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Extra Dimensions and Dark Matter

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Preface

So far the Standard Model of particle physics has given a remarkably successful description of known phenomena, proving to be one of the great successes in physics. The experiments carried out until the GeV range show no deviations from the predictions of the Standard Model. However there are hints for new physics.

First of all recent cosmological measurements severely constraint the amount of baryon, matter and dark energy in the universe. In units of the critical density, these energy densities are

$$\begin{aligned} \Omega_B &= 0.044 \pm 0.004 \\ \Omega_{\text{matter}} &= 0.27 \pm 0.04 \\ \Omega_\Lambda &= 0.73 \pm 0.04 \end{aligned}$$

The non-baryonic dark matter component (at a confidence level of 95%) is given by $0.094 < \Omega_{DM} h^2 < 0.129$. Here $h \simeq 0.71$ is the normalized Hubble expansion rate. These measurements clearly show that the known particles make up only for a small fraction of the total energy density of the universe. Also the microscopic properties of dark matter and dark energy are unconstrained by cosmological and astrophysical observations. Therefore particle physics must suggest candidates for dark matter and dark energy in order to identify experiments and observations that may confirm or exclude such speculations.

Besides cosmological observations, we know that a new framework must appear at the Planck scale ($\sim 10^{19}$ GeV) where gravitational effects become important. Since the ratio between this scale and the electroweak scale (~ 100 GeV) is huge, all quantum corrections will turn out to be many orders of magnitude greater than the tree values. This is the so called hierarchy problem, related to fine-tuning and naturalness. Quantum corrections are power-law divergent (usually quadratically) which means that the shortest-distance physics is most important. More technically, the question is why the Higgs boson is so much lighter than the Planck mass. Indeed one would expect that the large (quadratically divergent) quantum contributions to the square of the Higgs boson mass would inevitably make the mass huge, unless there is an incredible fine-tuning cancellation between the quadratic radiative corrections and the bare mass.

During the last years many appealing ideas were used to overcome this problem. The most popular theory but not the only one proposed to solve the hierarchy problem is supersymmetry, even if this was not the historical motivation for developing supersymmetry. In 1981 it was proposed the Minimal Supersymmetric Standard Model (MSSM), i.e. the minimal extension of the Standard Model. Supersymmetry pairs bosons with fermions, therefore every Standard Model particle has a partner (yet to be discovered). If these supersymmetric partners exist, it is likely that they will be observed at the Large Hadron Collider. In the MSSM, the Higgs has a fermionic superpartner, called the Higgsino (with the same mass if supersymmetry was an exact symmetry). Because fermion masses are radiatively stable, the Higgs mass inherits this stability. Supersymmetry removes the power-law divergences of the radiative corrections to the Higgs mass. However, there is no understanding of why the Higgs mass is so small from the beginning, which is known as the mu problem.

The only unambiguous way to claim discovery of supersymmetry is to produce superparticles in accelerators since we expected them to be 100 to 1000 times heavier than the proton. There are various particles which are superpartners of the Standard Model ones: squarks, gluinos, charginos, neutralinos, and sleptons. These superparticles have their interactions and subsequent decays described by the MSSM. The MSSM imposes R-parity to explain the stability of the proton. It adds supersymmetry breaking by introducing explicit soft supersymmetry breaking, at least at weak scale where we know for sure that supersymmetry is broken. Unfortunately there are 120 new parameters in the MSSM. Most of these parameters lead to unacceptable phenomenology such as large flavor changing neutral currents or large electric dipole moments for the neutron and electron. To avoid these problems, the MSSM requires all of the soft susy breaking terms to be diagonal in flavor space and all of the new CP violating phases to vanish.

But the most intriguing feature of the MSSM is that it naturally provides dark matter candidates with approximately the right relic density. In fact if R-parity is preserved, then the lightest supersymmetric particle (LSP) of the MSSM is stable and is a massive particle with weak interactions (WIMP). This makes the LSP a good cold dark matter (CDM) particle. This fact provides a strong and completely independent motivation for supersymmetric theories. In the following we will focus on dark matter, where the connections between supersymmetry and cosmology are concrete and rich.

On theoretical grounds, we can go also beyond the MSSM considering the most investigated proposal of grand unified theory (GUT), which merges the three fundamental gauge symmetries. Grand unification is based on the idea that at extremely high energies, all symmetries have the same gauge coupling strength. GUTs predict that at energies above 10^{14} GeV, the electromagnetic, weak nuclear, and strong nuclear forces are fused into a single unified field. A gauge theory where the gauge group is a simple group only has one gauge coupling constant, and since the fermions are now grouped together in larger representations, there are fewer Yukawa coupling coefficients as well.

In addition, the chiral fermion fields of the Standard Model unify into three generations of two irreducible representations ($10 \oplus \bar{5}$) in $SU(5)$, and three generations of an irreducible representation (16) in $SO(10)$. This is very significant since a generic combination of chiral fermions which are free of gauge anomalies will not be unified in a representation of some larger Lie group without adding additional matter fields.

GUT theory predicts also relations among the fermion masses but most of them don't hold, neither approximately. On the contrary, if we look at the renormalization group running of the three-gauge couplings we find that they meet at the same point if the hypercharge is normalized so that it is consistent with $SU(5)$ or $SO(10)$ GUTs, which are precisely the GUT groups which lead to a simple fermion unification. However, if the MSSM is used instead of the Standard Model, the match becomes much more accurate. The unification scale turns out to be $M_{GUT} \approx 2 \times 10^{16}$ GeV. This sets a natural energy scale for grand unification. Although this energy scale is lower than the Planck scale, it is yet too high to probe physics at low energies. Even indirect evidences, such as proton decay, are limited by this huge scale.

In [1] it was shown how to lower the unification scale. Indeed, it would be strange to consider a unification of the MSSM gauge groups at lower scale at which their couplings have not yet unified. There must exist a mechanism to achieve gauge coupling unification at scales $M \ll M_{GUT}$. If we add extra matter states to the MSSM we get worse. Indeed such extra matter tends to raise the unification scale, also driving the theory towards strong coupling. Let us then investigate why unification occurs at such a high energy scale.In the MSSM gauge couplings run logarithmically with energy scale. Since the gauge couplings have different values at weak scale they can unify (if it is the case) only after many orders of magnitude in energy. Then we need a way to change the running of the gauge couplings so that they run more quickly (say exponentially rather than linearly) to achieve a faster unification.

Outstandingly, there does exist a physical effect that causes such an exponential running: the appearance of extra spacetime dimensions. Already in the 20's Kaluza and Klein looked for a model to unify gravitation and electromagnetism. A five dimensional spacetime gives rise to Einstein equations and Maxwell equations in four dimensions. They proposed that the fourth spatial dimension is curled up in a circle of very small radius. The size of the dimension is given by the distance a particle can travel before reaching its initial position. Since this extra dimension is compact we refer to this mechanism as compactification. Extra spacetime dimensions are naturally predicted in string theory, so we expect that such a scenario will take part within a more fundamental theory. We will see that this scenario can also be discussed in terms of field theory only. The drawback of this picture is that in field theory, extra spacetime dimensions lead to a loss of renormalizability. All physical quantities such as gauge couplings do not run in the usual sense. We must rephrase the 'exponential running' as an exponential dependence on the cutoff pertaining a more fundamental theory (perhaps string theory). Nevertheless we will see that there exists a renormalizable theory essentially equivalent to the non-renormalizable one.

Given these motivations, in this thesis we shall analyze the consequences of extra spacetime dimensions on the MSSM. We shall begin by reviewing the MSSM in four dimensions. In chapter 1 we specify the superpotential and all the possible soft terms to break supersymmetry, showing their renormalization group equations. Chapter 2 contains a pedagogical introduction to the method of the effective potential. We apply this method to the MSSM to obtain the renormalization group equations of the previous chapter. We also do an analysis in terms of Feynman diagrams, which will be useful in the following. In chapter 3 we present the string-inspired model by Dienes and collaborators [1]. We treat different scenarios showing how extra dimensions give rise to 'power-law' running of the parameters. In chapter 4 we analyze the possible implications of extra dimensions related to the dark matter problem. The result of the computation is that the neutralino is still the lightest supersymmetric particle and it is higgsino like in most of the parameter space.

Acknowledgements

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Contents

Chapter 1

Review of the MSSM

Supersymmetry is an extension of the known spacetime symmetries. It emerges naturally in string theory and, in a sense, is the maximal possible extension of Poincare symmetry. The basic prediction of supersymmetry is, then, that for every known particle there is another particle, its superpartner, with spin differing by 1/2. One may show that no particle of the standard model is the superpartner of another. Supersymmetry therefore predicts a plethora of superpartners, none of which has been discovered. Mass degenerate superpartners cannot exist and so supersymmetry cannot be an exact symmetry. The only viable supersymmetric theories are therefore those with non-degenerate superpartners. This may be achieved by introducing supersymmetry-breaking contributions to superpartner masses to lift them beyond current search limits. At first sight, this would appear to be a loss in the appeal of supersymmetry. It turns out, however, that the main virtues of supersymmetry are preserved even if such mass terms are introduced. In addition, the presence of an electrically neutral LSP acting as supersymmetric dark matter emerges naturally in theories with broken supersymmetry. The treatment of the MSSM is entirely based on the work of Martin [2].

1.1 MSSM: the superpotential

As in any renormalizable supersymmetric field theory, the interactions and masses of all particles are determined just by their gauge transformation properties and by the superpotential W. W is an analytic function of the chiral complex fields ϕ_i. We recall here that a superfield is a single object which contains as components all of the bosonic, fermionic, and auxiliary fields within the corresponding supermultiplet. The gauge quantum numbers and mass dimension of a chiral superfield are given by that of its scalar component. In the superfield notation we can write the superpotential as

$$W = \frac{1}{2} M^{ij} \Phi_i \Phi_j + \frac{1}{6} y^{ijk} \Phi_i \Phi_j \Phi_k \tag{1.1}$$

The equation (1.1) shows that W determines not only the scalar interactions in the theory, but also the fermion masses and Yukawa couplings. At first sight, superfield methods might seem redundant; however they have the great advantage of making invariance under supersymmetry transformations manifest. The specification of the superpotential encodes the interactions present in the lagrangian,

The form of the superpotential is restricted by gauge invariance. As a result, only a subset of the couplings M^{ij} and y^{ijk} will be allowed to be non-zero. For example, mass terms M^{ij} can only be non-zero for i and j such that the supermultiplets Φ_i and Φ_j transform under the gauge group in representations which are conjugates of each other. Indeed, we will see that in the MSSM there is only one such term. Likewise, the Yukawa couplings y^{ijk} can only be non-zero when Φ_i, Φ_j, and Φ_k transform in representations which can combine to form a gauge singlet.

The superpotential for the MSSM is given by

$$W_{\mathrm{MSSM}} = \overline{u} \mathbf{y_u} Q H_u - \overline{d} \mathbf{y_d} Q H_d - \overline{e} \mathbf{y_e} L H_d + \mu H_u H_d \tag{1.2}$$

The objects H_u, H_d, Q, L, $\overline{u}$, $\overline{d}$, $\overline{e}$ appearing in equation (1.2) are chiral superfields corresponding to the chiral supermultiplets [1]. The dimensionless Yukawa coupling parameters $\mathbf{y_u}, \mathbf{y_d}, \mathbf{y_e}$ are 3×3 matrices in family space. The "μ term", can be written out explicitly as

$$\mu (H_u)_\alpha (H_d)_\beta \epsilon^{\alpha\beta} \tag{1.3}$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric symbol which makes the expression gauge-invariant. Likewise, the term $\overline{u}\mathbf{y_u} Q H_u$ can be written out as

$$\overline{u}^i_a \, (\mathbf{y_u})_i{}^j \, Q^a_{j\alpha} \, (H_u)_\beta \epsilon^{\alpha\beta} \tag{1.4}$$

[1] For simplicity, we have suppressed all of the gauge and family indices.

where $i = 1, 2, 3$ is a family index, and $a = 1, 2, 3$ is a color index which is raised (lowered) in the $\mathbf{3}$ ($\overline{\mathbf{3}}$) representation of $SU(3)_C$ group.

The μ term in equation (1.2) is the supersymmetric version of the Higgs boson mass in the Standard Model. It is unique, because other terms are not analytic and thus cannot be included in the superpotential. From equation (1.2) we can see that both H_u and H_d are needed to give Yukawa couplings, and thus masses, to all of the quarks and leptons. So we need both H_u and H_d, even without invoking anomaly cancellation [2].

The Yukawa matrices determine the masses and CKM mixing angles, after the occurrence of electroweak symmetry breaking and the neutral scalar components of H_u and H_d get VEVs. Since the top quark, bottom quark and tau lepton are the heaviest fermions in the Standard Model, it is often used the approximation in which only the $(3, 3)$ family components of each of $\mathbf{y_u}$, $\mathbf{y_d}$ and $\mathbf{y_e}$ are important:

$$\mathbf{y_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix} \qquad \mathbf{y_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b \end{pmatrix} \qquad \mathbf{y_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau \end{pmatrix} \tag{1.5}$$

It is useful to write the superpotential in terms of the $SU(2)_L$ components

$$\begin{aligned} W_{\mathrm{MSSM}} \approx \; & y_t(\bar{t}tH_u^0 - \bar{t}bH_u^+) - y_b(\bar{b}tH_d^- - \bar{b}bH_d^0) - y_\tau(\bar{\tau}\nu_\tau H_d^- - \bar{\tau}\tau H_d^0) \\ & + \mu(H_u^+ H_d^- - H_u^0 H_d^0). \end{aligned} \tag{1.6}$$

Here we used the notation: $Q_3 = (t\, b)$; $L_3 = (\nu_\tau\, \tau)$; $H_u = (H_u^+\, H_u^0)$; $H_d = (H_d^0\, H_d^-)$; $\overline{u}_3 = \bar{t}$; $\overline{d}_3 = \bar{b}$; $\overline{e}_3 = \overline{\tau}$. The minus signs inside the parentheses appear because of the antisymmetry of the $\epsilon^{\alpha\beta}$ symbol. These minus signs were chosen so that the terms proportional to y_t, y_b and y_τ have positive signs when they will become the top, bottom and tau masses.

The Yukawa interactions y^{ijk} must be completely symmetric under interchange of i, j, k; this implies that besides the Higgs-quark-quark and Higgs-lepton-lepton couplings of the Standard Model, there are also squark-Higgsino-quark and slepton-Higgsino-lepton interactions.

To see this more explicitly, one can look at Figs. 1.1 (a), (b), (c) which show some of the interactions which involve the top-quark Yukawa coupling y_t. Figure 1.1(a) is the Standard Model-like coupling of the top quark to the neutral complex scalar Higgs boson, which follows from the first term in equation (1.6). The symbols t_L and $t_R^\dagger$ stand for their synonyms t and $\bar{t}$.

[2] Cancellation of gauge anomalies requires $Tr[Y^3] = 0$ and $Tr[T_3^2 Y] = 0$, where T_3 and Y are the third component of weak isospin and the weak hypercharge, respectively. These can be satisfied if there are two Higgs supermultiplets, one with each of $Y = \pm 1/2$.

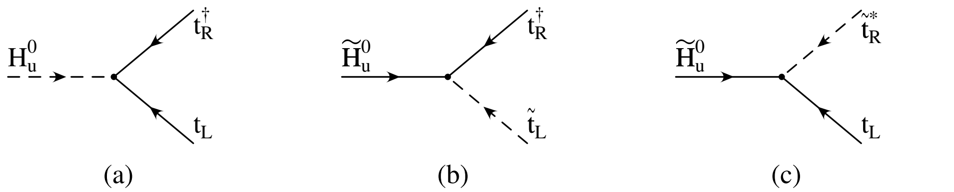


Figure 1.1: Interactions proportional to y_t: the top-quark Yukawa coupling (a) and its supersymmetrizations (b), (c).

In Fig. 1.1(b), we show the coupling of the left-handed top squark $\widetilde{t}_L$ to the neutral higgsino field $\widetilde{H}_u^0$ and right-handed top quark, while in Fig. 1.1(c) the right-handed top-squark field couples to $\widetilde{H}_u^0$ and t_L. For each of the three interactions, there is another with $H_u^0 \to H_u^+$ and $t_L \to -b_L$, corresponding to the second part of the first term in equation (1.6). All of these interactions are required by supersymmetry to have the same strength y_t.

There are also scalar quartic interactions with strength proportional to y_t^2 as can be seen from Fig. 1.2(b).

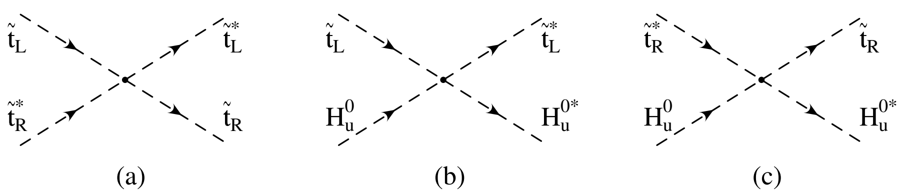


Figure 1.2: Some of the (scalar)^4 interactions with strength proportional to y_t^2.

They are twelve in total; three of them are shown in Fig. 1.2 while the other nine can be obtained by replacing $\widetilde{t}_L \to \widetilde{b}_L$ and/or $H_u^0 \to H_u^+$ in each vertex. This is a peculiar characteristic of supersymmetry; many interactions are determined by a single parameter.

It turns out that the dimensionless interactions play a minor role for phenomenology. This is because the Yukawa couplings are very small, except for those of the third family (top, bottom and tau). Decay and production processes for superpartners in the MSSM are instead dominated by the supersymmetric interactions which depend on gauge-coupling. The couplings of the Standard Model gauge bosons (photon, $W^\pm$, Z^0 and gluons) to the MSSM particles are determined as usual by the kinetic terms and by their gauge invariance. The gauginos then also couple to (squark, quark), (slepton, lepton) and (Higgs, higgsino) pairs. The Feynman diagram for the interaction squark-quark-gluino is shown in Fig. 1.3(a). In Figs. 1.3(b), (c)

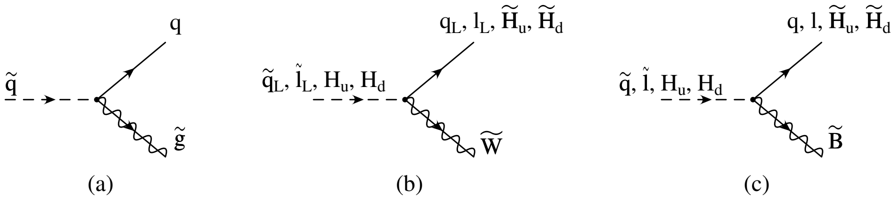


Figure 1.3: Couplings of the gluino, wino, and bino to MSSM (scalar, fermion) pairs.

we show the couplings of (squark, quark), (lepton, slepton) and (Higgs, higgsino) pairs to the winos and bino; their interactions are proportional to the electroweak gauge couplings g and g' respectively.

The winos only couple to the left-handed squarks and sleptons, while the (lepton, slepton) and (Higgs, higgsino) pairs do not couple to the gluino since they have no color number. The bino couplings for each (scalar, fermion) pair are proportional to the weak hypercharges Y. The interactions shown in Fig. 1.3 are responsible for decays $\widetilde{q} \to q\widetilde{g}$ and $\widetilde{q} \to \widetilde{W}q'$ and $\widetilde{q} \to \widetilde{B}q$. The complication arises because $\widetilde{W}$ and $\widetilde{B}$ states are not mass eigenstates: this is due to electroweak symmetry breaking.

There are also various scalar quartic interactions in the MSSM which are uniquely determined by gauge invariance and supersymmetry, Among them are (Higgs)^4 terms proportional to g^2 and g'^2 in the scalar potential. These can be thought as the direct generalization of the quartic term in the Standard Model Higgs potential, to the case of the MSSM.

The only dimensionful parameter appearing in the supersymmetric part of the MSSM lagrangian is the μ term. We find that μ gives rise to higgsino mass terms

$$\mathcal{L} \supset -\mu(\widetilde{H}_u^+\widetilde{H}_d^- - \widetilde{H}_u^0\widetilde{H}_d^0) + \text{c.c.} \tag{1.7}$$

as well as Higgs (mass)^2 terms in the scalar potential

$$-\mathcal{L} \supset V \supset |\mu|^2\big(|H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2\big) \tag{1.8}$$

It is clear that equation (1.8) is positive-definite. Then, to understand electroweak symmetry breaking we must include supersymmetry-breaking soft terms [3] of dimension two for the Higgs scalars, letting them to be neg-

[3]Soft terms break supersymmetry without introducing quadratic divergences. We will discuss them in detail in the next.

ative. However, this leads to a puzzle. Since μ is related to the Higgs VEV of order 174 GeV, we expect that μ should be roughly of order 10^2 or 10^3 GeV. But why should μ be roughly of the same order as m_{soft}? Why is so small compared to M_{Pl}? In this way, the scalar potential of the MSSM seems to depend on two distinct dimensionful parameters, namely the supersymmetry-respecting mass μ and the supersymmetry-breaking soft mass terms. The observed value for the electroweak breaking scale suggests that both these mass scales should be within an order of magnitude or so of 100 GeV. This puzzle is called "the μ problem". A way to solve this problem [4] is to postulate that μ is absent at tree-level, and must be replaced by the VEV(s) of some new field(s). This VEV is determined by minimizing a potential which depends on soft supersymmetry-breaking terms. Thus, the value of the effective parameter μ is strictly related to supersymmetry breaking. From the point of view of the MSSM, however, we can just treat μ as an independent parameter.

The μ-term and the Yukawa couplings in the superpotential equation (1.2) give also (scalar)^3 couplings of the form

$$\begin{aligned}\mathcal{L} \;\supset\; & \mu^*(\tilde{\bar{u}}\mathbf{y_u}\tilde{u}H_d^{0*} + \tilde{\bar{d}}\mathbf{y_d}\tilde{d}H_u^{0*} + \tilde{\bar{e}}\mathbf{y_e}\tilde{e}H_u^{0*} \\ & + \tilde{\bar{u}}\mathbf{y_u}\tilde{d}H_d^{-*} + \tilde{\bar{d}}\mathbf{y_d}\tilde{u}H_u^{+*} + \tilde{\bar{e}}\mathbf{y_e}\tilde{\nu}H_u^{+*}) + \text{c.c.}\end{aligned} \tag{1.9}$$

In Fig. 1.4 we show some of these couplings, proportional to $\mu^* y_t$, $\mu^* y_b$, and $\mu^* y_\tau$ respectively. These play an important role in determining the mixing of top squarks, bottom squarks, and tau sleptons.

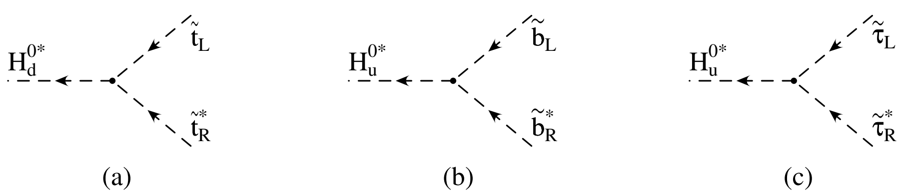


Figure 1.4: Some of the supersymmetric (scalar)^3 couplings proportional to $\mu^* y_t$, $\mu^* y_b$, and $\mu^* y_\tau$.

1.2 Soft supersymmetry breaking interactions

To obtain a realistic phenomenological model we must require supersymmetry breaking. Supersymmetry, if it exists at all, should be an exact symme-

[4] Some other attractive solutions for the μ problem are proposed in Refs.[3, 4, 5, 6, 7, 8].

try which is spontaneously broken. In other words, supersymmetry must be hidden at low energies, exactly like it happens with electroweak symmetry in the ordinary Standard Model. It turns out however, that spontaneous breaking of global supersymmetry doesn't give realistic models.

The hierarchy problem is solved if the theory is free of quadratic divergences. The absence of this kind of divergences does not however imply that the theory must be supersymmetric. It is possible to add some terms to the supersymmetric lagrangian, breaking supersymmetry without introducing quadratic divergences. These new terms are called soft breaking terms. At one loop it is simple to count the possible terms because all divergences are found in the wave function renormalization of chiral multiplets and vector multiplets (equivalent to the renormalization of gauge coupling constants).

Soft terms are actually present in the low energy effective lagrangian of spontaneously broken supergravity theories. This means that they can be derived from a more fundamental theory (i.e. valid at higher energies) such as $N = 1$ supergravity. We will see how it is possible to do this in the next section, when we will treat the so called gravity-mediated scheme. In general, these models extend the MSSM including new particles and interactions at very high mass scales. For practical purposes, it is extremely useful to parameterize these models just introducing extra terms which break supersymmetry explicitly in the effective MSSM lagrangian. In the context of a general renormalizable theory, the possible soft supersymmetry breaking terms in the lagrangian are [5]

$$\begin{aligned}\mathcal{L}_{\rm soft} &= -\frac{1}{2}\left(M_\lambda\,\lambda^a\lambda^a + \text{c.c.}\right) - (m^2)^i_j\phi^{j*}\phi_i \\ &\quad -\left(\frac{1}{2}b^{ij}\phi_i\phi_j + \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + \text{c.c.}\right)\end{aligned} \tag{1.10}$$

They consist of gaugino masses M_λ for each gauge group, scalar (mass)^2 terms $(m^2)^j_i$ and b^{ij}, and (scalar)^3 couplings a^{ijk}. It is possible to show that a softly-broken supersymmetric theory with $\mathcal{L}_{\rm soft}$ as given by equation (1.10) has no quantum corrections which have quadratic divergences to all orders in perturbation theory. For all the details see the work of Girardello and Grisaru [9].

The lagrangian $\mathcal{L}_{\rm soft}$ breaks supersymmetry, since it involves only scalars and gauginos, excluding their respective superpartners. In fact, the soft

[5] In the MSSM one might imagine that the D term for $U(1)_Y$ has a Fayet-Iliopoulos term as the principal source of supersymmetry breaking. Unfortunately, this cannot work, because the squarks and sleptons do not have superpotential mass terms.

terms in $\mathcal{L}_{\text{soft}}$ can give mass to all of the scalars and gauginos in the theory, even if the gauge bosons and fermions in chiral supermultiplets are massless. The gaugino masses M_λ are always allowed by gauge symmetry. The $(m^2)^i_j$ terms are allowed for i, j such that ϕ_i, ϕ^{j*} transform in complex conjugate representations under all gauge symmetries. In particular, when $i = j$ this is certainly true, so every scalar can get a mass in this way.

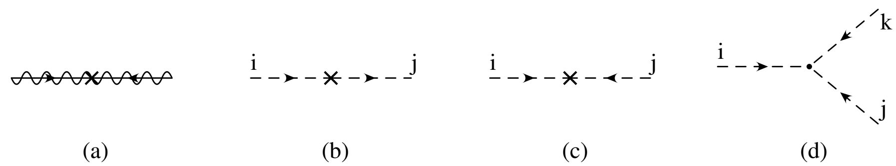


Figure 1.5: Soft supersymmetry-breaking terms: (a) gaugino mass insertion M_λ; (b) non-analytic scalar (mass)^2 $(m^2)^i_j$; (c) analytic scalar (mass)^2 b^{ij}; (d) (scalar)^3 coupling a^{ijk}.

The remaining soft terms are restricted by the symmetries. It is worth noting that the b^{ij} and a^{ijk} terms have the same form as the M^{ij} and y^{ijk} terms in the superpotential, so they will be allowed by gauge invariance if and only if a corresponding superpotential term is allowed. The Feynman diagram interactions corresponding to the allowed soft terms in equation (1.10) are shown in Fig. 1.5. For each of the interactions in Figs. 1.5 a,c,d there is one with all arrows reversed, which corresponds to the complex conjugate term in the lagrangian.

1.3 Gravity-mediated susy breaking models

In this section we systematically analyze the way in which the MSSM soft terms arise. The gravity-mediated models are characterized by the fact that the hidden sector of the theory communicates with the MSSM only (or primarily) through gravitational interactions. In an effective field theory, this means that the supergravity lagrangian contains non-renormalizable terms which are suppressed by powers of the Planck mass, since the gravitational coupling is proportional to $1/M_{Pl}$. These will include

$$\begin{aligned}
\mathcal{L}_{\rm NR} \;=\; & -\frac{1}{M_{Pl}} F_X \sum_a \frac{1}{2} f_a \lambda^a \lambda^a + \text{c.c.} \\
& -\frac{1}{M_{Pl}} F_X \left(\frac{1}{6} y'^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} \mu'^{ij} \phi_i \phi_j \right) \\
& -\frac{1}{M_{Pl}^2} F_X F_X^* \, k^i_j \phi_i \phi^{*j} + \text{c.c.}
\end{aligned} \tag{1.11}$$

where F_X is the auxiliary field for a chiral supermultiplet X in the hidden sector, and ϕ_i and λ^a are the scalar and gaugino fields of the MSSM. The terms in equation (1.11) are part of a non-renormalizable supersymmetric lagrangian which contains other terms that we may ignore. Now if one assumes that $\langle F_X \rangle \sim 10^{10}$ or 10^{11} GeV, then $\mathcal{L}_{\rm NR}$ will give us a lagrangian of the form $\mathcal{L}_{\rm soft}$ in equation (1.10), with MSSM soft terms of order a few hundred GeV.

The dimensionless parameters f_a, k^i_j, y'^{ijk} and the dimensionful μ'^{ij} in $\mathcal{L}_{\rm NR}$ are to be determined by the underlying theory, perhaps string theory. Even without knowing this theory, we can simplify the task assuming a "minimal" form for the normalization of kinetic terms and gauge interactions in the non-renormalizable supergravity lagrangian. In that case, we have a common coupling for the three gauginos ,$f_a = f$, and for the scalars $k^i_j = k\delta^i_j$. The other couplings are proportional to the corresponding superpotential parameters, giving $y'^{ijk} = \alpha y^{ijk}$ and $\mu'^{ij} = \beta \mu^{ij}$ with universal dimensionless constants α and β. Then all the soft terms in $\mathcal{L}^{\rm MSSM}_{\rm soft}$ can all be written in terms of just four parameters:

$$m_{1/2} = f \frac{\langle F_X \rangle}{M_{\rm P}} \qquad m_0^2 = k \frac{|\langle F_X \rangle|^2}{M_{\rm P}^2} \qquad A_0 = \alpha \frac{\langle F_X \rangle}{M_{\rm P}} \qquad B_0 = \beta \frac{\langle F_X \rangle}{M_{\rm P}} \tag{1.12}$$

In terms of these, one can write for the parameters appearing in the soft lagrangian

$$M_3 = M_2 = M_1 = m_{1/2} \tag{1.13}$$

$$\mathbf{m^2_Q} = \mathbf{m^2_u} = \mathbf{m^2_d} = \mathbf{m^2_L} = \mathbf{m^2_e} = m_0^2\, \mathbf{1} \quad m^2_{H_u} = m_0^2 = m^2_{H_d} \tag{1.14}$$

$$\mathbf{a_u} = A_0 \mathbf{y_u} \qquad \mathbf{a_d} = A_0 \mathbf{y_d} \qquad \mathbf{a_e} = A_0 \mathbf{y_e} \tag{1.15}$$

$$b = B_0 \mu \tag{1.16}$$

Even if this choice of parameterization is not completely well-motivated on theoretical grounds, from a phenomenological perspective it sounds very nice. Moreover, this framework avoids the most dangerous types of FCNC and CP-violation. Equations (1.13)-(1.15) are also highly predictive, while equation (1.16) is content-free unless one can relate B_0 to other parameters in the theory. They should be applied as RG boundary conditions at the scale M_{Pl}. We must then RG evolve the soft parameters down to the electroweak scale, predicting in this way the entire MSSM spectrum. This can be done in terms of just five parameters $m_{1/2}$, m_0^2, A_0, B_0, and μ (plus the already-measured gauge and Yukawa couplings of the MSSM). In practice, one starts this RG running from the unification scale [6] $M_U \approx 2 \times 10^{16}$ GeV instead of M_{Pl}. The reason is that unification of gauge couplings suggests that we have the theory under control only up to M_U, but we know little at scales between M_U and M_{Pl}. The error is proportional to a loop suppression factor times $\ln(M_{Pl}/M_U)$ and can be partially absorbed redefining m_0^2, $m_{1/2}$, A_0 and B_0. However, in some cases can lead to important effects. This framework represents the bulk of phenomenological studies of supersymmetry. It is sometimes referred to as the minimal supergravity (mSUGRA) or supergravity-inspired scenario for the soft terms.

1.4 Soft supersymmetry breaking in the MSSM

To obtain insights from the MSSM, we need to specify the soft supersymmetry breaking terms. In section 1.2, we learned how to write the most general terms in a given supersymmetric theory. Applying this recipe to the MSSM, we have

$$\begin{aligned}\mathcal{L}_{\rm soft}^{\rm MSSM} \quad = \quad & -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B}\right) + \text{c.c.} \\ & -\left(\widetilde{\overline{u}}\,\mathbf{a_u}\,\widetilde{Q}H_u - \widetilde{\overline{d}}\,\mathbf{a_d}\,\widetilde{Q}H_d - \widetilde{\overline{e}}\,\mathbf{a_e}\,\widetilde{L}H_d\right) + \text{c.c.} \\ & -\widetilde{Q}^\dagger\,\mathbf{m^2_Q}\,\widetilde{Q} - \widetilde{L}^\dagger\,\mathbf{m^2_L}\,\widetilde{L} - \widetilde{\overline{u}}\,\mathbf{m^2_{\overline{u}}}\,\widetilde{\overline{u}}^\dagger - \widetilde{\overline{d}}\,\mathbf{m^2_{\overline{d}}}\,\widetilde{\overline{d}}^\dagger - \widetilde{\overline{e}}\,\mathbf{m^2_{\overline{e}}}\,\widetilde{\overline{e}}^\dagger \\ & - m^2_{H_u}H_u^*H_u - m^2_{H_d}H_d^*H_d - (bH_uH_d + \text{c.c.}) \end{aligned} \tag{1.17}$$

In equation (1.17), M_3, M_2, and M_1 are the gluino, wino, and bino mass terms respectively [7]. The second line in equation (1.17) contains the (scalar)^3 couplings, of the type a^{ijk} in equation (1.10). Each of $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$ is a complex 3×3 matrix in family space, with mass dimensions. They are in one-to-one

[6] We will see in the following that the presence of extra dimensions will lower the unification scale by some order of magnitude.

[7] Here, and from now on, we suppress the adjoint representation gauge indices on the wino and gluino fields, and the gauge indices on all of the chiral supermultiplet fields.

correspondence with the Yukawa coupling matrices in the superpotential. The third line of equation (1.17) contains squark and slepton mass terms of the $(m^2)_i^j$ type in equation (1.10). Each of $\mathbf{m^2_Q}$, $\mathbf{m^2_{\bar{u}}}$, $\mathbf{m^2_{\bar{d}}}$, $\mathbf{m^2_L}$, $\mathbf{m^2_{\bar{e}}}$ is a 3×3 matrix in family space which can have complex entries, but they must be hermitian in order to have a real lagrangian [8]. Finally, in the last line of equation (1.17) we have supersymmetry-breaking contributions to the Higgs potential; $m^2_{H_u}$ and $m^2_{H_d}$ are $(\text{mass})^2$ terms of the $(m^2)_i^j$ type, while b is the only $(\text{mass})^2$ term of the type b^{ij} in equation (1.10) which can appear in the MSSM [9]. The expression (1.17) is the most general soft supersymmetry-breaking Lagrangian of the form (1.10) which respects gauge invariance and conserves matter parity.

Unlike the supersymmetry-preserving part of the lagrangian, $\mathcal{L}^{\text{MSSM}}_{\text{soft}}$ introduces many new parameters in a number which is much greater than that in the ordinary Standard Model. A careful count [10] shows that there are 105 masses, phases and mixing angles in the MSSM lagrangian which cannot be rotated away by redefining the phases and flavor basis for the quark and lepton supermultiplets. Thus, in principle, supersymmetry (or more precisely, supersymmetry breaking) appears to introduce more arbitrariness in the lagrangian than the Standard Model does. Fortunately, there are already experiments which limit this number. This is because most of the new parameters in equation (1.17) involve flavor mixing or CP violation, which is already restricted by experiment [11].

All of these potentially dangerous FCNC and CP-violating effects in the MSSM can be avoided assuming that supersymmetry breaking is "universal". In particular, one can suppose that the squark and slepton mass matrices are flavor-blind. This means that they should each be proportional to the 3×3 identity matrix in family space:

$$\begin{array}{l} \mathbf{m^2_Q} = m^2_Q \mathbf{1} \quad \mathbf{m^2_{\bar{u}}} = m^2_{\bar{u}} \mathbf{1} \quad \mathbf{m^2_{\bar{d}}} = m^2_{\bar{d}} \mathbf{1} \\ \mathbf{m^2_L} = m^2_L \mathbf{1} \quad \mathbf{m^2_{\bar{e}}} = m^2_{\bar{e}} \mathbf{1} \end{array} \tag{1.18}$$

If so, then all squark and slepton mixing angles are trivial, because squarks and sleptons with the same electroweak quantum numbers will be degenerate in mass and can be rotated into each other. Supersymmetric contributions to FCNC processes will therefore be very small, modulo the mixing due to $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$. One can make the further simplifying assumption that the $(\text{scalar})^3$ couplings are each proportional to the corresponding Yukawa

[8] To avoid confusion, we do not put tildes on the $\mathbf{Q}$ in $\mathbf{m^2_Q}$, etc.

[9] The parameter we call b often appears in the literature as $B\mu$.

coupling matrix:

$$\mathbf{a_u} = A_{u0}\,\mathbf{y_u} \qquad \mathbf{a_d} = A_{d0}\,\mathbf{y_d} \qquad \mathbf{a_e} = A_{e0}\,\mathbf{y_e} \tag{1.19}$$

This ensures that only the squarks and sleptons of the third family can have large (scalar)^3 couplings. To avoid large CP-violating effects it is assumed that the soft parameters do not introduce new complex phases. This is certainly true for Higgs, squarks and sleptons mass matrices if equations (1.18) are valid. One can also fix μ in the superpotential and b in equation (1.17) to be real, rotating the phase of H_u and H_d. One can also assume that the masses of the gauginos are real and that the matrices A_i appearing in equations (1.19) are real; then the only CP-violating phase in the theory will be the ordinary CKM phase of the Standard Model.

