundamental if and only if its corresponding cross-marked Satake diagram* $(\boldsymbol{\mathcal{S}}, \Phi_C)$ *has the property:*

$$\alpha \in \Phi_C \setminus \mathcal{R}_\bullet \Longrightarrow \varepsilon_C(\alpha) \notin \Phi_C \,. \tag{6.1}$$

Here ε_C is the involution in $\mathcal{B}(C)$ defined in Proposition 5.1.

PROOF. Assume that α_1 and $\alpha_2 = \varepsilon_C(\alpha_1)$ both belong to Φ_C, and let $\Psi_C = \{\alpha_1, \alpha_2\}$. Then $\Psi_C \subset \Phi_C$ and hence $\mathfrak{q}_{\Phi_C} \subset \mathfrak{q}_{\Psi_C}$. To show that $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is not fundamental, it is sufficient to check that $\mathfrak{q}_{\Psi_C} = \bar{\mathfrak{q}}_{\Psi_C}$. To this aim it suffices to verify that $\mathcal{Q}^n_{\Psi_C} = \overline{\mathcal{Q}}^n_{\Psi_C}$. Let $\mathcal{B}(C) = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\ell\}$. Every root $\alpha \in \mathcal{Q}^n_{\Psi_C}$ can be written in the form $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ with $k_1 + k_2 > 0$. Since C is adapted to the conjugation σ, using (5.1) we obtain:

$$\bar{\alpha} = \sum_{i=1}^{\ell} k_i \varepsilon_C(\alpha_i) + \sum_{\beta \in \mathcal{B}_\bullet(C)} k_{\alpha,\beta}\beta = \sum_{i=1}^{\ell} k'_i \alpha_i,$$

with $k'_1 + k'_2 = k_2 + k_1 > 0$, showing that also $\bar{\alpha} \in \mathcal{Q}^n_{\Psi_C}$. Thus condition (6.1) is necessary.

Assume vice versa that there exists a proper parabolic subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q}_{\Phi_C} \subset \mathfrak{q}' = \bar{\mathfrak{q}}'$. Then $\mathfrak{q}' = \mathfrak{q}_{\Psi_C}$ for some $\Psi_C \subset \Phi_C$, $\Psi_C \neq \emptyset$. Since $\overline{\mathcal{Q}}^n_{\Psi_C} = \mathcal{Q}^n_{\Psi_C} \subset \mathcal{R}^+(C)$, we have $\Psi_C \cap \mathcal{R}_\bullet = \emptyset$. Hence, again by (5.1), we obtain that $\varepsilon_C(\alpha) \in \Psi_C$ for all $\alpha \in \Psi_C$. □

From Theorem 6.1, Theorem 5.15, and Proposition 5.17 we obtain:

THEOREM 6.2. *Let $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ be an effective compact parabolic CR algebra and let $(\boldsymbol{\mathcal{S}}, \Phi_C)$ be the corresponding cross-marked Satake diagram. Let*

$$\Psi_C = \{\alpha \in \Phi_C \setminus \mathcal{R}_\bullet \,|\, \varepsilon_C(\alpha) \in \Phi_C\}.$$

Then

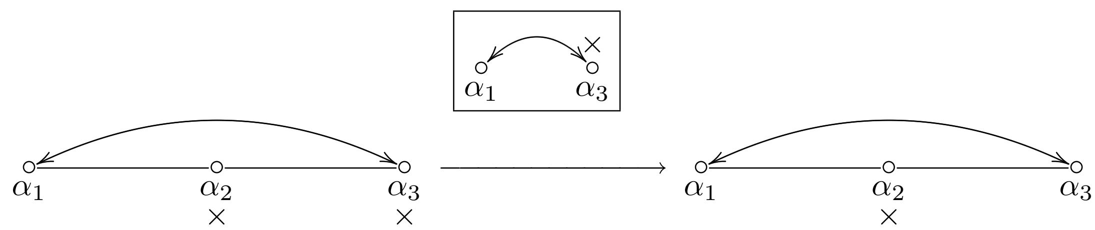
(*i*) *The diagram $\boldsymbol{\mathcal{S}}'$ obtained from $\boldsymbol{\mathcal{S}}$ by erasing all the nodes corresponding to the roots in Ψ_C and the lines and arrows issued from them is still a Satake diagram, corresponding to a semisimple real Lie algebra $\mathfrak{g}'$.*

(*ii*) *$(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ is a totally real effective compact parabolic CR algebra.*

(*iii*) *The natural map $(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Psi_C})$, defined by the inclusion $\mathfrak{q}_{\Phi_C} \subset \mathfrak{q}_{\Psi_C}$, is a $\mathfrak{g}$-equivariant CR fibration. The effective quotient of its fiber is the fundamental compact parabolic CR algebra $(\mathfrak{g}'', \mathfrak{q}_{\Phi'_C})$, associated to the cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}'', {\Phi_C}')$, where $\Phi'_C = \Phi_C \setminus \Psi_C$ and $\boldsymbol{\mathcal{S}}''$ is the union of the σ-connected components of $\boldsymbol{\mathcal{S}}'$ that contain some root of Φ'_C.* □

The map in (*iii*) is the fundamental reduction of $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ and the totally real CR algebra $(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ its *basis*.

EXAMPLE 6.1. Let $\mathfrak{g} \simeq \mathfrak{su}(2,2)$ and let $\Phi_C = \{\alpha_2, \alpha_3\}$ (we refer to the diagram below). We have $\varepsilon_C(\alpha_i) = \alpha_{4-i}$ for $i = 1,2,3$ and hence $\Psi_C = \{\alpha \in \Phi_C \,|\, \varepsilon_C(\alpha) \in \Phi_C\} = \{\alpha_2\}$. In particular $(\mathfrak{g}, \mathfrak{q}_{\{\alpha_2,\alpha_3\}})$ is not fundamental. We obtain by Theorem 6.2 a $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}_{\{\alpha_2,\alpha_3\}}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\{\alpha_2\}})$ with fundamental fiber $(\mathfrak{g}', \mathfrak{q}'_{\{\alpha_3\}})$, with $\mathfrak{g}' \simeq \mathfrak{sl}(2, \mathbb{C})$.



6.2 Weak nondegeneracy

In this section we characterize those compact parabolic CR algbras $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ that are weakly nondegenerate. We recall that this means that there is no nontrivial complex CR fibration $M(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow N$ with totally complex fibers. This is also equivalent to the fact that $M(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is not, locally, CR equivalent to the product of a CR manifold with the same CR codimension and of a complex manifold of positive dimension.

From Proposition 5.18 we obtain:

LEMMA 6.3. *A fundamental effective compact parabolic CR algebra $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is weakly degenerate if and only if there is $\Psi_C \subsetneq \Phi_C$ such that the $\mathfrak{g}$-equivariant fibration $(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ is a CR fibration with totally complex fiber.* □

LEMMA 6.4. *Let $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ be a compact fundamental effective parabolic CR algebra. A necessary and sufficient condition in order that $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ be weakly degenerate is that there exists $\Psi_C \subset \Phi_C$ satisfying conditions in Theorem 5.20 and such that $\mathfrak{q}_{\Psi_C} \subset \mathfrak{q}_{\Phi_C} + \bar{\mathfrak{q}}_{\Phi_C}$.* $\square$

We now give a characterization of the pairs (Φ_C, Ψ_C) for which (5.5) is a CR fibration with totally complex fiber in terms of properties of the roots α in $\Phi_C \setminus \Psi_C$.

LEMMA 6.5. *Let $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ be a compact fundamental effective parabolic CR algebra, with $\mathfrak{g}$ of the real type (i.e. $\hat{\mathfrak{g}}$ is also simple). Let $\emptyset \neq \Psi_C \subset \Phi_C$ and assume that (5.5) is a CR fibration with a totally complex fiber. Then each $\alpha \in \Phi_C \setminus \Psi_C$ satisfies one of the following conditions:*

(i) $\check{\Psi}_C(\alpha) \subset \mathcal{R}_\bullet$;

(ii) (a) $\check{\Psi}_C(\alpha) \cap \mathcal{R}_\bullet = \emptyset$ *and* (b) $\big(\check{\Psi}_C(\alpha)\big) \cap \varepsilon_C\big(\check{\Psi}_C(\alpha)\big) = \emptyset$;

(iii) (a) $\emptyset \neq \check{\Psi}_C(\alpha) \cap \mathcal{R}_\bullet \neq \check{\Psi}_C(\alpha)$ *and* (b) $\varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big) = \check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet$.

PROOF. Fix $\alpha \in \Phi_C \setminus \Psi_C$ with $\check{\Psi}_C(\alpha) \not\subset \mathcal{R}_\bullet$ and let $\delta = \delta\big(\check{\Psi}_C(\alpha)\big)$.

If $\beta \in \check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet$ and $\varepsilon_C(\beta) \in \check{\Psi}_C(\alpha)$, then $\operatorname{supp}_C(\bar{\delta}) \ni \varepsilon_C(\beta)$. Since it is connected and does not meet Ψ_C, we obtain $\operatorname{supp}_C(\bar{\delta}) \subset \check{\Psi}_C(\alpha)$. This implies that $\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet = \varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big)$. In this way we have shown that either $\varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big) \cap \check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet = \emptyset$, or $\varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big) = \check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet$

If $\check{\Psi}_C(\alpha) \cap \mathcal{R}_\bullet$ is not empty, then there exists $\beta \in \check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet$ such that $\partial_e\{\beta\} \cap \check{\Psi}_C(\alpha) \cap \mathcal{R}_\bullet \neq \emptyset$. Hence $\operatorname{supp}_C(\bar{\beta}) \cap \check{\Psi}_C(\alpha) \neq \emptyset$ and, by the same argument as above, $\varepsilon_C(\beta) \in \check{\Psi}_C(\alpha)$ and we get (iii.b).

Finally we consider the case where $\check{\Psi}_C(\alpha) \cap \mathcal{R}_\bullet = \emptyset$. The boundary $\partial_e\big(\check{\Psi}_C(\alpha)\big)$ is not empty, thus it contains a root $\beta \in \Psi_C$ and $\beta \notin \mathcal{R}_\bullet$ because of Theorem 5.20. The fact that $\mathfrak{g}, \mathfrak{q}_{\Psi_C}$ is fundamental implies that $\varepsilon_C(\beta) \notin \Psi_C$. In particular $\varepsilon_C(\beta) \notin \partial_e\big(\check{\Psi}_C(\alpha)\big)$. By applying again Theorem 5.20, we have $\varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big) \cap \Psi_C = \emptyset$, hence $\varepsilon_C(\beta) \notin \check{\Psi}_C(\alpha)$. Since $\varepsilon_C(\beta) \in \partial_e\big(\operatorname{supp}_C(\bar{\delta})\big)$ and $\operatorname{supp}_C(\bar{\delta}) \cap \mathcal{R}_\bullet \not\subset \check{\Psi}_C(\alpha)$, it follows that $\operatorname{supp}_C(\bar{\delta}) \cap \check{\Psi}_C(\alpha) = \emptyset$, thus $\check{\Psi}_C(\alpha) \cap \varepsilon_C\big(\check{\Psi}_C(\alpha)\big) = \emptyset$. $\square$

LEMMA 6.6. *With the same hypotheses of Lemma 6.5, the effective quotient of the fiber of the $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ has cross-marked Satake diagram $\boldsymbol{\mathcal{S}}', \Phi'_C)$ with:*

$$\boldsymbol{\mathcal{S}}' = \bigcup_{\alpha \in \Phi_C \setminus \Psi_C} \check{\Psi}_C(\alpha) \cup \varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big)$$

and $\Phi'_C = \Phi_C \cap \boldsymbol{\mathcal{S}}'$.

In particular the fiber is totally complex if and only if, for each $\alpha \in \Phi_C \setminus \Psi_C$, either condition ($i$) or condition ($ii$) of Lemma 6.5 holds.

PROOF. From Theorem 5.15 we know that $\boldsymbol{\mathcal{S}}' \subset \bigcup_{\alpha \in \Phi_C \setminus \Psi_C} \check{\Psi}_C(\alpha) \cup \varepsilon_C\big(\check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet\big)$. Equality then follows from the observation that if $\beta \in \check{\Psi}_C(\alpha) \setminus \mathcal{R}_\bullet$ then $\operatorname{supp}_C(\bar{\beta}) \cap \Psi_C = \emptyset$.

To prove the second statement, we can assume that there is exactly one root $\alpha \in \Phi_C \setminus \Psi_C$. In cases ($i$) and ($ii$) of Lemma 6.5 the cross-marked Satake diagram of the fiber is of the types described in Theorem 5.22 (i), (ii) and is totally complex. If we are in case (iii) of Lemma 6.5, then $\check{\Psi}_C(\alpha) \cap \mathcal{R}_\bullet \neq \emptyset$, and the fiber is totally complex if and only if $\big(\check{\Psi}_C(\alpha), \Phi_C \cap \check{\Psi}_C(\alpha)\big)$ is one of the diagrams in Theorem 5.22 (iii).

Since $\partial_e\big(\check{\Psi}_C(\alpha)\big) \cap \mathcal{R}_\bullet = \emptyset$ and $\varepsilon_C\big(\partial_e\check{\Psi}_C(\alpha)\big) \cap \big(\check{\Psi}_C(\alpha) \cup \partial_e\check{\Psi}_C(\alpha)\big) = \emptyset$, the involution ε_C is not the identity, hence $\boldsymbol{\mathcal{S}}$ must be of one of the types (cf. the Appendix) A III, A IV, D Ib, D IIIb, E II or E III. We exclude types A III, A IV, D Ib and E II because they do not contain subdiagrams of type A II or D II, so we are left with the types D IIIb and E III.

Type D IIIb must be excluded because in this case we have $\alpha = \alpha_1$ or $\alpha_{\ell-2}$, $\check{\Psi}_C(\alpha) = \{\alpha_1, \dots, \alpha_{\ell-2}\}$ and $\partial_e\big(\check{\Psi}_C(\alpha)\big) = \{\alpha_{\ell-1}, \alpha_\ell\} = \varepsilon_C\big(\partial_e\check{\Psi}_C(\alpha)\big)$.

Similarly type E III must be excluded because we have $\alpha = \alpha_3$ or α_5, $\check{\Psi}_C(\alpha) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $\partial_e\big(\check{\Psi}_C(\alpha)\big) = \{\alpha_1, \alpha_6\} = \varepsilon_C\big(\partial_e\check{\Psi}_C(\alpha)\big)$. $\square$

THEOREM 6.7. *Let $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ be a simple fundamental effective compact parabolic CR algebra and assume that it is not totally complex. Let Π be the set of simple roots α in Φ_C that satisfy either one of:*

(i) $\check{\Phi}_C(\alpha) \subset \mathcal{R}_\bullet$;

(ii) $\big(\check{\Phi}_C(\alpha) \cup \partial_e\check{\Phi}_C(\alpha)\big) \cap \mathcal{R}_\bullet = \emptyset$ *and* $\varepsilon_C\big(\check{\Phi}_C(\alpha)\big) \cap \Phi_C = \emptyset$.

Then $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is weakly nondegenerate if and only if $\Pi = \emptyset$.

Set $\Psi_C = \Phi_C \setminus \Pi$. Then $(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \to (\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ is a $\mathfrak{g}$-equivariant CR fibration with totally complex fiber and fundamental weakly nondegenerate base.

PROOF. Fix $\alpha \in \Phi_C \setminus \Psi_C$. Then the validity of either one of conditions (i) and (ii) is necessary and sufficient for $(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \to (\mathfrak{g}, \mathfrak{q}_{\Phi_C \setminus \{\alpha\}})$ to be a $\mathfrak{g}$-equivariant CR fibration with totally complex fiber. This observation, together with Lemma 6.6 and Lemma 6.4, yield our first statement.

To prove the last part of the Theorem, we make the following

Claim. Let $\alpha, \beta \in \Phi_C$ with $\alpha \in \Pi$. Then β satisfies either (i) or (ii) for Φ_C if and only if β satisfies either (i) or (ii) for $\Phi'_C = \Phi_C \setminus \{\alpha\}$.

Assuming that this claim is true, we conclude as follows. If $\Pi = \{\beta_1, \dots, \beta_k\}$, we have $\mathfrak{g}$-equivariant CR fibrations with totally complex fibers:

$$(\mathfrak{g}, \mathfrak{q}_{\Phi_C}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Phi_C \setminus \{\beta_1\}}) \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Phi_C \setminus \{\beta_1, \beta_2\}}) \longrightarrow \cdots \longrightarrow (\mathfrak{g}, \mathfrak{q}_{\Phi_C \setminus \Pi}).$$

Their composition is still a $\mathfrak{g}$-equivariant CR fibration with totally complex fiber, and the base $(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ is weakly nondegenerate.

Now we prove the claim. If $\beta \notin \partial_e\big(\check{\Phi}_C(\alpha)\big) \cup \partial_e\varepsilon_C\big(\check{\Phi}_C(\alpha)\big)$, then $\check{\Phi}_C(\beta) = \check{\Phi}'_C(\beta)$, $\varepsilon_C\big(\check{\Phi}_C(\beta)\big) = \varepsilon_C\big(\check{\Phi}'_C(\beta)\big)$, and there is nothing to prove.

Assume $\beta \in \partial_e\big(\check{\Phi}_C(\alpha)\big)$; then $\check{\Phi}'_C(\beta) = \check{\Phi}_C(\beta) \cup \check{\Phi}_C(\alpha)$. If $\check{\Phi}_C(\alpha) \subset \mathcal{R}_\bullet$, then $\check{\Phi}_C(\beta) \subset \mathcal{R}_\bullet$ if and only if $\check{\Phi}'_C(\beta) \subset \mathcal{R}_\bullet$.

If $\check{\Phi}'_C(\beta) \cap \mathcal{R}_\bullet = \emptyset$, we need to prove that, if β satisfies (i) or (ii), then $\check{\Phi}_C(\alpha) \cap \varepsilon_C\big(\check{\Phi}_C(\beta)\big) = \emptyset$. This is true because otherwise $\varepsilon_C(\beta) \in \partial_e\big(\check{\Phi}_C(\alpha)\big)$, and this yields a contradiction, because we assumed that $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is fundamental.

Finally, if $\beta \in \partial_e\varepsilon_C\big(\check{\Phi}_C(\alpha)\big)$, then $\beta \notin \mathcal{R}_\bullet$ and $\varepsilon_C(\beta) \in \partial_e\big(\check{\Phi}_C(\alpha)\big)$, again contradicting the assumption that $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is fundamental. $\square$

6.3 Strict nondegeneracy

In this section we give necessary and sufficient conditions for a weakly nondegenerate CR algebra to be strictly nondegenerate. We recall from the introduction that the CR geometry of strict nondegenerate homogeneous CR manifolds can be related to the so called "standard" models and investigated by using Levi-Tanaka algebras (cf. [MN97], [Tan67], [Tan70]). Therefore, by classifying the weakly degenerate compact orbits that do not have the strict nondegeneracy property, we single out a class of homogeneous CR manifolds with a highly non trivial CR structure that cannot be discussed by using the standard Levi-Tanaka models. This also explains the need to introduce CR algebras, as a generalization of the Levi-Tanaka algebras, in [MN05].

First we reformulate weak and strict nondegeneracy in terms of the root system:

LEMMA 6.8. *A fundamental effective compact parabolic CR algebra* $(\mathfrak{g},\mathfrak{q})$ *is weakly nondegenerate if and only if for every root* $\alpha\in\bar{\mathcal{Q}}\setminus\mathcal{Q}$ *there exist a sequence* $(\beta_i\in\mathcal{Q})_{1\le i\le n}$ *such that*

$$\alpha_j=\alpha+\sum_{i\le j}\beta_i\in\mathcal{R}\quad\forall j=1,\dots,n\,,\qquad \alpha_n\notin\mathcal{Q}\cup\bar{\mathcal{Q}}\,. \tag{6.2}$$

PROOF. The statement is an easy consequence of [MN05, Theorem 6.2]. □

Likewise we have:

LEMMA 6.9. *A fundamental effective compact parabolic CR algebra* $(\mathfrak{g},\mathfrak{q})$ *is strictly nondegenerate if and only if for every root* $\alpha\in\bar{\mathcal{Q}}\setminus\mathcal{Q}$ *there exists a root* $\beta\in\mathcal{Q}$ *such that* $\alpha+\beta\in\mathcal{R}$ *and* $\alpha+\beta\notin\mathcal{Q}\cup\bar{\mathcal{Q}}$. □

Next we prove that it suffices to check this condition on purely imaginary roots:

PROPOSITION 6.10. *A necessary and sufficient condition for a fundamental effective weakly nondegenerate compact parabolic CR algebra* $(\mathfrak{g},\mathfrak{q})$ *to be strictly nondegenerate is that for every root* $\alpha\in\mathcal{R}_\bullet\cap\bar{\mathcal{Q}}\setminus\mathcal{Q}$ *there exists* $\beta\in\mathcal{Q}$ *such that* $\alpha+\beta\in\mathcal{R}$ *and* $\alpha+\beta\notin\mathcal{Q}\cup\bar{\mathcal{Q}}$.

PROOF. The condition is obviously necessary. To prove sufficiency, consider a root $\alpha\in\bar{\mathcal{Q}}\setminus\mathcal{Q}$, $\alpha\notin\mathcal{R}_\bullet$; since $\alpha\prec 0$, we have $-\alpha\in\mathcal{R}^+\setminus\mathcal{R}_\bullet$. This implies that $-\bar{\alpha}\in\mathcal{R}^+\subset\mathcal{Q}$. Then $-\alpha\in\bar{\mathcal{Q}}$ and $\alpha\in\bar{\mathcal{Q}}^r$. By the assumption that $(\mathfrak{g},\mathfrak{q})$ is weakly nondegenerate, using Lemma 6.8 we can find a sequence of roots (β_i) satisfying (6.2). Take the sequence $(\beta_i)_{1\le i\le n}$ of minimal length; we claim that for every permutation τ of the indices, the sequence $(\beta_{\tau(i)})_{1\le i\le n}$ still satisfies (6.2).

Indeed, fix a Chevalley basis $\{X_\alpha\}_{\alpha\in\mathcal{R}}\cup\{H_\beta\}_{\beta\in\mathcal{B}_C}$. Then, for every transposition $(i,i+1)$:

$$\begin{aligned}\mathfrak{q}+\bar{\mathfrak{q}}\not\ni{}&[X_{\beta_n},\dots,X_{\beta_{i+1}},X_{\beta_i},\dots,X_{\beta_1},X_\alpha]=\\&=[X_{\beta_n},\dots,X_{\beta_i},X_{\beta_{i+1}},\dots,X_{\beta_1},X_\alpha]+[X_{\beta_n},\dots,[X_{\beta_{i+1}},X_{\beta_i}],\dots,X_{\beta_1},X_\alpha].\end{aligned}$$

The last addendum in the right hand side belongs to $\mathfrak{q}+\bar{\mathfrak{q}}$, by our assumption that $(\beta_i)_{1\le i\le n}$ has minimal length. Thus:

$$[X_{\beta_n},\dots,X_{\beta_i},X_{\beta_{i+1}},\dots,X_{\beta_1},X_\alpha]\in\hat{\mathfrak{g}}\setminus(\mathfrak{q}+\bar{\mathfrak{q}})\,.$$

In particular $\alpha+\beta_i\in\mathcal{R}$ for every i. At least one of the β_i's, say β_{i_0}, does not belong to $\bar{\mathcal{Q}}$, so $\alpha+\beta_{i_0}\notin\bar{\mathcal{Q}}$. Indeed, since $\alpha\in\bar{\mathcal{Q}}^r$, if $\alpha+\beta_i\in\bar{\mathcal{Q}}$, then also

$\beta_i = (\alpha+\beta_i)+(-\alpha) \in \bar{\mathcal{Q}}$. By a permutation, we can take $\beta_{i_0} = \beta_n$. Then we claim that $\alpha+\beta_n \notin \mathcal{Q}\cup\bar{\mathcal{Q}}$. Indeed we already choose β_n so that $\alpha+\beta_n \notin \bar{\mathcal{Q}}$. If $\alpha+\beta_n \in \mathcal{Q}$, we have $[X_{\beta_1},\dots,X_{\beta_{n-1}},X_{\beta_n},X_\alpha] = [X_{\beta_1},\dots,X_{\beta_{n-1}},[X_{\beta_n},X_\alpha]] \in \mathfrak{q}$, because $X_{\beta_i} \in \mathfrak{q}$ for every $i=1,\dots,n$, and hence $\alpha_n \in \mathcal{Q}$, contradicting (6.2). $\square$

THEOREM 6.11. *Let $(\mathfrak{g},\mathfrak{q}_{\Phi_C})$ be an effective compact parabolic CR algebra, with $\mathfrak{g}$ simple. If $(\mathfrak{g},\mathfrak{q}_{\Phi_C})$ is weakly nondegenerate, but is not strictly nondegenerate, then Φ_C is contained in a connected component of $\mathcal{B}\cap\mathcal{R}_\bullet$.*

The strictly nondegenerate $(\mathfrak{g},\mathfrak{q}_{\Phi_C})$ with $\mathfrak{g}$ simple and Φ_C contained in a connected component of $\mathcal{R}_\bullet$ are those listed below:

(B Ib / B II) $\quad \Phi_C = \{\alpha_{p+1}\}$

(C IIa / IIb) $\quad \Phi_C = \{\alpha_{2i-1}\},\ 1\le i\le p$

(D Ia) $\quad \Phi_C = \{\alpha_{p+1}\}$

(D II) $\quad \Phi_C = \{\alpha_2\}$

(E III) $$\Phi_C = \begin{cases} \{\alpha_4\} \\ \{\alpha_3,\alpha_4\} \\ \{\alpha_4,\alpha_5\} \\ \{\alpha_3,\alpha_4,\alpha_5\} \end{cases}$$

$\begin{pmatrix} \text{E IV} \\ \text{E VII} \\ \text{E IX} \end{pmatrix}$ $$\Phi_C = \begin{cases} \{\alpha_3\} \\ \{\alpha_5\} \\ \{\alpha_3,\alpha_4\} \\ \{\alpha_3,\alpha_5\} \\ \{\alpha_4,\alpha_5\} \\ \{\alpha_2,\alpha_3\} \\ \{\alpha_2,\alpha_5\} \end{cases}$$

(F II) $$\Phi_C = \begin{cases} \{\alpha_2\} \\ \{\alpha_4\} \end{cases}$$

PROOF. We prove the first statement. The proof of the second will be omitted, as it requires a straightforward case by case analysis, chasing over the different Satake diagrams.

Suppose that $(\mathfrak{g},\mathfrak{q}_{\Phi_C})$ is weakly, but not strictly, nondegenerate. Then there is some root $\alpha \in \bar{\mathcal{Q}}_{\Phi_C}\setminus\mathcal{Q}_{\Phi_C}$, $\alpha \prec 0$, such that $\alpha+\beta \in \mathcal{Q}_{\Phi_C}\cup\bar{\mathcal{Q}}_{\Phi_C}$ for all $\beta\in\mathcal{Q}_{\Phi_C}$ for which $\alpha+\beta\in\mathcal{R}$. By Proposition 6.10 we can take $\alpha\in\mathcal{R}_\bullet$. Let $\mathcal{B}'$ be the connected component of $\mathrm{supp}_C(\alpha)$ in $\mathcal{B}\cap\mathcal{R}_\bullet$. Since $\alpha\notin\mathcal{Q}_{\Phi_C}$, we have $\mathcal{B}'\cap\Phi_C\neq\emptyset$.

