Università degli Studi di Roma
"Tor Vergata"

Facoltà di Ingegneria

Dottorato di Ricerca in
Informatica ed Ingegneria dell'Automazione

XXII Ciclo del Corso di Dottorato

Analysis of Hybrid Systems and Design of Hybrid Controllers

Fulvio Forni

A.A. 2009/2010

Relatore: Prof. Alberto Pettorossi

Correlatore: Prof. Luca Zaccarian

Coordinatore: Prof. Daniel Pierre Bovet

Acknowledgments

I am grateful to my advisor, mentor and friend Alberto Pettorossi for having guided me through the challenges of my studies and for teaching me the importance of mathematical rigour in research.

I would like to express my gratitude to Sergio Galeani and Luca Zaccarian, for their friendship, advices and constant help throughout my doctorate program, and to Osvaldo Grasselli for his encouragement throughout the years.

I also want to thank the Control Group and the MAP Group of the Dipartimento di Informatica Sistemi e Produzione, Università di Roma Tor Vergata, and of the Istituto di Analisi dei Sistemi ed Informatica A. Ruberti, Consiglio Nazionale delle Ricerche, who directly and indirectly have given me a very interesting journey through the beauty of research.

I am thankful to Julian Bradfield for hosting me at the LFCS of the University of Edinburgh, Scotland, UK, and for having suggested me a lot of interesting readings on modal logics. I am also grateful to Andy Teel for introducing me to his rigorous and elegant mathematical approach to the theory of hybrid systems, during my visiting period at the CCDC of the University of California Santa Barbara, California, US.

Introduction

Hybrid systems define a common mathematical framework for combining continuous and discrete processes, like the case of processes defined by differential equations and by transition relations, respectively. Electrical circuits with both analog and digital components, models of impacts, computing devices running real-time applications, are all examples of processes defined by a combination of differential equations and transition relations. Thus, they can be modeled and studied as hybrid systems.

Hybrid systems have been studied in the last twenty years both by the computer science community and by the control community, and a lot of different definitions and results have been developed. Common to all of these definitions is the mathematical characterization of the *evolution* and of the *interaction* of continuous and discrete processes by way of the crucial notion of *state*. In this thesis, we consider a quite general definition of hybrid systems that, to the best of the knowledge of the candidate, subsumes classical definitions of a hybrid system in both computer science and control theory. Based on this general definition of a hybrid system, we study two classical problems: *stability problems* of control theory and *verification problems* of computer science, both generalized to hybrid systems. Indeed, in the first part of the thesis, we propose Lyapunov-like tools for the stability problem of a peculiar class of hybrid systems, and we propose a specific temporal logic, and a method for rewriting the formulas of this logic as fixpoint expressions, for the verification problem of hybrid systems. The synthesis problem on hybrid systems, namely the problem of synthesizing a hybrid system for achieving some predetermined goal, is a forward consequence of the studies on analysis of hybrid systems. In the second part of the thesis, we consider the framework of dynamical control systems, proposing non-hybrid controllers on continuous systems with bounds on the inputs, and hybrid controllers that, by virtue of their discrete dynamics, guarantee suitable properties of the closed loop. Is worth mentioning that the combination of a classical continuous process and of a hybrid controller results in a hybrid system that can be studied with the analysis tools developed in the first part of the thesis.

The first chapter of this thesis is an introduction to hybrid systems. It is based on the approach outlined in [61] for which several structural results has been developed [63, 126, 127] and partially summarized in [62]. In these works, a hybrid system is mainly defined by four components: a *flow map*, that defines the *continuous motion of the state* (it is used to characterize the dynamics of continuous processes); a *jump map* that defines the *discrete motion of the state.* (it is used to characterize the transition relation of discrete processes); a *flow set*, that defines the subset of the state space in which the continuous motion of the state *may occur*; a *jump set*, that defines the subset of the state space in which the transitions of a discrete process *may occur*. Hybrid systems of this kind are then compared with other approaches to hybrid systems in [68, 99] and several notions like the concept of solution, dependence on initial states, robustness are considered.

In the second chapter of this thesis, we study some stability problems [63] for a specific class of hybrid systems named *homogeneous* hybrid systems [152]. We address these problems by a Lyapunov-like approach and we propose sum-of-squares algorithms [113] to characterize automatically the stability of a hybrid system within the considered class. The contribution of Chapter 2 is in the definition of *local* Lyapunov-like conditions for inferring *global* stability properties (i.e. that apply to the whole state space) and in the definition of sum-of-squares algorithms to automatically decide whether or not a hybrid system satisfies those global stability properties. The results of the chapter have been partially developed during the visiting period at the CCDC of the University of California Santa Barbara, US.

In Chapter 3 we continue to study hybrid systems by defining a *temporal logic* [39] to express properties on hybrid systems. Temporal logics are frequently used in computer science on discrete processes and on simple hybrid systems [39] (e.g. automata equipped with *clocks*, to take into account the elapsing of time). The contribution of Chapter 3 is in the definition of a new semantics for TCTL, a branching temporal logic with constraints on time [73]. The proposed semantics allows for a generalization of this logic to express properties of hybrid systems when *Zeno phenomena* occur (namely when the motion of the state has infinitely many jump in a bounded interval of time). Based on this new semantics, we propose a method to reduce a formula to a fixpoint expression [73]. We show also that the proposed semantics coincides with the semantics of CTL [39] when a hybrid system is used to model a process that is only discrete. The results of the chapter are partially based on the studies on modal logics developed during the visiting period at the LFCS of the University of Edinburgh, Scotland, UK.

Synthesis problems are studied in Chapters 4 and 5. In Chapter 4 we consider the case of continuous systems with bounds on input. For specific planar cases we propose an approach that blends together linear and optimal controllers [11], that guarantees time- or fuel-optimal performance when the state of the system is far from the equilibrium, while it guarantees exponential stability when the state is close to the equilibrium. For general closed loops with linear plants, we propose controllers based on the anti-windup approach [58]. In particular, we consider the case of magnitude and rate saturation at the input of the plant and we design antiwindup controllers by solving BMI and LMI [24]. It is worth to note that the approaches in Chapter 4 produce non-hybrid controllers, based on the fact that no discrete transitions occur. The control authority is continuously moved between two control devices, a *global* one and a *local* one, based on a suitable relation that depends on the state of the system. These approaches can be related to hybrid control techniques by relaxing the requirement on continuity of the management of the control authority, introducing switching policies and resets of the controllers state.

Hybrid techniques in control problems are considered in Chapter 5. In the first part of the chapter we propose a technique for inducing passivity [130] on a class on nonlinear systems. In particular, we stabilize a passive system through the interconnection with a passive controller whose passivity is induced by resets. In the second part of the chapter we propose a technique for breaking the signals continuity in a feedback loop, namely the continuity of signals that bring the plant output measurements to the controller input. We propose a sampling mechanism based on intelligent sensors, that transmit the measurements samples based on a nonperiodic scheduling. Since the ultimate goal of the above policies is to reduce the data rate, we call lazy these intelligent sensors, to resemble the fact that they are reluctant to transmit, and that they do so only when it is required for preserving the stability of the closed-loop system.

The research activity carried out during the PhD studies of the candidate produced the following publications in international journals and international conferences:

– *A family of global stabilizers for quasi-optimal control of planar linear saturated systems.*
  F. Forni, S. Galeani, and L. Zaccarian,
  IEEE Transaction on Automatic Control, 2010, to appear.

- *Gain-scheduled, model based anti-windup for LPV systems.*
  F. Forni and S. Galeani.
  Technical communique, Automatica, 46(1):222-225, 2010.

- *Passification of nonlinear controllers via suitable time-regular reset map.*
  F. Forni, D. Nešić, and L. Zaccarian.
  Symposium on Nonlinear Control Systems (NOLCOS), August 2010.

- *An almost anti-windup scheme for plants with magnitude, rate and curvature saturation.*
  F. Forni, S. Galeani, and L. Zaccarian.
  American Control Conference, June 2010.

- *Model recovery anti-windup for plants with rate and magnitude saturation.*
  F. Forni, S. Galeani, and L. Zaccarian.
  European Control Conference, August 2009.

A number of additional publications related to the most recent research work are currently submitted and under review.

Contents

Notation

- $\mathbb{R}$ denotes the set of *real numbers*. $\mathbb{Z}$ denotes the set of *integer numbers*. For any given $a \in \mathbb{R}$, $\mathbb{R}_{\geq a}$ denotes the set of real numbers greater than or equal to a and $\mathbb{R}_{>a}$ denotes the set of real numbers strictly greater than a. Analogously for $\mathbb{Z}_{\geq a}$ and $\mathbb{Z}_{>a}$. We will use $\mathbb{Z}_{\geq 0}$ to denote $\mathbb{Z}_{\geq 0}$.
- *Euclidean norm* of a vector and the corresponding induced matrix norm are denoted by $|\cdot|$. For a vector $x \in \mathbb{R}^p$, $|x|_\infty = \max\{|x_i|, i = 1, \ldots, p\}$, where x_i is the ith component of x. The $\mathcal{L}_2$ *norm of a signal* is denoted by $\|\cdot\|_2$. The $\mathcal{L}_\infty$ *norm of a signal* is denoted by $\|\cdot\|_\infty$.
- Given a set $S \subseteq \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, for some $n \in \mathbb{Z}_{\geq 0}$, $|x|_S = \inf_{y \in S} |x - y|$.
- Given a square matrix X, $\mathrm{He}(X) = X + X^T$.
- Given a vector $x \in \mathbb{R}^n$, for some $n \in \mathbb{Z}_{\geq 0}$, $\mathrm{diag}(x)$ denotes *diagonal matrix having the entries of x on the main diagonal.* x^T denotes the transpose vector of x.
- Given two vectors $x, y \in \mathbb{R}^n$, for some $n \in \mathbb{Z}_{\geq 0}$, $\langle x, y \rangle = x^T y$.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$; it is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\lim_{r \to +\infty} \alpha(r) = +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if for each $t, s \in \mathbb{R}_{\geq 0}$ (i) $\beta(\cdot, t)$ is non decreasing and $\lim_{t \to 0} \beta(s, t) = 0$, and (ii) $\beta(s, \cdot)$ is non increasing and $\lim_{s \to \infty} \beta(s, t) = 0$. A function is $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KLL}$ if for each $r \geq 0$, $\gamma(\cdot, \cdot, r)$ and $\gamma(\cdot, r, \cdot)$ are $\mathcal{KL}$ functions.
- $\mathbb{B}(x, a)$ denotes the *ball of radius $a \in \mathbb{R}_{>0}$ centered in $x \in \mathbb{R}^n$*, that is, $\mathbb{B}(x, a) = \{y : |x - y| \leq a\}$. $\mathbb{B}(a)$ or $a\mathbb{B}$ denote $\mathbb{B}(0, a)$. $\mathbb{B}$ denotes $\mathbb{B}(0, 1)$.

- Given a set S, $\overline{S}$ denotes the *closure of* S (namely the union of S with its boundary) and $\overline{\text{co}}(S)$ denotes the *closed convex hull* of S.

- Given two sets S_1 and S_2, $S_1 \times S_2 = \{(s_1, s_2) \,|\, s_1 \in S_1 \text{ and } s_2 \in S_2\}$.

- Given two sets S_1 and S_2, subsets of $\mathbb{R}^n$, $S_1 + S_2 = \{s \,|\, s = s_1 + s_2 \text{ for some } s_1 \in S_1 \text{ and } s_2 \in S_2\}$.

- Given two sets S_1 and S_2, $S_1 \setminus S_2 = \{s \,|\, s \in S_1 \text{ and } s \notin S_2\}$.

- For a given set S, 2^S denotes the set of all subset of S.

- For any given sets S_1 and S_2, a *set-valued mapping* M from S_1 to S_2, denoted $M : S_1 \rightrightarrows S_2$, maps each element of S_1 to a subset of S_2.
  • The *domain* of M is the set: $\operatorname{dom} M = \{x \in S_1 \,|\, M(x) \neq \emptyset\}$.
  • The *range* of M is the set: $\operatorname{rng} M = \{y \in S_2 \,|\, \exists x \in S_1, y \in M(x)\}$.
  • The *graph* of M is the set: $\operatorname{gph} M = \{(x, y) \in S_1 \times S_2 \,|\, y \in M(x)\}$.
  We denote that set-valued mapping M also as $M : S_1 \to 2^{S_2}$.

- For any given mapping $M : S \to S$, and for any given element s of S, we write $M^i(s)$ to denote i applications of M to s, that is, $M^0(s) = s$ and $M^{i+1}(s) = M(M^i(s))$.