The conditions (1.18) and (1.19) reflect the so called assumption of soft-breaking universality. The soft-breaking universality relations have not a well established theoretical background. One can just think that they come from a specific model for the origin of supersymmetry breaking. The equations (1.18) and (1.19) must be taken as boundary conditions on the running soft parameters at a certain RG scale Q_0, much greater than the actual experiments can probe. We must then RG-evolve all of the soft parameters, the superpotential parameters, and the gauge couplings down to the electroweak scale.

At the electroweak scale, equations (1.18) and (1.19) will not be satisfied. However, RG corrections coming from gauge interactions will respect equations (1.18) and (1.19), while RG corrections coming from Yukawa interactions are small except for the ones of the third family. In particular, the (scalar)^3 couplings should be negligible for the squarks and sleptons of the first two families. We must stress that, if universality hold at the input scale, then supersymmetry will contribute only with a small amount to FCNC and CP-violating observables.

1.5 RG equations for the MSSM

In the next chapter we will describe in detail how to obtain the renormalization group (RG) equations. However here we want to anticipate the result for the MSSM. The 1-loop RG equations for the Standard Model gauge couplings g_1, g_2, g_3 are given by

$$\frac{d}{dt} g_a = \frac{1}{16\pi^2} b_a g_a^3 \quad \Rightarrow \quad \frac{d}{dt}\alpha_a^{-1} = -\frac{b_a}{2\pi} \qquad (a = 1, 2, 3) \tag{1.20}$$

where $t = \ln(Q/Q_0)$ with Q the RG scale. In the Standard Model, $b_a^{SM} = (41/10,\ -19/6,\ -7)$, while in the MSSM one finds $b_a^{MSSM} = (33/5,\ 1,\ -3)$. The latter coefficients are larger because of the presence of the extra MSSM particles in the loops. The normalization for g_1 here is chosen to agree with grand unified theories like $SU(5)$ or $SO(10)$. Thus in terms of the electroweak gauge couplings g and g' with $e = g \sin\theta_W = g' \cos\theta_W$, one has $g_2 = g$ and $g_1 = \sqrt{5/3}g'$. The quantities $\alpha_a = g_a^2/4\pi$ turn out to run linearly with RG scale at one-loop order. In Fig. 1.6 it is shown the RG evolution

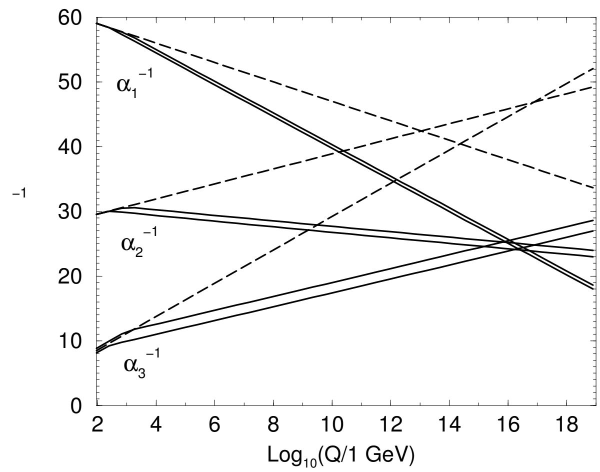


Figure 1.6: RG evolution of the inverse gauge couplings $\alpha_a^{-1}(Q)$ in the Standard Model (dashed lines) and the MSSM (solid lines). In the MSSM case, $\alpha_3(m_Z)$ is varied between 0.113 and 0.123, and the sparticle mass thresholds between 250 GeV and 1 TeV. Two-loop effects are included.

of the α_a^{-1}, including two-loop effects, in the Standard Model (dashed lines) and the MSSM (solid ones). Unlike the Standard Model, the content in particle of the MSSM is right to ensure the gauge couplings unification, at a scale [10] $M_U \sim 2 \times 10^{16}$ GeV. At first sight, the unification of gauge couplings at M_U might seem only an accident; but it may also be taken as a hint for grand unified theory (GUT) or superstring models. Indeed, both the theories predict gauge coupling unification below M_{Pl}. We then expect to apply a similar RG analysis to the other MSSM couplings and soft masses.

[10] We will see in the following that in presence of extra dimensions, the unification scale will lower towards values much smaller than the Planck scale.

The one-loop RG equations for the three gaugino mass parameters in the MSSM are determined by the same quantities b_a^{MSSM} which appear in the gauge coupling RG equations (1.20):

$$\frac{d}{dt}M_a = \frac{1}{8\pi^2}b_a g_a^2 M_a \qquad (b_a = 33/5, 1, -3) \tag{1.21}$$

for $a = 1, 2, 3$. As we will see in the next chapter these equations can not be achieved by means of effective potential alone, but to some extent they are generalization of the corresponding RGEs of [12] to an arbitrary content of particles. It is easy to show that the three ratios M_a/g_a^2 are each constant, in the sense that they are RG-scale independent [11]. In minimal supergravity models, we can therefore write

$$M_a(Q) = \frac{g_a^2(Q)}{g_a^2(Q_0)}m_{1/2} \qquad (a = 1, 2, 3) \tag{1.22}$$

at any RG scale $Q < Q_0$, where Q_0 is the input scale. Since the gauge couplings are observed to unify at a scale $M_U \sim 0.01 M_{Pl}$, one expects [12] that $g_1^2(Q_0) \approx g_2^2(Q_0) \approx g_3^2(Q_0)$.

Therefore, one finds that

$$\frac{M_1}{g_1^2} = \frac{M_2}{g_2^2} = \frac{M_3}{g_3^2} \tag{1.23}$$

at any RG scale if we neglect small two-loop effects. In minimal supergravity models, the common value in equation (1.23) is equal to $m_{1/2}/g_U^2$, where g_U is the unified gauge coupling at the input scale and $m_{1/2}$ is the common gaugino mass. Since the gauge couplings g_1, g_2, and g_3 are quite well known at the electroweak scale, the prediction (1.23) can be extrapolated up to M_U. Moreover, we will see later that the gaugino mass parameters enter into the RG equations for all of the other soft terms.

In the next chapter we will study the renormalization group equations for softly broken supersymmetric theory, showing how to derive them in the case of the MSSM. Here we simply present the complete set of equations. Let us consider the running of the superpotential parameters at one loop

[11] This is true up to small two-loop corrections.

[12] In a GUT model, it is automatic that the gauge couplings and gaugino masses are unified at all scales $Q > M_U$ and in particular at $Q \approx M_{Pl}$, because in the unified theory the gauginos all live in the same representation of the unified gauge group.

level. The Yukawa couplings run with scale according to [13]

$$\begin{aligned}
\frac{d}{dt}y_t &= \frac{y_t}{16\pi^2}\left[6|y_t|^2 + |y_b|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2\right] && (1.24)\\
\frac{d}{dt}y_b &= \frac{y_b}{16\pi^2}\left[6|y_b|^2 + |y_t|^2 + |y_\tau|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2\right] && (1.25)\\
\frac{d}{dt}y_\tau &= \frac{y_\tau}{16\pi^2}\left[4|y_\tau|^2 + 3|y_b|^2 - 3g_2^2 - \frac{9}{5}g_1^2\right] && (1.26)
\end{aligned}$$

Note that the β-function for each supersymmetric parameter is proportional to the parameter itself. This is a consequence of the famous non-renormalization theorem for susy theories. For the same reason, the equation for the μ parameter has the form

$$\frac{d}{dt}\mu = \frac{\mu}{16\pi^2}\left[3|y_t|^2 + 3|y_b|^2 + |y_\tau|^2 - 3g_2^2 - \frac{3}{5}g_1^2\right] \tag{1.27}$$

Next we consider the 1-loop RG equations for the analytic soft parameters $\mathbf{a_u}$, $\mathbf{a_d}$, $\mathbf{a_e}$. In models obeying equation (1.19), these matrices are proportional to the corresponding Yukawa couplings at the input scale. With the approximation of equation (1.5), one can therefore write

$$\mathbf{a_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_t \end{pmatrix} \qquad \mathbf{a_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_b \end{pmatrix} \qquad \mathbf{a_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_\tau \end{pmatrix} \tag{1.28}$$

which defines [14] running parameters a_t, a_b, and a_τ. We must stress however, that in this approximation we don't neglect the masses of the lightest squarks and sleptons. We instead assume a mass degeneracy for the first and second generation as we will see in (1.33). In this limit, only the third family and Higgs fields contribute to the MSSM superpotential.

[13]We will use the approximation, as is common, that only the third family Yukawa couplings are significant.

[14]Usually in the literature rescaled soft parameters $A_t = a_t/y_t$, $A_b = a_b/y_b$, and $A_\tau = a_\tau/y_\tau$ are commonly used.

The RG equations for the trilinear couplings are given by

$$\begin{aligned}
16\pi^2 \frac{d}{dt} a_t &= a_t \left[18|y_t|^2 + |y_b|^2 - \frac{16}{3} g_3^2 - 3g_2^2 - \frac{13}{15} g_1^2 \right] + 2 a_b y_b^* y_t \\
&\quad + y_t \left[\frac{32}{3} g_3^2 M_3 + 6 g_2^2 M_2 + \frac{26}{15} g_1^2 M_1 \right] \qquad (1.29) \\
16\pi^2 \frac{d}{dt} a_b &= a_b \left[18|y_b|^2 + |y_t|^2 + |y_\tau|^2 - \frac{16}{3} g_3^2 - 3g_2^2 - \frac{7}{15} g_1^2 \right] + 2 a_t y_t^* y_b \\
&\quad + 2 a_\tau y_\tau^* y_b + y_b \left[\frac{32}{3} g_3^2 M_3 + 6 g_2^2 M_2 + \frac{14}{15} g_1^2 M_1 \right] \qquad (1.30) \\
16\pi^2 \frac{d}{dt} a_\tau &= a_\tau \left[12|y_\tau|^2 + 3|y_b|^2 - 3g_2^2 - \frac{9}{5} g_1^2 \right] + 6 a_b y_b^* y_\tau \\
&\quad + y_\tau \left[6 g_2^2 M_2 + \frac{18}{5} g_1^2 M_1 \right] \qquad (1.31)
\end{aligned}$$

while the RG equation for the b parameter (the one that appears in the Higgs potential) is

$$\begin{aligned}
16\pi^2 \frac{d}{dt} b &= b \left[3|y_t|^2 + 3|y_b|^2 + |y_\tau|^2 - 3g_2^2 - \frac{3}{5} g_1^2 \right] \\
&\quad + \mu \left[a_t y_t^* + 6 a_b y_b^* + 2 a_\tau y_\tau^* + 6 g_2^2 M_2 + \frac{6}{5} g_1^2 M_1 \right] \qquad (1.32)
\end{aligned}$$

The β-function for each of these soft parameters is not proportional to the parameter itself. This is due to the fact that, parameters which violate supersymmetry are not protected by non-renormalization theorems.

Next let us consider the RG equations for the scalar masses in the MSSM. In the approximation of equations (1.5) and (1.28), the scalar masses satisfy boundary conditions like equation (1.18) at an input RG scale. When they are renormalized, they will stay almost diagonal

$$\mathbf{m_Q^2} \approx \begin{pmatrix} m_{Q_1}^2 & 0 & 0 \\ 0 & m_{Q_1}^2 & 0 \\ 0 & 0 & m_{Q_3}^2 \end{pmatrix} \qquad \mathbf{m_{\overline{u}}^2} \approx \begin{pmatrix} m_{\overline{u}_1}^2 & 0 & 0 \\ 0 & m_{\overline{u}_1}^2 & 0 \\ 0 & 0 & m_{\overline{u}_3}^2 \end{pmatrix} \qquad (1.33)$$

The first and second family squarks and slepton remain very nearly degenerate, but the third family squarks and sleptons will be affected by large Yukawa couplings. The one-loop RG equations for the first and second family squark and slepton squared masses can be written as

$$16\pi^2 \frac{d}{dt} m_\phi^2 = - \sum_{a=1,2,3} 8 g_a^2 C_a^\phi |M_a|^2 \qquad (1.34)$$

for each scalar ϕ. Here the sum is over the three gauge groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$. M_a are the corresponding running gaugino mass parameters and the constants C_a^ϕ are the quadratic Casimir invariants. Looking at the equation (1.34), we note that the right-hand sides are strictly negative: then scalar (mass)^2 parameters will grow in the running from the input scale (unification scale) down to the electroweak scale. The scalar masses will receive large positive contributions at the electroweak scale, given by the presence of the gaugino masses.

The RG equations for the mass parameters of the Higgs scalars receive contributions from the Yukawa ($y_{t,b,\tau}$) and the soft couplings ($a_{t,b,\tau}$). It turns out to be useful to introduce the following combinations

$$X_t = 2|y_t|^2(m_{H_u}^2 + m_{Q_3}^2 + m_{\bar{u}_3}^2) + 2|a_t|^2 \tag{1.35}$$
$$X_b = 2|y_b|^2(m_{H_d}^2 + m_{Q_3}^2 + m_{\bar{d}_3}^2) + 2|a_b|^2 \tag{1.36}$$
$$X_\tau = 2|y_\tau|^2(m_{H_d}^2 + m_{L_3}^2 + m_{\bar{e}_3}^2) + 2|a_\tau|^2 \tag{1.37}$$

In terms of these quantities, the RG equations for the soft Higgs mass parameters are

$$16\pi^2 \frac{d}{dt} m_{H_u}^2 = 3X_t - 6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2 \tag{1.38}$$
$$16\pi^2 \frac{d}{dt} m_{H_d}^2 = 3X_b + X_\tau - 6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2 \tag{1.39}$$

The third family mass parameters also get contributions from X_t, X_b and X_τ. Their RG equations are given by

$$16\pi^2 \frac{d}{dt} m_{Q_3}^2 = X_t + X_b - \frac{32}{3}g_3^2|M_3|^2 - 6g_2^2|M_2|^2 - \frac{2}{15}g_1^2|M_1|^2 \tag{1.40}$$
$$16\pi^2 \frac{d}{dt} m_{\bar{u}_3}^2 = 2X_t - \frac{32}{3}g_3^2|M_3|^2 - \frac{32}{15}g_1^2|M_1|^2 \tag{1.41}$$
$$16\pi^2 \frac{d}{dt} m_{\bar{d}_3}^2 = 2X_b - \frac{32}{3}g_3^2|M_3| - \frac{8}{15}g_1^2|M_1|^2 \tag{1.42}$$
$$16\pi^2 \frac{d}{dt} m_{L_3}^2 = X_\tau - 6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2 \tag{1.43}$$
$$16\pi^2 \frac{d}{dt} m_{\bar{e}_3}^2 = 2X_\tau - \frac{24}{5}g_1^2|M_1|^2 \tag{1.44}$$

Examining the RG equations (1.29)-(1.32) and (1.38)-(1.44) we note that if the gaugino mass parameters are non-zero at the input scale, then all of the other soft terms will be non null. While if the gaugino masses vanish at tree level, then they don't get any contributions to their masses at one-loop order. In that case the gaugino masses M_1, M_2 and M_3 will be very small.

Before concluding this section, we must comment on a peculiar property of mSUGRA models. There are also terms in the scalar (mass)^2 RG equations which are proportional to $Tr(Ym^2)$ (the sum of the weak hypercharge times the soft (mass)^2 for all scalars in the theory). However, these contributions vanish in the case of minimal supergravity boundary conditions for the soft terms, as one can see by explicitly calculating $Tr(Ym^2)$ in each case. If $Tr(Ym^2)$ is zero at the input scale, then it will remain zero under RG evolution, i.e. is a RG invariant. Therefore we neglect such terms in our discussion. We will see however that in presence of KK excitations $Tr(Ym^2)$ is no longer invariant and those terms will have an effect.

1.6 Mass spectrum: the neutralinos

The electroweak symmetry breaking is responsible for the mix of higgsinos and electroweak gauginos. The neutral higgsinos ($\widetilde{H}_u^0$ and $\widetilde{H}_d^0$) and the neutral gauginos ($\widetilde{B}$, $\widetilde{W}^0$) form four neutral mass eigenstates, called neutralinos. In this section we will denote [15] the neutralino mass eigenstates by $\widetilde{N}_i$ ($i = 1, 2, 3, 4$). The lightest neutralino, $\widetilde{N}_1$, is usually assumed to be the LSP, the lightest supersymmetric particle. As we will see this is the only MSSM particle which can make a good cold dark matter candidate. We now describe the mass spectrum and mixing of the neutralinos in the MSSM.

In the gauge-eigenstate basis $\psi^0 = (\widetilde{B}, \widetilde{W}^0, \widetilde{H}_d^0, \widetilde{H}_u^0)$, the neutralino mass terms in the lagrangian are

$$\mathcal{L} \supset -\frac{1}{2}(\psi^0)^T \mathbf{M}_{\widetilde{N}} \psi^0 + \text{c.c.} \tag{1.45}$$

where

$$\mathbf{M}_{\widetilde{N}} = \begin{pmatrix} M_1 & 0 & -c_\beta\, s_W\, m_Z & s_\beta\, s_W\, m_Z \\ 0 & M_2 & c_\beta\, c_W\, m_Z & -s_\beta\, c_W\, m_Z \\ -c_\beta\, s_W\, m_Z & c_\beta\, c_W\, m_Z & 0 & -\mu \\ s_\beta\, s_W\, m_Z & -s_\beta\, c_W\, m_Z & -\mu & 0 \end{pmatrix} \tag{1.46}$$

Here we have introduced the usual standard abbreviations $s_\beta = \sin\beta$, $c_\beta = \cos\beta$, $s_W = \sin\theta_W$ and $c_W = \cos\theta_W$. The gaugino masses M_1 and M_2 in this matrix come from the MSSM soft Lagrangian in equation (1.17), while μ is the supersymmetric higgsino mass terms appearing in equation (1.7). The

[15] Other common notations use $\widetilde{\chi}_i^0$ or $\widetilde{Z}_i$ for neutralinos.

terms proportional to m_Z are the result of Higgs-higgsino-gaugino couplings. The mass matrix $\mathbf{M}_{\widetilde{N}}$ can be diagonalized by a unitary matrix, giving rise to positive real eigenvalues.

In general, the parameters M_1, M_2, and μ can have arbitrary complex phases. In mSUGRA models satisfying the unification conditions equation (1.13), M_2 and M_1 will have the same complex phase preserved by RG evolution equation (1.21). In that case, redefining the phases of $\widetilde{B}$ and $\widetilde{W}$ we can make M_1 and M_2 both real and positive. The phase of μ then cannot be rotated away. However, if μ is not real, CP-violating effects in low-energy physics will arise. Therefore, it is common to assume that μ is real. The sign of μ remains still undetermined by this constraint and will be a parameter of the model.

In models which satisfy equation (1.23), one has the result

$$M_1 \approx \frac{5}{3}\tan^2\theta_W\, M_2 \approx 0.5 M_2 \tag{1.47}$$

at the electroweak scale. Then the neutralino masses and mixing angles depend on only three parameters. This assumption is usually made in almost all phenomenological models.

There is an interesting limit in which electroweak symmetry breaking effects can be treated as small perturbation on the neutralino mass matrix. If

$$m_Z \ll |\mu \pm M_1|, |\mu \pm M_2| \tag{1.48}$$

then the neutralino mass eigenstates are very nearly $\widetilde{N}_1 \approx \widetilde{B}$; $\widetilde{N}_2 \approx \widetilde{W}^0$; $\widetilde{N}_3, \widetilde{N}_4 \approx (\widetilde{H}_u^0 \pm \widetilde{H}_d^0)/\sqrt{2}$, with mass eigenvalues:

$$m_{\widetilde{N}_1} = M_1 - \frac{m_Z^2 s_W^2 (M_1 + \mu \sin 2\beta)}{\mu^2 - M_1^2} + \ldots \tag{1.49}$$

$$m_{\widetilde{N}_2} = M_2 - \frac{m_W^2 (M_2 + \mu \sin 2\beta)}{\mu^2 - M_2^2} + \ldots \tag{1.50}$$

$$m_{\widetilde{N}_3} = |\mu| + \frac{m_Z^2 (1 - \epsilon \sin 2\beta)(|\mu| + M_1 c_W^2 + M_2 s_W^2)}{2(|\mu| + M_1)(|\mu| + M_2)} + \ldots \tag{1.51}$$

$$m_{\widetilde{N}_4} = |\mu| + \frac{m_Z^2 (1 + \epsilon \sin 2\beta)(|\mu| - M_1 c_W^2 - M_2 s_W^2)}{2(|\mu| - M_1)(|\mu| - M_2)} + \ldots \tag{1.52}$$

Here we have assumed that μ is real, but its sign $\epsilon = \pm 1$ is undetermined. It turns out that a "bino-like" LSP $\widetilde{N}_1$ can have the right cosmological abundance to make a good dark matter candidate. For this reason, the large

$|\mu|$ limit is preferred. Moreover, this limit tends to emerge from mSUGRA boundary conditions on the soft parameters, which require $|\mu|$ to be larger than M_1 and M_2 to have electroweak symmetry breaking. In practice, the masses and mixing angles for the neutralinos can be computed numerically. The corresponding Feynman rules can be found in Refs. [13, 14, 15, 16].

Chapter 2

Renormalization group equations

In general supersymmetric gauge theories are renormalizable field theories, and renormalization can be performed to all orders in perturbation theory without breaking supersymmetry. Using the superfield formalism for quantum corrections it is possible to show that any perturbative contribution to the effective action must be expressed as one integral over the whole superspace. This leads to the celebrated non renormalization theorems. In particular, there are no quadratic divergences, which are responsible for the so called hierarchy problem. This is the main reason to introduce supersymmetry in the Standard Model.

In order to have a realistic model, however, it is necessary to add supersymmetry breaking terms. These terms can be chosen in such a way to avoid quadratic divergences even if susy is broken. These are the so called soft terms. The running of these terms doesn't follow the rules of supersymmetry. It turns out at this scope to use the method of effective potential. In this way it is possible to obtain the renormalization group equations for all the parameters: gauge couplings, Yukawa couplings and all the soft terms. Once the equations are obtained, one can rephrase all the contribution in terms of usual Feynman diagrams.

In the MSSM the gauge couplings unify at a scale near the Planck mass. When we will introduce extra dimensions in the next chapter, the KK towers will give rise to power-law behavior. These exotic properties, make the model more appealing because the unification of gauge couplings happens at energies much lower than the usual.

2.1 Method of effective potential

Before discussing the renormalization group equations (RGEs), we want to review a powerful method to obtain the running without making use of Feynman diagram calculations. This method, known as the effective potential method, will be useful even to treat soft breaking terms. We saw in the previous chapter that supersymmetry reduces the number of independent diagrams. However, when we introduce soft breaking terms, we must deal with a usual field theory. In this case the number of diagrams to analyze will grow fast, raising the amount of work to do in calculating the RGE. The method of the effective potential, avoiding the use of Feynman diagrams, is more suitable in this case. Indeed, in this framework, it will be sufficient only to calculate the mass matrices for the various fields involved. The quantum corrections will depend on these matrices.

To begin with, let us consider the simple case of a real scalar field with lagrangian

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi(x)) \\ V(\phi) &= \frac{\alpha_0}{4!}\phi^4 - \frac{m_0^2}{2}\phi^2 \end{aligned} \tag{2.1}$$

where m_0 is the bare mass and α_0 is the bare self-coupling. In this lagrangian we can put $\phi(x) = \rho + \phi_1(x)$ where ρ is the value that $\phi(x)$ assumes at the minimum of the potential. Then we have

$$\begin{aligned} S[\phi] &= \int d^4x \left[\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - V(\phi_1+\rho)\right] \\ &= -\frac{1}{2}\int d^4x\phi_1\left[\Box^2 + V''(\rho)\right]\phi_1 + \cdots \end{aligned} \tag{2.2}$$

where the omitted terms involve a constant piece plus higher powers of ϕ_1. From field theory, we know that the action is expressed in terms of the potential

$$S[\rho] = \int d^4x\mathcal{L}(x)|_{\phi=\rho} = -\Omega V(\rho) \tag{2.3}$$

where Ω is the total volume of space-time, while the effective action is defined as

$$\Gamma[\rho] = -\Omega U(\rho) \tag{2.4}$$

The function U is known as effective potential. The relation between the usual potential and the effective one can be inferred by second order terms in the saddle point expansion [1]

$$U(\rho) = V(\rho) - \frac{i\hbar}{2}\Omega^{-1} \ln det\{i\Delta^{-1}[\rho]\} \tag{2.5}$$

where $\Delta[\rho]$ is the propagator. To calculate the second term we note that

$$\begin{aligned} \ln det\{i\Delta^{-1}[\rho]\} &= Tr \ln\{i\Delta^{-1}[\rho]\} \\ &= \int \frac{d^4k}{(2\pi)^4} \ln\langle k|i\Delta^{-1}[\rho]|k\rangle \end{aligned} \tag{2.6}$$

Hence we obtain

$$U(\rho) = V(\rho) - \frac{i\hbar}{2}\int \frac{d^4k}{(2\pi)^4} \ln\langle k|i\Delta^{-1}[\rho]|k\rangle \tag{2.7}$$

We must now evaluate the propagator. From the usual definition we have

$$\begin{aligned} \langle x|i\Delta^{-1}[\rho]|y\rangle &= \left.\frac{\delta^2 S[\phi]}{\delta\phi_1(x)\delta\phi_2(y)}\right|_{\phi_1=0} \\ &= -\left[\Box^2 + V''(\rho)\right]\delta^4(x-y) \end{aligned} \tag{2.8}$$

In the formula (2.7) we need the propagator in momentum space. We find then

$$\Omega^{-1}\langle k|i\Delta^{-1}[\rho]|k\rangle = k^2 - V''(\rho) \tag{2.9}$$

We can now substitute this into (2.7) to obtain

$$U(\rho) = V(\rho) + \frac{\hbar}{2}\int \frac{d^4k_E}{(2\pi)^4} \ln[k_E^2 + V''(\rho)] + O(\hbar^2) \tag{2.10}$$

where we have rotated into Euclidean momentum space. The above integral is divergent, but we can cut it off at a scale $k_E^2 = \Lambda^2$, getting

$$\begin{aligned} \int d^4k_E \ln(k_E^2 + V'') = \pi^2 \Bigg[\Lambda^4\left(\ln\Lambda - \frac{1}{4}\right) \\ +\Lambda^2 V'' + \frac{1}{2}(V'')^2\left(\ln\frac{V''}{\Lambda^2} - \frac{1}{2}\right)\Bigg] + O\left(\frac{1}{\Lambda^2}\right) \end{aligned} \tag{2.11}$$

Using this result, we have

$$U(\rho) = V(\rho) + \hbar V_1(\rho) + O(\hbar^2) \tag{2.12}$$

[1] In this section we leave $\hbar$ to make the expressions more clear.

where

$$V_1(\rho) = \frac{\Lambda^2}{32\pi^2} V''(\rho) + \frac{(V''(\rho))^2}{64\pi^2} \left[\ln \frac{V''(\rho)}{\Lambda^2} - \frac{1}{2} \right] \tag{2.13}$$

In this expression we have dropped a constant of the order of $\Lambda^4 \ln \Lambda$ because this has no effect on the location of the minimum of $U(\rho)$. Note that the second derivative of the potential is just the (squared) mass of the scalar field. This will prove useful when we will deal with supersymmetric theories.

To separate $V_1(\rho)$ into its divergent part and its convergent one, we must introduce an arbitrary scale parameter μ, so that we can write

$$\ln \frac{V''}{\Lambda^2} = \ln \frac{V''}{\mu^2} + \ln \frac{\mu^2}{\Lambda^2} \tag{2.14}$$

This allows us to rewrite expression (2.13) as

$$V_1(\rho) = \frac{1}{32\pi^2} \left[\Lambda^2 V'' - \frac{1}{2} (V'')^2 \ln \frac{\Lambda^2}{\mu^2} \right] + \frac{(V'')^2}{64\pi^2} \left(\ln \frac{V''}{\mu^2} - \frac{1}{2} \right) \tag{2.15}$$

As we can see, the terms in square brackets are divergent, while the rest is finite. To eliminate the divergences in $V_1(\rho)$ we make use of the renormalization procedure. It is known that in order to do this, the divergent terms must have the same form as those present in the lagrangian. Then one can absorb the divergences defining new parameters. Indeed, this is what happens for any renormalizable theory.

For a potential $V(\rho)$ of polynomial degree n, $V''(\rho)$ is a polynomial of degree $n-2$ and subsequently the divergent part of $V_1(\rho)$ is a polynomial of degree $2n-4$. Thus we find the known result that the theory is renormalizable if $n \leq 4$.

To display the counter terms, we rewrite the parameters in $\mathcal{L}(x)$ as

$$\begin{aligned} \alpha_0 &= \alpha_1 + \delta\alpha \\ m_0^2 &= m_1^2 + \delta m^2 \end{aligned} \tag{2.16}$$

where $\delta\alpha$ and δm^2 may be divergent in perturbation theory, while the parameters α_1 and m_1^2 must be finite. The corrections $\delta\alpha$ and δm^2 will be of order $\hbar$. Introducing the expressions (2.16) into the Lagrangian, we can rewrite

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - V(\phi(x)) + \frac{\delta m^2}{4} \phi^2 - \frac{\delta\alpha}{4!} \phi^4 \\ V(\phi) &= \frac{\alpha_1}{4!} \phi^4 - \frac{m_1^2}{4} \phi^2 \end{aligned} \tag{2.17}$$

The last two terms in the lagrangian are the counter terms. Since they are of order $\hbar$, they can be simply added to the one loop effective potential, giving

$$U(\rho) = V(\rho) + \hbar V_1(\rho) - \frac{\delta m^2}{4}\phi^2 + \frac{\delta\alpha}{4!}\phi^4 \tag{2.18}$$

We choose $\delta\alpha$ and δm^2 so as to cancel the divergent part of $V_1(\rho)$:

$$\begin{aligned} \delta\alpha &= \frac{3\alpha_1^2\hbar}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2} \\ \delta m^2 &= \frac{2\alpha_1\hbar}{32\pi^2}\left(\Lambda^2 + \frac{m_1^2}{2}\ln\frac{\Lambda^2}{\mu^2}\right) \end{aligned} \tag{2.19}$$

The right hand side of these equations are ambiguous up to additive finite terms, but they can always be absorbed into the scale parameter μ. We note that from the first equation in (2.19) we can recover the usual beta function for a self interacting scalar field, which is[2]

$$\beta = \frac{3\alpha_1^2}{16\pi^2} \tag{2.20}$$

In the next section we will make use of this powerful method to renormalize supersymmetric theories.

2.2 Effective potential and supersymmetry

We can generalize the previous analysis to the case of supersymmetric theories [17]. At this end we note that the second derivatives of the potential are just the squared masses of the various particles. The mass matrices will involve bosons as well as fermions. In the last case, we must take into account the minus sign coming from the ordering of propagators in a fermion loop. Thus we introduce the supertrace

$$STr\mathcal{M}^2 = \sum_j (-)^j(2j+1)m_j^2 \tag{2.21}$$

where the sum is over all particles, j is the spin and $(2j+1)$ is the number of degrees of freedom for a massive particle of spin j. The formula (2.13) then becomes

$$\delta V = \frac{\Lambda^2}{32\pi^2}STr\mathcal{M}^2(z) + \frac{1}{64\pi^2}STr\mathcal{M}^4(z)\ln\left(\frac{\mathcal{M}^2(z)}{\Lambda^2}\right) \tag{2.22}$$

[2] To be consistent with the standard notation we have set $\hbar = 1$, as usual.

where Λ is the cut-off mass parameter [3] and z stands for a scalar field. For supersymmetric theories the quantity (2.21) vanishes and this ensures that there are no quadratic divergences at all. As we will see, it is even possible that a theory with broken susy has no quadratic divergences. This is peculiar of soft breaking terms.

We will consider the one-loop renormalization of the scalar potential, which contains a gauge part depending on the gauge coupling, to determine all renormalization constants of the theory. The one-loop divergent contributions to the scalar potential are given by the expression (2.22). We will let the scalar fields belong to a certain representation of a gauge group and we will denote them as z^a. Fields in the conjugate representation are denoted with a lower index z_a. The index a actually hides various indices such as flavour number, number generation and gauge index. To make the formulas less cumbersome, we will use only one collective index.

The supertrace $STr\mathcal{M}^2(z)$ must be considered as a function of the scalar fields and not only of their vacuum expectation values. The supertrace $STr\mathcal{M}^4(z)$ is the analogous for the fourth power of the mass matrices. The second term contains all logarithmic one-loop divergences. The problem of finding soft breaking terms corresponds to add new gauge invariant terms to the lagrangian such that $STr\mathcal{M}^2(z)$ does not receive any new field dependent contributions. These terms otherwise, would correspond to new quadratic divergences. With this simple recipe, we can investigate easily one-loop soft breaking terms. It is also possible to show that these terms are also soft to all orders, provided they do not give any field dependent contribution to the trace of $\mathcal{M}^2(z)$ for all states of a given spin [9].

We must outline that soft breaking terms arise naturally in the effective low-energy theory of spontaneously broken supergravity theories. In these theories, local supersymmetry can be spontaneously broken, using the super-Higgs mechanism. At energy scales much lower than the Planck scale, or equivalently in the limit $M_P \rightarrow \infty$, gravitation decouples. As a result, one has an effective gauge theory which is globally supersymmetric and embodies a set of soft breaking terms. In general, every soft term can be generated choosing the appropriate sector in which Higgs breaking occurs. However, these models are strongly constrained by phenomenology and not all the terms are allowed. For instance, all scalar mass terms are equal: $(m^2)^i_j = m^2\delta^i_j$. These constraints are useful to reduce the arbitrariness of the soft terms. However they are subject to runnings governed by the

[3] In the formula (2.22) we omitted a constant term proportional to Λ^2, which is irrelevant in dealing with the effective potential.

renormalization group. We will now derive these equations at the one-loop level.

We consider the most general [4] supersymmetric lagrangian with arbitrary soft terms, gauge group G and chiral multiplets in the representation R. For simplicity all the formulas will be given for a simple gauge group, with a unique gauge coupling constant. The generalization to non simple gauge group is straightforward. Indeed, for the MSSM it will be the usual standard model gauge group $SU(3) \times SU(2) \times U(1)$. The scalar potential is given by

$$V(z^a, z_a) = V_{susy}(z^a, z_a) + V_{soft}(z^a, z_a) \tag{2.23}$$

where V_{susy} is the standard supersymmetric potential and V_{soft} is the soft breaking piece. The susy potential, as usual, has the form

$$V_{susy}(z^a, z_a) = f_a f^a + \frac{1}{2} D^A D^A \tag{2.24}$$

where f is the superpotential. For a renormalizable theory, it is at most cubic in the fields

$$f = \frac{1}{2}\mu_{ab} z^a z^b + \frac{1}{6} f_{abc} z^a z^b z^c \tag{2.25}$$

We made use of the notation

$$f_a = \frac{df}{dz^a} \qquad f^a = \frac{df}{dz_a} \tag{2.26}$$

and similar expressions hold for higher order derivatives. The D-terms read instead

$$D^A = -g z_a T^{A\,a}_{\ \ b} z^b \tag{2.27}$$

where T^A are the generators of the gauge group acting on the chiral multiplets z_a. The most general soft terms are

$$V_{soft}(z^a, z_a) = (m^2)^a_b z_a z^b + \eta(z) + \bar{\eta}(\bar{z}) - \frac{1}{2}\Delta^{AB}\lambda^A\lambda^B - \frac{1}{2}\bar{\Delta}^{AB}\bar{\lambda}^A\bar{\lambda}^B \tag{2.28}$$

The first term gives masses to scalar particles, while $\eta(z)$ is an arbitrary gauge invariant polynomial of third degree in the scalar fields z^a. We have also included a gaugino mass term, that can always be chosen real by a phase redefinition of the gauginos. In the following we will diagonalize this mass matrix putting it in the form $\Delta^{AB} = \Delta^A \delta^{AB}$.

With the method of the effective potential one can obtain all the renormalized constants but the gaugino mass terms. The gaugino mass indeed

[4] We will assume that there are no Fayet-Iliopoulos terms.

involves spinors and thus cannot be studied through the analysis of the scalar potential. One then has to make use of Feynman diagrams or the method of the pure spinor [18].

The gauge coupling constant appears in the gauge potential, while the Yukawa couplings are either proportional to g or directly given by f_{abc}. Since all soft breaking terms are at most of dimension three, Yukawa couplings cannot be affected by them: their renormalization goes in the same way as in the supersymmetric case. This is an interesting result concerning the hierarchy problem, since the Yukawa couplings running is still proportional to the coupling itself. From the scalar sector only, is thus possible to obtain all the renormalization group (RG) equations [5]. We will now compute all renormalization constants and the corresponding RG equations to one loop order. The one loop contributions to the scalar potential are given in equation (2.22). Since, by definition of soft breaking terms, $STr\mathcal{M}^2$ does not receive any field dependent contribution, our theory will be free of quadratic divergences. All divergences are then logarithmic and we can define hatted quantities in the following way

$$\hat{V}(\hat{z}) = V(z) - kSTr\mathcal{M}^4(z) \tag{2.29}$$

where $k = \ln(\Lambda/Q)/32\pi^2$ and Q is an energy scale. Now the main task is to compute the supertrace of the quartic mass matrix for arbitrary values of the fields z^a. Let's start with the mass matrix for spin one. It is easy to see that

$$(\mathcal{M}_1^2)^{AB} = D_a^A D_a^B + D_a^B D_a^A \tag{2.30}$$

The trace of the fourth power is then

$$3Tr\mathcal{M}_1^4 = 6\left(D_a^A D^{Ba}\right)\left(D^{Ab} D_b^B\right)) + 6\left(D_a^A D^{Ba}\right)\left(D_b^A D^{Bb}\right) \tag{2.31}$$

The fermionic mass matrix contains the gaugino masses

$$\mathcal{M}_{\mathbf{1/2}} = \begin{pmatrix} f_{ab} & i\sqrt{2}D_a^A \\ -i\sqrt{2}D^{Ba} & \Delta^{AB} \end{pmatrix} \tag{2.32}$$

One can obtain

$$\begin{aligned} -2Tr\mathcal{M}_{1/2}^4 \;=\; & -2f_{ab}f^{bc}f_{cd}f^{da} - 16f_{ab}f^{ac}D_c^A D^{Ab} - 16(D_a^A D^{Ab})(D^{Ba}D_b^B) \\ & +8f_{ab}D^{Aa}\Delta^{AB}D^{Bb} + f^{ab}D_a^A\bar{\Delta}^{AB}D_b^B - 16\bar{\Delta}^{AC}\Delta^{CB}D_a^A D^{Ba} \\ & -2\Delta^{AB}\bar{\Delta}^{BC}\Delta^{CD}\bar{\Delta}^{DA} \end{aligned} \tag{2.33}$$

[5] Except for gaugino mass terms, as explained before.