Since we assumed that $(\mathfrak{g},\mathfrak{q}_{\Phi_C})$ is weakly nondegenerate, for each $\gamma\in\Phi_C$ the set $\check{\Phi}_C(\gamma)$ is not contained in $\mathcal{R}_\bullet$. As $\mathrm{supp}_C(\alpha)\cap\Phi_C\neq\emptyset$, this implies that there is some $\beta\in\mathcal{Q}_{\Phi_C}$, with $\beta\prec 0$, such that $\beta\notin\mathcal{R}_\bullet$ and $\alpha+\beta\in\mathcal{R}$. Since $\beta\in\mathcal{Q}^r_{\Phi_C}$ and $-\alpha\in\mathcal{Q}^n_{\Phi_C}$, we obtain that $\alpha+\beta\notin\mathcal{Q}_{\Phi_C}$. If $\mathcal{B}'\cap\Phi_C$ contains some α_i which does not belong to $\mathrm{supp}_C(\alpha)$, this α_i would belong to $\mathrm{supp}_C(\overline{\alpha+\beta})$. Indeed

$\alpha + \beta \notin \mathcal{R}_\bullet$, hence $\mathrm{supp}_C(\overline{\alpha+\beta})$ contains all simple imaginary roots γ that are not in $\mathrm{supp}_C(\alpha+\beta)$ and such that $\partial_e \check{\Xi}(\gamma) \cap \mathrm{supp}_C(\alpha+\beta) \neq \emptyset$. This shows that $\mathcal{B}' \cap \Phi_C = \mathrm{supp}_C(\alpha) \cap \Phi_C$.

Let $\mathcal{A} = (\mathcal{R}_\bullet \cap [\Phi_C \setminus \mathcal{B}']) \cup \varepsilon_C(\Phi_C \setminus \mathcal{R}_\bullet)$. We want to show that $\mathcal{A} = \emptyset$.

Assume by contradiction that $\mathcal{A}$ is not empty. Then there exists a segment S in $\mathcal{B} \setminus \Phi_C$ joining $\mathcal{A}$ to $\mathrm{supp}_C(\alpha)$, i.e. such that $\partial_e S \cap \mathcal{A} \neq \emptyset$, $\partial_e S \cap \mathrm{supp}_C(\alpha) \neq \emptyset$. By taking S of minimal lenght, we can also assume that $S \cap (\mathcal{A} \cup \mathrm{supp}_C(\alpha)) = \emptyset$.

Let $\beta = -\delta(S)$. Then $\beta \prec 0$, $\beta \in \mathcal{Q}^r_{\Phi_C}$ and $\beta \notin \mathcal{R}_\bullet$, so that $\alpha + \beta \in \mathcal{R} \setminus \mathcal{Q}_{\Phi_C}$.

If there is some α_i in $\partial_e S \cap \mathcal{A} \cap \mathcal{R}_\bullet \neq \emptyset$, then $\alpha + \beta \in \mathcal{R}$, $\mathrm{supp}_C(\overline{\alpha+\beta}) \ni \alpha_i$, and $\alpha + \beta \notin \bar{\mathcal{Q}}_{\Phi_C}$, contradicting our assumption.

If $\partial_e S \cap \mathcal{A} \cap \mathcal{R}_\bullet = \emptyset$, there is α_i in $\Phi_C \setminus \mathcal{R}_\bullet$ with $\varepsilon_C(\alpha_i) \in \partial_e S \cap \mathcal{A}$. Set $\beta' = \beta - \varepsilon_C(\alpha_i)$. Then $\beta' \in \mathcal{Q}_{\Phi_C}$, and $\alpha + \beta' \in \mathcal{R} \setminus \left(\mathcal{Q}_{\Phi_C} \cup \bar{\mathcal{Q}}_{\Phi_C}\right)$, yielding a contradiction; this shows that $\mathcal{A}$ is empty, completing the proof of our first claim.$\square$

CHAPTER 7

Essential pseudoconcavity for compact parabolic CR manifolds

Let (M, HM, J) be a CR manifold of finite kind. We say that (M, HM, J) is *essentially pseudoconcave* (see [HN96]) if it is possible to define a Hermitian symmetric smooth scalar product h on the fibers of HM such that for each $\xi \in H^0M$ the Levi form $\mathcal{L}_\xi$ has zero trace with respect to h. For a homogeneous CR manifold, this last condition is equivalent to the fact that for each $\xi \in H^0M$ the Levi form $\mathcal{L}_\xi$ is either 0 or has at least one positive and one negative eigenvalue.

The CR functions defined on essentially pseudoconcave CR manifolds enjoy some nice properties, like local smoothness and the local maximum modulus principle; CR sections of CR complex line bundles have the weak unique continuation property . When M is compact and essentially pseudoconcave, global CR functions are constant and CR-meromorphic functions form a field of finite transcendence degree.

In this chapter we classify the essentially pseudoconcave compact parabolic CR manifolds.

We keep the notation of the previous chapter. In particular, $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is an effective compact parabolic CR algebra, with associated cross-marked Satake diagram $(\boldsymbol{\mathcal{S}}, \Phi_C)$. Moreover, we introduce a Chevalley system for $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$, i.e. a family $(Z_\alpha)_{\alpha\in\mathcal{R}}$ with the properties ([Bou02, Ch. VIII, §2]):

- (i) $Z_\alpha \in \hat{\mathfrak{g}}^\alpha$ for all $\alpha \in \mathcal{R}$;
- (ii) $[Z_\alpha, Z_{-\alpha}] = -H_\alpha$, where H_α is the unique element of $[\hat{\mathfrak{g}}^\alpha, \hat{\mathfrak{g}}^{-\alpha}]$ such that $\alpha(H_\alpha) = 2$;
- (iii) the $\mathbb{C}$-linear map that transforms each $H \in \hat{\mathfrak{h}}$ into $-H$ and Z_α into $Z_{-\alpha}$ for every $\alpha \in \mathcal{R}$ is an automorphism of the complex Lie algebra $\hat{\mathfrak{g}}$.

In particular, $(Z_\alpha)_{\alpha\in\mathcal{R}} \cup (H_\alpha)_{\alpha\in\mathcal{B}}$ is a basis of $\hat{\mathfrak{g}}$ as a $\mathbb{C}$-linear space. We denote by $(\xi^\alpha)_{\alpha\in\mathcal{R}} \cup (\omega^\alpha)_{\alpha\in\mathcal{B}}$ the corresponding dual basis in $\hat{\mathfrak{g}}^*$.

Let $\mathfrak{M}$ be the complex flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$ and M the compact orbit $\mathbf{G}/\mathbf{G}_+$ of $\mathbf{G}$ in $\mathfrak{M}$. As usual, $o \simeq e \cdot \mathbf{G}_+ \simeq e \cdot \mathbf{Q}$ is the base point. We note that

$$T^{1,0}_{\mathbf{o}}\mathfrak{M} \simeq \hat{\mathfrak{g}}/\mathfrak{q} \simeq \langle Z_\alpha \mid -\alpha \in \mathcal{Q}^n \rangle_{\mathbb{C}} \ .$$

Therefore a Hermitian metric in $\mathfrak{M}$ is expressed at the point $\mathbf{o}$ by:

$$\tilde{h}_{\mathbf{o}} = \sum_{\alpha,\beta\in\mathcal{Q}^n} c_{\alpha,\bar{\beta}} \xi^{-\alpha} \otimes \bar{\xi}^{-\beta} \,.$$

where $(c_{\alpha,\bar{\beta}})$ is Hermitian symmetric and positive definite. For the compact orbit we have:

$$T^{1,0}_{\mathbf{o}}M \simeq \bar{\mathfrak{q}}/(\mathfrak{q} \cap \bar{\mathfrak{q}}) \simeq \left\langle Z_\alpha \mid -\alpha \in \mathcal{Q}^n \,,\ \alpha \in \bar{\mathcal{Q}} \right\rangle_{\mathbb{C}} \ .$$

Thus a Hermitian metric h in $T^{1,0}M$ can be represented at $\mathbf{o}$ by:

$$h_{\mathbf{o}} = \sum_{\alpha,\beta\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n} c_{\alpha,\bar{\beta}}\xi^{-\alpha}\otimes\bar{\xi}^{-\beta}.$$

where $(c_{\alpha,\bar{\beta}})$ is again Hermitian symmetric and positive definite.

The subspace $\hat{\mathfrak{t}} = \sum_{\alpha\in\mathcal{Q}^n\cap\bar{\mathcal{Q}}^n}\hat{\mathfrak{g}}^{-\alpha}$ is a nilpotent Lie subalgebra of $\hat{\mathfrak{g}}$, which is the complexification of a real subalgebra $\mathfrak{t} = \hat{\mathfrak{t}}\cap\mathfrak{g}$ of $\mathfrak{g}$. It can be identified to the quotient $T_{\mathbf{o}}M/H_{\mathbf{o}}M$ and hence its dual space $\mathfrak{t}^*$ to the stalk $H^0_{\mathbf{o}}M$ of the characteristic bundle of M at $\mathbf{o}$.

From this discussion we obtain the criterion:

PROPOSITION 7.1. *A necessary and sufficient condition for M to be essentially pseudoconcave is that there exists a positive definite Hermitian symmetric matrix $(c_{\alpha,\bar{\beta}})_{\alpha,\beta\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n}$ such that*

$$\sum_{\substack{\alpha,\beta\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n\\ \alpha+\bar{\beta}=\gamma}} c_{\alpha,\bar{\beta}}[Z_\alpha,\bar{Z}_\beta] = 0 \qquad \forall\gamma\in\mathcal{Q}^n\cap\bar{\mathcal{Q}}^n\,. \tag{7.1}$$

PROOF. Indeed (7.1) ie equivalent to the formula we obtain by changing α,β,γ into $-\alpha,-\beta,-\gamma$. $\square$

Denote by $\check{\mathfrak{T}}^{1,0}$ the $\mathbb{C}$-linear subspace of $\hat{\mathfrak{g}}$ with basis $(Z_\alpha)_{\alpha\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n}$. To each $\gamma\in\mathcal{Q}^n\cap\bar{\mathcal{Q}}^n$ we associate a complex-valued form of type $(1,1)$ in $\check{\mathfrak{T}}^{1,0}$:

$$\mathbf{L}_\gamma : \check{\mathfrak{T}}^{1,0}\times\check{\mathfrak{T}}^{1,0}\ni(Z,W)\longrightarrow\mathbf{L}_\gamma(Z,W) = (1/i)\kappa_{\hat{\mathfrak{g}}}(Z_{-\gamma},[Z,\bar{W}])\in\mathbb{C}\,, \tag{7.2}$$

where $\kappa_{\hat{\mathfrak{g}}}$ is the Killing form in $\hat{\mathfrak{g}}$. When $\gamma=\bar{\gamma}$ is real, we take $Z_{-\gamma}$ in $\mathfrak{g}$, to obtain a Hermitian symmetric $\mathbf{L}_\gamma$.

We have:

LEMMA 7.2. *The following are equivalent:*

(i) *$M = M(\mathfrak{g},\mathfrak{q})$ is essentially pseudoconcave;*

(ii) *There exists a Hermitian symmetric positive definite form $\mathbf{h}$ in $\check{\mathfrak{T}}^{1,0}$ such that all $\mathbf{L}_\gamma$, for $\gamma\in\mathcal{Q}^n\cap\bar{\mathcal{Q}}^n$ have zero trace with respect to $\mathbf{h}$;*

(iii) *For each $\gamma\in\mathcal{Q}^n\cap\bar{\mathcal{Q}}^n$ the Hermitian quadratic forms in $\check{\mathfrak{T}}^{1,0}$:*

$$\check{\mathfrak{T}}^{1,0}\ni Z\longrightarrow\Re\mathbf{L}_\gamma(Z,\bar{Z})\in\mathbb{R}\quad\text{and}\quad\check{\mathfrak{T}}^{1,0}\ni Z\longrightarrow\Im\mathbf{L}_\gamma(Z,\bar{Z})\in\mathbb{R} \tag{7.3}$$

are either 0 or have at least one positive and one negative eigenvalue.

PROOF. The equivalence was proved in [HN96]. $\square$

PROPOSITION 7.3. *Let $(\mathfrak{g},\mathfrak{q})$ be an effective compact parabolic fundamental CR algebra. A necessary and sufficient condition for $M = M(\mathfrak{g},\mathfrak{q})$ to be essentially pseudoconcave is that for all real roots $\gamma\in\mathcal{Q}^n\cap\bar{\mathcal{Q}}^n$ the Hermitian symmetric form $\mathbf{L}_\gamma$ is either zero or has at least one positive and one negative eigenvalue.*

PROOF. The condition is obviously necessary. We prove sufficiency. Let Γ be a subset of $\mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ and let $\mathcal{H}(\Gamma)$ the $\mathbb{R}$-linear space consisting of the Hermitian symmetric parts of all linear combinations $\sum_{\gamma\in\Gamma} a_\gamma \mathbf{L}_\gamma$ with $a_\gamma \in \mathbb{C}$. When $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ is not real, the Hermitian symmetric part h of $a\mathbf{L}_\gamma$, for $a \in \mathbb{C}$, satisfies $h(Z_\alpha, \bar{Z}_\alpha) = 0$ for all $\alpha \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$. More generally, if Γ_0 is the set of all $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ for which $\sum_{\alpha\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n} \mathbf{L}_\gamma(Z_\alpha, \bar{Z}_\alpha) = 0$, then the matrices $(h(Z_\alpha, \bar{Z}_\beta))_{\alpha,\beta\in\mathcal{Q}^n\setminus\bar{\mathcal{Q}}^n}$ corresponding to $h \in \mathcal{H}(\Gamma_0)$ have zero trace and thus every $h \in \mathcal{H}(\Gamma_0)$ that is $\neq 0$ has at least one positive and one negative eigenvalue.

Choose Γ as a maximal subset of $\mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ that contains Γ_0 and has the property that all non zero $h \in \mathcal{H}(\Gamma)$ have at least one positive and one negative eigenvalue.

If $\Gamma = \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$, then $M(\mathfrak{g}, \mathfrak{q})$ is essentially pseudoconcave. Assume by contradiction that there is $\gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n \setminus \Gamma$.

Then γ is real, $\mathbf{L}_\gamma$ is Hermitian symmetric and $\mathcal{H}(\Gamma \cup \{\gamma\}) = \mathcal{H}(\Gamma) + \mathbb{R}\cdot\mathbf{L}_\gamma$. Moreover, there is at least one root $\alpha_0 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ such that $\gamma = \alpha_0 + \bar{\alpha}_0$. Assume that there is another root $\alpha_1 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ with $\alpha_1 + \bar{\alpha}_1 = \gamma$ and $\mathbf{L}_\gamma(Z_{\alpha_0}, \bar{Z}_{\alpha_0}) \cdot \mathbf{L}_\gamma(Z_{\alpha_1}, \bar{Z}_{\alpha_1}) < 0$. If $h \in \mathcal{H}(\Gamma)$, then $h(Z_{\alpha_0}, \bar{Z}_{\alpha_0}) = h(Z_{\alpha_1}, \bar{Z}_{\alpha_1}) = 0$. Then the matrix associated in the basis (Z_α) to a linear combinations $h + c\mathbf{L}_\gamma$ with $c \in \mathbb{R}$, $c \neq 0$, has two entries of opposite sign on the main diagonal and therefore at least one negative and one positive eigenvalue. This would contradict the maximality of Γ. Hence we must assume that all terms $\mathbf{L}_\gamma(Z_\alpha, \bar{Z}_\alpha)$ have the same sign.

By the assumption that $\mathbf{L}_\gamma$ has at least one positive and one negative eigenvalue, we deduce that there are roots $\beta_1, \beta_2 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ such that $\beta_2 \neq \bar{\beta}_1$ and $\beta_1 + \bar{\beta}_2 = \bar{\beta}_1 + \beta_2 = \gamma$, so that $\mathbf{L}_\gamma(Z_{\beta_1}, \bar{Z}_{\beta_2}) \neq 0$. If $h(Z_{\beta_2}, \bar{Z}_{\beta_2}) = 0$ for all $h \in \mathcal{H}(\Gamma)$, then the matrix corresponding to $h + c\mathbf{L}_\gamma$, for $h \in \mathcal{H}(\Gamma)$, $c \in \mathbb{R}$, $c \neq 0$ in the basis (Z_α) contains a principal 2×2 minor matrix, corresponding to β_1, β_2, of the form

$$\begin{pmatrix} a & \lambda \\ \bar{\lambda} & 0 \end{pmatrix} \quad \text{with} \quad a \in \mathbb{R} \quad \text{and} \quad \lambda \in \mathbb{C},\ \lambda \neq 0\,.$$

Thus it would have at least one positive and one negative eigenvalue, contradicting the choice of Γ.

Therefore, if $\Gamma \neq \mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$, we have:

(i) there exists $\alpha_0 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ such that $\alpha_0 + \bar{\alpha}_0 = \gamma \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$;

(ii) there exists $\alpha_1, \alpha_2 \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ with $\alpha_2 \neq \alpha_1$, $\alpha_2 \neq \bar{\alpha}_1$ and $\alpha_1 + \bar{\alpha}_2 = \gamma$;

(iii) for all $\alpha, \beta \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$ with $\alpha \neq \beta$, $\beta \neq \bar{\alpha}$ and $\alpha + \bar{\beta} = \gamma$, we have $\alpha + \bar{\alpha} \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ and $\beta + \bar{\beta} \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$.

The roots $\alpha_0, \bar{\alpha}_0, \alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2$ generate a root system $\mathcal{R}'$ in their span in $\mathfrak{h}^*_{\mathbb{R}}$, that is closed under conjugation. Since we have the relations $\alpha_0 + \bar{\alpha}_0 = \alpha_1 + \bar{\alpha}_2 = \alpha_2 + \bar{\alpha}_0$, the span of $\mathcal{R}'$ has dimension ≤ 4. Moreover, $\alpha_0 + \bar{\alpha}_0$, $\alpha_1 + \bar{\alpha}_1$ and $\alpha_2 + \bar{\alpha}_2$ must be three distinct roots in $\mathcal{R}'$. Indeed, set $\alpha_1 + \bar{\alpha}_1 = \gamma_1$, $\alpha_2 + \bar{\alpha}_2 = \gamma_2$. By assumption $\gamma_1 \neq \gamma \neq \gamma_2$. Moreover we obtain $\alpha_1 - \alpha_2 = \gamma_1 - \gamma = \gamma - \gamma_2$, i.e. $\gamma_1 + \gamma_2 = 2\gamma$, which implies that $\gamma_1 \neq \gamma_2$ when $\gamma_1 \neq \gamma \neq \gamma_2$.

Thus the dimension of the span of $\mathcal{R}'$ is ≤ 4. An inspection of the Satake diagrams corresponding to bases of at most 4 simple roots shows that no such root system contains 3 distinct positive real roots that are sum of a root and its conjugate. Denote by Ω the set of positive real roots γ that are of the form $\gamma = \alpha + \bar{\alpha}$ with $\alpha \in \mathcal{R}$. To verify our claim, we only need to consider the diagrams with

$\ell = 3, 4$ and $\Omega \neq \emptyset$:

$$\text{(A IIIa, IIIb)} \qquad \begin{array}{ll} \mathfrak{su}(1,3): & \Omega = \{\alpha_1 + \alpha_2 + \alpha_3\} \\ \mathfrak{su}(2,2): & \Omega = \{\alpha_1 + \alpha_2 + \alpha_3\} \\ \mathfrak{su}(1,4): & \Omega = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\} \\ \mathfrak{su}(2,3): & \Omega = \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\} \end{array}$$

$$\text{(C IIa)} \qquad \begin{array}{ll} \mathfrak{sp}(1,2): & \Omega = \{\alpha_1 + 2\alpha_2 + \alpha_3\} \\ \mathfrak{sp}(1,3): & \Omega = \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4\} \end{array}$$

Thus we obtained a contradiction, proving our statement. □

THEOREM 7.4. *Let $(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ be a simple effective and fundamental compact parabolic CR algebra. Then $M(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is always essentially pseudoconcave if $\mathfrak{g}$ is either of the complex type, or compact, or of real type* A II, A IIIb, B, C IIb, D I, D II, D IIIa, E II, E IV, E VI, E VII, E IX. *In the remaining cases $M(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ is essentially pseudoconcave if and only if we have one of the following:*

$$\text{(A IIIa-IV)} \qquad \begin{cases} \Phi_C \subset \mathcal{R}_\bullet \\ \Phi_C \subset \{\alpha_i \,|\, i < p\} \cup \{\alpha_i \,|\, i > q\} \end{cases}$$

$$\text{(C IIa)} \qquad \begin{cases} \Phi_C \subset \{\alpha_{2h-1} \,|\, 1 \leq h \leq p\} \\ \Phi_C \subset \{\alpha_i \,|\, i > 2p\} \end{cases}$$

$$\text{(D IIIb)} \qquad \Phi_C \cap \{\alpha_{\ell-1}, \alpha_\ell\} = \emptyset$$

$$\text{(E III)} \qquad \begin{cases} \{\alpha_4\} \subset \Phi_C \subset \mathcal{R}_\bullet \\ \Phi_C = \{\alpha_3, \alpha_5\} \end{cases}$$

$$\text{(F II)} \qquad \Phi_C \subset \{\alpha_1, \alpha_2\}$$

[See the table of Satake diagrams in the Appendix for the types and the references to the roots in the statement.]

We first require two Lemmas.

LEMMA 7.5. *Let $\mathfrak{g}$ be a semisimple real Lie algebra, with a Cartan decomposition, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathfrak{h}$ a Cartan subalgebra which is invariant with respect to the corresponding Cartan involution ϑ and with maximal vector part. Denote by σ the conjugation of $\hat{\mathfrak{g}}$ with respect to the real form $\mathfrak{g}$ and let $\tau = \sigma \circ \vartheta$ the conjugation with respect the compact form $\mathfrak{k} \oplus i\,\mathfrak{p}$ of $\hat{\mathfrak{g}}$. Set $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$. Then there exists a Chevalley system $\{X_\alpha\}_{\alpha \in \mathcal{R}}$ with $X_\alpha \in \hat{\mathfrak{g}}^\alpha$ such that:*

$$\begin{cases} [X_\alpha, X_{-\alpha}] = -H_\alpha \quad \forall \alpha \in \mathcal{R} \\ [H_\alpha, X_\beta] = \beta(H_\alpha) X_\beta \\ [X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} \\ \tau(X_\alpha) = \sigma(X_\alpha) = \bar{X}_\alpha = X_{-\alpha} \quad \forall \alpha \in \mathcal{R}_\bullet \end{cases}$$

where the H_α and the coefficients $N_{\alpha,\beta}$ satisfy:

$$\begin{cases} \beta(H_\alpha) = q - p \\ N_{\alpha,\beta} = \pm(q+1) \\ N_{\alpha,\beta} \cdot N_{-\alpha,\alpha+\beta} = -p(q+1) \\ \text{if } \beta - q\alpha, \dots, \beta + p\alpha \text{ is the } \alpha\text{-string through } \beta. \end{cases}$$

Proof of Lemma 7.5. For the proof of this lemma we refer the reader to [Bou02, Ch.VIII], or [Hel01, Ch.III]. $\square$

Lemma 7.6. *With the notation of Lemma 7.5: let $\alpha, \beta \in \mathcal{R}$, with $\alpha \in \mathcal{R}_\bullet$, and $\alpha+\beta \in \mathcal{R}$, $\alpha-\beta \notin \mathcal{R}$, $\beta+\bar{\beta} \in \mathcal{R}$. Let*

$$\beta, \ldots, \beta + p\alpha \qquad \text{and} \qquad \beta+\bar{\beta}-q'\alpha, \ldots, \beta+\bar{\beta}+p'\alpha$$

be the α-strings through β and $\beta+\bar{\beta}$, respectively. Then we have:

$$\left[X_{\alpha+\beta}, \bar{X}_{\alpha+\beta}\right] = \left[[X_\alpha, X_\beta], \overline{[X_\alpha, X_\beta]}\right] = (p - p'(1+q'))\, [X_\beta, \bar{X}_\beta]\,. \tag{7.4}$$

Proof of Lemma 7.6. We observe that $[X_\alpha, X_\beta] = \pm X_{\alpha+\beta}$, because $\beta - \alpha \notin \mathcal{R}$. We have:

$$\begin{aligned}\left[X_{\alpha+\beta}, \bar{X}_{\alpha+\beta}\right] &= \left[[X_\alpha, X_\beta], \overline{[X_\alpha, X_\beta]}\right] = \left[[X_\alpha, X_\beta], [X_{-\alpha}, \bar{X}_\beta]\right] \\ &= [[[X_\alpha, X_\beta], X_{-\alpha}], \bar{X}_\beta] + [X_{-\alpha}, [[X_\alpha, X_\beta], \bar{X}_\beta]] \\ &= [[[X_\alpha, X_{-\alpha}], X_\beta], \bar{X}_\beta] + [X_{-\alpha}, [X_\alpha, [X_\beta, \bar{X}_\beta]]] \\ &= \left(-\beta(H_\alpha) + N_{\alpha,\beta+\bar{\beta}} N_{-\alpha,\beta+\bar{\beta}+\alpha}\right) [X_\beta, \bar{X}_\beta]\,,\end{aligned}$$

which, by Lemma 7.5, yields (7.4). $\square$

Proof of Theorem 7.4. We exclude in the statement the split forms, because in these cases $(\mathfrak{g}, \mathfrak{q})$ is not fundamental. When $\mathfrak{g}$ is compact, $(\mathfrak{g}, \mathfrak{q})$ is totally complex and thus essentially pseudoconcave, since the condition on the Levi form is trivially fulfilled.

For $\mathfrak{g}$ of the complex types or of the real types A II, A IIIb, B, C IIb, D I, D II, D IIIa, E II, E IV, E VI, E VII, E IX the statement follows from the fact that $\mathcal{Q}^n \cap \bar{\mathcal{Q}}^n$ cannot possibly contain a root of the form $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{Q}^n \setminus \bar{\mathcal{Q}}^n$.

We proceed by a case by case analysis of the simple real Lie algebras containing real roots γ of the form $\gamma = \alpha + \bar{\alpha}$.

$\boxed{\text{A IIIa} - \text{IV}}$ The positive real roots that are of the form $\alpha + \bar{\alpha}$ for some $\alpha \in \mathcal{R}$ are:

$$\gamma_h = \textstyle\sum_{j=h}^{p+q-h} \alpha_j \quad \text{for} \quad h = 1, \ldots, p.$$

(i) Assume that $\Phi_C \subset \mathcal{R}_\bullet$. All γ_h's belong to $\mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$ and are sums $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$. To prove that $\mathbf{L}_{\gamma_h}$ has at least one positive and one negative eigenvalue, we consider the roots $\beta = \sum_{j=h}^{q-1} \alpha_j$ and $\delta = \sum_{j=p+1}^{p+q-h}$. They both belong to $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ and $\beta + \bar{\beta} = \delta + \bar{\delta} = \gamma_h$. We have $\delta = \bar{\beta} + \eta$ with $\eta \in \mathcal{R}_\bullet$ and $\bar{\beta} - \eta \notin \mathcal{R}$. Since $\gamma_h \pm \eta \notin \mathcal{R}$, by Lemma 7.6 we obtain:

$$[X_\delta, \bar{X}_\delta] = [[X_{\bar{\beta}}, X_\eta], \overline{[X_{\bar{\beta}}, X_\eta]}] = -\,[X_\beta, \bar{X}_\beta]\,.$$

(ii) Assume that $\Phi_C \cap \mathcal{R}_\bullet = \emptyset$ and $\Phi_C \cap \{\alpha_p, \alpha_q\} = \emptyset$. Let $\Phi_C = \{\alpha_{j_1}, \ldots, \alpha_{j_r}, \alpha_{h_1}, \ldots, \alpha_{h_s}\}$ with $1 \le j_1 < \cdots < j_r < p < q < h_1 < \cdots < h_s \le \ell = p+q-1$. We can assume that $r \ge 1$ and, if $s \ge 1$, that $p - j_r < h_1 - q$. Let

$h'_1 = p+q-h_1$ if $s \geq 1$, and $h'_1 = 0$ otherwise. The real roots in $\mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$ are the γ_k's with $1 \leq k \leq j_r$. All $\mathbf{L}_{\gamma_k}$'s with $k \leq h'_1$ are 0. To show that the $\mathbf{L}_{\gamma_k}$'s with $h'_1 < k \leq j_r$ have at least one positive and one negative eigenvalues, we consider $\alpha = \sum_{i=k}^{j_r} \alpha_i$ and $\beta = \sum_{i=k}^{\ell-j_r} \alpha_i$. They both belong to $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$, are distinct, and $\alpha + \bar{\beta} = \gamma_k$.