- For any given set S, an ω-chain in 2^S is a set $\{S_i \,|\, S_i \subseteq S \text{ and } i \in \mathbb{Z}_{\geq 0}\}$ such that either $\forall i \in \mathbb{Z}_{\geq 0}$, $S_i \subseteq S_{i+1}$, or $\forall i \in \mathbb{Z}_{\geq 0}$, $S_i \supseteq S_{i+1}$.

- For any given expression $E[s]$ constraining the variable s ranging over a given set S, we denote by $\lambda s.E[s]$ the function that maps each value s of S to $E[s]$. For a function $\lambda s.E[s]$ and for a variable t, $\lambda s.E[s]t = E[t]$.

- Given a linearly ordered set I under the relation $\leq$, consider a sequence of sets $\{S_i\}_{i \in I}$. $\cup_{i \leq j} S_i$'s denotes the *union* of sets S_i's with $i \leq j$. $\cup_i S_i$ denotes the union of *all sets* S_i's. Analogously for the intersections $\cap_{i \leq j} S_i$ $\cap_i S_i$.

- For any given set S and any given mapping $M : 2^S \to 2^S$, we denote the *least fixpoint* of M as $\mu Z.M(Z) = \bigcap\{Z \,|\, M(Z) \subseteq Z\}$. Similarly, we denote the *greatest fixpoint* of M as $\nu Z.M(Z) = \bigcup\{Z \,|\, M(Z) \supseteq Z\}$.

- In the fixpoint expression $E = \mu X.\varphi$, φ is said to be the *scope* of $\mu X.$. We say that the fixpoint $\mu X.$ in the fixpoint expression $E = \mu X.\varphi$ *binds* the variable X. Analogously for $E = \nu X.\varphi$.

Chapter 1

The Hybrid Systems Framework

A *hybrid system* is a dynamical process whose dynamics can be reduced neither to a continuous motion nor to a discrete sequence of transitions An example of these systems is given by the interaction of a computer program and a robot, when a computer program is used to control the movement of the robot. In that case, the dynamics of the robot is continuous, that is, it can be described by equations of classical mechanics, while the behavior of the computer program is a discrete process based on suitable transitions between memory configurations, and those transitions are determined by the instructions of the program. Several new phenomena arise from the interaction of discrete and continuous processes and they must be considered during the analysis of the complete system, as well as during the synthesis of each part. Usually these phenomena do not occur or are unimportant for the characterization of the behavior of a purely continuous process or of a discrete process. For example, on the computer-robot case, the time between a *request* to the computer program and a possible *answer* must respect some time-constraints, due to the fact that during the computation time the robot is still in motion. This *real-time* requirement on the program, namely the ability to answer in a predetermined amount of time, is usually not a required feature on computer programs. Similar new phenomena occur also on the side of the robot. The motion of the robot satisfies a given differential equation, until some *event* occurs. For example, an electronic bumper may become aware of the presence of an obstacle near the robot, producing an interrupt to the computer program. In that case, the continuous motion of the robot is instantaneously modified by the computer program decision, that could force a fast safety-reaction either by a discontinuous variation of the inputs to the robot, or by enforcing a completely different equation of motion (i.e. a safety-brake).

Complex dynamics not reducible to classical continuous or discrete processes can be found also on much simpler examples. Consider the case of the fan of an electronic temperature control system. Suppose that the fan has two operative conditions: *on* - with a given *steady speed*, and *off* - with the *zero speed*. The dynamics of the fan depends in a non-continuous way on the particular switch event. It is also not discrete, by the fact that the variation of the speed of the fun is not instantaneous. Then, a model that takes into account startup and shutdown time intervals of the fan, namely the time interval that the fan needs to reach its steady-speed, when a switch from off to on occurs, and the time to reach the zero speed, when a switch from on to off occurs, would be necessarily a hybrid system.

Processes with variable dynamics, processes that involve logical variables or some logical reasoning, electronic circuits with digital and analog components, mechanical systems defined on dramatically different time-scales, like the case of mechanical systems with impacts, are all processes that cannot be characterized by relying either on a continuous model, like differential equations, or on a discrete model, like transitions relations. Hybrid systems provide a mathematical framework to characterize those kinds of processes. A mathematical definition of a hybrid system will be presented in the next section. Now we present some simple examples of hybrid systems that combine continuous dynamics and discrete transitions.

Example 1.1 Consider the *relay hysteresis* defined as $\dot{x} = H(x) + u$, where $x \in \mathbb{R}$, $\Delta \in \mathbb{R}_{>0}$, and $H(x) = 1$ for each $x < -\Delta$, while $H(x) = -1$ for each $x > \Delta$. When $-\Delta \leq x \leq \Delta$, the value of $H(x)$ depends on the history of x, as represented in Figure 1.1.

A model for the relay hysteresis can be constructed by adding a new variable q for taking into account the history of x. Indeed, the motion of x satisfies the following differential equations.

- $\dot{x} = 1 + u$ and $x \leq \Delta$ and $q = 1$,
- $\dot{x} = -1 + u$ and $x \geq -\Delta$ and $q = -1$.

Then, the transition relation on q can be defined as follows:

- q jumps from $q = 1$ to $q = -1$ and $x \geq \Delta$,

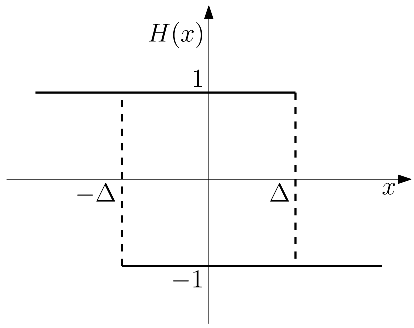


Figure 1.1: Relay type hysteresis

– q jumps from $q = -1$ to $q = 1$ and $x \leq -\Delta$.

We assume that the continuous motion of x satisfies both the differential equations and the conditions on the right-hand side of the equations, at the same time. Analogously for the motion of q. For example, suppose that $x \leq \Delta$, $u = 0$ and $q = 1$. Then, the value of x grows until $x = \Delta$. From $x = \Delta$ a discrete transition occurs, that is, q jumps from $q = 1$ to $q = -1$, and the value of x begins to decrease. An interesting case is $x < -\Delta$ and $q = -1$. None of the differential equations has a right-hand side compatible with this *initial states*. In this case, no continuous motion of x is possible. Nevertheless, a transition from $q = -1$ to $q = 1$ is indeed compatible, therefore a jump takes place.

Example 1.2 The motion of a ball in a circular pool table of radius r can be described by the following differential equation, where the position of the ball is defined by the variables x_1 and y_1, and the velocity of the ball is defined by the variables x_2 and y_2.

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & 0 \\ \dot{y}_1 & = & y_2 \\ \dot{y}_2 & = & 0 \end{array} \right. \qquad x_1^2 + y_1^2 \leq r^2 \tag{1.1}$$

The motion of the ball satisfies the differential equations above, provided that the ball remains inside the pool table, as defined by the inequality $x_1^2 + y_1^2 \leq r^2$ on the right-hand side of Equation (1.1). When the ball *hits* the border of the pool table, defined by the equation $x_1^2 + y_1^2 = r^2$, its motion must be instantaneously modified to take into account the occurrence of an *impact*.

An impact is modeled by the following transition relation, where the superscript + denotes the values of the position and of the velocity of the ball after the impact. α_x and α_y are functions in $\mathbb{R}^4 \to \mathbb{R}$, that define the new value of the velocity after the impact.

$$\left\{ \begin{array}{lcl} \dot{x}_1^+ & = & x_1 \\ \dot{x}_2^+ & = & \alpha_x(x_1, x_2, y_1, y_2) \\ \dot{y}_1^+ & = & y_1 \\ \dot{y}_2^+ & = & \alpha_y(x_1, x_2, y_1, y_2) \end{array} \right. \qquad x_1^2 + y_1^2 = r^2 \ . \tag{1.2}$$

Analogously to the continuous dynamics, the motion of the ball satisfies the transition relation above, provided that the conditions on the right-hand side of transition relation are verified. The impact of the ball to the border of the pool does not change the ball position x_1, y_1, but it instantaneously modifies the ball velocity x_2, y_2, as defined by α_x and α_y. See Figure 1.2.

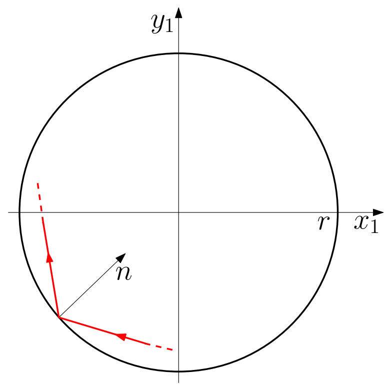


Figure 1.2: A circular pool table

Example 1.3 From [46, Page 37], a simplified model of a manual transmission can be defined as follows.

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \dfrac{-ax_2 + u}{1 + v} \end{array} \right. \tag{1.3}$$

where $v \in \{1,2,3,4,5\}$ is the gear shift position and u is the acceleration. a is a parameter of the system. The differential equation above is parameterized with respect to v, that is, there are five different differential equations each of which related to a specific gear shift position. Note that, for this case, a transition relation for v is not defined and we assume that v is driven by some external *events*.

Example 1.4 In the classical racetrack between Achilles and the tortoise, Achilles runs 10 times faster than the tortoise, while the tortoise begins its run with some distance a_0 ahead of Achilles. The race begins and, when Achilles reaches the starting point of the tortoise, a_0, the tortoise has moved to another point, say $a_1 > a_0$. Then, Achilles runs to that new point a_1 but, at that time, the tortoise has already moved to another point, say $a_2 > a_1$. This situation would repeat infinitely many times.

A model for the Achilles-tortoise racetrack can be defined by the following equations. Denote x_T, x_A and x_P respectively the current position of the tortoise, the current position of Achilles and the previous position of the tortoise. Then, using the notation adopted in Example (1.2),

$$\begin{array}{ll} \left\{ \begin{array}{rcl} \dot{x}_T &=& 1 \\ \dot{x}_A &=& 10 \\ \dot{x}_P &=& 0 \end{array} \right. & \qquad x_A \leq x_P \\ \left\{ \begin{array}{rcl} x_T^+ &=& x_T \\ x_A^+ &=& x_A \\ x_P^+ &=& x_T \end{array} \right. & \qquad x_A = x_P \end{array} \tag{1.4}$$

Both Achilles and the tortoise move at constant speed, satisfying the differential equations above, while x_P remains constant, preserving the value of position of the tortoise, stored in x_P. This continuous motion must satisfy the condition on the right-hand side of the differential equation, that is, the position of Achilles must be lower then the value stored in x_P for all time. Note that the lack of constraints on the position of the tortoise, in the right-hand side of the differential equations, means that the motion of the tortoise must satisfy the differential equation only.

The transition relation defines the discrete motion of x_P. When the condition on the right-hand side of the transition relation above is satisfied, a jump occurs and the value of x_P is updated to the current position of the tortoise, satisfying the transition relation.

By looking at Figure 1.3, from any initial state for which the tortoise has some advantage on Achilles, that is, $x_T > x_A$ and $x_P = x_T$, the hybrid system produces a motion of the state in which there are infinitely many transitions in a bounded interval of time. This behavior is usually called *Zeno phenomenon*.

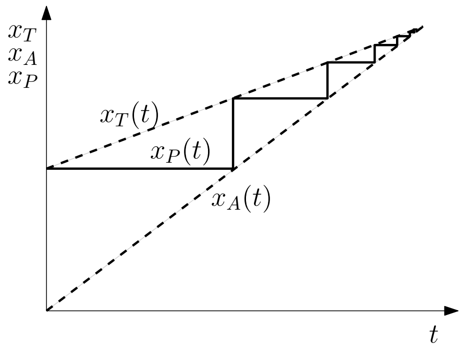


Figure 1.3: Achilles-tortoise racetrack.

1.1 Hybrid Systems: Models and Solutions

As we have indicated through several examples in the previous section, hybrid phenomena arise from the interaction of two different kinds of processes, (i) the processes that live in a continuous-time world, like a dynamical system defined by a differential equation, and (ii) the processes that live in a discrete-time world, like the case of a computing system whose computation depends on a specific transition relation. Hybrid systems can be considered as a framework for modeling that continuous and discrete processes together with their interaction, so that hybrid phenomena can be described and studied in a common mathematical framework.

The description of continuous and discrete processes as well as the characterization of their interaction can be developed in several ways [42, 46, 62, 68, 99, 100, 110]. In what follows we consider a notion of a hybrid system in which the description and the interaction of continuous and discrete processes are modeled towards the central notion of *state*, inherited from automata and from continuous dynamical systems as well. In general, the current configuration of a hybrid system is stored in the state and the dynamic behavior of the system (namely the motion of the state) is fully determined by suitable functions and relations that depend on state. A *solution* to a hybrid system, that is, the motion of the state of the hybrid system, is both continuous (flow intervals) and discontinuous (jumps): flow intervals and jumps satisfy specific differential inclusions and transitions relations that depend on the state, and their interleaving is defined by specific relations on the state.

