June 2010 RR-10.84

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Abstract

This report provides the structure of a policy minimizing a long term, average, expected, backlog/inventory cost for a fluid model, single machine, single product manufacturing system subject to a failure/repair Markov process, where the failure rate is a piecewise constant function of the production rate. This policy generalizes previous results and confirms several conjectures reported in the literature, providing an interesting insight into the problem.

1 Introduction

A large literature deals with the problem of failure prone manufacturing systems. A complete analytical solution has been given in [1] for a single machine characterized by a homogeneous Markov failure/repair process. In this case the control minimizing a long term average expected cost penalizing both surplus and backlog is the *hedging point* policy, according to which the machine is operated at full rate until the inventory level hits a non-negative hedging level (or safety stock) Z, which is then maintained until the next failure event occurs.

The problem becomes much more involved if the failure rate depends on the production rate. In [3] it has been proved that the hedging point policy remains optimal if and only if the dependence of the failure rate on the production rate is affine and it was conjectured for more general cases, e.g. when this dependence is quadratic, that *as the inventory level approaches a "hedging level", it may be beneficial to decrease the production rate to gain in reliability.* This conjecture was actually confirmed by the numerical results reported in [4].

An analytical increment in this direction, still confirming the conjecture in [3], was presented in [5] (an extended version of [5] is available in [6]), where it was considered a machine characterized by two failure rates: one for low and one for high production rates. In this report we generalize this problem by considering a machine with N different failure rates: more specifically, the failure rate is assumed to depend on the production rate through an increasing, piecewise constant function. This makes the proof of optimality much more involved with respect to the one given in [5] since it is not possible in this case to derive in closed form several parameters characterizing the optimal control.

The optimal policy, which is a multi-level, decreasing, piecewise constant, feedback function of the backlog/inventory level, allows to obtain the following interesting insight: the production rates providing the maximum expected long run buffer increment are convenient when the backlog/inventory level is far from the safety stock, while the production rates guaranteeing the maximum expected up times are better when approaching the safety stock. The shape of the optimal policy strongly depends on the convexity properties of the failure rate function, confirming also in this case the numerical findings of [4] and the analytical results of [3].

2 Notation and problem formulation

Let $x(t)$ denote the buffer content at time t, with $x(t) > 0$ representing an inventory surplus and $x(t) < 0$ a backlog of $-x(t)$. Let d be the constant demand rate to be met. Then the buffer level

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$x(t)$ at time t satisfies the following dynamical equation:

$$\dot{x} = u(t) - d \tag{1}$$

where the production rate $u(t) = 0$ if at time t the machine is in the down state (also referred to as state 0), and $u(t) \in [0, \mu]$ if at time t the machine is in the up state (also referred to as state 1). We assume a Markov failure/repair process: the repair rate q_{up} is constant while the failure rate $q_d(u)$ depends on the production rate u as follows:

$$q_d(u) := \begin{cases} q_1 & u \leq U_1 \\ q_2 & u \in (U_1, U_2] \\ \dots & \\ q_N & u \in (U_{N-1}, U_N] \end{cases} \tag{2}$$

where $0 < q_1 < q_2 < \ldots < q_N$ and $0 < U_1 < U_2 < \ldots < U_N =: \mu$. The piecewise constant function in (2) may result from the discrete approximation of a continuous function, for example of the type considered in [3]:

$$Q_d(u) = au^{\beta} + b, \tag{3}$$

with a, b and β non-negative constants. The state of the machine at time t will be denoted by $s(t)$, hence $s(t) \in \{0, 1\}$ for all t. The scheduling problem considered in this report is the determination of the optimal control $u^*(t)$ minimizing the long-term average expected cost

$$J = \limsup_{T \to \infty} \frac{1}{T} E\left[\int_0^T g[x(t)]dt\right], \tag{4}$$

where $g(x) = c_p x^+ + c_m x^-$, with $x^- = \max\{0, -x\}$, $x^+ = \max\{0, x\}$, c_p and c_m non-negative constants. We are interested only in admissible control laws, i.e. in non anticipative policies such that for all $t \geq 0$, $0 \leq u(t) \leq \mu \cdot s(t)$ (see e.g. [7] for more details on this). If the machine is operated at rate $u \in [0, \mu]$, the average production rate at steady state is $u\, q_{up}/(q_{up} + q_d(u))$. The following feasibility Assumption will be considered, that there exists at least one U_i such that the demand may be met in the average operating the machine at U_i. If this assumption were not satisfied, any rate $u \in [0, \mu]$ would be unfeasible and the backlog, under any policy, would increase with no bound over time.

Assumption 1 *Consider the N-level failure rate function $q_d(u)$ reported in (2) and let*

$$\Delta_i := U_i q_{up} - d(q_{up} + q_i). \tag{5}$$

Then, it will be assumed that $\Delta_i > 0$ for at least one $i \in \{1, \ldots, N\}$.

If, for some i, $\Delta_i > 0$, then also $U_i > d$. So, for each i, the following quantity often used in the sequel, is positive if U_i is feasible (i.e. if $\Delta_i > 0$):

$$\alpha_i := \frac{\Delta_i}{d(U_i - d)} \tag{6}$$

3 The structure of the optimal policy

The optimal policy for the considered problem has the following structure:

$$u(x) := \begin{cases} 0 & x > X_\ell \\ d & x = X_\ell \\ U_{i_\ell} & x \in [X_{\ell+1}, X_\ell) \\ U_{i_{\ell+1}} & x \in [X_{\ell+2}, X_{\ell+1}) \\ \dots & \\ U_{i_L} & x < X_L \end{cases} \tag{7}$$

with $1 \leq \ell \leq L \leq N$ and $\mathcal{S} := \{i_k\}_{k=\ell,\dots,L}$, $1 \leq i_\ell < i_{\ell+1} < \dots < i_L \leq N$, the sequence of integers derived with the procedure reported in Algorithm 1 below (the reason behind the fact that the first element of $\mathcal{S}$ is denoted by ℓ and not by 1 will be explained below). Notice that $Z := X_\ell$ in (7) is the hedging level, i.e. the safety stock to be maintained by the system, while all the X_i's, for $i > \ell$, are only thresholds where the production rate changes.

The procedure to derive the sequence $\mathcal{S}$ appearing in (7), reported in Algorithm 1, requires the definition of the following positive quantities:

$$\phi_{ij} = \frac{q_j - q_i}{U_j - U_i} \tag{8}$$

for $1 \leq i < j \leq N$. Also, we need to define, for any $j, l \in \{1, \dots, N\}$,

$$\Delta U_{j,l} := (q_{up} + q_l)U_j - (q_{up} + q_j)U_l. \tag{9}$$

The intuition behind these quantities and the structure in (7) will be given in Section 3.1. We first give the procedure for the computation of the sequence $\mathcal{S} = \{i_k\}_{k=\ell,\dots,L}$.

Algorithm 1 *Generate at first the following sequence $\mathcal{S}' = \{i_k\}_{k=1,\dots,L'}$ of indexes:*

$$i_k = \begin{cases} 1 & k = 1 \\ \arg\min_{j > i_{k-1}} \phi_{i_{k-1},j} & k = 2, \dots, L' \end{cases} \tag{10}$$

where $L' \leq N$ is the first index for which $i_{L'} = N$. Then the sequence $\mathcal{S}$ is given by the elements $\{i_\ell, i_{\ell+1}, \dots, i_L\}$ of $\mathcal{S}'$, where i_ℓ is the first index such that $U_{i_\ell} > d$ (under Assumption 1 it can be shown that such an i_ℓ is always comprised in $\mathcal{S}'$) and $L \geq \ell$ is the first index (if any) for which it happens that $\Delta U_{i_L,i_{L+1}} \geq 0$, while $L = L'$ if $\Delta U_{i_k,i_{k+1}} < 0$ for all $k = \ell, \dots, L' - 1$.

When using a policy in the class reported in (7), the cost index in (4) is a function of the levels X_k's, i.e. $J = J(X_\ell, \dots, X_L)$ and we will denote by X_k^*, the optimal value of X_k, and by J^* the corresponding optimal cost, i.e.:

$$J^* = J(X_\ell^*, \dots, X_L^*) = \min_{X_\ell \geq \dots \geq X_L} J(X_\ell, \dots, X_L).$$

The optimal value of the X_k's can be numerically derived (e.g. through a gradient descent method) using the analytical expression of $J(X_\ell, \dots, X_L)$, for some given $X_\ell \geq \dots \geq X_L$, reported in Appendix 7.1. It must be remarked that the optimal hedging level $Z^* = X_\ell^*$ can not be negative, as shown in Appendix 7.1.

3.1 General observations and some particular cases.

The optimal policy in (7) operates the machine by selecting for each buffer level the production rate providing the best trade off between expected long run buffer increment and expected uptime, giving more importance to the former when the inventory level is far from Z^* and to the latter when approaching the safety stock. This is formally established by the following lemma which provides also an explanation to the necessity of introducing the quantities $\Delta U_{j,l}$ in (9) (see also Remark 1).

Lemma 1 *Consider $\mathcal{S}$ and the corresponding sequence of production rates $\{U_{i_k}\}_{k=\ell,\dots,L}$. Then the expected uptimes and the expected long run buffer increments associated with the $\{U_{i_k}\}$ are respectively decreasing and increasing with k.*

Proof. First of all notice that $\mathcal{S}$ is an increasing sequence of indexes (i.e. $i_{k+1} > i_k$ for all k). Now, the expected uptime associated with U_{i_k} is $1/q_{i_k}$ while the expected buffer increment associated with U_{i_k} over a long time interval of duration T is $(E_{i_k} - d)T$, where, for any $h \in \{1, \dots, N\}$,

$$E_h := \frac{q_{up} U_h}{q_{up} + q_h} \tag{11}$$

is the average expected (or effective) long run production rate if the machine is operated at rate U_h. Since q_{i_k} is increasing with k, the expected uptimes are decreasing with k. As for the expected long run buffer increment, notice that the sequence $\mathcal{S}$ comprises all elements such that $\Delta U_{i_k,i_{k+1}} < 0$. Now, the condition $\Delta U_{i_k,i_{k+1}} < 0$ is equivalent to the condition $E_{i_k} < E_{i_{k+1}}$, that is, to the condition $(E_{i_k} - d)T < (E_{i_{k+1}} - d)T$, which is what has to be proved. $\square$

Remark 1 *The quantities $\Delta U_{j,l}$ play an important role in the search of the optimal policy: actually $\Delta U_{j,l}$ tells us if the average expected long run buffer increment associated with U_j is larger than the one associated with U_l. Notice also that if $j > l$, i.e., if $U_j > U_l$, not necessarily $E_j > E_l$, so $\Delta U_{j,l}$, with $j > l$, may have any sign.*

The result established in Lemma 1 confirms the conjecture reported in [3] where it is remarked how the optimal policy decreases the production rate approaching the hedging level to gain in reliability. Actually Lemma 1 says something more: when approaching the hedging level, the optimal policy, to gain in reliability, decreases not simply the production rate but the *effective* production rate.

Another interesting observation concerns a major difference arising in the optimal policy between the case the failure rate $q_d(u)$ is a convex function of the production rate and the case it is affine or concave. This major difference was actually highlighted, based on a numerical investigation, in [4]. This difference is confirmed by our derivation. As a matter of fact, if the considered failure rate function $q_d(u)$ is convex (i.e. $\phi_{i,i+1} < \phi_{i+1,i+2}$ for all $i = 1, \ldots, N-2$ - this happens if discretizing a convex function like the one in (3) when $\beta > 1$), the procedure above gives $L' = N$, $\mathcal{S}' = \{1, 2, \ldots, N\}$ and in general $L \leq L'$, depending on the steepness of the function $q_d(u)$. This aligns with the numerical results of [4] where it was observed how, in the convex case, the production rate is smoothly decreased when approaching the safety stock. If $Q_d(u)$ is affine, any discretization would provide $\phi_{i,i+1} = \phi_{i+1,i+2}$ for all $i = 1, \ldots, N-2$. The results of this report align then with the analytical findings of [3] according to which, in the affine case, the optimal policy is the hedging point policy. In fact, in this case, we obtain $L' = 2$, $\mathcal{S}' = \{1, N\}$ and, if $U_1 < d$, $\ell = L = 2$ and $\mathcal{S} = \{N\}$, which corresponds to the hedging point policy since the production rate is sharply reduced from the maximum production rate U_N to 0. Finally, if the $q_d(u)$ is concave (i.e. $\phi_{i,i+1} > \phi_{i+1,i+2}$ for all $i = 1, \ldots, N-2$ - this happens if discretizing a $Q_d(u)$ in (3) when $\beta < 1$), we still obtain $L' = 2$, $\mathcal{S}' = \{1, N\}$ and, if $U_1 < d$, $\ell = L = 2$ and $\mathcal{S} = \{N\}$. This seems to correspond, as in the affine case, to a hedging point policy. However now the obtained policy does not satisfy the conditions of optimality of Theorem 2 (see Remark 2). This also aligns with the numerical findings of [4], where it was conjectured that the optimal policy in the concave case is only asymptotic, and consists of a hedging point policy where the safety stock is maintained through an infinite switching of the production rate between 0 and μ.

4 Proof of optimality

As in [5], we use the following result to assess the optimality of (7) (with optimal levels X_i's). Its proof, under certain regularity and stability conditions imposed on the control (actually met, under Assumption 1, by any policy in the class (7)) is essentially like the one given in [1].

Theorem 1 *(Verification Theorem) If there exist a constant J^* and two continuously differentiable functions $V(x,0)$ and $V(x,1)$, $|V(x,i)| \leq c_1 x^2 + c_2$ for some constants c_1 and c_2 $(i = 0, 1)$, such that a control $u(x)$ satisfies the following HJB equations:*

$$\min_{u \in [0,\, \mu]} \left\{ (u - d)\frac{dV(x,1)}{dx} + q_d(u)[V(x,0) - V(x,1)] \right\} = J^* - g(x), \tag{12}$$

$$d\frac{dV(x,0)}{dx} + q_{up}[V(x,0) - V(x,1)] = g(x) - J^*, \tag{13}$$

then $u(x)$ is optimal and J^* is the corresponding optimal cost.

The following result states that, under Assumption 1, if some additional condition is met, the policy given in (7) (with suitable levels X_i's) is optimal.

Theorem 2 *If Assumption 1 is satisfied together with one of the following two conditions:*

i) $U_i > d$ *for all* i, *or*

ii) there exists $i_{j-1}, i_j, i_{j+1} \in \mathcal{S}'$ *such that* $U_{i_j} = d$ *and* $\frac{q_{i_{j+1}} - q_{i_j}}{U_{i_{j+1}} - U_{i_j}} = \frac{q_{i_j} - q_{i_{j-1}}}{U_{i_j} - U_{i_{j-1}}}$,

the optimal policy has the structure reported in (7).