(iii) When $\Phi_C \cap \{\alpha_p, \alpha_q\} \neq \emptyset$, we can assume, modulo a CR isomorphism, that $\alpha_p \in \Phi_C$. Then $\gamma_p \in \mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$ and all pairs (α, β) of roots in $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ with $\alpha + \bar{\beta} = \gamma_p$ are of the form (β_k, β_k) with $\beta_k = \sum_{i=p}^{k} \alpha_i$ for some $p \leq k < q$. By Lemma 7.6, we have

$$[X_{\beta_k}, \bar{X}_{\beta_k}] = [X_{\alpha_p}, \bar{X}_{\alpha_p}]\,,$$

and hence the corresponding $\mathbf{L}_{\gamma_p}$ is $\neq 0$ and semi-definite.

(iv) Assume that $\Phi_C \cap \mathcal{R}_\bullet \neq \emptyset$ and $\Phi_C \not\subset \mathcal{R}_\bullet$. We can assume, modulo a CR isomorphism, that there is $\alpha_j \in \Phi_C$ with $j \leq p$ and that $\alpha_i \notin \Phi_C$ if either $j < i \leq p$, or $q \leq i \leq p+q-j$. Let r be the largest integer $< q$ such that $\alpha_r \in \Phi_C$. We observe that $\gamma_j \in \mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$ and that all pairs (α, β) of roots in $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ with $\alpha + \bar{\beta} = \gamma_j$ are of the form (β_k, β_k) with $\beta_k = \sum_{i=j}^{k} \alpha_i$ for some $r \leq k < q$. As in the previous case, for all $p \leq k < q$:

$$[X_{\beta_k}, \bar{X}_{\beta_k}] = [X_{\beta_p}, \bar{X}_{\beta_p}]\,,$$

and hence $\mathbf{L}_{\gamma_j}$ is $\neq 0$ and semi-definite.

$\boxed{\text{C IIa}}$ The positive real roots that can be written a sum $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{R}$ are:

$$\gamma_h = \alpha_{2h-1} + \alpha_\ell + 2\sum_{i=2h}^{\ell-1} \alpha_i \quad \text{for} \quad h = 1, \ldots, p.$$

(i) Assume that $\Phi_C = \{\alpha_{2h_1-1}, \ldots, \alpha_{2h_r-1}\}$ with $1 \leq h_1 < \cdots < h_r \leq p$. The roots in $\mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$ that are of the form $\alpha + \bar{\alpha}$ are the γ_h with $1 \leq h \leq h_r$. The root γ_{h_r} is the only one that can be written as $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$. But this root can also be written as $\alpha + \bar{\beta}$ with $\alpha = \alpha_{2h_r-1} + \gamma_{h_r}$ and $\beta = \alpha_{2h_r-1}$, and therefore $\mathbf{L}_{\gamma_{h_r}}$ has at least one positive and one negative eigenvalue.

(ii) Assume that $\Phi_C = \{\alpha_{k_1}, \ldots, \alpha_{k_r}\}$ with $2p < k_1 < \cdots < k_r \leq \ell$. Then all γ_h belong to $\mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$. Fix $1 \leq h \leq p$, and consider the roots $\beta = \sum_{i=2h}^{2p} \alpha_i + \alpha_\ell + 2\sum_{i=2p+1}^{\ell-1} \alpha_i$ and $\alpha = \alpha_{2h-1}$. Then $\beta, \alpha+\beta \in \mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ and $\beta + \bar{\beta} = (\alpha+\beta) + \overline{(\alpha+\beta)} = \gamma_h$. By Lemma 7.6 we have:

$$[X_{\alpha+\beta}, \bar{X}_{\alpha+\beta}] = [[X_\alpha, X_\beta], [X_{-\alpha}, \bar{X}_\beta]] = -[X_\beta, \bar{X}_\beta]\,,$$

showing that $\mathbf{L}_{\gamma_h}$ has at least one positive and one negative eigenvalue.

(iii) Assume that $\Phi_C \supset \{\alpha_{2h-1}, \alpha_k\}$ with $1 \leq h \leq p$ and $k > 2p$. We can take h to be the largest integer $\leq p$ with $\alpha_{2h-1} \in \Phi_C$ and k to be the smallest integer $> 2p$ with $\alpha_k \in \Phi_C$. Then $\gamma_h \in \mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$. The set of pairs (α, β) of elements of $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ with $\alpha + \bar{\beta} = \gamma_h$ consists of the pairs (β_r, β_r), where:

$$\beta_r = \sum_{i=2h-1}^{r-1} \alpha_i + \alpha_\ell + 2\sum_{i=r}^{\ell-1} \alpha_i$$

for $r = 2p, \dots, k-1$. We observe that $\beta_r = \beta_{r+1} + \alpha_r$, and that $\gamma_h \pm \alpha_r \notin \mathcal{R}$. Hence by Lemma 7.6 we have:

$$[X_{\beta_r}, \bar{X}_{\beta_r}] = [[X_{\alpha_r}, X_{\beta_{r+1}}], [X_{-\alpha_r}, \bar{X}_{\beta_{r+1}}]] = [X_{\beta_{r+1}}, \bar{X}_{\beta_{r+1}}],$$

for all $r = 2p, \dots, k-2$. Hence $\mathbf{L}_{\gamma_h}$ is $\neq 0$ and semi-definite.

$\boxed{\text{D IIIb}}$ The positive real roots that can be written as $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{R}$ are:

$$\gamma_h = \alpha_{2h-1} + \alpha_{\ell-1} + \alpha_\ell + 2\sum_{i=1}^{\ell-2} \alpha_i, \quad h = 1, \dots, p, \quad \text{for} \quad p = \frac{\ell-1}{2}.$$

(i) Assume that $\Phi_C \cap \{\alpha_{\ell-1}, \alpha_\ell\} \neq \emptyset$. Then $\gamma_p \in \mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$, and the same discussion of case A IV shows that $\mathbf{L}_{\gamma_p}$ is $\neq 0$ and semi-definite.

(ii) Assume that $\Phi_C = \{\alpha_{2h_1-1}, \dots, \alpha_{2h_r-1}\}$ with $1 \leq h_1 < \cdots < h_r \leq p$. Then $\gamma_1, \dots, \gamma_{h_r} \in \mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$, but only γ_{h_r} can be represented as a sum $\alpha + \bar{\alpha}$ with $\alpha \in \mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$. If $h_r = p$, we reduce to the case of A IV. Assume that $h_r < p$. Then we consider the two distinct roots:

$$\beta = \alpha_{2h_r-1} + \alpha_{2h_r} \quad \text{and} \qquad \delta = \beta + \gamma_{h_{r+1}}.$$

They both belong to $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ and $\beta + \bar{\delta} = \gamma_{h_r}$, showing that $\mathbf{L}_{\gamma_{h_r}}$ has at least one positive and one negative eigenvalue.

$\boxed{\text{E III}}$ Set $\gamma_1 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. These are the real positive roots in $\mathcal{R}$ that can be written as a sum $\alpha + \bar{\alpha}$ for a root $\alpha \in \mathcal{R}$. Note that γ_1, γ_2 both belong to $\mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$ for every choice of Φ_C. The discussion of the signature of $\mathbf{L}_{\gamma_1}$ reduces to the one we did for A IV.

(i) Assume that $\Phi_C \cap \{\alpha_1, \alpha_6\} \neq \emptyset$. In this case the discussion for A IV shows that $\mathbf{L}_{\gamma_1}$ is $\neq 0$ and semi-definite.

(ii) Assume that $\Phi_C = \{\alpha_3\}$ (the case $\Phi_C = \{\alpha_5\}$ is analogous). Then the set of pairs (α, β) of roots of $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ such that $\alpha + \bar{\beta} = \gamma_2$ contains only the pair (α, α) with $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$. Hence $\mathbf{L}_{\gamma_2}$ has rank 1 and is $\neq 0$ and semi-definite.

(iii) Assume that either $\alpha_4 \in \Phi_C \subset \mathcal{R}_\bullet$, or $\Phi_C = \{\alpha_3, \alpha_5\}$. Then the set of pairs (α, β) of roots of $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ such that $\alpha + \bar{\beta} = \gamma_2$ is empty, so that $\mathbf{L}_{\gamma_2} = 0$. The discussion for A IV shows that in this case $\mathbf{L}_{\gamma_1}$ has one positive and one negative eigenvalue.

$\boxed{\text{F II}}$ The real root $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ is the only positive root which can be written in the form $\alpha + \bar{\alpha}$ for some $\alpha \in \mathcal{R}$. It belongs to $\mathcal{Q}^n_{\Phi_C} \cap \bar{\mathcal{Q}}^n_{\Phi_C}$ for every choice of Φ_C.

(i) Assume that $\alpha_3 \in \Phi_C$. Then (α_4, α_4) is the only pair (α, β) of roots in $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$ with $\alpha + \bar{\beta} = \gamma$. Thus $\mathbf{L}_\gamma$ has rank 1 and hencefore is $\neq 0$ and semi-definite.

(ii) Assume that $\Phi_C \subset \{\alpha_1, \alpha_2\}$. Set $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = \bar{\alpha}_4 - \alpha_3$ and $\alpha = \alpha_3$. Then β and $\beta + \alpha$ both belong to $\mathcal{Q}^n_{\Phi_C} \setminus \bar{\mathcal{Q}}^n_{\Phi_C}$. With the notation of Lemma 7.6, we have $p = 1$, $p' = 1$, $q' = 1$. Thus:

$$[X_{\alpha+\beta}, \bar{X}_{\alpha+\beta}] = [[X_\alpha, X_\beta], [X_{-\alpha}, \bar{X}_\beta]] = -[X_\beta, \bar{X}_\beta],$$

showing that $\mathbf{L}_\gamma$ has at least one positive and one negative eigenvalue. $\square$

Part 3

Applications

CHAPTER 8

The fundamental group of parabolic CR manifolds

In this chapter we compute the fundamental group of parabolic CR manifolds.

8.1 The isotropy subgroups

Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\hat{\mathfrak{g}}$ its complexification, ϑ a Cartan involution of $\mathfrak{g}$. Let $\mathfrak{h}$ be a ϑ-invariant Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ the root system of $\hat{\mathfrak{g}}$ with respect to the complexification $\hat{\mathfrak{h}}$ of $\mathfrak{h}$. Denote by $\Lambda(\mathcal{R})$ the additive subgroup of $\mathfrak{h}^*_{\mathbb{R}}$ generated by $\mathcal{R}$ and by $\Pi(\mathcal{R})$ the *lattice of weights*, consisting of all $\eta \in \mathfrak{h}^*_{\mathbb{R}}$ for which $(\eta|\alpha^\vee) = 2(\eta|\alpha)/\|\alpha\|^2 \in \mathbb{Z}$. We have $\Lambda(\mathcal{R}) \subset \Pi(\mathcal{R})$.

Given a lattice (i.e. a free Abelian group) $\mathcal{L}$, a *character* of $\mathcal{L}$ is a homomorphism $\chi : \mathcal{L} \longrightarrow \mathbb{C}^*$ of $\mathcal{L}$ into the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of non-zero complex numbers. If $b_1, \dots, b_\ell \in \mathcal{L}$ is a basis of $\mathcal{L}$ over $\mathbb{Z}$, a character $\chi \in \mathrm{Hom}(\mathcal{L}, \mathbb{C}^*)$ is completely determined by its values $\lambda_i = \chi(b_i)$ (for $1 \leq i \leq \ell$) on the basis, so that $\mathrm{Hom}(\mathcal{L}, \mathbb{C}^*) \simeq [\mathbb{C}^*]^\ell$, where $\ell = \dim \mathfrak{h}$ is the rank of $\hat{\mathfrak{g}}$.

We keep the notation of the previous sections, in particular $\hat{\mathbf{G}}$ is a connected and simply connected complex Lie group with Lie algebra $\hat{\mathfrak{g}}$ and $\mathbf{G}$ its analytic subgroup with Lie algebra $\mathfrak{g}$. It is a covering group of any linear group with Lie algebra $\mathfrak{g}$. Let $\hat{\mathbf{H}}$ be the Cartan subgroup of $\hat{\mathbf{G}}$ corresponding to $\hat{\mathfrak{h}}$:

$$\hat{\mathbf{H}} = \mathbf{Z}_{\hat{\mathbf{G}}}(\hat{\mathfrak{h}}) = \{ z \in \hat{\mathbf{G}} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(z)(H) = H \,,\ \forall H \in \hat{\mathfrak{h}} \} . \tag{8.1}$$

All finite dimensional $\mathbb{C}$-linear representations of the complex semisimple Lie algebra $\hat{\mathfrak{g}}$ are differentials of representations of the complex Lie group $\hat{\mathbf{G}}$. Each element h of $\hat{\mathbf{H}}$ defines a character $\chi_h \in \mathrm{Hom}(\Pi(\mathcal{R}), \mathbb{C}^*)$, and vice versa, a character $\chi \in \mathrm{Hom}(\Pi(\mathcal{R}), \mathbb{C}^*)$ defines the element $h_\chi \in \hat{\mathbf{H}}$. To explain this correspondence, take first a faithful representation $\boldsymbol{\rho} : \hat{\mathbf{G}} \hookrightarrow \mathbf{SL}_{\mathbb{C}}(V)$, corresponding to $\rho : \hat{\mathfrak{g}} \hookrightarrow \mathfrak{sl}_{\mathbb{C}}(V)$, for a finite dimensional complex linear space V, and define $\boldsymbol{\rho}(h_\chi)(v) = \chi(\omega)\, v$ for $v \in V^\omega = \{ v \in V \mid \rho(H)(v) = \omega(H)\, v \,,\ \forall H \in \hat{\mathfrak{h}} \}$, for $\omega \in \Pi(\mathcal{R})$. The Cartan subgroup $\hat{\mathbf{H}}$ is analytic and $\exp\colon \hat{\mathfrak{h}} \longrightarrow \hat{\mathbf{H}}$ is onto, so that the correspondence with $\mathrm{Hom}(\Pi(\mathcal{R}), \mathbb{C}^*)$ can also be described by $\chi_{\exp(H)}(\omega) = \exp(\omega(H))$ for $H \in \hat{\mathfrak{h}}$. With $\ell = \dim_{\mathbb{C}}(\hat{\mathfrak{h}})$=rank of $\hat{\mathfrak{g}}$ we have:

$$\hat{\mathbf{H}} = \mathbf{Z}_{\hat{\mathbf{G}}}(\hat{\mathfrak{h}}) = \{ h_\chi \mid \chi \in \mathrm{Hom}(\Pi(\mathcal{R}), \mathbb{C}^*) \} \simeq \mathrm{Hom}(\Pi(\mathcal{R}), \mathbb{C}^*) \simeq [\mathbb{C}^*]^\ell . \tag{8.2}$$

The Cartan subgroup $\mathbf{H}$ of $\mathbf{G}$ corresponding to $\mathfrak{h}$ is the centralizer of $\mathfrak{h}$ in $\mathbf{G}$:

$$\mathbf{H} = \mathbf{Z}_{\mathbf{G}}(\mathfrak{h}) = \{ h \in \mathbf{G} \mid \mathrm{Ad}_{\mathfrak{g}}(h)(H) = H \,,\ \forall H \in \mathfrak{h} \} . \tag{8.3}$$

For a lattice $\mathcal{L} \subset \mathfrak{h}^*_{\mathbb{R}}$, with $\sigma(\mathcal{L}) = \mathcal{L}$, we set

$$\mathrm{Hom}^\sigma(\mathcal{L}, \mathbb{C}^*) = \{ \chi \in \mathrm{Hom}(\mathcal{L}, \mathbb{C}^*) \mid \chi(\bar{\eta}) = \overline{\chi(\eta)} \,,\ \forall \eta \in \mathcal{L} \}.$$

We obtain:

LEMMA 8.1. *Let* $\mathfrak{g}$ *be a real semisimple Lie algebra,* $\mathfrak{h}$ *a Cartan subalgebra of* $\mathfrak{g}$. *Then:*

$$\mathbf{H} = \mathbf{Z_G}(\mathfrak{h}) = \{ h_\chi \mid \chi \in \operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*) \} . \tag{8.4}$$

PROOF. The action of σ on $\operatorname{Hom}(\Pi(\mathcal{R}), \mathbb{C}^*)$, given by $\sigma(\chi)(\eta) = \overline{\chi(\bar{\eta})}$ coincides, under the correspondence (8.1), with the action of σ on $\hat{\mathbf{H}}$. Then the statement follows from the fact that $\mathbf{H} = \hat{\mathbf{H}} \cap \mathbf{G}$ and $\mathbf{H} = \hat{\mathbf{H}}^\sigma$ because of (2.18). $\square$

In view of the preceding Lemma, we now give a more explicit description of $\operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*)$. Fix a Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$ that is S-adapted to the conjugation σ. Set $\mathcal{B}(C) = \{\beta_1, \dots, \beta_a\} \cup \{\tau_1, \dots, \tau_b\}$, with $\{\beta_1, \dots, \beta_a\} = \mathcal{B}(C) \setminus \mathcal{R}_{\mathrm{im}}$ and $\{\tau_1, \dots, \tau_b\} = \mathcal{B}(C) \cap \mathcal{R}_{\mathrm{im}}$. By Theorem 3.1, the conjugation σ is described in $\mathcal{B}(C)$ by an involutive permutation $\jmath : \beta_i \longrightarrow \jmath(\beta_i) = \beta_{i'}$ of $\{\beta_1, \dots, \beta_a\}$ and a matrix of nonnegative integers $(k_{i,q})$, $1 \le i \le a$, $1 \le q \le b$, such that:

$$\begin{cases} \bar{\beta}_i = \beta_{i'} + \sum_{q=1}^b k_{i,q} \tau_q \,, \\ \jmath(\beta_i) = \beta_{i'} \,, \quad k_{i',p} = k_{i,p} \quad \text{for } 1 \le i \le a \,,\ 1 \le p \le b \,. \end{cases} \tag{8.5}$$

In $\Pi(\mathcal{R})$ we consider the basis $\mathcal{B}^*(C) = \{\omega_1, \dots, \omega_a, \theta_1, \dots, \theta_b\}$, adjoint of $\mathcal{B}(C)$, defined by:

$$\begin{cases} (\omega_i | \beta_j^\vee) = \delta_{i,j} & (\omega_i | \tau_q^\vee) = 0 \\ (\theta_p | \beta_j^\vee) = 0 & (\theta_p | \tau_q^\vee) = \delta_{p,q} \end{cases} \quad \text{for } 1 \le i, j \le a \,, \quad 1 \le p, q \le b \,. \tag{8.6}$$

The conjugation σ is described in $\mathcal{B}^*(C)$ by:

$$\begin{cases} \bar{\omega}_i = \omega_{i'} & \text{for } 1 \le i, i' \le a \,,\ \jmath(\beta_i) = \beta_{i'} \\ \bar{\theta}_p = -\theta_p + \sum_{j=1}^a k'_{j,p} \omega_j & \text{for } 1 \le p \le b \,, \end{cases} \tag{8.7}$$

where $k'_{j,p} = k_{j,p} \|\tau_p\|^2 / \|\beta_j\|^2$.

The characters $\chi \in \operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*)$ are those satisfying:

$$\begin{cases} \chi(\omega_i) \in \mathbb{C}^* \,,\ \chi(\theta_p) \in \mathbb{C}^* \,, & \text{for } 1 \le i \le a \,,\ 1 \le p \le b \,, \\ \overline{\chi(\omega_i)} \ = \ \chi(\omega_{i'}) & \text{for } 1 \le i, i' \le a \,,\ \jmath(\beta_i) = \beta_{i'} \\ |\chi(\theta_p)|^2 = \prod_{j=1}^a [\chi(\omega_j)]^{k'_{j,p}} & \text{for } 1 \le p \le b \,. \end{cases} \tag{8.8}$$

Each lattice $\mathcal{L}$ in $\mathfrak{h}^*_{\mathbb{R}}$, with $\Lambda(\mathcal{R}) \subset \mathcal{L} \subset \Pi(\mathcal{R})$, is the set of weights of all finite dimensional linear representations of an essentially unique connected complex semisimple Lie group $\hat{\mathbf{G}}_{\mathcal{L}}$. For instance, we have $\hat{\mathbf{G}}_{\Pi(\mathcal{R})} = \hat{\mathbf{G}}$ (the simply connected complex Lie group with Lie algebra $\hat{\mathfrak{g}}$) and $\hat{\mathbf{G}}_{\Lambda(\mathcal{R})} = \mathbf{Int}_{\mathbb{C}}(\hat{\mathfrak{g}})$ (the group of inner automorphisms of the complex semisimple Lie algebra $\hat{\mathfrak{g}}$) (see e.g. [Vin94, Ch.3, Theorem 2.11]). When moreover $\sigma(\mathcal{L}) = \mathcal{L}$, the analytic Lie subgroup of $\hat{\mathbf{G}}_{\mathcal{L}}$ with Lie algebra $\mathfrak{g}$ is a real form $\mathbf{G}_{\mathcal{L}}$ of $\hat{\mathbf{G}}_{\mathcal{L}}$. Vice versa, every connected linear semisimple Lie group with Lie algebra $\mathfrak{g}$ can be obtained in this way. The Cartan subgroup:

$$\mathbf{H}_{\mathcal{L}} = \mathbf{Z}_{\mathbf{G}_{\mathcal{L}}}(\mathfrak{h}) \tag{8.9}$$

of $\mathbf{G}_{\mathcal{L}}$, relative to $\mathfrak{h}$, is a real Lie subgroup of the complex Cartan subgroup:

$$\hat{\mathbf{H}}_{\mathcal{L}} = \mathbf{Z}_{\hat{\mathbf{G}}_{\mathcal{L}}}(\hat{\mathfrak{h}}) = \{ h_\chi \mid \chi \in \operatorname{Hom}(\mathcal{L}, \mathbb{C}^*) \} \tag{8.10}$$

of $\hat{\mathbf{G}}_{\mathcal{L}}$, relative to $\hat{\mathfrak{h}}$. From Lemma 8.1 we obtain:

Corollary 8.2. *Let $\mathcal{L}$ be a lattice in $\mathfrak{h}^*_{\mathbb{R}}$ with $\Lambda(\mathcal{R}) \subset \mathcal{L} \subset \Pi(\mathcal{R})$ and $\sigma(\mathcal{L}) = \mathcal{L}$. Denote by $\chi \longrightarrow \chi^\flat$ the restriction homomorphism* $\operatorname{Hom}(\Pi(\mathcal{R}), \mathbb{C}^*) \longrightarrow \operatorname{Hom}(\mathcal{L}, \mathbb{C}^*)$. *Then:*

$$\mathbf{H}_{\mathcal{L}} = \left\{ h_{\chi^\flat} \,\middle|\, \chi \in \operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*) \right\} . \tag{8.11}$$

Proof. The covering map $\mathbf{G} \longrightarrow \mathbf{G}_{\mathcal{L}}$ transforms the Cartan subgroup $\mathbf{H}$ of $\mathbf{G}$ into the Cartan subgroup $\mathbf{H}_{\mathcal{L}}$ of $\mathbf{G}_{\mathcal{L}}$. □

From Corollary 8.2 we obtain the exact sequence:

$$\mathbf{1} \longrightarrow \operatorname{Hom}^\sigma(\Pi(\mathcal{R})/\mathcal{L}, \mathbb{C}^*) \longrightarrow \operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*) \longrightarrow \mathbf{H}_{\mathcal{L}} \longrightarrow \mathbf{1} . \tag{8.12}$$

We can utilize (8.12) to compute the group $\pi_0(\mathbf{H}_{\mathcal{L}})$ of the connected components of the Cartan subgroup $\mathbf{H}_{\mathcal{L}}$. We have indeed:

Theorem 8.3. *Keep the previous notation. Let $C \in \mathfrak{C}(\mathcal{R})$ be S-adapted to the conjugation σ. Denote by $\omega_1, \ldots, \omega_a$ the weights in $\mathcal{B}^*(C)$ that vanish on $\mathcal{B}(C) \cap \mathcal{R}_{\mathrm{im}}$, and by $\{\theta_1, \ldots, \theta_b\}$ those vanishing on $\mathcal{B}(C) \setminus \mathcal{R}_{\mathrm{im}}$. By reordering, we assume that $\{\omega_1, \ldots, \omega_c\}$ is the set of weights in $\mathcal{B}^*(C)$ with $\bar{\omega}_i = \omega_i$. Define the non negative integers $k'_{i,p}$, for $1 \le i \le a$, $1 \le p \le b$, by $\bar{\theta}_p = -\theta_p + \sum_{i=1}^a k'_{i,p}\omega_i$. Consider the subgroups of the free Abelian group $\mathbb{Z}_2^c$:*

$$\begin{aligned} \mathbf{A} &= \{(\eta_1, \ldots, \eta_c) \in \mathbb{Z}_2^c \mid \textstyle\sum_{i=1}^c k'_{i,p}\eta_i \equiv 0 \mod 2\,,\ \forall\, 1 \le p \le b\}\,, \\ \mathbf{A}_{\mathcal{L}} &= \{(\eta_1, \ldots, \eta_c) \in \mathbf{A} \mid \textstyle\sum_{i=1}^c k_i\eta_i \equiv 0 \mod 2\,,\ \textit{if } \sum_{i=1}^c k_i\omega_i \in \mathcal{L}\}\,. \end{aligned} \tag{8.13}$$

Then:

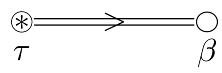
$$\pi_0(\mathbf{H}_{\mathcal{L}}) \cong \mathbf{A}/\mathbf{A}_{\mathcal{L}} . \tag{8.14}$$

Proof. The exact sequence (8.12) yields the exact sequence for the groups of the connected components:

$$\pi_0(\operatorname{Hom}^\sigma(\Pi(\mathcal{R})/\mathcal{L}, \mathbb{C}^*)) \longrightarrow \pi_0(\operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*)) \longrightarrow \pi_0(\mathbf{H}_{\mathcal{L}}) \longrightarrow \mathbf{1} .$$

The statement follows because $\pi_0(\operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*)) \cong \mathbf{A}$, and, in this isomorphism, the image of $\pi_0(\operatorname{Hom}^\sigma(\Pi(\mathcal{R})/\mathcal{L}, \mathbb{C}^*))$ in $\pi_0(\operatorname{Hom}^\sigma(\Pi(\mathcal{R}), \mathbb{C}^*))$ is identified with $\mathbf{A}_{\mathcal{L}}$. □

Example 8.1. Consider the B_2 system of roots $\mathcal{R} = \{\pm e_1 \pm e_2\} \cup \{\pm e_1, \pm e_2\} \subset \mathbb{R}^2$, with σ defined by $\sigma(e_1) = e_2$ and $\sigma(e_2) = e_1$. In the Weyl chamber C with simple roots $\{e_1 - e_2, e_2\}$, that is S-adapted to σ, it can be represented by the diagram:



We have, for the adjoint basis, $\theta = e_1$ and $\omega = (e_1 + e_2)/2$. We obtain $\bar{\theta} = -\theta + 2\omega$ and $\bar{\omega} = \omega$. Then $k' = 2$, $m = 1$, and $\mathbf{A} = \mathbb{Z}_2$. The quotient $\Pi(\mathcal{R})/\Lambda(\mathcal{R})$ is a group of order 2, with the generator $[\omega]$ and $2[\omega] \equiv 0$. Hence $\pi_0(\mathbf{H}) \simeq \mathbb{Z}_2$, while $\pi_0(\mathbf{H}_{\Lambda(\mathcal{R})}) \cong \mathbb{Z}_2/\mathbb{Z}_2 \cong \mathbf{1}$.