By following [61, 62, 63], we consider a model of a hybrid system defined as follows.

Definition 1.1 A *hybrid system* is a 5-tuple $\mathcal{H} = (O, C, D, F, G)$ where, for some $n \in \mathbb{Z}_{\geq 0}$, O is an open subset of $\mathbb{R}^n$, C and D are subset of $\mathbb{R}^n$ denoted as *flow set* and *jump set* and $F : O \rightrightarrows \mathbb{R}^n$ and $G : O \rightrightarrows O$ are set-valued mappings denoted as *flow map* and *jump map*.

Hybrid systems can be represented as follows

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \cap O \\ x^+ \in G(x) & x \in D \cap O \end{cases} \tag{1.5}$$

The upper part of (1.5) defines the continuous dynamics of the hybrid system. The motion of the state must satisfy both the differential inclusion on the left-hand side, and the relation on the right-hand side, that is, the state moves in accordance with the *flow map* and it must remains within the *flow set*, for all times. Analogously, the lower part of (1.5) defines the discrete dynamics of the hybrid system. Indeed, the state moves in accordance with the *jump map*, in the left-hand side, provided that it satisfies the relation on the right-hand side, that is, a jump occurs when the state belongs to the *jump set*. Note that we will use $\dot{x} = F(x)$ instead of $\dot{x} \in F(x)$ whenever F is a single-valued mapping. Analogously for G.

A convenient parameterization for the sequence of jumps and flow intervals that characterize the motion of the state relies on a generalized notion of time, called *hybrid time*. The hybrid time is defined by two parameters $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. $t \in \mathbb{R}_{\geq 0}$ can be interpreted as the usual *time* variable of continuous process and $j \in \mathbb{Z}_{\geq 0}$ can be interpreted as the usual *counter* variable of discrete processes. As usual, we denote the value of the state x of a hybrid system $\mathcal{H}$ (1.5) at time (t, j) as $x(t, j)$.

We define the hybrid time in Definition 1.2 below, following [63]. Similar notions can be found in [42, Definition 2.1] and in [99, Definition II.2]. It is worth mentioning that parameterizing the motion of the state by the hybrid time allows for the use of several powerful tools of set-valued analysis on hybrid systems. For example, graphical convergence of sets can be conveniently applied to discontinuous solutions of hybrid systems, to characterize convergence of sequences of solutions to some given limit solution.

Definition 1.2 A subset S of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if

$$S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\}) \tag{1.6}$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. S is a *hybrid time domain* if for all $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

$$S \cap ([0, T] \times \{0, 1, \ldots, J\}) \tag{1.7}$$

is a compact hybrid time domain.

Equivalently, a subset S of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a hybrid time domain if it is the union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$. If the sequence is finite, say of $J \in \mathbb{Z}_{\geq 0}$ intervals, then the J-th interval is possibly of the form $[t_{J-1}, t_J) \times \{J - 1\}$ with $t_J = +\infty$. An example of hybrid time is in Figure 1.4. Note that the usual ordering $(t, j) \leq (t', j')$ if $t \leq t'$ and $j \leq j'$ induces a total ordering on each hybrid time domain, given by $(t, j) \leq (t', j')$ if and only if $t + j \leq t' + j'$.

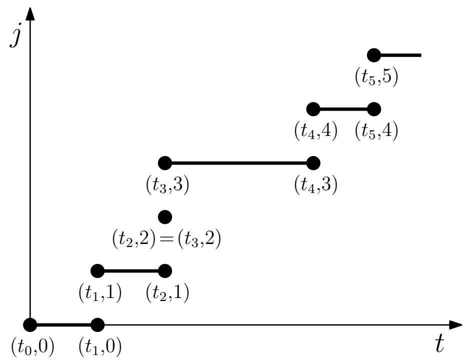


Figure 1.4: An example of hybrid time. Filled circles denote the border of each interval and characterize the occurrence of a jump.

Several different characterizations of the motion of the state of a hybrid system can be found in literature and several different words like execution, solution, trajectory are used to denote the motion of the state of a hybrid system. In what follows we use the word *solution* and we consider the definition of *solution to a hybrid system* given in [63]. A comparison between different concepts of solution to a hybrid system can be found in [127].

Definition 1.3 A *hybrid arc* x is a map $x : \operatorname{dom} x \to \mathbb{R}^n$ such that

1. $\operatorname{dom} x$ is a hybrid time domain, and
2. for each j, the function $t \mapsto x(t, j)$ is a locally absolutely continuous function on the interval $I_j = \{t : (t, j) \in \operatorname{dom} x\}$.

Definition 1.4 A hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in O$ and

1. for each $j \in \mathbb{Z}_{\geq 0}$ such that I_j has a nonempty interior,

$$\begin{array}{ll} \dot{x}(t, j) \in F(x(t, j)) & \text{for almost all } t \in I_j \\ x(t, j) \in C & \text{for all } t \in [\min I_j, \sup I_j) \end{array} \tag{1.8}$$

2. for each $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$\begin{array}{rcl} x(t, j + 1) & \in & G(x(t, j)) \\ x(t, j) & \in & D \end{array} \tag{1.9}$$

where we used "almost all" for denoting the one-dimensional Lebesgue measure on $\operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \{j\})$.

The relationship between hybrid time domains and solutions to a hybrid system is more complicated than the usual relationship between solutions to differential equations and time. Usually, for a hybrid arc x, $\operatorname{dom} x$ is not mentioned explicitly but it is always assumed to be defined exactly as the set of points (t, j) for which a given hybrid arc is defined. In fact, a possible alternative interpretation of a hybrid arc, say x, is to consider it as a set-valued mapping $x : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \rightrightarrows \mathbb{R}^n$, that maps each $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ either to a single value of $\mathbb{R}^n$ or to the empty set. In that case, what we called $\operatorname{dom} x$ would be the subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ of points (t, j) for which $x(t, j)$ is nonempty. Intuitively, a hybrid arc carries its own hybrid domain, namely, the subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ on which it is defined. An example of a solution graph is in Figure 1.5.

In general, a solution stays in O, it flows satisfying $\dot{x} \in F(x)$ when it is in C, and it jumps satisfying $x^+ \in G(x)$ when it is in D. Note that C and D may overlap. In that case, a solution x to $\mathcal{H}$ either flows or jumps, nondeterministically. Nondeterminism of solutions to hybrid systems also occur when flow and

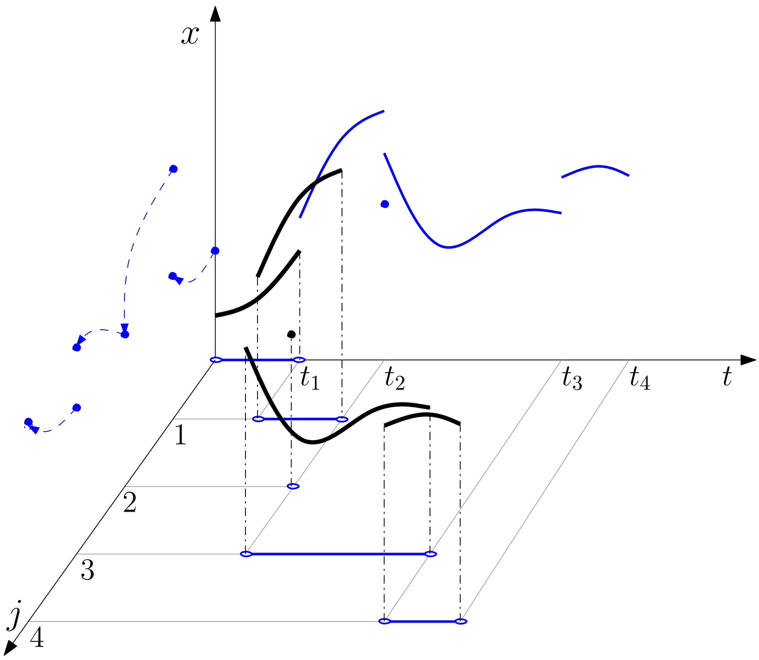


Figure 1.5: A possible solution x to a hybrid system $\mathcal{H}$

jump sets do not overlap. In fact, from a given initial state, a differential equation with continuous right-hand sides have *at least* one solution and, to have unicity of solution, further conditions must be considered, like Lipschitzianity. An example is given by a hybrid system of equation

$$\mathcal{H} : \begin{cases} \dot{x} &= x^{\frac{1}{3}} & x \in \mathbb{R} \\ x^{+} &= 0 & x \in \emptyset \end{cases} \tag{1.10}$$

Then, both the hybrid arcs ξ_1 and ξ_2 defined by $\xi_1(t,0) = (\frac{2t}{3})^{\frac{3}{2}}$ and $\xi_2(t,0) = 0$, for all $t \in \mathbb{R}_{\geq 0}$, are solutions to the hybrid system above, from the initial state 0, [87, Section 3.1]. Nondeterminism of solutions is also related to the use of set-valued mappings. For example, consider the following hybrid system

$$\mathcal{H} : \begin{cases} \dot{x} &\in [-1,1] & x \in \mathbb{R} \\ x^{+} &\in \mathbb{R} & x \in \emptyset \end{cases} \tag{1.11}$$

Then, each hybrid arc ξ defined by $\xi(t,0) \in [x_0 - t, x_0 + t]$, for all $t \in \mathbb{R}_{\geq 0}$, is a solution to the hybrid system from the initial state $x_0 \in \mathbb{R}$. An example of a solution to a given hybrid system is represented in Figure1.6.

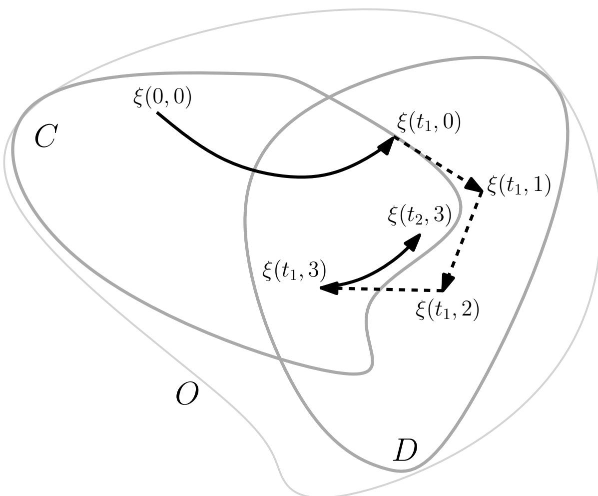


Figure 1.6: A possible solution x to a hybrid system $\mathcal{H}$

In what follows we refer to the following types of solutions.

Definition 1.5 A solution x to a hybrid system $\mathcal{H}$ is called
- *nontrivial*, if $\operatorname{dom} x$ has at least two points;
- *complete*, if $\operatorname{dom} x$ is unbounded, that is, it does not exist a compact set $\mathcal{K} \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that $\operatorname{dom} x$ is a subset of $\mathcal{K}$;
- *Zeno*, if $\operatorname{dom} x$ is complete and $\sup_t \operatorname{dom} x < \infty$;
- *maximal*, if it cannot be extended, that is, it does not exist a solution y to $\mathcal{H}$ such that $\operatorname{dom} x$ is a proper subset of $\operatorname{dom} y$ and $y(t,j) = x(t,j)$ for each $(t,j) \in \operatorname{dom} x$.
- *discrete*, if there are no flow intervals.
- *continuous*, if there are no jumps.

Using the notion of solution to a hybrid system, we can define the set of reachable states from a given set $X \subseteq O$.

Definition 1.6 Consider a hybrid system $\mathcal{H}$. A state x is *reachable* from x_0 if there exists a solution ξ to $\mathcal{H}$ such that $\xi(0,0) = x_0$ and $\xi(T,J) = x$ for some (T,J) in $\operatorname{dom}\xi$. A solution ξ to $\mathcal{H}$ *reaches* a point x if there exists a $(T,J) \in \operatorname{dom}\xi$ such that $\xi(T,J) = x$. The set of *reachable states* of $\mathcal{H}$ from a set $X \subseteq O$ is defined as

$$\begin{aligned} \mathit{Reach}(X) = \{x \mid \exists x_0 \in X, \exists \xi \text{ solution to } \mathcal{H} \text{ such that} \\ \xi(0,0) = x_0 \text{ and } \exists (T,J) \in \operatorname{dom}\xi,\ \xi(T,J) = x\} \end{aligned} \tag{1.12}$$

1.2 Relations to Other Models

Several different definitions of a hybrid system can be found in literature. Among others, [68] present a definition of a hybrid system, called *hybrid automata*, that as been widely accepted and used, with minor modifications, by the computer science community. A similar definition of a hybrid system can be found in [99]. This definition is widely used by the control community. The definition in [68, 99] is based on the following data.