The mass matrix squared of scalar fields, as usual, can be expressed in terms of second derivatives of the potential

$$\mathcal{M}_{\mathbf{0}}^{\mathbf{2}} = \begin{pmatrix} V_b^a & V^{ac} \\ V_{db} & V_d^c \end{pmatrix} \tag{2.34}$$

where we have defined

$$\begin{aligned} V_{ab} &= f_{abc} f^c + D_a^A D_b^A + \eta_{ab} && (2.35) \\ V^{ab} &= f^{abc} f_c + D^{Aa} D^{Ab} + \bar{\eta}^{ab} && (2.36) \\ V_b^a &= f^{ac} f_{bc} + D^{A}{}^{a}_{b} D^A + D^{Aa} D_b^A + (m^2)_b^a && (2.37) \end{aligned}$$

Putting together all the mass matrix squared we can verify our assertion about the renormalization properties of the soft breaking terms. Forgetting susy terms we have

$$Tr\mathcal{M}_{soft}^2 = 2(m^2)_a^a - 2\Delta^{AB}\bar{\Delta}^{BA} \tag{2.38}$$

This trace is actually field independent, and thus we have no quadratic divergences.

Our theory is invariant under gauge transformations as well as $STr\mathcal{M}^4$ is. Then we can express all the terms by means of group invariants. The gauge transformations act over the fields as $\delta_g z^a \propto (T^A)_b^a z^b$. We can thus exploit the gauge invariance of the superpotential, obtaining

$$f_a D^{Aa} = -g f_a T^{A}{}^{a}_{b} z^b = 0 \tag{2.39}$$

Differentiating once we have

$$f_{ab} D^{Aa} = -f_a D^{A}{}^{a}_{b} \tag{2.40}$$

and differentiating twice

$$f_{abc} D^{Aa} = -f_{ab} D^{A}{}^{a}_{c} - f_{ac} D^{A}{}^{a}_{b} \tag{2.41}$$

The invariance of the gauge potential reads

$$D^A D_a^A D^{Ba} - D^A D^{Aa} D_a^B = 0 \tag{2.42}$$

At the same time we have for the polynomial $\eta(z)$

$$\eta_a D^{Aa} = 0 \tag{2.43}$$

These identities help us to rewrite $STr(\mathcal{M}^4)$ in terms of group invariants of the representation R

$$\begin{aligned} Tr(T^A T^B) &= T(R)\delta^{AB} &(2.44)\\ T^{A}{}^{a}_{c} T^{A}{}^{c}_{b} &= C(R)\delta^a_b &(2.45)\end{aligned}$$

and the quadratic Casimir of the adjoint representation

$$f^{ACD} f^{BCD} = C_2(G)\delta^{AB} \tag{2.46}$$

The mass terms (m^2) gauge invariance leads to

$$(m^2)^a_b z_a D^{Ab} - (m^2)^a_b z^b D^A_a = 0 \tag{2.47}$$

By combining these various identities, one gets the following expressions involving the superpotential. From the equations (2.40) and (2.41) we obtain

$$2 f^{ac} f_{bc} D^{A}{}^{b}_{a} = -f_{abc} f^{ab} D^{Ac} = -f^{abc} f_{ab} D^A_c \tag{2.48}$$

$$f^{ac} f_{bc} D^A_a D^{Ab} = g^2 C(R) f_a f^a \tag{2.49}$$

Many of the terms appearing in $STr(\mathcal{M}^4)$ can now be expressed in a different form. For instance

$$f_{abc} f^c D^{Aa} D^{Ab} = g^2 C(R) f^a f_a - g^2 C(R) z^a f^b f_{ab} \tag{2.50}$$

$$f^{abc} f_c D^A_a D^A_b = g^2 C(R) f^a f_a - g^2 C(R) z_a f_b f^{ab} \tag{2.51}$$

The equation (2.51) is also obtained from (2.50) by complex conjugation. Besides we have

$$f^{ab} D^A_a D^A_b = -g^2 C(R) f^a z_a \tag{2.52}$$

$$f_{ab} D^{Aa} D^{Ab} = -g^2 C(R) f_a z^a \tag{2.53}$$

For the soft terms involving the polynomial and the scalar masses we obtain

$$\begin{aligned} \eta_{ab} D^{Aa} D^{Ab} &= -g^2 C(R)\eta_a z^a \\ \bar{\eta}^{ab} D^A_a D^A_b &= -g^2 C(R)\bar{\eta}^a z_a \\ (m^2)^a_b D^A_a D^{Ab} &= g^2 C(R)(m^2)^a_b z_a z^b \end{aligned} \tag{2.54}$$

Finally, we have identities involving group invariants

$$\begin{aligned} D^A\left(D^A_a D^{B}{}^a_b D^{Bb}\right) &= g^2 C(R) D^A D^A \\ D^A\left(D^B_a D^{A}{}^a_b D^{Bb}\right) &= g^2\left[C(R) - \frac{1}{2}C(G)\right] D^A D^A \\ D^A D^B\left(D^{A}{}^a_b D^{B}{}^b_a\right) &= g^2 T(R) D^A D^A \end{aligned} \tag{2.55}$$

After some calculations and making use of equations (2.48)- (2.55), we finally obtain

$$\begin{aligned} STr\mathcal{M}^4 = &+2g^2[T(R) + 2C(R) - 3C(G)]D^A D^A - 8g^2 C(R) f^a f_a \\ &+4 f^{ac} f_{bc} D^{A}{}^b_a D^A + 2 f^{abc} f_{abd} f^d f_c - 16 g^2 C(R) \Delta^2_A z^a z_a \\ &-2g^2 C(R)[z^a f^b f_{ab} + z_a f_b f^{ab}] + 2\eta_{ab}\bar{\eta}^{ab} \\ &-8g^2 C(R)\Delta_A (f_a z^a + f^a z_a) - 2\Delta^4_A \\ &+2 f^{abc} f_c \eta_{ab} + 2 f_{abc} f^c \eta^{ab} - 2g^2 C(R)(\bar{\eta}^a z_a + \eta_a z^a) \\ &+4 f^{ab} f_{ac} (m^2)^c_b + 4g^2 C(R)(m^2)^a_b z_a z^b + 2(m^2)^a_b (m^2)^b_a \\ &+4(m^2)^a_b (D^A)^b_a D^A \end{aligned} \tag{2.56}$$

Notice that the last term vanishes as a consequence of Schur's lemma for traceless generators of semi-simple groups [6]. This is not generally true for possible $U(1)$ factors present in the gauge group. This fact is important for the MSSM, which contains indeed a $U(1)$ group. One can rewrite this expression in terms of the scalar fields, the coupling constants and the mass parameters by inserting the expansions in the superpotential. In $STr\mathcal{M}^4$ the following terms are present:

- terms of order $zz\bar{z}\bar{z}$;

$$\begin{aligned} &+ 2g^2[T(R) - 3C(G)]D^A D^A + g z_a X, T^{A}{}^a_b z^b D^A \\ &+ 4g^2 C(R) D^A D^A - 4g^2 C(R) f^{abc} f_{ade} z_b z_c z^d z^e \\ &+ \frac{1}{2} X^a_b f_{acd} f^{bc'd'} z^c z^d z_{c'} z_{d'} \end{aligned} \tag{2.57}$$

- terms of order $zz\bar{z}$;

$$\begin{aligned} &- 7g^2 C(R) f_{abc}\mu^{ad} z^b z^c z_d + h.c. \\ &+ X^a_b f_{acd}\mu^{be} z_e z^c z^d + h.c. \end{aligned} \tag{2.58}$$

[6] Actually we have $STr(Tm^2) = m^2 STr(T)$.

- terms of order zzz;

$$\begin{aligned} &- 2g^2 C(R)\eta_{(3)abc} z^a z^b z^c + h.c. \\ &+ 2f^{abc} f_{cde} \eta_{(3)abd'} z^d z^e z^{d'} + h.c. \\ &- 4g^2 C(R) \sum_A \Delta^A f_{abc} z^a z^b z^c + h.c. \end{aligned} \tag{2.59}$$

- terms of order $z\bar{z}$;

$$\begin{aligned} &- 12g^2 C(R)\mu_{ab}\mu^{bc} z^a z_c + 2X^a_b \mu^{bc}\mu_{ad} z_c z^d \\ &+ 4g^2 C(R)(m^2)^a_b z_a z^b + 8\eta_{(3)abc}\eta^{(3)abd} z^c z_d \\ &+ 4g^2 Tr(T^A m^2)(T^A)^b_a z^a z_b \\ &- 16g^2 C(R) \sum_A (\Delta^A)^2 (z_a z^a) + 4(m^2)^a_b f_{acd} f^{bce} z_e z^d \end{aligned} \tag{2.60}$$

- terms of order zz;

$$\begin{aligned} &+ f^{abc} f_{cde} \eta_{(2)ab} z^d z^e + h.c. \\ &+ 4f^{abc} \mu_{cd} \eta_{(3)abe} z^d z^e + h.c. \\ &- 2g^2 C(R)\eta_{(2)ab} z^a z^b + h.c. \\ &- 8g^2 C(R) \sum_A \Delta^A \mu_{ab} z^a z^b + h.c. \end{aligned} \tag{2.61}$$

- terms of order z;

$$\begin{aligned} &+ 4f_{bcd}\mu^{ac}(m^2)^b_a z^d + h.c. \\ &+ 2f^{abc}\mu_{cd}\eta_{(2)ab} z^d + h.c. \\ &- 2g^2 C(R)\eta_{(1)a} z^a + h.c. \\ &+ 4\eta_{(3)abc}\eta^{ab}_{(2)} z^c + h.c. \end{aligned} \tag{2.62}$$

Here, to avoid cumbersome notation, we have defined the useful quantity

$$X^a_b = f^{acd} f_{bcd} \tag{2.63}$$

With the help of these formulas, we can renormalize our theory. In supersymmetric theories all divergences are found in the wave function renormalization of chiral multiplets and in the renormalization of gauge coupling constants (which is equivalent to wave function renormalization of vector multiplets). We then introduce a parameter ρ to exploit the running of the gauge couplings, while the wave function renormalization is governed by ϵ^a_b

$$\hat{z}^a = \left(\delta^a_b - \frac{1}{2}\epsilon^a_b\right) z^b \tag{2.64}$$

$$\hat{g} = (1+\rho)g \tag{2.65}$$

The dimension four terms $zz\bar{z}\bar{z}$ are not affected by soft terms, so the renormalization of the gauge coupling constant g as well as the Yukawa couplings f_{abc} are the same as with unbroken supersymmetry

$$\hat{f}_{abc} = f_{abc} + \frac{1}{2}\epsilon^{a'}_a f_{a'bc} + \frac{1}{2}\epsilon^{b'}_b f_{ab'c} + \frac{1}{2}\epsilon^{c'}_c f_{abc'} + f^{a'b'c'}_{abc} f_{a'b'c'} \tag{2.66}$$

$$\hat{\mu}_{ab} = \mu_{ab} + \frac{1}{2}\epsilon^{a'}_a \mu_{a'b} + \frac{1}{2}\epsilon^{b'}_b \mu_{ab'} + \mu^{a'b'}_{ab}\mu_{a'b'} \tag{2.67}$$

Inserting expressions (2.64) into the formula (2.29) we obtain

$$\rho = -2kg^2[T(R) - 3C(G)] \tag{2.68}$$

$$\epsilon^a_b = 4kg^2 C(R)\delta^a_b - 2kX^a_b \tag{2.69}$$

while for the other parameters we have

$$\mu^{a'b'}_{ab} = 2kg^2[C(A) + C(B)]\delta^{a'}_a \delta^{b'}_b \tag{2.70}$$

$$f^{a'b'c'}_{abc} = 2kg^2[C(A) + C(B) + C(C)]\delta^{a'}_a \delta^{b'}_b \delta^{c'}_c \tag{2.71}$$

In these last two formulas we can recognize the famous renormalization properties of supersymmetry

$$\mu_{ab} = (Z^{1/2})^{a'}_a (Z^{1/2})^{b'}_b \mu_{a'b'} \tag{2.72}$$

$$f_{abc} = (Z^{1/2})^{a'}_a (Z^{1/2})^{b'}_b (Z^{1/2})^{c'}_c f_{a'b'c'} \tag{2.73}$$

The renormalization constant Z corresponds to the wave function renormalization applied to chiral superfield in the case of unbroken supersymmetry

$$(Z^{1/2})^a_b = \delta^a_b + 4kg^2 C(A)\delta^a_b - kX^a_b \tag{2.74}$$

We note that $Z^{1/2} \neq 1 + \epsilon/2$. This difference is due to the presence of the vector multiplet. We recall here that z^a is just the scalar component of the chiral multiplet and not the chiral field itself. All other parameters will not be multiplicatively renormalized, because of the presence of soft breaking terms. For the masses one gets the corrections

$$\begin{aligned}(\hat{m}^2)^a_b &= (m^2)^a_b - kX^a_c(m^2)^c_b - kX^c_b(m^2)^a_c - 4kf^{ade}f_{bce}(m^2)^c_d \\ &\quad -8k\eta_{(3)acd}\eta^{bcd}_{(3)} + 8kg^2[C(A)\Delta^2_A + C(B)\Delta^2_B]\delta^a_b \\ &\quad +4g^2Tr(T^Am^2)(T^A)^a_b \end{aligned} \tag{2.75}$$

The last term is present only for $U(1)$ group. The function $\eta(z)$ is a polynomial of degree three and every power of fields must be analyzed separately. For the cubic terms we obtain

$$\begin{aligned}\hat{\eta}_{(3)abc} &= \eta_{(3)abc} + 4kg^2[C(A)+C(B)+C(C)]\eta_{(3)abc} \\ &\quad -2k\left[f_{bcd}f^{b'c'd}\eta_{(3)ab'c'} + f_{acd}f^{a'c'd}\eta_{(3)a'bc'} + f_{abd}f^{a'b'd}\eta_{(3)a'b'c}\right] \\ &\quad -k\left[X^{a'}_a\eta_{(3)a'bc} + X^{b'}_b\eta_{(3)ab'c} + X^{c'}_c\eta_{(3)abc'}\right] \\ &\quad +4kg^2[C(A)\Delta_A + C(B)\Delta_B + C(C)\Delta_C]f_{abc} \end{aligned} \tag{2.76}$$

while for the quadratic pieces we have

$$\begin{aligned}\hat{\eta}_{(2)ab} &= \eta_{(2)ab} + 4kg^2[C(A)+C(B)]\eta_{(2)ab} - k[X^{a'}_a\eta_{(2)a'b} + X^{b'}_b\eta_{(2)ab'}] \\ &\quad -2kf_{abc}f^{cde}\eta_{(2)de} - 4kf^{cde}[\eta_{(3)acd}\mu_{eb} + \eta_{(3)acd}\mu_{eb}] \\ &\quad +8kg^2[C(A)\Delta_A + C(B)\Delta_B]\mu_{ab} \end{aligned} \tag{2.77}$$

Finally for the linear terms

$$\begin{aligned}\hat{\eta}_{(1)a} &= \eta_{(1)a} + 4kg^2C(A)\eta_{(1)a} - kX^b_a\eta_{(1)b} - 2kf^{bcd}\mu_{ab}\eta_{(2)cd} \\ &\quad -4k\eta_{(3)abc}\eta^{bc}_{(2)} - 4kf_{abc}\mu^{cd}(m^2)^b_d \end{aligned} \tag{2.78}$$

Now it is easy to obtain the renormalization group equations. To properly renormalize our theory we need to specify a energy scale Q, using the quantity k defined after (2.29). For example, for the gauge coupling we have

$$Q\frac{d}{dQ}g = -\frac{g^3}{16\pi^2}[3C(G) - T(R)] \tag{2.79}$$

which is the usual formula for the one-loop beta function that can be derived with standard field theory techniques, taking into account the content of multiplets. The full set of renormalization group equations is listed below

$$
\begin{aligned}
Q\frac{d}{dQ}f_{abc} &= -\frac{1}{32\pi^2}\Big[4g^2[C(A)+C(B)+C(C)]f_{abc} \\
&\quad -(X_a^{a'}f_{a'bc}+X_b^{b'}f_{ab'c}+X_c^{c'}f_{abc'})\Big] && (2.80)\\
Q\frac{d}{dQ}\mu_{ab} &= -\frac{1}{32\pi^2}\Big[4g^2[C(A)+C(B)]\mu_{ab}-(X_a^{a'}\mu_{a'b}+X_b^{b'}\mu_{ab'})\Big] && (2.81)\\
Q\frac{d}{dQ}(m^2)_b^a &= -\frac{1}{32\pi^2}\Big[-X_c^a(m^2)_b^c-X_b^c(m^2)_c^a-4f^{ade}f_{bce}(m^2)_d^c+ \\
&\quad -8\eta_{(3)acd}\eta_{(3)}^{bcd}+8g^2[C(A)\Delta_A^2+C(B)\Delta_B^2]\delta_b^a \\
&\quad -4g^2Tr(T^Am^2)(T^A)_b^a\Big] && (2.82)\\
Q\frac{d}{dQ}\eta_{(3)abc} &= -\frac{1}{32\pi^2}\Big[+4g^2[C(A)+C(B)+C(C)]\eta_{(3)abc} \\
&\quad -2\left(f_{bcd}f^{b'c'd}\eta_{(3)ab'c'}+f_{acd}f^{a'c'd}\eta_{(3)a'bc'}+f_{abd}f^{a'b'd}\eta_{(3)a'b'c}\right) \\
&\quad -X_a^{a'}\eta_{(3)a'bc}-X_b^{b'}\eta_{(3)ab'c}-X_c^{c'}\eta_{(3)abc'} \\
&\quad +4g^2[C(A)\Delta_A+C(B)\Delta_B+C(C)\Delta_C]f_{abc}\Big] && (2.83)\\
Q\frac{d}{dQ}\eta_{(2)ab} &= -\frac{1}{32\pi^2}\Big[+4g^2[C(A)+C(B)]\eta_{(2)ab}-[X_a^{a'}\eta_{(2)a'b}+X_b^{b'}\eta_{(2)ab'}] \\
&\quad -2f_{abc}f^{cde}\eta_{(2)de}-4f^{cde}[\eta_{(3)acd}\mu_{eb}+\eta_{(3)acd}\mu_{eb}] \\
&\quad +8g^2[C(A)\Delta_A+C(B)\Delta_B]\mu_{ab}\Big] && (2.84)\\
Q\frac{d}{dQ}\eta_{(1)a} &= -\frac{1}{32\pi^2}\Big[+4g^2C(A)\eta_{(1)a}-X_a^b\eta_{(1)b}-2f^{bcd}\mu_{ab}\eta_{(2)cd} \\
&\quad -4\eta_{(3)abc}\eta_{(2)}^{bc}-4f_{abc}\mu^{cd}(m^2)_d^b\Big] && (2.85)
\end{aligned}
$$

To generalize these equations to non simple gauge groups we must simply replace combinations like $g^2[C(A)+C(B)]$ by a sum over all factors in the gauge groups, including also $U(1)$ factors. For every gauge group we will have a gauge coupling constant and the respective group invariant $C(R)$.

2.3 RGEs for the MSSM

In this section we show how to derive the RG equations of section (1.5). In order to do this we recognize that f_{abc} are just the Yukawa couplings y_t, y_b and y_τ. For every fermion only the respective Higgs will be involved:

up-type for the top and down-type for the bottom and the tau. The μ-term is given by the parameter μ_{ab} when $a = H_u$ and $b = H_d$ or vice versa. The soft scalar masses $(m^2)^a_b$ are diagonal: there is a term for every quark and lepton and one for every Higgs. To obtain the equations (1.29)-(1.31) for the trilinear terms attention must be paid. The parameters a_t, a_b and a_τ are given by $2\eta_{(3)abc}$. Similar considerations discussed for the Yukawa couplings hold in this case. The b parameter must be identified with the only non vanishing $\eta_{(2)ab}$, where a and b stand for the up-type Higgs and the down-type or vice versa. As can be seen from the equation (1.6) of the previous chapter, in the case of the MSSM there is no linear term in the superpotential. We note also that there is a change in sign for the gaugino masses: we have $\Delta^A = -M^A_\lambda$ for every gauge group. At this point we need the quadratic Casimir for the various gauge groups. For $SU(N)$ groups the following formula holds

$$C_2(SU(N)) = \frac{N^2-1}{2N} \tag{2.86}$$

We then have a value of 3/4 for $SU(2)$, and 4/3 for $SU(3)$. For $U(1)$ we have simply $C_2 = 3/5\ Y^2$ where Y is the weak hypercharge and the factor 3/5 comes from the normalization in grand unified theories. The value of the hypercharges are $1/6, 2/3, 1/6, -1/3, -1/2, -1, 1/2, -1/2$ for $t, \bar{t}, b, \bar{b}, \tau, \bar{\tau}, H_u, H_d$ respectively. The quantities defined in equation (2.63) are not null only when the two indices are the same. In particular we have for the quarks of the third family

$$X^{\bar{t}}_{\bar{t}} = 4|y_t|^2 \tag{2.87}$$
$$X^{\bar{b}}_{\bar{b}} = 4|y_b|^2 \tag{2.88}$$
$$X^t_t = 2|y_t|^2 + 2|y_b|^2 \tag{2.89}$$

The corresponding X^a_b for b is the same as for t since they belong to the same weak doublet. For the leptons we have

$$X^\tau_\tau = 2|y_\tau|^2 \tag{2.90}$$
$$X^{\bar{\tau}}_{\bar{\tau}} = 4|y_\tau|^2 \tag{2.91}$$

while for the Higgs

$$X^{H_u}_{H_u} = 6|y_t|^2 \tag{2.92}$$
$$X^{H_d}_{H_d} = 6|y_b|^2 + 2|y_\tau|^2 \tag{2.93}$$

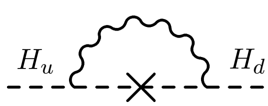
It is now a simple task to retrieve the equations for the MSSM of the previous chapter, taking care of the multiplicities due to color charge.

2.4 Feynman diagrams analysis of the soft terms

In the previous section we obtained all the renormalization group equations. Here we want to translate those equations in terms of Feynman diagrams. This will prove useful when we will deal with higher dimensional theory to manage soft terms in presence of Kaluza-Klein particles.

We will not show a generic treatment of the renormalization for all the parameters, but we will limit to the most interesting case of the MSSM. In section 1.1 there are all the vertices of the supersymmetric extension of the Standard Model. From those vertices we can construct all the diagrams that renormalize the soft parameters.

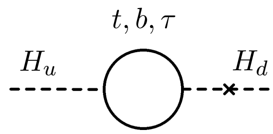
Let's start with the b parameter which plays a crucial role in the electroweak breaking sector. Looking at the equation (2.84) we note that there are five different kind of terms. However, the first on the second line involving two f_{abc} and $\eta_{(2)ab}$ doesn't give any contribution in the case of the MSSM. Indeed, the only non vanishing $\eta_{(2)ab}$ is the one where both a and b stand for Higgs fields; since there is no Yukawa coupling with two Higgs fields, we conclude that this term is not present. Thus we are left with four kind of diagrams. Every diagram must be logarithmically divergent and the nature of the couplings will suggest us which interaction vertex must be take into account. The first terms in (2.84) contain the square of the gauge coupling and a Higgs mass insertion. We can immediately recognize that the corresponding diagram is



$$b(-3g_2^2 - \frac{3}{5}g_1^2) \qquad (2.94)$$

One propagator in the loop is the one of the gauge boson, which can belong to the $SU(2)$ or the $U(1)$ gauge group, since the Higgs are colorless. The other propagator is the Higgs one with a "mass" insertion. In four dimensions both the propagators carries a factor of $1/p^2$, where p is the internal momentum. In the limit of large p, the integrand goes like $1/p^4$. Then, we have a logarithmically divergent integral, as we expected.

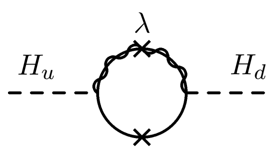
In (2.84) there is another piece proportional to the b parameter and it comes form the following diagram



$$b(3y_t^2 + 3y_b^2 + y_\tau^2) \tag{2.95}$$

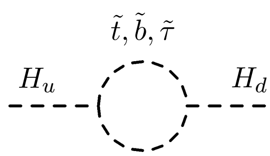
Clearly, there is another diagram in which the b (mass) insertion stays on the other external leg. Both the contributions are included in (2.95). The particles which circulate in the loop are quarks and leptons. Since we work in the third family approximation only the bottom, top and tau will give rise to significant contribution. Even if at first sight this diagram might seem quadratically divergent (two fermion propagators which carry a factor $1/p$ each), it is actually logarithmic divergent, thanks to gauge invariance.

The other two diagrams responsible for the renormalization of b are proportional to the soft term μ. The first involves a gaugino and its mass insertion



$$\mu(6g_2^2 M_2 + \frac{6}{5} g_1^2 M_1) \tag{2.96}$$

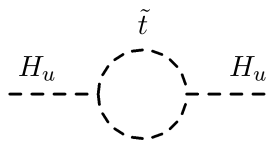
Note that this diagram is related to the diagram (2.94). The four fermion propagators provide a logarithmically divergent integral even in this case. The second diagram involving the μ term contains three-scalar vertices



$$2\mu(3a_t y_t + 3a_b y_b + a_\tau y_\tau) \tag{2.97}$$

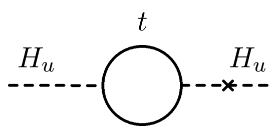
However, the nature of the two vertices is quite different. One comes directly from the trilinear couplings a_i, while the other is due to the superpotential (μy_i). Again, two scalar propagators give rise to $1/p^4$ in the loop integral. As usual, only squarks and sleptons of the third family which can circulate in the loop. Putting together all these pieces (2.94)-(2.97), we easily retrieve the equation (1.32) of chapter 1.

We now move to discuss the renormalization of the Higgs masses. The equation (2.82) contains a term proportional to the trilinear couplings. The diagram responsible for this term is



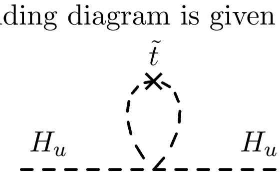
$$6a_t^2 \tag{2.98}$$

In (2.98) we show the case of the up-type Higgs where only the stop enters in the loop. In the diagram for the down-type, instead, both the sbottom and the stau can circulate. In the first line of (2.82) all the terms contain the soft squared masses. The first two are proportional to the Higgs mass itself since the quantity X_b^a is always diagonal in the MSSM. These terms correspond to a mass insertion in the external leg



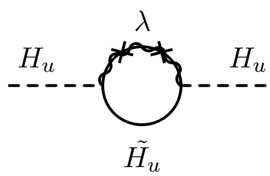
$$6y_t^2 m_{H_u}^2 \tag{2.99}$$

Clearly, the mass insertion can be put the same way on the other external leg. As explained above, in the case of the up-type Higgs only the top can circulate in the loop, while for the down-type Higgs both the bottom and the tau can circulate. The last term in the first line of (2.82) involves mass insertions within the loop and it is proportional to the Yukawa coupling. The corresponding diagram is given by



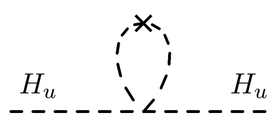
$$6y_t^2(m_t^2 + m_{\tilde{t}}^2) \tag{2.100}$$

The quartic vertex needed for the loop comes from the superpotential; thus it is proportional to the Yukawa couplings. There are similar diagrams for the down-type Higgs: we need only to substitute the top squark with the sbottom and stau particles. In the second line of (2.82) there are pieces containing the gaugino masses. The corresponding diagram is



$$-\left(\frac{6}{5}g_1^2M_1^2 + 6g_2^2M_2^2\right) \tag{2.101}$$

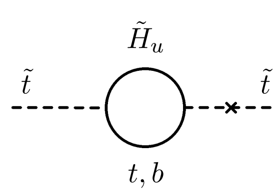
Actually we have two gaugino mass insertions within the loop. This guarantees the presence of four fermionic propagators which give rise to a logarithmically divergent diagram, as it should be. The diagram with the down-type Higgs gives the same contribution. Finally the last line in (2.82) is present only for the $U(1)$ gauge group



$$\frac{3}{5}g_1^2Tr(Ym^2) \tag{2.102}$$

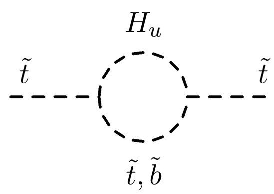
In this case the quartic interaction comes from the presence of the gauge bosons in the kinetic terms of the various particles. Indeed, the trace $Tr(Ym^2)$ defined in paragraph 1.5 is over all the matter states and the Higgses as well. The contribution of the corresponding diagram for the down-type is opposite with respect to (2.102). Collecting the terms (2.98)-(2.102) we obtain the equations (1.38) and (1.39).

The analysis for the squark and slepton soft squared masses is much the same as for the Higgs. Indeed the renormalization are governed by the same equation (2.82). We will present the diagrams for the stop, following the same order we used for the Higgs mass. The first diagram is



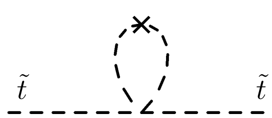
$$(2y_t^2 + 2y_b^2)m_t^2 \tag{2.103}$$

This diagram is valid for the $SU(2)$ left doublet. For the singlets every diagram will contribute with its own Yukawa. For the sbottom, as for the stau, we must substitute the up-type Higgs with the down-type one. In every case, the term turns out to be proportional to the mass of the particle involved. The second diagram which is built from three-scalar vertices is



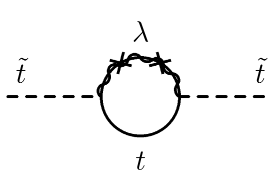
$$2(a_t^2 + a_b^2) \tag{2.104}$$

Similar arguments discussed for the diagram (2.103) hold for this diagram too. The diagram proportional to the Yukawa couplings is instead



$$2y_t^2(m_{H_u}^2 + m_{\tilde{t}}^2) + 2y_b^2(m_{H_d}^2 + m_{\tilde{b}}^2) \tag{2.105}$$

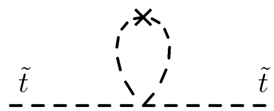
The quartic vertex comes directly from the superpotential. In the loop both the Higgs can circulate. The diagram with two gaugino mass insertions for squarks and sleptons is



$$-\left(\frac{32}{3}g_3^2M_3^2 + 6g_2^2M_2^2 + \frac{2}{15}g_1^2M_1^2\right) \tag{2.106}$$

We must take care of the Casimir under the full gauge group. For the stau there is no term proportional to g_3, while there is no g_2 for the singlets.

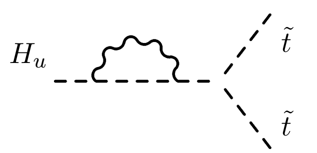
The last diagram is present only for $U(1)$ gauge group.



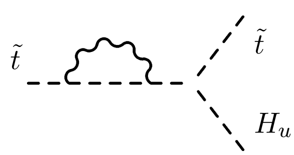
$$\frac{1}{5} g_1^2 Tr(Y m^2) \tag{2.107}$$

Note the different factor with respect to (2.102). This resembles the hypercharge of the various particles. Combining all the diagrams (2.103)-(2.107) we arrive at the formula (1.40). The other formulas (1.41)-(1.44) can be derived in the same way.

Finally we analyze the diagrams which renormalize the trilinear couplings. First we look for those terms in (2.83) which are proportional to the gauge coupling and to the trilinear coupling itself. The relative diagrams are



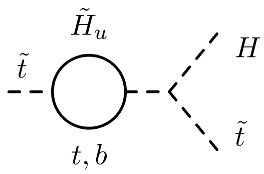
$$-a_t \left(\frac{3}{10} g_1^2 + \frac{3}{2} g_2^2 \right) \tag{2.108}$$



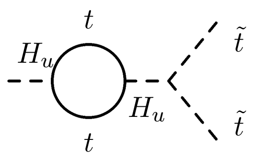
$$-a_t \left(\frac{17}{30} g_1^2 + \frac{3}{2} g_2^2 + \frac{16}{3} g_3^2 \right) \tag{2.109}$$

It is easy to see that on the left we have one of the diagram which renormalizes the wave function of the Higgs or the top. This "subdiagram" is proportional to the square of the gauge coupling: for the Higgs clearly we have no term regarding $SU(3)$. Attached to this subdiagram there is a three-scalar vertex, precisely the one appearing at tree level. For the other trilinear couplings we must substitute the up-type Higgs with the down-type one.

There are other two diagrams which are connected to those diagrams before



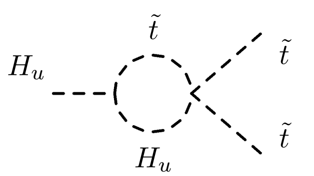
$$a_t(3y_t^2 + y_b^2) \tag{2.110}$$



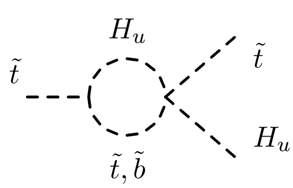
$$3a_t y_t^2 \tag{2.111}$$

Indeed we simply replace the subdiagram on the left with the other responsible for the renormalization of the wave functions. We must take into account that the trilinear coupling involve a squark and its complex conjugate. This explains the factor y_b^2 in the first diagram. Of course there are the corresponding diagram involving the sbottom and the stau.

There are other two diagrams which are proportional to the trilinear coupling and to the Yukawa couplings



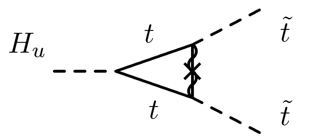
$$6a_t y_t^2 \tag{2.112}$$



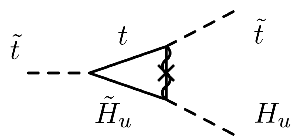
$$6a_t y_t^2 + 2a_b y_t y_b \tag{2.113}$$

The quartic interactions come from the superpotential and thus their strengths are given by the product of two Yukawas. In the first case we have only stop and up-type Higgs, while in the other case we must take into account the complex nature of the squark. When there is a three-scalar vertex with the sbottom, the quartic coupling includes also the sbottom.

Lastly, we have two diagrams proportional to the gaugino masses



$$-y_t\left(\frac{16}{3}g_3^2M_3+3g_2^2M_2+\frac{13}{15}g_1^2M_1\right) \quad (2.114)$$



$$-y_t\left(\frac{16}{3}g_3^2M_3+3g_2^2M_2+\frac{13}{15}g_1^2M_1\right) \quad (2.115)$$

In this case one of the internal line in the triangle loop is a gaugino with its mass insertion. On the left we can note the Yukawa vertex. The difference between the two is just a reshuffle of the external legs. Again, putting together all the terms (2.108)-(2.115) we find the equation (1.29). For the sbottom and the stau, using the same arguments one can obtain the equations (1.30) and (1.31) respectively.

Chapter 3

Extra dimensions

It is possible to incorporate extra spacetime dimensions into a field-theoretic analysis of the MSSM. In this case, there will be the appearance of Kaluza-Klein modes and this leads to the resulting lack of renormalizability that afflicts higher-dimensional field theories, where the couplings are not dimensionless. This kind of scenario is called a bottom-up approach: one starts from a four dimensional theory and then adds extra dimensions in such a way to reproduce the physical observables measured in the actual experiments. This is in contrast with the top-down approach which starts from a higher dimensional theory to fall to our world. This is what string theory tries to do, even if a phenomenologically interesting model is not available so far. In the bottom-up, there is a "minimal" scenario that consistently embeds the MSSM into higher dimensions. In the future, this model would be derived from a more complete high-energy theory (perhaps string theory).

3.1 Kaluza-Klein model

Actual experiments show that our world consists of only four flat dimensions. It is thus clear that the only way to discuss extra spacetime dimensions is to assume that they are little enough. Then, they would be invisible at the energies accessible so far. Let us study in detail the case of extra dimensions compactified on a circle of fixed radius R. Extra dimensions become important at the energy scale $\mu_0 \equiv R^{-1}$. Here we review the simple case of a complex scalar field $\Phi(x)$. This field depends on the usual four dimensional spacetime $\mathbf{x} \equiv (x_0, x_1, x_2, x_3)$, but also on the additional (space) coordinates $\mathbf{y} \equiv (y_1, y_2, \cdots, y_\delta)$ where $\delta = D-4$ is the number of extra dimensions. Since we are compactifying on a circle we must demand periodicity of $\Phi(x)$ under

$$y_i \to y_i + 2\pi R \tag{3.1}$$

which implies that $\Phi(x)$ takes the form

$$\Phi(x) = \sum_{n_i=-\infty}^{+\infty} \Phi^{(n)}(x) \exp(i\mathbf{n} \cdot \mathbf{y}/R) \tag{3.2}$$

where $\mathbf{n} \equiv (n_1, n_2, \cdots, n_\delta)$ with $n_i \in \mathbb{Z}$. The four dimensional fields $\Phi^{(n)}(x)$ are called Kaluza-Klein (KK) modes, and n_i are the corresponding Kaluza-Klein excitation numbers. The mass of each KK mode can be inferred substituting the expansion (3.2) into the equations of motion. This gives

$$m_n^2 = m_0^2 + \frac{\mathbf{n} \cdot \mathbf{n}}{R^2} \tag{3.3}$$

where m_0 is the mass of the zero-mode. At energies below R^{-1} only the zero-mode can be observed: it corresponds to the usual four dimensional state. Raising the energy, extra dimensions become significant through the appearance of an infinite tower of associated KK states. Their mass increases with the KK number. As explained before, we must assumed that R^{-1} is much greater than the observable energies. Then the mass of the zero-mode m_0 will be completely negligible.

All the massive KK states look like copies of the zero-mode state. However, in the case of the MSSM, KK states must fall into given representation. Then it turns out that not every state can have KK excitations. Indeed, chiral MSSM state alone cannot form a KK mass; we must introduce its chiral conjugate mirror.