Remark 2 *Condition (i) in the theorem agrees with the result reported in [5] for* $N = 2$ *and also with the classical solution of the problem with a constant failure rate function* $q_d(u) = q_d$, *solved in [1]. Condition (ii) allows to explain several (analytical and numerical) results reported in the literature. In particular Condition (ii) is met if the considered* $q_d(u)$ *comes from the discretization of an affine failure rate function* $Q_d(u)$ *(if* $U_i = d$ *is selected in the discretization), which allows to understand the optimality of the hedging point policy for affine failure rate functions, as also proved in [3]. If* $Q_d(u)$ *is convex, it is always possible to meet Condition (ii) as the discretization step goes to zero around the value* $u = d$. *This would match with the numerical results reported in [4]. If the continuous failure rate function* $Q_d(u)$ *is concave, there is no way of satisfying Condition (ii) if* $U_1 < d$, *and this would confirm the discussion in [4] where it was conjectured that an optimal feedback control does not exist in the concave case. Considering Condition (ii), and not simply that there exists* $i_j \in \mathcal{S}'$ *such that* $U_{i_j} = d$, *allows to prove the continuity of* $V_x(x, 1)$ *needed in the Verification Theorem (see Appendix 7.3): this is identical to the reason reported in [8] for not including* d *in the capacity set of the machine (Assumption A5 in [8]). The last equality in Condition (ii) ensures in fact that* d *can be optimal only on a point and not over a finite interval (even if this interval would be a transient set).*

To prove the theorem we need the following two lemmas. Lemma 2 states that the segments $(-\infty, \phi_{i_1 i_2})$, $(\phi_{i_1 i_2}, \phi_{i_2 i_3})$, …, $(\phi_{i_{L'-1} i_{L'}}, \infty)$, provide a partition of the real axis. In Lemma 3, it is proved that, if some $i_k \in \mathcal{S}'$ is more efficient than the next element $i_{k+1} \in \mathcal{S}'$ (i.e. $\Delta U_{i_k, i_{k+1}} > 0$) then this holds for all subsequent elements of $\mathcal{S}'$.

Lemma 2 *Let* $q_d(u)$ *as defined in (2),* ϕ_{ij} *as in (8) and consider the sequence* $\mathcal{S}' = \{i_1, \ldots, i_{L'}\}$, *defined in Section 3. Then this sequence is such that:*

$$i_k = \arg \min_{j < i_{k+1}} \left(-\phi_{j, i_{k+1}}\right) \tag{14}$$

for all $k = 1, \ldots, L' - 1$, *hence:*

$$m_{k+1} := \min_{j < i_{k+1}} \left(-\phi_{j, i_{k+1}}\right) = -\phi_{i_k, i_{k+1}} = \max_{j > i_k} \left(-\phi_{i_k, j}\right) =: M_k \tag{15}$$

for all $k = 1, \ldots, L' - 1$, *and the intervals:*

$$(-\infty, m_{L'}],\ [M_{L'-1}, m_{L'-1}],\ [M_{L'-2}, m_{L'-2}], \ldots, [M_1, \infty)$$

provide a partition of the real axis and, for all $j \notin \mathcal{S}'$,

$$\min_{i<j} \left(-\phi_{i,j}\right) < \max_{i>j} \left(-\phi_{j,i}\right). \tag{16}$$

Proof. The proof follows directly by observing that the sequence $\mathcal{S}'$ defines the convex envelop of the function $q_d(u)$. More details are given in Appendix 7.6. □

Lemma 3 *Let* $i_k \in \mathcal{S}'$, *with* $k \le L' - 2$. *Then* $\Delta U_{i_k, i_{k+1}} \ge 0$ *implies* $\Delta U_{i_{k+1}, i_{k+2}} > 0$.

Proof. From the definition of ϕ_{ij}, it is possible to write:

$$q_{i_{k+2}} = q_{i_{k+1}} + \phi_{i_{k+1}, i_{k+2}} (U_{i_{k+2}} - U_{i_{k+1}}). \tag{17}$$

Substituting in the definition of $\Delta U_{i_{k+1}, i_{k+2}}$ we obtain:

$$\Delta U_{i_{k+1}, i_{k+2}} = (U_{i_{k+2}} - U_{i_{k+1}}) \left[\phi_{i_{k+1}, i_{k+2}} U_{i_{k+1}} - (q_{up} + q_{i_{k+1}})\right] \tag{18}$$

Similarly, it holds:

$$\Delta U_{i_k,i_{k+1}} = (U_{i_{k+1}} - U_{i_k}) \left[\phi_{i_k,i_{k+1}} U_{i_k} - (q_{up} + q_{i_k})\right] \tag{19}$$

Since we have that $\Delta U_{i_k,i_{k+1}} \geq 0$, the previous equation implies:

$$\left[\phi_{i_k,i_{k+1}} U_{i_k} - (q_{up} + q_{i_k})\right] \geq 0 \tag{20}$$

Applying the definition used in (17) at $k+1$, we obtain:

$$q_{i_{k+1}} = q_{i_k} + \phi_{i_k,i_{k+1}} (U_{i_{k+1}} - U_{i_k}) \tag{21}$$

which, substituted in (18), gives:

$$\Delta U_{i_{k+1},i_{k+2}} = (U_{i_{k+2}} - U_{i_{k+1}}) \left[\phi_{i_{k+1},i_{k+2}} U_{i_{k+1}} - q_{up} - q_{i_k} - \phi_{i_k,i_{k+1}} U_{i_{k+1}} + \phi_{i_k,i_{k+1}} U_{i_k}\right] \tag{22}$$

Using (20), (22) becomes:

$$\Delta U_{i_{k+1},i_{k+2}} \geq (U_{i_{k+2}} - U_{i_{k+1}}) \left[\phi_{i_{k+1},i_{k+2}} - \phi_{i_k,i_{k+1}}\right] U_{i_{k+1}} \tag{23}$$

which is positive, being $\phi_{i_{k+1},i_{k+2}} > \phi_{i_k,i_{k+1}}$. In fact, by definition, $\phi_{i_k,i_{k+1}} < \phi_{i_k,j}$ for all $j > i_k$, in particular

$$\phi_{i_k,i_{k+1}} < \phi_{i_k,i_{k+2}} \tag{24}$$

Similarly, $\phi_{i_{k+1},i_{k+2}} > \phi_{j,i_{k+2}}$ for all $j < i_{k+2}$, hence

$$\phi_{i_{k+1},i_{k+2}} > \phi_{i_k,i_{k+2}} \tag{25}$$

From (24) and (25), it follows $\phi_{i_{k+1},i_{k+2}} > \phi_{i_k,i_{k+1}}$. □

Proof of Theorem 2. Let $v(x,0)$ and $v(x,1)$ denote the *differential costs* associated with the optimal policy starting from $x(0) = x$ with a down and with an up machine respectively (see e.g. [10] for a definition of these functions, also reported in Appendix 7.5). Following the procedure in Appendix H of [9], it is possible to show that $V(x,0) := v(x,0) + c$ and $V(x,1) := v(x,1) + c$ (where c is any constant) satisfy (12)-(13) at least in the viscosity sense. Take for simplicity $c = 0$: $V(x,i)$ will then represent from now on the differential costs associated with the optimal policy. Now, $V(x,0)$ is continuously differentiable for all x while $V(x,1)$ may be not differentiable only on the switching levels, i.e. where the control is discontinuous (see e.g. [10], Section IV or [2], ch. 9.3). It will be shown below however that either under Condition (i) or (ii) of the theorem, $V(x,1)$ is differentiable also on the switching levels. This will imply that $V(x,0)$ and $V(x,1)$ are a classical ($\mathcal{C}^1$) solution to (12) and (13). Let:

$$V_x(x,1) := \frac{dV(x,1)}{dx}, \qquad H(x) := V(x,0) - V(x,1).$$

We first prove that the differential cost starting with a down machine is larger than the differential cost starting with an up machine, i.e. $H(x) > 0$ for all x. This is obtained directly from the HJB equations (12) and (13) (considered in the classical sense between the switching levels) which allow to prove that for all x where $V(x,1)$ is differentiable $H_x(x) := \frac{dH}{dx} \geq -\frac{q_{up}+q_d(0)}{d} H(x)$. In addition, from (13), using the quadratic lower bound on $V(x,0)$ established in Appendix 7.5, it is possible to show that there exists a sequence x_k, with $x_k \to -\infty$ as $k \to \infty$, where $H(x_k) > 0$. These two results imply that $H(x) > 0$ for all x. More details are reported in Appendix 7.2.

We then proceed by showing that $V(x,1)$ is continuously differentiable also on the switching levels. This is performed (under Condition (i) or (ii) of the theorem) by using a procedure similar to the one adopted for the same purpose in [9], ch. 3.3. The details are given in Appendix 7.3.

Finally, it is possible to prove that $|V(x,i)|$, $i = 0,1$, are bounded as requested in the Verification Theorem (i.e. $|V(x,i)| \leq c_1 x^2 + c_2$, $i = 0,1$): this can be shown using a result in [8] according to which if τ denotes the time necessary for going from any given state (x',i) to another state (x,j) (with $x > x'$) by working at a rate $U_k > d$, then τ is such that $E[\tau^\ell] \leq c_a + c_b |x - x'|^\ell$ (where $\ell = 1,2,\ldots$). More details are given in Appendix 7.4.

To conclude the proof, it must be shown that the policy $u(x)$ satisfying the HJB equations (12)-(13) (in the classical sense) with the aforementioned properties of the functions $V(x,i)$ (quadratically bounded in modulus, $\mathcal{C}^1$, with $H(x) > 0$), is policy (7). Now, from (12), we immediately have:

$$u^*(x) = \arg \min_{u \in [0,\, \mu]} \left[u\, V_x(x,1) + q_d(u) H(x) \right]. \tag{26}$$

Since $H(x) > 0$ for all x:

- $u^*(x) = 0$ if $V_x(x,1) > 0$;
- $u^*(x) = U_i$ for some i if $V_x(x,1) < 0$;
- $u^*(x) \in [0, U_1]$ if $V_x(x,1) = 0$.

Let $Z_g := \inf\{x : V_x(y,1) < 0 \; \forall\, y < x,\ V_x(x,1) = 0$ and there exists $\epsilon > 0 :\ V_x(y,1) > 0 \, \forall\, y \in (x, x+\epsilon)\}$ (this Z_g exists thanks to the fact that $V(x,1)$ is continuously differentiable and goes to infinity as $|x| \to \infty$ as shown in Appendix 7.5). Based on (26), the region $\{x > Z_g\}$ is transient (the optimal control is 0 over a non-zero interval $(Z_g, Z_g + \epsilon)$ and eventually the buffer will drop below Z_g). To complete the proof that policy in (7) is candidate to solve the HJB equations, we need to further develop the second item above, that is, which U_i is optimal when $x < Z_g$ (where $V_x(x,1) < 0$). From (26) it can be seen that a rate U_k is optimal at $x < Z_g$ if and only if

$$V_x(x,1) U_k + H(x) q_k \le V_x(x,1) U_i + H(x) q_i$$

for all i, i.e. if and only if

$$V_x(x,1) \le -H(x) \phi_{ik} \tag{27}$$

for all $i < k$ (where ϕ_{ik} has been defined in (8)) and

$$V_x(x,1) \ge -H(x) \phi_{ki} \tag{28}$$

for all $i > k$. Exploiting the fact that $H(x) > 0$ for all x, and introducing the function

$$T(x) := \frac{V_x(x,1)}{H(x)},$$

the relations above will result in the following conditions:

- U_1 is optimal at x if and only if
$$T(x) \ge \max_{i>1} (-\phi_{1i}); \tag{29}$$
- U_k, $k = 2, \ldots, N-1$ is optimal at x if and only if
$$\min_{i<k} (-\phi_{ik}) \ge T(x) \ge \max_{i>k} (-\phi_{ki}); \tag{30}$$
- U_N is optimal at x if and only if
$$T(x) \le \min_{i<N} (-\phi_{iN}). \tag{31}$$

Fig. 1(a) helps to figure out what is happening. In the figure $-\phi_{mk} = \min_{i<k}(-\phi_{ik})$ and $-\phi_{kM} = \max_{i>k}(-\phi_{ki})$. According to (30), if $\min_{i<k}(-\phi_{ik}) < \max_{i>k}(-\phi_{ki})$ for some k, U_k can never be optimal.

It remains to show that (29)-(31) result in the sequence $\mathcal{S}$ defined in Section 3. This can be proved as follows: thanks to Lemma 2, we know that the intervals $[-\phi_{kM}, -\phi_{mk}]$ provide a partition of the real axis. So, to conclude the proof, it is enough to show, for all $x \le Z_g$, the following facts: 1) $T(x)$ is a continuous and negative function, with $T(Z_g) = 0$; 2) $T(x)$ intersects each m_j, for $j = 2, \ldots, L'$, at most once and, 3) the last element $i_L \in \mathcal{S}$ is the first element in $\mathcal{S}'$ such that $\Delta U_{i_L, i_{L+1}} \ge 0$.

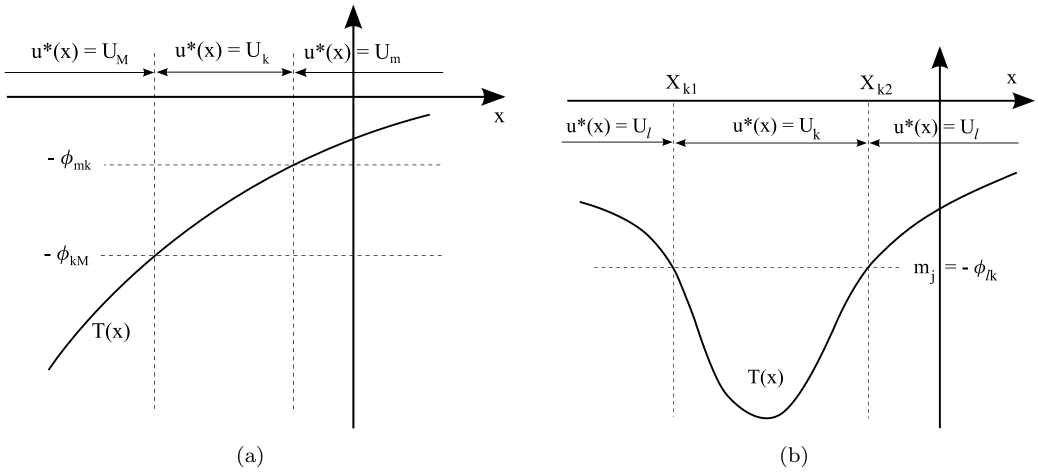


Figure 1: A graphical representation of (30) (left) and the case of multiple intersections between $T(x)$ and a given m_j (right)

1) *Continuity and negativity of $T(x)$, with $T(Z_g) = 0$.*

Since $H(x) > 0$ for all x, $V_x(x,1) < 0$ for all $x < Z_g$ and $V_x(Z_g,1) = 0$, it immediately follows that $T(x) < 0$ for all $x < Z_g$ and $T(Z_g) = 0$. In addition, the continuous differentiability of $V(x,0)$ and $V(x,1)$ implies that $V_x(x,1)$ and $H(x)$ are continuous functions for all x. Hence $T(x)$ is continuous for all x.