1. Q is a finite set of *nodes*

2. X is a finite set of continuous variables, each of them ranging over $\mathbb{R}$.

3. $\mathit{dom} : Q \rightrightarrows \mathbb{R}^n$ maps each $q \in Q$ to a subset of $\mathbb{R}^n$. For each $q \in Q$, $\mathit{dom}(q)$ defines the set in which the system *must* flow.

4. *flow* : $Q \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map of the hybrid system. In [99], for each $q \in Q$, *flow*$(q, \cdot)$ is a single-valued mapping denoted by $f_q(\cdot)$. In [68], for each $q \in Q$ and $x \in \mathbb{R}^n$, flow is a particular predicate on x and $\dot{x}$ that denotes a subset of $\mathbb{R}^n$.

5. $E \subseteq Q \times Q$ is a set of *edges* that defines the *transitions* between modalities.

6. *guard* : $E \rightrightarrows \mathbb{R}^n$ maps each edge to a subset or $\mathbb{R}^n$. For each $e \in E$, *guard*(e) defines the set from which a jump may occur.

7. *reset* : $E \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ nondeterministically maps each transition E and each point $x \in \mathbb{R}^n$ to a point $y \in \mathbb{R}^n$. *reset*(e, x) defines the possible values of the state after a jump.

Hybrid systems in [68, 99] are usually represented by nodes and arcs, following the classical representation of finite automata [81]. An example is given in Figure 1.7. Usually, the *dom* and the *flow* functions are represented inside a node or linked to a node, the *guard* functions *enable a jump* and are placed at the beginning of an edge, and the *reset* functions defines the value of the state *after a jump*, and are usually placed at the end of an edge.

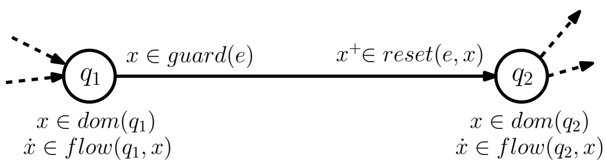


Figure 1.7: Example of a edge $e = (q_1, q_2)$ that connects two modes of a hybrid system in [68, 99].

Solutions to these hybrid systems are usually named *executions* and are quite similar, with minor differences, to the notion of solution in Definition 1.4. An execution ξ is a sequence of continuous functions $x_i \in I_i \to \mathbb{R}^n$, where $I_i \subseteq \mathbb{R}_{\geq 0}$ and $i \in \mathbb{Z}_{\geq 0}$. For each i, x_i maps $t \in I_i \subseteq \mathbb{R}_{\geq 0}$ to $x_i(t)$ and it characterizes the continuous motion of the state of the system. Indeed, for each i, a function x_i is associated to a mode $q_i \in Q$ and it satisfies the differential inclusion $\dot{x_i}(t) \in$ *flow*$(q_i, x_i(t))$, provided that $x_i(t)$ belongs to *dom*(q_i) for all $t \in I_i$, until a jump occurs. For each $i \in \mathbb{Z}_{\geq 0}$, if x_i enters *guard*(e), that is, for some $t \in \mathbb{R}_{\geq 0}$ and some $q \in Q$, $x_i(t) \in$ *guard*(q_i, q), then a jump may occur. If x_i enters *guard*(e) and no further continuous motion of the state is allowed by

$dom(q_i)$, then a jump must occur. When a jump occurs, the mode $q_i \in Q$ is updated to a new mode $q_{i+1} \in Q$ and a reset occurs on ξ, satisfying *reset*.

These hybrid systems, say $H = (Q, X, E, dom, flow, guard, reset)$, can be rewritten to hybrid systems in Definition 1.1, say $\mathcal{H} = (\mathbb{R}^{n+1}, C, D, F, G)$, where n is the dimension of the state $x \in X$, as follows:

- $C = \bigcup_{q \in Q} dom(q) \times \{q\}$;
- $D = \bigcup_{q \in Q} \bigcup_{(q,q') \in E} guard(q, q') \times \{q\}$;
- $F(x, q) = flow(q, x) \times \{0\}$;
- $G(x, q) = \bigcup_{\{q' \mid x \in guard(q,q')\}} reset((q, q'), x) \times \{q'\}$.

Then, by considering $(x, q) \in \mathbb{R}^n \times \mathbb{R}$ the state vector of $\mathcal{H}$, it follows that

$$\mathcal{H} : \begin{cases} \dot{\begin{bmatrix} x \\ q \end{bmatrix}} \in F(x, q) & (x, q) \in C \\ \begin{bmatrix} x \\ q \end{bmatrix}^+ \in G(x, q) & (x, q) \in D \end{cases} \tag{1.13}$$

Example 1.5 Consider the hybrid system $H = (Q, X, E, dom, flow, guard, reset)$ represented in Figure 1.8, whose data are

$Q = \{q_1, q_2\}$, $E = \{q_1, q_2\}$ and $X = \{x\}$, where $x \in \mathbb{R}$,
$dom(q_1) = [0, 1]$ and $dom(q_2) = [0, 2]$,
$flow(q_1, x) = 1$ and $flow(q_2, x) = -1$,
$guard(q_1, q_2) = 1$ and $guard(q_2, q_1) = 0$,
$reset(q_1, q_2, x) = reset(q_2, q_1, x) = x$.

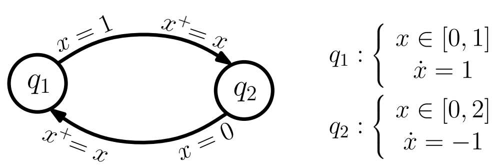


Figure 1.8: Representation by graphs of the hybrid system in Example 1.5

Then, by encoding q_1 with 1 and q_2 with 0, we can rewrite $\mathcal{H}$ as a hybrid system $\mathcal{H} = (\mathbb{R}^2, C, D, F, G)$, where

$$
\begin{aligned}
&C = [0,1] \times \{1\} \cup [0,2] \times \{0\},\\
&D = \{(1,1),(0,0)\},\\
&F = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}^T & \text{if } (x,q) \in [0,1] \times \{1\} \\ \begin{bmatrix} -1 & 0 \end{bmatrix}^T & \text{if } (x,q) \in [0,2] \times \{0\} \end{cases}\\
&G = \begin{cases} \begin{bmatrix} x & 0 \end{bmatrix}^T & \text{if } (x,q) \in [0,1] \times \{1\} \\ \begin{bmatrix} x & 1 \end{bmatrix}^T & \text{if } (x,q) \in [0,2] \times \{0\} \end{cases}
\end{aligned}
$$

that is

$$
\mathcal{H} : \begin{cases} \dot{\begin{bmatrix} x \\ q \end{bmatrix}} \in \begin{bmatrix} 1 \\ 0 \end{bmatrix} q + \begin{bmatrix} -1 \\ 0 \end{bmatrix} (1-q) & (x,q) \in [0,1] \times \{1\} \cup [0,2] \times \{0\} \\ \begin{bmatrix} x \\ q \end{bmatrix}^+ \in \begin{bmatrix} x \\ \mod (q+1, 2) \end{bmatrix} & (x,q) \in \{(1,1),(0,0)\} \end{cases} \tag{1.14}
$$

An execution is parameterized with respect to a *hybrid time trajectory*[1] [99]. A hybrid time trajectory can be easily associated to the notion of hybrid time domain of a solution, in Definition 1.2, where more importance is given to the information on jump instants, stored in the variable j. In fact, a hybrid time trajectory of an execution is a sequence of intervals $I_i = [t_i, t'_i]$, $i \in \mathbb{Z}_{\geq 0}$, characterized by the fact that $t'_i = t_{i+1}$ and such that the last interval I_N, $N \in \mathbb{Z}_{\geq 0}$, when it exists, can be of the form $I_N = [t_N, \infty)$. Basically, hybrid time trajectories can be considered as the projection of hybrid time domains to the t axis. Moreover, executions ξ to a hybrid system of the form in [99] are usually assumed CADLAG - continue a droit, limite a gauche:

- the left limit to the point of discontinuity t'_i of a CADLAG solution ξ is different from the value of the execution ξ at that point, say $\xi(t'_i)$;
- the right limit to the point of discontinuity t'_i of a CADLAG solution ξ coincides with $\xi(t'_i)$.

Indeed, suppose that for some $i \in \mathbb{Z}_{\geq 0}$ an execution ξ is defined by two function x_i and x_{i+1}, defined on the intervals $I_i = [t_i, t'_i]$ and $I_{i+1} = [t_{i+1}, t'_{i+1}]$, with $t'_i = t_{i+1}$. Then, $\lim_{t \to t'^-_i} \xi(t) = \lim_{t \to t'^-_i} x_i(t) \neq \xi(t'_i)$, while $\lim_{t \to t'^+_i} \xi(t) = \lim_{t \to t'^+_i} x_{i+1}(t) = x_{i+1}(t'_i) = x_{i+1}(t_{i+1}) = \xi(t_{i+1})$ It follows that ξ satisfies the differential equation $\dot{\xi}(t) \in \mathit{flow}(q_i, \xi(t))$ for each $t \in [\inf(I_i), \sup(I_i))$, leaving

[1] The relation between an execution and a hybrid time trajectory is similar to the relation between a solution and its hybrid time domain for hybrid systems in Definition 1.1

the time $t'_i = t_{i+1}$ unused and ready for the initial point $\xi(t_{i+1}) = x_{i+1}(t_{i+1})$ of the solution x_{i+1} associated to the next interval I_{i+1}. Each execution can be easily associated to hybrid arc in Definition 1.3, and the intricacies of the use of CADLAG functions can be ruled out by using the hybrid time in Definition 1.2. In fact, for the same execution ξ, index i and functions x_i and x_{i+1} considered above, we can define a hybrid arc $\overline{\xi}$ such that

- $\overline{\xi}(t, i) = \xi(t)$ for $t_i \leq t < t'_i$;
- $\overline{\xi}(t'_i, i) = \lim_{t \to t'^-_i} \xi(t)$;
- $\overline{\xi}(t, i+1) = \xi(t)$, for $t_{i+1} \leq t < t'_{i+1}$.

Thus, $\overline{\xi}(t, i) = x_i(t)$, for $t_i \leq t \leq t'_i$, and $\overline{\xi}(t, i+1) = x_{i+1}(t)$, for $t_{i+1} \leq t \leq t'_{i+1}$.

1.3 Basic Conditions

Although hybrid systems (1.5) are defined as the juxtaposition of a continuous dynamics and of a discrete dynamics, several new phenomena arise from the interaction of these different kinds of dynamics, and it is not immediate to generalize results on continuous and discrete systems to hybrid systems. For example, non-unicity of solutions from a given initial state is avoided on continuous systems by enforcing some mild conditions on the right-hand side of differential equations while, on hybrid systems, the possibility of $C \cap D \neq \emptyset$ inherently entails it.

One important feature on hybrid systems (1.5) is their fragility to small state-perturbations. New solutions, completely unrelated to the solutions of the nominal system, can appear on a perturbed hybrid system, no matter how small the perturbation magnitude is. Moreover, solutions can fail to exists. This is related to the lack of sequential compactness of the space of solutions to a hybrid system (namely when the limits of sequences of solutions are themselves solutions), that has effects also on the continuity of solutions with respect to the initial state, on the possibility of numerically simulate solutions to hybrid systems and, in general, it is one of the obstacles to the generalization of classical results of nonlinear systems to hybrid systems.

Following [125, Chapter 3], we can define an admissible state perturbation as follows.

Definition 1.7 A mapping e is an *admissible state perturbation* if $\operatorname{dom} e$ is a hybrid time domain and, for each $j \in \mathbb{Z}_{\geq 0}$, the function $\lambda t.e(t, j)$ is measurable on $\operatorname{dom} e \cap (\mathbb{R}_{\geq 0} \times \{j\})$, Then, a hybrid system $\mathcal{H} = (O, C, D, F, G)$ with a state perturbation e, denoted by $\mathcal{H}_e$, can be defined as

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x+e) & x+e \in C \cap O \\ x^+ \in G(x+e) & x+e \in D \cap O. \end{cases} \tag{1.15}$$

The concept of solution to $\mathcal{H}_e$ is quite similar to concept of solution to $\mathcal{H}$ in Definition 1.4.