3.2 Extra dimensions in the MSSM

In this section we follow the notation of Dienes and collaborators [1, 19]. We denote by η the number of generations of MSSM chiral fermions that will have KK excitations. Of course, the simplest scenario is the one with $\eta = 0$, i.e. no chiral MSSM fermions have KK excitations. Thus, in this scenario, gauge bosons and Higgs fields will be the only fields with KK excitations, while the quark and lepton representations will not have KK excitations. This scenario is said the "minimal scenario". In the following we will also consider the other cases in which $\eta = 1, 2, 3$. This scenarios are perfectly consistent with the framework of string theory, as we shall see in the following.

We now want to consider the spectrum of KK excitations for the various kind of particles that appear in the MSSM. Let's start with the Higgs fields. We saw in section 1 that the MSSM contains two kind of Higgs fields, thus for each KK mass level there will be two massive chiral $N = 1$ supermultiplets $H_{u,d}$. In [1] it was convenient to assemble the two $N = 1$ Higgs chiral fields into an $N = 2$ hypermultiplet. To be invariant under $N = 2$ SUSY, besides the kinetic terms for the vector multiplet and the hypermultiplet the lagrangian must also contain an interaction term of the following form

$$\int d^2\theta \left(\sqrt{2} H_u \Phi H_d + \mu H_u H_d\right) + h.c. \tag{3.4}$$

where μ plays the role of the usual μ-term appearing in the MSSM, while Φ is the chiral multiplet contained in the $N = 2$ vector multiplet. These are the only admissible terms for an $N = 2$ theory. The first term is a coupling between the hypermultiplet and the vector multiplet, and it is needed to enlarge the amount of supersymmetry. To check how robust the scenario is with respect to the presence of this term, we have studied two different cases which we call Higgs $N = 1$ and $N = 2$. In the former case we assume that the KK states behave as their zero-modes and that the first term in (3.4) is absent. In the latter case this term is switched on.

We then pass to discuss gauge bosons. A massless gauge boson is an $N = 1$ vector supermultiplet. On the contrary, a massive gauge boson is represented by a massive $N = 1$ vector supermultiplet. This is equivalent to join an $N = 1$ massless vector supermultiplet $\mathcal{A} \equiv (A, \lambda)$ with an $N = 1$ chiral supermultiplet $\mathcal{A}' \equiv (\phi, \psi)$. Taken together, these form a massive $N = 2$ vector supermultiplet:

$$V^{(n)} = \begin{pmatrix} A^{(n)} & \phi^{(n)} \\ \lambda^{(n)} & \psi^{(n)} \end{pmatrix} \tag{3.5}$$

The chiral supermultiplet contains two real fields (or a complex one). One of the real field of $\mathcal{A}'$ represents the longitudinal component of the massive gauge boson, while the other field and the Weyl fermion remain massive. Thus, these KK towers of states are effectively $N = 2$ supersymmetric. This is true for $\delta = 1$ or 2, i.e. for five or six dimensions. For higher dimensionality the situation can be different. In general the minimal number of supercharge grows with the dimension [1].

Finally, in the cases $\eta \geq 1$ some of the chiral fermions have KK excitations and these excitations will have the form

$$F^{(n)} = \begin{pmatrix} \phi_1^{(n)} & \phi_2^{(n)} \\ \psi_1^{(n)} & \psi_2^{(n)} \end{pmatrix} \tag{3.6}$$

where $\mathcal{F}_1^{(n)} \equiv (\phi_1^{(n)}, \psi_1^{(n)})$ are the KK excitations of the MSSM fermions and $\mathcal{F}_2^{(n)} \equiv (\phi_2^{(n)}, \psi_2^{(n)})$ are the corresponding KK excitations of the mirrors. Taken together they form an $N = 2$ hypermultiplet.

It will not be necessary to use the $N = 2$ formalism to describe these towers of KK states. What we must bear in mind is that we shall give KK excitations to the gauge bosons, to the Higgs fields, and to only η generations of the MSSM fermions. The effects of extra dimensions are constrained to this subset of the MSSM spectrum.

There are two important points. The first is how to avoid KK excitations only for some of the fields. The second concerns the fact that starting from a higher dimensional theory, we must be faced with a extended four-dimensional theory, i.e. a theory with at least $N = 2$ supersymmetry. To explain both the points we discuss the simplest case. In five dimensions, as in every odd dimension, it is indeed impossible to define a chiral projector. Then, there is no Weyl fermion at all. Constructing all the matrices needed for the Clifford algebra, and in particular the conjugacy matrix, it is even possible to demonstrate that there are no Majorana fermions in five dimensions. A Dirac charge will split into two four-dimensional Weyl fermions giving rise to a $N = 2$ theory. For higher dimensions the situation is much the same. Then, how can we have KK towers of gauge bosons falling into $N = 2$ representations, while their zero-modes are only $N = 1$ supersymmetric?

At this end, let us consider the case of a single additional dimension. The spacetime now consists of $\mathbf{x} \equiv (x_0, x_1, x_2, x_3)$ and the extra y coordinate.

[1] This is due to the nature of the spinor fields, whose number of components grows with the dimension.

The complex scalar field can be written as $\Phi(x) = \Phi_+(x) + i\Phi_-(x)$, where

$$\begin{aligned} \Phi_+(x,y) = & \sum_{n=0}^{\infty}[\Phi^{(n)}(x) + \Phi^{(-n)}(x)] \cos(ny/R) \\ \Phi_-(x,y) = & \sum_{n=0}^{\infty}[\Phi^{(n)}(x) - \Phi^{(-n)}(x)] \sin(ny/R) \end{aligned} \tag{3.7}$$

Since $\Phi(x)$ is a complex field, in general even $\Phi_\pm(x)$ are complex. However it is possible to distinguish between Φ_+ and Φ_- through their properties under the $\mathbb{Z}_2$ transformation $y \to -y$. In particular we have

$$\Phi_+(x,-y) = +\Phi_+(x,y) \tag{3.8}$$

$$\Phi_-(x,-y) = -\Phi_-(x,y) \tag{3.9}$$

The decomposition (3.7) has a special property: Φ_- lacks a zero-mode. When we compactify a $N = 1$ theory from five to four dimensions, we end up with a $N = 2$ theory. However, the MSSM is only $N = 1$ supersymmetric. If a field of a given multiplet transforms as an odd function under reflection of the extra coordinate, then the zero-modes are only $N = 1$ supersymmetric even though the KK tower is $N = 2$.

By demanding specific properties for our wave function, we are actually not compactified on a circle. The identification $y \approx -y$ changes the circle into a so-called $\mathbb{Z}_2$ orbifold. This is a simple example of a quotient manifold that is natural from the string theory point of view. The compactification on a orbifold explains how to have $3-\eta$ generations with no KK excitations. In our orbifold there are two particular points: $y = 0$ and $y = \pi R$, which are invariant under the action $y \to -y$. Such special points are called fixed points of the orbifold. The presence of these points changes the expansion of the fields, which now takes the form

$$\Phi(x) = \Phi^{A}(\mathbf{x})\delta(y) + \Phi^{(B)}(\mathbf{x})\delta(y - \pi R) \tag{3.10}$$

Such mode expansion respects the symmetries of the orbifold. However, this expansion does not give rise to infinite KK towers, because these states exist only on the fixed points. In this way a given MSSM fermion will have no KK tower simply by requiring that it only sits on the fixed points. In string theory such states are called twisted states, because of the twisting caused by the orbifold action [2]. Summarizing, we have a higher dimensional theory

[2] These considerations apply to closed string theory. However, for open string, D-branes allow a different mechanism.

consisting of four flat dimensions and a certain number of extra dimensions compactified on orbifolds. Our zero-modes will consist of the usual MSSM particle. For what concern the higher KK mode, we will have infinite towers for the gauge bosons and the Higgs fields, while for the chiral sector we will have η generations with KK excitations. The remaining $3-\eta$ generations will not have KK states and they will be restricted to fixed points.

Let us summarize our framework. The higher-dimensional MSSM is described in terms of a spacetime consisting of four flat dimensions and some extra dimensions compactified on orbifolds of radius R. The massless zero-mode states form the particle content of the MSSM. The higher levels consist of an infinite towers of KK states associated with the Higgs field, the gauge-boson states, and η generations of the chiral MSSM fermions. The variable η can assume the values $0, 1, 2$ or 3. The remaining $3-\eta$ generations instead lie on the fixed points of the orbifold and thus they don't have KK states.

The non minimal scenarios will be understood in the following way. We distinguish three different cases: $\eta = 1$ in which only the third family has a KK tower, $\eta = 2$ in which only the first two families have a KK tower and $\eta = 3$ in which all the families have KK tower. This choice is dictated by the usual third family approximation (see section 1) and by a constraint imposed by the ISASUGRA code [20] concerning the mass degeneracy of the first two families.

3.3 “Running” of the gauge couplings

The non-chiral sector of the MSSM, gauge and Higgs bosons, live in D flat spacetime dimensions, where $D = 4+\delta$ and δ is the number of extra dimensions. This spacetime consists of four flat dimensions and δ circles of fixed radius $R = \mu_0^{-1}$ which is actually different from a flat $(4+\delta)$-dimensional spacetime. Increasing the energy scale μ is equivalent to decrease the effective length which will become much shorter than the radius R. From a purely four-dimensional point of view we can evaluate the vacuum polarization diagram including the effects of the MSSM particles as well as the KK excitations circulating in the loops. For pedagogical reason, we will consider the most simple case of a single Dirac fermion, compactified on a single extra dimension ($\delta = 1$). The KK excitations will contribute the same as the zero-modes for each particle.

For a single Dirac fermion with KK excitations, the vacuum polarization diagram is given by

$$\Pi_{\mu\nu}(k) = -\sum_{n_i=-\infty}^{\infty} g^2 \int_0^\infty \frac{d^4q}{(2\pi)^4} Tr\left(\gamma_\mu \frac{1}{\not{q} - m_n}\gamma_\nu \frac{1}{\not{k} + \not{q} - m_n}\right) \quad (3.11)$$

We took the ground state mass to be zero for simplicity. Indeed, the zero-mode mass in the MSSM is of the order of 100 GeV, while the compactification scale will be taken greater than 10^5 GeV and thus all the MSSM masses can be neglected. The term in equation (3.11) with $n_i = 0$ amounts to consider only the original fermionic state without its KK excitations. Opposite values of n_i correspond instead to momenta (in the extra dimension) which have opposite directions. The minus sign comes from the fermionic nature of the loop. All our calculations will suppose a suitable ultraviolet regulator with cutoff Λ.

Making use of gauge invariance

$$\Pi_{\mu\nu}(k) = \left(k_\mu k_\nu - g_{\mu\nu}k^2\right)\Pi(k^2) \quad (3.12)$$

and contracting the Lorentz indices we can write

$$\Pi(k^2) = -\frac{8g^2}{3k^2}\sum_{n_i=-\infty}^{\infty}\int_0^\infty \frac{d^4q}{(2\pi)^4}\left\{\frac{-(k+q)\cdot q + 2m_n^2}{(q^2-m_n^2)[(k+q)^2-m_n^2]}\right\} \quad (3.13)$$

To evaluate the integral we must now pass to euclidean momenta. As usual we combine the propagators by means of the Feynman x parameter

$$\frac{1}{AB} = \int_0^1 dx[Ax + B(1-x)]^{-2} \quad (3.14)$$

The terms in the integrand that are odd in q vanish and thus only terms that are even in q will contribute, yielding

$$\Pi(k^2) = -\frac{8g^2}{3k^2}\sum_{n_i=-\infty}^{\infty}\int_0^1 dx \int_0^\infty \frac{d^4q}{(2\pi)^4}\left\{\frac{q^2 - x(1-x)k^2 + 2m_n^2}{[q^2 + x(1-x)k^2 + m_n^2)]^2}\right\} \quad (3.15)$$

We can rewrite this expression in terms of a Schwinger parameter t using the following identity

$$\frac{1}{A^2} = \int_0^\infty dt\ t\ e^{-At} \quad (3.16)$$

This yields

$$\begin{aligned}\Pi(k^2) \quad = \quad & -\frac{8g^2}{3k^2}\sum_{n_i=-\infty}^{\infty}\int_0^1 dx\int_0^\infty dt\ t\int_0^\infty \frac{d^4q}{(2\pi)^4}[q^2 - x(1-x)k^2 + 2m_n^2] \\ & \times \exp\left\{-t[q^2 + x(1-x)k^2 + m_n^2]\right\} \end{aligned} \quad (3.17)$$

We make use of the following identities

$$\int_0^\infty \frac{d^4q}{(2\pi)^4} e^{-tq^2} = \frac{1}{16\pi^2 t^2}\ , \qquad \int_0^\infty \frac{d^4q}{(2\pi)^4} q^2 e^{-tq^2} = \frac{1}{8\pi^2 t^3} \tag{3.18}$$

to obtain

$$\begin{aligned}\Pi(k^2) \quad &= \quad -\frac{g^2}{6\pi^2 k^2} \sum_{n_i=-\infty}^{\infty} \int_0^1 dx \int_0^\infty \frac{dt}{t} \left[\frac{2}{t} - x(1-x)k^2 + 2m_n^2\right] \\ &\qquad \times \exp\left\{-t[x(1-x)k^2 + m_n^2]\right\}\end{aligned} \tag{3.19}$$

After integration by parts we have

$$\Pi(k^2) = \frac{g^2}{2\pi^2} \sum_{n_i=-\infty}^{\infty} \int_0^1 dx\ x(1-x) \int_0^\infty \frac{dt}{t} \exp\left\{-t[x(1-x)k^2 + m_n^2]\right\} \tag{3.20}$$

We can now express the sum over KK states appearing in (3.20). We first recall the definition of the Jacobi θ_3 function:

$$\theta_3(\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2) \tag{3.21}$$

where τ is a complex number. By means of the Poisson resummation formula we can infer a nice property of this function that will be useful in the following

$$\theta_3(-1/\tau) = \sqrt{-i\tau}\,\theta_3(\tau) \tag{3.22}$$

In order to avoid polidromy, one chooses the branch of the square root with non negative real part. The expression (3.20) can be put in terms of the Jacobi function as

$$\Pi(k^2) = \frac{g^2}{2\pi^2} \int_0^1 dx\ x(1-x) \int_0^\infty \frac{dt}{t} e^{-tx(1-x)k^2} \left\{\theta_3\left(\frac{it}{\pi R^2}\right)\right\}^\delta \tag{3.23}$$

In conclusion, we find that $\Pi(0)$ is given by

$$\Pi(0) = \frac{g^2}{12\pi^2} \int_0^\infty \frac{dt}{t} \left\{\theta_3\left(\frac{it}{\pi R^2}\right)\right\}^\delta \tag{3.24}$$

This expression is clearly divergent. Actually there are problems both in the lower limit as in the upper one. To regulate (3.24) we must introduce both infrared and ultraviolet regulators in order to eliminate these divergences. To make (3.24) finite, we modify the integral in the following way

$$\int_0^\infty dt \rightarrow \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} dt \tag{3.25}$$

Here Λ is a ultraviolet cutoff, while μ_0 is the infrared one. The numerical coefficient r is defined as

$$r \equiv \pi (X_\delta)^{-2/\delta} \tag{3.26}$$

The quantities X_δ and r cannot be deduced in the framework of a non-renormalizable theory. We will discuss this issue in the next chapter. However here we anticipate that X_δ is just the volume of a δ-dimensional sphere of unitary radius and take into account the number of states included in this region.

In the limit $R \rightarrow 0$ with only one extra dimension, we retrieve the usual result. Indeed, in this case we can safely substitute θ_3 with one to find

$$\Pi(0) = \frac{g^2}{6\pi^2} \ln \frac{\Lambda}{\mu_0} = \frac{g^2 b}{8\pi^2} \ln \frac{\Lambda}{\mu_0} \tag{3.27}$$

where $b = 4/3$ is the beta-function for a single Dirac fermion (2/3 is for a Weyl one). We can generalize the formula (3.24) to the full MSSM. At this end we denote by $\tilde{b}_i$ the beta-function coefficients for the KK modes but the zero-mode. We find

$$\begin{aligned} \Pi(0) &= \frac{g_i^2 b_i}{8\pi^2} \ln \frac{\Lambda}{\mu_0} + \frac{g_i^2 \tilde{b}_i}{16\pi^2} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{dt}{t} \left\{ \left[\theta_3 \left(\frac{it}{\pi R^2} \right) \right]^\delta - 1 \right\} \\ &= \frac{g_i^2 (b_i - \tilde{b}_i)}{8\pi^2} \ln \frac{\Lambda}{\mu_0} + \frac{g_i^2 \tilde{b}_i}{16\pi^2} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{dt}{t} \left\{ \theta_3 \left(\frac{it}{\pi R^2} \right)^\delta \right\} \end{aligned} \tag{3.28}$$

In the first line we put in evidence the term coming from the zero-mode; since the theta function contains this contribution also, we must subtract it in the integral. This term can be easily integrated and gives rise to the first term in the second line of (3.28). This results in a modified beta-function coefficients, while the contribution of the KK modes is given by the second term.

From (3.28) we can easily infer

$$\alpha_i^{-1}(\Lambda) = \alpha_i^{-1}(\mu_0) - \frac{b_i - \tilde{b}_i}{2\pi} \ln \left(\frac{\Lambda}{\mu_0} \right) - \frac{\tilde{b}_i}{4\pi} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{dt}{t} \left\{ \theta_3 \left(\frac{it}{\pi R^2} \right) \right\}^\delta \tag{3.29}$$

This result is exact in the case of δ extra dimensions compactified on circles of equal radii R. This is also valid for any mass scales Λ and μ_0; only at the end we will identify the scale μ_0 with R^{-1}. The presence of KK modes is incorporated in the θ_3 function.

Let us now suppose that both Λ and μ_0 are greater than R^{-1}. In this case we have $t/R^2 \ll 1$ and the θ_3 function can be approximated by means of (3.22)

$$\theta_3\left(\frac{it}{\pi R^2}\right) \approx R\sqrt{\frac{\pi}{t}} \tag{3.30}$$

If we insert this approximation in (3.29) we obtain

$$\alpha_i^{-1}(\Lambda) = \alpha_i^{-1}(\mu_0) - \frac{b_i - \tilde{b}_i}{2\pi} \ln\left(\frac{\Lambda}{\mu_0}\right) - \frac{\tilde{b}_i X_\delta}{2\pi\delta} R^\delta \left(\Lambda^\delta - \mu_0^\delta\right) \tag{3.31}$$

Finally, if we identify μ_0 with R^{-1} we have

$$\alpha_i^{-1}(\Lambda) = \alpha_i^{-1}(\mu_0) - \frac{b_i - \tilde{b}_i}{2\pi} \ln\left(\frac{\Lambda}{\mu_0}\right) - \frac{\tilde{b}_i X_\delta}{2\pi\delta} \left[\left(\frac{\Lambda}{\mu_0}\right)^\delta - 1\right] \tag{3.32}$$

This is the relation we were searching for. We conclude with a comment about the approximation that $\mu_0 \gg R^{-1}$. At the end we identified μ_0 with R^{-1}. It is not possible to manage in an analytical way integrand of theta functions. However numerical results show that actually the equation (3.32) holds with good approximation [3].

3.4 More on power-law behavior

In this section we will further analyze the approximations adopted in the previous section [21]. We start by rewriting the evolution of α_i

$$\alpha_i^{-1}(\mu_0) = \alpha_i^{-1}(\Lambda) + \frac{b_i - \tilde{b}_i}{2\pi} \ln\left(\frac{\Lambda}{\mu_0}\right) + \frac{\tilde{b}_i}{4\pi} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{dt}{t} \left\{\theta_3\left(\frac{it}{\pi R^2}\right)\right\}^\delta \tag{3.33}$$

Assuming $\Lambda \gg \mu_0$ we can deduce the numerical factor r appearing in the integral. We can compare the limit with the usual renormalization group analysis once we have decoupled all the excited states with masses above Λ; we also assume that the number of KK states below a certain scale μ

[3] See [1] for details.

between μ_0 and Λ is well approximated by the volume of a δ-dimensional sphere of radius μ/μ_0

$$N(\mu, \mu_0) = X_\delta \left(\frac{\mu}{\mu_0}\right)^\delta \tag{3.34}$$

where X_δ is given by

$$X_\delta = \frac{\pi^{\delta/2}}{\Gamma(1+\delta/2)} \tag{3.35}$$

Here Γ is the Euler gamma function satisfying $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1) = 1$. Thus, as expected we find $X_0 = 1$. For higher values of δ we have $X_1 = 2$, $X_2 = \pi$, $X_3 = 4\pi/3$, and so forth. The result is then a power-law behavior of the gauge coupling constants given by

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(\mu_0) - \frac{b_i - \tilde{b}_i}{2\pi} \ln\left(\frac{\mu}{\mu_0}\right) - \frac{\tilde{b}_i}{2\pi}\frac{X_\delta}{\delta}\left[\left(\frac{\mu}{\mu_0}\right)^\delta - 1\right] \tag{3.36}$$

When the mass of the zero-mode can be neglected, which is the most common case, the mass of a KK mode is well approximated by

$$\mu_n^2 = \mu_0^2 \sum_{i=1}^{\delta} n_i^2 \tag{3.37}$$

At each mass level μ_n, the number of KK modes is given by the solutions of the equation (3.37). In the case of a single extra dimension, each KK level will have two KK states of opposite KK number. On the contrary, the zero-mode is not degenerate and corresponds to a particle of the MSSM. In higher extra dimensions the KK level are not equally spaced and the spectra contains the excited levels with energy in the δ-dimensional box. Let us consider in detail the case of a single extra dimension. The one-loop renormalization group equations for energies just above the n-th level ($\mu > n\mu_0$) are

$$\frac{d}{d\ln\mu}\alpha_i^{-1} = -\frac{b_i + 2n\tilde{b}_i}{2\pi} \tag{3.38}$$

This formula can be easily understood taking into account that all the low energy particles contribute through b_i (which include the zero-modes) and all the excited states in the first n KK levels contribute twice. The boundary conditions imply that

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(n\mu_0) - \frac{b_i + 2n\tilde{b}_i}{2\pi} \ln\left(\frac{\mu}{n\mu_0}\right) \tag{3.39}$$

With a similar argument we obtain

$$\alpha_i^{-1}(n\mu_0) = \alpha_i^{-1}((n-1)\mu_0) - \frac{b_i + 2(n-1)\tilde{b}_i}{2\pi} \ln\left(\frac{n}{n-1}\right) \tag{3.40}$$

and so on, up to

$$\alpha_i^{-1}(2\mu_0) = \alpha_i^{-1}(\mu_0) - \frac{b_i + 2\tilde{b}_i}{2\pi} \ln 2 \tag{3.41}$$

Combining the equations together we finally get

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(\mu_0) - \frac{b_i}{2\pi} \ln\left(\frac{\mu}{\mu_0}\right) - \frac{\tilde{b}_i}{2\pi} \cdot 2\left[n \ln\left(\frac{\mu}{\mu_0}\right) - \ln n!\right] \tag{3.42}$$

This equation shows a logarithmic behavior corrected by the presence of n threshold states below μ.

In the limit of large n we can make use of Stirling's formula

$$n! \approx n^n e^{-n}\sqrt{2\pi n} \tag{3.43}$$

The expression (3.42) then takes the form of a power-law running

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(\mu_0) - \frac{b_i}{2\pi} \ln\left(\frac{\mu}{\mu_0}\right) - \frac{\tilde{b}_i}{2\pi} \cdot 2\left[\left(\frac{\mu}{\mu_0}\right) - \ln\sqrt{2\pi}\right] \tag{3.44}$$

The term inside the square brackets is $\ln\sqrt{2\pi} \approx 0.9189$. Thus we recover the usual result (3.32) with a good approximation. The small discrepancy could be corrected by high energy thresholds or second order corrections. Now we can move to the case of higher dimensions. In these case each level is characterized by a set of numbers $n_1, ..., n_\delta$ which satisfy equation (3.37). While the zero-mode is unique, each KK level is $(2^\delta \delta!)$-fold degenerate. Indeed there are $\delta!$ ways of distributing these δ (absolute) values between the δ numbers n_i, and there are 2^δ different combinations of the signs for each one of the combinations. However, we must take into account that some of these numbers could be equal or even zero, then the degeneracy of each level is

$$g_N = 2^{\delta - p}\frac{\delta!}{k_1!k_2!\cdots k_l!p!} \tag{3.45}$$

where k_i is the number of times that the value (without sign) of n_i appears in the array $n_1, ..., n_\delta$, and p is the number of zero elements in the same array. The index N stands for the label of the level corresponding to the squared ratio of masses

$$\sum_{i=1}^{\delta} n_i^2 = \left(\frac{\mu_{N-1}}{\mu_0}\right)^2 \tag{3.46}$$

where N is an integer number. Some levels have additional degeneracies. For instance, for $\delta = 2$, we have $5^2+0 = 4^2+3^2 = 25$, thus level 25 is 12-fold degenerated (4 times from the first plus 8 times from the second one), while level 5 is just 8-fold degenerated ($5 = 2^2 + 1$) and level 3 does not exist at all. The renormalization group equations for energies above the N-th level receive contributions from b_i and of all the KK excited states in the levels below, giving

$$f_\delta(N) = \sum_{n=1}^{N} g_\delta(n) \tag{3.47}$$

where $g_\delta(n)$ represents the total degeneracy of the level n. The evolution of the coupling then looks like

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(\mu_{N-1}) - \frac{b_i + f_\delta(N)\tilde{b}_i}{2\pi} \ln\left(\frac{\mu}{\mu_{N-1}}\right) \tag{3.48}$$

We can iterate this result for all the first N levels. Combining all of them together and considering equation (3.46), we get the logarithmic running

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(\mu_0) - \frac{b_i}{2\pi} \ln\left(\frac{\mu}{\mu_0}\right) - \frac{\tilde{b}_i}{2\pi}\left[f_\delta(N)\ln\left(\frac{\mu}{\mu_0}\right) - \frac{1}{2}\sum_{n=1}^{N} g_\delta(n)\ln n\right] \tag{3.49}$$

where now the correction of the N thresholds appears in a explicit way. As it should be, for $\delta = 1$ we recover the equation (3.42). We now show that for large N these expression reduces to a power-law running. Let us consider only the terms in parentheses in (3.49). We define those terms as

$$F_\delta\left(\frac{\mu}{\mu_0}\right) \equiv f_\delta(N)\ln\left(\frac{\mu}{\mu_0}\right) - \frac{1}{2}\sum_{n=1}^{N} g_\delta(n)\ln n \tag{3.50}$$

Regarding that $g_\delta(n) = f_\delta(n) - f_\delta(n-1)$ we can write

$$\begin{aligned} F_\delta\left(\frac{\mu}{\mu_0}\right) &= f_\delta(N)\left[\ln\left(\frac{\mu}{\mu_0}\right) - \frac{1}{2}\ln N\right] \\ &\quad -\frac{1}{2}\left[\sum_{n=1}^{N}\left(f_\delta(n) - f_\delta(n-1)\right) - f_\delta(N)\ln N\right] \end{aligned} \tag{3.51}$$

In the limit when N is large we have $N \approx (\mu/\mu_0)^2$. Hence the first term vanishes and the rest becomes

$$F_\delta\left(\frac{\mu}{\mu_0}\right) \approx -\frac{1}{2}\left[\int_1^N dn\frac{df_\delta}{dn}(n)\ln n - f_\delta(N)\ln N\right] = \int_{\mu_0}^{\mu}\frac{dt}{t}f_\delta(n(t)) \quad (3.52)$$

In this limit we can assume that $f_\delta(n(\mu) \approx X_\delta(\mu/\mu_0)^\delta - 1$ and we recover the customary approximation for large N

$$F_\delta\left(\frac{\mu}{\mu_0}\right) \approx \frac{X_\delta}{\delta}\left[\left(\frac{\mu}{\mu_0}\right)^\delta - 1\right] - \ln\left(\frac{\mu}{\mu_0}\right) \quad (3.53)$$

We can easily extend our analysis to the case where the δ compactification radii are not all equal. In this case the masses of the excited KK states are given by

$$m_n^2 = \sum_{i=1}^{\delta} n_i^2\mu_i^2 \quad (3.54)$$

where we have defined $\mu_i = 1/R_i$. Let us also define the scale $\mu_0 = 1/R_{max}$, where $1/R_{max}$ is the inverse of the largest radius. We have a new threshold each time that μ reaches a level in the tower characterized by the squared ratio of masses

$$M_n \equiv \left(\frac{m_n}{\mu_0}\right)^2 = \sum_{i=1}^{\delta} n_i^2\left(\frac{\mu_i}{\mu_0}\right)^2 \quad (3.55)$$

We can follow the same steps as before, but now F_δ is given by

$$F_\delta(\mu,\mu_0,\cdots,\mu_\delta) = f_\delta(N)\ln\left(\frac{\mu}{\mu_0}\right) - \frac{1}{2}\sum_{n=1}^{N} g_\delta(n)\ln M_n \quad (3.56)$$

for μ just above the N-th level. In the continuum limit we may assume that the number of states below the energy scale μ is well approximated by the volume of the δ dimensional ellipsoid defined by

$$N(\mu,\mu_0,\cdots,\mu_\delta) \approx X_\delta\prod_{i=i}^{\delta}\left(\frac{\mu}{\mu_i}\right) \quad (3.57)$$

In this limit we have

$$F_\delta(\mu,\mu_0,\cdots,\mu_\delta) \approx \frac{X_\delta}{\delta}\left[\prod_{i=i}^{\delta}\left(\frac{\mu}{\mu_i}\right) - \prod_{i=i}^{\delta}\left(\frac{\mu_0}{\mu_i}\right)\right] - \ln\left(\frac{\mu}{\mu_0}\right) \quad (3.58)$$

with the explicit extraction of the zero-modes. Clearly, when all the radii are equal we reproduce the result (3.53).

Ghilencea and Ross [22] pointed out that in the MSSM the energy range between μ_0 and Λ is relatively small due to the power-law behavior in the evolution of the couplings. For instance, in the case of one single extra dimension we have an upper limit Λ/μ_0 of order 30, which decreases for higher δ to be less than 6. This fact seems to clash with the assumption which justifies the volume approximation. Only the first few levels appear to be relevant for $\mu \approx \mu_0$. However, we can do a careful analysis defining the integral

$$I = \sum_{n=-\infty}^{+\infty} \int_{r(\mu_0/\Lambda)^2}^{r} \frac{dx}{x} e^{-n^2 x} \tag{3.59}$$

We can compute this integral with the help of the Poisson resummation formula

$$\sum_{n=-\infty}^{+\infty} e^{-n^2 x} = \sqrt{\frac{\pi}{x}} \sum_{n=-\infty}^{+\infty} e^{-\pi^2 n^2/x} \tag{3.60}$$

Then the integral (3.59) can be approximated by

$$\begin{aligned} I &= 4\left(\frac{\Lambda}{\mu_0} - 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1 - Erf(2n\sqrt{\pi})}{n}\right) \\ &\approx 4\left(\frac{\Lambda}{\mu_0} - 1 + \frac{1}{4\pi} e^{-4\pi}\right) \end{aligned} \tag{3.61}$$

where Erf is the error function. In the second line we took into account only the $n = 1$ dominant mode. Numerically we have $e^{-4\pi}/4\pi \approx 10^{-7}$. This shows how good the approximation is.

3.5 Extra dimensions: gauge couplings

In this section we analyze the effect of extra dimensions on the gauge couplings. In four spacetime dimensions, the gauge couplings g_i are dimensionless. In D spacetime dimensions, however, the gauge couplings $\tilde{g}_i$ [4] have the engineering mass dimension

$$[\tilde{g}_i] = 2 - \frac{D}{2} \tag{3.62}$$

[4] The tilde is to distinguish this coupling with the standard one, obtained after passing to four dimensions.

and thus the structure constants α_i^{-1} have dimension $D-4=\delta$. The relation between the higher- and lower-dimensional couplings can be easily inferred. Since the extra spacetime dimensions have a fixed radius R, the standard compactification gives

$$\alpha_i = R^{-\delta}\tilde{\alpha}_i \tag{3.63}$$

From standard field theory technique it is known that higher-dimensional field theories are non-renormalizable. From the four dimensional point of view, this non-renormalizability, at the end, comes from the presence of infinite towers of KK states. These states can circulate in the loops, giving rise to new quantum corrections.

In four dimensions there exists an elegant result, which shows the running of gauge couplings at one-loop level. Let us define the structure constants $\alpha_i \equiv g_i^2/4\pi$. Then we have

$$\frac{d}{d\ln\mu}\alpha_i^{-1}(\mu) = -\frac{b_i}{2\pi} \tag{3.64}$$

This equation can be easily integrated to provide

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(M_Z) - \frac{b_i}{2\pi}\ln\frac{\mu}{M_Z} \tag{3.65}$$

where the coefficients for the MSSM are given by

$$(b_Y, b_2, b_3) = (11, 1, -3) \tag{3.66}$$

This is the famous logarithmic running, where we have chosen the mass of the Z boson as the input scale. To be consistent with grand unified theory [5], like $SU(5)$ or $SO(10)$, it is common to define $\alpha_1 \equiv (5/3)\alpha_Y$ and $b_1 \equiv (3/5)b_Y$. Unlike the standard model case, in the MSSM the evolution of the couplings leads to a unification at a scale of the order of 10^{16} GeV, with a common gauge coupling given by

$$\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) \;=\; \alpha_3(M_{GUT}) \;\approx\; \frac{1}{24} \tag{3.67}$$

In a extra dimensions model the corrections are given by the presence of KK towers. After resumming vacuum polarization diagrams we obtain the gauge coupling as a function of the cutoff

$$g_i(\Lambda) = \left(\frac{1}{1-\Pi(0)}\right)^{1/2} g_i \tag{3.68}$$

[5] In the case of grand unified theories the MSSM gauge groups come from the unified gauge group. All the generators of a given group must have the same normalization.

From (3.24) we obtain

$$\alpha_i^{-1}(\Lambda) = \alpha_i^{-1} - \frac{b_i - \tilde{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} - \frac{\tilde{b}_i X_\delta}{2\pi\delta} \left[\left(\frac{\Lambda}{\mu_0} \right)^\delta - 1 \right] \tag{3.69}$$

The beta-function coefficients b_i, given in (3.66), correspond to the contribution of the zero-mode states. The beta-function coefficients $\tilde{b}_i$ are instead given by

$$(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = (3/5, -3, -6) + \eta(4, 4, 4) \tag{3.70}$$

These beta-function coefficients come from the contributions of the KK states: gauge bosons, Higgs fields and η generation of chiral fermions with the appropriate mirrors [6].

To be consistent, we must impose matching conditions on α_i, which represent the uncorrected value of the effective four-dimensional couplings. This must coincide with the value of the coupling at the scale μ_0, paying attention that under the scale μ_0 equation (3.65) holds. We then obtain

$$\alpha_i^{-1}(\Lambda) = \alpha_i^{-1}(M_Z) - \frac{b_i}{2\pi} \ln \frac{\Lambda}{M_Z} + \frac{\tilde{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} - \frac{\tilde{b}_i X_\delta}{2\pi\delta} \left[\left(\frac{\Lambda}{\mu_0} \right)^\delta - 1 \right] \tag{3.71}$$

valid for all $\Lambda \geq \mu_0$.

The equation (3.71) must be not confused with a renormalization group equation. It simply expresses the dependence of the couplings on the value of the cutoff Λ. The gauge couplings depend on the parameters of the theory (μ_0, δ and η) and the scale Λ at which a new fundamental theory must appear. The presence of the extra dimensions makes the unification of couplings possible for any given values of Λ. Moreover, the unification will take place for every parameters μ_0, δ and η.

In Fig. 3.1 we show the dependence of the gauge couplings as a function of the compactification scale μ_0. Before μ_0 we have the usual logarithmic running, while for $\mu > \mu_0$ the couplings evolve in a power-law way. The matching conditions ensure that the couplings evolve in a smooth way, giving rise to continuous curves. As can be seen, the unification happens a decade after the compactification scale. This is a common feature of models with extra dimensions. If we increase the number of extra dimensions the unification will take place sooner.

In Fig. 3.2 we show the dependence on η, once we fix the compactification scale. If the number of families with KK towers increases, the common value

[6] These are necessary to give mass to KK excitations.

of the couplings at the unification scale increases too. For this value of μ_0, we see that the unification remains perturbative for all η; we can then trust in our one-loop equations.

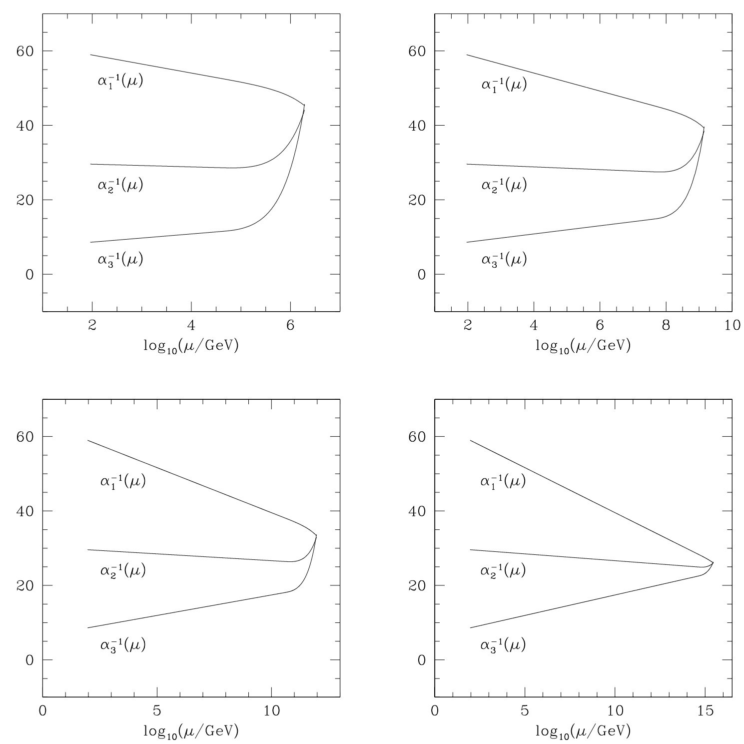


Figure 3.1: Unification of gauge couplings in the minimal scenario for $\delta = 1$. Four typical cases: $\mu_0 = 10^5$ GeV (top left), $\mu_0 = 10^8$ GeV (top right), $\mu_0 = 10^{11}$ GeV (bottom left), and $\mu_0 = 10^{15}$ GeV (bottom right).