2) *Unique intersection of $T(x)$ with the m_j's.*

Now we show that $T(x)$ can intersect a given m_j, $j = 2,\ldots,L'$, at most once. This is done by contradiction. Assume that there are two (or more) intersections of $T(x)$ with a given m_j and, to simplify notation, let $i_j = k$ and $i_{j-1} = l$, with $m_j = -\phi_{lk}$ and $U_k > U_l$ (see Fig. 1(b)). Notice that this is the most general case being $T(Z_g) = 0$ and $T(x) < 0$ for all $x < Z_g$, hence $T(x)$ is certainly increasing in a neighbor on the left of Z_g. So the first time (starting from Z_g and going left) we would observe the intersection of $T(x)$ with a level m_j already met, it must be of the type reported in Fig. 1(b), where $T(X_{k1}) = T(X_{k2}) = m_j$ (see Appendix 7.7 for more details).

Now, due to the continuity of $V_x(x,1)$, from the HJB equation (12) at X_{k1} and at X_{k2} (i.e. from $\left.\frac{dV(x,1)}{dx}\right|_{x=X_{k_i}^-} = \left.\frac{dV(x,1)}{dx}\right|_{x=X_{k_i}^+}$, $i = 1,2$) we obtain:

$$H(X_{k_1}) = -\frac{(J^* - g(X_{k_1}))(U_l - U_k)}{B_{kl}} \tag{32}$$

and

$$H(X_{k_2}) = -\frac{(J^* - g(X_{k_2}))(U_k - U_l)}{B_{lk}} \tag{33}$$

where $B_{lk} = q_k(U_l - d) - q_l(U_k - d)$. Notice that $B_{kl} = -B_{lk}$, and that it can have any sign even if $U_k > U_l$. Also, $B_{kl} \neq 0$ since $H(x)$ is a continuous function, and at least one between $J^* - g(X_{k_2})$ and $J^* - g(X_{k_1})$ must be different from 0. This depends on the fact that $X_{k1} < X_{k2} < Z_g$ and, as shown afterwards, $J^* > g(Z_g)$. So, to be $J^* - g(X_{k_2}) = J^* - g(X_{k_1}) = 0$ it must be $X_{k1} < X_{k2} < 0$. But then it is not possible that $-c_m X_{k1} = -c_m X_{k2} = J^*$.

Now, the fact that $B_{kl} = -B_{lk} \neq 0$, together with (32)-(33) and the fact that $H(x) > 0$ for all x, implies that $J^* - g(X_{k_1})$ and $J^* - g(X_{k_2})$ have the same sign, positive if $B_{kl} > 0$ and negative otherwise. Denote by R_s the set of all $x \le Z_g$ where the cost $g(x)$ is smaller than J^*. Then we have $R_s = (x_s, Z_g)$, where $x_s < 0$ is such that $g(x_s) = -c_m x_s \equiv J^*$. In fact, as mentioned above, from equation (12), with $x = Z_g$, it follows that (thanks to the positivity of $H(Z_g)$), $J^* > c_p Z_g$, i.e., $J^* > g(x)$ for all $x \in (0, Z_g)$. For this reason R_s ends at Z_g.

Now, assume first that $B_{kl} < 0$ (later we will see the other case). This implies that $X_{k2} < x_s$ (and $X_{k1} < x_s$ as well). A direct computation allows to show that

$$T'_{k1} := \lim_{x \to X_{k1}^-} T'(x) = \frac{1}{[H(X_{k1})]^2} \frac{N_0 + N_1 H(X_{k1})}{d(U_l - d)} \tag{34}$$

where $N_0 = U_l(J^* - g(X_{k1}))^2$ and $N_1 = (J^* - g(X_{k1}))\Delta_l + c_m d(U_l - d)$. From (34), using (32), it is possible to obtain the following expression for the left derivative of $T(x)$ as $x \to X_{k1}$:

$$T'_{k1} = \frac{J_{k1}}{[H(X_{k1})]^2} \frac{J_{k1}\Delta U_{kl} + c_m d(U_k - U_l)}{d\, B_{kl}} \tag{35}$$

where $J_{k1} := J^* - g(X_{k1})$ is a negative quantity, being $X_{k1} < x_s$. Similarly, the left derivative of $T(x)$ as $x \to X_{k2}$ can be expressed as:

$$T'_{k2} = \frac{J_{k2}}{[H(X_{k2})]^2} \frac{J_{k2}\Delta U_{kl} + c_m d(U_k - U_l)}{d\, B_{kl}} \tag{36}$$

where $J_{k2} := J^* - g(X_{k2})$, and $0 > J_{k2} > J_{k1}$, being $X_{k1} < X_{k2} < x_s$. It must be $T'_{k1} < 0$ and $T'_{k2} > 0$ (see Fig. 1(b)). Now, if $\Delta U_{kl} \leq 0$, it is clear that both T'_{k1} and T'_{k2} are positive, and this would exclude the possibility of a double intersection. Assume then that $\Delta U_{kl} > 0$ (notice that ΔU_{kl} can take any sign even if $B_{kl} < 0$). In this case, as explained below (see item (3) and Lemma 3), the last element of the optimal sequence is some U_i, with $i \geq k > l$, so the function $T(x)$ should intersect again m_j at some $X_{k0} < X_{k1}$ with a positive left derivative T'_{k0} given by:

$$T'_{k0} = \frac{J_{k0}}{[H(X_{k0})]^2} \frac{J_{k0}\Delta U_{kl} + c_m d(U_k - U_l)}{d\, B_{kl}}$$

Now, $J_{k1} > J_{k0}$, hence we have that if T'_{k1} is negative, negative must also be T'_{k0}. Hence, the possibility of a multiple intersection is excluded also in this case.

Assume now that $B_{kl} > 0$. From (32) and (33), it follows that $I_k := (X_{k1}, X_{k2}) \subset R_s$. In this case, either $X_{k1} < 0$ or $X_{k1} \geq 0$.

If $X_{k1} \geq 0$ (hence also $X_{k2} > 0$), the expression of the left derivative of $T(x)$ as $x \to X_{k2}$ can be expressed similarly to (36) and is given by:

$$T'_{k2} = \frac{J_{k2}}{[H(X_{k2})]^2} \frac{J_{k2}\Delta U_{kl} - c_p d(U_k - U_l)}{d\, B_{kl}}. \tag{37}$$

To have, as required, a positive T'_{k2}, it must be $\Delta U_{kl} > 0$ (being in this case $J_{k2} > 0$ and $B_{kl} > 0$). Since $J_{k2} < J_{k1}$ in this case, it follows that $J_{k2}\Delta U_{kl} < J_{k1}\Delta U_{kl}$, that is, being

$$T'_{k1} = \frac{J_{k1}}{[H(X_{k1})]^2} \frac{J_{k1}\Delta U_{kl} - c_p d(U_k - U_l)}{d\, B_{kl}}, \tag{38}$$

also T'_{k1} should be positive, in contrast with the behavior assumed (see Fig. 1(b)).

If, on the other hand, $X_{k1} < 0$, from (35), to have, as required, a negative T'_{k1}, it should be $\Delta U_{kl} < 0$. But then also T'_{k2} will be negative, both if $X_{k2} \geq 0$ (just consider (37)), both if $X_{k2} < 0$ (being in this case $J_{k1} < J_{k2}$, hence $J_{k2}\Delta U_{kl} < J_{k1}\Delta U_{kl}$ and comparing (36) with (35)). Hence, also if $B_{kl} > 0$, the possibility of a multiple intersection is excluded.

3) *Determination of the last element of $\mathcal{S}$*

To prove that the last element of $\mathcal{S}$ is the first $L \geq \ell$ in $\mathcal{S}'$ such that $\Delta U_{i_L, i_{L+1}} \geq 0$, we compute $T_\infty := \lim_{x \to -\infty} T(x)$ and show that it is a finite quantity with $T_\infty \in [M_L, m_L]$.

As for the computation of T_∞, we proceed as follows. Let U_k be the value of $u(x)$ on a certain interval $I_k := (X_{k_1}, X_{k2})$ (notice that this corresponds to have $-\phi_{kM} < T(x) < -\phi_{mk}$ for all $x \in I_k$, for some $m < k < M$, see Fig. 1(a)). The solutions $V(x, 0)$ and $V(x, 1)$ to the HJB equations in I_k can then be obtained and are reported in Appendix 7.8 (see expressions (62) and (63)). Collecting the terms in (62) and (63), we obtain:

$$V_x(x,1) = A_g x + B_g e^{-\alpha_k(x-X_{k2})} + C_g \tag{39}$$

and

$$H(x) = A_h x + B_h e^{-\alpha_k(x-X_{k2})} + C_h \tag{40}$$

where A_g, B_g, C_g, A_h, B_h and C_h are suitable constants. Since $|V(x,0)|$ and $|V(x,1)|$ are bounded by a quadratic function for all x (see Appendix 7.4), it is clear that $B_g = 0$ and $B_h = 0$ if we are considering the most negative interval (i.e. the one with $X_{k1} = -\infty$). From this, it is possible to see that:

$$T_\infty := \lim_{x\to-\infty} T(x) = \lim_{x\to-\infty} \frac{A_g x + C_g}{A_h x + C_h} = \frac{A_g}{A_h}$$

Substituting the analytical expression of A_g and A_h (which can be obtained from (62) and (63)) and simplifying, allows to show that:

$$T_\infty = \frac{A_g}{A_h} = -\frac{q_{up} + q_k}{U_k} < 0. \tag{41}$$

Let i_R, $R \le L'$, be the last element of $\mathcal{S}$' used by the optimal policy and let for convenience $m_{L'+1} = -\infty$. It must be:

$$m_{R+1} \le T_\infty < m_R.$$

i.e., using the expression of T_∞ in (41),

$$m_{R+1} \le -\frac{q_{up} + q_{i_R}}{U_{i_R}} < m_R.$$

From the expression of m_R and m_{R+1} (see (15)), it is possible to show that this condition corresponds to $\Delta U_{i_{R-1},i_R} < 0$ and $\Delta U_{i_R,i_{R+1}} \ge 0$. Since, according to Lemma 3, $\Delta U_{i_k,i_{k+1}} \ge 0$ implies $\Delta U_{i_{k+1},i_{k+2}} > 0$, this means that all the elements i_k, with $k < R$, are such that $\Delta U_{i_k,i_{k+1}} < 0$ while all the i_k, with $k > R$, are such that $\Delta U_{i_k,i_{k+1}} > 0$. This allows to conclude that, when constructing the sequence $\mathcal{S}$ from $\mathcal{S}'$, we have to stop at L, which is in fact the first element of $\mathcal{S}'$ such that $\Delta U_{i_L,i_{L+1}} \ge 0$.

Now, if $U_i > d$ for all i (Condition (i) of Theorem 2) the proof is complete, with $Z^* := X^*_\ell = Z_g$. On the contrary, if $U_i < d$ for some i (but Condition (ii) of Theorem 2 holds), we have proved so far the optimality of a policy which uses some $U_{i_j} < d$ for $x \in (Z^*, Z_g)$. However, if in a given region it is optimal to apply a control $U_i < d$, this region is transient and we have decided (for simplicity of notation in (7)) to replace all these U_i with $u = 0$. This does not influence the steady state average performance index considered in this report but only the transient behavior. □

5 Numerical Examples

5.1 Example 1

Consider a system with a $q_d(u)$ given by (2) with $N = 5$ and the parameters as follows: $d = 1$, $c_m = 1000$, $c_p = 1$, $q_{up} = 0.5$, $\mathbf{q_d} := \{q_1, \ldots, q_N\} = \{0.002,\ 0.003,\ 0.008,\ 0.01,\ 0.02\}$, $\mathbf{U} := \{U_1, \ldots, U_N\} = \{5,\ 20,\ 25,\ 40,\ 50\}$. Since $U_i > d$ for all i and all U_i's are feasible, this example meets Condition (i) of Theorem 2.

It is interesting to observe (see Fig. 2) how the point (U_3, q_3) is a non convex point of the function $q_d(u)$, in the sense that $-\phi_{23} = \min_{i<3}(-\phi_{i,3}) < \max_{i>3}(-\phi_{3,i}) = -\phi_{34}$, condition for the absence of U_3 in the optimal policy, according to Lemma 2.

As a matter of fact, the optimal sequence defined in Section 3 turns out to be in this case $\mathcal{S}' = \{U_1, U_2, U_4, U_5\}$, i.e., $L' = 4$ and $U_3 \notin \mathcal{S}$. Here we have $U_1 > d$, $\Delta U_{1,2} = -7.5$, $\Delta U_{2,4} = -9.9$ and $\Delta U_{4,5} = -4.7$: for this reason $\mathcal{S} \equiv \mathcal{S}'$ with $\ell = 1$ and $L \equiv L'$.

The optimal levels X^*_i, $i = 1, \ldots, L$, numerically computed through a gradient descent method applied starting from tentative levels (a possible initialization procedure which provides interesting

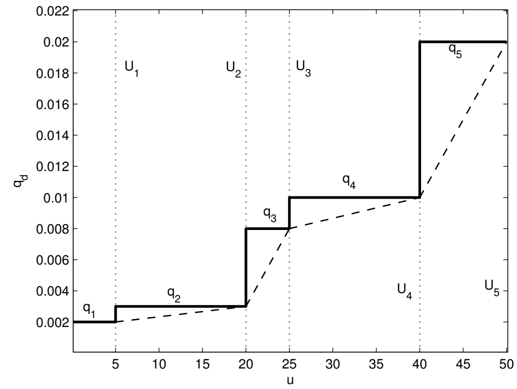


Figure 2: The failure rate function $q_d(u)$ for the $N = 5$ level example of Section 5.1

results is sketched in Appendix 7.10) and using the expression of $J(X_1, \ldots, X_L)$ reported in (42) of Appendix 7.1, are given by $\mathbf{X}^* := \{X_1^*, \ldots, X_L^*\} = \{2.81, 1.55, -0.02, -0.131\}$ with a total cost $J^* = 4.8$.

Notice that the function $J(X_1, \ldots, X_L)$ is not convex as it is possible to verify considering the plot of $J(X_1, \ldots, X_L)$ vs X_1 with $X_2, \ldots, X_L$ fixed: the plot tends to become flat as $X_1 \to -\infty$. Nevertheless, in all the examples considered, the gradient method, started with several different initial conditions randomly taken, successfully attained the optimal cost (as also observed in [4]). This probably depends on a quasi-convexity property of $J(X_1, \ldots, X_L)$.