Definition 1.8 [125, Definition 3.2] A hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *solution to the hybrid system* $\mathcal{H}_e$ *with admissible state perturbation* e, if

- $\operatorname{dom} x = \operatorname{dom} e$,
- $x(0,0) + e(0,0) \in \overline{C} \cup D$,
- for all $(t,j) \in \operatorname{dom} x$, $x(t,j) + e(t,j) \in O$, and

  1. for each $j \in \mathbb{Z}_{\geq 0}$ such that $I_j = \{t : (t,j) \in \operatorname{dom} x\}$ has a nonempty interior,

  $$\begin{array}{ll} \dot{x}(t,j) \in F(x(t,j) + e(t,j)) & \text{for almost all } t \in I_j \\ x(t,j) + e(t,j) \in C & \text{for all } t \in [\min I_j, \sup I_j); \end{array} \tag{1.16}$$

  2. for each $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

  $$\begin{array}{rcl} x(t,j+1) & \in & G(x(t,j) + e(t,j)) \\ x(t,j) + e(t,j) & \in & D. \end{array} \tag{1.17}$$

To see the differences between solutions to $\mathcal{H}$ and solutions to $\mathcal{H}_e$, let us consider the following example.

Example 1.6 [Effect of small state-perturbations]
Consider a hybrid system with state vector $x = [\,x_1\ x_2\,]^T \in \mathbb{R}^2$ defined by

$$\mathcal{H} = \begin{cases} \dot{x} \;=\; \begin{bmatrix} 0 & -\frac{\pi}{2} \\ \frac{\pi}{2} & 0 \end{bmatrix} x & x \in C \\ x^+ \;=\; \begin{bmatrix} 0 & -\frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix} x & x \in D \end{cases} \tag{1.18}$$

where $D = \{x \,|\, x_1 = 0\}$ and $C = \mathbb{R}^2 \setminus D$. The unique solution to the system $\mathcal{H}$, from $x_0 = [1, 0]^T$, rotates counterclockwise for 1 unit of time, until it hits the set D. From there, it jumps to $[-\frac{1}{2}, \frac{1}{2}]^T$, decreasing its norm by a factor 2 and rotating counterclockwise of $\frac{\pi}{4}$. Then, this dynamics is repeated, alternating flow intervals and jumps. Note that at each jump the norm of the state vector decreases and, during flows, it remains constant. Therefore, the system converges to the point in 0.

Consider now the set $T = \{(1+2k, 0) \mid k \in \mathbb{Z}_{\geq 0}\}$ and consider a perturbation e defined as $e(t,j) = [\varepsilon, 0]^T$ for each $(t,j) \in T$, for some $\varepsilon > 0$, and 0 otherwise. From Definition 1.8, the unique solution x_e to the perturbed system $\mathcal{H}_e$, from $x_0 = [1, 0]^T$, coincides with x until x hits D. Then, x jumps while $x_e(1,0) + e(1,0) \notin D$, that is, x_e does not jump. Moreover, looking at the time instants at which e is not zero, it turns out that x_e never jumps. See Figure 1.9.

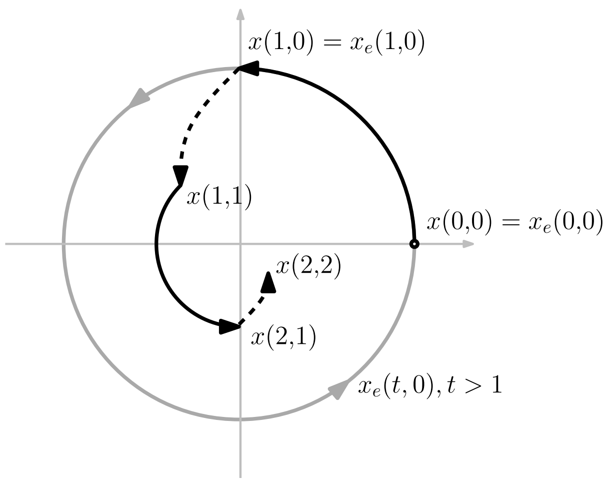


Figure 1.9: Solutions x and x_e to the nominal hybrid system $\mathcal{H}$ and to the perturbed hybrid system $\mathcal{H}_e$, respectively.

The solution x_e is quite different from x. Indeed, it is different from each solution to $\mathcal{H}$, with initial state in the neighborhood of $x_0 = [1, 0]^T$, and this is independent from the particular magnitude $\varepsilon > 0$ of the perturbation. Note that such a situation would not occur if the set C was closed. In that case, a solution that flows only would appear also in $\mathcal{H}$ and x_e would remain close to this solution, at least for an interval of time that depends on the perturbation magnitude.

By considering an intuitive notion of convergence of a solution ξ to a point x defined as $\lim_{(t,j)\in \operatorname{dom}\xi,\,(t+j)\to\infty} \xi(t,j) = x$, this example shows also that convergence of a solution to a point can be easily disrupted by arbitrarily small state-perturbations. This has effects also on stability properties of hybrid systems. In fact, intuitively from usual stability concepts on nonlinear systems, we can see that a possible result on asymptotic stability of the point 0 would be not robust to some arbitrarily small state-perturbations.

We will not go into details of robustness problems but interesting references on this topic, on hybrid systems, are [61, 63, 125, 127]. The main contribution of these articles is in the definition of a set of minimal conditions on the data of a hybrid system, called *basic conditions*, to get rid of phenomena showed above. More precisely, the satisfaction of the basic conditions guarantees that a properly defined limit of a sequence of solutions, generated by perturbations that decrease in magnitude, would be itself a solution to the nominal hybrid system. Indeed, a hybrid system $\mathcal{H}$ that satisfies the basic conditions exhibits a sort of regularity of solutions that leads to several important results. For these kind of systems, sequential compactness of the space of solutions holds, [63, Theorem 4.4], there is outer semicontinuous dependence of solutions on initial states, [62, Theorem 5], and it is possible to directly relate the solutions to $\mathcal{H}$ with the solutions to state-perturbed hybrid systems $\mathcal{H}_\delta$, constructed from $\mathcal{H}$ by a suitable state-perturbation of magnitude δ, [62, Theorem 8].

In [61, 62, 63, 125, 127] that arguments are developed in an exhaustive and clear way, and several example are presented. In what follows, we simply present the *basic conditions*, pointing out the goodness of working with *regular* hybrid systems, namely, hybrid systems that satisfy the basic conditions.

Definition 1.9 A *regular hybrid systems* $\mathcal{H} = (O, C, D, F, G)$ satisfies the following *basic conditions*:

(i) $O \subseteq \mathbb{R}^n$ is an open set

(ii) $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ are relatively closed sets in O.

(iii) $F : O \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on O and, for each $x \in C$, $F(x)$ is nonempty and convex.

(iv) $G : O \rightrightarrows O$ is an outer semicontinuous set-valued mapping, locally bounded and, for each $x \in D$, $G(x)$ is nonempty.

The sets C and D are *relatively closed* in O if $D = O \cap \overline{D}$ and $C = O \cap \overline{C}$. F is *outer semicontinuous* if for all $x \in O$ and all sequences $x_i \to x$, $y_i \in F(x_i)$ such that $y_i \to y$, we have that $y \in F(x)$. F is *locally bounded* if for any compact set $K \subseteq O$, there exists a radius $m > 0$ such that $F(K) = \{F(x) \mid x \in K\} \subseteq \mathbb{B}(m)$. Analogously for G.

Remark 1.1 The outer semicontinuity of F can be studied by considering its graph. F is outer semicontinuous in O if and only if $\operatorname{gph} F = \{(x,y) \mid x \in O, y \in \mathbb{R}^n, y \in F(x)\}$ is relatively closed in $O \times \mathbb{R}^n$. Analogously for G. See [123, Theorem 5.7] and [63, Page 578].

In the following example we compare a hybrid system $\mathcal{H}$ that does not satisfy the basic conditions and a hybrid system $\overline{\mathcal{H}}$, closely related to $\mathcal{H}$, that satisfies them.

Example 1.7 [Existence of solutions]
Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ where $O = C = D = \mathbb{R}$, $G(x) = \emptyset$, and for $x \in \mathbb{R}$, F is defined as follows

$$F(x) = \begin{cases} -1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases} \tag{1.19}$$

A solution x to $\mathcal{H}$ from $x_0 = m$, $m \in \mathbb{R}_{>0}$ is defined by the following expression: for each $t \in [0, m]$, $x(t, 0) = x_0 - t$. Then, at time m, we have that $x(t, 0) = 0$. From that point, the solution cannot be continued. In fact, $F(0) = -1$ but $F(-\varepsilon) = 1$, for any arbitrarily small $\varepsilon > 0$. Indeed, there is no solution y to $\mathcal{H}$ from the point 0.

Consider now the same system but with the continuous dynamics replaced by the following set-valued mapping $\overline{F}$, that satisfies the basic conditions:

$$\overline{F}(x) = \begin{cases} -1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x < 0 \end{cases} \tag{1.20}$$

In such a case, $0 \in F(0)$ therefore the solution $\overline{x}$ from $x_0 = m$, $m \in \mathbb{R}_{>0}$, can be defined as $\overline{x}(t, 0) = x_0 - t$ for each $t \in [0, m]$, and $\overline{x}(t, 0) = 0$ for each $t \geq m$. For the same reason, for each $t \geq 0$, $y(t, 0) = 0$ is a solution to $\mathcal{H}$ from 0.

Problems on solutions to differential equation with discontinuous right-hand side, like (1.19), are usual in sliding mode control. To overcome issues on solutions, one can approximate the equation $\dot{x} = F(x)$ by $\dot{x} = -\mathrm{sat}(\frac{x}{\epsilon})$, that has a continuous right-hand side and is close to (1.19) for $\epsilon > 0$ sufficiently small. In fact, imperfections in switching devices and delays would force solutions that suffer from *chattering*, and $\dot{x} = -\mathrm{sat}(\frac{x}{\epsilon})$ characterizes well that phenomena. See

[87, Section 14.1]. Note that $\overline{F}(x)$ can be considered as the limit function that arises from $-\mathrm{sat}(\frac{x}{\epsilon})$ when ϵ converges to 0.

Remark 1.2 It is worth mentioning that by working on $\dot{x} = -\mathrm{sat}(\frac{x}{\epsilon})$ instead of $\dot{x} = F(x)$ we are implicitly assuming that solutions to $\dot{x} = -\mathrm{sat}(\frac{x}{\epsilon})$ and to $\dot{x} = F(x)$ are *close* (w.r.t. a pointwise distance between solutions) when some imperfections in switching or delays occur. Roughly speaking, this is based on the intuition that *close* equations have *close* solutions and on the intuition that solutions from *close* initial conditions are *close*. This intuition, which is correct for nonlinear systems, cannot be generalized easily to hybrid systems, as shown in Example 1.6.

With the basic conditions satisfied, we can characterize the following result on existence of solutions.

Definition 1.10 The *tangent cone* to C at $x \in C$, denoted by $T_C(x)$, is the set of all $v \in \mathbb{R}^n$ for which there exists a sequence $\alpha_i \to 0$, with $\alpha_i \in \mathbb{R}_{>0}$ for each $i \in \mathbb{Z}_{\geq 0}$, and a sequence of vectors $v_i \to v$, with $v_i \in \mathbb{R}^n$ for each $i \in \mathbb{Z}_{\geq 0}$, such that, for all $i \in \mathbb{Z}_{\geq 0}$, $x + \alpha_i v_i \in C$.

Proposition 1.1 *[63, Proposition 2.4] Consider a hybrid system $\mathcal{H}$* (1.5) *that satisfies the basic conditions. If $x_0 \in D$ or*

(VC) $x_0 \in C$ and there exists a neighborhood U of x_0 such that, for all $x' \in U \cap C$, $T_C(x') \cap F(x') \neq \emptyset$,

then there exists a solution x to $\mathcal{H}$ with $x(0,0) = x_0$ and $\mathrm{dom}\, x \neq (0,0)$.

If (VC) holds for all x_0 in $C \setminus D$, then for any maximal solution x at least one of the following statements is true:

(i) x is complete;

(ii) x eventually leaves every compact subset of O, that is, for any compact $K \subseteq O$, there exists $(T, J) \in \mathrm{dom}\, x$ such that $\forall (t,j) \in \mathrm{dom}\, x$ such that $t + j > T + J$, $x(t,j) \notin K$;

(iii) for some $(T, J) \in \mathrm{dom}\, x$, $(T, J) \neq (0,0)$, we have $x(T, J) \neq C \cup D$.

Case (iii) does not occur if

(VD) for all $x_0 \in D$, $G(x_0) \subseteq C \cup D$.