Let us now explain this unification in a physical way. First suppose that all the MSSM states have KK excitations. In this case we have $\tilde{b}_i = b_i$ for all i. Then at each level we have heavier copies of the entire MSSM spectrum. In the context of the MSSM we know that the unification is independent of the number of generations. Indeed, every generation fits into complete grand unification group multiplets (whatever the gauge group is). We conclude that the KK modes don't spoil the unification if this would occur in the MSSM case.

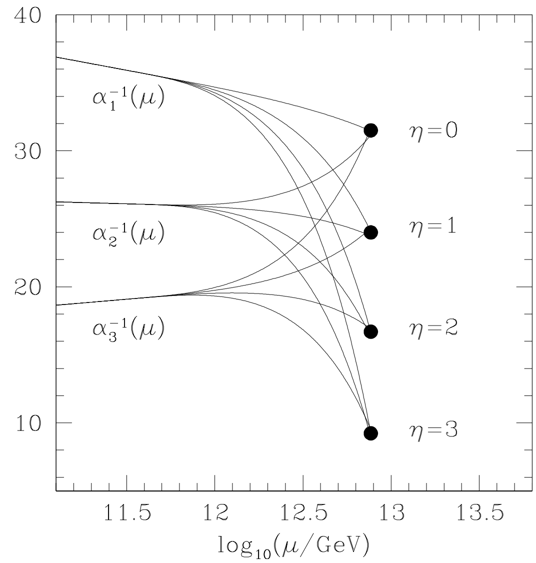


Figure 3.2: Unification of gauge couplings for different values of η. We fix $\mu_0 = 10^{12}$ GeV and $\delta = 1$.

In general, we have $\tilde{b}_i \neq b_i$. To achieve unification we must require that

$$B_{ij} \equiv \frac{\tilde{b}_i - \tilde{b}_j}{b_i - b_j} \tag{3.72}$$

be independent of i and j. Equivalently, we must require

$$\frac{B_{12}}{B_{13}} = \frac{B_{13}}{B_{23}} = 1 \tag{3.73}$$

Even tough in our case these relations are not exactly satisfied, we have the approximate result

$$\frac{B_{12}}{B_{13}} = \frac{72}{77} \approx 0.94 \qquad \frac{B_{13}}{B_{23}} = \frac{11}{12} \approx 0.92 \tag{3.74}$$

This result is independent of the value of η because it simply shifts all the $\tilde{b}_i$ by the same amount: then the unification occurs as well. The experimental uncertainties (although smaller than in the past) ensure that the unification is preserved for all values of μ_0, δ, and η.

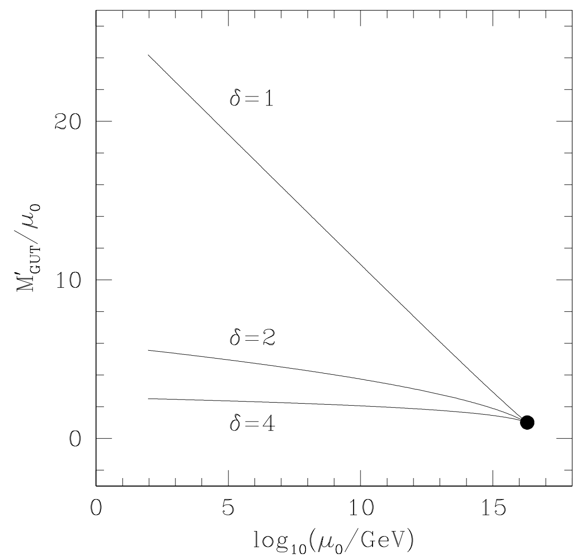


Figure 3.3: The ratio of the unification scale M'_{GUT} to the scale μ_0, as a function of μ_0. The dot stands for the limit of the usual four-dimensional MSSM.

It is also easy to explain why the unification scale is independent of η. When we increase the value of η we simply add complete multiplets to the spectrum at each KK mass level. These extra complete multiplets always preserve the unification scale to one-loop order shifting the unified coupling towards higher values. This behavior can be seen in Fig. 3.2.

This gauge coupling unification will occur at a scale Λ, which we identified as a new unification scale M'_{GUT}. Indeed, in the previous, we interpret the cutoff Λ as the hint of a new fundamental theory, such as grand unified theory (GUT). In the framework of our renormalizable truncated KK theory, M'_{GUT} can be interpreted as the scale of unification [7]. This scenario then predicts the appearance of a higher $(4+\delta)$ dimensional GUT at the scale M'_{GUT}.

This new unification scale M'_{GUT} is generally much lower than the usual unification scale $M_{GUT} \equiv 2 \times 10^{16}$ GeV in the ordinary MSSM. In order to solve equation (3.71) at the unification point, we make the approximation

[7] This is not possible for the non abelian case, see [23].

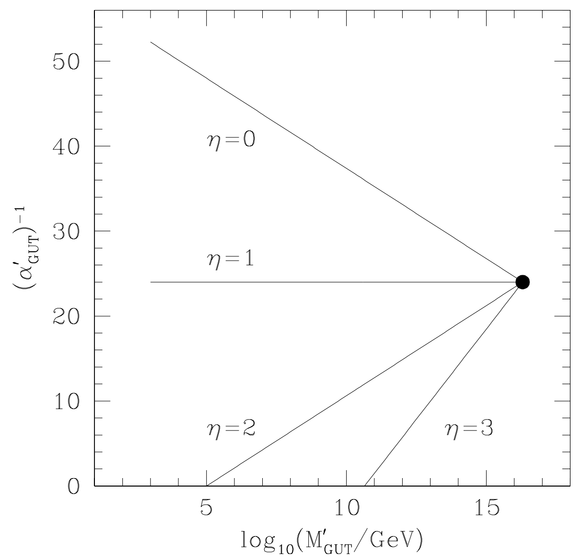


Figure 3.4: The unified coupling $(\alpha'_{GUT})^{-1}$ as a function of the unification scale M'_{GUT}, for $\eta = 0, 1, 2, 3$.

that the B_{ij} in (3.72) are equal for all i and j. We denote the common value as $B_{ij} = B$. We then find

$$M'_{GUT} \;\approx\; \mu_0 \, f^{1/\delta} \tag{3.75}$$

for any value of μ_0 and δ. The factor f is given by

$$f = 1 + \frac{\delta}{X_\delta B} \ln \frac{M_{GUT}}{\mu_0} \geq \; 1 \tag{3.76}$$

In the limit $\delta \to 0$ the compactification scale μ_0 tends towards M_{GUT} and we recover the MSSM, since $f \to 1$ and M'_{GUT} will coincide with M_{GUT}.

However, in general, we see that the new unification scale M'_{GUT} can be much lower than the usual M_{GUT}. The ratio of the unification scale M'_{GUT} to the scale μ_0 for different values of δ is shown in Fig. 3.3. For $\delta > 1$ we see that the unification is very quick, being M'_{GUT} at most six times the compactification scale.

Of course, also the new unified coupling α'_{GUT} will differ from its MSSM value. We find the approximate result

$$(\alpha'_{GUT})^{-1} \;\approx\; \alpha_{GUT}^{-1} \;+\; \frac{2}{\pi B}\,(1-\eta)\ln\frac{M_{GUT}}{M'_{GUT}} \tag{3.77}$$

where $\alpha_{GUT} \approx 1/24$ is the unified MSSM coupling. The behavior (3.77) is plotted in Fig. 3.4. We see that in the $\eta = 0$ case, α'_{GUT} is always less than α_{GUT}, i.e. the higher dimensional theory is more perturbative than the usual MSSM.

For $\eta = 1$ the theory is always perturbative as the MSSM, and the unified coupling is invariant under changes in the unification scale. For $\eta = 2$ the theory is less perturbative than the usual MSSM, and the theory can be applied only for values of the compactification scale greater than 10^5 GeV. There is then a lower bound on the radii of the extra dimensions. In a similar way, for $\eta = 3$ we have a lower bound on the radii of extra dimensions of 10^{10} GeV.

We pointed out that higher-loop corrections could be particularly significant because the gauge couplings have power-law behavior. Thus, higher-loop effects might be particularly large. However, the massive KK excitations fall into $N = 2$ supermultiplets. In this way, we are guaranteed that power-law corrections to the gauge couplings must vanish at higher-loop. Thus, the only corrections beyond one-loop order are the logarithmic corrections due to zero-modes. All the higher-loop corrections will be small and we then expect that gauge coupling unification will be achieved also beyond one-loop order.

Now that we have established that gauge coupling unification occurs at scales lower than the usual GUT theories, we face with the issue of fermion mass hierarchy.

3.6 Extra dimensions: Yukawa couplings

In this section we describe the evolution of Yukawa couplings. These couplings are strictly related to the fermion mass through the electroweak symmetry breaking mechanism. Following what we have done with the gauge couplings we can define a sort of Yukawa structure constants $\alpha_F \equiv y_F^2/4\pi$, where F stands for the various quarks and leptons ($F = u, d, s, c, b, t, e, \mu, \tau$). When we deal with the fermion masses we face with values that differ by many orders of magnitude. Indeed, α_F goes from 1 for the top quark to 10^{12} for the electron. The hope is that the power-law behavior (as we will see in a moment) of these couplings in presence of extra dimensions could solve this problem, providing the observed mass hierarchy.

As in the previous section, we start by reviewing how the Yukawa couplings run in the usual (4-dimensional) MSSM. The electroweak symmetry breaking provides mass to the fermion in the form

$$m_F = y_F \, v \, \cos\beta \tag{3.78}$$

for down-type quarks and leptons, while for the up-type quarks we have

$$m_F = y_F \, v \, \sin\beta \tag{3.79}$$

Here $v \approx 174$ GeV enters in the Higgs potential and $\tan\beta$ is the ratio of up-type and down-type Higgs VEV's. As we saw in paragraph 1.1, the superpotential has the generic form

$$W = \sum_F y_F \, F\overline{F} \, H_{u,d} \tag{3.80}$$

In the MSSM, these Yukawa couplings run logarithmically with the energy. Their RG evolution is given by

$$\frac{d}{d\ln\mu} \, \alpha_F^{-1}(\mu) = -\frac{b_F(\mu)}{2\pi} \tag{3.81}$$

We must note that here the one-loop beta-function coefficients $b_F(\mu)$ are not constant as in the gauge coupling case, but they depend on the energy scale. For example for the top quark we have

$$b_t = 6 + \frac{1}{\alpha_t}\left(\alpha_b + 3\alpha_u + 3\alpha_c - \frac{16}{3}\alpha_3 - 3\alpha_2 - \frac{13}{15}\alpha_1\right) \tag{3.82}$$

Only the first term on the right hand side is constant, while the others evolve with the energy scale μ. The situation is similar for the other fermions. In order to generalize the equation (3.81), we review how it was derived in section 2.2.

As can be seen in (3.80) the Yukawa coupling y_F is responsible for the supersymmetric triple vertex involving a Higgs field and a fermion-antifermion pair. The supersymmetric non-renormalization theorem ensures that the renormalization of α_F depends only on the wave function renormalization factors Z_i

$$\alpha_F^{-1}(\mu) = Z_H Z_F Z_{\overline{F}} \, \alpha_F^{-1}(\mu_0) \tag{3.83}$$

The factors Z_i can be calculated through the formula (2.74). The general result is

$$Z_i = 1 - \frac{\gamma_i}{2\pi} \ln\frac{\mu}{\mu_0} \tag{3.84}$$

where γ_i is the anomalous dimension of the field i. Specifically we have for the quarks

$$\begin{aligned}\gamma_t &= \alpha_t + \alpha_b - \frac{1}{30}\alpha_1 - \frac{3}{2}\alpha_2 - \frac{8}{3}\alpha_3 \\ \gamma_{\overline{b}} &= 2\alpha_b - \frac{2}{15}\alpha_1 - \frac{8}{3}\alpha_3 \\ \gamma_{\overline{t}} &= 2\alpha_t - \frac{8}{15}\alpha_1 - \frac{8}{3}\alpha_3\end{aligned} \tag{3.85}$$

while for the leptons we have

$$\begin{aligned}\gamma_\tau &= \alpha_\tau - \frac{3}{10}\alpha_1 - \frac{3}{2}\alpha_2 \\ \gamma_{\overline{\tau}} &= 2\alpha_t - \frac{6}{5}\alpha_1\end{aligned} \tag{3.86}$$

Finally the anomalous dimensions of the Higgs fields are

$$\begin{aligned}\gamma_{H_u} &= 3\alpha_t - \frac{3}{10}\alpha_1 - \frac{3}{2}\alpha_2 \\ \gamma_{H_d} &= 3\alpha_b + \alpha_\tau - \frac{3}{10}\alpha_1 - \frac{3}{2}\alpha_2\end{aligned} \tag{3.87}$$

Taking into account equation (3.83) and limiting to linear terms in the logarithms we obtain the running (3.81), with beta-function coefficients given by

$$\alpha_F b_F \equiv \gamma_F + \gamma_{\overline{F}} + \gamma_{H_i} \tag{3.88}$$

We can now pass to discuss the situation in presence of extra dimensions. The δ extra spacetime dimensions appear at the energy scale $\mu_0 \equiv R^{-1}$. Below this scale there is no effect due to KK states and the Yukawa couplings run logarithmically according to (3.81). When we go over the energy μ_0 we have finite one-loop corrections as functions of the cutoff Λ. We have

$$\alpha_F^{-1}(\Lambda) = Z_H Z_F Z_{\overline{F}}\ \alpha_F^{-1}(\mu_0) \tag{3.89}$$

Following what we obtained for the gauge couplings, we expect the form of the anomalous dimensions to be

$$Z_i = 1 - \frac{\gamma_i(\mu_0) - \tilde{\gamma}_i(\mu_0)}{2\pi} \ln\frac{\Lambda}{\mu_0} - \frac{\tilde{\gamma}_i(\mu_0)}{2\pi}\frac{X_\delta}{\delta}\left[\left(\frac{\Lambda}{\mu_0}\right)^\delta - 1\right] \tag{3.90}$$

The power-law term arise from the summation over KK states which enter the loops. As we will see, the anomalous dimensions $\tilde{\gamma}_i$ corresponding to the excited KK modes differ in general from the anomalous dimensions γ_i of the zero-modes. Only in that case a logarithm term will appear in (3.90).

It is worth noting that opposite to the case of gauge coupling, these anomalous dimensions γ_i depend on the scale μ_0. In (3.89) and (3.90), these coefficients must be evaluated at the (fixed) scale μ_0. This is due to the nature of this theory as an effective theory.

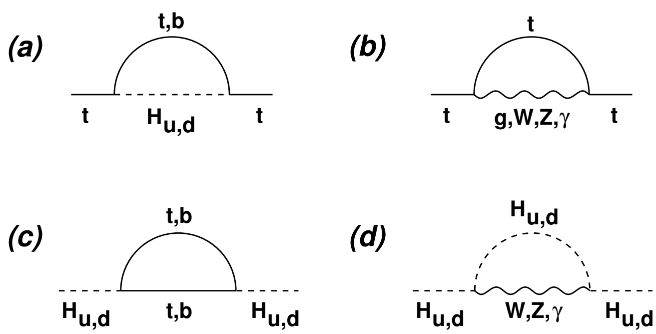


Figure 3.5: Wave function renormalization diagrams in the MSSM. Diagrams (a,b) contribute to Z_t and $Z_{\bar{t}}$, while diagrams (c,d) contribute to $Z_{H_{u,d}}$.

Our goal is now to evaluate the functions $\tilde{\gamma}_i$, which are responsible of the one-loop corrections coming from the KK massive levels. For convenience we collect the diagrams responsible for the anomalous dimensions in Fig. 3.5. We will deal first with the minimal scenario. Let us then consider the diagram (a). In the loop a fermion and a Higgs field circulate. In the minimal scenario the MSSM fermions do not have KK towers, so the fermion in the loop must be a zero-mode. Regarding the Higgs tower, we note that each massive level contribute the same as the zero-mode of the Higgs fields itself, in both the cases $N = 1$ and $N = 2$. As a consequence $\tilde{\gamma}_F$ is the same as the four dimensional one. Looking at the diagram (b) in Fig. 3.5, we see that the situation is much the same. The fermion must be a zero-mode again, while the massive gauge bosons can be KK modes. Even though the towers are $N = 2$ supersymmetric, they couple with the fermions only through their $N = 1$ components. This is due to the symmetry properties of the wave function (see section 3.2). Even in this case we find that the contribution to the fermion anomalous dimension coming from this diagram is the same as the usual four-dimensional one. In conclusion, for the minimal scenario we find

$$\tilde{\gamma}_F = \gamma_F \qquad \tilde{\gamma}_{\overline{F}} = \gamma_{\overline{F}} \tag{3.91}$$

for all fermions.

The anomalous dimensions of the Higgs fields are given by the diagrams (c) e (d) shown in Figs. 3.5. In Fig. 3.5(c), the only particles that propagate in the loop are fermions. Since the fermions have no KK states, this diagram does not affect the anomalous dimension $\tilde{\gamma}_H$.

For what regards the diagram in Fig. 3.5(d) we note that the full $N = 2$ set of KK states for the gauge boson can circulate in the loop . Now we must separate the two cases: Higgs zero-modes $N = 1$ and $N = 2$. In the first case we have a $N = 1$ diagram which contributes to the anomalous dimensions. Then $\tilde{\gamma}_{H_u}$ and $\tilde{\gamma}_{H_d}$ have only the gauge terms, which are the same in this case as can be seen from equation (3.87).

In the case $N = 2$ after imposing KK momentum conservation at each of the vertices, we find that there is no contribution to $\tilde{\gamma}_H$. This is a consequence of $N = 2$ theorems, once we take into account also the term (3.4). In conclusion, for the minimal scenario, we find that both $\tilde{\gamma}_{H_u}$ and $\tilde{\gamma}_{H_d}$ are null. The presence of extra dimensions has no effect on the Higgs wave function renormalization.

Let us now discuss the non minimal scenarios. In section 3.2, we isolated the three different cases. When $\eta = 2$ only the first two families will have KK towers and we fall in the case of the minimal scenario. In the third family approximation indeed the Yukawa couplings of the first two families are completely negligible. In the other two cases, $\eta = 1$ or $\eta = 3$, the third family will have KK states.

Let us take again the diagrams in Fig. 3.5. In diagram (a) also the fermions can propagate in the loop. Since we must impose KK momentum conservation, we have a single sum over KK modes. Indeed, the external states are zero-modes [8] and KK momentum conservation implies that the levels of the particles in the loop must be opposite. Then we have a single sum. We must outline that this diagram cannot vanish due to $N = 2$ properties, since the external states are not $N = 2$ and the whole diagram is a $N = 1$ diagram. The situation is much the same in the case of Fig. 3.5(b). Then we have the same result (3.91) found in the minimal scenario.

Finally we consider the Higgs fields. In diagram (c) both the fermions in the loop will have KK towers. After imposing KK momentum conservation we end again with a single sum. For what regards the last diagram in Fig. 3.5 we must distinguish between the two cases. If we treat the Higgs

[8] Recall that we are calculating corrections to the MSSM states, i.e. zero-modes.

(zero-modes) as belonging to a full $N = 2$ multiplet, then this diagram is a $N = 2$ diagram and thus vanishes. In this case the anomalous dimensions $\tilde{\gamma}_{H_i}$ have only the terms containing the couplings in (3.87). In the other case, when the Higgs (zero-modes) are $N = 1$ particles, the diagram (d) in Fig. 3.5 does not vanish and gives rise to a single sum. Then the anomalous dimensions due to KK modes are the same as the zero-modes.

3.6.1 Higgs N=1: minimal scenario

In this case the anomalous dimension receives a contribution from the diagrams in which particles from the KK tower of the gauge bosons circulate in the loop leading to a power-law contribution. For the Higgs fields we have

$$\tilde{\gamma}_{H_u} = -\frac{3}{10}\alpha_1 - \frac{3}{2}\alpha_2 \tag{3.92}$$

$$\tilde{\gamma}_{H_d} = -\frac{3}{10}\alpha_1 - \frac{3}{2}\alpha_2 \tag{3.93}$$

For what the Yukawa couplings are concerned, we get

$$16\pi^2 \frac{d}{dt} y_t = y_t \left[3|y_t|^2 + X_\delta \left(\frac{\Lambda}{\mu_0}\right)^\delta \left(3|y_t|^2 + |y_b|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2\right)\right] \tag{3.94}$$

$$16\pi^2 \frac{d}{dt} y_b = y_b \left[3|y_b|^2 + |y_\tau|^2 + X_\delta \left(\frac{\Lambda}{\mu_0}\right)^\delta \left(|y_t|^2 + 3|y_b|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2\right)\right] \tag{3.95}$$

$$16\pi^2 \frac{d}{dt} y_\tau = y_\tau \left[|y_\tau|^2 + 3|y_b|^2 + X_\delta \left(\frac{\Lambda}{\mu_0}\right)^\delta \left(3|y_\tau|^2 - 3g_2^2 - \frac{9}{5}g_1^2\right)\right] \tag{3.96}$$

Finally we analyze the (supersymmetric) Higgs mass

$$16\pi^2 \frac{d}{dt} \mu = \mu \left[\left(3|y_t|^2 + 3|y_b|^2 + |y_\tau|^2\right) + X_\delta \left(\frac{\Lambda}{\mu_0}\right)^\delta \left(-3g_2^2 - \frac{3}{5}g_1^2\right)\right] \tag{3.97}$$

3.6.2 Higgs N=1: non minimal scenarios

In the non minimal scenarios we introduce KK towers also for the matter fields. As we saw in section 3.2 we must distinguish among three different cases: $\eta = 1$ in which only the third family has a KK tower, $\eta = 2$ in which only the first two families have a KK tower and $\eta = 3$ in which all the families have KK tower.

One of the consequences of having KK towers for the matter fields is that they no longer live at the orbifold fixed point and they are allowed to be in the bulk. The anomalous dimensions of the Higgs fields in the $\eta = 2$ case is the same as in the minimal scenario. For $\eta = 1, 3$ we find

$$\tilde{\gamma}_{H_u} = \gamma_{H_u} \tag{3.98}$$
$$\tilde{\gamma}_{H_d} = \gamma_{H_d} \tag{3.99}$$

For the chiral fields t, b, τ we have a similar situation, the contribution of the KK towers is the same of the zero-modes. Then the coefficients of the β-function have only a power-law contribution.

3.6.3 Higgs N=2 scenarios

We discuss first the minimal scenario. As in [1] the wave function renormalization of the Higgs fields are given by

$$\tilde{\gamma}_{H_u} = 0 \tag{3.100}$$
$$\tilde{\gamma}_{H_d} = 0 \tag{3.101}$$

The one-loop beta-functions for the Yukawa couplings are then

$$16\pi^2 \frac{d}{dt} y_t = y_t \Bigg[3|y_t|^2 - \frac{3}{10} g_1^2 - \frac{3}{2} g_2^2 + X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Big(3|y_t|^2 + |y_b|^2 - \frac{16}{3} g_3^2 - \frac{3}{2} g_2^2 - \frac{17}{30} g_1^2 \Big) \Bigg] \tag{3.102}$$

$$16\pi^2 \frac{d}{dt} y_b = y_b \Bigg[3|y_b|^2 + |y_\tau|^2 - \frac{3}{10} g_1^2 - \frac{3}{2} g_2^2 + X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Big(|y_t|^2 + 3|y_b|^2 - \frac{16}{3} g_3^2 - \frac{3}{2} g_2^2 - \frac{1}{6} g_1^2 \Big) \Bigg] \tag{3.103}$$

$$16\pi^2 \frac{d}{dt} y_\tau = y_\tau \Bigg[|y_\tau|^2 + 3|y_b|^2 - \frac{3}{10} g_1^2 - \frac{3}{2} g_2^2 + X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Big(3|y_\tau|^2 - \frac{3}{2} g_2^2 - \frac{3}{2} g_1^2 \Big) \Bigg] \tag{3.104}$$

For the (supersymmetric) Higgs mass, in virtue of the non-renormalization theorem, we have

$$\frac{d}{dt}\mu = 0 \tag{3.105}$$

This equation holds for any η. For what concerns the non minimal scenarios, the results we found in section 3.7.2 for $N = 1$ are still valid.

3.7 Extra dimensions: soft terms

In presence of KK modes the effect of the thresholds is to contribute to the running of the soft terms with power-like terms. The general form of the running of a parameter [9] g will be of the type

$$Q \frac{dg}{dQ} = (b - \tilde{b_g}) \ln \frac{\Lambda}{Q} + X_\delta \tilde{b_g} \left(\frac{\Lambda}{\mu_0}\right)^\delta \tag{3.106}$$

where b_g is the standard one-loop value of the coefficient, while $\tilde{b_g}$ is the contribution of the thresholds.

While the effective potential method is powerful in determining the runnings for the parameters in the SUSY Lagrangian, it is difficult to apply it to

[9] Here we used g for a generic soft term and b_g for the relative one-loop coefficient.

the computation of the soft terms in presence of the KK states. This is why we have decided to also carry on an analysis in terms of Feynman diagrams in paragraph 2.4. In section 2 we saw the general form of the running of the couplings and masses. All the contributions to the renormalization of these parameters are collected in paragraph 2.4.

The computation of the effect of extra dimensions is now straightforward: each time that in a Feynman diagram KK states are allowed to circulate in the loop (external states can only be zero-modes), the contribution of the diagram must be added to the $\tilde{b}$ coefficient in (3.106). We finally remind the reader that in an orbifold compactification the KK momentum is conserved only in the bulk. The wave function in the compactified dimension must, in fact, be expanded in a basis which is invariant under the discrete group acting on the compactified dimension. Furthermore the orbifold fixed points break translation invariance along the extra dimensions.

Following the previous prescription we can obtain the β-functions for all the soft terms. As usual, we must distinguish between the minimal and non minimal scenarios in both the cases, “Higgs $N = 1$” and “Higgs $N = 2$”.

3.7.1 Higgs N=1: minimal scenario

We start with the trilinear couplings

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} a_t &= 9a_t|y_t|^2 + X_\delta(\frac{\Lambda}{Q_0})^\delta \Bigg\{ a_t \left[9|y_t|^2 + |y_b|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2 \right] \\
&\quad + 2a_b y_b^* y_t + y_t \left[\frac{32}{3}g_3^2 M_3 + 6g_2^2 M_2 + \frac{26}{15}g_1^2 M_1 \right] \Bigg\} \qquad (3.107)
\end{aligned}
$$

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} a_b &= 9a_b|y_b|^2 + a_b|y_\tau|^2 + 2a_\tau y_\tau^* y_b + X_\delta(\frac{\Lambda}{Q_0})^\delta \Bigg\{ a_b \Big[9|y_b|^2 + |y_t|^2 \\
&\quad - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2 \Big] + 2a_t y_t^* y_b \\
&\quad + y_b \left[\frac{32}{3}g_3^2 M_3 + 6g_2^2 M_2 + \frac{14}{15}g_1^2 M_1 \right] \Bigg\} \qquad (3.108)
\end{aligned}
$$

$$\begin{aligned}16\pi^2\frac{d}{dt}a_\tau &= 3a_\tau|y_b|^2+3a_\tau|y_\tau|^2+X_\delta(\frac{\Lambda}{Q_0})^\delta\Big\{a_\tau\Big[9|y_\tau|^2\\&-3g_2^2-\frac{9}{5}g_1^2\Big]+6a_by_b^*y_\tau\\&+y_\tau\left[6g_2^2M_2+\frac{18}{5}g_1^2M_1\right]\Big\}\end{aligned}\qquad(3.109)$$

The β-function for the b parameter (crucial for the electroweak symmetry breaking mechanism), is given by

$$\begin{aligned}16\pi^2\frac{d}{dt}b &= b\left(3|y_t|^2+3|y_b|^2+|y_\tau|^2\right)+\mu\left[6a_ty_t^*+6a_by_b^*+2a_\tau y_\tau^*\right]\\&+X_\delta\left(\frac{\Lambda}{Q_0}\right)^\delta\Big\{b\left(-3g_2^2-\frac{3}{5}g_1^2\right)\\&+\mu\left(+6g_2^2M_2+\frac{6}{5}g_1^2M_1\right)\Big\}\end{aligned}\qquad(3.110)$$

Now we can show the β-function for all the soft squared masses. For the two Higgs mass parameters we have

$$\begin{aligned}16\pi^2\frac{d}{dt}m_{H_u}^2 &= 6|y_t|^2\left(m_{H_u}^2+m_t^2+m_{\bar t}^2\right)+6|a_t|^2+\frac{3}{5}g_1^2Tr'(Ym^2)\\&+X_\delta\left(\frac{\Lambda}{Q_0}\right)^\delta\Big\{-6g_2^2|M_2|^2-\frac{6}{5}g_1^2|M_1|^2\\&+\frac{3}{5}g_1^2\left(m_{H_u}^2-m_{H_d}^2\right)\Big\}\end{aligned}\qquad(3.111)$$

$$\begin{aligned}16\pi^2\frac{d}{dt}m_{H_d}^2 &= 6|y_b|^2\left(m_{H_d}^2+m_t^2+m_{\bar b}^2\right)+6|a_b|^2+2|a_\tau|^2\\&+2|y_\tau|^2\left(m_{H_d}^2+m_L^2+m_{\bar\tau}^2\right)-\frac{3}{5}g_1^2Tr'(Ym^2)\\&+X_\delta\left(\frac{\Lambda}{Q_0}\right)^\delta\Big\{-6g_2^2|M_2|^2-\frac{6}{5}g_1^2|M_1|^2\\&-\frac{3}{5}g_1^2\left(m_{H_u}^2-m_{H_d}^2\right)\Big\}\end{aligned}\qquad(3.112)$$

In order to avoid cumbersome expressions, we have introduced the quantity

$$Tr'(Ym^2)\equiv Tr(Ym^2)-m_{H_u}^2+m_{H_d}^2\qquad(3.113)$$

where the usual trace over the hypercharge, defined in the MSSM, is given by

$$\begin{aligned} Tr(Ym^2) &= m_{H_u}^2 - m_{H_d}^2 + m_Q^2 - 2\left(m_{\bar{u}}^2 + m_{\bar{c}}^2 + m_{\bar{t}}^2\right) \\ &\quad + m_{\bar{d}}^2 + m_{\bar{s}}^2 + m_{\bar{b}}^2 - m_L^2 + m_{\bar{e}}^2 + m_{\bar{\mu}}^2 + m_{\bar{\tau}}^2 \end{aligned} \tag{3.114}$$

Moreover, in our computations we have assumed the first two families to be degenerate in mass, such that

$$\begin{aligned} m_Q^2 &= 2m_{Q_1}^2 + m_{Q_3}^2 \\ m_L^2 &= 2m_{L_1}^2 + m_{L_3}^2 \\ m_{\bar{e}}^2 &= m_{\bar{\mu}}^2 \\ m_{\bar{u}}^2 &= m_{\bar{c}}^2 \end{aligned} \tag{3.115}$$

The equations for the squarks mass terms read

$$\begin{aligned} 16\pi^2 \frac{d}{dt} m_t^2 &= 2|y_t|^2 m_{\bar{t}}^2 + 2|y_b|^2 m_{\bar{b}}^2 + \frac{1}{5} g_1^2 Tr'(Ym^2) \\ &\quad + X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Bigg\{ -\frac{32}{3} g_3^2 |M_3|^2 - 6 g_2^2 |M_2|^2 - \frac{2}{15} g_1^2 |M_1|^2 \\ &\quad + 2|a_t|^2 + 2|a_b|^2 + 2|y_t|^2 \left(m_{H_u}^2 + m_t^2\right) + 2|y_b|^2 \left(m_{H_d}^2 + m_t^2\right) \\ &\quad + \frac{1}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2\right) \Bigg\} \end{aligned} \tag{3.116}$$

$$\begin{aligned} 16\pi^2 \frac{d}{dt} m_{\bar{t}}^2 &= 4|y_t|^2 m_t^2 - \frac{4}{5} g_1^2 Tr'(Ym^2) \\ &\quad + X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Bigg\{ -\frac{32}{3} g_3^2 |M_3|^2 - \frac{32}{15} g_1^2 |M_1|^2 + 4|a_t|^2 \\ &\quad + 4|y_t|^2 \left(m_{H_u}^2 + m_{\bar{t}}^2\right) - \frac{4}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2\right) \Bigg\} \end{aligned} \tag{3.117}$$

$$\begin{aligned} 16\pi^2 \frac{d}{dt} m_{\bar{b}}^2 &= 4|y_b|^2 m_t^2 + \frac{2}{5} g_1^2 Tr'(Ym^2) \\ &\quad + X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Bigg\{ -\frac{32}{3} g_3^2 |M_3|^2 - \frac{8}{15} g_1^2 |M_1|^2 + 4|a_b|^2 \\ &\quad + 4|y_b|^2 \left(m_{H_d}^2 + m_{\bar{b}}^2\right) + \frac{2}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2\right) \Bigg\} \end{aligned} \tag{3.118}$$

Finally, for the sleptons masses we have

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} m_L^2 &= 2|y_\tau|^2 m_{\tilde{\tau}}^2 - \frac{3}{5} g_1^2 Tr'(Ym^2) \\
&+ X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Bigg\{ - 6g_2^2|M_2|^2 - \frac{6}{5} g_1^2 |M_1|^2 + 2|a_\tau|^2 \\
&+ 2|y_\tau|^2 \left(m_{H_d}^2 + m_L^2\right) - \frac{3}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2\right) \Bigg\} \qquad (3.119) \\
16\pi^2 \frac{d}{dt} m_{\tilde{\tau}}^2 &= 4|y_\tau|^2 m_L^2 + \frac{6}{5} g_1^2 Tr'(Ym^2) \\
&+ X_\delta \left(\frac{\Lambda}{Q_0}\right)^\delta \Bigg\{ - \frac{24}{5} g_1^2 |M_1|^2 + 4|a_\tau|^2 \\
&+ 4|y_\tau|^2 \left(m_{H_d}^2 + m_{\tilde{\tau}}^2\right) + \frac{6}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2\right) \Bigg\} \qquad (3.120)
\end{aligned}
$$

3.7.2 Higgs N=1: non minimal scenarios

In this case for the b parameter we must distinguish between the $\eta = 2$ and $\eta = 1, 3$ cases. In the former case the running is the same as in the minimal case due to the third family approximation. In the latter case, once again, the β-functions coefficients have only a power-law behavior.