In any case, it is possible to verify that the levels obtained above are indeed optimal by using the Verification Theorem (i.e. Theorem 1). In fact, with the numerical values of all the parameters and of the optimal cost, it is possible to derive the explicit expression of the differential cost-to-go functions $V(x, i)$ (which turn out to be as requested continuous differentiable functions) and to show that policy (7) with the thresholds just computed satisfies the HJB equations (12)-(13) and meets all the conditions of the Verification Theorem. The details of this example are given in Appendix 7.9.

5.2 Example 2

Consider now a system with a $q_d(u)$ given by (2) with $N = 4$ and the parameters as follows: $d = 6$, $c_m = 10$, $c_p = 1$, $q_{up} = 0.2$, $\mathbf{q_d} := \{q_1, \ldots, q_N\} = \{0.05, 0.08, 0.09, 0.13\}$, $\mathbf{U} := \{U_1, \ldots, U_N\} = \{7, 8, 9, 10\}$.

Here only U_3 and U_4 are feasible, since $\Delta_1 = -0.1$ and $\Delta_2 = -0.08$ are negative. Again, since $U_i > d$ for all i and there are some feasible U_i, also this example meets Condition (i) of Theorem 2. Applying Algorithm 1, we obtain $\mathcal{S}' = \{1, 3, 4\}$, with $L' = 3$. Since $U_1 > d$ and $\Delta U_{3,4} = 0.07 > 0$, $\mathcal{S} = \{1, 3\}$, with $\ell = 1$ and $L = 2$. There are then only two thresholds, which optimal value is $Z^* = X_1^* = 691.15$ and $X_2^* = 630.26$. The optimal cost is $J^* = 715.15$. A procedure, similar to the one reported in Appendix 7.9 for the example of Section 5.1, can be followed to show that also in this case all the conditions of the Verification Theorem are met by the considered policy and that the differential costs $V(x, i)$ are convex.

5.3 Example 3

Consider now a system which meets Condition (ii) of Theorem 2, with $N = 4$ and the parameters as follows: $d = 10$, $c_m = 1$, $c_p = 1$, $q_{up} = 1$, $\mathbf{q_d} := \{q_1, \ldots, q_N\} = \{0.05, 0.07, 0.09, 0.13\}$, $\mathbf{U} := \{U_1, \ldots, U_N\} = \{7, 10, 13, 15\}$. It is straightforward to verify that, as requested by Condition (ii) of the theorem,

$$\frac{q_3 - q_2}{U_3 - U_2} \equiv \frac{q_2 - q_1}{U_2 - U_1} = 0.0067.$$

Here only U_3 and U_4 are feasible, since $\Delta_1 = -3.5$ and $\Delta_2 = -0.7$ are negative. Applying Algorithm 1, we obtain $\mathcal{S}' = \{1, 2, 3, 4\}$, with $L' = 4$. Since only U_3 and U_4 are larger than d and $\Delta U_{3,4} = -1.66 < 0$, we have $\mathcal{S} = \{3, 4\}$, with $\ell = 3$ and $L = 4$.

The optimal thresholds are $Z^* = X_3^* = 0$ and $X_4^* = -1.51$: the optimal policy is here JIT (Just In Time), being $Z^* = 0$. The optimal cost is $J^* = 2.98$. A procedure, similar to the one reported in Appendix 7.9 for the example of Section 5.1, can be followed to show that also in this case all the conditions of the Verification Theorem are met by the considered policy and that the differential costs $V(x, i)$ are convex. In this case $V_x(Z^*, 1) = -0.28$ is not zero, since we are in Case (ii) of Theorem 2, and $V_x(x, 1)$ becomes 0 at $Z_g = 1.07$. So the policy satisfying the Verification Theorem would use $U_1 = 7$ for $x \in (X_3^*, Z_g)$ and $u^*(x) = 0$ only for $x \geq Z_g$. However, since the region above Z^* is transient and we are interested in an average, long term cost, the same performance is achieved by setting $u(x) = 0$ for all $x > Z^*$ (as mentioned in the proof of Theorem 2).

5.4 Example 4

This example shows that the sufficient Conditions (i) and (ii) of Theorem 2 are not necessary for the optimality of policy (7). In fact, even if the conditions of the theorem are not met in this case, it is possible to show that policy (7) is optimal by directly applying the Verification Theorem.

Let $d = 6$, $c_m = 10$, $c_p = 1$, $q_{up} = 0.2$, $\mathbf{q_d} := \{q_1, \ldots, q_N\} = \{0.01, 0.02, 0.05, 0.08, 0.09, 0.13\}$, $\mathbf{U} := \{U_1, \ldots, U_N\} = \{4, 6, 7, 8, 9, 10\}$. Here only U_5 and U_6 are feasible. Applying Algorithm 1, we obtain $\mathcal{S}' = \{1, 2, 5, 6\}$, with $L' = 4$. We have $U_1 < d$ and $U_2 = d$ but Condition (ii) of Theorem 2 is not satisfied:

$$\frac{q_5 - q_2}{U_5 - U_2} = 0.0233 \neq 0.005 = \frac{q_2 - q_1}{U_2 - U_1}.$$

Nevertheless, we will show that the Verification Theorem applies also in this case.

Since $U_1 < d$, $U_2 = d$ and $\Delta U_{5,6} = 0.07 > 0$, we have $\mathcal{S} = \{5\}$, with $\ell = 3$ and $L = 3$. The optimal threshold is $Z^* = X_3^* = 633.1$ and the optimal cost is $J^* = 708.1$. A procedure, similar to the one reported in Appendix 7.9 for the example of Section 5.1, can be followed to show that also in this case all the conditions of the Verification Theorem are met by the considered policy and that the differential costs $V(x, i)$ are convex. As in the example of Section 5.3, $V_x(Z^*, 1) = -87.5$ is not zero and $V_x(x, 1)$ becomes 0 at $Z_g = 700.96$. In this case, however, we have that the rate d is optimal over an interval and not just on a point (i.e. the hedging level is an interval): this is a consequence of the fact that the conditions of Theorem 2 are not met. So, the policy meeting all the conditions of the Verification Theorem is:

$$u(x) := \begin{cases} 0 & x \geq Z_g = 700.96 \\ U_1 & x \in [676.5, 700.96) \\ d & x \in [633.1, 676.5) \\ U_5 & x < Z^* = 633.1 \end{cases}$$

Since the transient part does not influence the optimal cost, removing this part from this policy, results in (7), which is then optimal and is given by

$$u(x) := \begin{cases} 0 & x > Z^* = 633.1 \\ d & x = Z^* \\ U_5 & x < Z^* \end{cases}$$

5.5 Example 5

This example also shows that the sufficient Conditions (i) and (ii) of Theorem 2 are not necessary for the optimality of policy (7). In fact, also in this case, even if the conditions of the theorem are not met, it is possible to show that policy (7) is optimal by directly applying the Verification Theorem. In this case we have all $U_i \geq d$ but Condition (i) of Theorem 2 is not met because $U_1 = d$. Such a situation will imply that the rate d is optimal over an interval and not just over a point.

The parameters here are like in Section 5.4, we only remove element (U_1, q_1), i.e.: $d = 6$, $c_m = 10$, $c_p = 1$, $q_{up} = 0.2$, $\mathbf{q_d} := \{q_1, \ldots, q_N\} = \{0.02, 0.05, 0.08, 0.09, 0.13\}$, $\mathbf{U} := \{U_1, \ldots, U_N\} = \{6, 7, 8, 9, 10\}$. So $U_1 = d$ and Conditions (i) and (ii) of the theorem are not satisfied. Nevertheless, we will show that the Verification Theorem applies also in this case.

Here only U_4 and U_5 are feasible. Applying Algorithm 1, we obtain $\mathcal{S}' = \{1, 4, 5\}$, with $L' = 3$. Since $U_1 = d$ and $\Delta U_{4,5} = 0.07 > 0$, we have $\mathcal{S} = \{4\}$, with $\ell = 2$ and $L = 2$.

It is interesting to observe that the optimal threshold and the optimal cost are like in the example of Section 5.4, i.e. $Z^* = X_2^* = 633.1$ and $J^* = 708.1$: actually, the only difference concerns the transient behavior, as shown subsequently.

We can apply again a procedure similar to the one reported in Appendix 7.9 for the example of Section 5.1 to show that also in this case all the conditions of the Verification Theorem are met by the considered policy and that the differential costs $V(x, i)$ are convex. As in the example of Section 5.4, $V_x(Z^*, 1) = -87.5$ (the same value of Section 5.4) and $V_x(x, 1)$ becomes 0 at $Z_g = 680.8$. As anticipated, also in this case we have that the rate d is optimal over an interval and not just on a point: this is a consequence of the fact that the conditions of Theorem 2 are not met. In fact, such conditions were requested to guarantee that the rate d could be optimal only over a point, allowing to prove the differentiability of $V(x, 1)$. This example, together with the one of Section 5.4, shows that the conditions of the theorem are not necessary for the differentiability of such a function (hence to prove the optimality of the feedback law in (7)). However, even if these conditions are not necessary for optimality, it seems that, if the U_i's are not all larger than d, a necessary condition for the existence of the optimal control is that the rate d must appear in $\mathcal{S}'$ (which implies that, if U_i are not all larger than d, the rate d must be present among the levels U_k's). The policy meeting all the conditions of the Verification Theorem is in this case:

$$u(x) := \begin{cases} 0 & x \geq Z_g = 680.8 \\ d & x \in [633.1, 680.8) \\ U_4 & x < Z^* = 633.1 \end{cases}$$

Since the transient part does not influence the optimal cost, removing this part from this policy, results in (7), which is also optimal and is given by

$$u(x) := \begin{cases} 0 & x > Z^* = 633.1 \\ d & x = Z^* \\ U_4 & x < Z^* \end{cases}$$

6 Conclusions

The problem of minimizing a long term average expected backlog/inventory cost for a manufacturing system comprising a machine characterized by a Markovian, production dependent failure rate process has been considered in this report. The dependence of the failure rate on the production rate has been described through a piecewise function which can be thought of as the discrete approximation of a continuous failure rate function. The discretization step can be selected to obtain the desired degree of approximation. The structure of the optimal policy has been given in the report: even if under the discrete approximation considered, this policy confirms several analytical findings and conjectures reported in the literature.

7 Appendix

7.1 Analytical expression of the probability density function and of the total cost

Consider the piecewise constant feedback control given in (7) for some fixed (not necessarily optimal) levels $X_\ell, \ldots, X_L$ and let for convenience of notation $X_{L+1} := -\infty$.

The expression of the steady state probability density function $p_{X_\ell \ldots X_L}(x)$ of the buffer level corresponding to the application of this feedback policy can be derived as indicated below and the cost in (4) can be evaluated as follows:

$$J(X_\ell \dots X_L) = \int_{-\infty}^{Z} g(x) p_{X_\ell \dots X_L}(x) dx + \gamma(X_\ell \dots X_L) g(Z), \tag{42}$$

where, as remarked above, we denote with Z the largest threshold X_ℓ. In (42), $\gamma(X_\ell \dots X_L)$ is the point mass probability in Z. Now, using the general procedure reported in [4], it is possible to determine the expression of $p_{X_\ell \dots X_L}(x)$ and of $\gamma(X_\ell \dots X_L)$. By denoting with $p^{(j)}_{X_\ell X_{\ell+1} \dots X_L}(x)$ the steady state probability density function of the buffer level for $x \in [X_{j+1}, X_j)$ (where $u(x) = U_{i_j}$) $j = \ell, \ell+1, \dots, L$, we have (notice that, according to (7), $U_{i_j} > d$ for all $j = \ell, \dots, L$):

$$p^{(j)}_{X_\ell X_{\ell+1} \dots X_L}(x) = K_j \frac{U_{i_j}}{U_{i_j} - d} e^{\alpha_{i_j}(x - X_j)}, \tag{43}$$

$$\gamma(X_\ell \dots X_L) = K_\ell \frac{d}{q_d(d)} \tag{44}$$

where the constants K_j in (43) and (44) can be computed through the following recursive relation:

$$K_j = K_{j-1} e^{\alpha_{i_{j-1}}(X_j - X_{j-1})} \tag{45}$$

for $j = \ell+1, \dots, L$, while K_ℓ can be derived through the normalization condition:

$$\int_{-\infty}^{Z} p_{X_\ell X_{\ell+1} \dots X_L}(x) dx + \gamma(X_\ell \dots X_L) = 1$$

obtaining

$$\frac{1}{K_\ell} = \frac{d}{q_d(d)} + \sum_{j=\ell}^{L} \frac{U_{i_j}}{U_{i_j} - d} F(j) \frac{1 - e^{\alpha_{i_j}(X_{j+1} - X_j)}}{\alpha_{i_j}}$$

where $F(\ell) = 1$ and $F(j) = e^{\alpha_{i_{j-1}}(X_j - X_{j-1})} F(j-1)$ for $j = \ell+1, \dots, L$.

Using (42)-(44), it is quite straightforward but cumbersome for any selected X_i, $i = \ell, \dots, L$, to derive the cost $J(X_\ell \dots X_L)$. The optimal hedging level $Z^* = X^*_\ell$ can not be negative: assume by contradiction $Z^* < 0$. Then, by shifting all the X^*_k of $+\delta$ (where $\delta > 0$ is such that $Z^* + \delta$ is still non positive), we obtain from (42) a cost $J = J^* - c_m \delta < J^*$, which contradicts that J^* is optimal.

7.2 Positivity of $H(x) := V(x, 0) - V(x, 1)$

Considering equation (12) with $u = 0$ at any x where $V(x, 1)$ is differentiable gives

$$J^* - g(x) \le -d \;\; V_x(x, 1) + q_d(0) H(x).$$

Summing this equation to (13), we obtain

$$H_x(x) := \frac{dH}{dx} \ge -\frac{q_{up} + q_d(0)}{d} H(x). \tag{46}$$

Let X_a and X_b denote two neighboring switching levels and remember that, as mentioned in Section 4, $V(x, 1)$ is differentiable for $x \in (X_a, X_b)$ (since the control is constant, hence continuous, in (X_a, X_b)). If we know that $H(X_a) > 0$, then (46) allows to conclude that also $H(X_b) > 0$. This fact, together with the continuity of the differential costs (hence of $H(x)$), implies that if we know that $H(x_0) > 0$ for some x_0, $H(x) > 0$ for all $x \ge x_0$ (i.e., even if between x_0 and x there are some switching levels).