Condition "x_0 in D" guarantees that a jump from x_0 can occur. In a similar way, Condition (VC) guarantees that the system can flow for a possibly small interval of time. In fact, Condition (VC) requires that the derivative of the system, that is, the instantaneous motion of the state, is compatible with the constraints that the set C induces on the motion of the state. Then, a solution from x_0 has either a domain with at least two values $(0,0)$, $(0,1)$, defined by a possible jump (x_0 in D), or a domain defined by a small interval $[0,t] \times \{0\}$, that take into account a possible flow (Condition (VC)). Moreover, under the hypothesis that (VC) is true for each point x_0 of C, Conclusions (i)-(iii) characterize each possible solution to the hybrid system $\mathcal{H}$. (i) refers to solutions with unbounded domains. This case parallels the usual behavior of solutions to continuous systems $\dot{x} = f(x)$ under the assumption of global Lipschitzianity of $f(x)$. Note that, on hybrid systems, unboundedness of the domain can occur also on the j direction: for discrete solutions and Zeno solutions. Conclusion (ii) takes into account unbounded solutions, namely, solutions that escapes any compact subset of O (note that O can be $\mathbb{R}^n$). This is an effect of the fact that C and D are relatively closed in O. If C and D are closed sets, (ii) does not occur. Conclusion (iii) refers to the case of solutions that leaves the sets C and D. From the definition of solution to a hybrid system, a solution can leave $C \cup D$ only by a jump. Therefore, if Condition (VD) holds, Conclusion (iii) does not occur.

Proposition 1.1 underlines two important results. It defines some local conditions on a point x_0 that guarantee the existence of nontrivial solutions from that point x_0, and it defines a set of global conditions that guarantee that each maximal solution is complete, or it is unbounded, that is, it moves to the border of O, or it escapes $C \cup D$ by a jump. (when (VD) does not hold).

By Proposition 1.1 we have some conditions on regular hybrid systems that guarantee the existence of a solution from a given initial state, say x_0. In what follows we go beyond the characterization of the existence of a solution to a hybrid system, by studying the relation between solutions and small perturbations of the initial state. Indeed, we study the effects on solutions to a hybrid system when we add a small perturbation ε to the initial state x_0, that is, we study the dependence of solutions from the initial state $x_0 + \varepsilon$, for ε that goes to zero. Moreover, we consider the case of a sequence of solutions to $\mathcal{H}$ that converges to some set-valued mapping $x : \operatorname{dom} x \rightrightarrows \mathbb{R}^n$, for some properly defined convergence notion on solutions, and we try to understand if this limit set-valued mapping is itself a solution to $\mathcal{H}$. Useful mathematical tools for studying these problems involve topics like sets-convergence and set-valued mappings. A very interesting reference on these subjects is [123, Chapters 4 and 5]. Then, using these tools, we can properly characterize notions of *convergence* of solutions to a limit hybrid arc, and of *closeness* of a solution to another.

Consider a sequence of points $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of $\mathbb{R}^n$. We write $x_i \to x$ whenever x is the limit of the sequence of x_i, as i goes to ∞. Consider a sequence of sets $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of subset of $\mathbb{R}^n$. We write $\{S_{i_k}\}_{i_k\in N\subseteq\mathbb{Z}_{\geq 0}} \subseteq \{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$ to denote the infinite subsequence $\{S_{i_k}\}_{i_k\in N\subseteq\mathbb{Z}_{\geq 0}}$ of the sequence of sets $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$. We write $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}} \in \{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$ to denote a sequence of points $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ constructed by taking a point x_i from each set S_i of the sequences of sets $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$. Then, from [123, Definitions 4.1 and 5.32], we can characterize the following convergence concepts.

Definition 1.11 Consider a sequence of sets $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$.
The *inner limit* of $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is

$$\liminf_{i\to\infty} S_i = \{x \,|\, \exists \{x_i\}_{i\in\mathbb{Z}_{\geq 0}} \in \{S_i\}_{i\in\mathbb{Z}_{\geq 0}}, x_i \to x\}. \tag{1.21}$$

The *outer limit* of $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is

$$\limsup_{i\to\infty} S_i = \\ = \{x \,|\, \exists \{S_{i_k}\}_{i_k\in N\subseteq\mathbb{Z}_{\geq 0}} \subseteq \{S_i\}_{i\in\mathbb{Z}_{\geq 0}}, \exists \{x_{i_k}\}_{i\in N\subseteq\mathbb{Z}_{\geq 0}} \in \{S_{i_k}\}_{i_k\in N\subseteq\mathbb{Z}_{\geq 0}}, x_{i_k} \to x\}. \tag{1.22}$$

A sequence of sets $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}}$ *converges* to a set S, denoted by $S_i \to S$, if $S = \liminf_{i\to\infty} S_i = \limsup_{i\to\infty} S_i$.

Definition 1.12 Consider a sequence of set-valued mappings $\{F_i\}_{i\in\mathbb{Z}_{\geq 0}}$ where each mapping F_i belongs to $\mathbb{R}^m \rightrightarrows \mathbb{R}^n$, $m, n \in \mathbb{Z}_{\geq 0}$. The sequence $\{F_i\}_{i\in\mathbb{Z}_{\geq 0}}$ *converges graphically* to the set-valued mapping F, denoted by $F_i \xrightarrow{g} F$, if $\operatorname{gph} F_i \to \operatorname{gph} F$.

Remark 1.3 For a given sequence of mappings $\{F_i\}_{i\in\mathbb{Z}_{\geq 0}}$, if the graphical limit exists, the limit set-valued mapping F is defined as the set of pairs (x, y), i.e. $y \in F(x)$, for which there exist two sequences $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}} \in \{\operatorname{dom} F_i\}_{i\in\mathbb{Z}_{\geq 0}}$ and $\{y_i\}_{i\in\mathbb{Z}_{\geq 0}} \in \{F_i(x_i)\}_{i\in\mathbb{Z}_{\geq 0}}$, such that $x_i \to x$ and $y_i \to y$.

Note that graphical limit and pointwise limit of sequences of set-valued mappings $\{F_i\}_{i\in\mathbb{Z}_{\geq 0}}$ differ. Indeed, for each $x \in \mathbb{R}^m$, the pointwise limit F of these $\{F_i\}_{i\in\mathbb{Z}_{\geq 0}}$, denoted by $F_i \xrightarrow{p} F$, can be defined as the set $F(x)$ such that $F_i(x) \to F(x)$. Intuitively, the construction of the graphical limit uses a different x_i for each F_i, with $x_i \to x$, while the construction of the pointwise limit uses a fixed x. Graphical convergence is more convenient than pointwise convergence for studying sequences of functions with different domains, as in the case of sequences of hybrid arcs.

Set convergence can be used to study the convergence of a sequence of domains of hybrid arcs, and graphical convergence of set-valued mappings can be used to study the convergence of a sequence of hybrid arcs. Then, we can characterize the following result on *sequential compactness* of the space of solutions.

Definition 1.13 A sequence of hybrid arcs $x_i : \operatorname{dom} x_i \to \mathbb{R}^n$, $i \in \mathbb{Z}_{\geq 0}$ is *locally eventually bounded with respect to O* if for any $m > 0$ there exists $i_0 > 0$ and a compact set $K \subseteq O$ such that for all $i > i_0$, all $(t, j) \in \operatorname{dom} x_i$ with $t + j < m$, $x_i(t, j) \in K$.

Theorem 1.1 *[63, Theorem 4.4 and Lemma 4.3] Consider a hybrid system $\mathcal{H}$, (1.5), satisfying the basic conditions. Let $x_i : \operatorname{dom} x_i \to \mathbb{R}^n$, $i \in \mathbb{Z}_{\geq 0}$, be locally eventually bounded with respect to O sequence of solutions to $\mathcal{H}$.*

- *Then, there exists a subsequence of x_i's graphically converging to a solution x of $\mathcal{H}$.*
- *Moreover, if $\{x_i\}_{i \in \mathbb{Z}_{\geq 0}}$ converges graphically to some set-valued mapping $x : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$, then x is a solution to $\mathcal{H}$.*

By Theorem 1.1, if a sequence of solutions to a hybrid system $\mathcal{H}$ graphically converges to some hybrid arc, then this hybrid arc is itself a solution to $\mathcal{H}$. A different version of Theorem 1.1, for a slightly simpler class of hybrid systems, can be found in [62, Theorem 4] . Let us consider the following example.

Example 1.8 Consider a hybrid system with state $x = [x_1 \ x_2]^T$ and defined by

$$\mathcal{H} = \begin{cases} \dot{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} & x \in \{x \mid x_2 > 0\} \\ x^+ = \begin{bmatrix} -1 \\ -1 \end{bmatrix} & x \in \{x \mid x_2 \leq 0\}, \end{cases} \tag{1.23}$$

and consider an initial state $\begin{bmatrix} 0 & \frac{1}{i} \end{bmatrix}^T$, parameterized with respect to $i \in \mathbb{Z}_{>0}$. Define a sequence hybrid arc $\{x_i\}_{i \in \mathbb{Z}_{\geq 0}, i>0}$ as follows: for each $t \geq 0$ and each $i \in \mathbb{Z}_{>0}$, $x_i(t, 0) = \begin{bmatrix} t & \frac{1}{i} \end{bmatrix}^T$. Then, x_i is a solution to $\mathcal{H}$ from the initial state $\begin{bmatrix} 0 & \frac{1}{i} \end{bmatrix}^T$, for each $i \in \mathbb{Z}_{>0}$.

The sequence $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}, i>0}$ converges graphically to the hybrid arc x defined as $x(t,0) = \left[\begin{array}{cc} t & 0 \end{array}\right]^T$, for each $t \in \mathbb{R}_{\geq 0}$, but x is not a solution to $\mathcal{H}$. In fact, the unique solution from the initial state $x_0 = \left[\begin{array}{cc} 0 & 0 \end{array}\right]^T$ jumps to the point $\left[\begin{array}{cc} -1 & -1 \end{array}\right]^T$. See Figure 1.10. Note that the set C is open, therefore $\mathcal{H}$ does

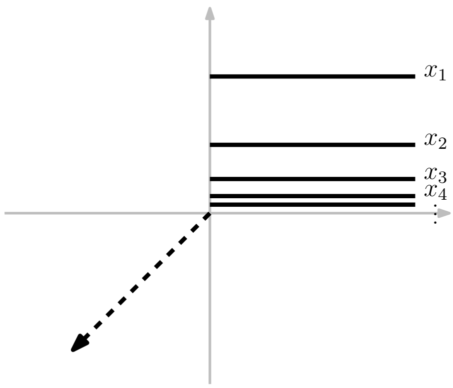


Figure 1.10: Example of possible solutions x to the system $\mathcal{H}$ of Example 1.8.

not satisfy the basic conditions and the space of solutions is not, in general, sequentially compact.

Let us regularize the hybrid system in (1.23) by considering a flow set defined by $\overline{C}$ instead of C, that is, $\{x \,|\, x_2 \geq 0\}$. With a closed flow set, the hybrid system satisfies the basic conditions. As expected in this case, the hybrid arc x is a solution to the regularized hybrid system.

Continuous dependence on initial states is related to the sequential compactness of the space of the solutions. Example 1.8 shows well the lack of continuity of solutions with respect to the initial states for hybrid systems that do not satisfy the basic assumption. In fact, the solution x from $x_0 = \left[\begin{array}{cc} 0 & 0 \end{array}\right]^T$ jumps to $\left[\begin{array}{cc} -1 & -1 \end{array}\right]^T$, while the solutions from $\left[\begin{array}{cc} 0 & \varepsilon \end{array}\right]^T$, $\varepsilon > 0$ flow only, growing unbounded. Therefore, for any $\varepsilon > 0$, the solutions from the initial state $x_0 + \varepsilon$ differ from the solutions from the initial state x_0.

To state formally the result on continuity of solutions with respect to initial states, for regular hybrid systems, we need to define a concept of distance between solutions. The following concept of (T, J, ε)-*closeness* of solutions is related to the concept of graphical convergence defined above and allows to compare solutions that jump at different time instants, that is, to compare hybrid arcs with different domains.

Definition 1.14 [63, Page 579], Hybrid arcs $x : \operatorname{dom} x \to \mathbb{R}^n$ and $y : \operatorname{dom} y \to \mathbb{R}^n$ are (T, J, ε)-*closeness of hybrid arcs* if

(i) for all $(t, j) \in \operatorname{dom} x$ with $t \leq T$ and $j \leq J$, there exists s such that $(s, j) \in \operatorname{dom} y$, $|t - s| < \varepsilon$, and $|x(t, j) - y(s, j)| < \varepsilon$;

(ii) for all $(t, j) \in \operatorname{dom} y$ with $t \leq T$ and $j \leq J$, there exists s such that $(s, j) \in \operatorname{dom} x$, $|t - s| < \varepsilon$, and $|y(t, j) - x(s, j)| < \varepsilon$.

Remark 1.4 From [62, Page 46], an equivalent concept is the (T, ε)-*closeness* of solutions. Intuitively, in this case, we require that the solutions remain ε-close for each $t + j \leq T$.