Finally the running of soft scalar masses in the non minimal cases is given by a logarithmic term multiplied by

$$Tr'(Ym^2) \qquad (3.121)$$

and a power-law term given by

$$Tr(Ym^2) - Tr'(Ym^2) \qquad (3.122)$$

In the $\eta = 1$ case the primed trace is defined as

$$Tr'_{\eta=1}(Ym^2) = 2m_{Q_1}^2 - 2\left(m_{\tilde{u}}^2 + m_{\tilde{c}}^2\right) + m_{\tilde{d}}^2 + m_{\tilde{s}}^2 - 2m_{L_1}^2 + 2m_{\tilde{e}}^2 \qquad (3.123)$$

while for $\eta = 3$

$$Tr'_{\eta=3}(Ym^2) = 0 \qquad (3.124)$$

because all the families have KK tower and there is no logarithmic contribution. Finally for the case $\eta = 2$ we have

$$Tr'_{\eta=2}(Ym^2) = m_{Q_3}^2 - 2\left(m_{\tilde{t}}^2\right) + m_{\tilde{b}}^2 - m_{L_3}^2 + m_{\tilde{\tau}}^2 \qquad (3.125)$$

3.7.3 Higgs N=2: minimal scenario

We begin with the trilinear terms, which look like

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} a_t &= a_t \left(9|y_t|^2 - \frac{3}{10} g_1^2 - \frac{3}{2} g_2^2\right) + X_\delta (\frac{\Lambda}{Q_0})^\delta \Bigg\{ a_t \left[-\frac{16}{3} g_3^2 - \frac{3}{2} g_2^2 - \frac{17}{30} g_1^2\right] \\
&+ a_t \left(9|y_t|^2 + |y_b|^2\right) + 2 a_b y_b^* y_t \\
&+ y_t \left[\frac{32}{3} g_3^2 M_3 + 6 g_2^2 M_2 + \frac{26}{15} g_1^2 M_1\right] \Bigg\} \qquad (3.126) \\
16\pi^2 \frac{d}{dt} a_b &= a_b \left(9|y_b|^2 + |y_\tau|^2 - \frac{3}{10} g_1^2 - \frac{3}{2} g_2^2\right) + 2 a_\tau y_\tau^* y_b \\
&+ X_\delta (\frac{\Lambda}{Q_0})^\delta \Bigg\{ a_b \Big[9|y_b|^2 + |y_t|^2 - \frac{16}{3} g_3^2 - \frac{3}{2} g_2^2 - \frac{1}{6} g_1^2\Big] + 2 a_t y_t^* y_b \\
&+ y_b \left[\frac{32}{3} g_3^2 M_3 + 6 g_2^2 M_2 + \frac{14}{15} g_1^2 M_1\right] \Bigg\} \qquad (3.127) \\
16\pi^2 \frac{d}{dt} a_\tau &= a_\tau \left(3|y_b|^2 + 3|y_\tau|^2 - \frac{3}{10} g_1^2 - \frac{3}{2} g_2^2\right) + 6 a_b y_b^* y_\tau \\
&+ X_\delta (\frac{\Lambda}{Q_0})^\delta \Bigg\{ a_\tau \Big[9|y_\tau|^2 - \frac{3}{2} g_2^2 - \frac{3}{2} g_1^2\Big] \\
&+ y_\tau \left[6 g_2^2 M_2 + \frac{18}{5} g_1^2 M_1\right] \Bigg\} \qquad (3.128)
\end{aligned}
$$

The β-functions for the b parameter reads

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} b &= b\left(3|y_t|^2 + 3|y_b|^2 + |y_\tau|^2\right) + \mu\left(6 a_t y_t^* + 6 a_b y_b^* + 2 a_\tau y_\tau^*\right) \\
&+ b\left(-3 g_2^2 - \frac{3}{5} g_1^2\right) + \mu\left(+6 g_2^2 M_2 + \frac{6}{5} g_1^2 M_1\right) \qquad (3.129)
\end{aligned}
$$

The β-functions for the soft squared masses of the Higgs fields are

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} m_{H_u}^2 &= 6|y_t|^2 \left(m_{H_u}^2 + m_t^2 + m_{\bar{t}}^2 \right) + 6|a_t|^2 + \frac{3}{5} g_1^2 Tr(Ym^2) \\
&\quad -6g_2^2|M_2|^2 - \frac{6}{5} g_1^2 |M_1|^2 \qquad (3.130) \\
16\pi^2 \frac{d}{dt} m_{H_d}^2 &= 6|y_b|^2 \left(m_{H_d}^2 + m_t^2 + m_{\bar{b}}^2 \right) + 6|a_b|^2 + 2|a_\tau|^2 \\
&\quad +2|y_\tau|^2 \left(m_{H_d}^2 + m_L^2 + m_{\bar{\tau}}^2 \right) - \frac{3}{5} g_1^2 Tr(Ym^2) \\
&\quad -6g_2^2|M_2|^2 - \frac{6}{5} g_1^2 |M_1|^2 \qquad (3.131)
\end{aligned}
$$

while for the squarks we have

$$
\begin{aligned}
16\pi^2 \frac{d}{dt} m_t^2 &= 2|y_t|^2 m_{\bar{t}}^2 + 2|y_b|^2 m_{\bar{b}}^2 + \frac{1}{5} g_1^2 Tr'(Ym^2) \\
&\quad + X_\delta \left(\frac{\Lambda}{Q_0} \right)^\delta \Bigg\{ -\frac{32}{3} g_3^2 |M_3|^2 - 6g_2^2|M_2|^2 - \frac{2}{15} g_1^2 |M_1|^2 \\
&\quad +2|a_t|^2 + 2|a_b|^2 + 2|y_t|^2 \left(m_{H_u}^2 + m_t^2 \right) + 2|y_b|^2 \left(m_{H_d}^2 + m_t^2 \right) \\
&\quad + \frac{1}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2 \right) \Bigg\} \qquad (3.132) \\
16\pi^2 \frac{d}{dt} m_{\bar{t}}^2 &= 4|y_t|^2 m_t^2 - \frac{4}{5} g_1^2 Tr'(Ym^2) \\
&\quad + X_\delta \left(\frac{\Lambda}{Q_0} \right)^\delta \Bigg\{ -\frac{32}{3} g_3^2 |M_3|^2 - \frac{32}{15} g_1^2 |M_1|^2 + 4|a_t|^2 \\
&\quad +4|y_t|^2 \left(m_{H_u}^2 + m_{\bar{t}}^2 \right) - \frac{4}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2 \right) \Bigg\} \qquad (3.133) \\
16\pi^2 \frac{d}{dt} m_{\bar{b}}^2 &= 4|y_b|^2 m_t^2 + \frac{2}{5} g_1^2 Tr'(Ym^2) \\
&\quad + X_\delta \left(\frac{\Lambda}{Q_0} \right)^\delta \Bigg\{ -\frac{32}{3} g_3^2 |M_3|^2 - \frac{8}{15} g_1^2 |M_1|^2 + 4|a_b|^2 \\
&\quad +4|y_b|^2 \left(m_{H_d}^2 + m_{\bar{b}}^2 \right) + \frac{2}{5} g_1^2 \left(m_{H_u}^2 - m_{H_d}^2 \right) \Bigg\} \qquad (3.134)
\end{aligned}
$$

Finally the equations for the leptons read

$$\begin{aligned}16\pi^2\frac{d}{dt}m_L^2 &= 2|y_\tau|^2 m_{\bar\tau}^2 - \frac{3}{5}g_1^2 Tr'(Ym^2)\\ &+X_\delta\left(\frac{\Lambda}{Q_0}\right)^\delta\Bigg\{-6g_2^2|M_2|^2 - \frac{6}{5}g_1^2|M_1|^2 + 2|a_\tau|^2\\ &+2|y_\tau|^2\left(m_{H_d}^2 + m_L^2\right) - \frac{3}{5}g_1^2\left(m_{H_u}^2 - m_{H_d}^2\right)\Bigg\} \qquad (3.135)\\ 16\pi^2\frac{d}{dt}m_{\bar\tau}^2 &= 4|y_\tau|^2 m_L^2 + \frac{6}{5}g_1^2 Tr'(Ym^2)\\ &+X_\delta\left(\frac{\Lambda}{Q_0}\right)^\delta\Bigg\{-\frac{24}{5}g_1^2|M_1|^2 + 4|a_\tau|^2\\ &+4|y_\tau|^2\left(m_{H_d}^2 + m_{\bar\tau}^2\right) + \frac{6}{5}g_1^2\left(m_{H_u}^2 - m_{H_d}^2\right)\Bigg\} \qquad (3.136)\end{aligned}$$

We must outline that our results (3.132)-(3.136) for the masses of the squarks and sleptons differ from those reported in [24] in two respects: on the one hand the terms proportional to Y and $m_{H_u}^2 - m_{H_d}^2$, due to the $U(1)$ gauge factor [10], are absent in [24]. On the other hand in (3.132)-(3.136) there are terms coming from the diagrams (2.105) which do not get contributions from the KK states and therefore do not have a power-law type running.

3.7.4 Higgs N=2: non minimal scenarios

The results we found in section 3.7.2 for $N = 1$ are still valid. The only difference concerns the soft mass terms: for the Higgs masses, $m_{H_u}^2, m_{H_d}^2$ no diagrams will contribute to the power-law. The equations are then the same as in the MSSM. On the other hand the squark masses have only power-law contributions. For the other soft terms the equations are the same of the minimal scenario, regardless of the value of η.

3.8 Connection with string theory

So far the calculations of the beta functions were made using field theory methods only. In this section we want to elucidate the connection of extra

[10] For an explicit evaluation of this term, given by the diagram (2.107), see for example (4.12) in [25].

space-time dimensions with the framework of string theory. For example, the notion of orbifold is borrowed from strings, where it is a symmetry of the entire theory.

In string theory it is known that beside the KK momentum there are also winding numbers, which are related to nature of the string itself. A string can wrap around the compactified dimensions. In our model however we can safely neglect the winding modes since our radii are presumed large. Besides there are also extra states coming from string theory. However the nature of these states depends on the model and it is not possible to take into account their contribution without a full understanding of the string model.

Usually in GUT theories, the unification scale is close to the perturbative heterotic string scale $M_{string} \approx 5 \times 10^{17}\ GeV$. If we could identify these two scales then we could think of those model as embedded into string theory. In our model, the power-law running however lowers the scale of unification.

Note that the appearance of the GUT symmetry in string theory has a profound effect. From a field theory point of view we can say that the gauge couplings unification signals a grand unified theory at the scale M'_{GUT}. In the meanwhile, GUT symmetry is broken in string theory through the action of the orbifold. As a consequence, there are extra GUT states which are not present in the standard MSSM. For example, in the $SU(5)$ model, there appear the X and Y gauge bosons, which are responsible for proton decay. These states exist in the string spectrum, with masses $m \sim n/R$, $n \in \mathbb{Z}$, with $n \geq 1$.

We must outline that the presence of such states with masses $n\mu_0$ means that these states, together with their KK towers, should influence the gauge couplings between μ_0 and M'_{GUT}. The effects of the X and Y bosons must be included below the unification scale as a consequence of GUT breaking via an orbifold projection. Breaking GUT symmetry in string theory through an orbifold doesn't mean that the GUT symmetry is restored above M'_{GUT}. Actually GUT symmetry is restored at the scale μ_0 since the states with masses $m \geq \mu_0$ fall into GUT multiplets. These states are then called GUT precursors [26]. Since the KK towers corresponding to the X and Y bosons appear in complete GUT multiplets the rapid gauge coupling unification is still preserved. This property is common to all the GUT group.

String theory can also address the question of whether it is possible to have such a small scale $\mu_0 = R^{-1}$. There are two ways of answering. Let us start with a perturbative description, i.e. a weakly coupled heterotic string with a large radius of compactification. As was done in Ref. [27, 28], the Planck scale remains at the perturbative heterotic string value 5×10^{17} GeV.

In this perturbative framework, it is not possible to explain why the scale of gauge coupling unification should be lower than the Planck scale. Indeed, weakly coupled heterotic strings [29] lead to gauge coupling unification near the Planck scale. To escape this, we can interpret the unification scale as the string scale embedding directly our scenario into string theory.

In weakly coupled heterotic strings, the tree-level string scale M_{string} is related to the Planck scale in the following way

$$M_{string} \ \sim \ g_{string}\, M_{Pl} \tag{3.137}$$

where g_{string} is the unified string coupling. This relation does not depend on the number of extra dimensions nor the radii of compactification.

String dualities change the situation. Heterotic strings at strong coupling can indeed be described as (open) Type I strings at weak coupling. We can study the non-perturbative regime of heterotic string theory by analyzing weakly coupled Type I strings. In Type I string theory there exists the intriguing possibility [30, 31] of lowering the fundamental string scale. In this case equation (3.137) is no longer valid and it is substituted by the relation

$$M_{string} \ \sim \ e^{\phi/2}\, g_{gauge}\, M_{Pl} \tag{3.138}$$

where ϕ is the so-called ten-dimensional dilaton field and g_{gauge} is the Type I gauge coupling. Choosing the VEV of the dilaton in a suitable one, we can lower M_{string} with respect to M_{Pl}.

Let us recall the ten dimensional action

$$S = \int d^{10}x \left\{ e^{-2\phi}\mathcal{R} + e^{-\phi}F^2 \right\} \tag{3.139}$$

It is then possible to relate M_{string} and M_{Pl}, eliminating the dependence on the dilaton. We find

$$M_{string} \ \sim \ \sqrt{\frac{1}{\alpha_{gauge} M_{Pl}}}\, V^{-1/4} \tag{3.140}$$

where $\alpha_{gauge} \equiv g^2_{gauge}/(4\pi)$ and where $(2\pi)^6 V$ is the six-dimensional volume of the compactification. The relation (3.140) is just an order of magnitude estimate, which could be deduced on dimensional grounds.

We will try to identify α_{gauge} with α'_{GUT} and M_{string} with $M'_{GUT} = 10$ TeV $\ll 10^{16}$ GeV. Let us assume, as usual, δ extra dimensions of radius $R \equiv \mu_0^{-1}$. From the relation (3.140) we can determine the (common) radius

r for the $6-\delta$ compactified dimensions. The normalized compactification volume is given by

$$V \sim R^{\delta} r^{6-\delta} \tag{3.141}$$

thus we find

$$\frac{M'_{GUT}}{M_{Pl}} \sim \alpha'_{GUT} (M'_{GUT} R)^{\delta/2} (M'_{GUT} r)^{3-\delta/2} \tag{3.142}$$

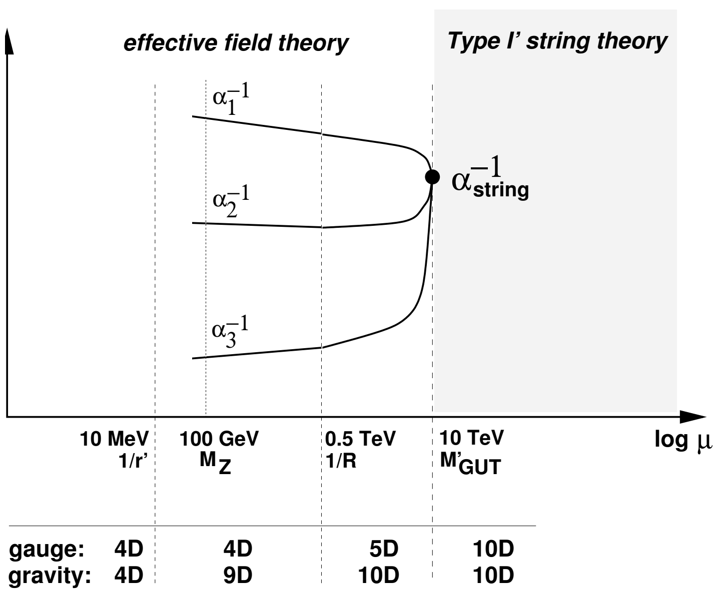


Figure 3.6: Sketch of the evolution of the gauge couplings within a Type I′ realization of our scenario. The corresponding effective number of spacetime dimensions felt by the gauge and gravitational couplings are indicated.

We start with the case $\delta = 1$. If we take $M'_{GUT} = 10$ TeV, from Figs. 3.3 and 3.4 we see that $M'_{GUT} R \approx 20$ and $\alpha'_{GUT} \approx 1/50$. This implies that $M'_{GUT} r \approx 10^{-6}$. Thus we find that the radius r of the five extra dimensions must be smaller than the string length scale. This signals that we must make use of the so called Type I′ description. In string theory this is possible thanks to T-duality [11]. In general a Type I theory with a compactified radius r is equivalent (in the sense of the T-duality) to a Type I′ theory

[11] T-duality is a symmetry which links the large scale behavior of string theory to its small scale structure.

with a compactified radius $r' \equiv (M_{string}^2 r)^{-1}$

$$M_{string} r \;\leftrightarrow\; (M_{string} r')^{-1} \tag{3.143}$$

We therefore pass to a Type I$'$ description with

$$(r')^{-1} \;\sim\; 10^{-6}\, M'_{GUT} \;\sim\; 10 MeV \tag{3.144}$$

We thus found a way to associate the scale of gauge coupling unification at 10 TeV with the string scale of a Type I$'$ theory: one dimension has radius $R^{-1} \approx 0.5$ TeV and the five remaining dimensions have radii $r' \sim (10MeV)^{-1}$. This scenario is sketched in Fig. 3.6. Below the unification scale we have an effective field theory and above the Type I$'$ string description is adequate. We identify the string scale with the unification scale $M'_{GUT} \approx 10$ TeV. The gauge couplings feel a new dimension at $R^{-1} \equiv \mu_0 \approx 0.5$ TeV. Gravity instead feels five new dimensions.

A similar calculation for the case $\delta = 2$ provides the result $r' \sim (0.1\, GeV)^{-1}$ for the remaining four dimensions. We point out that these results depend on the string scale. Indeed, if we take $M'_{GUT} = 10^{12}$ GeV we have now the same result $r' \sim (10^9 GeV)^{-1}$ both for the case $\delta = 1$ and $\delta = 2$.

Chapter 4

Susy and dark matter

In astronomy there is overwhelming evidence that most of the mass in the universe is some non luminous dark matter of yet unknown composition. The bulk of this dark matter is of non baryonic nature. The presence of an exact discrete symmetry, R-parity, in the MSSM guarantees that the lightest supersymmetric particle (LSP) is stable. Such a weakly interacting massive particle (WIMP) would have a cosmological abundance $\Omega \sim 1$ today. It is then natural to consider the possibility that this LSP is the dark matter. In most cases, this particle is the neutralino, a linear combination of the susy partners of the photon, the neutral Z^0 and the Higgs bosons. Although dark, in the sense that they cannot emit nor absorb electromagnetic radiation, WIMPs must have nonzero coupling to ordinary matter, because they must annihilate into it during the freeze out period in the early universe. The presence of extra dimensions will modify the expansion law of the universe, which is crucial in studying Boltzmann equation for the relic density. In this way the resultant relic density can be altered and the allowed region for the MSSM can be dramatically modified from the one in standard cosmology.

4.1 Neutralino cosmology

The most of dark matter candidates are of non-baryonic nature. The main distinction is between "hot" and "cold" dark matter. A dark matter is called "hot" if it was moving at relativistic speeds at the time galaxies could just start to form. It is called "cold" if it was moving non-relativistically at that time. This categorization has important ramifications for structure formation. Experimental studies on galaxy formation may provide an important hint on whether dark matter is hot or cold. Hot dark matter can cluster only when it has cooled to non-relativistic speeds. N-body simulations of structure formation in a universe dominated by hot dark matter, fail to reproduce the observed structure.

The non-baryonic cold dark matter candidates are basically elementary particles which have not yet been discovered. The leading candidates are axions and weakly-interacting massive particles (WIMPs). These are stable particles which arises in supersymmetric extension of the Standard Model. WIMP masses are typically in the range 10 GeV-10 TeV, and they have interactions with ordinary matter which are characteristic of the weak interactions. The most promising WIMP candidate is the neutralino [32, 33], and in the rest of the chapter we will focus on this possibility.

Then, among the dark matter candidates, the neutralino in supersymmetric models is a suitable one. Two conditions must be satisfied: the neutralino must be the lightest supersymmetric particle (LSP) and R-parity must be conserved. The first one occurs in a broad region of the space parameter of minimal supergravity (mSUGRA) model, while the second is required to ensure the neutralino to be a stable particle.

Such a particle exists in thermal equilibrium and in abundance in the early universe, when the temperature of the universe exceeds the mass m_χ of the particle. The equilibrium abundance is maintained by annihilation of the particle with its antiparticle $\bar{\chi}$ into lighter particles l ($\chi\bar{\chi} \to l\bar{l}$) and vice versa ($l\bar{l} \to \chi\bar{\chi}$). In many cases, the particle is a Majorana particle in which case $\chi = \bar{\chi}$. As the universe cools to a temperature less than the mass of the particle, the equilibrium abundance drops exponentially until the rate for the annihilation reaction $\chi\bar{\chi} \to l\bar{l}$ falls below the expansion rate H, at which point the interactions which maintain thermal equilibrium freeze out, and a relic cosmological abundance freeze in.

This idea was used in the late 70's to constrain the mass of a heavy neutrino and subsequently to suggest that the dark matter could be composed of weakly interacting massive particle (WIMPs). Since then many calculations have been done and improved. The result of the cosmological abundance calculation for a thermal relic is crucial to the argument for WIMP dark matter.

At this scope let's do a simple estimate. Suppose that in addition to the known particles of the Standard Model there exist a new, yet undiscovered, stable particle χ. In thermal equilibrium, the number density of χ particles is

$$n_{eq}(\chi) = \frac{g}{(2\pi)^3} \int f(p) d^3p \tag{4.1}$$

where g is the number of internal degrees of freedom of the particle and $f(p)$ is the familiar Fermi-Dirac or Bose-Einstein distribution [1]. At high temperatures ($T >> m_\chi$), we find the usual relation $n_{eq}(\chi) \propto T^3$. At low temperatures, instead, we have

$$n_{eq}(\chi) \simeq g \left(\frac{m_\chi T}{2\pi} \right)^{3/2} e^{-m_\chi/T} \tag{4.2}$$

so that their density is Boltzmann suppressed. If the expansion of the universe were so slow that thermal equilibrium was always maintained, the number of WIMPs today would be exponentially suppressed. Essentially, there would be no WIMP at all. However the universe is not static, so equilibrium thermodynamics is not the whole story.

At high temperatures, χs are abundant and rapidly converting to lighter particles and vice versa. After T drops below m_χ, the number density of χs drops exponentially, and the rate for annihilation drops below the expansion rate. This rate is $\Gamma =< \sigma_A v > n_\chi$, where $< \sigma_A v >$ is the thermally averaged total cross section for annihilation of $\chi\bar{\chi}$ into lighter particles times relative velocity v.

4.2 Freeze out: the Boltzmann equation

In the very early universe, when the temperature is very high, all particles are in thermal equilibrium. As the universe expands, however, it cools and the interaction rates become too low to maintain this equilibrium. We say that the particles "freeze out." Unstable particles that freeze out will decay and thus will disappear from the universe. However, the number of stable particles will tend towards a non-vanishing constant, and their thermal relic density survives to nowadays.

[1] Actually, for a neutralino the Fermi-Dirac statistic is the suitable one.

This kinetic process is described quantitatively by the Boltzmann equation

$$\frac{dn_\chi}{dt} = -3Hn_\chi - \langle\sigma_A v\rangle \left(n_\chi^2 - {n_{eq}}^2\right) \tag{4.3}$$

where n_χ is the number density of the dark matter particle, H is the Hubble parameter and n_{eq} is the dark matter number density in thermal equilibrium. The quantity $\langle\sigma_A v\rangle$ is the thermally averaged annihilation cross section for annihilation of $\chi\bar{\chi}$ into lighter particles times relative velocity v.

A detailed derivation of this equation can be found in appendix A. Anyway (4.3) can be easily understood. The second term on the left-hand side accounts for the expansion of the universe. In the absence of number changing interactions, the right-hand side would be zero, and we would find $n_\chi \propto a^{-3}$, as we should. The first term in brackets on the right-hand side of equation (4.3) accounts for depletion of WIMPs due to annihilation and the second term arises from creation of WIMPs from inverse reaction. It can be naively derived by noting that, in equilibrium, the rate for depletion and creation of particles is equal. This equation both describes Dirac as well as Majorana particles, such as neutralinos. For the case of Majorana particles, the annihilation rate is $< \sigma_A v > n_\chi^2/2$, but in each annihilation two particles are removed, which cancels the factor of 2 in the annihilation rate.

It turns out to be convenient to change variables from time to temperature,

$$t \to x \equiv \frac{m_\chi}{T} \tag{4.4}$$

where m_χ is the χ mass, and to replace the number density by the co-moving number density

$$n \to Y \equiv \frac{n}{s} \tag{4.5}$$

where $s \simeq 0.4 g_* T^3$ is the entropy density. Here g_* is the effective number of relativistic degrees of freedom. Since s scales inversely with the volume of the universe when entropy is conserved, the expansion of the universe has no effect on Y. In terms of these new variables, the Boltzmann equation (4.3) becomes

$$\frac{x}{Y_{eq}}\frac{dY}{dx} = -\frac{n_{eq}\langle\sigma_A v\rangle}{H}\left(\frac{Y^2}{{Y_{eq}}^2} - 1\right) \tag{4.6}$$

It is now clear that before freeze out, when the annihilation rate is large compared with the expansion rate, Y trails the equilibrium value Y_{eq}. After freeze out, Y approaches a constant. This constant is determined by the annihilation cross section $\langle\sigma_A v\rangle$. Y will follow its exponentially decreasing equilibrium value. This behavior is shown in Fig. 4.1.

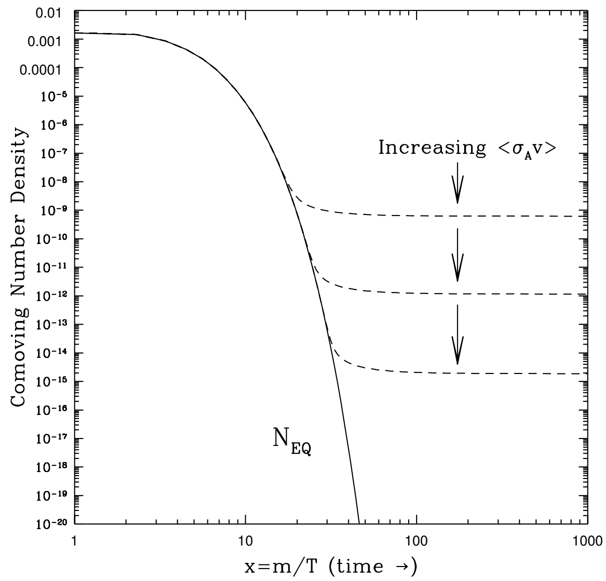


Figure 4.1: The co-moving number density Y of a dark matter particle as a function of temperature and time. From Ref. [34].

If we consider the neutralino as our WIMP, the mass and annihilation cross section are set by the weak scale: $m_\chi^2 \sim \langle\sigma_A v\rangle^{-1} \sim {M_{weak}}^2$. Freeze out takes place when the decay width is of the order of the Hubble parameter

$$n_{eq}\langle\sigma_A v\rangle \sim H \ . \tag{4.7}$$

For a non-relativistic particle we have the behavior (4.2) and the Hubble expansion rate falls with temperature as $H \equiv 8\pi G/3 \simeq 1.66 g_*^{1/2} T^2/M_{Pl}$. From these relations, we find that WIMPs freeze out when

$$\frac{m_\chi}{T} \sim \ln\left[\langle\sigma_A v\rangle m_\chi M_{Pl}\left(\frac{m_\chi}{T}\right)^{1/2}\right] \sim 30 \tag{4.8}$$

Since $m_\chi v^2 = 3T$, WIMPs freeze out with velocity $v \sim 0.3$.

At first sight one can think that freeze out should occur at $T \sim m_\chi$. But this is not the case: gravity is weak since M_{Pl} is large. The expansion rate is then extremely slow, and freeze out occurs much later than one might expect. For $m_\chi \sim 300\ GeV$, freeze out occurs at temperature $T \sim 10\ GeV$ and the corresponding time is $t \sim 10^{-8}\ s$. In the book of Kolb and Turner [35], one can find also the freeze out density

$$\Omega_\chi = m_\chi s Y_\infty \sim \frac{10^{-10}\ GeV^{-2}}{\langle \sigma_A v \rangle} \tag{4.9}$$

An estimate of the weak cross section gives

$$\langle \sigma_A v \rangle \sim \frac{\alpha_2^2}{M_{weak}^2} \sim 10^{-9}\ GeV^{-2} \tag{4.10}$$

where α_2 is the weak coupling. Equation (4.10) corresponds to a thermal relic density of $\Omega_\chi h^2 \sim 0.1$. This analysis has ignored many numerical factors, but the orders of magnitude is correct. WIMPs therefore naturally have thermal relic densities of the right order of magnitude to close the universe. This coincidence is a hint that connects the problems of electroweak symmetry breaking and the topic of dark matter.

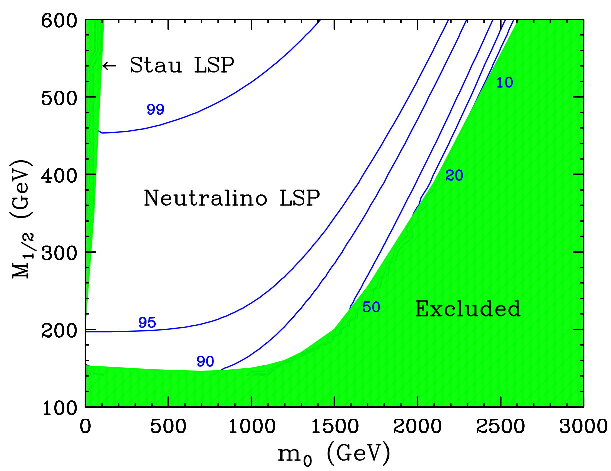


Figure 4.2: Regions of the $(m_0, M_{1/2})$ parameter space in mSUGRA with $A_0 = 0$, $\tan\beta = 10$, and $\mu > 0$. The lower shaded region is excluded by the LEP chargino mass limit. The stau is the LSP in the narrow upper shaded region. In the rest of parameter space, the LSP is the lightest neutralino, and contours of its gaugino fraction Z_g (in percent) are shown. From Ref. [36].

4.3 Neutralino annihilation

The two processes which govern nearly all of neutralino cosmology are annihilation of neutralino pairs and scattering of neutralinos off ordinary matter. In this section we discuss the annihilation cross sections. We provide results for all the final states that appear at tree level and for those one loop final states that are also important. The annihilation cross sections are needed in cosmology for calculations of the cosmological neutralino relic abundance, the flux of energetic neutrinos from neutralino annihilation in the sun and earth.

On general theoretical grounds the annihilation cross section should have the velocity dependence $\langle \sigma_A v \rangle \propto v^p$. It is then sufficient to expand the annihilation cross section in the non-relativistic ($v \ll 1$) limit [2]

$$\langle \sigma_A v \rangle = a + bv^2 + O(v^4) \tag{4.11}$$

where a is the s-wave contribution at zero relative velocity and b contains contributions from both the s and p waves. In the simplest case, the s wave is unsuppressed and $\langle \sigma_A v \rangle$ is almost energy independent. Thus, only the a term is necessary. However, it often happens that χ is a Majorana particle and the annihilation into light fermions is suppressed by helicity. In this case also the b term is needed. For practical purposes, the first two terms in (4.11) are sufficient. We can then find the quantity Y_∞ appearing in (4.9). The result for the relic abundance is

$$Y_\infty^{-1} = 0.264 g_* M_{Pl} m_\chi \left[\frac{a}{x_f} + \frac{3(b - \frac{1}{4}a)}{x_f^2} \right] \tag{4.12}$$

where $x_f = T_f/m_\chi$ is the freeze out epoch and g_* is the number of degrees of freedom evaluated at T_f.

For what regards neutralinos in the galactic halo, sun and earth they move with velocities $O(10^{-3})$ (in units where $c = 1$), so only the a term in equation (4.11) is needed for calculations involving relic neutralinos. As we saw in equation (4.8), when neutralino interactions freeze out in the early universe, their relative velocity are approximately $v \simeq 0.3$ so both the a and b terms are generally needed for relic abundance calculations. Results for the a and b terms can be found in [32, 37, 38].

Neutralino can annihilate into numerous final states. The processes which dominate are those at tree level (lowest order in perturbation theory), i.e. the two body final states. Other than fermion-antifermion pairs

[2] A detailed derivation of (4.11) can be found in the appendix B.

(neutrinos, leptons and quarks), there are

$$W^+W^-,\ Z^0Z^0,\ W^+H^-,\ W^-H^+,\ Z^0A^0,\ Z^0H^0,\ Z^0h^0,\ H^+H^-$$

and all the combinations (six) of the neutral states A^0, h^0 and H^0. Many Feynman diagrams contribute to each of these processes [39], so the computation of the total annihilation cross section turns out to be a difficult task. In one of the following sections we shall concentrate on the $\gamma\gamma$ final states, since they can be a proof of the existence of neutralino dark matter [3].

The analytic results for the a term are not cumbersome expressions. On the other hand, calculations of the b terms are more involved. They can be obtained by "brute force" evaluation of Feynman diagrams, giving rise to results which are valid for any value of the center of mass energy, including non relativistic limit. Since there is a large number of diagrams involved [4], a simple check about the calculation is the high energy behaviour of the cross sections, which must be consistent with unitarity. The complete analytic results for the a and b terms can be found in several works [39, 41, 42].

Almost all the parameters of the MSSM are involved in the calculation. Given this large freedom in adjusting the parameters, there is a plethora of ways to achieve the desired relic density for neutralino dark matter. We will see that many of these different ways may be found in minimal supergravity (mSUGRA). In various regions of mSUGRA parameter space it is possible to obtain the desired thermal relic density.

4.4 Thermal relic density: the bulk region

Numerical analysis show that the LSP is a bino-like neutralino in much of mSUGRA parameter space. We can thus consider the limit in which the neutralino is a pure bino. In this case, all processes with final state gauge bosons vanish [5].

The process $\chi\chi \to f\bar{f}$ can now occur through a sfermion exchange (t-channel). As susy partners of Standard Model bosons, neutralinos are Majorana fermions. If the initial state neutralinos are in an S-wave state, the Pauli exclusion principle implies that the initial state is CP-odd, with total spin $S = 0$ and total angular momentum $J = 0$. If the neutralinos are gauginos, the vertices preserve chirality, and the final state $f\bar{f}$ has spin $S = 1$.

[3] We must stress, however, that this contribution is negligible for abundance calculations.

[4] In order to manage this complicated task, we made use of a publicly available code [40].

[5] This follows from supersymmetry and the absence of 3-gauge boson vertices for abelian gauge group.

In order to have $J = 0$ we need a mass insertion on the fermion line. This process is therefore either P-wave-suppressed (no a term in equation 4.11) or chirality suppressed. This conclusion actually holds also for mixed gaugino-Higgsino neutralinos and can be applied to all other processes that contribute to the $f\bar{f}$ final state [39].

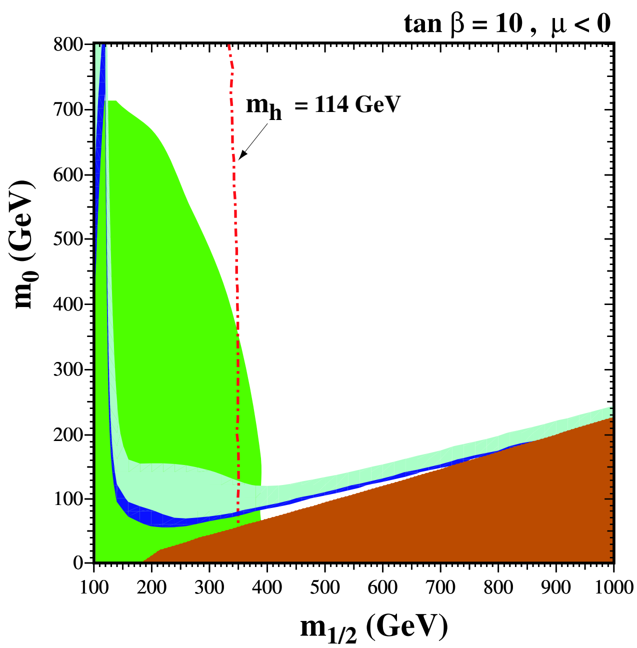


Figure 4.3: The bulk and co-annihilation regions of mSUGRA with $A_0 = 0$, $\tan\beta = 10$ and $\mu < 0$. In the light blue region, the thermal relic density satisfies the pre-WMAP constraint $0.1 < \Omega_{DM}h^2 < 0.3$. In the dark blue region, the neutralino density is in the post-WMAP range $0.094 < \Omega_{DM}h^2 < 0.129$. The bulk region is the dark blue region with $(m_0, M_{1/2}) \sim (100\ GeV, 200\ GeV)$. The stau LSP region is given in dark red, and the co-annihilation region is the dark blue region along the stau LSP border. Current bounds on $b \to s\gamma$ exclude the green shaded region, and LEP limit to the Higgs mass is shown ($m_h = 114\ GeV$). From Ref. [43].

The region of mSUGRA parameter space with a bino-like neutralino where $\chi\chi \to f\bar{f}$ yields the right relic density is the region shown in Fig. 4.3 with $(m_0,\ M_{1/2}) \sim (100\ GeV,\ 200\ GeV)$. This region is called the "bulk region". In the past, due to uncertainty in the measurement, there was a wide range of parameters with $(m_0, M_{1/2}) \lesssim 300\ GeV$ that predicted dark matter within the observed range. After WMAP data, the "bulk region" has been reduced to a thin ribbon of acceptable parameter space.

Moving from the bulk region by increasing m_0 and keeping all other parameters fixed, one finds too much dark matter. This behavior is shown in Fig. 4.3. In the bulk region, a large sfermion mass suppresses the annihilation cross section ($\langle \sigma_A v \rangle \sim m_{\tilde{f}}^{-2}$), which implies a large Ω_{DM}. In fact, sfermion masses not far above current bounds are required to offset the P-wave suppression of the annihilation cross section. In a sense, it seems that an over-closed universe can provide upper bounds on superpartner masses.

In the above we assumed that $\chi\chi \to f\bar{f}$ is the only annihilation channel. However, for neutralinos which are not bino-like, there are many other contributions. In the next, we will shortly describe this possibility studying the so called focus point region.

We want to stress that the bulk region, is also severely constrained by other data. The existence of a light superpartner spectrum in the bulk region implies a light Higgs boson mass, and typically significant deviations in low energy observables such as $b \to s\gamma$. Current bounds on the Higgs boson mass, as well as concordance between experiments and standard model predictions for $b \to s\gamma$, therefore disfavor this region. This is shown in Fig. 4.3. For this reason, we will consider also other possibilities.

4.4.1 Focus point region

As can be seen in Fig. 4.2, in mSUGRA the neutralino can be gaugino-like or higgsino-like; thus a bino-like LSP is not the ultimate answer. When m_0 is very large, to ensure electroweak symmetry breaking the Higgsino mass parameter $|\mu|$ must be small. In this case, the LSP becomes a gaugino-Higgsino mixture. The allowed region for this kind of LSP is called the focus point region, a name which resembles peculiar properties of the renormalization group equations [44, 45, 46].

In the focus point region sfermions are very heavy and thus the related diagram are suppressed. However, the existence of Higgsino components in the LSP implies that diagrams with exchange of charged higgsino/gaugino, like in $\chi\chi \to W^+W^-$, are no longer suppressed.

Neutralinos may annihilate efficiently enough as to produce the desired thermal relic density. Cosmological constraints limit the regions with the right relic densities (see Fig. 4.4). However, since the right relic density can be achieved with arbitrarily heavy sfermions, cosmology doesn't supply upper bound on superpartner masses.

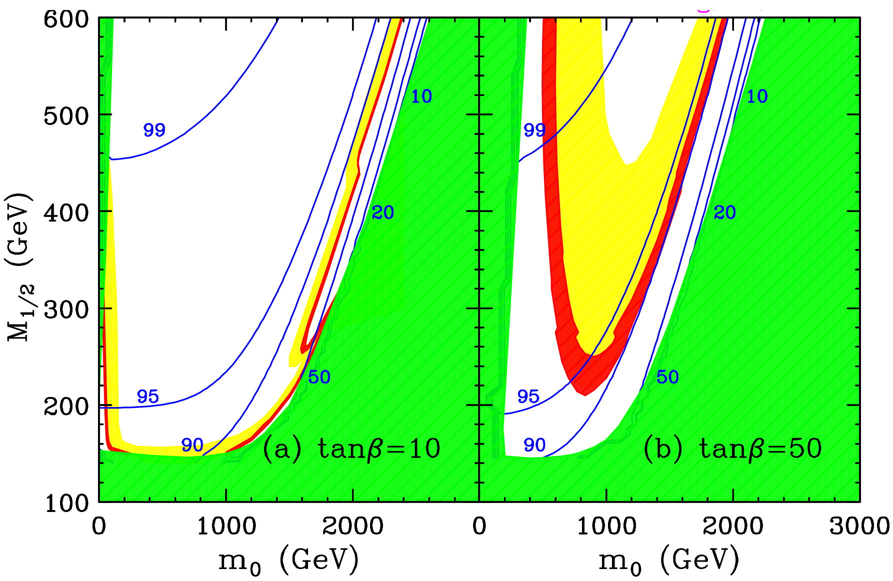


Figure 4.4: Focus point region of minimal supergravity for $A_0 = 0$, $\mu > 0$, and $\tan\beta$ as indicated. The excluded regions and contours are as in Fig. 4.2. In the light yellow region, the thermal relic density satisfies the pre-WMAP constraint $0.1 < \Omega_{DM}h^2 < 0.3$. In the medium red region, the neutralino density is in the post-WMAP range $0.094 < \Omega_{DM}h^2 < 0.129$. The focus point region is the cosmologically favored region with $m_0 \gtrsim 1\ TeV$. From Ref. [36].