Let $\mathbf{G}_{\mathcal{L}}$ be any connected real linear Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. The Cartan subgroup $\mathbf{H}_{\mathcal{L}}$ is a normal subgroup of the normalizer

$$\mathbf{N}_{\mathcal{L}} = \mathbf{N}_{\mathbf{G}_{\mathcal{L}}}(\mathfrak{h}) = \{g \in \mathbf{G}_{\mathcal{L}} \mid \mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h}\} \tag{8.15}$$

of $\mathfrak{h}$ in $\mathbf{G}_{\mathcal{L}}$. The quotient

$$\mathbf{W}_{\mathcal{L}} = \mathbf{N}_{\mathcal{L}}/\mathbf{H}_{\mathcal{L}} \tag{8.16}$$

is the *analytic Weyl group* corresponding to $\mathfrak{h}$ and $\mathcal{L}$.

We know (see e.g. [War72, Ch.1,§4,p.115]) that actually the *analytic Weyl group* only depends, modulo natural isomorphisms, upon the real Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$; to stress this fact, we shall write $\mathbf{W}(\mathfrak{g},\mathfrak{h})$ instead of $\mathbf{W}_{\mathcal{L}}$.

Let us consider now an effective parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$.

We keep the notation introduced in previous sections, in particular we fix a Cartan pair $(\vartheta,\mathfrak{h})$ adapted to $(\mathfrak{g},\mathfrak{q})$.

We have the decomposition

$$\mathfrak{g}_+ = \mathfrak{q}\cap\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{g}_0 = \mathfrak{r}\oplus\mathfrak{s}_0 = \mathfrak{n}\oplus\mathfrak{z}_0\oplus\mathfrak{s}_0 \tag{8.17}$$

where:

$\mathfrak{r} = \mathfrak{n}\oplus\mathfrak{z}_0$ is the radical of $\mathfrak{g}_+$,

$\mathfrak{n}$ is the ideal of the $\mathrm{ad}_{\mathfrak{g}}$-nilpotent elements of the radical of $\mathfrak{g}_+$,

$\mathfrak{z}_0$ is a maximal Abelian subalgebra of ad-semisimple elements of $\mathfrak{r}$,

$\mathfrak{g}_0 = \mathfrak{z}_o\oplus\mathfrak{s}_0$ is the ϑ-invariant reductive complement of $\mathfrak{n}$ in $\mathfrak{g}_+$,

$\mathfrak{s}_o = [\mathfrak{g}_0,\mathfrak{g}_0]$ is the semisimple ideal and $\mathfrak{z}_0$ is the center of $\mathfrak{g}_0$.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, contained in $\mathfrak{g}_+$, decomposes into the direct sum:

$$\mathfrak{h} = \mathfrak{z}_0\oplus\mathfrak{h}_0 \tag{8.18}$$

where $\mathfrak{h}_0 = \mathfrak{h}\cap\mathfrak{s}_0$ is a Cartan subalgebra of $\mathfrak{s}_0$. The subalgebra $\mathfrak{z}_0$ is characterized by:

$$\mathfrak{z}_0 = \{H\in\mathfrak{h} \mid \alpha(H) = 0\,,\ \forall\alpha\in\mathcal{Q}^r\cap\bar{\mathcal{Q}}^r\}\,. \tag{8.19}$$

Indeed its complexification $\hat{\mathfrak{z}}_0$ is the center of the reductive complex Lie algebra $\mathfrak{q}^r\cap\bar{\mathfrak{q}}^r$.

The isotropy subgroup

$$\mathbf{G}_+ = \mathbf{N}_{\mathbf{G}}(\mathfrak{q}) = \{g\in\mathbf{G} \mid \mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}\} \tag{8.20}$$

is a closed real-algebraic subgroup of $\mathbf{G}$, with Lie algebra $\mathfrak{g}_+$. Thus we have the Chevalley decomposition:

$$\mathbf{G}_+ = \mathbf{G}_0 \rtimes \mathbf{N} \tag{8.21}$$

where $\mathbf{G}_0$ is a closed, real-algebraic, reductive subgroup of $\mathbf{G}$, with Lie algebra $\mathfrak{g}_0$, and $\mathbf{N}$ is a unipotent closed connected and simply connected subgroup of $\mathbf{G}$, diffeomorphic to its Lie algebra $\mathfrak{n}$.

We also define $\mathbf{S}_0$ to be the analytic semisimple Lie subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{s}_0$.

PROPOSITION 8.4. *The group* $\mathbf{G}_0$ *is the normalizer in* $\mathbf{G}$ *of* $\mathfrak{g}_0$ *and the centralizer in* $\mathbf{G}$ *of* $\mathfrak{z}_0$ *:*

$$\begin{aligned}\mathbf{G}_0 &= \mathbf{N}_{\mathbf{G}}(\mathfrak{g}_0) = \{g \in \mathbf{G} \mid \mathrm{Ad}(g)(\mathfrak{g}_0) = \mathfrak{g}_0\} \\ &= \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0) \;=\; \{g \in \mathbf{G} \mid \mathrm{Ad}(g)(H) = H\,,\ \forall H \in \mathfrak{z}_0\,\}\,.\end{aligned} \tag{8.22}$$

PROOF. While proving this Proposition we can assume that $\mathfrak{h}$ is maximally noncompact among the Cartan subalgebras of $\mathfrak{g}$ that are contained in $\mathfrak{g}_+$. Indeed, all admissible Cartan subalgebras of $(\mathfrak{g}, \mathfrak{q})$ are conjugate to the direct sum of $\mathfrak{z}_0$ and a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{s}_0$. Thus our assumption means that we have chosen a maximally noncompact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{s}_0$.

We first prove that

$$\mathbf{G}_0 = \{g \in \mathbf{G}_+ \mid \mathrm{Ad}(g)(\mathfrak{g}_0) = \mathfrak{g}_0\}\,. \tag{8.23}$$

If $g \in \mathbf{G}_+$, then $\mathrm{Ad}(g)(\mathfrak{g}_0)$ is a maximal reductive subalgebra of $\mathfrak{g}_+$. By [Vin94, Ch.I,§6.5] there is an element $n \in \mathbf{N}$ such that $\mathfrak{g}_0 = \mathrm{Ad}(g\,n^{-1})(\mathfrak{g}_0)$. This yields a decomposition $g = g_0\,n$, with $g_0 \in \mathbf{N}_{\mathbf{G}_+}(\mathfrak{g}_0)$ and $n \in \mathbf{N}$. Since $\mathfrak{z}_0 \subset \mathfrak{g}_0$, and $[\mathfrak{z}_0, \mathfrak{n}] = \mathfrak{n}$, we have $\mathbf{N} \cap \mathbf{N}_{\mathbf{G}_+}(\mathfrak{g}_0) = \{1\}$. This yields the decomposition $\mathbf{G}_+ = \mathbf{N}_{\mathbf{G}_+}(\mathfrak{g}_0) \rtimes \mathbf{N}$. Since we also have (8.21) and $\mathfrak{g}_0$ is the Lie algebra of both $\mathbf{G}_0$ and $\mathbf{N}_{\mathbf{G}_+}(\mathfrak{g}_0)$, we obtain that $\mathbf{G}_0 = \mathbf{N}_{\mathbf{G}_+}(\mathfrak{g}_0)$.

Next we show that $\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0) = \mathbf{N}_{\mathbf{G}}(\mathfrak{g}_0)$. If $g_0 \in \mathbf{N}_{\mathbf{G}}(\mathfrak{g}_0)$, then $\mathrm{Ad}(g_0)(\mathfrak{z}_0) = \mathfrak{z}_0$. Moreover, $\mathrm{Ad}(g_0)(\mathfrak{h}_0)$ is another maximally noncompact Cartan subalgebra of $\mathfrak{s}_0$. Therefore there is an inner automorphism of $\mathfrak{s}_0$, and thus also an element $g_1 \in \mathbf{S}_0$ such that $g_1 \circ g_0(\mathfrak{h}) = \mathfrak{h}$ (see e.g. [Sug59], [Sug71], [Vin94, Ch.4,§4.7]). Let $g = g_1 \circ g_0$. Since $\mathbf{S}_0 \subset \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$, it suffices to show that $g \in \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$. We have $\mathrm{Ad}(g)(\mathfrak{z}_0) = \mathfrak{z}_0$ and $\mathrm{Ad}(g)(\mathfrak{h}_0) = \mathfrak{h}_0$. In particular, $g \in \mathbf{N}_{\mathbf{G}}(\mathfrak{h})$. Since $\mathrm{Ad}(g)$ commutes with the conjugation σ in $\hat{\mathfrak{g}}$, and $\mathrm{Ad}(g)(\mathfrak{q}^n) = \mathfrak{q}^n$, $\mathrm{Ad}(g)(\mathfrak{q}^r) = \mathfrak{q}^r$, we also have:

$$\mathrm{Ad}(g)(\mathfrak{q}^r \cap \bar{\mathfrak{q}}^{\,r}) = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^{\,r} \quad \text{and} \quad \mathrm{Ad}(g)\left(\mathfrak{q}^n + [\bar{\mathfrak{q}} \cap \mathfrak{q}^r]\right) = \mathfrak{q}^n + [\bar{\mathfrak{q}} \cap \mathfrak{q}^r]\ . \tag{8.24}$$

The analytic Weyl group $\mathbf{W}(\mathfrak{g}, \mathfrak{h}) = \mathbf{N}_{\mathbf{G}}(\mathfrak{h})/\mathbf{H}$ can be identified to a subgroup of the Weyl group $\mathbf{W}(\mathcal{R}) = \mathbf{N}_{\hat{\mathbf{G}}}(\hat{\mathfrak{h}})/\hat{\mathbf{H}}$. The element $s_g \in \mathbf{W}(\mathcal{R})$ defined by g satisfies: $\mathrm{Ad}(g)(\hat{\mathfrak{g}}^{\alpha}) \subset \hat{\mathfrak{g}}^{s_g(\alpha)}$ for all $\alpha \in \mathcal{R}$. Because of (8.24), g normalizes the complex parabolic subalgebra $\mathfrak{v} = \mathfrak{q}^n + (\mathfrak{q}^r \cap \bar{\mathfrak{q}})$, with $\mathfrak{v}^r = \mathfrak{q}^r \cap \bar{\mathfrak{q}}^{\,r}$. Hence s_g is the composition of symmetries s_α with $\alpha \in \mathcal{Q}^r \cap \bar{\mathcal{Q}}^{\,r}$. If $\alpha \in \mathcal{Q}^r \cap \bar{\mathcal{Q}}^{\,r}$ and $H \in \mathfrak{z}_0$ we obtain for $X_\beta \in \hat{\mathfrak{g}}^{\beta}$:

$$\begin{aligned}[\mathrm{Ad}(g)(H), \mathrm{Ad}(g)(X_\beta)] &= \mathrm{Ad}(g)([H, X_\beta]) \\ &= \beta(H)\,\mathrm{Ad}(g)(X_\beta) = s_\alpha(\beta)(H)\mathrm{Ad}(g)(X_\beta)\end{aligned}$$

because $s_\alpha(\beta)(H) = (\beta - (\beta|\alpha^\vee)\alpha)(H) = \beta(H)$, since $\alpha(H) = 0$ by (8.19). This shows that $[\mathrm{Ad}(g)(H), X_\beta] = \beta(H)X_\beta$ for all $\beta \in \mathcal{R}$ and hence that $\mathrm{Ad}(g)(H) = H$. Moreover $\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0) \subset \mathbf{N}_{\mathbf{G}}(\mathfrak{g}_0)$ because $\mathfrak{g}_0$ is the centralizer of $\mathfrak{z}_0$ in $\mathfrak{g}$, and hence the equality follows.

Finally we observe that $\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0) \subset \mathbf{Q} = \mathbf{N}_{\hat{\mathbf{G}}}(\mathfrak{q})$. Indeed, if $g_0 \in \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$, we can find $g_1 \in \mathbf{S}_0$ such that $g = g_1 g_0$ normalizes our Cartan subalgebra $\mathfrak{h}$, still centralizing $\mathfrak{z}_0$. Being $\mathcal{Q} = \{\alpha \in \mathcal{R} \,|\, \alpha(A) \geq 0\}$ for an element $A \in \mathfrak{h}_{\mathbb{R}} \cap \hat{\mathfrak{z}_0}$, clearly $\mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$. Hence also $\mathrm{Ad}(g_0)(\mathfrak{q}) = \mathfrak{q}$. This shows that $\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0) \subset \mathbf{G}_+$, completing the proof of the Theorem. $\square$

Next we prove that our choice of $\mathbf{G} = \mathbf{G}_{\Pi(\mathcal{R})}$ brings that also $\mathbf{S}_0$ is the semisimple real group associated to the full weights lattice of $\hat{\mathfrak{s}}_0$. Indeed we have:

Proposition 8.5. *The complexification $\hat{\mathbf{S}}_0$ of the linear Lie group $\mathbf{S}_0$ is simply connected.*

First we prove the following:

Lemma 8.6. *Let E be a linear hyperplane of $\mathfrak{h}^*_{\mathbb{R}}$ and let $\mathcal{R}' = E \cap \mathcal{R}$. Then every basis of simple roots $\{\alpha_1, \ldots, \alpha_m\}$ of $\mathcal{R}'$ is a subset of a basis of simple roots $\{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_\ell\}$ of $\mathcal{R}$.*

Proof. Let $A \in \mathfrak{h}_{\mathbb{R}}$ be such that $E = \{\eta \in \mathfrak{h}^*_{\mathbb{R}} \,|\, \eta(A) = 0\}$. Next we take a regular element $B \in \mathfrak{h}_{\mathbb{R}}$ such that $\{\alpha_1, \ldots, \alpha_m\}$ is the system of simple roots in $\mathcal{R}'^+ = \{\alpha \in \mathcal{R}' \,|\, \alpha(B) > 0\}$. For small $\varepsilon > 0$ the element $A + \varepsilon B$ is regular and $\alpha_1, \ldots, \alpha_m$ are simple in $\mathcal{R}^+ = \{\alpha \in \mathcal{R} \,|\, \alpha(A + \varepsilon B) > 0\}$. Indeed, we take $\varepsilon > 0$ such that $\varepsilon\,|\alpha(B)| < |\alpha(A)|$ when $\alpha(A) \neq 0$. Assume by contradiction that, for some $1 \leq i \leq m$, the root α_i is not simple in $\mathcal{R}^+$, i.e. that we have $\alpha_i = \beta + \gamma$, with $\beta(A + \varepsilon B) > 0$, $\gamma(A + \varepsilon B) > 0$, $\beta(B) \geq \gamma(B)$. Since α_i is simple in $\mathcal{R}'^+$, we have $\gamma(B) < 0 < \alpha_i(B) < \beta(B)$. Hence $\gamma(A) > 0$. Moreover $\beta(A) \geq 0$, because otherwise $\beta(A + \varepsilon B) < 0$ by our choice of ε. Thus we obtain:

$$\alpha_i(A) = \beta(A) + \gamma(A) \geq \gamma(A) > 0\,,$$

contradicting $\alpha_i \in E$. $\square$

Proof of Proposition 8.5. We keep the notation introduced in the previous discussion. While applying Lemma 8.6 to our situation, we observe that, if our parabolic $\mathcal{Q}$ is $\{\alpha \in \mathcal{R} \,|\, \alpha(A) \geq 0\}$ for some $A \in \mathfrak{h}_{\mathbb{R}}$, then the set of roots α with $\hat{\mathfrak{g}}^\alpha \subset \hat{\mathfrak{s}}_0$ is $\mathcal{R}' = \mathcal{Q}^r \cap \bar{\mathcal{Q}}^r = \{\alpha \in \mathcal{R} \,|\, \alpha(A + \varepsilon \bar{A}) = 0\}$, where ε is any positive real number with $\varepsilon\,|\bar{\alpha}(A)| < \alpha(A)$ for $\alpha \in \mathcal{Q}^n$. This set $\mathcal{R}'$ is naturally isomorphic to the root system $\mathcal{R}(\hat{\mathfrak{s}}_0, \hat{\mathfrak{h}}_0)$ of the semisimple complex Lie algebra $\hat{\mathfrak{s}}_0$ with respect to its Cartan subalgebra $\hat{\mathfrak{h}}_0$. We identify $\Pi(\mathcal{R}')$ to the set of elements ω in $\langle\mathcal{R}'\rangle_{\mathbb{R}} \subset \mathfrak{h}^*_{\mathbb{R}}$ for which $(\omega|\alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \mathcal{R}'$. We have a natural projection $\varpi : \Pi(\mathcal{R}) \to \Pi(\mathcal{R}')$, that is defined by $(\varpi(\eta)|\alpha^\vee) = (\eta|\alpha^\vee)$ for all $\alpha \in \mathcal{R}'$, and coincides with the orthogonal projection onto $\langle\mathcal{R}'\rangle_{\mathbb{R}}$.

The lattice $\mathcal{L} = \varpi(\Pi(\mathcal{R}))$ satisfies $\Lambda(\mathcal{R}') \subset \mathcal{L} \subset \Pi(\mathcal{R}')$ and is the set of weights of the finite dimensional linear representations of $\mathbf{S}_0$.

According to Lemma 8.6 we can fix $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, with $\mathcal{B}(C) = \{\alpha_1, \ldots, \alpha_\ell\}$, in such a way that $\mathcal{B}' = \{\alpha_1, \ldots, \alpha_m\}$, with $m \leq \ell$, is a basis of $\mathcal{R}'$. Let $\mathcal{B}^*(C) = \{\mu_1, \ldots, \mu_\ell\} \subset \Pi(\mathcal{R})$ and $\mathcal{B}'^* = \{\nu_1, \ldots, \nu_m\} \subset \Pi(\mathcal{R}') \subset \langle\mathcal{R}'\rangle_{\mathbb{R}}$ be the corresponding adjoint basis. Then $\varpi(\mu_i) = \nu_i$ for $i = 1, \ldots, m$. This shows that actually $\mathcal{L} = \Pi(\mathcal{R}')$, proving our statement. $\square$

Now we give a more accurate description of the group $\pi_0(\mathbf{G}_+)$ of the connected components of $\mathbf{G}_+$.

THEOREM 8.7. *Let* $(\mathfrak{g},\mathfrak{q})$ *be an effective parabolic CR algebra. Keep the notation of* (8.17), (8.18), (8.20) *and* (8.21). *Then* $\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h}_0)$ *is a closed normal Lie subgroup of* $\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$. *The natural inclusion* $\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)\hookrightarrow\mathbf{G}_+$ *passes to the quotient to define a one-to-one correspondence of connected components:*

$$\pi_0\left(\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)/\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h})\right)\longleftrightarrow\pi_0\left(\mathbf{G}_+\right). \tag{8.25}$$

PROOF. The inclusion $\mathbf{G}_0\hookrightarrow\mathbf{G}_+$ defines an isomorphism $\pi_0(\mathbf{G}_0)\xrightarrow{\sim}\pi_0(\mathbf{G}_+)$.

We showed in the proof of Proposition 8.4 that every connected component of $\mathbf{G}_0$ contains an element of $\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{G}_0$ and that $\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{G}_0=\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$.

Since $\mathfrak{s}_0$ is an ideal of $\mathfrak{g}_0=\mathfrak{s}_0\oplus\mathfrak{z}_0$, the analytic subgroup $\mathbf{S}_0$ is normal in $\mathbf{G}_0$ and we already noticed that is contained in $\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$. Moreover, $\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h})=\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{S}_0$.

Let $\mathbf{T}_0$ be the analytic Lie subgroup of $\mathbf{G}_0$ with Lie algebra $\mathfrak{z}_0$. Then $\mathbf{S}_0\bowtie\mathbf{T}_0$ is the connected component of the identity in $\mathbf{G}_0$. We have:

$$\mathbf{T}_0\subset\mathbf{H}\subset\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0).$$

Hence we obtain isomorphisms:

$$\frac{\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)}{\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h})\bowtie\mathbf{T}_0}\xrightarrow{\ \cong\ }\frac{\mathbf{G}_0}{\mathbf{S}_0\bowtie\mathbf{T}_0}\xrightarrow{\ \cong\ }\pi_0(\mathbf{G}_+), \tag{8.26}$$

yielding the isomorphism in (8.25). $\square$

We denote by $\mathbf{H}_0$ the Cartan subalgebra of $\mathbf{S}_0$ corresponding to $\mathfrak{h}_0$. Since $\mathbf{S}_0\subset\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$, we have:

$$\mathbf{H}_0=\{g\in\mathbf{S}_0\mid\operatorname{Ad}(g)(H)=H,\ \forall H\in\mathfrak{h}_0\}=\mathbf{H}\cap\mathbf{S}_0. \tag{8.27}$$

By (8.27), we obtain for the analytic Weyl group $\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)=\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h}_0)/\mathbf{H}_0$:

$$\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)\cong\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h})/\mathbf{Z}_{\mathbf{S}_0}(\mathfrak{h})=\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h})/\left(\mathbf{H}\cap\mathbf{S}_0\right). \tag{8.28}$$

Thus we can identify $\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)$ with a subgroup of the centralizer $\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)$ of $\mathfrak{z}_0$ in $\mathbf{W}(\mathfrak{g},\mathfrak{h})$. Since $\mathbf{S}_0$ is normal in $\mathbf{G}_0$, it turns out that $\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)$ is normal in $\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)$. We have:

THEOREM 8.8. *Keep the notation and assumptions of Theorem 8.7. We have a short exact sequence of groups and homomorphisms:*

$$\mathbf{1}\to\frac{\mathbf{H}}{\mathbf{H}_0}\to\frac{\mathbf{N}_{\mathbf{G}}(\mathfrak{h})\cap\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)}{\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h}_0)}\to\frac{\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)}{\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)}\to\mathbf{1}. \tag{8.29}$$

We obtain an exact sequence of finite groups:

$$\mathbf{1}\to\pi_0\left(\frac{\mathbf{H}}{\mathbf{H}_0}\right)\to\pi_0\left(\mathbf{G}_+\right)\to\frac{\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)}{\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)}\to\mathbf{1}. \tag{8.30}$$

When $\mathfrak{h}$ *is maximally compact in* $\mathfrak{g}_+$, *then* $\mathbf{H}_0$ *is connected and* (8.30) *yields the exact sequence:*

$$\mathbf{1}\to\pi_0\left(\mathbf{H}\right)\to\pi_0\left(\mathbf{G}_+\right)\to\frac{\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)}{\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)}\to\mathbf{1}. \tag{8.31}$$

When $\mathfrak{h}$ *is maximally noncompact in* $\mathfrak{g}_+$, *then* $\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)\cong\mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)$ *and* (8.30) *yields the isomorphism:*

$$\pi_0\left(\frac{\mathbf{H}}{\mathbf{H}_0}\right)\cong\pi_0\left(\mathbf{G}_+\right). \tag{8.31}$$

PROOF. Since $\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)$ is a finite group, (8.30) is a consequence of the exact homotopy sequence of a fiber bundle, of (8.29) and of Theorem 8.7. Thus a proof of (8.29) also provides a proof of (8.30).

We observe that $\mathbf{H}$ is a normal subgroup of $\mathbf{N}_{\mathbf{G}}(\mathfrak{h}) \cap \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$ and that the quotient $(\mathbf{N}_{\mathbf{G}}(\mathfrak{h}) \cap \mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0))/\mathbf{H}$ is naturally isomorphic to the centralizer $\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0)$ of $\mathfrak{z}_0$ in the analytic Weyl group $\mathbf{W}(\mathfrak{g},\mathfrak{h})$. Moreover, $\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h}_0)$ is contained in $\mathbf{Z}_{\mathbf{G}}(\mathfrak{z}_0)$ and $\mathbf{N}_{\mathbf{S}_0}(\mathfrak{h}_0) \cap \mathbf{H} = \mathbf{H}_0$. Thus (8.29) follows from the general homomorphism theorems of groups.

If $\mathfrak{h}$ is maximally compact, then $\mathfrak{h}_0$ is a maximally compact Lie subalgebra of $\mathfrak{s}_0$. Then (see e.g. [Kna02, Proposition 7.90, p.488]) the Cartan subalgebra $\mathbf{H}_0$ is connected and (8.31) follows from (8.30).

If $\mathfrak{h}$ is maximally noncompact, then, by [Kna02, Proposition 7.90, p.488] the Cartan subalgebra $\mathbf{H}$ intersects every connected component of $\mathbf{G}_+$. This implies that the map $\pi_0(\mathbf{H}/\mathbf{H}_0) \to \pi(\mathbf{G}_+)$ in (8.30) is onto and hence (8.31) holds true and $\mathbf{Z}_{\mathbf{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{z}_0) \cong \mathbf{W}(\mathfrak{s}_0,\mathfrak{h}_0)$. □

We describe now $\pi_0(\mathbf{G}_+)$ in terms of characters:

PROPOSITION 8.9. *Let $\mathfrak{h}$ be a maximally noncompact Cartan subalgebra of $\mathfrak{g}_+$. Fix a Weyl chamber $C \in \mathfrak{C}(\mathcal{R})$ such that $\mathcal{B}(C) = \{\alpha_1,\dots,\alpha_\ell\}$ contains a basis $\mathcal{B}_0(C) = \{\alpha_1,\dots,\alpha_m\}$ of simple roots of $\mathcal{Q}^r \cap \bar{\mathcal{Q}}^r$. Let $E = \langle \alpha_{m+1},\dots,\alpha_\ell\rangle^\perp$. Then:*

$$\pi_0(\mathbf{G}_+) \simeq \pi_0\left(\{\chi \in \mathrm{Hom}^\sigma(\Pi(\mathcal{R}),\mathbb{C}^*) \mid \chi(\mu) = 1\,,\ \forall \mu \in \Pi(\mathcal{R}) \cap E\}\right)\,. \tag{8.32}$$

PROOF. We denote by ϖ_0 the orthogonal projection onto $\langle \alpha_{m+1},\dots,\alpha_\ell\rangle^\perp$. Let $\mathcal{B}^*(C) = \{\omega_1,\dots,\omega_\ell\}$ be the adjoint basis of $\mathcal{B}(C)$. Then we have $\varpi_0(\eta) = \sum_{i=1}^m (\eta|\alpha_i^\vee)\omega_i$. We can identify $\mathcal{R}' = \mathcal{R}(\hat{\mathfrak{s}}_0,\hat{\mathfrak{h}}_0)$ with $\varpi_0(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^r)$. The advantage of this identification is that the corresponding weight space $\Pi(\mathcal{R}')$ becomes a subspace of $\Pi(\mathcal{R})$, and moreover:

$$\Pi(\mathcal{R}') \;=\; \varpi_0(\Pi(\mathcal{R})) \;=\; \Pi(\mathcal{R}) \cap \langle \alpha_{m+1},\dots,\alpha_\ell\rangle^\perp\,. \tag{8.33}$$

In this way, for each character $\chi \in \mathrm{Hom}^\sigma(\Pi(\mathcal{R}),\mathbb{C}^*)$, the composition $\chi \circ \varpi_0$ is still a character of $\mathrm{Hom}^\sigma(\Pi(\mathcal{R}),\mathbb{C}^*)$, whose restriction to $\Pi(\mathcal{R}')$ is a character in $\mathrm{Hom}^\sigma(\Pi(\mathcal{R}'),\mathbb{C}^*)$, and we have:

$$\mathbf{H}_0 = \{h_\chi \mid \chi \in \mathrm{Hom}^\sigma(\Pi(\mathcal{R}'),\mathbb{C}^*)\} = \{h_{\chi\circ\varpi} \mid \chi \in \mathrm{Hom}^\sigma(\Pi(\mathcal{R}),\mathbb{C}^*)\}\,. \tag{8.34}$$

Thus, setting:

$$\mathbf{H}_1 = \{h_\chi \mid \chi \in \mathrm{Hom}^\sigma(\Pi(\mathcal{R}),\mathbb{C}^*)\,,\ \chi(\eta) = 1\ \forall \eta \in \Pi(\mathcal{R}')\}\,, \tag{8.35}$$

the map:

$$\mathbf{H} \ni h_\chi \longrightarrow h_{\chi\circ\varpi}^{-1} \circ h_\chi \in \mathbf{H}_1 \tag{8.36}$$

yields, by passing to the quotient, the isomorphism $\mathbf{H}/\mathbf{H}_0 \xrightarrow{\sim} \mathbf{H}_1$, which implies (8.32). □

COROLLARY 8.10. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be an effective parabolic totally real CR algebra. Let* $\mathfrak{h}$ *be a maximally noncompact Cartan subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{g}_+$ *and let* $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ *be S-fit for* $(\mathfrak{g}, \mathfrak{q})$ *and hence S-adapted to the conjugation* σ. *Then* $\pi_0(\mathbf{G}_+)$ *is isomorphic to* $\mathbb{Z}_2^e$, *where* e *is the number of roots in* $\Phi_C \cap \mathcal{R}_{\mathrm{re}}$.