Now, the differential costs $V(x, i)$, $i = 0, 1$, go to infinity as $|x| \to \infty$ (see Appendix 7.5). Since $V(x, 0)$ is continuously differentiable and goes to infinity as $x \to -\infty$, there must be at least a decreasing sequence of x_k ($k = 0, 1, \dots$) with $x_k \to -\infty$ as $k \to \infty$, where $V_x(x_k, 0) := dV(x_k, 0)/dx < 0$. Take x_0 such that $g(x_0) > J^*$, hence $g(x_k) > J^*$ for all k. Then, from (13), $H(x_k)$ must be positive for all k. The two properties (i.e. $H(x_k) > 0$ for all k and $\frac{dH}{dx} \ge -\frac{q_{up} + q_d(0)}{d} H(x)$ for all x where $V(x, 1)$ is differentiable) allow to conclude that $H(x) > 0$ for all x. In fact, for any given $\bar{x}$, there always exists a value x_k in the sequence such that $x_k < \bar{x}$. The fact that $H(x_k) > 0$, implies, as mentioned, $H(x) > 0$ for all $x > x_k$, i.e. $H(\bar{x}) > 0$.

7.3 Continuous differentiability of the differential cost $V(x,1)$ on the switching levels

To show the continuity of $V_x(x,1)$ on the switching levels (i.e. where the control is discontinuous), we proceed as follows. Let X be a switching level and assume U_i is optimal on a left neighborhood of X and U_j on the right. If $D^-(X)$ denotes the left derivative of $V(x,1)$ at X (i.e. $\lim_{x\uparrow X} V_x(x,1)$) and $D^+(X)$ the right derivative of $V(x,1)$ at X (i.e. $\lim_{x\downarrow X} V_x(x,1)$), it is possible to write from (12):

$$(U_i - d)D^-(X) + q_i\, H(X) = J^* - g(X), \tag{47}$$

and

$$(U_j - d)D^+(X) + q_j H(X) = J^* - g(X). \tag{48}$$

Also:

$$(U_i - d)D^-(X) + q_i\, H(X) \le (U_j - d)D^-(X) + q_j\, H(x), \tag{49}$$

being U_i optimal on the left of X and, similarly

$$(U_j - d)D^+(X) + q_j\, H(X) \le (U_i - d)D^+(X) + q_i\, H(X). \tag{50}$$

Combining (47), (48) and (50) allows to write:

$$(U_i - d)D^-(X) \le (U_i - d)D^+(X), \tag{51}$$

Similarly, combining (47), (48) and (49):

$$(U_j - d)D^+(X) \le (U_j - d)D^-(X). \tag{52}$$

If $U_i - d$ and $U_j - d$ have the same sign, (51) and (52) allow to conclude $D^+(X) \equiv D^-(X)$. This holds at all the switching levels except possibly if X is the hedging level where $u^*(X) = d$, $U_i > d$ and $U_j < d$. In this case, the proof of the differentiability follows a reasoning similar to the one adopted for the same purpose in [9], ch. 3.3.

Since $U_i > d$, equation (51) implies $D^-(X) \le D^+(X)$. If $V(x,1)$ is not $\mathcal{C}^1$ at X, (12) should be intended (as mentioned in Section 4) in the viscosity sense, i.e., since $D^-(X) \le D^+(X)$:

$$\min_{u\in[0,\,\mu]} \{(u-d)r + q_d(u)H(X)\} \le J^* - g(X), \tag{53}$$

for all $r \in D_{sub}(X)$, where $D_{sub}(X)$ denotes the set of all subdifferentials of $V(x,1)$ at X (see e.g. [9], Appendix F, for a definition of $D_{sub}(X)$) and is given in this case by $[D^-(X), D^+(X)]$. To simplify the notation, let

$$S(r) := \min_{u\in[0,\,\mu]} \{(u-d)r + q_d(u)H(X)\}\,.$$

With this notation, equation (53) becomes:

$$S(r) \le J^* - g(X) \tag{54}$$

for all $r \in D_{sub}(X)$. For $r = D^-(X)$ and for $r = D^+(X)$, according to (47) and (48), we have:

$$S(D^-(X)) = J^* - g(X), \qquad S(D^+(X)) = J^* - g(X) \tag{55}$$

Now, the function $S(r)$ is concave. In fact, for any $\alpha \in (0,1)$:

$$\begin{aligned}
S(\alpha r_1 + (1-\alpha)r_2) &= \min_{u\in[0,\,\mu]} \{(u-d)(\alpha r_1 + (1-\alpha)r_2) + q_d(u)H(X)\} \\
&= \min_{u\in[0,\,\mu]} \{\alpha\,[(u-d)r_1 + q_d(u)H(X)] \\
&\quad + (1-\alpha)\,[(u-d)r_2 + q_d(u)H(X)]\} \\
&\ge \min_{u\in[0,\,\mu]} \{\alpha\,[(u-d)r_1 + q_d(u)H(X)]\} \\
&\quad + \min_{u\in[0,\,\mu]} \{(1-\alpha)\,[(u-d)r_2 + q_d(u)H(X)]\} \\
&= \alpha S(r_1) + (1-\alpha)S(r_2).
\end{aligned}$$

Since $D_{sub}(X) = [D^-(X), D^+(X)]$, any $r \in D_{sub}(X)$ can be obtained as the convex combination of $D^-(X)$ and $D^+(X)$. In view of the concavity of $S(r)$ and of (55),

$$S(r) \geq J^* - g(X) \tag{56}$$

for all $r \in D_{sub}(X)$. Comparing (54) with (56) implies that $S(r)$ is constant on $D_{sub}(X)$. Now, since U_i is optimal for $r = D^-(X)$ and U_j is optimal for $r = D^+(X)$, if $D_{sub}(X)$ was not a singleton, in view of Conditions (i) or (ii) of Theorem 2, at least one of the following situations occurs: a) there exists an $r_0 > D^-(X)$ such that U_i is optimal for $r \in [D^-(X), r_0]$, b) there exists an $r_0 < D^+(X)$ such that U_j is optimal for $r \in [r_0, D^+(X)]$, c) there exists $U_k \in \mathcal{S}$ between the elements U_i and U_j, $U_k \neq d$ and a finite interval $[r_a, r_b] \subset [D^-(X), D^+(X)]$ such that U_k is optimal for $r \in [r_a, r_b]$ (this could occur e.g. under Condition (i) of the Theorem if U_i was not U_1 but some $U_k > U_1$). In each of these three situations, $S(r)$ should be constant over a finite interval, e.g., in the first situation, $S(r) = (U_i - d)r + q_i H(X)$ should give the same value for all $r \in [D^-(X), r_0]$ (and similarly in the other two cases). This is clearly not possible and implies that $D_{sub}(X)$ is a singleton, i.e. $D^-(X) = D^+(X)$, and d is optimal at X (this would correspond under Condition (i) of Theorem 2 to $U_i = U_1$ and $U_j = 0$ and, under Condition (ii), to $U_i = U_{i_{j+1}}$, $U_{i_j} = d$ and $U_j = U_{i_{j-1}}$).

7.4 Quadratic bound on the modulus of the differential costs $V(x, 0)$ and $V(x, 1)$

Let $x_n \in \Re$ be a large amount of backlog such that for all $x \leq x_n < 0$ the optimal action is to select a production rate larger than d (this x_n exists otherwise the optimal cost would be infinite, while it is simple to define policies providing a finite cost - see also [2], ch. 9.3). Similarly, let $x_p \in \Re$ be a large amount of surplus such that for all $x \geq x_p > 0$ the optimal action is to select a 0 production rate.

Now, due to the continuity of $V(x,i)$ for all x and $i = 0, 1$ (which derives from the fact that $V(x,i)$ is a viscosity solution to (12)-(13), as mentioned in Section 4), $V(x,i)$ is bounded over this finite interval $[x_n, x_p]$, i.e.

$$|V(x,i)| \leq K \tag{57}$$

for all $x \in [x_n, x_p]$, $i = 0, 1$.

Consider now any initial state (x,i), with $x < x_n$ and let τ denote the time necessary to reach the inventory x_n from x under the optimal control (i.e. the time necessary to go from the state (x,i) to $(x_n, \cdot)$ under the optimal control). Then, if $E_{x,i}$ is the expectation taken assuming that $x(0) = x$, $s(0) = i$ and that the optimal control is applied, it is possible to write:

$$\begin{aligned}
|V(x,i)| &:= \left| \lim_{T\to\infty} E_{x,i} \left[\int_0^T (g[x(t)] - J^*)\, dt \right] \right| \\
&= \left| \lim_{T\to\infty} E_{x,i} \left[\int_0^\tau (g[x(t)] - J^*)\, dt + \int_\tau^T (g[x(t)] - J^*)\, dt \right] \right| \\
&\leq \left| E_{x,i} \left[\int_0^\tau (g[x(t)] - J^*)\, dt \right] \right| + \left| \lim_{T\to\infty} E_{x,i} \left[\int_\tau^T (g[x(t)] - J^*)\, dt \right] \right| \\
&\leq \left| E_{x,i} \left[\int_0^\tau (g[x(t)] - J^*)\, dt \right] \right| + K \\
&\leq E_{x,i} \left[\int_0^\tau (g[x(t)] + J^*)\, dt \right] + K.
\end{aligned}$$

Now, $0 \leq g(x) \leq C|x|$ (where $C = \max\{c_p, c_m\}$), and $|x(t)| \leq |x| + (\mu + d)t$. So we can write:

$$|V(x,i)| \leq (C|x| + J^*) E_{x,i}[\tau] + C(\mu + d) E_{x,i}[\tau^2/2]) + K. \tag{58}$$

Based on a result of [8] (Lemma 7.1 of [8], properly adapted to our case), we have $E[\tau^\ell] \leq c_a + c_b |x - x_n|^\ell$ (where $\ell = 1, 2, \ldots$). Substituting this in (58) allows to obtain the quadratic bound on the $|V(x,i)|$. The case $x > x_p$ is straightforward being in this case everything deterministic and $\tau = (x - x_p)/d$, which still gives a quadratic bound on $|V(x,i)|$.

7.5 Quadratic lower bound on the differential costs $V(x,0)$ and $V(x,1)$

We show that the differential costs $V(x,0)$ and $V(x,1)$ are also bounded from below by a quadratic function, that is, $V(x,i) \geq c_3x^2 + c_4$, $i = 0,1$, for some constants $c_3 > 0$ and c_4 (and so go to infinity as $|x| \to \infty$). This can be proved with a procedure similar to the one used in Appendix 7.4. So, consider now $x_n < 0$ and $x_p > 0$ such that $g(x_n) > J^*$ and $g(x_p) > J^*$. Then we have, as above, for the continuity of $V(x,i)$, $i = 0,1$, that $V(x,i) \geq G$ for some constant G (possibly negative) for all $x \in [2x_n, 2x_p]$.

Consider now any initial state (x,i), with $x < 2x_n$ and let τ denote the time necessary to reach the level $x/2$ from x under the optimal control (i.e. the time necessary to go from the state (x,i) to $(x/2,\cdot)$ under the optimal control). Then, if $E_{x,i}$ is the expectation taken assuming that $x(0) = x$, $s(0) = i$ and that the optimal control is applied, it is possible to write:

$$
\begin{aligned}
V(x,i) &:= \lim_{T\to\infty} E_{x,i}\left[\int_0^T (g[x(t)] - J^*)\,dt\right] \\
&= \lim_{T\to\infty} E_{x,i}\left[\int_0^\tau (g[x(t)] - J^*)\,dt + \int_\tau^T (g[x(t)] - J^*)\,dt\right] \\
&= E_{x,i}\left[\int_0^\tau (g[x(t)] - J^*)\,dt\right] + \lim_{T\to\infty} E_{x,i}\left[\int_\tau^T (g[x(t)] - J^*)\,dt\right] \\
&\geq E_{x,i}\left[\int_0^\tau (g[x(t)] - J^*)\,dt\right] + G.
\end{aligned}
$$

The last inequality holds both if $x/2 \geq 2x_n$ (since $V(x,i) \geq G$ in $[2x_n, 2x_p]$), both if $x/2 < 2x_n$ (since in this case the contribution of the integral between τ and the time to reach $2x_n$ is positive, being in this interval $g[x(t)] > J^*$, and a smaller quantity is obtained by neglecting it). Now, for all $t \in [0,\tau]$, $g[x(t)] = -c_m x(t)$, and $x(t) \leq x/2$. So we can write:

$$
V(x,i) \geq E_{x,i}\left[\int_0^\tau \left(-c_m\frac{x}{2} - J^*\right)dt\right] + G = (-c_m\frac{x}{2} - J^*)E_{x,i}[\tau] + G.
$$

Now, since $x/2 < x_n$, $-c_m x/2 > J^*$, i.e. $-c_m x/2 - J^* > 0$. In addition, for all realizations, $\tau \geq -x/(2(\mu - d)) > 0$ (being $x < 2x_n < 0$) and hence $E_{x,i}[\tau] \geq -x/(2(\mu - d))$. Substituting this in the equation above allows to obtain:

$$
V(x,i) \geq (c_m\frac{x}{2} + J^*)\frac{x}{2(\mu - d)} + G \geq c_3x^2 + c_4
$$

for suitable constants $c_3 > 0$ and c_4. The case $x > 2x_p$ is similar and allows to obtain the same quadratic lower bound on the $V(x,i)$. This lower bound, together with the upper bound determined in Appendix 7.4, allows to conclude that $V(x,i)$, $i = 0,1$, goes to infinity when $|x| \to \infty$ as a quadratic function.

7.6 Proof of Lemma 2

The proof can be performed by induction. The case $N = 2$ is trivial with $\mathcal{S}' = \{1,2\}$ (i.e. $L' = N = 2$). Consider now also the case $N = 3$ since it is needed in the induction step. If $N = 3$, either $\phi_{12} < \phi_{23}$ (convex case), either $\phi_{12} > \phi_{23}$ (concave case). In the first case the ordered sequence of ϕ_{jk} is: $-\phi_{12} > -\phi_{13} > -\phi_{23}$ as it is immediate to verify and the sequence $\mathcal{S}' = \{1, 2, 3\}$, with $L' = N = 3$ satisfies all the properties stated by the lemma, with $M_1 = -\phi_{12} \equiv m_2$, $M_2 \equiv m_3 = -\phi_{23}$. Similarly, in the second case, the ordered sequence[1] of ϕ_{jk} is: $-\phi_{23} > -\phi_{13} > -\phi_{12}$ and the sequence $\mathcal{S}' = \{1, 3\}$, with $L' = 2$ satisfies all the properties stated by the lemma, with $M_1 = -\phi_{13} \equiv m_2$ and $\min_{i<2}(-\phi_{i,2}) < \max_{i>2}(-\phi_{2,i})$.