4.4.2 The A funnel region

The coupling with the A Higgs boson opens the possibility that the dark matter annihilates to fermion pairs through an s-channel pole. This process is efficient when $2m_\chi \approx m_A$. The A resonance region occurs in mSUGRA for $\tan\beta \gtrsim 40$ [47, 48] and is shown in Fig. 4.5. This resonance is so efficient that the relic density may be reduced too much. Therefore, to obtained a suitable relic density the process must be near the resonance, but not exactly on it.

4.4.3 Co-annihilation region

In the previous cases we neglect the possibility that there are other particles present in significant numbers when the LSP freezes out. In this case the

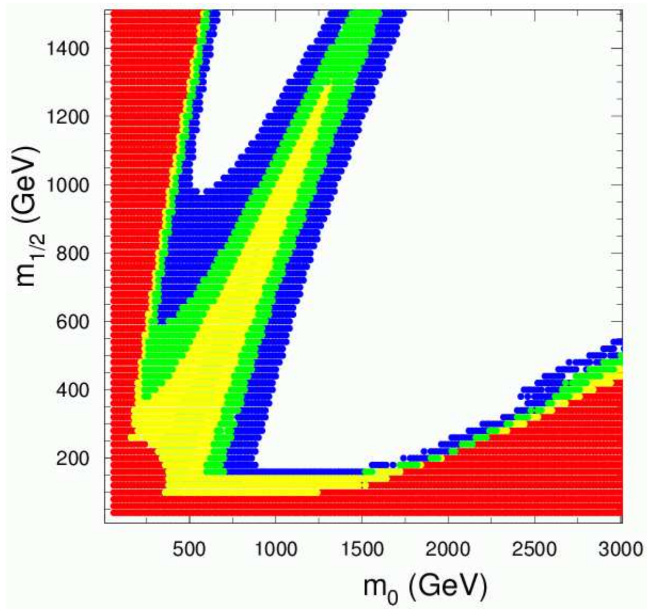


Figure 4.5: The A funnel region of minimal supergravity with $A_0 = 0$, $\tan\beta = 45$, and $\mu < 0$. The red region is excluded. The other shaded regions have $\Omega_{DM}h^2 < 0.1$ (yellow), $0.1 < \Omega_{DM}h^2 < 0.3$ (green), and $0.3 < \Omega_{DM}h^2 < 1$ (blue). From Ref. [48].

$\chi\chi$ annihilation is no more efficient.

The neutralino density may drop due to co-annihilation with the other species [49, 50]. These particles must be mass degenerate with the neutralino at the freeze out temperature, $T \approx m_\chi/30$. Co-annihilation can dominate over P-wave-suppressed $\chi\chi$ annihilation cross section and thus may be important even with mass splittings much larger than T.

As can be seen in Fig. 4.3, the co-annihilation possibility is realized in mSUGRA along the $\tilde{\tau}$ LSP (χ LSP border). In the acceptable region, we can thus find a narrow finger extending up to masses $m_\chi \sim 600\ GeV$.

4.5 LSP: limits and constraints

We saw that the most well motivated WIMP candidate is the lightest supersymmetric particle (LSP). In most of the parameter space of the MSSM the LSP is the neutralino, a linear combination of the supersymmetric partners of the photon, the Higgs bosons and the neutral Z^0 boson.

Extensive calculations have shown that the cosmological abundance of the LSP is close to unity and suitable for solving the dark matter problem, independent of the specific composition of the LSP. In particular, we must outline that models with neutralino as LSP are favourite for cosmological reasons, giving rise to acceptable galaxy formation (cold dark matter).

Lower limits to the age of the universe are often used to provide constraints to $\Omega_\chi h^2$. It is often misstated that if $\Omega_\chi > 1$, then neutralino will over close the universe. However, adding matter to the universe doesn't change its geometry, but causes the universe to expand more rapidly. In this case the universe will reach the actual size in a shorter period of time. Thus, lower limits to the age of the universe provide upper bounds to Ωh^2, and therefore to $\Omega_\chi h^2$. The limits to Ωh^2 for a given age of the universe are obtained assuming the universe is matter dominated [6].

Obviously, the mass density of neutralinos must be less than the total mass density, so for a universe older than 10 billion years, we must have

$$\Omega_\chi h^2 \simeq 1 \ . \tag{4.13}$$

Experimentally, there is evidence that the universe is closed. Data collected until now suggest that the universe is actually flat. Even the inflation model suggests that the density of the universe is the critical one. If the universe is open or flat constraints are even stronger: for a universe 10 billion years old we must have $\Omega_\chi h^2 \lesssim 0.5$, while for a value of 13 billions years the constraint is $\Omega_\chi h^2 \lesssim 0.25$.

Now we can give a limit on the WIMP mass. On dimensional arguments annihilation cross section is generally expected to decrease as the neutralino mass is increased, so the relic abundance should increase. Therefore, heavier neutralino should be more likely to dominate the mass of the universe and partial-wave unitarity provides an upper limit $\propto m_\chi^{-2}$ to the coefficient a and b in the non-relativistic expansion of the cross section. This can be used to put a model independent lower bound

$$\Omega_\chi h^2 \gtrsim \left(\frac{m_\chi(TeV)}{300}\right)^2 \ , \tag{4.14}$$

to the cosmological density of any stable thermal relic. The age of the universe constraint then leads to the limit of the mass of a stable dark matter particle $m_\chi \lesssim 300TeV$. This conclusion will not change if the relic

[6] We must point out that these limits cannot be relaxed by introducing a cosmological constant.

density is determined by annihilation near a threshold or co-annihilation (see section 4.4.3). We must outline that this limit is in general two orders of magnitude greater than the masses usually considered, but it is referred to a model with a coupling of order one. Instead, in the MSSM, the cross section are proportional to a coupling of the order $\alpha^2 \sim 10^{-4}$. So the largest cosmological acceptable masses are

$$m_\chi \lesssim 3TeV \tag{4.15}$$

For a flat universe, neutralino with an abundance $\Omega_\chi h^2 \simeq 0.25$ is the most attractive. Summarizing the results we have that

$$0.025 \lesssim \Omega_\chi h^2 \lesssim 1 \ , \tag{4.16}$$

is a conservative range for the relic density of neutralino which could take into account dark matter in the galactic halo.

4.6 Dark matter detection

We saw that one of the successes of supersymmetry with exact R-parity conservation is the prediction of an electrically neutral LSP, which represents a good candidate for the dark matter. The most attractive prospects for direct detection of susy dark matter comes from the idea that the lightest neutralino is the LSP, as is the case of mSUGRA models.

We saw in section 4.1 that once the universe cooled and expanded, the very massive sparticles can no longer be produced and they all annihilate or decay into neutralino. The remaining neutralinos can annihilate through processes $\chi\chi \to f\overline{f}$ with the exchange of squarks and sleptons or the exchange of Higgs scalars or a Z boson. For a high massive neutralino new channels open like $\chi\chi \to W^+W^-$, ZZ, Zh^0, h^0h^0 or even $W^\pm H^\mp$, ZA^0, h^0A^0, h^0H^0, H^0A^0, H^0H^0, A^0A^0, or H^+H^-. When the density of LSPs decreased, the annihilation rate became very small, and the χ relic density is determined by the Boltzmann equation, i.e. the annihilation rate and the dilution caused by the expansion of the universe.

The predicted density of a bino-like (or perhaps higgsino-like [7]) neutralino LSP obtained by doing these calculations can have the right range to constitute a fraction of the critical density of the universe, and perhaps to explain the rotation curves of galaxies [34, 51, 52].

[7] In presence of extra dimensions we will see that both the situations can happen.

It is also necessary to require that the density of surviving LSPs not be too large. Otherwise, the universe could have not reached its present size and age of at least 10^{10} years. This puts an upper limit on the LSP mass, but this limit is not parameter independent, because if the masses are varied in the right way, the LSP may happen to annihilate very efficiently through a resonance.

If neutralino LSPs were really the cold dark matter, then their mass density in our neighborhood must be at least about 0.1 GeV/cm^3 in order to explain the rotation curves of galaxies. In this case, they should be detectable by means of weak interactions with ordinary matter, or through annihilations.

The direct detection of χ depends on their elastic scattering of heavy nuclei in a detector. The elementary process is the scatter of a quark by virtual exchange of squarks, a Z boson, or Higgs scalars; another possibility is the scatter of gluons through one-loop diagrams [8]. The energy transferred to the nucleus in these collisions is typically of order tens of keV. However, there are important backgrounds from radioactivity and cosmic rays. It turns out that the optimal detector material (e.g. germanium, silicon, or niobium) depends on the details of the χ-nucleus interaction.

Another, more indirect, way to detect neutralino LSPs is through their annihilations. This can occur in regions of space where the density is very high in order to increase the possibility of the interaction. This can occur if the LSPs lose energy by repeated scattering off of nuclei, eventually becoming concentrated inside massive astronomical bodies like the earth or the sun. In this case the annihilation of neutralino pairs into neutrinos is the most important process, since all the other known particles cannot escape from the center of the massive object where the annihilation takes place. In particular, muon neutrinos and antineutrinos from $\chi\chi \to \nu_\mu \overline{\nu}_\mu$ will travel large distances, finally undergoing a charged-current interaction leading to energetic muons. There are also interesting possible signatures from neutralino LSP annihilation in the galactic halo which might produce detectable quantities of high-energy photons, positrons, and antiprotons [34, 51, 52].

4.7 Neutralino dark matter in higher dimensions

Cosmological observations, in particular the Wilkinson Microwave Anisotropy Probe (WMAP) satellite [53, 54], have established the CDM cosmological

[8]We will see this process in detail in a following section.

model [9] with great accuracy. The relic abundance of the cold dark matter is estimated to be

$$\Omega_{CDM}h^2 = 0.1126^{+0.00161}_{-0.0181} \tag{4.17}$$

These observations, which have enhanced the precision in relic density measurements, have greatly reduced the parameter space of the MSSM. We saw in section 4.2 that the thermal relic density of a dark matter particle can be obtained by solving the Boltzmann equation

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma_A v \rangle \left[(n_\chi)^2 - (n_{eq})^2 \right] \tag{4.18}$$

As we can see, the thermal relic density depends on the underlying cosmological model as well as its annihilation cross section. If a non standard cosmological model is taken into account, the resultant relic density of the dark matter can be altered from the one in standard cosmology. For what concerns the cross section, this will be modified by the appearance of other particles (KK modes and other exotic ones).

To evaluate the situation in presence of extra dimensions we must analyze the differences with respect to the standard framework. In section 3 we found that the behavior of the Hubble constant changes as a result of a modified Friedmann equation. This will change the left hand side of (4.18), which takes care of the expansion of the universe. In the standard setup, this term is responsible for the density $n_\chi \propto a^{-3}$, in absence of interactions. The brane world model is instead a well known example of a non standard cosmological model. The Friedmann equation for spatially flat spacetime is now [55, 56, 57, 58]

$$H^2 = \frac{8\pi G}{3}\rho \left(1 + \frac{\rho}{\rho_0} \right) \tag{4.19}$$

where

$$\rho_0 = 96\pi G M_5^6 \tag{4.20}$$

We recall here that H is the Hubble parameter, ρ is the energy density of matter, while M_5 is the five dimensional Planck mass and we have neglected the four dimensional cosmological constant. Moreover, we have omitted the so called "dark radiation" term, which is constrained by nucleosynthesis analysis. The second term proportional to ρ^2 leads to a non standard behavior. At a high energy regime ($\rho \gg \rho_0$), this term dominates and the universe obeys a non standard expansion law.

[9] CDM stands for cold dark matter. Cold dark matter is composed of objects sufficiently massive that they move at sub-relativistic velocities. The low velocities of cold dark matter allow the formation of structures on small scales.

As we did in section 4.2, we rewrite equation (4.18) into the form

$$\begin{aligned} \frac{dY}{dx} &= -\frac{s}{xH}\langle\sigma v\rangle(Y^2 - Y_{EQ}^2) \\ &= -\lambda\frac{x^{-2}}{\sqrt{1+\left(\frac{x_t}{x}\right)^4}}\langle\sigma v\rangle(Y^2 - Y_{EQ}^2), \end{aligned} \tag{4.21}$$

in terms of the number density to entropy ratio $Y = n/s$. We also defined the variable $x = m/T$, where m is a dark matter particle mass and $\lambda = 0.24 g_*^{1/2} M_{Pl} m$, as can be inferred from the estimates in section 4.2. The quantity x_t is defined as

$$x_t^4 \equiv \left.\frac{\rho}{\rho_0}\right|_{T=m}. \tag{4.22}$$

When $x \ll x_t$, the ρ^2 term dominates in equation (4.19), while the ρ^2 term becomes negligible after $x \gg x_t$ and we recover the expansion law in the standard cosmology. The temperature defined as $T_t = m x_t^{-1}$ is often called "transition temperature" since the expansion law of the early universe changes from the non-standard one to the standard one. We focus on the effect of the ρ^2 term for the dark matter relic density. Then we consider the case that the decoupling temperature of the dark matter (let us call it T_d) is higher than the transition temperature, namely $x_t \geq x_d = m/T_d$. Using the definition of ρ_0 and x_t in equations (4.20) and (4.22), this condition gives

$$\begin{aligned} M_5 &\leq \left(\frac{\pi^2 g_*}{30} m^4 \frac{1}{96\pi G} x_d^{-4}\right)^{1/6} \\ &\simeq 4.6 \times 10^3\,\mathrm{TeV}\left(\frac{m}{100\,\mathrm{GeV}}\right)^{2/3}\left(\frac{20}{x_d}\right)^{2/3}. \end{aligned} \tag{4.23}$$

Here we have normalized the decoupling temperature by its typical value: in our case the estimate is $x_d \equiv m/T_d \simeq 20$. For $M_5 \lesssim 10^3$ TeV, we can expect significant brane world cosmological effects.

We can now analyze our model, checking where this contribution dominates, i.e. where $\rho >> \rho_0$. In this case we have a simple relation between the five dimensional Planck mass and the compactification radius, given by

$$(M_5)^3 R = M_P^2 \tag{4.24}$$

where $\mu_0 \equiv 1/R$ is the compactification scale.

We then easily obtain the table 4.1. As one can see, the ρ^2 term will never dominate, so we can use the usual Friedmann equation. Unfortunately

$\mu_0 \equiv 1/R$ Gev	$M_5 = (\mu_0 M_{Pl}^2)^{1/3}$ GeV	$\rho_0 = 96\pi G M_5^6$ GeV^4
10^5	$2.15 \cdot 10^{14}$	$1.2 \cdot 10^{49}$
10^6	$4.64 \cdot 10^{14}$	$1.2 \cdot 10^{51}$
10^8	$2.15 \cdot 10^{15}$	$1.2 \cdot 10^{55}$
10^{10}	$1 \cdot 10^{16}$	$1.2 \cdot 10^{59}$
10^{12}	$4.64 \cdot 10^{16}$	$1.2 \cdot 10^{63}$

Table 4.1: Values of the $D = 5$ Planck mass M_5 and of ρ_0 for different values of the compactification scale μ_0.

in the literature there is not a general expression of the Hubble expansion rate H for an arbitrary number [10] of extra dimensions δ. There are some interesting issues concerning the $D = 6$ dimensional case in [62, 63] and also in [64, 65]. We considered explicitly only the case $D = 5$ and we argued that the conclusion still holds in the higher dimensional case [66].

At this point we can analyze the changes in the cross section. This involves only tree level processes. A typical one will be of the form

$$\chi + \chi \rightarrow X + X \tag{4.25}$$

where X are KK modes. Since the neutralino is the lightest particle of the MSSM, its mass will never exceed some TeV. However, KK modes will have at least a mass of 100 TeV for the minimal scenario. So energy conservation will forbid any annihilation. However, we will see in the following that KK modes can play a role in a loop process, since in the loop the energy must be not conserved.

4.8 Phenomenology

In this section we present the results for the low energy phenomenology of our scenario. To compute the weak scale parameters we used the publicly available codes ISASUGRA [20] and DarkSUSY [40, 67]. We modified the ISAJET routines in order to take into account the power-law running of the gauge couplings, of the Yukawa couplings and of all the soft terms following

[10] See [59, 60, 61] for general reviews on the subject.

the procedure outlined in section 3. We solved the differential equations [11] given by the running of the couplings and masses imposing boundary conditions as in the usual mSUGRA scenario [68, 69]. In this way we have, at the new unification scale M'_{GUT}, a universal scalar mass m_0, a universal gaugino mass $m_{1/2}$ and a common trilinear coupling A_0. The only other parameters which need to be specified are $\tan\beta$ and $sign(\mu)$.

For a fixed choice of the extra-dimensional parameters μ_0, δ and η we performed a detailed scan of the (m_0,$m_{1/2}$) parameter space keeping fixed $\tan\beta$, A_0 and $sign(\mu)$. We computed for every model the thermal relic density

$$\Omega_\chi h^2 = \frac{m_\chi n_\chi}{\rho_c} \tag{4.26}$$

solving the Boltzmann equation (4.3) for the neutralino number density. We considered only tree-level annihilation processes in the computation of $\langle\sigma_A v\rangle$. This implies that we do not have to worry about the presence of the KK states. In fact KK modes are too heavy to be produced in the intermediate tree-level annihilation channels. This is guaranteed by the fact that the neutralino mass (i.e. the mass of the lightest supersymmetric particle) is at most of order of 1 TeV for every model we considered. For the relic density computation we have taken into account all the possible coannihilations with other sparticles. Because the mass of the first KK excited level is of order μ_0 there is no possible coannihilation between the neutralino (a 0-mode particle) and the KK particles. In this way the standard treatment of the coannihilations is still valid. As discussed in the previous paragraph, the possible difference with the standard scenario may arise only from the Hubble expansion rate H.

Let us fix the notation defining the lightest neutralino as the linear combination

$$\tilde{\chi} = N_{11}\tilde{B} + N_{12}\tilde{W} + N_{13}\tilde{H}_u + N_{14}\tilde{H}_d \tag{4.27}$$

where $\tilde{B}$ and $\tilde{W}$ are the bino and wino fields, while $\tilde{H}_u$ and $\tilde{H}_d$ are the two higgsinos. The gaugino fraction, as usual is defined as

$$Z_g = |N_{11}|^2 + |N_{12}|^2 \tag{4.28}$$

We say that a neutralino is gaugino-like (in particular in our case bino-like) if $Z_g > 0.9$ while it is higgsino-like when $Z_g < 0.1$. In all the intermediate cases we denote the neutralino as mixed-like.

[11] These differential equations can be solved numerically by mean of the so called Runge-Kutta method, which is described in appendix C.

As explained in section 3 we considered two different scenarios: one in which the two zero-mode Higgs fields are chiral $N = 1$ superfields as in the MSSM and the other in which they form an $N = 2$ matter hypermultiplet.

We first analyze the $N = 1$ Higgs case in the minimal scenario $\eta = 0$ in which the matter fermions do not have KK towers. We performed a detailed scan in the parameter space $(m_0,\, m_{1/2})$ fixing all the other parameters. In Fig. 4.6 we show the regions already excluded due to either theoretical or experimental reasons for a given model ($\mu_0 = 10^5$ GeV, $\delta = 1$ and $\eta = 0$). The dark gray region is excluded because the $\tilde{\tau}$ is the LSP rather than the neutralino, the red region is excluded because the models do not achieve electroweak symmetry breaking (EWSB) while the blue region is excluded because the models do not satisfy the current accelerator bounds (limits on the chargino masses, $b \to s\gamma$, etc.). In almost all the parameter space

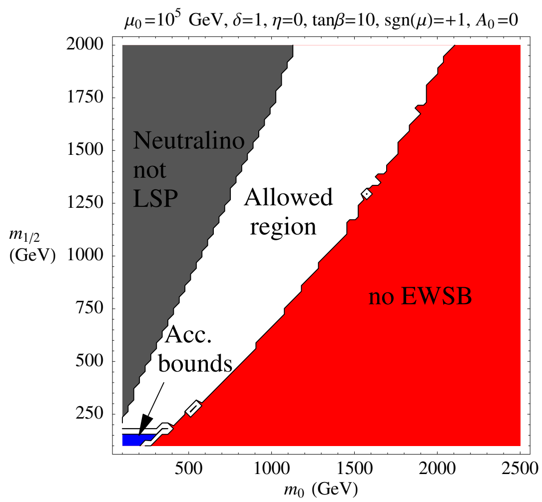


Figure 4.6: Excluded regions in the plane $(m_0,\, m_{1/2})$

the neutralino is still the LSP. One of the main result is that, unlike the standard mSUGRA case, the neutralino is no longer bino-like (as we saw in section. 4.4) but it tends to be a very pure higgsino (see Fig. 4.7). This conclusion strongly depends from the value of the compactification scale μ_0. In fact the regions in which the neutralino is higgsino-like get smaller for higher values of μ_0 (see the left panel of Fig. 4.7 for the case $\mu_0 = 10^5$ GeV and Fig. 4.9 for the two cases $\mu_0 = 10^8$ GeV and $\mu_0 = 10^{10}$ GeV). For

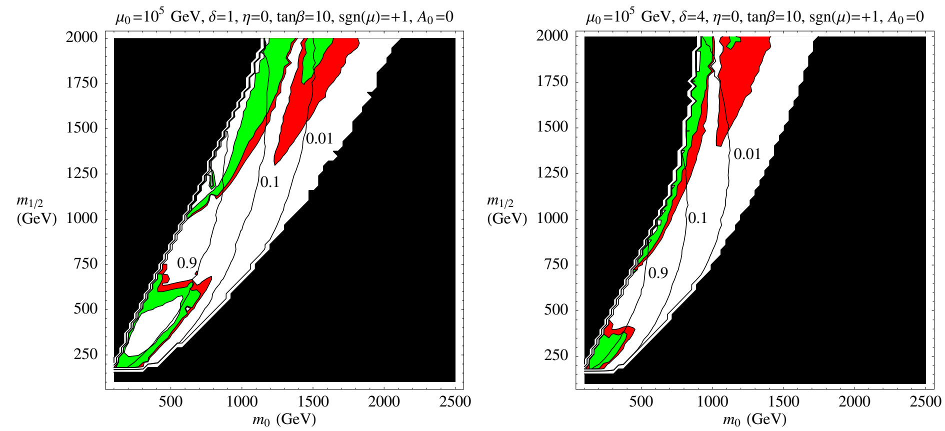


Figure 4.7: Contour plots of the gaugino fraction in the plane (m_0, $m_{1/2}$) for different values of δ.

higher values of μ_0 the higgsino-like region approaches the region excluded due to an incorrect EWSB. The previous result is only slightly dependent from the number of extra-dimensions δ as can be seen by comparing the two panels of Fig. 4.7. The only difference is a change in the shape of the excluded regions. In other words the crucial property of the β-functions is the power-law behavior rather than the effective power-law index, i.e. δ.

In every contour plot for the gaugino fraction we have shown the cosmologically allowed regions. The red regions are those for which the relic density Ωh^2 satisfies the WMAP constraints [54, 70]

$$0.09 \leq \Omega h^2 \leq 0.13 \tag{4.29}$$

while the green regions denote the pre-WMAP constraints

$$0.13 < \Omega h^2 \leq 0.30 \tag{4.30}$$

The cosmologically allowed regions have a huge overlap with the pure higgsino-like region, especially for low values of μ_0, i.e. $\mu_0 \lesssim 10^8$ GeV. For higher values of μ_0 the cosmologically allowed regions tend in general to overlap with a mixed-like neutralino region.

We also present in Fig. 4.8 the isomass contour plots for the neutralino for increasing μ_0 and for fixed δ. As in the case of the gaugino fraction we show the cosmologically allowed regions. For higher values of μ_0 high neutralino mass contours (around 1 TeV) are shifted upwards so that models that possess the right relic density have $m_\chi \lesssim 500$ GeV. Increasing the value of A_0, for example $A_0 = 2500$, implies a shift of the cosmologically allowed regions towards higher values of the neutralino mass of about $m_\chi \gtrsim 800$ GeV and a wider region excluded by the accelerator bounds. Moreover, in this case, the neutralino is a very pure higghsino $Z_g < 0.1$ in all the allowed parameter space.

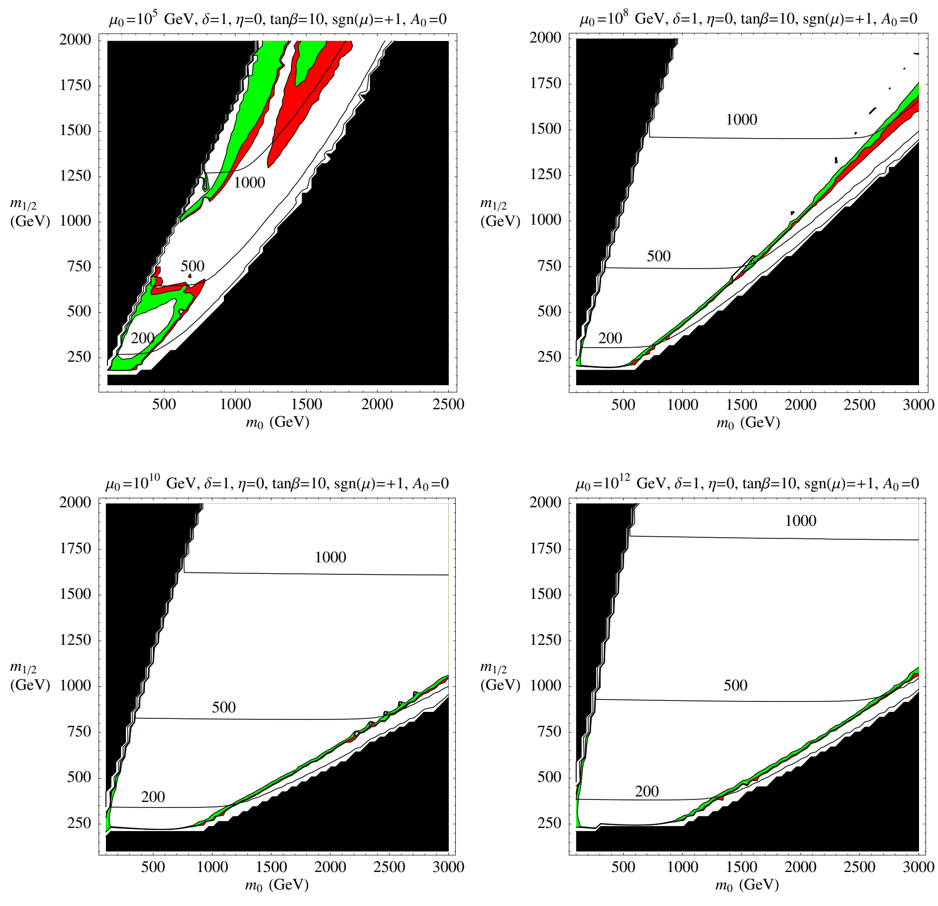


Figure 4.8: Contour plots of the neutralino mass (in GeV) in the plane (m_0, $m_{1/2}$) for increasing values of μ_0.

For increasing values of $\tan\beta$ we obtain the same shift of the cosmologically allowed regions as in the case of high A_0. Moreover the excluded region in which the neutralino is not the LSP grows towards higher values of m_0.

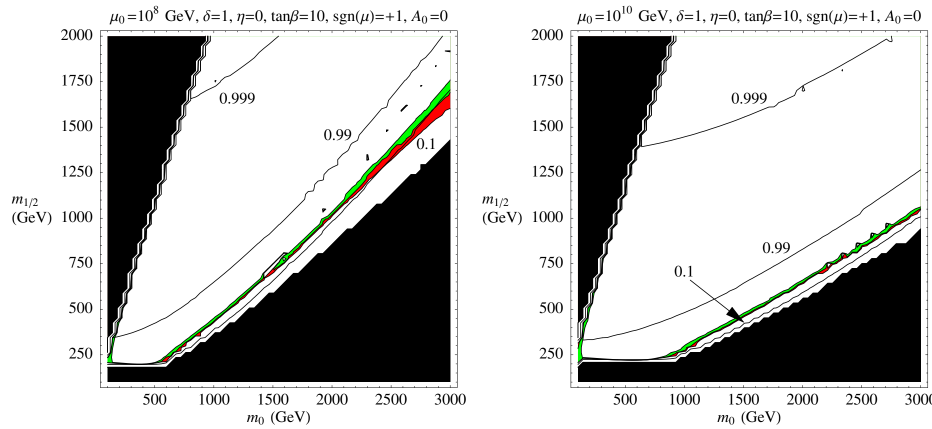


Figure 4.9: Contour plots of the gaugino fraction in the plane (m_0, $m_{1/2}$) for increasing values of μ_0.

Let us now analyze the non minimal models with $\eta > 0$. As explained in paragraph 3.2 we considered three different cases: $\eta = 1$ in which only the third family have a KK tower, $\eta = 2$ in which only the first two families have a KK tower and $\eta = 3$ in which all the families have KK tower.

For $\eta = 2$ we must have $\mu_0 \geq 10^8$ GeV in order to avoid non perturbative gauge couplings at the unification scale (see for example Fig. 4 of [1]). In this case, regions with $m_0 \gtrsim 3000$ GeV are excluded due to an incorrect EWSB while for $\tan\beta \lesssim 30$ there is no region in which the neutralino is not the LSP. For higher $\tan\beta$ the parameter space develops a region, for small m_0, excluded by EWSB constraints and by the stau being the LSP. In almost all the allowed region the neutralino turns out to be a very pure bino (as in mSUGRA) though there is still a significant overlap between the cosmologically favoured region and the region in which the neutralino is higgsino or mixed-like. The possible values of the neutralino mass are lower than in the minimal scenario, with typically $m_\chi \lesssim 500$ GeV.

In the case $\eta = 1$ in which only the third family has KK tower almost all the parameter space is allowed for low $\tan\beta$. There are only two small regions excluded by the accelerator bounds and by the EWSB constraints. The neutralino turns out to be always a bino. The cosmologically allowed region is very small and in correspondence with a low neutralino mass region, i.e. $m_\chi \lesssim 50$ GeV. For higher $\tan\beta$ we have the same behaviour except for the presence of the two excluded regions (EWSB constraints and the stau being the LSP) for small m_0.

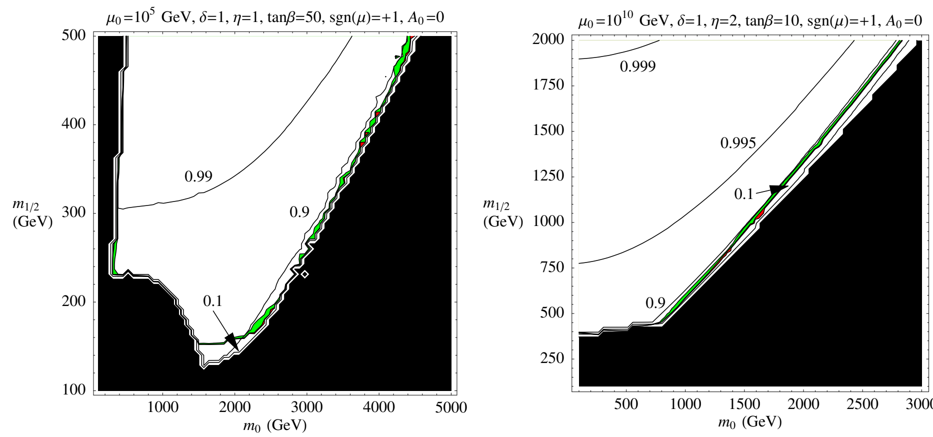


Figure 4.10: Gaugino fraction in the plane (m_0, $m_{1/2}$). Left panel: non minimal scenario $\eta = 1$. Right panel: non minimal scenario $\eta = 2$.

Finally we consider the $\eta = 3$ case in which all the families possess KK tower. In these models the region for which $m_{1/2} \lesssim 750$ GeV is always excluded for any value of $\tan\beta$. For $\tan\beta \lesssim 30$ there is only a very small region in which the neutralino is not the LSP, namely for m_0 and $m_{1/2}$ quite null. For higher values of $\tan\beta$ there is another region in which the neutralino is not the LSP together with a region excluded by the EWSB on the left of the parameter space. The neutralino mass for this kind of models is lower, $m_\chi \lesssim 200$ GeV , with respect to the minimal case (left panel of Fig. 4.11). The neutralino composition is essentially that of a pure bino except for a region close to the excluded regions in which $Z_g \leq 0.1$ (right panel of Fig. 4.11). Once again the relic density is in the right range in the regions where the neutralino is higgsino or mixed-like.

In general non minimal models are disfavoured from the point of view of the cosmological relic density because the allowed regions are very small.

In the scenario in which the Higgs form an $N = 2$ matter hypermultiplet the results about the neutralino composition and the corresponding relic density remain essentially unchanged. However there are some differences in the shape of the allowed and excluded regions in the parameter space. For example in the minimal scenario $\eta = 0$ (see Fig. 4.12(a)) the regions excluded due to EWSB and to the accelerator bounds are smaller than in

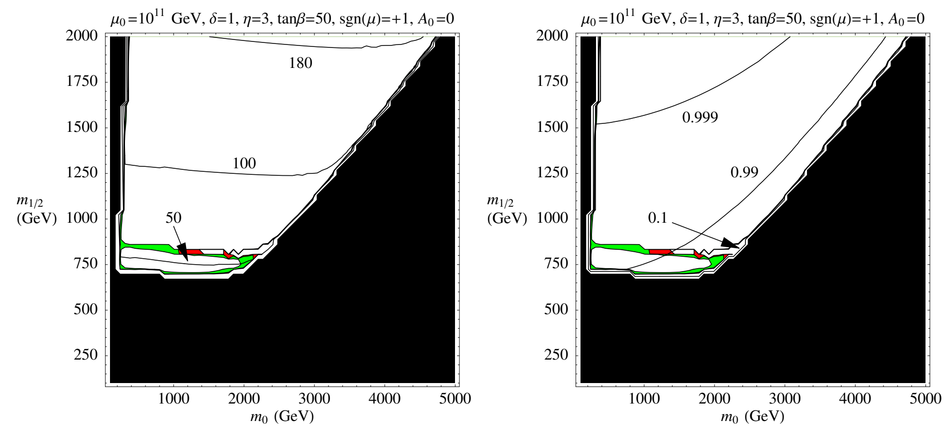
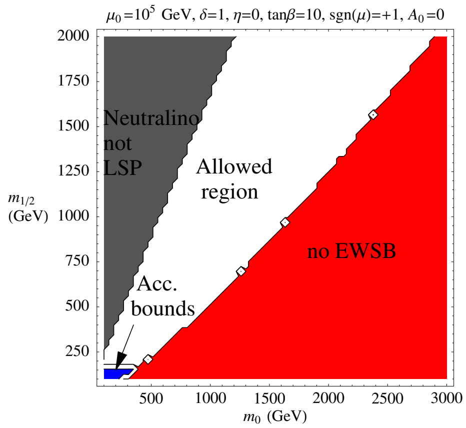
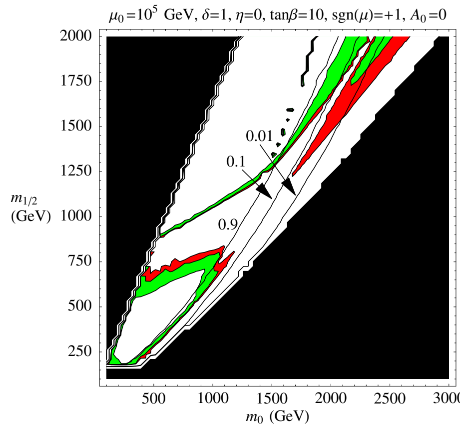


Figure 4.11: Left panel: neutralino isomass contours (m_0, $m_{1/2}$) for the non minimal scenario $\eta = 3$. Right panel: gaugino fraction in the plane (m_0, $m_{1/2}$) for the non minimal scenario $\eta = 3$.

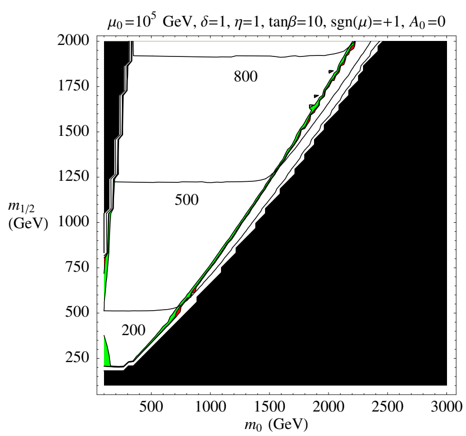
$N = 1$ case. This holds for all $\tan\beta$ and A_0. Keeping fixed all the other parameters, there are in general allowed regions in the right part of the parameter space with $m_0 > 2500$ GeV. The cosmologically allowed regions get shifted in order to "follow" the regions in which $Z_g \lesssim 0.1$. In the non minimal cases $\eta > 0$ the behaviour is the opposite and the parameter space is in general more constrained, although there are cases in which the region excluded by the neutralino not being the LSP is absent (see Fig. 4.12(c),(d)).



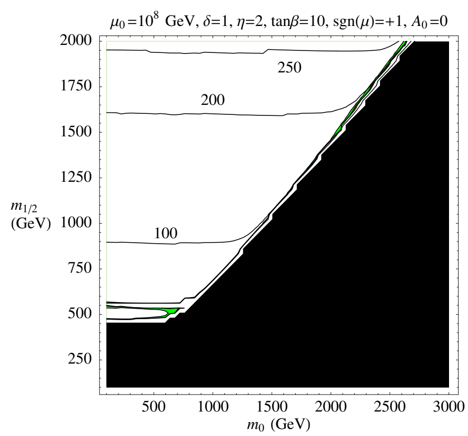
(a) Excluded regions in the plane (m_0, $m_{1/2}$) for the minimal scenario $\eta = 0$.



(b) Gaugino fraction in the plane (m_0, $m_{1/2}$) for the minimal scenario $\eta = 0$.



(c) Neutralino isomass contour in the plane (m_0, $m_{1/2}$) for the non minimal scenario $\eta = 1$.



(d) Neutralino isomass contour in the plane (m_0, $m_{1/2}$) for the non minimal scenario $\eta = 2$.

Figure 4.12: Contour plots for $N = 2$ Higgs models.

4.9 Cosmic gamma rays

Neutralino annihilation in the halo may produce a gamma-ray flux, with both the continuum and the line contributions. We must outline however, that since the gamma-ray background in the universe is not well understood, it is difficult to obtain precise measurements, any inference of dark matter in the halo from gamma-ray observations must come from distinct gamma ray signatures.