PROOF. By inspecting the Satake diagrams of the simple real Lie algebras, we obtain that $\chi(\omega_i)$ is a positive real number when ω_i is a real weight corresponding to a complex root $\alpha_i \in \mathcal{B}(C)$. Then the statement follows from Proposition 8.9.

Another independent proof can be given, using [DKV83] and [Wig98]. Because $\mathbf{G}$ is connected, we have an exact sequence:

$$\cdots \longrightarrow \pi_1(\mathbf{G}/\mathbf{G}_+) \longrightarrow \pi_0(\mathbf{G}_+) \longrightarrow \mathbf{1}\,. \tag{8.37}$$

Take $\Phi'_C = \Phi_C \setminus \mathcal{R}_{\mathrm{re}}$. Then also $\mathfrak{q}' = \mathfrak{q}_{\Phi'_C}$ is the complexification of a real parabolic subalgebra $\mathfrak{g}'_+$ of $\mathfrak{g}$. By [Wig98] and (8.37), the stabilizer $\mathbf{G}'_+$ of $\mathfrak{q}'$ in $\mathbf{G}$ is connected. By Proposition 8.9, this implies that $\chi(\omega_i) > 0$ on the real $\omega_i \in \mathcal{B}^*(C)$ corresponding to a complex root α_i in $\mathcal{B}(C)$. Again, the statement follows from Proposition 8.9. $\square$

For each $\omega \in \Pi(\mathcal{R})$ there is, modulo isomorphisms, a unique irreducible finite dimensional complex linear representation $\rho_\omega : \hat{\mathfrak{g}} \to \mathfrak{gl}_{\mathbb{C}}(V_\omega)$, for which ω is an extremal weight. For each weight η we denote by:

$$V_\omega^\eta = \{v \in V_\omega \mid \rho_\omega(H)(v) = \eta(H)\, v\,,\ \forall H \in \hat{\mathfrak{h}}\} \tag{8.38}$$

the eigenspace of V_ω corresponding to the weight η. Let:

$$\hat{\mathfrak{g}}_\omega = \{Z \in \hat{\mathfrak{g}} \mid \rho_\omega(V_\omega^\omega) \subset V_\omega^\omega\}\,. \tag{8.39}$$

Since the eigenspace V_ω^ω of ω is one-dimensional, for each $Z \in \hat{\mathfrak{g}}_\omega$ there is a unique complex number, that we shall denote by $\omega(Z)$, such that:

$$\rho_\omega(Z)(v) = \omega(Z)\, v\,, \qquad \forall Z \in \hat{\mathfrak{g}}_\omega\,,\ \forall v \in V_\omega^\omega\,. \tag{8.40}$$

This agrees with the natural definition of $\omega(Z)$ by the duality pairing when $Z \in \hat{\mathfrak{h}}$.

PROPOSITION 8.11. *Let* $\omega_1\,, \omega_2 \in \Pi(\mathcal{R})$ *and* $\omega = \omega_1 + \omega_2$. *Then:*

$$\hat{\mathfrak{g}}_{\omega_1} \cap \hat{\mathfrak{g}}_{\omega_2} \subset \hat{\mathfrak{g}}_\omega \tag{8.41}$$

and:

$$\omega(Z) = \omega_1(Z) + \omega_2(Z)\,,\ \forall Z \in \hat{\mathfrak{g}}_{\omega_1} \cap \hat{\mathfrak{g}}_{\omega_2}\,. \tag{8.42}$$

PROOF. There is an injective homomorphism of $\hat{\mathfrak{g}}$-modules $V_\omega \to V_{\omega_1} \otimes V_{\omega_2}$ that maps V_ω^ω into $V_{\omega_1}^{\omega_1} \otimes V_{\omega_2}^{\omega_2}$. For $Z \in \hat{\mathfrak{g}}$, we have $(\rho_{\omega_1} \otimes \rho_{\omega_2})(v_1 \otimes v_2) = \rho_{\omega_1}(Z)(v_1) \otimes v_2 + v_1 \otimes \rho_{\omega_2}(Z)(v_2)$ for all $v_1 \in V_{\omega_1}$, $v_2 \in V_{\omega_2}$. This implies (8.41) and (8.42). $\square$

We also have:

LEMMA 8.12. *Let* $\omega \in \Pi(\mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}))$ *and* $\hat{\mathfrak{h}}'$ *another Cartan subalgebra of* $\hat{\mathfrak{g}}$ *contained in* $\hat{\mathfrak{g}}_\omega$. *Let* V_ω *be an irreducible* $\hat{\mathfrak{g}}$*-module with extremal weight* ω. *Then:*

(*i*) *There is a weight* $\omega' \in \Pi(\mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}'))$ *that is an extremal weight for* V_ω;

(*ii*) *we can choose* $\omega' \in \Pi(\mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}'))$ *in such a way that* $V_\omega^\omega = V_\omega^{\omega'}$;

(*iii*) $\hat{\mathfrak{g}}_\omega = \hat{\mathfrak{g}}_{\omega'}$ *and* $\omega(Z) = \omega'(Z) \quad \forall Z \in \hat{\mathfrak{g}}_\omega$. □

Since we took a simply connected $\hat{\mathbf{G}}$, for each $\omega \in \Pi(\mathcal{R})$, the representation $\rho_\omega : \hat{\mathfrak{g}} \longrightarrow \mathfrak{gl}_{\mathbb{C}}(V_\omega)$ lifts to a representation $\rho_\omega : \hat{\mathbf{G}} \rightarrow \mathbf{GL}_{\mathbb{C}}(V_\omega)$. The stabilizer of the line V_ω^ω in $\hat{\mathbf{G}}$ contains the Cartan subgroup $\hat{\mathbf{H}}$ of $\hat{\mathbf{G}}$ and therefore is the closed connected subgroup $\hat{\mathbf{G}}_\omega$ of $\hat{\mathbf{G}}$ with Lie algebra $\hat{\mathfrak{g}}_\omega$. The map $\omega : \hat{\mathfrak{g}}_\omega \longrightarrow \mathbb{C}$ also lifts to a character $\varphi_\omega : \hat{\mathbf{G}}_\omega \longrightarrow \mathbb{C}^*$, with $\varphi_\omega(\exp(Z)) = \exp(\omega(Z))$ for all $Z \in \hat{\mathfrak{g}}_\omega$.

We observe that every parabolic $\mathfrak{q}$ containing $\hat{\mathfrak{h}}$ is equal to some $\hat{\mathfrak{g}}_\omega$, for a suitable choice of $\omega \in \Pi(\mathcal{R})$. This choice is not unique. It can be done in the following way. First we choose a Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$. Let $\mathfrak{q} = \mathfrak{q}_{\Phi_C}$ for $\Phi_C \subset \mathcal{B}(C)$. Let $\mathcal{B}^*(C)$ be the adjoint basis in $\Pi(\mathcal{R})$ and $\Phi^*(C) = \{\omega_1, \ldots, \omega_k\}$ the set of elements of $\mathcal{B}^*(C)$ that vanish on $\mathcal{B}(C) \setminus \Phi_C$. Then for any k-tuple of strictly positive integers $(t_1, \ldots, t_k)$ we have:

$$\mathfrak{q} = \hat{\mathfrak{g}}_{t_1\omega_1 + \cdots + t_k\omega_k} = \bigcap_{1 \le i \le k} \hat{\mathfrak{g}}_{\omega_i} . \tag{8.43}$$

PROPOSITION 8.13. *Let* $\mathbf{Q}$ *be the parabolic subgroup of* $\hat{\mathbf{G}}$ *of the parabolic complex Lie subalgebra* $\mathfrak{q}$. *Then, with the notation above:*

$$\mathbf{Q} = \bigcap_{1 \le i \le k} \hat{\mathbf{G}}_{\omega_i} \quad \textit{and} \quad \mathrm{Hom}_{\mathbb{C}}(\mathbf{Q}, \mathbb{C}^*) = \{\varphi_\omega \,|\, \omega \in \Pi(\mathcal{R}) \cap \langle \mathcal{Q}^r \rangle_{\mathbb{R}}^{\perp}\} \tag{8.44}$$

is the free Abelian group generated by $\varphi_{\omega_1}, \ldots, \varphi_{\omega_k}$.

PROOF. The inclusion $\{\varphi_\omega \,|\, \omega \in \Pi(\mathcal{R}) \cap \langle \mathcal{Q}^r \rangle_{\mathbb{R}}^{\perp}\} \subset \mathrm{Hom}_{\mathbb{C}}(\mathbf{Q}, \mathbb{C}^*)$ is a consequence of the previous discussion.

To prove the opposite inclusion, we fist observe that a character in $\mathrm{Hom}_{\mathbb{C}}(\mathbf{Q}, \mathbb{C}^*)$ restricts to a character in $\mathrm{Hom}_{\mathbb{C}}(\hat{\mathbf{H}}, \mathbb{C}^*)$, and hence is of the form φ_ω for some $\omega \in \Pi(\mathcal{R})$. Moreover, being connected, $\mathbf{Q}$ admits a Levi decomposition $\mathbf{Q} = \mathbf{Q}_{\mathrm{ss}}\mathbf{Q}_{\mathrm{rad}}$ and $\varphi_\omega(g) = 1$ for $g \in \mathbf{Q}_{\mathrm{ss}}$(= the Levi subgroup of $\mathbf{Q}$). This yields $(\omega|\alpha^\vee) = 0$ for $\alpha \in \mathcal{Q}^r$. □

LEMMA 8.14. *If* $\omega \in \Pi(\mathcal{R})$ *is a real weight, then* φ_ω *is real valued on* $\hat{\mathbf{G}}_\omega \cap \mathbf{G}$.

PROOF. From $\varphi_\omega(\exp(Z)) = \exp(\omega(Z))$ for $Z \in \hat{\mathfrak{g}}_\omega$ we obtain that $\varphi_\omega(g)$ is real when g belongs to a neighborhood of the identity in $\hat{\mathbf{G}}_\omega \cap \mathbf{G}$. Since $\varphi_\omega(g)$ is an algebraic function of g, the statement follows because any neighborhood of the identity is Zariski dense in $\hat{\mathbf{G}}_\omega \cap \mathbf{G}$. □

Let $(\vartheta, \mathfrak{h})$ be a Cartan pair adapted to the effective parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$, choose $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ and set:

$$\Phi_C^* = \{\omega_1, \ldots, \omega_k\} = \mathcal{B}^*(C) \cap \left(\mathcal{Q}^r \cap \bar{\mathcal{Q}}^r\right)^{\perp} .$$

Then $\mathbf{G}_+ \subset \bigcap_{1 \le i \le k} \hat{\mathbf{G}}_{\omega_i}$, and we can define the map:

$$\boldsymbol{\varphi} : \mathbf{G}_+ \ni g \longrightarrow (\varphi_{\omega_1}(g), \ldots, \varphi_{\omega_k}(g)) \in (\mathbb{C}^*)^k . \tag{8.45}$$

We have:

THEOREM 8.15. *The map* (8.45) *yields, by passing to the quotients, a group isomorphism:* $\pi_0(\mathbf{G}_+) \xrightarrow{\boldsymbol{\varphi}_*} \pi_0(\boldsymbol{\varphi}(\mathbf{G}_+))$.

PROOF. By using Lemma 8.12, we can restrain to proving the Theorem in the case $\mathfrak{h}$ is maximally noncompact among the admissible Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{g}_+$. Moreover, by passing to the weakest CR model, we can as well assume that $\mathcal{Q}^r = \bar{\mathcal{Q}}^r$. By Proposition 8.9, we obtain a commutative diagram:

$$\begin{array}{ccc} \pi_0(\mathbf{H}/\mathbf{H}_0) & \xrightarrow{\sim} & \pi_0(\mathbf{G}_+) \\ \wr\big\downarrow & & \big\downarrow \boldsymbol{\varphi}_* \\ \pi_0(\boldsymbol{\varphi}(\mathbf{H})) & \xrightarrow{\sim} & \pi_0(\boldsymbol{\varphi}(\mathbf{G}_+)) \end{array} \tag{8.46}$$

showing that also $\boldsymbol{\varphi}_*$ is an isomorphism. □

In particular, when $(\mathfrak{g},\mathfrak{q})$ is totally real, i.e. $\mathfrak{q}=\bar{\mathfrak{q}}$, we obtain, by using Corollary 8.10:

THEOREM 8.16. *Let* $(\mathfrak{g},\mathfrak{q})$ *be a totally real parabolic CR algebra,* $\mathfrak{h}$ *a maximally noncompact Cartan subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{g}_+$, $C \in \mathfrak{C}(\mathcal{R},\mathcal{Q})$ *an S-fit (and S-adapted) Weyl chamber. Set:*

$$\mathcal{B}(C) = \{\alpha_1,\dots,\alpha_\ell\}\,, \quad \{\alpha_1,\dots,\alpha_\mu\} = \Phi_C \cap \mathcal{R}_{\mathrm{re}}\,, \quad \mathcal{B}^*(C) = \{\omega_1,\dots,\omega_\ell\}\,.$$

Then $\varphi_{\omega_i}(g) \in \mathbb{R}^*$ *for* $1 \le i \le \mu$. *The map:*

$$\boldsymbol{\varphi}^\flat \colon \mathbf{G}_+ \ni g \longrightarrow \left(\frac{\varphi_{\omega_1}(g)}{|\varphi_{\omega_1}(g)|},\dots,\frac{\varphi_{\omega_\mu}(g)}{|\varphi_{\omega_\mu}(g)|}\right) \in \{-1,1\}^\mu = \mathbb{Z}_2^\mu \tag{8.47}$$

defines, by passing to the quotient, an isomorphism:

$$\boldsymbol{\varphi}^\flat_* \colon \pi_0(\mathbf{G}_+) \ni [g] \longrightarrow \boldsymbol{\varphi}^\flat(g) \in \mathbb{Z}_2^\mu\,.\square \tag{8.48}$$

THEOREM 8.17. *We keep the notation and assumptions of Theorem* 8.16. *For each* $\alpha \in \Phi_C \cap \mathcal{R}_{\mathrm{re}}$, *let* $\mathbf{S}_\alpha$ *be the simple analytic real subgroup of* $\mathbf{G}$, *with Lie algebra* $\mathfrak{s}_\alpha = \mathfrak{g} \cap (\hat{\mathfrak{g}}^\alpha \oplus \hat{\mathfrak{g}}^{-\alpha} \oplus [\hat{\mathfrak{g}}^\alpha, \hat{\mathfrak{g}}^{-\alpha}])$. *Then there exists a set* $\{t_\alpha \mid \alpha \in \Phi_C \cap \mathcal{R}_{\mathrm{re}}\}$ *of generators of* $\pi_1(\mathbf{G}/\mathbf{G}_+)$, *and the homomorphism* $\boldsymbol{\delta} : \pi_1(\mathbf{G}/\mathbf{G}_+) \to \pi_0(\mathbf{G}_+)$ *of the exact homotopy sequence* (8.37) *of the principal bundle* $\mathbf{G} \to \mathbf{G}/\mathbf{G}_+$ *is given by:*

$$\boldsymbol{\delta} \colon \pi_1(\mathbf{G}/\mathbf{G}_+) \ni t_{\alpha_i} \longrightarrow (1,\dots,\underbrace{-1}_{i},\dots,1) \in \mathbb{Z}_2^\mu \simeq \pi_0(\mathbf{G}_+)\,. \tag{8.49}$$

PROOF. By using results from [DKV83], in [Wig98, §2] it is shown that the fundamental group of $\mathbf{G}/\mathbf{G}_+$ is generated by the images of the generators of the $\pi_1(\mathbf{S}_\alpha)$'s, some elements $\{t_\alpha\}$ for simple real roots $\alpha \in \Phi_C \cap \mathcal{R}_{\mathrm{re}}$. In fact the generalization of the Bruhat decomposition in [DKV83] yields a cell decomposition of $\mathbf{G}/\mathbf{G}_+$ where the open 1-cells correspond to roots $\alpha \in \Phi_C \cap \mathcal{R}_{\mathrm{re}}$. Let $\rho_\alpha : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{s}_\alpha$ be the representation with:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{\rho_\alpha} X_{-\alpha}\,, \quad \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\rho_\alpha} X_\alpha\,, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xrightarrow{\rho_\alpha} H_\alpha\,.$$

We still denote by ρ_α the corresponding group representation $\mathbf{SL}(2,\mathbb{R}) \longrightarrow \mathbf{S}_\alpha$. The open 1-cell corresponding to α can be parametrized by:

$$\left\{ g(t) = \rho_\alpha \left[\left(\begin{smallmatrix} \tan t & 1 \\ -1 & 0 \end{smallmatrix} \right) \right] \,\middle|\, -\pi/2 < t < \pi/2 \right\}.$$

Since $g_+(t) = \rho_\alpha \left[\left(\begin{smallmatrix} \cos t & -\sin t \\ 0 & 1/\cos t \end{smallmatrix} \right) \right] \in \mathbf{G}_+$ for all $|t| < \pi/2$, the closure in $\mathbf{G}/\mathbf{G}_+$ of this 1-cell is the loop t_α that is the image in $\mathbf{G}/\mathbf{G}_+$ of:

$$[-\pi/2, \pi/2] \ni t \longrightarrow g_0(t) = g(t)g_+(t) = \rho_\alpha \left[\left(\begin{smallmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{smallmatrix} \right) \right].$$

We note that $g_0(\pi/2) = e$, while $g_0(-\pi/2) = \exp(i\pi H_\alpha)$. Hence $\varphi_{\omega_j}(g_0(\pi/2)) = 1$, and:

$$\varphi_{\omega_j}(g_0(-\pi/2)) = \exp(i\pi\omega_j(H_\alpha)) = \exp(i\pi(\omega_j|\alpha^\vee)) = (-1)^{(\omega_j|\alpha^\vee)}.$$

This proves (8.49). □

8.2 The fundamental group

In this chapter we give an explicit combinatorial description of the fundamental group of parabolic CR manifolds. We keep the notation of the previous chapters. By using Proposition 4.17 and Theorem 8.16, we obtain:

THEOREM 8.18. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra and let $M = M(\mathfrak{g}, \mathfrak{q})$ be the corresponding homogeneous CR manifold. Then there exists a totally real parabolic CR algebra $(\mathfrak{g}, \mathfrak{q}')$ such that*

$$\mathbf{G}_+ = \{g \in \mathbf{G} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}) \subset \mathfrak{q}\} \subset \mathbf{G}'_+ = \{g \in \mathbf{G} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}') \subset \mathfrak{q}'\}$$

and the $\mathbf{G}$-equivariant map

$$f : M = \mathbf{G}/\mathbf{G}_+ \longrightarrow M' = M(\mathfrak{g}, \mathfrak{q}') = \mathbf{G}/\mathbf{G}'_+$$

has simply connected complex fibers. With $F = \mathbf{G}'_+/\mathbf{G}_+$, we obtain exact sequences:

$$\mathbf{1} \longrightarrow \pi_0(\mathbf{G}_+) \longrightarrow \pi_0(\mathbf{G'}_+) \longrightarrow \pi_0(F) \longrightarrow \mathbf{1}\,, \tag{8.50}$$

$$\mathbf{1} \longrightarrow \pi_1(M) \longrightarrow \pi_1(M') \longrightarrow \pi_0(F) \longrightarrow \mathbf{1}\,. \tag{8.51}$$

The induced map in homotopy $f_* : \pi_1(M) \longrightarrow \pi_1(M')$ is injective and $f_*(\pi_1(M))$ is a normal subgroup with finite index in $\pi_1(M')$.

PROOF. Let $\mathfrak{q}'$ be the parabolic $\mathfrak{q}_m$ of Proposition 4.17. We proved that the typical fiber $F = \mathbf{G}'_+/\mathbf{G}_+$ is simply connected. Thus (8.50) and (8.51) are consequences of Serre's long exact sequence for the homotopy groups of a fiber bundle. By Theorem 8.16, $\pi_0(\mathbf{G}'_+)$ is a finite Abelian group. Hence $\pi_0(\mathbf{G}_+)$ is normal in $\pi_0(\mathbf{G}'_+)$ and $\pi_0(F) \simeq \pi_0(\mathbf{G}'_+)/\pi_0(\mathbf{G}_+)$ has a natural structure of Abelian group. Then also the maps in (8.51) are group homomorphisms and therefore $f_*(\pi_1(M))$ is a normal subgroup of $\pi_1(M')$. $\square$

COROLLARY 8.19. *If $\mathfrak{g}$ is a simple real Lie algebra of the complex type or of real type* A II, A IIIa, A IV, B II, C II, D II, D IIIb, E III, E IV, F II, *then all orbits $M = M(\mathfrak{g}, \mathfrak{q})$ are simply connected.*

PROOF. The multiplicities of the simple real roots of $\hat{\mathfrak{g}}$ are always different from one, hence $M' = M(\mathfrak{g}, \mathfrak{q}')$ is simply connected, forcing M to be simply connected. $\square$

Note that the $M(\mathfrak{g}, \mathfrak{q}')$ for a totally real parabolic CR algebra $(\mathfrak{g}, \mathfrak{q}')$ are the *real flag manifolds*, and that there is a precise formula to compute their fundamental groups (see e.g. [DKV83], [Wig98]). Let $\mathfrak{h}'$ be a maximally noncompact Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}'_+$, fix an S-fit (and S-adapted) $C \in \mathfrak{C}(\mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}'), \mathcal{Q}')$ and let $\{\alpha_1, \ldots, \alpha_m\} = \mathcal{B}(C) \cap \mathcal{R}_{\mathrm{re}}$. Then $\pi_1(M')$ is described by generators $t_{\alpha_1}, \ldots, t_{\alpha_m}$ that satisfy the relations:

$$t_{\alpha_i} t_{\alpha_j} = t_{\alpha_j} t_{\alpha_i}^{(-1)^{(\alpha_i|\alpha_j^\vee)}} \quad \text{for} \quad 1 \le i, j \le m\,, \qquad t_{\alpha_i} = e \quad \text{if} \quad \alpha_i \notin \Phi'_C\,. \tag{8.52}$$

By using Theorem 8.17 we can now give a description of $\pi_1(M)$:

THEOREM 8.20. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra and $(\vartheta, \mathfrak{h})$ an adapted Cartan pair for $(\mathfrak{g}, \mathfrak{q})$ with $\mathfrak{h}$ maximally noncompact in $\mathfrak{g}_+$. Let $\mathbf{H}$ be the corresponding Cartan subgroup of $\mathbf{G}$.*

Let $(\mathfrak{g}, \mathfrak{q}')$ be the totally real parabolic CR algebra of Theorem 8.18, $(\vartheta, \mathfrak{h}')$ an adapted Cartan pair for $(\mathfrak{g}, \mathfrak{q}')$ with $\mathfrak{h}'$ maximally noncompact in $\mathfrak{g}$.

Fix a Weyl chamber $C' \in \mathfrak{C}(\mathcal{R}', \mathcal{Q}')$, where $\mathcal{R}' = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}')$ and $\mathcal{Q}'$ is the parabolic set of $\mathfrak{q}'$ in $\mathcal{R}'$, that is S-fit to $(\mathfrak{g}, \mathfrak{q}')$ and S-adapted. With $\mathcal{B}(C') = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\mathcal{B}^*(C') = \{\omega_1, \ldots, \omega_\ell\}$ (defined by $(\omega_i|\alpha_j^\vee) = \delta_{i,j}$), if $\{\omega_1, \ldots, \omega_\mu\} = \mathcal{B}^*(C') \cap \left[\mathcal{Q}'^{\,r}\right]^\perp$, we have $\{\alpha_1, \ldots, \alpha_\mu\} = \Phi_{C'}(\mathcal{Q}') \cap \mathcal{R}'_{\mathrm{re}}$. Let us consider the maps

$$\boldsymbol{\varphi}^\flat \colon \mathbf{G}'_+ \longrightarrow \mathbb{Z}_2^\mu \text{ of (8.47)} \qquad \text{and} \qquad \boldsymbol{\delta} : \pi_1(M') \to \mathbb{Z}_2^\mu \text{ of (8.49)}.$$

Then we have:

$$\pi_1(M) \;=\; \boldsymbol{\delta}^{-1}\left(\boldsymbol{\phi}^\flat(\mathbf{H})\right)\,. \qquad \square \tag{8.53}$$

COROLLARY 8.21. *Let $(\mathfrak{g}, \mathfrak{q})$, be an effective parabolic CR algebra, $M = M(\mathfrak{g}, \mathfrak{q})$ the corresponding parabolic CR manifold. Let $M' = M(\mathfrak{g}, \mathfrak{q}')$ and $f : M \longrightarrow M'$ be defined as in Theorem 8.18. If there is a Cartan subalgebra $\mathfrak{h}$ adapted to both $(\mathfrak{g}, \mathfrak{q})$ and $(\mathfrak{g}, \mathfrak{q}')$ and maximally noncompact in $\mathfrak{g}'_+$, such that moreover $\mathbf{H}/\mathbf{Z}(\mathbf{G})$ is connected, then $f_* : \pi_1(M) \longrightarrow \pi_1(M')$ is an isomorphism.*

Proof. For the isotropy subgroup $\mathbf{G}'_+$ of M' we have, by (8.31), that $\pi_0(\mathbf{G}'_+) \simeq \pi_0\left(\mathbf{H}/\left(\mathbf{H}\cap \mathbf{S}'_0\right)\right)$, where $\mathbf{S}'_0$ is an analytic semisimple subgroup of $\mathbf{G}'_+$. Since $\mathbf{Z}(\mathbf{G}) \subset \mathbf{G}_+$, the inclusion $\mathbf{H} \hookrightarrow \mathbf{G}'_+$ defines, passing to the quotients, a surjective map $\pi_0(\mathbf{H}/\mathbf{Z}(\mathbf{G})) \longrightarrow \pi_0\left(\mathbf{G}'_+/\mathbf{G}_+\right)$. Thus the fiber of $f\colon M \longrightarrow M'$ is connected and, by Theorem 8.18, $f_* : \pi_1(M) \longrightarrow \pi_1(M')$ is an isomorphism. □

Corollary 8.22. *With the notation of Corollary 8.21, if* $\mathfrak{g}$ *is a simple Lie algebra of real type* AIIIb, DIIIa, *then the map* $f_*\colon \pi_1(M) \longrightarrow \pi_1(M')$ *is an isomorphism.*

Proof. In fact in these cases $\mathbf{H}/\mathbf{Z}(\mathbf{G})$ is connected for every choice of $\mathfrak{h}$, and we can apply Corollary 8.21 to obtain an isomorphism of the fundamental group of $M(\mathfrak{g},\mathfrak{q})$ with the fundamental group of a totally real $M(\mathfrak{g},\mathfrak{q}')$. □

8.3 The case of compact parabolic CR manifolds

We apply the results of the previous sections to the case of compact parabolic CR manifolds.