[1]Notice that, with $N = 3$, $-\phi_{12} > -\phi_{13} > -\phi_{23}$ and $-\phi_{23} > -\phi_{13} > -\phi_{12}$ are the only two sequences which can be observed among the ϕ_{jk}

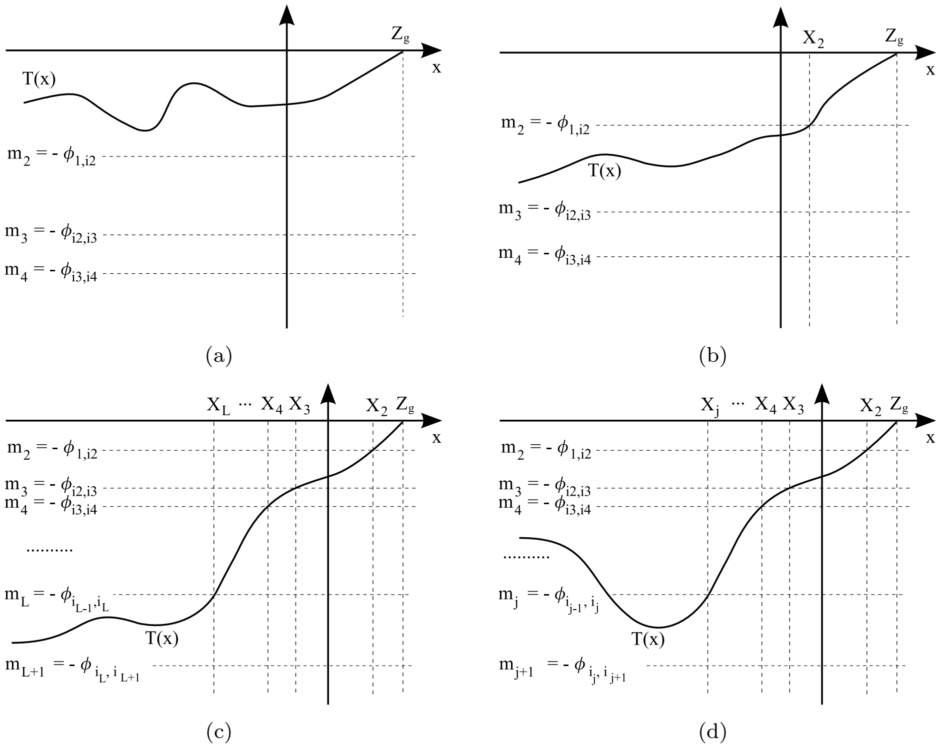


Figure 3: Possible intersections of $T(x)$ with the m_j's: the right-down plot in (d), where $T(x)$ intersects two (or more) times a level m_j, is excluded from Theorem 2 of the paper

Assume now the result holds true up to N, and let $\mathcal{S}'_N = \{i_1, \ldots, i_{L'}\}$ be the sequence for the N case. So $\mathcal{S}'_N$ satisfies all the conditions in the lemma and in particular for all $j \notin \mathcal{S}'_N$, Condition (16) holds.

Consider the $N+1$ case by adding a couple (U_{N+1}, q_{N+1}) to the $q_d(u)$. Clearly, the $\min_{l<i_{k+1}} \left(-\phi_{l,i_{k+1}}\right)$ remains unchanged for all $k = 1, \ldots, L'-1$ since we have added U_{N+1}. So, consider the $\max_{l>i_k}(-\phi_{i_k,l})$, $k = 1, \ldots, L'-1$. Nothing changes until this maximum is satisfied as before by $i_{k+1} < N+1$. If for some i_v this maximum is satisfied by $N+1$ then, exploiting the property expressed above (in the $N=3$ case) that given any $U_{k1} < U_{k2} < U_{k3}$ the only possible sequences are $-\phi_{k_1,k_2} > -\phi_{k_1,k_3} > -\phi_{k_2,k_3}$ or $-\phi_{k2,k3} > -\phi_{k1,k3} > -\phi_{k1,k2}$, it is possible to prove that:

$$N+1 = \arg\max_{l>i_k} \left(-\phi_{i_k,l}\right), \text{for all } k \geq v; \tag{59}$$

$$-\phi_{i_k,N+1} > -\phi_{i_{k-1},i_k}, \text{for all } k \geq v \tag{60}$$

$$i_v = \arg\min_{i<N+1} \left(-\phi_{i,N+1}\right). \tag{61}$$

This allows to verify that the sequence $\mathcal{S}'_{N+1} = \{i_1, i_2, \ldots, i_v, N+1\}$ satisfies all the properties stated in the Lemma: in fact the first $v-1$ elements of $\mathcal{S}'_{N+1}$ clearly satisfy the Lemma for the inductive assumption; as for the last two elements i_v and $N+1$, by definition of i_v it is $\arg\max_{l>i_v}\left(-\phi_{i_v,l}\right) = N+1$ and, using (61), all the properties in the Lemma are verified for i_v and $N+1$. Finally, by using (60), for all $j = i_{v+1}, i_{v+2}, \ldots, N$ (elements not belonging to $\mathcal{S}'_{N+1}$) equation (16) holds, since the term on the left of (60) represents, according to (59), the $\max_{i>i_k}\left(-\phi_{i_k,i}\right)$ while the term on the right of (60) represents the minimum (thanks again to the inductive assumption).

7.7 Unique intersection of $T(x)$ with the m_j's

In this appendix we give more details to explain why the case of Figure 1(b) is the most general case in the sense that, as mentioned above, "*the first time (starting from Z_g and going left) we* *would observe the intersection of $T(x)$ with a level m_j already met, it must be of the type reported in Fig. 1(b)*".

First of all, notice that, as shown above in item (1) of the Proof of Theorem 2, $T(x)$ is continuous, $T(Z_g) = 0$ and $T(x) < 0$ for all $x < Z_g$. So, let us move from Z_g toward $-\infty$ and check the intersections of $T(x)$ with the m_j's, $j = 2, 3, \ldots, L'$, which, according to Lemma 2, are all negative quantities and form a decreasing sequence, i.e. $0 > m_2 > m_3 > \ldots$ (notice that, according to the defined notation, see e.g. Lemma 2, the sequence of the m_j starts with $j = 2$ but we may consider, for completeness of notation, $m_1 = 0$). If $T(x)$ does not become negative enough and remains above m_2 for all x (see Fig. 3(a)), then $u^*(x) = U_1$ for all x: in this case we do not have any multiple intersection with the m_j's.

If it intersects m_2 at some point X_2 (see Fig. 3(b)), the optimal control on the left of X_2 becomes U_{i_2}, where i_2 denotes the second element of $\mathcal{S}'$ ($i_1 = 1$) and $u^*(x) = U_{i_k}$ when $T(x) \in (m_{k+1}, m_k)$, $k = 1, 2, \ldots, L$.

Let us continue to move toward $-\infty$. There are two possibilities:

a) $T(x)$ intersects only new (i.e. more negative) m_i's until it will remain in an interval (m_L, m_{L+1}) for all $x < X_L$ (like in Fig. 3(c), notice in fact that, as shown in item (3) of the Proof of Theorem 2, $\lim_{x\to-\infty} T(x)$ is bounded);

b) $T(x)$ returns to a level m_j already met, as shown in Fig. 3(d).

In case (a) (see Fig. 3(c)) we do not have multiple intersections and this actually is the situation proved in Theorem 2. In this theorem, in fact, it is proved that the intersections are like in Fig. 3(c), where L denotes the index of the last m_i intersected, i.e. $T(x) \in (m_L, m_{L+1})$ for all $x < X_L$ and we let $m_{L+1} := -\infty$ in the case $L \equiv L'$.

In case (b) (see Fig. 3(d)), on the contrary, there are multiple intersections. If this occurs, then the first time (starting from Z_g and going toward $-\infty$) we would observe a second intersection of $T(x)$ with some level m_j, is as illustrated in Fig. 3(d), which in fact coincides with Fig. 1(b). This situation can never happen, as proved in Theorem 2.

7.8 Solution to the HJB equations between the switching levels

Assume the optimal control $u^*(x) = U_k$ over the interval $I = (X_a, X_b)$ and assume for simplicity that $0 \notin I$ (otherwise the interval I should be partitioned into two sub-intervals $(X_a, 0)$ and $(0, X_b)$). Then the solution $V(x,0)$ and $V(x,1)$ to the HJB equations (12)-(13) with boundary conditions $V(X_b,0)$ and $V(X_b,1)$ is given by:

$$\begin{aligned} V(x,0) &= \frac{1}{\alpha_k(U_k-d)}\left[\frac{J^*(q_k+q_{up})(x-X_b)}{d} - \frac{c_x(q_{up}+q_k)(x^2-X_b^2)}{2d} - \right. \\ &- \left. \frac{J^*U_k q_{up}(1-e^{-\alpha_k(x-X_b)})}{\alpha_k d^2} + \frac{c_x U_k q_{up}(\alpha_k x - 1 - (\alpha_k X_b - 1)e^{-\alpha_k(x-X_b)})}{\alpha_k^2 d^2}\right] + \\ &+ \frac{q_k}{\alpha_k(U_k-d)}\left[\left(\frac{q_{up}(U_k-d)e^{-\alpha_k(x-X_b)}}{dq_k} - 1\right)V(X_b,0) + \right. \\ &+ \left. \frac{q_{up}(U_k-d)(1-e^{-\alpha_k(x-X_b)})}{dq_k}V(X_b,1)\right] \end{aligned} \tag{62}$$

$$\begin{aligned} V(x,1) &= \frac{1}{\alpha_k(U_k-d)}\left[\frac{J^*(q_k+q_{up})(x-X_b)}{d} - \frac{c_x(q_{up}+q_k)(x^2-X_b^2)}{2d} - \right. \\ &- \left. \frac{J^*U_k q_k(1-e^{-\alpha_k(x-X_b)})}{\alpha_k d(U_k-d)} + \frac{c_x U_k q_k(\alpha_k x - 1 - (\alpha_k X_b - 1)e^{-\alpha_k(x-X_b)})}{\alpha_k^2 d(U_k-d)}\right] + \\ &+ \frac{q_k}{\alpha_k(U_k-d)}\left[\left(e^{-\alpha_k(x-X_b)} - 1\right)V(X_b,0)\right. \\ &+ \left. \left(\frac{q_{up}(U_k-d)}{dq_k} - e^{-\alpha_k(x-X_b)}\right)V(X_b,1)\right] \end{aligned} \tag{63}$$

where $c_x = c_p$ if $X_a > 0$ and $c_x = -c_m$ if $X_b < 0$.

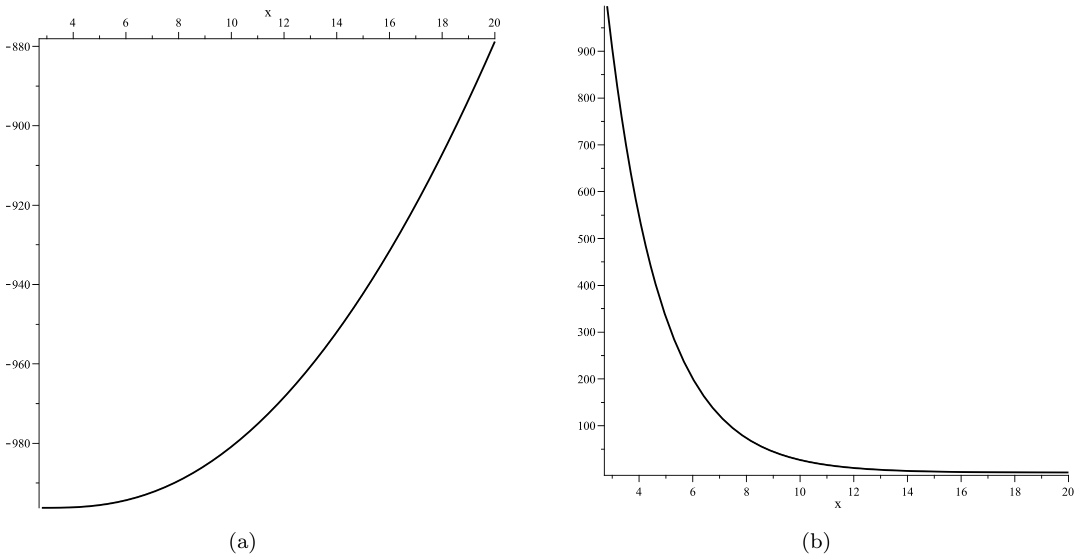


Figure 4: The plot of $V(x,1)$ (left) and of $H(x) = V(x,0) - V(x,1)$ for $x > Z^*$ (right)

7.9 Details of the Example in Section 5.1

Consider the example of Section 5.1 with the optimal levels X_i^*'s. In this section we want to show how the policy $u^*(x)$ in (7) with the levels X_i^* specified meets all the conditions of the Verification Theorem.

First of all, we determine the solutions to the HJB equations associated with the considered policy. This corresponds to solve (12) and (13) after removing the minimum from (12) and using in (12) as u the $u^*(x)$ of the considered policy. Assigning an arbitrary initial condition to $V(x,0)$ for an arbitrary x (notice in fact that if $V(x,i)$ solves the HJB equations, also $V(x,i)+c$, where c is a constant is a solution of these equations), e.g. $V(Z^*,0) = 0$, we have that, thanks to (12) (with $u^*(Z^*) = d$): $V(Z^*,1) = (c_p Z^* - J^*)/q_1$. Then we integrate the equations using as *initial* conditions in each interval the *final* conditions of the other (or vice versa). The solutions $V(x,i)$ are continuous on the borders of the intervals (obviously by construction). Now we show that all the conditions of the Verification Theorem are met, namely: that $V(x,0)$ and $V(x,1)$ are continuously differentiable, that they satisfy with $u^*(x)$ the HJB equations (12) and (13) (in particular, with these functions, $u^*(x)$ attains the minimum in (12)) and that they are quadratically bounded. We also show that $H(x) > 0$ for all x and, even if this is not required in our analysis, that $V(x,0)$ and $V(x,1)$ are convex.