By using the concept of (T, J, ε)-closeness, we can easily compare solutions with different domains, as in the case of solutions with not-synchronized jumps. Consider Figure 1.11, in which we represent only the t component of the domain of the hybrid arcs x and y. The hybrid arc y is converging (graphically) to x but a pointwise analysis would fail to recognize it. In fact, consider the point a. In a pointwise analysis, this point of y would be compared to the point b of x, and the distance between a and b does not decrease for y converging to x (it grows!), until y and x overlap. When they overlap the distance between a and b becomes instantaneously 0. Fortunately, x and y are $(T, J, \varepsilon) - close$. In fact, there exists a point c of x that belongs to a ball centered in y and of radius smaller than ε. This is the most important feature of the concept of (T, J, ε)-closeness, that is, the possibility of comparing points of solutions that are ε-close also in time.

Note that the concept of (T, J, ε)-closeness has a strong relation with the concept of graphical convergence, as stated in the following lemma.

Lemma 1.1 *[63, Lemma 4.5]*
Consider a sequence of hybrid arcs $x_i : \operatorname{dom} x \to \mathbb{R}$ that is locally eventually bounded, and a hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$. The sequence $\{x_i\}_{i \in \mathbb{Z}_{\geq 0}}$ converges graphically to x if and only if, for all $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and $\varepsilon > 0$, there exists $i_o \in \mathbb{Z}_{\geq 0}$ such that, for all $i > i_0$, the hybrid arcs x and x_i are (T, J, ε)-close.

Using the concept of (T, J, ε)-closeness of solutions we can finally characterize the dependence of solutions to initial states.

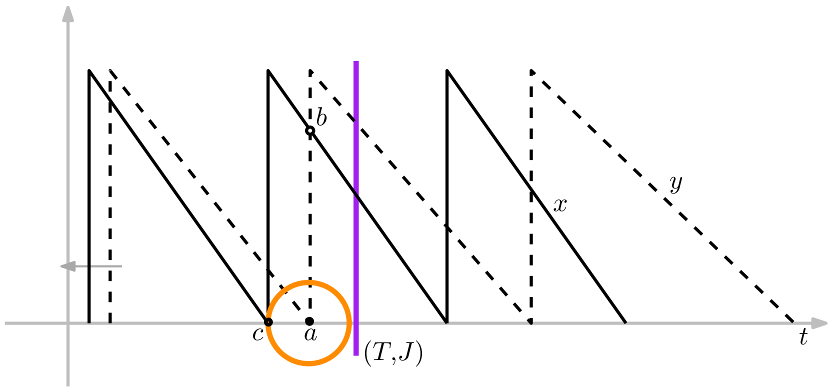


Figure 1.11: Two hybrid arcs (T, J, ε)-close.

Definition 1.15 A hybrid system $\mathcal{H}$ is *forward complete* at x_0 if every maximal solution to $\mathcal{H}$ from x_0 is complete.

Theorem 1.2 *[63, Corollary 4.8] Consider a hybrid system $\mathcal{H}$ satisfying the basic conditions. Suppose that $\mathcal{H}$ is forward complete at every $x_0 \in K$, for some compact set K. For any $\varepsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ there exists a $\delta > 0$ with the following property: for any maximal solution x_δ from $K + \mathbb{B}(\delta)$, there exists a solution x to $\mathcal{H}$ with $x(0, 0) \in K$ such that x_δ and x are (T, J, ε)-close.*

By Theorem 1.2, two solutions to a hybrid system $\mathcal{H}$ from close initial states will remain close in the future, at least for a given interval of time. Moreover, for any given (T, J), we can always find a sufficiently small perturbation to the initial state that guarantees that the distance between two solutions is less then ε, for each $(t, j) \leq (T, J)$.

Graphical convergence and (T, J, ε)-closeness of solutions can be used also to state the following result on the relation between Zeno solutions and discrete solutions.

Definition 1.16 A set of solutions X is *uniformly non-Zeno* if there exists $T > 0$ and $J \in \mathbb{Z}_{\geq 0}$ so that, for each solution $x \in X$ if $(t, j), (t', j') \in \operatorname{dom} x$, then $|t - t'| \leq T$ implies $|j - j'| \leq J$.

Definition 1.17 A set K is *forward invariant* if each maximal solution x originating in K is such that $\forall (t,j) \in \operatorname{dom} x$, $x(t,j) \in K$.

Theorem 1.3 *[63, Corollary 4.9] Consider a hybrid system $\mathcal{H}$ satisfying the basic conditions. Let $K \subseteq O$ be a compact set that is forward invariant. Then, either the set of maximal solutions is uniformly non-Zeno, or there exists a complete solution x with $\operatorname{dom} x = 0 \times \mathbb{Z}_{\geq 0}$ starting in K.*

Theorem 1.3 underline an important fact on the dynamics of hybrid systems, that relates Zeno solutions to discrete solution. Indeed, if a hybrid system $\mathcal{H}$ exhibits some Zeno solutions than such a system has also discrete solutions.

Remark 1.5 Results in Theorem 1.2 and in 1.3 can be found in [62, Theorem 5 and Proposition 6], for a slightly simpler class of hybrid systems. Both Theorems 1.2 and 1.3 are corollaries of Theorem 1.1.

We conclude this section by considering again the problem of state perturbations. A general approach for analyzing hybrid systems under state perturbation is to consider constant perturbation levels and to prove, for small enough perturbations, that properties of interest are preserved.

Definition 1.18 [Perturbed hybrid system]
Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and consider a continuous function $\sigma : O \to \mathbb{R}_{\geq 0}$ such that, for all $x \in O$, $\{x\} + \sigma(x)\mathbb{B} \subset O$. Then, for $\delta \in (0, 1)$, the *perturbed hybrid system* $\mathcal{H}_{\delta\sigma} = (O, C_{\delta\sigma}, D_{\delta\sigma}, F_{\delta\sigma}, G_{\delta\sigma})$ as follows.

- $C_{\delta\sigma} = \{x \in O \,|\, \big(\{x\} + \delta\sigma(x)\mathbb{B}\big) \cap C \neq \emptyset\}$
- $D_{\delta\sigma} = \{x \in O \,|\, \big(\{x\} + \delta\sigma(x)\mathbb{B}\big) \cap D \neq \emptyset\}$
- $F_{\delta\sigma} : O \rightrightarrows \mathbb{R}^n$ such that
  $F_{\delta\sigma}(x) = \overline{\text{co}}\{f \,|\, y \in \{x\} + \delta\sigma(x)\mathbb{B}, f \in F(y) + \delta\sigma(x)\}$
- $G_{\delta\sigma} : O \rightrightarrows \mathbb{R}^n$ such that
  $G_{\delta}(x) = \{g \,|\, y \in \{x\} + \delta\sigma(y)\mathbb{B}, f \in F(y) + \delta\sigma(x)\}$

Solutions to $\mathcal{H}_\delta$ can be interpreted as the solutions that appear in the hybrid system $\mathcal{H}$ when parameter variations, uncertainties in the model, measurement noise in control systems, or external disturbance occur. Note that, if $\mathcal{H}$ is a regular hybrid system, $C_{\delta\sigma}$ and $D_{\delta\sigma}$ in $\mathcal{H}_{\delta\sigma}$ converge as sets to C and D in $\mathcal{H}$, as δ goes to 0, and $F_{\delta\sigma}$ and $G_{\delta\sigma}$ in $\mathcal{H}_{\delta\sigma}$ graphically converge to F and G in $\mathcal{H}$, as δ goes to zero. Otherwise, if $\mathcal{H}$ does not satisfy the basic conditions, then $(O, C_{\delta\sigma}, D_{\delta\sigma}, F_{\delta\sigma}, G_{\delta\sigma})$ of $\mathcal{H}_{\delta\sigma}$ converge to a system $\overline{\mathcal{H}}$ which satisfies the basic conditions. For an example, see $\overline{F}$ in Example 1.7. In this case, we say that $\overline{\mathcal{H}}$ is the *regularization* of $\mathcal{H}$.

The following result, from [63, Theorem 5.1] and from [62, Theorem 8], relates solutions to $\mathcal{H}_{\delta\sigma}$ with solutions to $\mathcal{H}$, for δ converging to 0.

Theorem 1.4 *Consider a hybrid system $\mathcal{H}$ that satisfies the basic conditions. Let $x_\delta : \operatorname{dom} x_\delta \to O$ be a solution to the hybrid system $\mathcal{H}_{\delta\sigma}$. Consider a sequence $\{\delta_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converging to 0 as i goes to ∞ and suppose that a sequence $\{x_{\delta_i}\}_{i\in\mathbb{Z}_{\geq 0}}$ is locally eventually bounded with respect to O and converges graphically to a set-valued mapping x. Then, x is a solution to $\mathcal{H}$.*

From Theorem 1.4, small state-perturbations on a regular hybrid systems produces solutions that are close, in a graphical sense, to the solutions of the unperturbed system. For instance, in Example 1.7 we replaced $F(x)$ with $-\mathrm{sat}(\frac{x}{\epsilon})$ and we claimed that, under the hypothesis of uncertainties in the model, solutions to $\dot{x} = F(x)$ would be similar to the solutions to $\dot{x} = -\mathrm{sat}(\frac{x}{\epsilon})$, for small enough $\epsilon > 0$. Then, in Remark 1.2, we claimed that that approach cannot be generalized to hybrid systems. Now, by Theorem 1.4, *regular* hybrid systems exhibit a specific property of continuity with respect to small perturbations, that allows us to extend to regular hybrid system the approach of Example 1.7.

Example 1.9 Consider the hybrid system $\mathcal{H} = (O, C, D, F, G)$ of Example 1.6. That hybrid system does not satisfy the basic conditions and, as expected, an arbitrarily small perturbation produces new solutions completely unrelated to the solutions of the unperturbed system. For that example, consider now a flow set defined by the closure of C. The hybrid system $\overline{\mathcal{H}} = (O, \overline{C}, D, F, G)$, satisfies the basic conditions. Then an arbitrarily small perturbation e still produces solutions that rotates without converging but, by Theorem 1.4, they are close in a graphical sense to the solutions to the unperturbed system $\overline{\mathcal{H}}$, that rotates without converging. Moreover, the distance between solutions to the perturbed systems and solutions to the unperturbed system depends on the perturbation magnitude.

Remark 1.6 It is worth mentioning that robustness issues are related also to the problem of the simulation of a hybrid system. Classical numerical methods can be used to study the solutions to a differential equations but they introduce approximation errors due to numerical approximations. These errors can be reduced in magnitude by considering smaller simulations steps but they can never be eliminated. Therefore, if *basic conditions do not hold*, the use of numerical methods for the study of the dynamical behavior of a hybrid system may introduce new solutions (arising from the approximation errors of numerical computation) that are not solutions to the hybrid system. Fortunately, hybrid systems that *satisfy the basic conditions* guarantee that a small enough state-perturbation produces small effects on solutions. It follows that, under the basic conditions, classical numerical methods on hybrid systems produces approximated solutions that are *close* in graphical sense to some solution to the hybrid system. See [129].

1.4 Stability

Stability theory for hybrid systems parallels classical stability theory of continuous dynamical systems, with some differences related to the following problems.

1. Concepts of stability and convergence of solutions to a given set must take into account the case of solutions with bounded domains, that is, maximal solutions that are not complete. This issue does not occur on classical stability theory for which, under mild conditions on the systems data, solutions are defined for all times.

2. Consider the case of a hybrid system whose state $x = [\,\eta^T \; q\,]^T$ collects informations on the *operating condition* q of the system and informations on the *continuous motion* η of the system (see Section 1.2). In such a case, q does not converge to any specific value, ranging over over a finite set of integers. Intuitively, classical asymptotic stability would require that also q converges to some q_e. It follows that stability concepts on isolated points do not suits well on systems with logical modes.

The first problem can be addressed by considering two different notions of stability: (i) stability concepts that do not depend on completeness of solutions and (ii) stability concepts that take into account only complete solutions. The second problem can be addressed by developing a stability theory based on the central notion of set instead of a theory based on isolated points. It turns out that for a stability theory based on sets and such that solutions with bounded and unbound domains are considered together, allows to generalize classical Lyapunov theory to hybrid systems.