We say that an effective compact parabolic CR algebra $(\mathfrak{g},\mathfrak{q}_{\Phi_C})$ has *property* (F) if Φ_C contains no real roots.

From Theorem 8.17, we obtain a rigidity result for homogeneous CR manifolds which are locally equivalent to compact parabolic CR manifolds:

Corollary 8.23. *Let* $\mathbf{G}$ *be a semisimple real Lie group and* M *a connected* $\mathbf{G}$*-homogeneous* CR *manifold. If the associated* CR *algebra* $(\mathfrak{g},\mathfrak{q})$ *is compact parabolic and has property* (F)*, then* M *is simply connected and* CR*-diffeomorphic to* $M(\mathfrak{g},\mathfrak{q})$. □

If the associated CR algebra has not property (F), instead we have:

Proposition 8.24. *Let* $\mathbf{G}$ *be a semisimple Lie group and* M *a* $\mathbf{G}$*-homogeneous* CR *manifold. Assume that the* CR *algebra* $(\mathfrak{g},\mathfrak{q}_{\Phi_C})$ *associated to* M *is compact parabolic. Let* $(\mathfrak{g},\mathfrak{q}_{\Psi_C})$ *be the basis of its fundamental reduction. Then there exists a (totally real)* $\mathbf{G}$*-homogeneous* CR *manifold* N*, with associated* CR *algebra* $(\mathfrak{g},\mathfrak{q}_{\Psi_C})$*, and a* $\mathbf{G}$*-equivariant submersion* $\omega : M \longrightarrow N$ *such that the induced map* $\omega_* : \pi_1(M) \longrightarrow \pi_1(N)$ *is an isomorphism.*

Proof. Let o be a point of M and let $\mathbf{G}_+$ the stabilizer of o in $\mathbf{G}$. Let $\mathbf{H}$ be the analytic subgroup of $\mathbf{G}$ generated by $\mathfrak{g}\cap\mathfrak{q}_{\Psi_C}$. Then $\mathbf{H}$ contains $\mathbf{G}^\circ_+$. We claim that $\mathbf{H}\cdot\mathbf{G}_+ = \mathbf{G}'_+$ is a Lie subgroup of $\mathbf{G}$. Indeed, for all $g \in \mathbf{G}_+$, we have $\mathrm{Ad}(g)(\mathfrak{q}_{\Phi_C}) = \mathfrak{q}_{\Phi_C}$. Since g is real, we also have $\mathrm{Ad}(g)(\bar{\mathfrak{q}}_{\Phi_C}) = \bar{\mathfrak{q}}_{\Phi_C}$ and therefore $\mathrm{Ad}(g)(\mathfrak{q}_{\Psi_C}) = \mathfrak{q}_{\Psi_C}$ because $\mathfrak{q}_{\Psi_C}$ is generated by $\mathfrak{q}_{\Phi_C}+\bar{\mathfrak{q}}_{\Phi_C}$. This implies that $\mathrm{ad}(g)(\mathbf{H}) = \mathbf{H}$ for all $g \in \mathbf{G}_+$, and hence $\mathbf{G}'_+$ is a subgroup of $\mathbf{G}$. It is a Lie subgroup because its Lie algebra is real parabolic. Then $N = \mathbf{G}/\mathbf{G}'_+$ is a $\mathbf{G}$-homogeneous manifold. By the inclusion $\mathbf{G}_+ \subset \mathbf{G}'_+$ we obtain a $\mathbf{G}$-equivariant submersion $\omega : M \longrightarrow N$. By construction the fiber is connected. It has a natural structure of CR manifold, associated to a fundamental CR algebra $(\mathfrak{g}'',\mathfrak{q}_{\Phi'_C})$, as in Theorem 6.2, which is parabolic and compact. By Corollary 8.23 the fiber is simply connected. Hence $\omega_* : \pi_1(M) \longrightarrow \pi_1(N)$ is an isomorphism. □

For minimal parabolic CR manifold, Proposition 8.24 specializes to:

COROLLARY 8.25. *Let $M = M(\mathfrak{g}, \mathfrak{q})$ be a compact parabolic CR manifold and $\phi : M \longrightarrow M' = M(\mathfrak{g}, \mathfrak{q}')$ its fundamental reduction. Then the induced map $\phi_* : \pi_1(M) \longrightarrow \pi_1(M')$ is a group isomorphism.*

In particular if M is fundamental then it is simply connected. $\square$

CHAPTER 9

Examples

In this chapter we apply results from previous chapters to several examples.

EXAMPLE 9.1. Consider the semisimple real Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ of real 3×3 matrices with zero trace. Let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3 \subset \mathbb{C}^3$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ the basis of $\mathbb{C}^3$ given by:

$$\varepsilon_1 = e_1 + ie_2\,, \qquad \varepsilon_2 = e_1 - ie_2\,, \qquad \varepsilon_3 = e_3\,.$$

Let $\mathfrak{q}$ be the complex Borel subalgebra of complex matrices $Z \in \mathfrak{sl}(3,\mathbb{C})$ such that $Z(\langle\varepsilon_1\rangle) \subset \langle\varepsilon_1\rangle$ and $Z(\langle\varepsilon_1,\varepsilon_2\rangle) \subset \langle\varepsilon_1,\varepsilon_2\rangle$. Let $\mathfrak{h} \subset \mathfrak{q}$ be the Cartan subalgebra of traceless matrices that are diagonal in the basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. The corresponding Cartan subgroup:

$$\mathbf{H} = \{\mathrm{diag}(\lambda, \bar\lambda, |\lambda|^{-2}) \in \mathbf{SL}(3,\mathbb{R}) \mid \lambda \neq 0\}$$

is connected, hence also $\mathbf{G}_+$ is connected. There exists a unique Weyl chamber $C \in \mathfrak{C}(\mathcal{R}(\hat{\mathfrak{g}},\hat{\mathfrak{h}}))$ adapted to $(\mathfrak{g},\mathfrak{q})$, which is both S-fit and V-fit. The corresponding diagram is:

$$\begin{array}{ccc} \circledast & \text{———} & \oplus \\ \alpha_1 & & \alpha_2 \\ \times & & \times \end{array}$$

and we see that $M = M(\mathfrak{g},\mathfrak{q})$ is weakly degenerate. The basis of the weakly nondegenerate reduction is the totally real parabolic CR manifold $M' = M(\mathfrak{g},\mathfrak{q}')$, where $\mathfrak{q}' = \{Z \in \mathfrak{sl}(3,\mathbb{C}) \mid Z(\langle\varepsilon_1,\varepsilon_2\rangle) \subset \langle\varepsilon_1,\varepsilon_2\rangle\}$. Its diagrams, with respect to the Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, where $\mathfrak{h}'$ is the maximally noncompact Cartan subalgebra of traceless diagonal matrices in the basis (e_1, e_2, e_3), are:

$$\begin{array}{ccc} \circledast & \text{———} & \oplus \\ \alpha_1 & & \alpha_2 \\ & & \times \end{array} \quad \simeq \quad \begin{array}{ccc} \bigcirc & \text{———} & \bigcirc \\ \alpha_1 & & \alpha_2 \\ & & \times \end{array}$$

By [Wig98] the fundamental group of M' is $\pi_1(M') = \mathbb{Z}_2$. On the other hand, by Corollary 8.10 the fiber of the weakly nondegenerate reduction has two connected components. Hence the exact sequence (8.50) implies that M is simply connected.

EXAMPLE 9.2. Let us compute the fundamental group of $M(\mathfrak{g},\mathfrak{q})$ in the case of Example 4.1. Since in this case $\mathfrak{q}$ is Borel and the Cartan subalgebra $\mathfrak{h}$ is maximally compact, we know that the isotropy $\mathbf{G}_+ = \{X \in \mathfrak{g} \mid \mathrm{ad}_{\hat{\mathfrak{g}}}(X)(\mathfrak{q}) = \mathfrak{q}\}$ is connected (see [Kna02, Prop.7.90]). Consider the fiber F over the point $(\langle e_1, e_4\rangle, \langle e_1, e_2, e_4, e_5\rangle)$. We can verify that F has 4 connected components and $\pi_0(F) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. The fundamental group of $M(\mathfrak{g},\mathfrak{q}_3)$ can be computed using [Wig98]. We have $\pi_1(M(\mathfrak{g},\mathfrak{q}_3)) \simeq \mathbb{Z}_2\times\mathbb{Z}_2$ and thus, from the exact sequence:

$$\mathbf{1} \to \pi_1(M(\mathfrak{g},\mathfrak{q})) \to \pi_1(M(\mathfrak{g},\mathfrak{q}_3)) \simeq \mathbb{Z}_2\times\mathbb{Z}_2 \to \pi_0(F) \simeq \mathbb{Z}_2\times\mathbb{Z}_2 \to \mathbf{1}$$

we obtain that $M(\mathfrak{g},\mathfrak{q})$ is simply connected.

Example 9.3. Consider the complex flag manifold $\mathfrak{M}$ of $\mathbf{SO}(5,\mathbb{C})$, consisting of the complex projective lines contained in the quadric $\{z_2^2+2z_0z_4+2z_1z_3=0\}\subset\mathbb{CP}^4$. We identify the Lie algebra $\mathfrak{so}(5,\mathbb{C})$ to the matrix algebra:

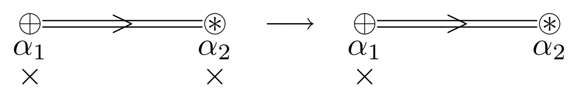
$$\mathfrak{so}(5,\mathbb{C})\simeq\hat{\mathfrak{g}}=\left\{Z\in\mathfrak{sl}(5,\mathbb{C})\,\middle|\,{}^tZS_5+S_5Z=0\right\}\quad\text{for}\quad S_5=\begin{pmatrix} &&&&1\\ &&&1&\\ &&1&&\\ &1&&&\\ 1&&&& \end{pmatrix}.$$

We consider the real form $\mathfrak{g}\simeq\mathfrak{so}(2,3)$ of $\mathfrak{so}(5,\mathbb{C})$ defined by:

$$\mathfrak{so}(2,3)\simeq\mathfrak{g}=\{Z\in\hat{\mathfrak{g}}\mid Z^*K+KZ=0\}\quad\text{for}\quad K=\begin{pmatrix} &&&&1\\ &1&0&0&\\ &0&-1&0&\\ &0&0&1&\\ 1&&&& \end{pmatrix}.$$

Let M be the orbit of the projective line corresponding to the plane $\ell_2=\langle e_1,e_2\rangle$ of $\mathbb{C}^5$ by the action of the analytic subgroup $\mathbf{G}$ with Lie algebra $\mathfrak{g}$: M is the submanifold of the Grassmannian of the complex 2-planes in $\mathbb{C}^5$, consisting of those that are totally isotropic for the symmetric form S_5 and degenerate, with signature $(+,0)$, with respect to the Hermitian symmetric form K. Denoting by $\mathfrak{q}$ the stabilizer of $\langle e_1,e_2\rangle$ in $\mathbf{SO}(5,\mathbb{C})$, we have $M=M(\mathfrak{g},\mathfrak{q})$. Take the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting of the diagonal matrices. With e_1,e_2 also denoting the value of the first and the second diagonal entry, we note that the conjugation σ is defined in $\mathfrak{h}^*_{\mathbb{R}}$ by $\sigma(e_i)=-(-1)^ie_i$. Take the Weyl chambers $C,C'\in\mathfrak{C}(\mathcal{R},\mathcal{Q})$ associated to the basis $\mathcal{B}(C)=\{e_1-e_2,e_2\}$ and $\mathcal{B}(C')=\{e_2-e_1,e_1\}$. Then C is S-fit and C' V-fit for $(\mathfrak{g},\mathfrak{q})$. We can describe M by the cross-marked diagrams:

From the first diagram we see that $(\mathfrak{g},\mathfrak{q})$ is fundamental, since $\Phi_C=\{\alpha_2\}$ and $\bar{\alpha}_1=\alpha_1+2\alpha_2\succ_C\alpha_2$; from the second we see that $(\mathfrak{g},\mathfrak{q})$ is weakly non-degenerate, because $\Phi_{C'}=\{\alpha'_2\}$ and $\bar{\alpha}'_2=\alpha'_2\succ_{C'}0$. The weakest CR model of $(\mathfrak{g},\mathfrak{q})$ is the parabolic CR algebra $(\mathfrak{g},\mathfrak{v})=(\mathfrak{g},\mathfrak{q}_{\Phi_C^\sharp})$, with $\Phi_C^\sharp=\{\alpha_1,\alpha_2\}$. The weakly non-degenerate reduction of $(\mathfrak{g},\mathfrak{v})$ is the totally real parabolic CR algebra $(\mathfrak{g},\mathfrak{q}_{\Psi_C})$, with $\Psi_C=\{\alpha_1\}$:



By composition we obtain the $\mathbf{G}$-equivariant projection:

$$M(\mathfrak{g},\mathfrak{q})\xrightarrow{\sim}M(\mathfrak{g},\mathfrak{v})\longrightarrow M'=M(\mathfrak{g},\mathfrak{q}_{\Psi_C=\{\alpha_1\}})$$

of M onto a totally real parabolic CR manifold M'. This projection associates to each $\ell_2\in M$ the isotropic line $\ell_2\cap\ell_2^\perp$, where $\perp$ is taken with respect to the Hermitian symmetric form K. The fiber over $\ell_1=\langle e_1\rangle$ consists of the planes generated by the columns of the matrices

$$\begin{pmatrix}1&0\\0&z_0\\0&z_1\\0&z_2\\0&0\end{pmatrix}\qquad\text{with}\qquad\begin{cases}(z_0:z_1:z_2)\in\mathbb{CP}^2\\ 2z_0z_2+z_1^2=0\\ z_0\bar{z}_0+z_2\bar{z}_2>z_1\bar{z}_1\,.\end{cases}$$

Then we see that the fiber is biholomorphic to $\mathbb{CP}^1 \setminus \mathbb{RP}^1$, that is the disjoint union of two disks in $\mathbb{C}$. Thus the fiber F of the projection $M \to M'$ has two connected components and $\pi_0(F) \simeq \mathbb{Z}_2$. Note that, by [Wig98], $\pi_1(M') \simeq \mathbb{Z}$. Thus the Serre's exact sequence:

$$\mathbf{1} \longrightarrow \pi_1(M) \longrightarrow \pi(M') \simeq \mathbb{Z} \longrightarrow \pi_0(F) \simeq \mathbb{Z}_2 \longrightarrow \mathbf{1}$$

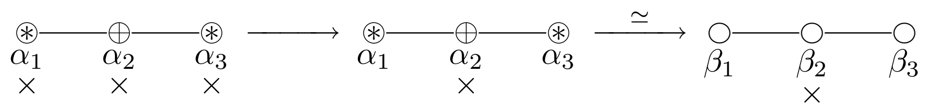
shows that $\pi_1(M) \simeq 2\mathbb{Z} \simeq \mathbb{Z}$.

Example 9.4. Let $(\varepsilon_i)_{1\leq i\leq 4}$ be the canonical basis of $\mathbb{C}^4$, and $\mathfrak{g} \simeq \mathfrak{sl}(4,\mathbb{R})$ consist of the elements of $\mathfrak{sl}(4,\mathbb{C})$ that have real entries in the basis $(\varepsilon_i)_{1\leq i\leq 4}$. We introduce the basis

$$e_1 = \varepsilon_1 + i\varepsilon_2\,,\ e_2 = \varepsilon_1 - i\varepsilon_2\,,\ e_3 = \varepsilon_3 + i\varepsilon_4\,,\ e_4 = \varepsilon_3 - i\varepsilon_4\,.$$

We take the complex flag manifold $\mathfrak{M}$ whose points are the pairs (ℓ_1, ℓ_3) of a complex line ℓ_1 of $\mathbb{C}^4$ and a complex 3-plane ℓ_3 with $\ell_1 \subset \ell_3 \subset \mathbb{C}^4$, and consider the $\mathbf{G}$-orbit M that contains the point $\mathfrak{o} = \langle e_1\rangle \subset \langle e_1, e_2, e_3\rangle$: we have $M = M(\mathfrak{g}, \mathfrak{q})$ where $\mathfrak{q}$ is the stabilizer in $\hat{\mathfrak{g}} \simeq \mathfrak{sl}(4,\mathbb{C})$ of $\mathfrak{o}$. Consider the Cartan subalgebra $\mathfrak{h}$ of the elements of $\mathfrak{g}$ that are diagonal matrices in the basis $(e_i)_{1\leq i\leq 4}$. With $e_i(H)$ also denoting the value of the i-th entry of $H \in \mathfrak{h}_{\mathbb{R}}$, we note that $\mathcal{B}(C) = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq 3\}$ is the system of simple roots for an S-fit Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$, and $\mathcal{B}(C) = \{\alpha'_1 = e_1 - e_3\,,\ \alpha'_2 = e_3 - e_2\,,\ \alpha'_3 = e_2 - e_4\}$ is the system of simple roots for a V-fit Weyl chamber. The corresponding cross-marked diagrams are:

Since $\bar{\alpha}_2 = \alpha_1 + \alpha_2 + \alpha_3$, from the first we see that $(\mathfrak{g}, \mathfrak{q})$ is fundamental, while the second shows that $(\mathfrak{g}, \mathfrak{q})$ is weakly nondegenerate. Its weakest CR model $(\mathfrak{g}, \mathfrak{v})$ has a totally real weakly nondegenerate reduction $(\mathfrak{g}, \mathfrak{q}')$, corresponding to diagrams:



where the last diagram is obtained by utilizing the Cartan subalgebra of real diagonal matrices of $\mathfrak{g}$ with respect to the canonical basis $(\varepsilon_i)_{1\leq i\leq 4}$. Using [Wig98], we obtain that $\pi_1(M(\mathfrak{g}, \mathfrak{q}')) \simeq \mathbb{Z}_2$. The isotropy subgroup $\mathbf{G}_+$ of $M(\mathfrak{g}, \mathfrak{q}')$ is isomorphic to the group of matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with A, B, C real 2×2 matrices with $\det(A) \cdot \det(C) = 1$, and hence has two connected components. The isotropy subgroup $\mathbf{G}_+$ of $M(\mathfrak{g}, \mathfrak{q})$ is connected: indeed $\mathbf{G}_+ = \mathbf{N} \ltimes \mathbf{H}$ for an Euclidean $\mathbf{N} = \exp(\mathfrak{n})$ and a Cartan subgroup $\mathbf{H} = \mathbf{Z}_{\mathbf{G}}(\mathfrak{h})$ that is connected because $\mathfrak{h}$ is maximally noncompact (cf. [Kna02, Prop.7.90]). Thus the fiber $\mathbf{G}'_+/\mathbf{G}_+$ has two connected components. Thus, from the exact sequence:

$$\mathbf{1} \longrightarrow \pi_1(M(\mathfrak{g}, \mathfrak{q})) \longrightarrow \underbrace{\pi_1(M(\mathfrak{g}, \mathfrak{q}'))}_{\simeq \mathbb{Z}_2} \longrightarrow \underbrace{\pi_0(\mathbf{G}'_+/\mathbf{G}_+)}_{\simeq \mathbb{Z}_2} \longrightarrow \mathbf{1}$$

we obtain that $M(\mathfrak{g}, \mathfrak{q})$ is simply connected.

EXAMPLE 9.5. Let $(\varepsilon_i)_{1\leq i\leq 6}$ be the canonical basis of $\mathbb{C}^6$. Let $\mathbf{G}\simeq\mathbf{SL}(6,\mathbb{R})$, with Lie algebra $\mathfrak{g}$, be the subgroup of $\mathbf{SL}(6,\mathbb{C})$ consisting of the matrices with real entries. Consider the basis

$$e_1'=\varepsilon_1+i\varepsilon_4\,,\ e_2'=\varepsilon_2\,,\ e_3'=\varepsilon_3+i\varepsilon_6\,,\ e_4'=\varepsilon_1-i\varepsilon_4\,,\ e_5'=\varepsilon_5\,,\ e_6'=\varepsilon_3+i\varepsilon_6\,.$$

Let $\mathfrak{q}$ be the stabilizer of $\langle e_1',e_2'\rangle\subset\langle e_1',e_2',e_3',e_4'\rangle$ in $\mathfrak{sl}(6,\mathbb{C})$ and consider the parabolic CR algebra $(\mathfrak{g},\mathfrak{q})$. The matrices in $\mathfrak{g}$ that are diagonal in the basis $e_1',\ldots,e_6'$ form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ adapted to $(\mathfrak{g},\mathfrak{q})$. Then $\mathfrak{h}_{\mathbb{R}}$ consists of the matrices of $\hat{\mathfrak{g}}=\mathfrak{sl}(6,\mathbb{C})$ that are real and diagonal in the basis $e_1',\ldots,e_6'$. We identify e_i' to the evaluation function of the i-th diagonal term of $H\in\mathfrak{h}_{\mathbb{R}}$. Then the $\alpha_i'=e_i'-e_{i+1}'$ are the simple root of a $C'\in\mathfrak{C}(\mathcal{R},\mathcal{Q})$ that is V-fit for $(\mathfrak{g},\mathfrak{q})$. We have the cross-marked diagram for $(\mathfrak{g},\mathfrak{q})$:

$$\begin{array}{ccccccccc} \ominus & \text{---} & \oplus & \text{---} & \ominus & \text{---} & \oplus & \text{---} & \ominus \\ \alpha_1' & & \alpha_2' & & \alpha_3' & & \alpha_4' & & \alpha_5' \\ & & \times & & & & \times & & \end{array}$$

Since $\Phi_{C'}=\{\alpha_2',\alpha_4'\}$ and $\bar{\alpha}_2'=\alpha_2'+\alpha_3'+\alpha_4'+\alpha_5'\succ_{C'}0$, $\bar{\alpha}_4'=\alpha_1'+\alpha_2'+\alpha_3'+\alpha_4'\succ_{C'}0$, the CR algebra $(\mathfrak{g},\mathfrak{q})$ is weakly nondegenerate. We obtain an S-fit Weyl chamber for $(\mathfrak{g},\mathfrak{q})$ by reordering the basis $e_1',\ldots,e_6'$. Set :

$$e_1=e_2'\,,\ e_2=e_1'\,,\ e_3=e_4'\,,\ e_4=e_3'\,,\ e_5=e_6'\,,\ e_6=e_5'\,.$$

Then $\alpha_i=e_i-e_{i+1}$ $(1\leq i\leq 5)$ is the basis $\mathcal{B}(C)$ of an S-fit Weyl chamber $C\in\mathfrak{C}(\mathcal{R},\mathcal{Q})$ yielding the cross-marked diagram :

$$\begin{array}{ccccccccc} \oplus & \text{---} & \circledast & \text{---} & \oplus & \text{---} & \circledast & \text{---} & \oplus \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 \\ & & \times & & & & \times & & \end{array}$$

Since $\Phi_C=\{\alpha_2,\alpha_4\}$ and $\bar{\alpha}_3=\alpha_2+\alpha_3+\alpha_4\succ_C\alpha_2,\alpha_4$, the CR algebra $(\mathfrak{g},\mathfrak{q})$ is also fundamental. The weakest CR model $(\mathfrak{g},\mathfrak{v})$ of $(\mathfrak{g},\mathfrak{q})$ is obtained by taking the complex Borel subalgebra of $\hat{\mathfrak{g}}$ associated to the chamber C. We have $\mathfrak{v}=\mathfrak{q}_{\Phi_C^\sharp}$ with $\Phi_C^\sharp=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$. The weakly nondegenerate reduction of $(\mathfrak{g},\mathfrak{q})$ is the totally real CR algebra $(\mathfrak{g},\mathfrak{q}_{\Psi_C})$ with $\Psi_C=\{\alpha_1,\alpha_3,\alpha_5\}$:

$$\begin{array}{ccccccccc} \oplus & \text{---} & \circledast & \text{---} & \oplus & \text{---} & \circledast & \text{---} & \oplus \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 \\ \times & & \times & & \times & & \times & & \times \end{array} \quad\longrightarrow\quad \begin{array}{ccccccccc} \oplus & \text{---} & \circledast & \text{---} & \oplus & \text{---} & \circledast & \text{---} & \oplus \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 \\ \times & & & & \times & & & & \times \end{array}$$

By composition we obtain a $\mathbf{G}$-equivariant fibration :
$M=M(\mathfrak{g},\mathfrak{q})\to M'=M(\mathfrak{g},\mathfrak{q}_{\Psi_C})$, with $(\ell_2\subset\ell_4)\to(\ell_2\cap\bar{\ell}_2\subset\ell_4\cap\bar{\ell}_4\subset\ell_4+\bar{\ell}_4)$.

The fiber F over $\langle e_1\rangle\subset\langle e_1,e_2,e_3\rangle\subset\langle e_1,e_2,e_3,e_4,e_5\rangle$ consists of the pairs $(\ell_2\subset\ell_4)$ where ℓ_2 is a complex 2-plane with $\langle e_1\rangle\subset\ell_2\subset\langle e_1,e_2,e_3\rangle$ and ℓ_4 is a complex 4-plane with $\langle e_1,e_2,e_3\rangle\subset\ell_4\subset\langle e_1,e_2,e_3,e_4,e_5\rangle$, with $\ell_2\neq\bar{\ell}_2$ and $\ell_4\neq\bar{\ell}_4$. Thus $\pi_0(F)\simeq\mathbb{Z}_2\times\mathbb{Z}_2$. On the other hand, by [Wig98], we have $\pi_1(M')\simeq\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. From the exact sequence

$$\mathbf{1}\longrightarrow\pi_1(M)\longrightarrow\pi_1(M')\simeq\mathbb{Z}_2^3\longrightarrow\pi_0(F)\simeq\mathbb{Z}_2^2\longrightarrow\mathbf{1}$$

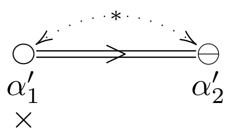
we obtain that $\pi_1(M)\simeq\mathbb{Z}_2$.