Now, for $x > Z^* = 2.81$, the solutions to the HJB equations associated to the considered policy is given by:

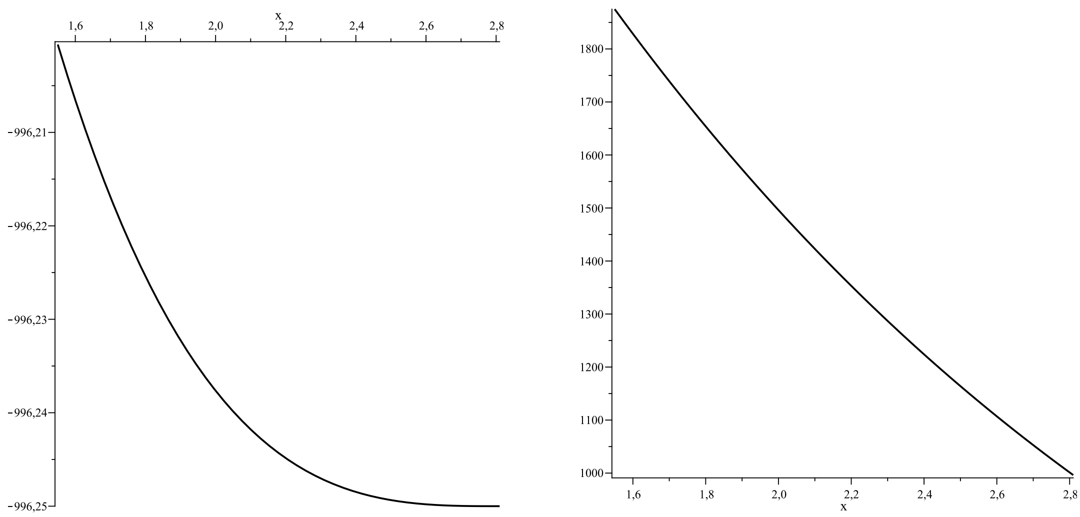
$$V(x,1) = -982.7 - 3.97e^{-0.5x+1.41} + 0.5x^2 - 4.8x \tag{64}$$
$$V(x,0) = -982.7 + 992.3e^{-0.5x+1.41} + 0.5x^2 - 4.8x \tag{65}$$

The right derivative of $V(x,1)$ at $x = Z^*$ is

$$\left.\frac{dV(x,1)}{dx}\right|_{x=Z^*,+} = 0 \tag{66}$$

and it is easy to verify that $V(x,0)$ and $V(x,1)$ are convex for $x > Z^* = 2.81$: this is trivial for $V(x,0)$. As for $V(x,1)$ we have, in this interval, $V''(x,1) = 1 - 4\,e^{-0.5\,x}$ which is positive for all $x \geq Z^*$. For $x \in (X_2^*, Z^*) = (1.55, 2.81)$:

$$V(x,1) = 1.21x - 999.66 - 0.126x^2 + 1.01e^{-0.5x+1.4} \tag{67}$$
$$V(x,0) = 3.7x - 1016.7 - 0.126x^2 + 1007.3e^{-0.5x+1.4} \tag{68}$$



(a) (b)

Figure 5: The plot of $V(x,1)$ (left) and of $H(x) = V(x,0) - V(x,1)$ for $x \in (X_2^*, Z^*)$ (right)

The left derivative of $V(x,1)$ at $x = Z^*$, given by

$$\left.\frac{dV(x,1)}{dx}\right|_{x=Z^{*,-}} = 0,$$

is like the right derivative at Z^* (see (66)), i.e. $V(x,1)$ is continuously differentiable at Z^*. In addition, also for $x \in (X_2^*, Z^*)$, the two functions are convex. We have, in fact, in this interval, $V''(x,1) = -0.252+1.024e^{-0.5\,x}$ which is positive for all $x \in (X_2^*, Z^*)$ and also implies the convexity of $V(x,0)$ (since the coefficient of the exponential term in $V(x,0)$ is much larger than the coefficient of the exponential term in $V(x,1)$).

For $x \in (0, X_2^*) = (0, 1.55)$:

$$V(x,1) = 0.26x - 997.13 - 0.03x^2 + 0.6e^{-0.5x+0.77} \tag{69}$$

$$V(x,0) = 2.36x - 1011.5 - 0.03x^2 + 1886.7e^{-0.5x+0.77} \tag{70}$$

It is:

$$\left.\frac{dV(x,1)}{dx}\right|_{x=X_2^{*,-}} = -0.1248683095 \tag{71}$$

$$\left.\frac{dV(x,1)}{dx}\right|_{x=X_2^{*,+}} = -0.1243746466, \tag{72}$$

i.e., $V(x,1)$ is differentiable at X_2^* (the slight difference between the two values depends on the numerical approximations, like the one regarding the optimal levels X_i^*'s and the optimal cost J^*). Also for $x \in (0, X_2^*)$, the two functions are convex. We have, in fact, in this interval, $V''(x,1) = -0.06 + 0.324e^{-0.5\,x}$ which is positive for all $x \in (0, X_2^*)$ and also implies the convexity of $V(x,0)$ (for the same reason mentioned above).

For $x \in (X_3^*, 0) = (-0.02, 0)$:

$$V(x,1) = -0.41x - 995.8 + 26.5x^2 - 0.04e^{-0.5x} \tag{73}$$

$$V(x,0) = -2106.34x + 3207.3 + 26.5x^2 - 124.6e^{-0.5x} \tag{74}$$

It is:

$$\left.\frac{dV(x,1)}{dx}\right|_{x=0^-} = -0.3912172607 \tag{75}$$

$$\left.\frac{dV(x,1)}{dx}\right|_{x=0^+} = -0.3912172603 \tag{76}$$

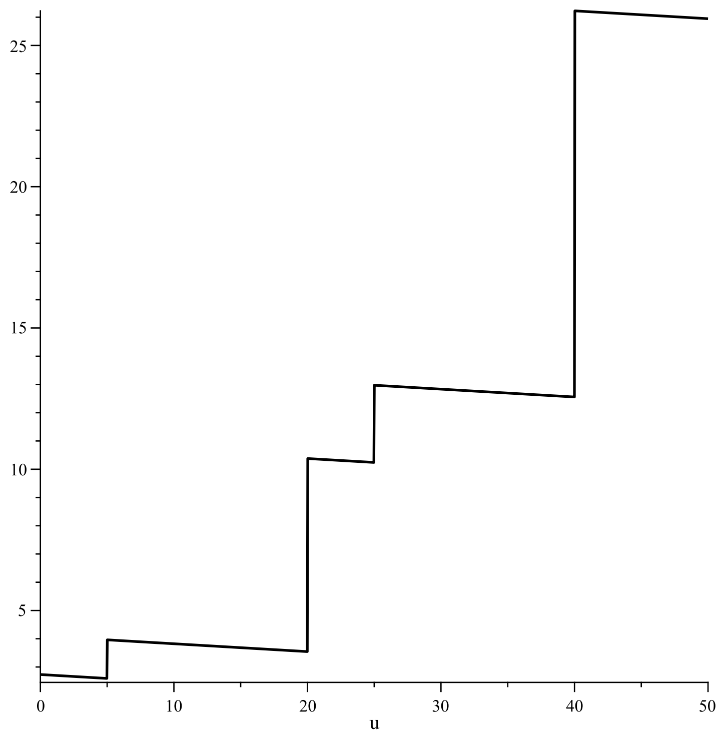
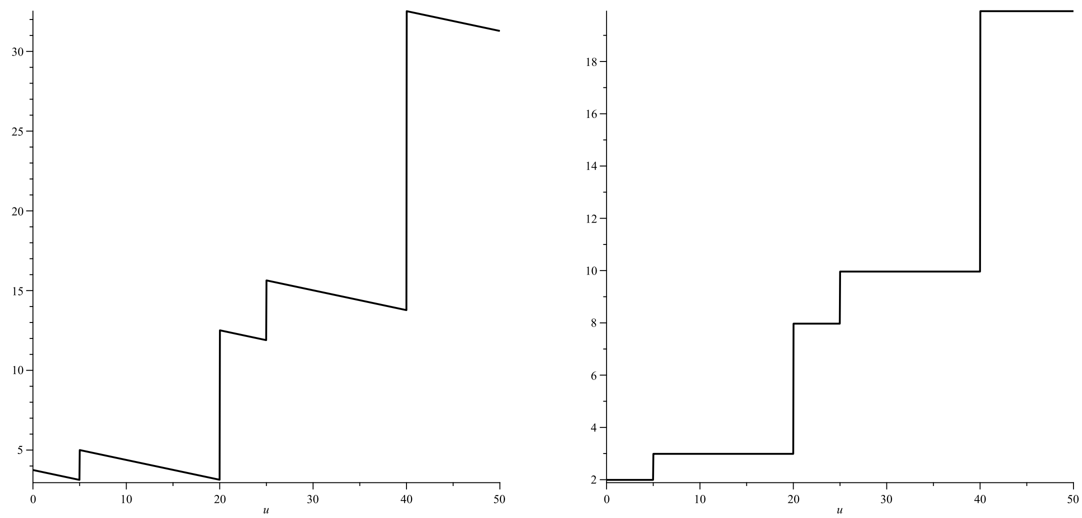
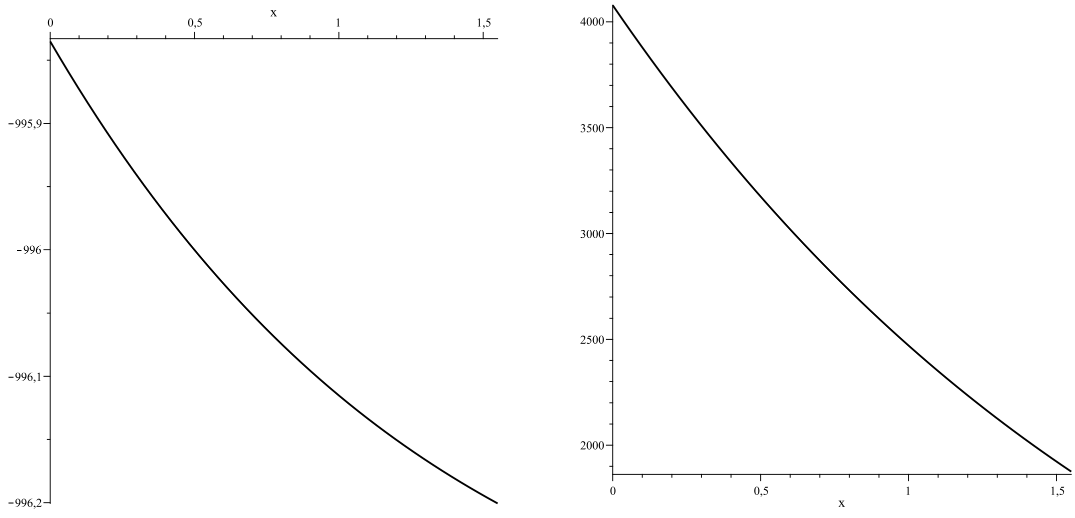


Figure 6: The plot of $dV(x,1)/dx \cdot u + q_d(u)H(x)$ for $x = (X_2^* + Z^*)/2$ showing that it is correct to take in the interval (X_2^*, Z^*) $u = U_1$ as the argument of the minimum in (12) (see also Figs. 7(a) and 7(b)).



(a) (b)

Figure 7: The plot of $dV(x,1)/dx \cdot u + q_d(u)H(x)$ for $x = X_2^*$ (left) and for $x = Z^*$ (right), showing that it is correct to take in the interval (X_2^*, Z^*) $u = U_1$ as argument of the minimum in (12) and that on the left of X_2^* it starts to become optimal $u = U_2$ (left) and on the right of Z^* the optimal control starts to be $u = 0$ (right)



(a) (b)

Figure 8: The plot of $V(x,1)$ (left) and of $H(x) = V(x,0) - V(x,1)$ for $x \in (0, X_2^*)$ (right)

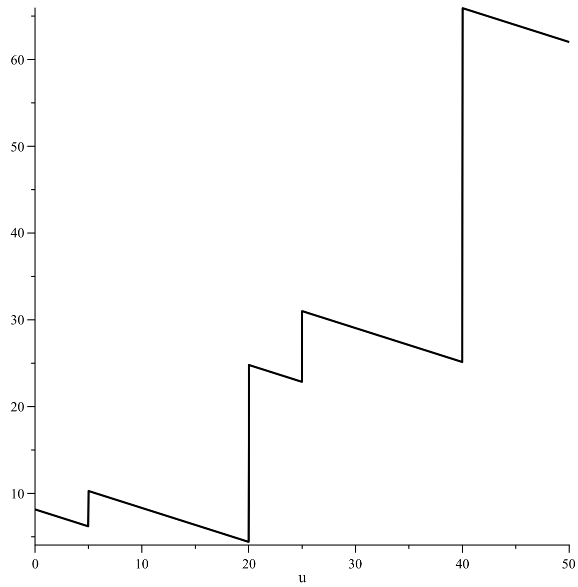


Figure 9: The plot of $dV(x,1)/dx \cdot u + q_d(u)H(x)$ for $x = 0$ showing that it is correct to take in the interval $(0, X_2^*)$ $u = U_2$ as minimum in (12)

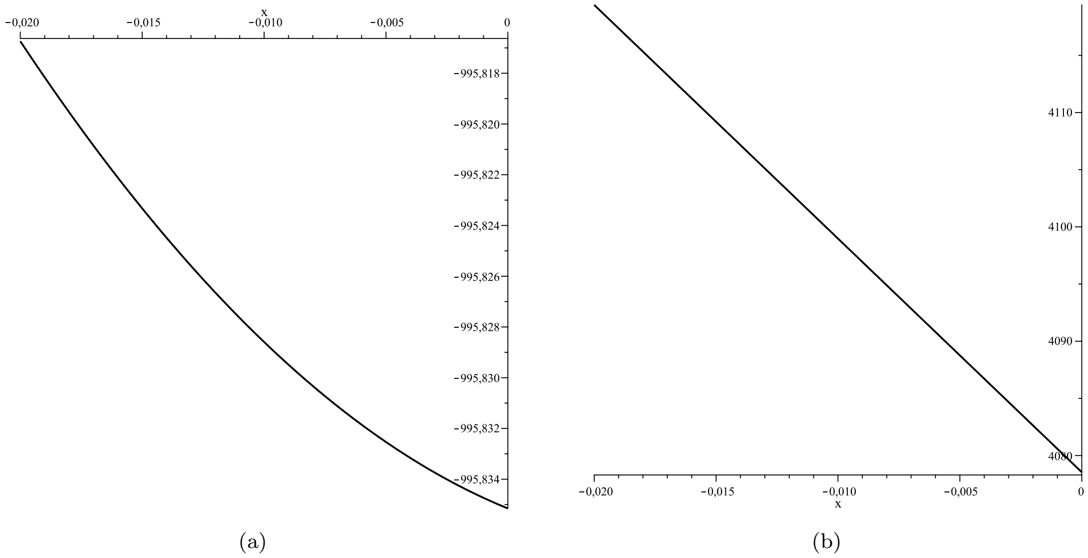


Figure 10: The plot of $V(x, 1)$ (left) and of $H(x) = V(x, 0) - V(x, 1)$ for $x \in (X_3^*, 0)$ (right)