Even if neutralinos, as neutral particles, have no direct coupling to photons, they can couple to ordinary matter (otherwise annihilations cannot provide $\Omega_\chi h^2 \lesssim 1$). In this way we can have coupling to photons through loop diagrams. Therefore, there will be a small cross section for direct annihilation of two neutralinos into gamma rays. The typical velocity of neutralinos in the halo ($\sim 300km/s$) is very small compared with the speed of light. So, photons produced by annihilation of neutralinos will be monochromatic, with an energy equal to the neutralino mass. Since, there is no known astrophysical source capable of producing monochromatic gamma rays in the range $10 \div 1000$ GeV, observation of such gamma rays would provide a "smoking gun" signal for the existence of neutralinos in the halo.

When neutralinos annihilate to quarks and leptons in the galactic halo, the hadronic shower will produce gamma rays with a broad energy distribution centered around one tenth of the neutralino mass. However, this signal cannot be easily distinguished from the background. The gamma-ray flux produced in neutralino annihilations through π^0 decays can be large but in general lacks distinctive features: it is quite impossible to isolate it from the background signal. When a pion decays into two γs, the spectrum is independent of the pion energy. It is peaked at half of the π^0 mass($\sim$70 MeV) and it is symmetric with respect to this peak (if plotted in logarithmic variables). The trouble is that this is true both for pions produced in neutralino annihilations and for those generated by cosmic ray protons interacting with the interstellar medium (or other possible sources). When we consider the gamma-ray background induced by cosmic ray, the neutralino induced gamma-ray flux looks like a secondary flux due to nucleon nucleon interactions; however it can be dominant for energies above 1 GeV or so.

Let us now consider the characteristic angular dependence of the gamma-ray intensity from neutralino annihilation in the galactic halo. Annihilation of neutralinos in an isothermal halo with core radius a leads to a gamma ray flux of

$$\frac{d\Phi_\gamma}{d\Omega} = \frac{\sigma_{\gamma\gamma} v}{4\pi m_\chi^2} \int_0^\infty \rho^2(r) dr(\psi) \tag{4.31}$$

where ψ is the angle between the line of sight and the galactic center, $r(\psi)$ is the distance along that line of sight. The quantity $\sigma_{\gamma\gamma} v$ is the cross section times the relative velocity v of neutralinos. The total cross section of neutralinos into gamma rays gives rise to the continuum signal.

We can also make an order of magnitude estimate of the expected gamma ray flux. For example, for the two photon annihilation cross section arising from diagrams with slepton loops, we have

$$\sigma_{\gamma\gamma} v \simeq \frac{\alpha^4 m_\chi^2}{m_{\tilde{f}}^4} \tag{4.32}$$

where α is a typical coupling constant and $m_{\tilde{f}}$ is the mass of the slepton. In general the slepton is the heaviest particle in the loop, so its propagator leads to a suppression $m_{\tilde{f}}^{-4}$ in the cross section. The factor of α^4 in equation (4.32) comes from the four couplings in a loop diagram (which must be squared to obtain the cross section) and the factor of m_χ^2 in the numerator can be understood on dimensional ground.

For purposes of illustration, let us focus on the case that the neutralino is a pure bino, which turns out to be the lightest susy particle in much of the parameter space as we saw in section 4.4. In this case, the relic abundance turns out to be [71]

$$\Omega_{\tilde{B}} h^2 \simeq 7 \cdot 10^{-3} \left(\frac{m_{\tilde{q}}}{m_\chi}\right)^2 \left(\frac{m_{\tilde{q}}}{100\ GeV}\right)^2 \tag{4.33}$$

Assuming that the universe is flat and that the binos constitute dark matter, then $\Omega_{\tilde{B}} h^2 \simeq 0.25$ and $\sigma_{\gamma\gamma} \simeq 3 \cdot 10^{-31} cm^3/s$. If we insert this estimate into equation (4.31) we find that these kind of signals lie at the limit of the observations of current detectors. The standard isothermal halo is broad and flat, and so we need local density enhancements to raise the gamma ray signal level.

4.9.1 Sources and fluxes

Following the discussion in [72], the monochromatic gamma-ray flux (in units of $cm^{-2}s^{-1}sr^{-1}$) measured in a detector with angular acceptance $\Delta\Omega$

is

$$\Phi_\gamma(\psi, \Delta\Omega) = 0.94 \cdot 10^{-11} \left(\frac{N_\gamma \ v\sigma_{X^0\gamma}}{10^{-29}\ \mathrm{cm}^3\mathrm{s}^{-1}} \right) \left(\frac{10\,\mathrm{GeV}}{M_\chi} \right)^2 \langle \, J\,(\psi) \rangle_{\Delta\Omega} \times \Delta\Omega \tag{4.34}$$

where ψ is the angle of observation and where N_γ is the number of outgoing photons: 2 for $\chi\,\chi \to \gamma\,\gamma$ and 1 for $\chi\,\chi \to Z\,\gamma$. Here the dimensionless function $J(\psi)$ is defined as

$$J\,(\psi) = \frac{1}{8.5\,\mathrm{kpc}} \cdot \left(\frac{1}{0.3\,\mathrm{GeV/cm}^3} \right)^2 \int_{line\ of\ sight} \rho^2_\chi(l)\ d\,l(\psi) \tag{4.35}$$

and its angular average over the resolution solid angle $\Delta\Omega$ is

$$\langle \, J\,(\psi) \rangle_{\Delta\Omega} = \frac{1}{\Delta\Omega} \int_{\Delta\Omega} d\Omega' J\left(\psi'\right) \ , \tag{4.36}$$

Analogously, the gamma-ray flux with continuum energy spectrum is obtained by replacing the term in parentheses $N_\gamma \ v\sigma_{X^0\gamma}$ with a sum over all tree level final states. This formalism can be used also to estimate the flux in the simple case of a single source which can be considered as point-like. If such a source is in the direction ψ at a distance d, equation (4.36) becomes

$$\langle \, J\,(\psi) \rangle_{\Delta\Omega} = \frac{1}{8.5\,\mathrm{kpc}} \cdot \left(\frac{1}{0.3\,\mathrm{GeV/cm}^3} \right)^2 \cdot \frac{1}{d^2} \cdot \frac{1}{\Delta\Omega} \int d^3r\ \rho^2_\chi(\vec{r}) \tag{4.37}$$

where the integral is over the extension of the source (which is supposed to be much smaller than d).

There are many sources of gamma-rays coming from the annihilation of dark matter particles. An obvious source is the dark halo of our galaxy and in particular the galactic center. The reason is that it is believed that the dark matter density profile is peaked towards it, possibly with huge enhancements close to the central black hole. In particular, the galactic center is ideal both for ground and space-based gamma-ray telescopes. As satellite experiments provide a full sky coverage, they will test the hypothesis of gamma-rays emitted in clumps of dark matter which may be present in the halo [12]. Also fluxes coming from external nearby galaxies can be take into account.

The DarkSUSY package is suitable to compute the gamma-ray flux from all these sources. The continuum gamma flux from all annihilation channels can be computed and may be easily obtained for a given energy or energy threshold. Two cases are included in the package:

[12] See for example [73].

a) assuming that neutralinos are smoothly distributed in the galactic halo with ρ_χ equal to the dark matter density profile, in DarkSUSY equation (4.36) is computed for a specified halo profile and any given ψ and $\Delta\Omega$ [72].

b) a portion of dark matter can be in the form of clumps, each of which is treated as a non-resolvable source in the detector. These clumps are distributed in the galaxy according to a probability distribution which can be specified by the user. In DarkSUSY the default choice is that they follow the dark matter density profile. It is then straightforward to extend this to all other astrophysical sources. In all the calculation one must take care of redshift effects and absorption on starlight and infrared background.

In the previous section we count the possibility of an enhancement, a "spike" in the vicinity of the galactic center. However, there is no consensus in the literature, and thus DarkSUSY does not include routines for these effects.

4.9.2 A "smoking gun": $\chi\chi \to \gamma\gamma$

A relevant gamma-ray contribution may arise directly (at one-loop level) in two body final states. Although such photons are much fewer than those from π^0 decays,we saw that they have a much better signature. Neutralinos annihilating in the galactic halos have a velocity of the order $v/c \sim 10^{-3}$; hence the outgoing photons will then be nearly monochromatic, with energy of the order of the neutralino mass. Since there is no other known astrophysical source with such a signature, this detection would be a spectacular confirmation of the existence of dark matter in form of exotic massive particles. Unfortunately, the processes are loop-suppressed and to enhance the flux we probably need a halo with a large central concentration, or clumps of dark matter to detect such a signal. The branching ratio for neutralino annihilations into 2γ is typically not larger than 1%, and the largest values of $v\sigma_{2\gamma}$ lies in the range $10^{-29} \div 10^{-28}$ cm^{3}s^{-1}. Such values allow the discovery of this signal in upcoming measurements.

In the DarkSUSY package the full expression for the annihilation cross section of the process

$$\chi + \chi \to \gamma + \gamma \tag{4.38}$$

is computed at the one loop level, in the limit of vanishing relative velocity of the neutralino pair, i.e. the case of interest for neutralinos in galactic halos. The outgoing photons have an energy equal to the mass of χ:

$$E_\gamma = m_\chi. \tag{4.39}$$

The total amplitude is implemented as the sum of the contributions obtained from four different classes of diagrams:

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_{f\tilde{f}} + \tilde{\mathcal{A}}_{H^+} + \tilde{\mathcal{A}}_W + \tilde{\mathcal{A}}_G \tag{4.40}$$

where the indices label the particles in the internal loops: fermions and sfermions, charged Higgs and charginos, W-bosons and charginos, and in the gauge we chose, charginos and Goldstone bosons. For every $\tilde{\mathcal{A}}$ term, the real and imaginary parts are computed separately; the full set of analytic formulas are given in [74, 75, 76], following the notation of [77], where some of the contributions were first computed. They are rather lengthy expressions with non trivial dependences on various combinations of parameters in the MSSM. We didn't complete the calculation: we reserve it for the next future.

Appendix A

Approximate solution to the Boltzmann equation

In this appendix, following [78], we will obtain an approximate solution to the Boltzmann equation

$$\frac{dn}{dt} = -3Hn - \langle \sigma v \rangle \left(n^2 - n_0^2 \right) \tag{A.1}$$

To this end, we change variable from the time to the temperature scaled by the mass of the lepton produced in the annihilation $\chi\chi \to l\,\bar{l}$

$$x = T/M \tag{A.2}$$

It turns out to be convenient also to scale the equilibrium density by the photon number density

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 \tag{A.3}$$

and define the following quantities

$$G(x) = \frac{n}{n_\gamma} \qquad G_0(x) = \frac{n_0}{n_\gamma} \tag{A.4}$$

The equilibrium density can be easy calculated from the formula

$$n_0 = 2 \int d^3p \frac{1}{e^{E/T} + 1} \tag{A.5}$$

valid for any fermion. The factor two is the number of degrees of freedom. It is then easy to obtain the expressions for $G_0(x)$ in two limiting cases. For temperatures high in comparison with the heavy lepton mass we can assume $E \simeq p$. From the useful formula

$$\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^3} = \frac{3}{4}\zeta(3) \tag{A.6}$$

we conclude that

$$G_0(x) = 3/4 \qquad x \gg 1 \tag{A.7}$$

In the opposite limit, i.e. for temperatures low in comparison with the mass we can apply the thermodynamic limit and put $E \simeq M + p^2/2M$ into the equation (A.5) obtaining

$$G_0(x) \simeq \frac{1}{2\zeta(3)} \sqrt{\frac{\pi}{2}} \frac{1}{x^{3/2}} e^{-1/x} \qquad x \ll 1 \tag{A.8}$$

The lepton freezes out while the universe is radiation dominated: in this case the dynamics is governed by

$$H \equiv \frac{\dot{R}}{R} = -\frac{\dot{T}}{T} \tag{A.9}$$

Using x as independent variable and making use of equations (A.3), (A.4) and (A.9) we find that the Boltzmann equation (A.1) can be expressed as

$$\frac{dG(x)}{dx} = \lambda[G(x)^2 - G_0(x)^2] \tag{A.10}$$

where the constant λ is given by

$$\lambda = \frac{3\zeta(3)}{\pi^3} \left[\frac{5}{2\pi N_F}\right]^{1/2} M M_{Pl} \langle \sigma v \rangle \tag{A.11}$$

Here M_{Pl} is the Planck mass and N_F is the number of massless degrees of freedom, with each spin state of a boson contributing 1/2 to N_F and each spin state of a fermion contributing 7/16 to N_F. When weak interactions dominate we have a simple estimate for the cross section ($\simeq G_F^2 M^2$) and thus for λ

$$\lambda \simeq 2 \times 10^8 \; M^3/GeV^3 \tag{A.12}$$

Since we expect that $M > 1 \; GeV$, we see that λ is larger than 10^8. The size of λ allows us to obtain an approximate analytic solution with good accuracy.

We will be more general as possible, solving a equation in which the parameter λ is not constant. This corresponds to a cross section which depends on the temperature, as is the case. The equation then looks like

$$\frac{dG(x)}{dx} = \lambda(x)\left[G(x)^2 - G_0(x)^2\right] \tag{A.13}$$

The equation (A.13) is a non linear first order Riccati differential equation. However, it may be converted into a second order linear equation. At this end we define a new variable

$$\xi(x) = \frac{1}{\lambda_0}\int_{x_0}^{x} dx'\lambda(x') \tag{A.14}$$

where λ_0 and x_0 are constants. When λ is constant, we can set $\lambda = \lambda_0$ and $x_0 = 0$. Then the variable ξ will be simply equal to x.

We now introduce the function $f(\xi)$

$$G(x) = -\frac{1}{\lambda_0 f(\xi)}\frac{df(\xi)}{d\xi} \tag{A.15}$$

which, in contrast to $G(x)$, satisfies a linear equation. Indeed, substituting the expression (A.15) into equation (A.13) we obtain

$$\left[\frac{d^2}{d\xi^2} - \lambda_0 G_0(x(\xi))^2\right] f(\xi) = 0 \tag{A.16}$$

Since λ is very large, we can apply a WKB solution over most of the range of x. The boundary condition is that $G \to G_0$ when $x \to \infty$. The solution is then given by

$$f(\xi(x)) \simeq G_0(x)^{-1/2}\exp\left[-\int^{x} dx'\lambda(x')G_0(x')\right] \tag{A.17}$$

The function $G(x)$ has the approximate form

$$G(x) \simeq G_0(x) + \frac{G_0'(x)}{2\lambda(x)G_0(x)} \tag{A.18}$$

where the prime denotes a derivative with respect to x. Since λ is very large, the number density $G(x)$ will follow the equilibrium density $G_0(x)$ up to the point at which the second term in (A.18) becomes comparable to the first term. This happens when the temperature T is much less than the lepton mass ($x \ll 1$). So we can use the approximation (A.8), obtaining the condition

$$\frac{\lambda(x)}{\zeta(3)}\sqrt{\frac{\pi}{2}}x^{1/2}e^{-1/x} > 1 \tag{A.19}$$

for $G(x)$ to be close to $G_0(x)$. This is satisfied if $x \gtrsim 1/\ln\lambda(x)$. When x decreases below this value, $G(x)$ moves away from $G_0(x)$. Then the WKB approximation fails at a point

$$\frac{\pi\lambda(x_0)^2}{2\zeta(3)^2}x_0 e^{-2/x_0} = 1 \tag{A.20}$$

The equation (A.20) determines the value of x_0, fixing also the constant

$$\lambda_0 = \lambda(x_0) \tag{A.21}$$

We note that in this way $\xi \simeq x - x_0$ for x near x_0. Since $\lambda(x_0)$ is very large $1/x_0$ will be too. In the region near $x = 0$ we have the following equation

$$\left[\frac{d^2}{d\xi^2} - \frac{1}{4x_0^4}\exp\left(2\xi/x_0^2\right)\right] f(\xi) = 0 \tag{A.22}$$

This has a solution in terms of the modified Bessel function

$$f(\xi) = K_0\left(e^{\xi/x_0^2}\right) \tag{A.23}$$

which matches the WKB approximation around the transition region. The argument of the modified Bessel function becomes very small as $x \to 0$ and we can apply the approximation

$$f(\xi) = -\xi/x_0^2 + \ln 2 - \gamma \tag{A.24}$$

where $\gamma \simeq 0,577$ is Euler's constant. The scaled number density (A.15) is then given by

$$G(0)^{-1} = \int_0^{x_0} dx\lambda(x) + \lambda_0 x_0^2(\ln 2 - \gamma) \tag{A.25}$$

Since x_0 is very small, we can write this result as

$$G(0)^{-1} = \int_0^{x_1} dx\lambda(x) \tag{A.26}$$

where

$$\frac{1}{x_1} = \frac{1}{x_0} - \ln 2 + \gamma \tag{A.27}$$

When λ is constant, the approximation (A.25) is a leading logarithmic approximation valid to order $1/\ln^2\lambda$. From (A.4), the number density of leptons is given by

$$n(t) = n_\gamma(t)G(0) \tag{A.28}$$

This result is valid down to 1 MeV. Below this temperature, photons are reheated by electron-positron annihilation, increasing their number density by a factor of 11/4. Thus, at present we have

$$\left(\frac{n}{n_\gamma}\right)_0 = \frac{4}{11}G(0) \simeq \frac{6.9}{\lambda} \tag{A.29}$$

Limit on the mass density implies that particles with mass μ and the density of 400 per cm^3 (just like the photons of the microwave background), have a mass bounded by $\mu \lesssim 20$ eV. This translates into a limit on the mass of leptons and antileptons

$$\frac{8}{11}G(0)M \simeq \frac{14M}{\lambda} \lesssim 20eV \tag{A.30}$$

Recalling equation (A.12), we obtain the bound

$$M \gtrsim 2\ GeV \tag{A.31}$$

Appendix B

Thermal averaged cross section

In this appendix we shall deal with the thermal averaging, showing that equation (4.11) is the most general one. Let $|\mathcal{M}|^2$ be the absolute square of the reduced matrix element for the annihilation of two χ particles, summed over final spins and averaged over initial spins and particle-antiparticle states. For particles with incoming momenta p_1 and p_2 we define

$$\sigma_A v = \frac{1}{4E_1E_2}\int d\Omega|\mathcal{M}|^2 \tag{B.1}$$

where $d\Omega$ is the Lorentz invariant phase space

$$d\Omega = (2\pi)^4\delta^4(p_1+p_2-\sum_j p_j)\prod_i \frac{d^3p_i}{(2\pi)^3 2p_i^0} \tag{B.2}$$

The sum and product are over the outgoing particles. The quantity $\int d\Omega|\mathcal{M}|^2$ is manifestly Lorentz invariant and thus must depend only on the Mandelstam variable $s=(p_1+p_2)^2$. It turns out to be convenient to introduce

$$w(s) \equiv \frac{1}{4}\int d\Omega|\mathcal{M}|^2 = E_1E_2\sigma_A v \tag{B.3}$$

We now suppose that the initial χ particles have an energy distribution $f(E)$, where $E^2=\vec{p}^2+m_\chi^2$. We define the thermal average as

$$\langle\sigma_A v\rangle \equiv \frac{1}{n_0^2}\int d^3p_1 d^3p_2 f(E_1)f(E_2)\frac{1}{E_1E_2}w(s) \tag{B.4}$$

where $n_0 = \int d^3p f(E)$. The distribution function in the Boltzmann limit is given by

$$f(E) = \frac{k}{(2\pi)^3} e^{-E/T} \tag{B.5}$$

Here T is the temperature and k is the number of spin states. Note that (B.4) does not depend on the overall normalization of $f(E)$. We saw in section 4.2 that the density of the χ particles will trail its equilibrium value until the freeze out temperature $T_f \simeq m_\chi/30$. We therefore need to know $\langle\sigma_A v\rangle$ only at temperatures where we can apply the Boltzmann approximation (B.5) regardless of the statistics of the particles involved. The integrals in (B.4) cannot be done analytically. So we expand $\langle\sigma_A v\rangle$ in powers of $x = T/m_\chi$. Since we are interested in regimes where $x \lesssim 1/30$, we need only take few terms to obtain a good approximation. We will limit to order x^2. We start by writing

$$s = (p_1 + p_2)^2 = 2\left(m_\chi^2 + E_1E_2 - p_1p_2\cos\theta\right) \tag{B.6}$$

where θ is the angle between $\vec{p}_1$ and $\vec{p}_2$. All the dependence on the angle is given by this factor. Using the thermodynamic limit (B.5) we can write

$$\begin{aligned}
\langle\sigma_A v\rangle &= \frac{k^2}{(2\pi)^6 n_0^2}\int d^3p_1 d^3p_2 e^{-E_1/T} e^{-E_2/T} \frac{1}{E_1E_2} w(s) \\
&= \frac{k^2}{8\pi^4 n_0^2}\int_0^\infty dp_1 dp_2 \frac{p_1^2 p_2^2}{E_1E_2} e^{-E_1/T} e^{-E_2/T} \int_{-1}^{+1} d\cos\theta\, w(s) \\
&= \frac{k^2}{8\pi^4 n_0^2}\int_{m_\chi}^\infty dE_1 dE_2 p_1 p_2 e^{-E_1/T} e^{-E_2/T} \int_{-1}^{+1} d\cos\theta\, w(s)
\end{aligned} \tag{B.7}$$

where we used $pdp = EdE$. In this case we chose one of the momenta as the reference axis. To manage equation (B.7) we change variables in the following way

$$E_a = m_\chi(1 + xy_a) \tag{B.8}$$

$$p_a = m_\chi (2x)^{1/2}\left(y_a + \frac{1}{2}xy_a^2\right)^{1/2} \tag{B.9}$$

where $a = 1, 2$ stands for the two incoming particles. We then have

$$\langle\sigma_A v\rangle = C\int_0^\infty dy_1 dy_2 \prod_{i=1,2}\left(y_i + \frac{1}{2}xy_i^2\right)^{1/2} e^{-y_i}\int_{-1}^{+1} d\cos\theta\, w(s) \tag{B.10}$$

where we introduced the constant

$$C = \frac{k^2 m_\chi^4 x^3 e^{-2/x}}{4\pi^4 n_0^2} \tag{B.11}$$

to avoid a more cumbersome expression. We can also rewrite the Mandelstam variable (B.6) as

$$\frac{s}{4m_\chi^2} = 1 + \frac{1}{2}x(y_1+y_2) + \frac{1}{2}x^2 y_1 y_2 - x\cos\theta \left(y_1 + \frac{1}{2}xy_1^2\right)^{1/2} \left(y_2 + \frac{1}{2}xy_2^2\right)^{1/2} \tag{B.12}$$

to expand the integrand of (B.10) in powers of x. We need also to expand $w(s)$ about $s/4m_\chi^2 = 1$. The result at order x^2 is

$$\begin{aligned} \int d\cos\theta w(s) &= 2w + xw'(y_1+y_2) \\ &\quad + x^2\left[w' y_1 y_2 + w''\left(\frac{1}{4}y_1^2 + \frac{1}{4}y_2^2 + \frac{5}{6}y_1 y_2\right)\right] \end{aligned} \tag{B.13}$$

where primes denote derivatives with respect to $s/4m_\chi^2$ and w and its derivatives are all to be evaluated at $s/4m_\chi^2 = 1$. We must also compute n_0, which appears in the definition of $\langle \sigma_A v \rangle$. We get

$$\begin{aligned} n_0 &= \int d^3 p f(E) \\ &= \left[k(2\pi x)^{-3/2} e^{-1/x} T^3\right] \frac{2}{\sqrt{\pi}} \int_0^\infty dy (1+xy)\left(y + \frac{1}{2}xy^2\right)^{1/2} e^{-y} \\ &= \left[k(2\pi x)^{-3/2} e^{-1/x} T^3\right]\left(1 + \frac{15}{8}x + \frac{105}{128}x^2 + \dots\right) \end{aligned} \tag{B.14}$$

and thus

$$\frac{1}{n_0^2} = \frac{8\pi^3 e^{2/x}}{k^2 x^3 m_\chi^6}\left(1 - \frac{15}{4}x + \frac{285}{32}x^2 + \dots\right) \tag{B.15}$$

The general formula for the averaged cross section is then

$$\langle \sigma_A v \rangle = \frac{1}{m_\chi^2}\left[w - \frac{3}{2}(2w - w')x + \frac{3}{8}(16w - 8w' + 5w'')x^2 + \dots\right] \tag{B.16}$$

Remembering that $m_\chi v^2 = 3T$ we have $v^2 = 3x$. We then recognize equation (B.16) as the general expression (4.11) in section 4.3, after the identification

$$a = w/m_\chi^2 \tag{B.17}$$

$$b = \left(\frac{1}{2}w' - w\right)/m_\chi^2 \tag{B.18}$$

Appendix C

Runge-Kutta method

C.1 Generalities

Integrating RG equations by hand is a practically impossible task. So we must force to use numerical methods. Isasugra package contains a subroutine, called RKSTP, which makes this possible. This subroutine is accessible form the CERN program library [1]. It can solve a system of $n \geq 1$ first-order differential equations

$$\frac{dy_i}{dx} = f_i\left(x, y_1, ..., y_n\right) \ , \qquad (i = 1, 2, ..., n) \tag{C.1}$$

This system can be integrated by means of Runge-Kutta method, a powerful numerical method. Before discussing this method, we want to revisit the general strategy of finding numerical solution. In this appendix we shall concentrate on the methods of solving a single first-order ordinary (no partial derivative) differential equation with one initial condition

$$y' = f(x, y) \tag{C.2}$$

$$y(x_0) = y_0 \tag{C.3}$$

The method can be easily generalized to handle systems of simultaneous first-order equations. Then there is broader application than the simplicity of the fundamental equation might suggest.

What we really mean by a solution to (C.2) and to (C.3)? Equation (C.2) is a relation that define a curve in the $x - y$ plane, giving the derivative at

[1] Visit the link http://wwwasd.web.cern.ch/wwwasd/cernlib/.

each point on the curve. There is a family of such curves and the condition (C.3) will specifies which curve is.

A solution is an expression for y in terms of x. To find numerical values of the function, we simply substitute any particular values of x into the expression and compute the corresponding values of y.

This method is a so called one-step method. We use information about the curve at one point and do not iterate the solution. This is a direct method, which seems to imply less effort, but in practical cases require more evaluations of the function. A method that theoretically provides a solution to any differential equation, but is nevertheless of little practical computational value, is the Taylor expansion. We write the expansion of the solution $y(x)$, about some point $x = x_m$. In other words, we assume that the process of solution has proceeded to some specified point, and we ask what happens in going to the next point

$$\begin{aligned} y(x) &= y_m + y'_m(x - x_m) \\ &\quad + \frac{y''_m}{2}(x - x_m)^2 + \frac{y'''_m}{3!}(x - x_m)^3 + \cdots \end{aligned} \tag{C.4}$$

The difficulty of this method is that it may be hard -in fact, in some cases impossible- to find derivatives. It is therefore generally impractical from a computational point of view. Its importance lies in the fact that it provides a basis for evaluating and comparing methods of practical worth: we have a yardstick for judging them.

There exists a broad class of techniques known as Runge-Kutta methods. These methods have three peculiar properties:

1. they are one-step methods: to find y_{m+1}, we need only the information available at the preceding point (x_m, y_m);

2. they agree with Taylor expansion up to h^p, where p is different for different methods and is called the order of the method[2];

3. they do not require the evaluation of any derivatives of $f(x, y)$, but only the function itself.

The third property makes Runge-Kutta method more practical than a simple Taylor expansion. The price we have to pay is to evaluate the function $f(x, y)$ for more than one value of x and y. It is a price well worth paying for avoiding derivatives.

[2]In Isasugra it is employed the fourth order method.

C.2 Generalized second order Runge-Kutta

Before passing to see the second order Runge-Kutta method, let us first analyze the Taylor expansion. We will limit to second order. Given two points x_m and x_{m+1} at a distance h, we have the solution

$$y_{m+1} = y_m + hf + \frac{h^2}{2}\left(f_x + ff_y\right) + O(h^3) \tag{C.5}$$

where the notation f_x stands for the partial derivative with respect to x and so on. The formula (C.5) is equivalent to approximate the curve which represents the solution with a parabola. This is certainly better than the simple approximation with a straight line, but it can be improved. In the next section we will see a more powerful method, i.e. the fourth order Runge-Kutta. Let us now evaluate the function f at $x = x_m + h$ and $y = y_m + hy'_m$, which are the first order corrections to the initial values. We obtain

$$f\left(x_m + h, y_m + hy'_m\right) = f + hf_x + hff_y + O(h^2) \tag{C.6}$$

where all the derivatives are evaluated at $x_m,\ y_m$. At this point, following geometrical insights, we can generalize this procedure defining

$$y_{m+1} = y_m + h\Phi\left(x_m, y_m, h\right) \tag{C.7}$$

where h is the step, defined as the difference $x_{m+1} - x_m$. The function Φ is of the form

$$\Phi\left(x_m, y_m, h\right) = a_1 f\left(x_m, y_m\right) + a_2 f\left(x_m + b_1 h, y_m + b_2 hy'_m\right) \tag{C.8}$$

The coefficients can be deduced by comparing the Taylor expansion up to second order. Repeating the expansion in (C.6), we obtain

$$f\left(x_m + b_1 h, y_m + b_2 hf\right) = f + b_1 hf_x + b_2 hff_y + O(h^2) \tag{C.9}$$

where, as usual, the functions on the right hand side are evaluated at $x_m,\ y_m$. Then we can express equation (C.7) as

$$y_{m+1} = y_m + h\left(a_1 f + a_2 f + h(a_2 b_1 f_x + a_2 b_2 ff_y)\right) + O(h^3) \tag{C.10}$$

We can compare this with the Taylor series (C.5). If terms in hf are to agree, then we must require

$$a_1 + a_2 = 1 \tag{C.11}$$

From comparing terms in $h^2 f_x$, we must have

$$a_2 b_1 = 1/2 \tag{C.12}$$

And finally, from comparing terms in $h^2 f f_y$, we require that

$$a_2 b_2 = 1/2 \tag{C.13}$$

Since we have three equations in four parameters, we may choose one of the parameter arbitrarily, excluding zero perhaps, depending on which parameter is taken as the free one. For instance, let

$$a_2 = \omega \neq 0 \tag{C.14}$$

Then we have

$$\begin{aligned} a_1 &= 1 - \omega \\ b_1 &= b_2 = \frac{1}{2\omega} \end{aligned} \tag{C.15}$$

Collecting together the various pieces we then have

$$\begin{aligned} y_{m+1} &= y_m + h\Bigg[(1-\omega) f\left(x_m, y_m\right) \\ &+ \omega f\left(x_m + \frac{h}{2\omega}, y_m + \frac{h}{2\omega} f(x_m, y_m)\right)\Bigg] + O(h^3) \end{aligned} \tag{C.16}$$

This is the most general second-order Runge-Kutta method. For $\omega = 1/2$ we recover the famous improved Euler method (Heun's method)

$$y_{m+1} = y_m + \frac{h}{2} f\left(x_m, y_m\right) + \frac{h}{2} f\left(x_m + h, y_m + h f(x_m, y_m)\right) \tag{C.17}$$

while for $\omega = 1$ we get the so called modified Euler method. The truncation error for any nonzero choice of ω is proportional to h^3. We will enter in detail on this in another section.

Before concluding this section and going on to higher order Runge-Kutta method, we may consider a simple example:

$$y' = y \tag{C.18}$$

with the initial condition

$$y(0) = 1 \tag{C.19}$$

The exact solution is $y = \exp(x)$. The second-order Runge-Kutta, following (C.16), is

$$y_{m+1} = y_m + h\left[(1-\omega)y_m + \omega\left(y_m + \frac{h}{2\omega}y_m\right)\right] \tag{C.20}$$

which, for any nonzero ω, reduces to

$$y_{m+1} = y_m\left(1 + h + \frac{h^2}{2}\right) \tag{C.21}$$

It follows that

$$y_{m+1} = \left(1 + h + \frac{h^2}{2}\right)^{m+1} \tag{C.22}$$

The term in parentheses is the same as the first three terms of the Taylor expansion for e^h. So, as we should expect, we have

$$y_{m+1} \backsimeq e^{h(m+1)} = e^{x_{m+1}} \tag{C.23}$$

C.3 Fourth order Runge-Kutta

We shall not show the derivation, but content ourselves with stating the fourth-order formula, which is one of the most commonly used methods of integrating differential equations. It is so widely used, in fact, that in the literature of numerical computation it is often referred to simply as "the Runge-Kutta method", without any qualification of the order or type.

Let's view this method in the simple case in which we have only a single differential equation. Introducing a small parameter h, we set

$$k_1 = hf(x, y) \tag{C.24}$$

This expression, contrary to the others, is exact to all orders in h. From this we can construct the second step k_2, which is

$$k_2 = hf\left(x + h/2, y + hk_1/2\right) \tag{C.25}$$

The third and fourth step are built in the following way

$$k_3 = hf\left(x + h/2, y + hk_2/2\right) \tag{C.26}$$
$$k_4 = hf\left(x + h, y + hk_3\right) \tag{C.27}$$

As you can see, these quantities are defined iteratively. Once you know k_1 you can construct k_2, and so on. The solution is then given by

$$y(x+h) = y(x) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{C.28}$$

The error per step is proportional to h^5. We will discuss this item in detail in the following section.

Now we want to prove that this method is actually a fourth order method. We start by reviewing the Taylor expansion. Assuming that the step-length is h, we have

$$\begin{aligned} y(x+h) &= y(x) + hy'(x) + \frac{1}{2!}h^2 y''(x) \\ &+ \frac{1}{3!}y'''(x) + \frac{1}{4!}y''''(x) + O(h^5) \end{aligned} \tag{C.29}$$

Then we calculate all the derivatives, taking into account that $f(x,y)$ is a two-variables function. The first derivative of y is simply the function f; if we differentiate twice we obtain

$$\frac{d^2y}{dx^2} = f_x + f f_y \tag{C.30}$$

where we have introduced the common notation $f_x \equiv \partial f/\partial x$. With some algebra we can obtain the expressions for the third derivative

$$\frac{d^3y}{dx^3} = f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2 \tag{C.31}$$

and the fourth one

$$\begin{aligned} \frac{d^4y}{dx^4} &= f_{xxx} + 3f_x f_{xy} + 5f f_y f_{xy} + 3f f_{xxy} + 3f^2 f_{xyy} \\ &+ f^3 f_{yyy} + 3f f_x f_{yy} + 4f^2 f_y f_{yy} + f_{xx} f_y + f_x f_y^2 + f f_y^3 \end{aligned} \tag{C.32}$$

Now that we have the general expression up to fourth order, we can pass to analyze the Runge-Kutta method. With some efforts, it is possible to calculate all the functions k_i up to fourth order in h. As explained earlier, the function k_1 is exact to all orders. So, let's start with the function k_2:

$$\begin{aligned} k_2 &= hf + \frac{h^2}{2}\left(f_x + f f_y\right) + \frac{h^3}{8}\left(f_{xx} + 2f f_{xy} + f^2 f_{yy}\right) \\ &+ \frac{h^4}{48}\left(f_{xxx} + 3f f_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}\right) + \cdots \end{aligned} \tag{C.33}$$

where dots stand for higher order terms. The expressions for the other functions are more lengthy; for k_3 we obtain

$$\begin{aligned}
k_3 &= hf + \frac{h^2}{2}(f_x + ff_y) + \frac{h^3}{8}\left(f_{xx} + 2ff_{xy} + f^2 f_{yy} + 2f_x f_y + 2ff_y^2\right) \\
&+ \frac{h^4}{48}\Big[f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} + 6f_x f_{xy} \\
&+ 3f_y f_{xx} + 12ff_y f_{xy} + 3f^2 f_y f_{yy} + 6ff_x f_{yy} + 6f^2 f_y f_{yy}\Big] + \cdots
\end{aligned} \tag{C.34}$$

while for k_4 we have

$$\begin{aligned}
k_4 &= hf + \frac{h^2}{2}(f_x + ff_y) + \frac{h^3}{2}\left(f_x f_y + ff_y^2 + f_{xx} + f^2 f_{yy} + ff_{xy}\right) \\
&+ \frac{h^4}{24}\Big[3f_{xx}f_y + 15f^2 f_y f_{yy} + 18ff_y f_{xy} + 6f_x f_y^2 + 6ff_y^3 + 12ff_x f_{xx} \\
&+ 12f_x f_{xy} + 4f_{xxx} + 12ff_{xxy} + 12f^2 f_{xyy} + 4f^3 f_{yyy}\Big] + \cdots
\end{aligned} \tag{C.35}$$

Now we can compare this result with the expression (C.29) obtained by Taylor expansion. We can see that, putting all these things together, the Runge-Kutta method is exact up to fourth order in h, as we anticipated at the beginning.

C.4 Error analysis

In general, the truncation error in a p-th order Runge-Kutta method is Kh^{p+1}, where K is some constant. The derivation of this constant is not a simple matter. One of the serious drawbacks of Runge-Kutta methods is indeed the lack of simple means for estimating the error. Without some measure of the truncation error, it is difficult to choose the proper step size h. Here we give only a rough rule. If

$$\frac{|k_2 - k_3|}{|k_1 - k_2|} \tag{C.36}$$

becomes large (more than a few hundredths), then h should be decreased. We want to stress that, even if the truncation error is small, a Runge-Kutta method may produce extremely inaccurate results under unfavorable conditions. Such erroneous results can arise because small errors (roundoff or truncation) may become magnified as the solution is carried out for larger and larger x. For example let us consider the simple equation

$$y' = -10y \tag{C.37}$$

with the initial condition $y(0) = 1$. The exact solution is

$$y(x) = e^{-10x} \tag{C.38}$$

From an analysis completely analogous to that unfolded for the example (C.18), we find that a second order method leads to

$$y_{m+1} = \left(1 - 10h + 50h^2\right)^m \tag{C.39}$$

But now notice that the term in parentheses is greater than 1 if $h > 0.2$. For large m, therefore, y becomes indefinitely large. The exact solution, on the other hand, becomes small!

This phenomenon is called partial instability. It is distinguished from other instabilities by the fact that it depends on h. We conclude stressing that this partial instability exists for the Runge-Kutta method even when the exact solution does not decay exponentially, as in the example just discussed.

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