EXAMPLE 9.6. Let $\hat{\mathfrak{g}} \simeq \mathfrak{so}(5,\mathbb{C})$ be as in Example 9.3. We take now $\mathfrak{g} \simeq \mathfrak{so}(2,3)$, defined by:

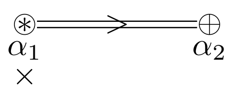
$$\mathfrak{g} = \{Z \in \hat{\mathfrak{g}} \mid Z^*K + KZ = 0\} \qquad \text{with} \quad K = \begin{pmatrix} & & \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right] \\ & 1 & \\ \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right] & & \end{pmatrix}.$$

Let $\mathfrak{q}$ be the stabilizer of the line $\langle e_1 \rangle \subset \mathbb{C}^5$ and consider the orbit $M = M(\mathfrak{g},\mathfrak{q})$. Our M is one of the two connected components of the manifold M^+ the nonreal lines ℓ_1, contained in the quadric cone $\{2z_0z_4 + 2z_1z_3 + z_2^2 = 0\} \subset \mathbb{C}^5$, for which $(\ell_1 + \bar{\ell}_1)$ is a totally isotropic complex 2-plane for the Hermitian symmetric form K. The involution $\ell_1 \longrightarrow \bar{\ell}_1$ interchanges the two connected components of M^+.

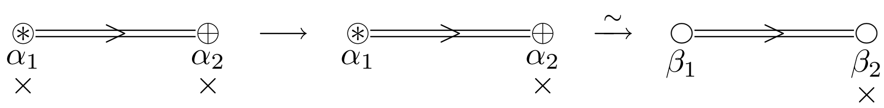
The diagonal matrices in $\mathfrak{g}$ define a Cartan subalgebra of $\mathfrak{g}$ adapted to $(\mathfrak{g},\mathfrak{q})$. As usual we denote again by e_i the value of the i-th entry in the diagonal of $\mathfrak{h}_{\mathbb{R}}$. Then we define a V-fit C' and an S-fit C by taking $\mathcal{B}(C') = \{\alpha_1' = e_1 + e_2,\ \alpha_2' = -e_2\}$ and $\mathcal{B}(C) = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2\}$. The associated cross-marked diagrams are:



and

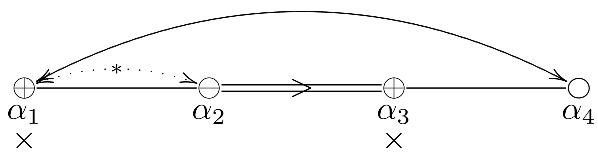


From the first, as $\mathfrak{q} = \mathfrak{q}_{\Phi_{C'}}$ with $\Phi_{C'} = \{\alpha_1'\}$ and $\bar{\alpha}_1' = \alpha_1' \succ_{C'} 0$, we see that $(\mathfrak{g},\mathfrak{q})$ is weakly nondegenerate; from the second, as $\mathfrak{q} = \mathfrak{q}_{\Phi_C}$ with $\Phi_C = \{\alpha_1\}$ and $\bar{\alpha}_2 = \alpha_1 + \alpha_2 \succ_C \alpha_1$, we see that $(\mathfrak{g},\mathfrak{q})$ is also fundamental. The weakest CR model of $(\mathfrak{g},\mathfrak{q})$ is $(\mathfrak{g},\mathfrak{q}_{\Psi_C^\sharp})$ with $\Psi_C^\sharp = \{\alpha_1,\alpha_2\}$. The basis of its weakly nondegenerate reduction is the totally real CR algebra $(\mathfrak{g},\mathfrak{q}_{\Psi_C})$ with $\Psi_C = \{\alpha_2\}$. We can represent these maps by the diagram:



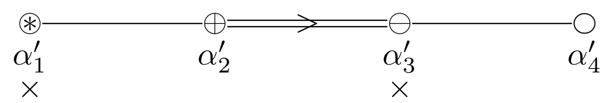
where the last is the cross-marked Satake diagram of $(\mathfrak{g},\mathfrak{q}_{\Psi_C})$, i.e. the cross-marked diagram for an S-fit and S-adapted C and a maximally noncompact $\mathfrak{h}$. We have, by [Wig98, Theorem 1.1], $\pi_1(M(\mathfrak{g},\mathfrak{q}_{\Psi_C})) \simeq \mathbb{Z}_2$. We observe that the stabilizer in the connected component of the identity of $\mathbf{SO}(2,3)$ of a totally isotropic 2-plane $\ell_2 \subset \mathbb{R}^5$ keeps its orientation and is connected. Thus the the fiber F of the projection $M(\mathfrak{g},\mathfrak{q}) \longrightarrow M(\mathfrak{g},\mathfrak{q}_{\Psi_C})$ is connected because the isotropy subgroup of $M(\mathfrak{g},\mathfrak{q}_{\Psi_C})$ is connected. Therefore $\pi_1(M(\mathfrak{g},\mathfrak{q})) \simeq \pi_1(M(\mathfrak{g},\mathfrak{q}_{\Psi_C})) \simeq \mathbb{Z}_2$.

EXAMPLE 9.7. Let $\mathfrak{g}$ be a simple real Lie algebra of type FI (split real form). We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, such that the conjugation σ defined in $\hat{\mathfrak{g}}$ by the real form $\mathfrak{g}$ restricts in $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle_{\mathbb{R}}$ to the linear involution that is defined on the canonical basis by: $\sigma(e_1) = -e_3$, $\sigma(e_2) = e_4$, $\sigma(e_3) = -e_1$, $\sigma(e_4) = e_2$. The vectors $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$ and $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ are a basis of simple roots of $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}},\hat{\mathfrak{h}})$. We consider the parabolic CR manifold $M(\mathfrak{g},\mathfrak{q})$ that corresponds to the cross-marked diagram:



This is a representation of $(\mathfrak{g}, \mathfrak{q})$ in a V-fit Weyl chamber $C \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$. Since $\bar{\alpha}_1 \succ_C 0$, $\bar{\alpha}_3 \succ_C 0$, by Theorem 3.9 the CR algebra $(\mathfrak{g}, \mathfrak{q})$ is weakly nondegenerate.

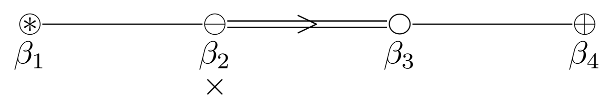
Consider the Weyl chamber C_1 obtained from C by the symmetry s_{α_2}. We have $\mathcal{B}(C_1) = \left\{e_2 - e_4, e_4 - e_3, e_3, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\right\} = \{\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4\}$ and the cross-marked diagram for $(\mathfrak{g}, \mathfrak{q})$ for the chamber $C_1 \in \mathfrak{C}(\mathcal{R}, \mathcal{Q})$ is:



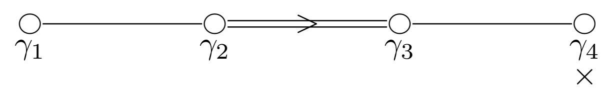
This diagram is S-fit. Since $\bar{\alpha}'_2 = 2\alpha'_1 + 3\alpha'_2 + 4\alpha'_3 + 2\alpha'_4$, by Theorem 3.5 the parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ is also fundamental.

Our next aim is to construct the fibration of Proposition 4.17. First we observe that the weakest CR model of $(\mathfrak{g}, \mathfrak{q})$ is $(\mathfrak{g}, \mathfrak{v}_1)$ with $\mathfrak{v}_1 = \mathfrak{q}_{\{\alpha'_1, \alpha'_2, \alpha'_3\}}$. Its weakly nondegenerate reduction is $(\mathfrak{g}, \mathfrak{q}_2)$ with $\mathfrak{q}_2 = \mathfrak{q}_{\{\alpha'_2\}}$.

To compute its weakest CR model, we need to find an S-fit Weyl chamber C_2 for $(\mathfrak{g}, \mathfrak{q}_2)$. This is provided by the basis of simple roots $\mathcal{B}(C) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ with $\beta_1 = e_2 - e_4$, $\beta_2 = e_1 - e_2$, $\beta_3 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\beta_4 = -\frac{1}{2}(e_1 - e_2 + e_3 - e_4)$, that is obtained from $\mathcal{B}(C_1)$ by the rotation $s_{\frac{1}{2}(e_1 - e_2 + e_3 - e_4)} \circ s_{e_3}$ of $\mathbf{W}(\mathcal{R})$. The corresponding cross-marked diagram for $(\mathfrak{g}, \mathfrak{q}_2)$ is:



Since $\bar{\beta}_4 = \beta_1 + 2\beta_2 + 2\beta_3 + \beta_4$, the weakest CR model of $(\mathfrak{g}, \mathfrak{q}_2)$ is $(\mathfrak{g}, \mathfrak{v}_2)$ with $\mathfrak{v}_2 = \mathfrak{q}_{\{\beta_2, \beta_4\}}$. The CR algebra $(\mathfrak{g}, \mathfrak{v}_2)$ has the weakly nondegenerate reduction $(\mathfrak{g}, \mathfrak{q}_{\{\beta_4\}})$. The element $A = (-1, -1, +1, -1)$ of $\mathfrak{h}_{\mathbb{R}}$ defines the parabolic set of $\mathfrak{v}_2$ and $\bar{A} = A$ shows then that $(\mathfrak{g}, \mathfrak{v}_2)$ is totally real. Thus, by choosing a maximally noncompact Cartan subalgebra $\mathfrak{h}'$ adapted to $(\mathfrak{g}, \mathfrak{v}_2)$, we can associate to $(\mathfrak{g}, \mathfrak{v}_2)$ its cross-marked Satake diagram as a totally real parabolic minimal CR algebra:



Then the isotropy subgroup $\mathbf{G}'_+$ of $M(\mathfrak{g}, \mathfrak{v}_2)$ has two connected components, and the fundamental group $\pi_1(M(\mathfrak{g}, \mathfrak{v}_2))$ is isomorphic to $\mathbb{Z}_2$ and is generated by any simple path joining the two connected components of $\mathbf{G}'_+$. By using Lemma 8.1 and (8.31) of Theorem 8.8, we find that $\mathbf{H}$ has four connected components and $\boldsymbol{\varphi}^\flat(\mathbf{H})$ has two connected components. Hence Theorem 8.20 yields $\pi_1(M) \simeq \mathbb{Z}_2$.

CHAPTER 10

Global CR functions

In this chapter we describe the space $\mathcal{O}(M)$ of smooth global CR functions on a parabolic CR manifold $M = M(\mathfrak{g}, \mathfrak{q})$.

To this aim we introduce two notions of CR separability. We say that M is (globally) *weakly CR separable* if global CR functions separate points of M, that is if for every pair $x, y \in M$, with $x \neq y$ there exists $f \in \mathcal{O}(M)$ such that $f(x) \neq f(y)$. We also say that M is *weakly locally CR separable* if every point $x \in M$ has a neighborhood U such that global CR functions on M separate points of U. In other words, a CR manifold M is weakly (locally) CR separable if and only if there exists a (locally) injective CR map into a complex Euclidean space.

Next we introduce the notion of strict local CR separability. Let $\mathcal{S} = \mathcal{S}(M) \subset \mathcal{C}^\infty(M, \mathbb{C}TM)$ be the space of complex vector fields X on M such that $X(f) = 0$ for all $f \in \mathcal{O}(M)$, and let $S = S(M) \subset \mathbb{C}TM$ be the vector distribution defined at $x \in M$ by:

$$S_x = \{X_x \mid X \in \mathcal{S}\}.$$

By the definition of CR functions, $T^{0,1}M \subset S$. We say that M is *strictly locally CR separable* at a point x is $S_x = T^{0,1}_x M$, and that M is strictly locally CR separable if it is strictly locally CR separable at every point. We have:

Lemma 10.1. *If M is strictly locally CR separable then M is weakly locally CR separable.*

Proof. Assume that there exists a point $p \in M$ and two sequences x_n and y_n , with $x_n \neq y_n$ and converging to p, such that for every CR function $f \in \mathcal{O}(M)$ we have $f(x_n) = f(y_n)$ for all n. Let d be the distance function on M defined by some Riemanniann metric g on M. The functionals ξ_n, defined on a smooth function f on M by:

$$\xi_n(f) = (f(x_n) - f(y_n))/d(x_n, y_n),$$

converge, up to the choice of a subsequence, to a unit real tangent vector $X \in T_pM$. Clearly $X(f) = 0$ for every $f \in \mathcal{O}(M)$, thus M is not strictly locally CR separable.$\square$

Local strict CR separability is an open condition, because dim S_x is upper semicontinuous with respect to x, and is actually equivalent to the existence, for each point $x \in M$, of a global CR map of M into a complex Euclidean space $\mathbb{C}^n$ that is a CR embedding in a neighborhood of x.

We return to the case of parabolic CR manifolds. Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic CR algebra, $M = M(\mathfrak{g}, \mathfrak{q})$ the associated parabolic CR manifold, and $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{G}_+ \simeq M$ the quotient projection. Since $\mathcal{S}$ is invariant under CR automorphisms of M, we have that S is a G-homogeneous complex vector bundle on M. We have:

LEMMA 10.2. *The vector subspace of* $\hat{\mathfrak{g}}$*:*
$$\mathfrak{s} = (\mathrm{d}\hat{\pi}_e)^{-1}(S_{\mathbf{o}})$$
is a parabolic complex Lie subalgebra of $\hat{\mathfrak{g}}$*, containing* $\mathfrak{q}$.

PROOF. Consider the sheaf $\mathcal{T}$ on $\mathbf{G}$ of germs of complex vector fields X on $\mathbf{G}$ such that $X(\pi^* f) = 0$ for all $f \in \mathcal{O}(M)$. Then $\mathcal{T}$ is invariant for the left action of $\mathbf{G}$, hence it is generated at every point by the global left invariant complex vector fields which belong to $\mathcal{T}$ near the identity $e \in \mathbf{G}$, and $\mathcal{T}$ is the sheaf of germs of smooth sections of a $\mathbf{G}$-homogeneous vector subbundle T of the complexified tangent bundle $\mathbb{C}T\mathbf{G}$. Since under the identification $\mathbb{C}T_e\mathbf{G} \simeq \hat{\mathfrak{g}}$, we have that $T_e \simeq s$, it follows that $\mathfrak{s} = (\mathrm{d}\hat{\pi}_e)^{-1}(S_{\mathbf{o}})$. The statement follows because $\mathcal{T}$ is involutive, hence $\mathfrak{s}$ is a subalgebra. It contains $\mathfrak{q}$, thus it is parabolic. □

Since $\mathfrak{q} \subset \mathfrak{s}$, we may consider the $\mathbf{G}$-equivariant fibration:
$$\rho : M = M(\mathfrak{g}, \mathfrak{q}) \longrightarrow M' = M(\mathfrak{g}, \mathfrak{s}). \tag{10.1}$$
Every CR function on M' defines, via the pullback by ρ, a CR function on M. Indeed more is true:

THEOREM 10.3. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be an effective parabolic* CR *algebra and* $M = M(\mathfrak{g}, \mathfrak{q})$ *the associated parabolic* CR *manifold. Then there exists a unique* $\mathbf{G}$*-equivariant fibration* $\rho : M \longrightarrow M^*$ *onto a strictly locally* CR *separable* $\mathbf{G}$*-homogeneous* CR *manifold* M^**, such that* ρ *induces an isomorphism on the space of* CR *functions, that is:*
$$\mathcal{O}(M) = \rho^* \mathcal{O}(M^*).$$
The $\mathbf{G}$*-homogeneous* CR *manifold* M^* *admits a* $\mathbf{G}$*-equivariant covering map onto* $M' = M(\mathfrak{g}, \mathfrak{s})$*, where* $\mathfrak{s} = (\mathrm{d}\pi_e)^{-1}(S_{\mathbf{o}})$ *is the Lie subalgebra of* $\hat{\mathfrak{g}}$ *defined in Lemma 10.2*

PROOF. From the definition of $\mathfrak{s}$ it follows that $M' = M(\mathfrak{g}, \mathfrak{s})$ is strictly locally CR separable. Furthermore $\mathfrak{s}$ is the smallest complex Lie subalgebra of $\hat{\mathfrak{g}}$, containing $\mathfrak{q}$, such that $M(\mathfrak{g}, \mathfrak{s})$ is strictly locally CR separable.

Let $\mathbf{G}_+^*$ be the Lie subgroup generated by $\mathbf{G}_+$ and by the analytic subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{g}'_+ = \mathfrak{g} \cap \mathfrak{s}$. Then $M^* = \mathbf{G}/\mathbf{G}_+^*$ is a finite cover of M', and we endow it with the unique CR structure such that the covering map $M^* \longrightarrow M'$ is a local CR isomorphism. Denote by $\rho : M \longrightarrow M^*$ the natural $\mathbf{G}$-equivariant projection.

Then global CR functions on M are invariant for the right $\mathbf{G}_+^*$-action, hence they factor through ρ: if $f \in \mathcal{O}(M)$ then there exists $f^* \in \mathcal{C}^\infty(M^*)$ such that $f = f^* \circ \rho$. Moreover, since $T^{0,1}_{\mathbf{o}} M^* \simeq \mathfrak{s}/\hat{\mathfrak{g}}_+^*$, we also have $f^* \in \mathcal{O}(M^*)$. □

We call the $\mathbf{G}$-equivariant fibration (10.1), or the corresponding $\mathfrak{g}$-equivariant fibration of CR algebras, the strictly CR separable reduction of $M(\mathfrak{g}, \mathfrak{q})$, or of $(\mathfrak{g}, \mathfrak{q})$.

We can consider only simple parabolic CR manifolds. Indeed we have:

THEOREM 10.4. *Let* $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ *be the decomposition of the real semisimple Lie algebra* $\mathfrak{g}$ *into the direct product of its simple ideals, and let* $\mathfrak{g}_i = \hat{\mathfrak{g}}_i$*,* $\mathfrak{q}_i = \hat{\mathfrak{g}}_i \cap q$*. Then each* $\mathfrak{q}_i$ *is parabolic in* $\hat{\mathfrak{g}}_i$*,* $\mathfrak{q} = \sum_i \mathfrak{q}_i$ *and* $M(\mathfrak{g}, \mathfrak{q})$ *is weakly (weakly locally, strictly locally)* CR *separable if and only if all* $M(\mathfrak{g}_i, \mathfrak{q}_i)$*'s are weakly (weakly locally, strictly locally)* CR *separable.*

PROOF. The parabolic CR manifold $M(\mathfrak{g},\mathfrak{q})$ is isomorphic to the Cartesian product $\Pi_i M(\mathfrak{g}_i,\mathfrak{q}_i)$. ☐

10.1 Restriction to manifolds of finite type

The following Theorem shows that we can restrict our consideration to parabolic CR manifolds of finite type.

THEOREM 10.5. *Let $M = M(\mathfrak{g},\mathfrak{q})$ be a parabolic CR manifold, denote by M' the fiber and by $M'' = M(\mathfrak{g},\mathfrak{q}'')$ the base of a $\mathbf{G}$-equivariant CR fibration $f : M \longrightarrow M''$ onto a totally real parabolic CR manifold M''. Then M is weakly (weakly locally, strictly locally) CR separable if and only if M' is weakly (weakly locally, strictly locally) CR separable.*

To prove the theorem, we first need a lemma:

LEMMA 10.6. *Let $M = M(\mathfrak{g},\mathfrak{q})$ be a parabolic CR manifold, M' the fiber and $M'' = M(\mathfrak{g},\mathfrak{q}'')$ the base of a $\mathbf{G}$-equivariant CR fibration $f : M \longrightarrow M''$ onto a totally real parabolic CR manifold M''. Then every $x \in M''$ has an open neighborhood $U \subset M''$ such that $f^{-1}(U)$ is CR diffeomorphic by a real analytic map to $U \times M'$.*

In particular Theorem 10.5 and Lemma 10.6 apply when ϕ is the fundamental reduction [MN05, § 5] of M.

PROOF OF LEMMA 10.6. Let $\mathfrak{g}_+ = \mathfrak{g}\cap\mathfrak{q}$ be the isotropy subalgebra of M and $\mathfrak{g}''_+ = \mathfrak{g}\cap\mathfrak{q}''$ that of M''. Since M'' is totally real, $\mathfrak{g}''_+$ is a parabolic subalgebra of $\mathfrak{g}$, hence there exists a nilpotent subalgebra $\mathfrak{n}$ complementary to $\mathfrak{g}''_+$. Let $\mathbf{G}_+$, $\mathbf{G}''_+$, $\mathbf{N}$ be the analytic subgroups of $\mathbf{G}$ with Lie algebras $\mathfrak{g}_+$, $\mathfrak{g}''_+$, $\mathfrak{n}$, respectively and $\pi : \mathbf{G} \longrightarrow M'' = \mathbf{G}/\mathbf{G}''_+$ the projection onto the quotient. The restriction of π to $\mathbf{N}$ is a real analytic local diffeomorphism. Choose an open neighborhood W of the identity in $\mathbf{N}$ such that $\pi|_W$ is a diffeomorphism onto an open subset $\pi(W) = U$ of M''. Then the map:

$$\psi : U \times M' \ni (z, g\mathbf{G}_+) \longrightarrow \left((\pi|_W)^{-1}(z)l\right)\mathbf{G}_+ \in M$$

is a real analytic CR trivialization in a neighborhood of $e\mathbf{G}''_+$. The result follows because of the homogeneity of M''. ☐

PROOF OF THEOREM 10.5. Let $x \neq y$ be two distinct points of M. If $\phi(x) \neq \phi(y)$ then we can choose any function f on M'' such that $f(\phi(x)) \neq f(\phi(y))$, and $f\circ\phi$ is CR, and separates x and y. If $\phi(x) = \phi(y)$ then by Lemma 10.6 we can find a CR function f on M that separates x and y if and only if we can find such an f on $\phi^{-1}(\phi(x))$. Thus M is weakly (weakly locally) CR separable if and only if M' is weakly (weakly locally) CR separable.

Fix a point $x \in M$, let $M' = \phi^{-1}(\phi(x))$ and denote by $\iota : M' \longrightarrow M$ the inclusion map. Let $X \in \mathbb{C}T_xM$ be a complex tangent vector at x with $\mathrm{d}\hat{\phi}(X) \neq 0$ and f a real analytic function on M'' such that $\mathrm{d}\hat{\phi}(X)(f) \neq 0$. Then $f\circ\phi$ is a CR function on M and $X(f\circ\phi) \neq 0$. This shows that $S(M) = \iota^*(S(M'))$, hence M is strictly locally CR separable if and only if M' is strictly locally CR separable. ☐

10.2 Extension to Levi-flat orbits

The case of totally complex parabolic CR manifolds, was discussed by Wolf in [Wol69]. There he proved (see [FHW06, Thm. 4.4.3]) the following:

PROPOSITION 10.7. *Let $(\mathfrak{g}, \mathfrak{q})$ be a simple totally complex parabolic effective CR algebra and $M = M(\mathfrak{g}, \mathfrak{q})$ the corresponding totally complex parabolic CR manifold. Then M is weakly locally CR separable if and only if M is a bounded symmetric domain. In this case M is also weakly CR separable and strictly locally CR separable.* $\square$

We recall that a CR manifold M is *Levi-flat* if the analytic tangent distribution HM is integrable. For parabolic CR manifolds this is equivalent to the condition that the fibers of their fundamental reduction are totally complex. We have:

THEOREM 10.8. *Let $M = M(\mathfrak{g}, \mathfrak{q})$ be a $\mathbf{G}$-orbit in the complex flag manifold $Z = \mathbf{G}/\mathbf{Q}$. Then there exists a Levi-flat $\mathbf{G}$-orbit N in Z, with $M \subset \overline{N}$, such that every CR function f on M continuously extends to a function $\tilde{f}$, continuous on $M \cup N$ and CR on N with $\|\tilde{f}\|_N \leq \|f\|_M$.*

If M is of finite type, then N is totally complex, hence open in Z.

PROOF. If M is Levi-flat we take $N = M$. Otherwise, a theorem of Tumanov [Tum90] asserts that there exists a complex wedge W, with edge contained in M, such that every CR function f on M extends, continuously and uniquely, to a continuous function $\check{f}$ on $M \cup W$ that is holomorphic on W and that satisfies the estimate $\|\check{f}\|_W \leq \|f\|_M$. Here by a complex wedge W with edge in M we mean a connected open subset W of a complex submanifold V of positive dimension of Z such that $M \cap V$ is CR generic in V and contained in the closure $\overline{W}$.

Let $x \in W$ and define, for all $g \in \mathbf{G}$, a new function $\tilde{f}$ by setting:

$$\tilde{f}(g \cdot x) = (f \circ m_g)^{\vee}(x),$$

where $m_g : M \longrightarrow M$ denotes the action of g on M. The function $\tilde{f}$ is well defined and CR on the whole $\mathbf{G}$-orbit $M' = \mathbf{G} \cdot x$ through x. By choosing x close enough to M, we may arrange that $M \subset \overline{M'}$ and $\tilde{f}$ is continuous on $M \cup M'$.

By iterating this construction, we obtain a sequence of $\mathbf{G}$-orbits $M^{(i)}$ of nondecreasing dimension, each contained in the closure of the next. This sequence must necessarily stabilize to a term $M^{(k)} = N$, that satisfies the first assertion of the theorem.

If M is of finite type, then also N is of finite type and, being Levi-flat and CR generic, is open in Z. $\square$

As a corollary, we obtain:

COROLLARY 10.9. *If $M = M(\mathfrak{g}, \mathfrak{q})$ is a strictly locally CR separable parabolic CR manifold, embedded in the complex flag manifold $Z = \mathbf{G}/\mathbf{Q}$, then:*

(1) *there exists a strictly locally CR separable Levi-flat $\mathbf{G}$-orbit $N \subset Z$ with $M \subset \overline{N}$;*
(2) *M is (globally) weakly CR separable.*

If M is of finite type then N is a bounded symmetric domain.

PROOF. We may assume that M is the $\mathbf{G}$-orbit in Z through the point $\mathbf{o} = e\mathbf{Q}$.

Let N be the Levi-flat $\mathbf{G}$-orbit defined in Theorem 10.8. Let $\phi : Z \longrightarrow Z' = \mathbf{G}/\mathbf{Q}'$ be the $\hat{\mathbf{G}}$-equivariant fibration of complex flag manifolds that induces, by restriction to N, the strictly separable CR reduction $\phi|_N : N \longrightarrow N' \subset Z'$. Then every CR function f on M extends continuously to a function $\tilde{f}$, continuous on $M \cup N$ and CR on N, constant along the fibers of ϕ. By continuity also f is constant along the fibers of ϕ and furthermore $f = \phi \circ f'$ for some CR function f' on $M' = \phi(M)$. This shows that:

$$S_{\mathbf{o}}(M) \supset (\mathrm{d}\hat{\phi})^{-1} T^{0,1}_{\mathbf{o}} M'.$$

Since M is strictly CR separable, we obtain that:

$$T^{0,1}_{\mathbf{o}} M = (\mathrm{d}\hat{\phi})^{-1} T^{0,1}_{\mathbf{o}} M',$$

which in turn implies that $\mathfrak{q} = \mathfrak{q}'$. Thus $N = N'$, that is N is strictly locally CR separable.

If M is of finite type, by Proposition 10.7, N is a bounded symmetric domain and $M \subset \overline{N}$. This fact also implies that M is weakly CR separable, thus the Theorem is proved if M is of finite type.