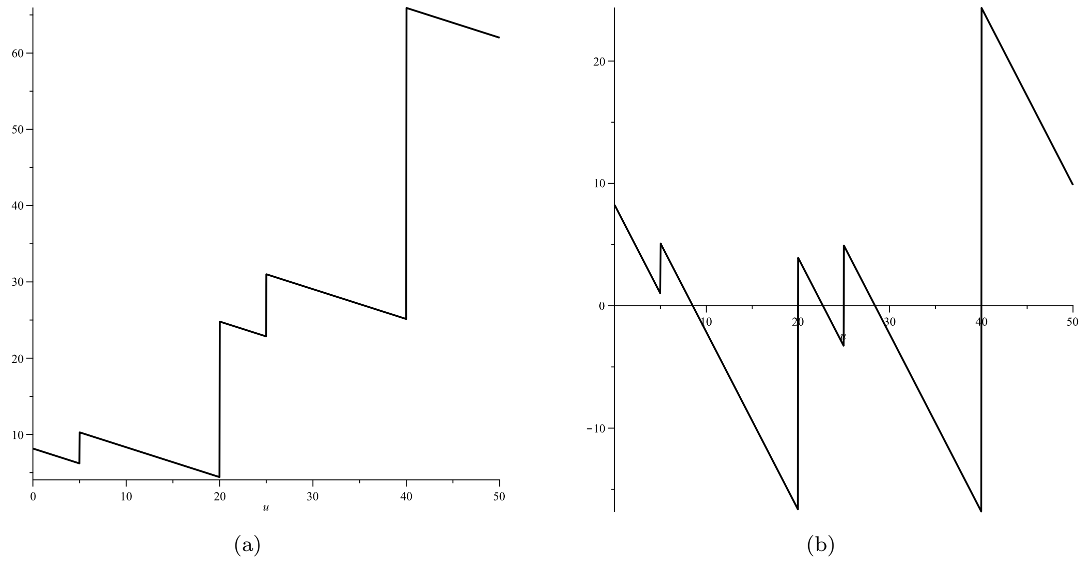
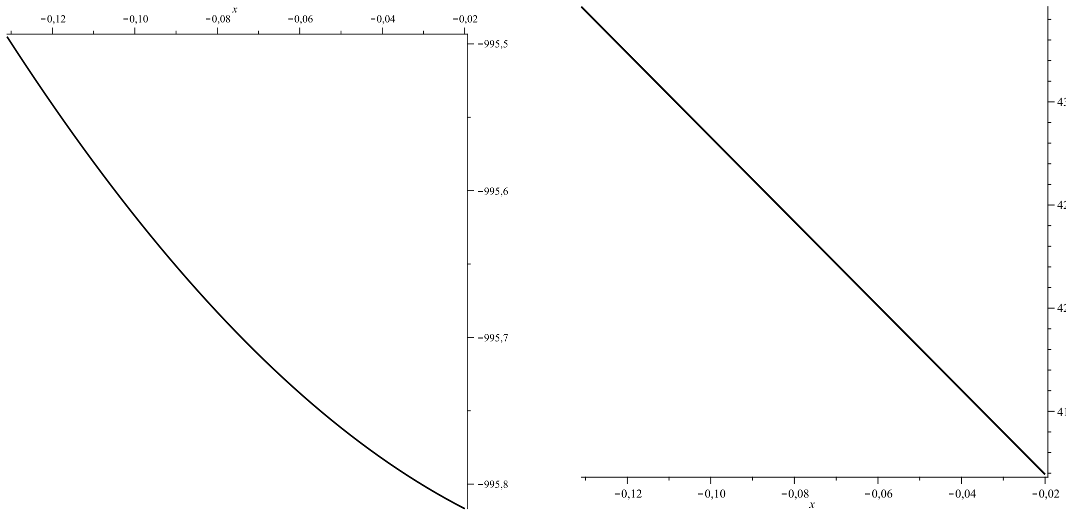


Figure 11: The plot of $dV(x,1)/dx \cdot u + q_d(u)H(x)$ for $x = 0$ (left) and for $x = X_3^*$ (right), showing that it is correct to take in the interval $(X_3^*, 0)$ $u = U_2$ as minimum in (12), but U_4 at X_3^* starts to be better (right). Notice also that U_3 can never be optimal (right).



(a) (b)

Figure 12: The plot of $V(x,1)$ (left) and of $H(x) = V(x,0) - V(x,1)$ for $x \in (X_4^*, X_3^*)$ (right)

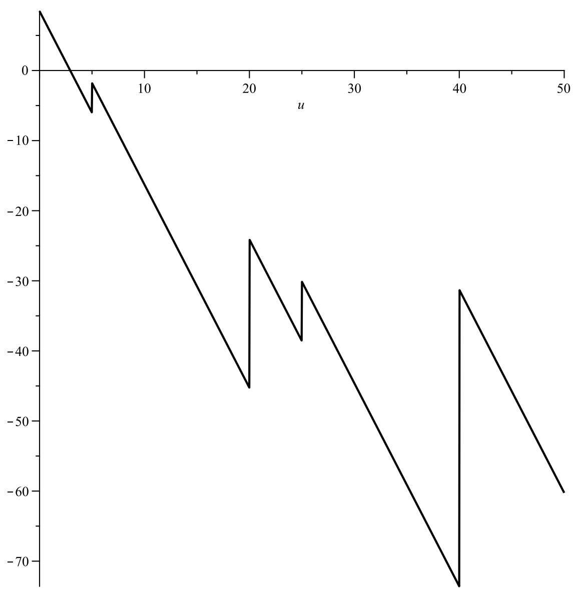
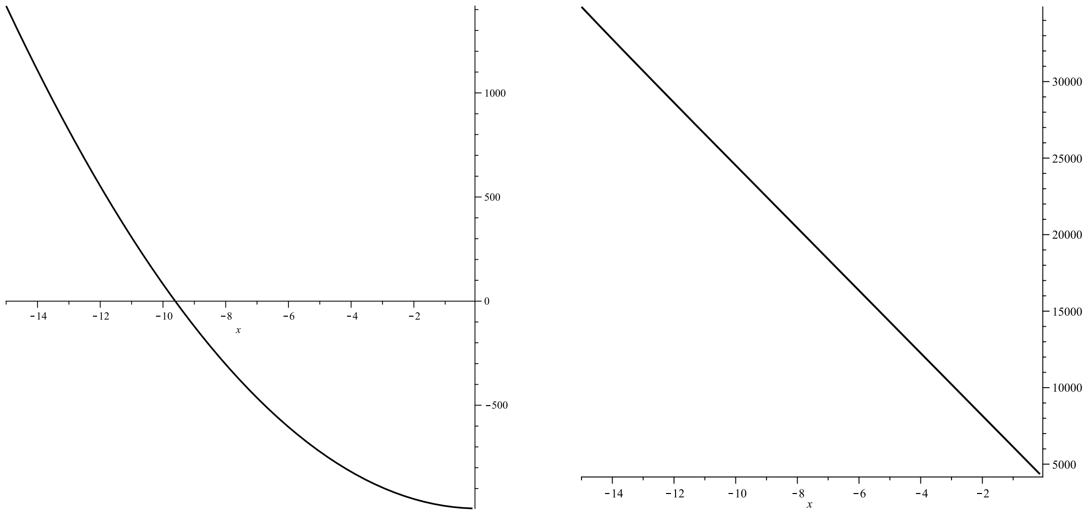
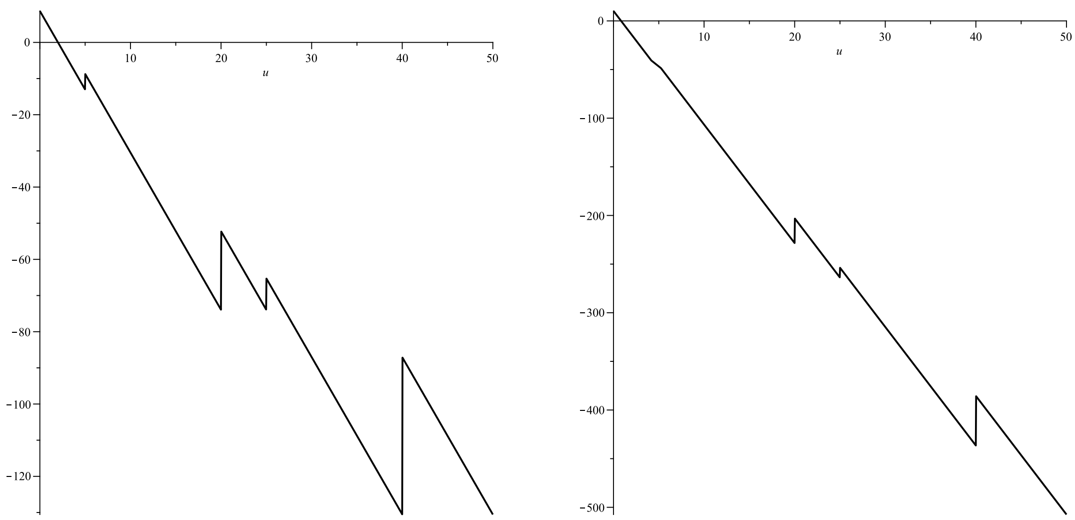


Figure 13: The plot of $dV(x,1)/dx \cdot u + q_d(u)H(x)$ for $x = (X_4^* + X_3^*)/2$ showing that it is correct to take in the interval (x_4^*, X_3^*) $u = U_4$ as minimum in (12)



(a) (b)

Figure 14: The plot of $V(x, 1)$ (left) and of $H(x) = V(x, 0) - V(x, 1)$ for $x < X_4^*$ (right)



(a) (b)

Figure 15: The plot of $dV(x, 1)/dx \cdot u + q_d(u)H(x)$ for $x = X_4^*$ (left) and for $x = -0.5$ (right) showing that it is correct to take $u = U_5$ as minimum in (12) for $x < X_4^*$. Notice again that at $X = X_4^*$ (left), U_4 is equivalent to U_5.

i.e., $V(x,1)$ is differentiable at 0. Also for $x \in (X_3^*, 0)$, the two functions are convex. We have, in fact, in this interval, $V''(x,1) = 53 - 0.01e^{-0.5\,x}$ and $V''(x,0) = 53 - 31.15e^{-0.5\,x}$ which are positive for all $x \in (X_3^*, 0)$.

For $x \in (X_4^*, X_3^*) = (-0.131, -0.02)$:

$$\begin{aligned} V(x,1) &= -0.93x - 995.8 + 13.1x^2 - 0.01e^{-0.5x-0.01} \quad (77) \\ V(x,0) &= -2053.26x + 3101.1 + 13.1x^2 - 18.6e^{-0.5x-0.01} \quad (78) \end{aligned}$$

It is:

$$\begin{aligned} \left.\frac{dV(x,1)}{dx}\right|_{x=X_3^{*,-}} &= -1.445938477 \quad (79) \\ \left.\frac{dV(x,1)}{dx}\right|_{x=X_3^{*,+}} &= -1.450301588 \quad (80) \end{aligned}$$

i.e., $V(x,1)$ is differentiable at X_3^*. Also for $x \in (X_4^*, X_3^*)$, the two functions are convex. We have, in fact, in this interval, $V''(x,1) = 26.2 - 0.0025e^{-0.5\,x}$ and $V''(x,0) = 26.2 - 4.6e^{-0.5\,x}$ which are positive for all $x \in (X_4^*, X_3^*)$.

For $x < X_4^* = -0.131$:

$$\begin{aligned} V(x,1) &= -1.57x - 995.9 + 10.62x^2 + 0.00008e^{-0.5x} \quad (81) \\ V(x,0) &= -2044.1x + 3082.6 + 10.62x^2 + 0.09e^{-0.5x} \quad (82) \end{aligned}$$

It is:

$$\begin{aligned} \left.\frac{dV(x,1)}{dx}\right|_{x=X_4^{*,-}} &= -4.349402371 \quad (83) \\ \left.\frac{dV(x,1)}{dx}\right|_{x=X_4^{*,+}} &= -4.350233541 \quad (84) \end{aligned}$$

i.e., $V(x,1)$ is differentiable at X_4^*. Also for $x < X_4^*$, the two functions are convex. We have, in fact, in this interval, $V''(x,1) = V''(x,0) = 21.24 > 0$ (the small coefficients of the exponential terms in (81) and (82) should be in fact intended as 0: they are different from 0 for the numerical approximations on the X_i^*'s and on J^*). Also, from the expressions of $V(x,0)$ and $V(x,1)$, it is possible to see how they can be bounded (from above and from below) by a quadratic function.

In Figs. 4-15 it is possible to visualize the convexity of $V(x,1)$ (Figs. 4(a), 5(a), 8(a), 10(a), 12(a) and 14(a)), that $H(x) > 0$ for all x (Figs. 4(b), 5(b), 8(b), 10(b), 12(b) and 14(b)) and that the considered policy achieves the minimum in (13) (Figs. 6, 7(a), 7(b), 9, 11(a), 11(b), 13, 15(a)) and 15(b)).

7.10 A way to initialize the value of the X_i's in the gradient descent method

In this section we present a procedure based on the $N = 2$ case of [5] to initialize the value of the X_i's in the gradient descent method mentioned in Section 5.1 which determines the optimal thresholds X_i^*'s. This procedure can be applied only if the U_i's are all feasible (this is because the results of [5] have been derived under this assumption).

In [5], for a $N = 2$ case, the optimal policy has been shown to have the same structure of (7), i.e., using the notation $U := U_1$ and $\mu := U_2$, is given by:

$$u(x) := \begin{cases} 0 & x > Z \\ d & x = Z \\ U & x \in [X, Z) \\ \mu & x < X \end{cases} \quad (85)$$

The optimal values X^* and Z^* respectively of X and Z have been analytically derived in [5].

The basic idea of the method is to determine the levels X_i's by using the thresholds X^* and Z^* which come from the application of the results presented in [5] to each pair $(U_{i_k}, U_{i_{k+1}})$, where $i_k, i_{k+1} \in \mathcal{S}$. This is performed as follows.

For each $i_k, i_{k+1} \in \mathcal{S}$, apply the results of [5] with $U = U_{i_k}$, $\mu = U_{i_{k+1}}$, $q_{d2} = q_{i_k}$ and $q_{d1} = q_{i_{k+1}}$ (the other parameters in [5] being as in this report). Let X^* and Z^* be the resulting optimal levels and denote them by $X^*_{i_k,i_{k+1}}$ and $Z^*_{i_k,i_{k+1}}$ respectively.

Then, the initial value of the thresholds X_i's in a gradient descent method can be selected as follows:

$$X_k(0) = X^*_{i_{k-1},i_k}, \; k = \ell+1, \dots L, \tag{86}$$

$$X_\ell(0) = Z^*_{i_\ell,i_{\ell+1}}. \tag{87}$$

This procedure, tested on several examples, provided really satisfactory results. For example, considering again the $N = 5$ case of Section 5.1, the initialization procedure just described provides a guess $\mathbf{X}(\mathbf{0}) := \{X_1(0), \dots, X_L(0)\} = \{2.8704, 1.606, -0.0207, -0.134\}$ not far from the optimal $\mathbf{X}^* := \{X_1^*, \dots, X_L^*\} = \{2.81, 1.55, -0.02, -0.131\}$ indicated in Section 5.1. This initialization greatly reduced the execution time of the gradient search.

The initialization procedure can not be applied to the other examples of Section 5 since the U_i's are not all feasible there.

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