If M is not of finite type, we apply two times Theorem 10.5 to the fiber M' of its fundamental reduction and obtain:

$$\begin{aligned} M \text{ is strictly locally } CR \text{ separable} \Longrightarrow \\ \Longrightarrow M' \text{ is strictly locally } CR \text{ separable} \Longrightarrow \\ \Longrightarrow M' \text{ is weakly } CR \text{ separable} \Longrightarrow \\ \Longrightarrow M \text{ is weakly } CR \text{ separable,} \end{aligned}$$

completing the proof. □

10.3 Examples

In this paragraph we discuss some examples. We will not need to utilize Tumanovs results, but more elementary extension theorems will suffice. In particular we recall the following statement. Let $S^3 = \{z \in \mathbb{C}^2 \mid |z| = 1\}$ be the three-dimensional sphere, endowed with the usual CR structure, $B^2 = \{z \in \mathbb{C}^2 \mid |z| < 1\}$ the two-dimensional complex ball, Σ a real two dimensional linear subspace of $\mathbb{C}^2$ (that may or may not be a complex line) and set: $\check{S}^3 = S^3 \setminus \Sigma$. Then every CR function on S^3 extends continuously to a function continuous on $\overline{B}^2 \setminus \Sigma$ and holomorphic on $\check{B}^2 = B^2 \setminus \Sigma$.

For totally complex parabolic CR manifolds the only obstruction to CR separability is the esistence of embedded compact complex submanifold. The general case is quite different. Indeed we exhibit two examples of parabolic CR manifold that are not weakly locally CR separable, but do not contain any compact complex submanifold.

EXAMPLE 10.1. Let $H(u, v) = u^* A v$ be the Hermitian form on $\mathbb{C}^3$ associated to the matrix $A = \mathrm{diag}(-1, 1, 1)$, and $\hat{\mathbf{G}} = \mathbf{SL}(3, \mathbb{C})$ the group of unimodular complex matrices. The subgroup $\mathbf{G}$ of matrices in $\hat{\mathbf{G}}$ that leave H invariant is a real form of G, isomorphic to $\mathbf{SU}(1, 2)$.

Let Z be the complex flag manifold:

$$Z = \{\ell^1 \subset \ell^2 \subset \mathbb{C}^3 \mid \dim \ell^i = i\}$$

and M the parabolic CR manifold:

$$M = \{(\ell^1, \ell^2) \in Z \mid \ell^1 \text{ is } H\text{-isotropic, } \ell^2 \text{ is } H\text{-hyperbolic}\}.$$

Consider the $\hat{\mathbf{G}}$-equivariant fibration

$$\phi : Z \ni (\ell^1, \ell^2) \longrightarrow \ell^1 \in W$$

onto the complex flag manifold:

$$W = \{\ell^1 \subset \mathbb{C}^3 \mid \dim \ell^1 = 1\}.$$

Then ϕ restricts to a $\mathbf{G}$-equivariant fibration $\phi : M \longrightarrow N$ onto the parabolic CR manifold:

$$N = \{\ell^1 \subset W \mid \ell^1 \text{ is } H\text{-isotropic}\}.$$

The fiber of $\phi|_M$ over a point $\ell^1 \in N$ is the set $\{\ell^2 \subset \mathbb{C}^3 \mid \ell^1 \subset \ell^2 \not\subset (\ell^1)^\perp\}$, which is biholomorphic to $\mathbb{C}$. The CR manifold N is the boundary of the open domain:

$$D = \{\ell^1 \in W \mid \ell^1 \text{ is } H\text{-negative}\}.$$

Fix an H-positive line $\ell^1_+ \subset \mathbb{C}^3$ and define:

$$M_{\ell^1_+} = \{(\ell^1, \ell^2) \in M \mid \ell^2 = \ell^1 + \ell^1_+\} \simeq \check{S}^3.$$

Any CR function f on $M_{\ell^1_+}$ extends continuously to a function $\tilde{f}$, continuous on $M_{\ell^1_+} \cup U_{\ell^1_+}$ and holomorphic on $U_{\ell^1_+}$, where:

$$U_{\ell^1_+} = \{(\ell^1, \ell^2) \in Z \mid \ell^1 \text{ is } H\text{-negative}, \ell^2 = \ell^1 + \ell^1_+\} \simeq \check{B}^2.$$

By letting ℓ^1_+ vary among all H-positive lines, we obtain that every CR function f on M extends continuously to a function $\tilde{f}$ continuous on $M \cup U$ and holomorphic on U, where:

$$U = \{(\ell^1, \ell^2) \in Z \mid \ell^1 \text{ is } H\text{-negative, } \ell^2 \text{ is } H\text{-hyperbolic}\}.$$

Let:

$$\begin{aligned} V &= \phi^{-1}(D) \setminus U \\ &= \{(\ell^1, \ell^2) \in Z \mid \ell^1 \text{ is } H\text{-negative, } H \text{ has signature } (0, -) \text{ on } \ell^2\}. \end{aligned}$$

Then V has real codimension two in $\phi^{-1}(D)$ and is not complex analytic. Hence, by a theorem of Hartogs ([Har09], see [Nar71, § 4, Thm. 3]), there is a point $x \in V$ with the property that $\tilde{f}$ holomorphically extends to a full neighborhood U_x of x in Z. It follows that $\tilde{f}$ is constant on $\phi^{-1} \circ \phi(y)$ for all $y \in U_x$, and by unique continuation $\tilde{f}$ is constant along the fibers of ϕ, hence M is not weakly locally CR separable.

EXAMPLE 10.2. Let $B(u,v) = u^t A v$ and $H(u,v) = u^* A v$ be the bilinear and Hermitian forms on $\mathbb{C}^5$ associated to the matrix $A = \operatorname{diag}(-1,-1,1,1,1)$, and $\hat{\mathbf{G}} = \mathbf{SO}(3,\mathbb{C})$ the group of unimodular complex matrices that preserve B. The connected component of the identity $\mathbf{G}$ in the subgroup of the real matrices in $\hat{\mathbf{G}}$ is isomorphic to $\mathbf{SO}^0(2,3)$, and the elements of $\mathbf{G}$ also preserve the Hermitian form H.

Let Z be the complex flag manifold:

$$Z = \{\ell^1 \subset \ell^2 \subset \mathbb{C}^5 \mid \dim \ell^i = i, \ell^i \text{ is } B\text{-isotropic}\}$$

and M the parabolic CR manifold:

$$M = \{(\ell^1,\ell^2) \in Z \mid \ell^1 \text{ is } H\text{-isotropic, } \ell^1 \neq \bar{\ell}^1,\ \ell^2 \text{ is } H\text{-hyperbolic}\}.$$

Consider the $\hat{\mathbf{G}}$-equivariant fibration

$$\phi : Z \ni (\ell^1,\ell^2) \longrightarrow \ell^1 \in W$$

onto the complex flag manifold:

$$W = \{\ell^1 \subset \mathbb{C}^3 \mid \dim \ell^1 = 1, \ell^1 \text{ is } B\text{-isotropic}\}.$$

Then ϕ restricts to a $\mathbf{G}$-equivariant fibration $\phi : M \longrightarrow N$ onto the parabolic CR manifold:

$$N = \{\ell^1 \subset W \mid \ell^1 \text{ is } H\text{-isotropic}, \ell^1 \neq \bar{\ell}^1\}.$$

The CR manifold N is an open stratum in the boundary of the open domain:

$$D = \{\ell^1 \in W \mid \ell^1 \text{ is } H\text{-negative}\}.$$

Fix a B-isotropic and H-positive line $\ell^1_+ \subset \mathbb{C}^5$ and define:

$$M_{\ell^1_+} = \{(\ell^1,\ell^2) \in M \mid \ell^2 = \ell^1 + \ell^1_+\} \simeq \check{S}^3.$$

Any CR function f on $M_{\ell^1_+}$ extends continuously to a function $\tilde{f}$, continuous on $M_{\ell^1_+} \cup U_{\ell^1_+}$ and holomorphic on $U_{\ell^1_+}$, where:

$$U_{\ell^1_+} = \{(\ell^1,\ell^2) \in Z \mid \ell^1 \text{ is } H\text{-negative}, \ell^2 = \ell^1 + \ell^1_+\} \simeq \check{B}^2.$$

By letting ℓ^1_+ vary among all B-isotropic and H-positive lines, we obtain that every CR function f on M extends continuously to a function $\tilde{f}$ continuous on $M \cup U$ and holomorphic on U, where:

$$U = \{(\ell^1,\ell^2) \in Z \mid \ell^1 \text{ is } H\text{-negative, } \ell^2 \text{ is } H\text{-hyperbolic}\}.$$

Let:

$$\begin{aligned} V &= \phi^{-1}(D) \setminus U \\ &= \{(\ell^1,\ell^2) \in Z \mid \ell^1 \text{ is } H\text{-negative, } \ell^2 \text{ is } H\text{-degenerate}\}. \end{aligned}$$

Then V has real codimension two in $\phi^{-1}(D)$ and is not complex analytic. By using the same argument of Example 10.1, it follows that any CR function on M is constant along the fibers of ϕ, hence M is not weakly locally CR separable.

The next example consists of a parabolic CR manifold that is weakly, but not strictly, locally CR separable.

EXAMPLE 10.3. Let $\hat{\mathbf{G}} = \mathbf{SL}(4, \mathbb{C})$ be the group of unimodular 4×4 complex matrices and $\mathbf{G} \simeq \mathbf{SU}(2,2)$ the subgroup of matrices preserving the Hermitian form associated to the matrix $A = \operatorname{diag}(-1,-1,1,1)$.

Let Z be the complex flag manifold:

$$Z = \{\ell^1 \subset \ell^2 \subset \mathbb{C}^4 \mid \dim \ell^i = i\}$$

and M the parabolic CR manifold:

$$M = \{(\ell^1, \ell^2) \in Z \mid \ell^1 \text{ is } H\text{-isotropic, } H|_{\ell^2} \text{ has signature } (0,-)\}.$$

Consider the $\hat{\mathbf{G}}$-equivariant fibration

$$\phi : Z \ni (\ell^1, \ell^2) \longrightarrow \ell^2 \in W$$

onto the complex flag manifold:

$$W = \{\ell^2 \subset \mathbb{C}^4 \mid \dim \ell^2 = 2\}.$$

Then ϕ restricts to a $\mathbf{G}$-equivariant fibration $\phi : M \longrightarrow N$, which is a CR map and a smooth diffeomorphism, but not a CR fibration, onto the parabolic CR manifold:

$$N = \{\ell^2 \subset W \mid H|_{\ell^1} \text{ has signature } (0,-)\}.$$

The CR manifold N is an open stratum in the boundary of the open domain:

$$D = \{\ell^2 \in W \mid \ell^2 \text{ is } H\text{-negative}\}.$$

Fix an H-negative line $\ell^1_- \subset \mathbb{C}^4$ and define:

$$M_{\ell^1_-} = \{(\ell^1, \ell^2) \in M \mid \ell^1 \text{ is } H\text{-isotropic}, \ell^1 \perp_H \ell^1_-, \ell^2 = \ell^1 + \ell^1_-\} \simeq S^3.$$

Any CR function f on $M_{\ell^1_-}$ extends continuously to a function $\tilde{f}$, continuous on $M_{\ell^1_-} \cup U_{\ell^1_-}$ and holomorphic on $U_{\ell^1_-}$, where:

$$U_{\ell^1_+} = \{(\ell^1, \ell^2) \in Z \mid \ell^1 \text{ is } H\text{-negative}, \ell^1 \perp_H \ell^1_-, \ell^2 = \ell^1 + \ell^1_-\} \simeq B^2.$$

By letting ℓ^1_- vary among all H-negative lines, we obtain that every CR function f on M extends continuously to a function $\tilde{f}$ continuous on $M \cup U$ and holomorphic on U, where:

$$U = \{(\ell^1, \ell^2) \in Z \mid \ell^2 \text{ is } H\text{-negative}\}.$$

Since each fiber of the restriction of ϕ to U is biholomorphic to $\mathbb{CP}^1$, every CR function f on M can be extended to a CR function $\tilde{f}$ on $\phi^{-1}(N)$, which is constant along the fibers of ϕ. This shows that f is also CR on N, hence M is not strictly locally CR separable. On the other hand N is strictly locally CR separable, hence by Lemma 10.6 M is weakly locally CR separable.

10.4 Global CR functions on compact parabolic CR manifolds

Now we apply the results obtained in previous sections to compact parabolic CR manifolds of finite type. We have:

THEOREM 10.10. *Let $(\mathfrak{g}, \mathfrak{q})$ be a simple effective compact parabolic CR algebra, and $M = M(\mathfrak{g}, \mathfrak{q})$ the associated compact parabolic CR manifold. Then M is globally weakly CR separable if and only if M is the Bergman-Shilov boundary of an irreducible bounded symmetric domain not of tube type.*

The compact parabolic CR manifolds that are Bergman-Shilov boundaries of irreducible bounded symmetric domains not of tube type are described in Examples 10.4, 10.5, and 10.6.

PROOF. From Theorem 10.8 and Corollary 10.9, we obtain that M is contained in the boundary of a bounded symmetric domain N and every CR function f on M extends to a function $\tilde{f}$ continuous on $M \cup N$, holomorphic on N, that satisfies the estimate $\|\tilde{f}\|_N \leq \|f\|_M$. This shows that the Bergman-Shilov boundary of N is contained in M. Since $\mathbf{G}$ acts on N by biholomorphisms and is transitive on M, then M coincides with the Shilov boundary of N. Finally, N is not of tube type because M is of finite type, while the Bergman-Shilov boundary of a bounded symmetric domain of tube type is totally real. $\square$

EXAMPLE 10.4. Fix positive integers $p < q$ and let $n = p + q$. We identify the simple real Lie algebra $\mathfrak{g} \simeq \mathfrak{su}(p, q)$ with the set of $(n \times n)$ complex matrices Z with zero trace that satisfy:

$$Z^* K \,+\, K\, Z \,=\, 0 \quad \text{where} \quad K = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}.$$

Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and let $\mathfrak{q}_{\alpha_\mathfrak{p}} \subset \hat{\mathfrak{g}} \simeq \mathfrak{sl}(n, \mathbb{C})$ be the set of $(n \times n)$ matrices in $\mathfrak{sl}(n, \mathbb{C})$ such that:

$$Z(\langle e_1 + e_{p+1}\,, \ldots\,, e_p + e_{2p}\rangle) \subset \langle e_1 + e_{p+1}\,, \ldots\,, e_p + e_{2p}\rangle\,.$$

Then $(\mathfrak{g}, \mathfrak{q}_{\alpha_\mathfrak{p}})$ is parabolic minimal.

The corresponding CR manifold $M = M(\mathfrak{g}, \mathfrak{q}_{\alpha_\mathfrak{p}})$ is the Grassmannian of p-planes ℓ_p in $\mathbb{C}^n$ which are totally isotropic for K (i.e. $v^* K v = 0$ for all $v \in \ell_p$). We have:

$$M \simeq \left\{\ell_p = \{(v, u(v)) \in \mathbb{C}^n \,|\, v \in \mathbb{C}^p\} \,\middle|\, u \in \mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)\right\} \simeq \mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$$

where $\mathbf{U}(\mathbb{C}^p, \mathbb{C}^q) = \{u \in \mathcal{M}_{q\times p}(\mathbb{C}) \,|\, u^* u = I_p\}$ is the set of unitary $q \times p$ matrices.

Give $\mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$ the CR structure induced by the embedding in $\mathcal{M}_{q\times p}(\mathbb{C})$. The compact subgroup $\mathbf{K}^{(1)} \simeq \mathbf{SU}(p) \times \mathbf{SU}(q)$ of matrices of $\mathbf{SU}(p, q)$ of the form $\begin{pmatrix} A_p & 0 \\ 0 & B_q \end{pmatrix}$ acts transitively by CR automorphisms on $\mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$, the action being given by: $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot u = B u A^{-1}$.

The associated CR algebra is $(\mathfrak{k}^{(1)}, \mathfrak{q}')$ where $\mathfrak{k}^{(1)} \simeq \mathfrak{su}(p) \oplus \mathfrak{su}(q)$ and $\mathfrak{q}'$ is the set of matrices in $\mathfrak{sl}(p) \oplus \mathfrak{sl}(q)$ of the form $\begin{pmatrix} A_p & 0 & 0 \\ 0 & A_p & D \\ 0 & 0 & C_{q-p} \end{pmatrix}$.

The group $\mathbf{K}^{(1)}$ acts transitively on M, and the associated CR algebra is $(\mathfrak{k}^{(1)}, \hat{\mathfrak{k}}^{(1)} \cap \mathfrak{q}) = (\mathfrak{k}^{(1)}, \mathfrak{q}')$. Thus the diffeomorphism $M \simeq \mathbf{U}(\mathbb{C}^p, \mathbb{C}^q)$ is in fact a CR isomorphism.

EXAMPLE 10.5. Fix a positive integer p and let $n = 2p+1$. We identify the simple real Lie algebra $\mathfrak{g} = \mathfrak{so}^*(2n)$ with the set of $(2n \times 2n)$ complex matrices Z with zero trace that satisfy:

$$\begin{cases} ZJ = J\bar{Z}\,, \\ {}^tZK + KZ = 0\,, \end{cases}$$

where:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let $\mathfrak{q}_{\alpha_1}$ be the parabolic subalgebra of matrices in $\mathfrak{g}$ that stabilize the subspace

$$V_n = \langle e_1 + e_{n+2p}, \dots, e_p + e_{n+p+1}, e_{p+1} - e_{n+p}, \dots, e_{2p} - e_{n+1}, e_{2p+1} \rangle.$$

Then $(\mathfrak{g}, \mathfrak{q}_{\alpha_1})$ is parabolic minimal.

The maximal compact subgroup $\mathbf{K} \simeq \mathbf{U}(n)$ of $\mathbf{G}$ of matrices of the form $\begin{pmatrix} A_n & 0 \\ 0 & {}^tA_n{}^{-1} \end{pmatrix}$, $A_n \in \mathbf{U}(n)$, acts transitively by CR isomorphisms on $M(\mathfrak{g}, \mathfrak{q}_{\alpha_1})$. The associated CR algebra is $(\mathfrak{k}, \mathfrak{q}')$ where $\mathfrak{k} \simeq \mathfrak{u}(n)$ and $\mathfrak{q}' = \hat{\mathfrak{k}} \cap \mathfrak{q}_{\alpha_1}$. This is the subalgebra of matrices in $\mathfrak{so}(2n, \mathbb{C})$ of the form $\begin{pmatrix} A_n & 0 \\ 0 & -{}^tA_n \end{pmatrix}$ where $A_n \in \mathfrak{gl}(n, \mathbb{C})$ is of the form $\begin{pmatrix} B_p & C_p & v_p \\ D_p & -{}^tB_p & w_p \\ 0 & 0 & is \end{pmatrix}$ with $B_p = {}^tB_p$, $D_p = {}^tD_p$.

We let $\mathbf{K}$ act on $\mathfrak{so}(n, \mathbb{C})$ by: $k \cdot X = A_n\, X\, {}^tA_n$ if $k = \begin{pmatrix} A_n & 0 \\ 0 & {}^tA_n{}^{-1} \end{pmatrix}$. Let N be the $\mathbf{K}$-orbit of $o = \begin{pmatrix} 0 & -I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The associated CR algebra is $(\mathfrak{k}, \mathfrak{q}')$ and the isotropy is connected and contains a generator of $\pi_1\big(\mathbf{U}(n)\big)$. Thus M is CR isomorphic to N.

EXAMPLE 10.6. Let D be the exceptional bounded symmetric domain of type V. Its Shilov boundary S is a real flag manifold (see [FKK+00, Part III,Ch.IV§2.8]) for the group E III and is compact, hence it is a minimal orbit $M(\mathfrak{g}, \mathfrak{q})$ where $\mathfrak{g}$ is of type E III. Furthermore it has CR dimension 8 and CR codimension 8 (see [KZ00, p. 180]), hence $\mathfrak{q} = \mathfrak{q}_{\alpha_1}$ or $\mathfrak{q} = \mathfrak{q}_{\alpha_6}$. Thus $M(\mathfrak{g}, \mathfrak{q}_{\alpha_1}) \simeq S$ is an embedded CR submanifold of $\mathbb{C}^{16}$.

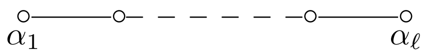
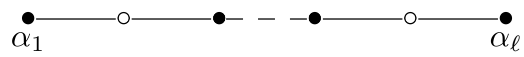
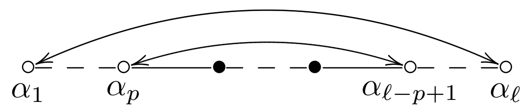
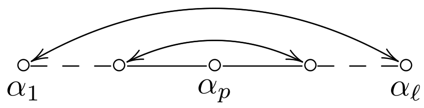
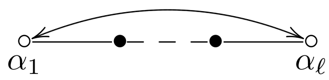
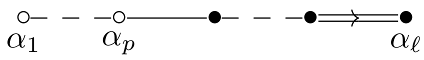
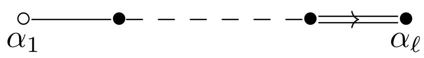
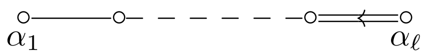
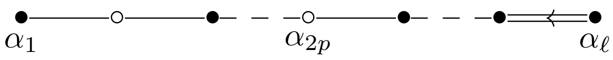
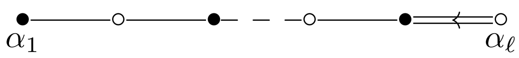
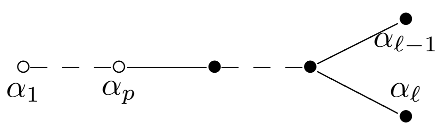
In terms of cross-marked Satake diagrams, we obtain (see the Appendix for the notation):

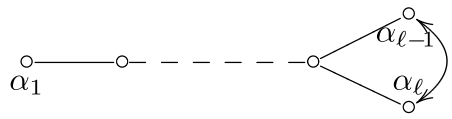
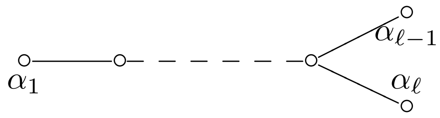
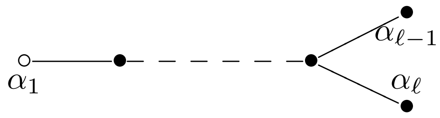
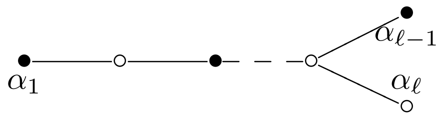
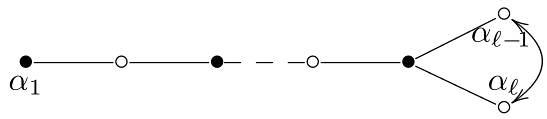
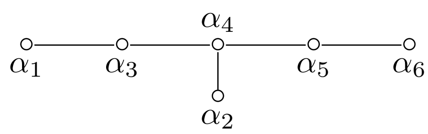
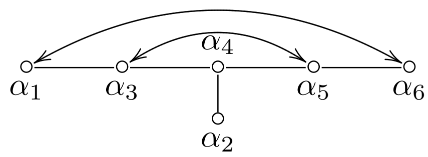
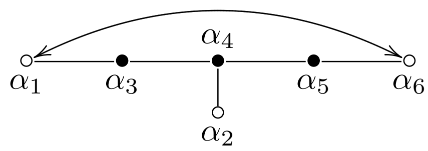
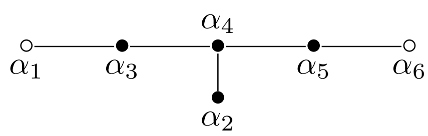
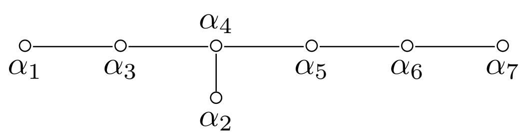
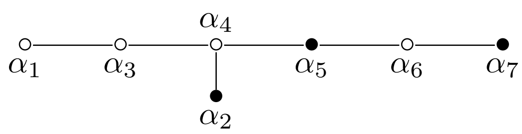
COROLLARY 10.11. *Let $(\mathfrak{g}, \mathfrak{q}_\Phi)$ be a simple effective fundamental compact parabolic CR algebra and $M = M(\mathfrak{g}, \mathfrak{q}_{\Phi_C})$ the associated compact parabolic CR manifold. Then there exists $\Psi_C \subset \Phi_C$ and a $\mathbf{G}$-equivariant fibration $\rho : M \longrightarrow M' = M(\mathfrak{g}, \mathfrak{q}_{\Psi_C})$ such that M' is globally weakly CR separable and $\mathcal{O}(M) = \rho^*\big(O(M')\big)$.*

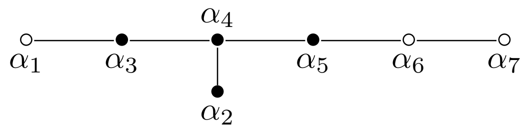
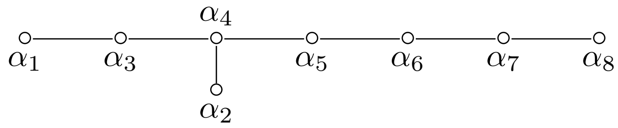
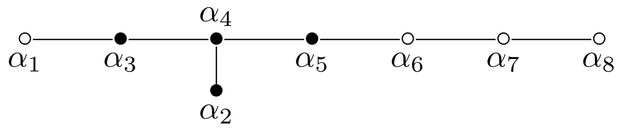
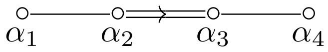
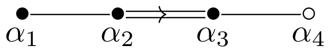
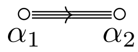
Furthermore $\Psi_C = \Phi_C \cap \Sigma_C$, where Σ_C is defined according to the type of $\mathfrak{g}$:

Type A IIIa : $\Sigma_C = \{\alpha_p, \alpha_q\}$;
Type D IIIb : $\Sigma_C = \{\alpha_{\ell-1}, \alpha_\ell\}$;
Type E III : $\Sigma_C = \{\alpha_1, \alpha_5\}$;
All other types : $\Sigma_C = \emptyset$. □

Appendix. Table of Satake Diagrams

Name	$\mathfrak{g}$	Satake diagram
A I	$\mathfrak{sl}(\ell+1,\mathbb{R})$	
A II	$\mathfrak{sl}(p,\mathbb{H})$ $2p+1=\ell$	
A IIIa	$\mathfrak{su}(p,\ell+1-p)$ $2\leq p\leq \ell/2$	
A IIIb	$\mathfrak{su}(p,p)$ $1\leq p=(\ell+1)/2$	
A IV	$\mathfrak{su}(1,\ell)$	
B I	$\mathfrak{so}(p,2\ell+1-p)$ $2\leq p\leq \ell$	
B II	$\mathfrak{so}(1,2\ell)$	
C I	$\mathfrak{sp}(2\ell,\mathbb{R})$	
C IIa	$\mathfrak{sp}(p,\ell-p)$ $2p<\ell$	
C IIb	$\mathfrak{sp}(p,p)$ $2p=\ell$	
D Ia	$\mathfrak{so}(p,2\ell-p)$ $2\leq p\leq \ell-2$	

Name	$\mathfrak{g}$	Satake diagram
D Ib	$\mathfrak{so}(\ell-1,\ell+1)$	
D Ic	$\mathfrak{so}(\ell,\ell)$	
D II	$\mathfrak{so}(1,2\ell-1)$	
D IIIa	$\mathfrak{so}*(2\ell)$ $\ell=2p$	
D IIIb	$\mathfrak{so}*(2\ell)$ $\ell=2p+1$	
E I		
E II		
E III		
E IV		
E V		
E VI		

Name	$\mathfrak{g}$	Satake diagram
E VII		
E VIII		
E IX		
F I		
F II		
G		

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