By following [62, 126],

Definition 1.19 Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and a compact set $\mathcal{A} \subseteq O$, then

- $\mathcal{A}$ is *stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$.
- $\mathcal{A}$ is *pre-attractive* for $\mathcal{H}$ if there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq \delta$ is bounded and for any complete solution $x(t,j)$ converges to $\mathcal{A}$, that is, $|x(t,j)|_{\mathcal{A}} \to 0$, as $t+j \to \infty$, where $(t,j) \in \operatorname{dom} x$. $\mathcal{A}$ is *attractive* if each solution from $|x(0,0)|_{\mathcal{A}} \leq \delta$ is also complete.
- $\mathcal{A}$ is *pre-asymptotically stable* if it is both stable and pre-attractive. $\mathcal{A}$ is *asymptotically stable* if it is stable and attractive.
- The *basin of pre-attraction* $\mathcal{B}_{\mathcal{A}}$ is the set of points in O from which each solution is bounded, and the complete solutions converge to $\mathcal{A}$. $\mathcal{B}_{\mathcal{A}}$ is the *basic of attraction* if is the *basin of pre-attraction* and each solution from $\mathcal{B}_{\mathcal{A}}$ is complete.
- By assuming $O \setminus (C \cup D) \subseteq \mathcal{B}_{\mathcal{A}}$, if the basin of pre-attraction $\mathcal{B}_{\mathcal{A}} = O$ then $\mathcal{A}$ is *globally pre-asymptotically stable*. In this case we will say that $\mathcal{H}$ is globally pre-asymptotically stable. $\mathcal{H}$ is *globally asymptotically stable* if it is globally pre-asymptotically stable and each solution to $\mathcal{H}$ is complete.
- $\mathcal{A}$ is *unstable* if it is not stable.

Remark 1.7 In general, the prefix *pre* is dropped from the properties above when each solution that satisfies the conditions of the property is also a complete solution. In this sense, the notion of *stability* should be denoted as *pre-stability*. We preferred to call it *stability* following the usual approach in literature [62]. Note that attractivity implies pre-attractivity and asymptotic stability implies pre-asymptotic stability.

Stability of sets can be used to exclude some element of state vector from the stability analysis. This case may occur when hybrid systems characterize processes whose behavior depends on particular *operating conditions* or modes. For example, the gear-shift of a car or a mobile robot that reacts to possible interrupts generated by its sensors.

Example 1.10 Consider a process that switches between to operative modes each second, that is, $\dot{x} = f_1(x)$ if $\exists k$, $2k \leq t \leq 2k+1$ and $\dot{x} = f_2(x)$ if

$\exists k$, $2k+1 \leq t \leq 2k+2$, where $x \in \mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq 0}$, and suppose we are interested in the asymptotic stability of the point x_e. We can characterize this process by the following hybrid system. The state of the system is in $\mathbb{R}^{n+2}$ and

$$\mathcal{H} = \begin{cases} \begin{bmatrix} \dot{x} \\ q \\ \tau \end{bmatrix} = \begin{bmatrix} F(q,x) \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} x \\ q \\ \tau \end{bmatrix} \in C \\ \begin{bmatrix} x \\ q \\ \tau \end{bmatrix}^+ = \begin{bmatrix} x \\ 2 - \mathrm{mod}(q+1,2) \\ 0 \end{bmatrix} & \begin{bmatrix} x \\ q \\ \tau \end{bmatrix} \in D \end{cases} \tag{1.24}$$

where $C = \{[\, x^T \;\; q \;\; \tau \,]^T \,|\, 0 \leq \tau \leq 1 \text{ and } 0 \leq q \leq 2\}$, $D = \{[\, x^T \;\; q \;\; \tau \,]^T \,|\, \tau \geq 1 \text{ and } 0 \leq q \leq 2\}$,

$$F(q,x) = \begin{cases} f_1(x) & \text{if } q = 1 \\ f_2(x) & \text{if } q = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1.25}$$

The analysis of stability of the point x_e of the switching process can be developed by analyzing the stability of the set

$$\mathcal{A} = \{x_e\} \times [0,1] \times [0,2] \tag{1.26}$$

of $\mathcal{H}$. For any given initial state in $C \cup D = \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times [0,2]$ the system lets time pass for at most one second and then it jumps resetting τ to zero and changing q either from 1 to 2 or from 2 to 1. During the flows period, τ grows as the time, while the motion of the state x satisfies the dynamics given by $F(q,x)$. See Figure 1.12. Thus, if the set A is asymptotically stable, then the switching process converges to x_e.

Pre-stability concepts relate classical stability concepts to solutions with bounded domains. Indeed, the pre-asymptotic stability of a set enforces the classical convergence argument only on complete solutions to hybrid systems, without requiring any convergence of solutions with bounded domains, for which it is only required a classical δ, ε boundedness. It follows that, for example, a hybrid system $\mathcal{H}$ can be stable despite the fact that its continuous dynamics is defined by a linear system whose eigenvalues are positive. In fact, it could be the case that the flow set forces each solution to $\mathcal{H}$ to have a bounded domain. In Figure 1.13 we represent the case of a planar hybrid system $\mathcal{H}$ whose continuous dynamics produces solutions that rotate and grow, and whose discrete dynamics produces solutions that converges to x_e. The point x_e is pre-asymptotically

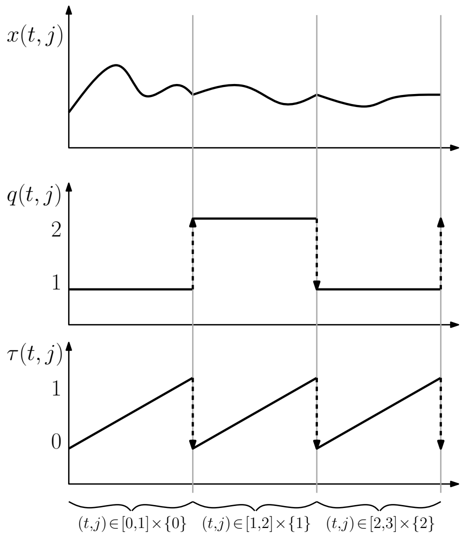


Figure 1.12: A possible solution to $\mathcal{H}$ in Example 1.10.

stable. In fact, initial states closer to x_e produce solutions that are closer to x_e. Moreover, each solution is bounded and complete solutions converge to x_e.

The stability of a compact set $\mathcal{A}$ of a hybrid system $\mathcal{H} = (O, C, D, F, G)$ can be analyzed by using Lyapunov-like tools. Following [62, 126], a function $V : O \to \mathbb{R}$ is a *Lyapunov-function candidate* for $(\mathcal{H}, \mathcal{A})$ if

(i) V is continuous and positive definite in $(C \cup D) \setminus \mathcal{A} \subseteq O$ and

(ii) V is continuously differentiable in a neighborhood of C subset of O, and

(iii) $\lim_{x\to\mathcal{A}, x\in O\cap(C\cup D)} V(x) = 0$.

Remark 1.8 Point (ii) can be relaxed by requiring V locally Lipschitz. In that case, generalized directional derivative and generalized gradient in the sense of Clarke must be used [40]. See [126].

The interpretation and the use of Lyapunov functions candidates on hybrid systems parallels the classical approach on continuous systems. In fact, conditions on function V above generalize classical Lyapunov functions candidates in the

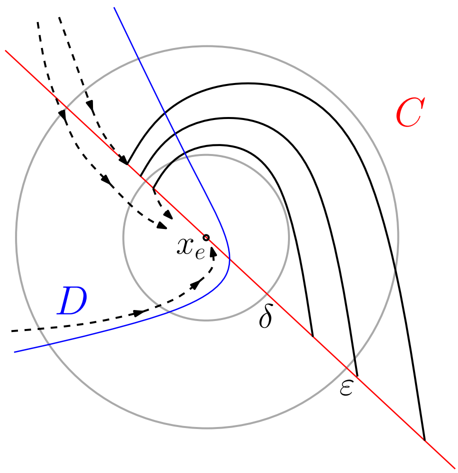


Figure 1.13: Possible solutions to the hybrid system $\mathcal{H}$ whose point x_e is pre-asymptotically stable.

following directions: (i) solutions to a hybrid system must stay within $C \cup D \setminus \mathcal{A}$ or they have a bounded domain. Therefore, continuity and nonegativity of V can be restricted to $C \cup D \setminus \mathcal{A}$; (ii) differentiability (local Lipschitzianity) of V is required only in a neighborhood of the flow set, where V and the continuous part of solutions to hybrid systems are used together; finally, (iii) V is zero on the whole set $\mathcal{A}$ following the set based approach of the definitions of stability.

As usual, the characterization of stability properties by Lyapunov-like tools is related to the analysis of the increment of the function V along a solution x to $\mathcal{H}$. With this aim, let us define $t(j) = \inf\{t \mid (t,j) \in \operatorname{dom} \xi\}$ and $j(t) = \inf\{j \mid (t,j) \in \operatorname{dom} \xi\}$. Let $x(t,j)$ be a solution to a hybrid system $\mathcal{H}$ and let $(\underline{t},\underline{j}), (\overline{t},\overline{j}) \in \operatorname{dom} x$ such that $(\underline{t},\underline{j}) \leq (\overline{t},\overline{j})$. Then, the increment of $V(x(\overline{t},\overline{j})) - V(x(\overline{t},\overline{j}))$ is given by the following equation.

$$V(x(\overline{t},\overline{j}))-V(x(\underline{t},\underline{j})) = \int_{\underline{t}}^{\overline{t}} \frac{d}{dt}V(x(t,j(t)))dt + \sum_{j=\underline{j}+1}^{\overline{j}} \left[V(x(t,j))-V(x(t(j),j-1))\right] \tag{1.27}$$

It is worth mentioning that by defining the increment of V as the sum of a *continuous increment*, the integral, and of a *discrete increment*, the sum, we can take into account the increment induced on V by any kind of solution, regardless of the particular jumping or flowing behavior of the solution. Note that the discrete increment is zero for solutions that flow only, and the continuous increment is zero for solutions that jump only.

We can now present the following theorem, that summarizes the results in [62, Theorem 20], [126, Theorem 7.6 and Corollary 7.7] and [125, Theorem 4.2 and Theorem 4.13]. For completeness of the exposition, we present the proof in Section 6.1.

Theorem 1.5 *Consider the hybrid system $\mathcal{H} = (O, C, D, F, G)$ satisfying the basic conditions 2.1 and consider a compact set $\mathcal{A} \subseteq O$. Suppose that there exists a Lyapunov-function candidate V for $(\mathcal{H}, \mathcal{A})$ and define two functions $u_C : C \to \mathbb{R}$ and $u_D : D \to \mathbb{R}$ such that*

$$\begin{aligned} u_C(x) &= \max_{f \in F(x)} \langle \nabla V(x), f \rangle && \text{for } x \in C \cap O \\ u_D(x) &= \max_{g \in G(x)} V(g) - V(x) && \text{for } x \in D \cap O. \end{aligned}$$

Then, consider a neighborhood $\mathcal{U}$ of $\mathcal{A}$

- $\mathcal{A}$ *is* stable *if*
  $u_C(x) \leq 0$ *for each* $x \in C \cap \mathcal{U}$ *and* $u_D(x) \leq 0$ *for each* $x \in D \cap \mathcal{U}$*;*
- $\mathcal{A}$ *is* pre-asymptotically stable *if*
  $\mathcal{A}$ *is stable and* $u_C < 0$ *in* $(C \setminus \mathcal{A}) \cap \mathcal{U}$ *and* $u_D < 0$ *in* $(D \setminus \mathcal{A}) \cap \mathcal{U}$*;*

Theorem 1.5 encompasses classical Lyapunov theorems for continuous systems. Consider a hybrid system $\mathcal{H} = (\mathbb{R}^n, \mathbb{R}^n, \emptyset, f, \emptyset)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ characterizes some nonlinear continuous dynamics. Then, Theorem 1.5 applied on a set $\mathcal{A} = \{x_e\}$ requires that V is a positive and continuously differentiable function, zero at zero, and such that the directional derivative $\langle \nabla V(x), f(x) \rangle$ is less then or equal to zero for each x in a neighborhood of the point $x_e = 0$. Then, x_e is a locally stable point [87, Theorem 4.1].

Remark 1.9 Conditions on functions u_C and u_D can be directly related to the increment of the function V along a solution x, in Equation (1.27). With the intuitive interpretation of Lyapunov functions as *energy* functions, negative values for $u_C(x)$ and for $u_D(x)$ force $V(x(\overline{t}, \overline{j}))$ to be lower than $V(x(\underline{t}, \underline{j}))$, that is, they guarantee that the energy of the solution is decreasing.

A slightly different version of Theorem 1.5 can be used to study global pre-asymptotic stability of sets. Following [62], consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and a compact set $\mathcal{H}$. Suppose that

– either $C \cup D$ is a compact set,

– or the sublevel sets of $V|_{O\cap(C\cup D)}$, defined by $\{x \in O \cap (C \cup D) \,|\, V(x) \leq c\}$ with $c \in \mathbb{R}_{\geq 0}$, are compact.

Then, $\mathcal{A}$ is *globally pre-asymptotically stable* if $\mathcal{A}$ is stable and $u_C < 0$ in $(C \setminus \mathcal{A}) \cap O$ and $u_D < 0$ in $(D \setminus \mathcal{A}) \cap O$.

Indeed, the compactness of $C \cup D$ or the compactness of sublevels sets of V, force each solution to be bounded, therefore to converge to $\mathcal{A}$, based on the fact that $u_C(x)$ is negative for $x \in C$ and $u_D(x)$ is negative for $x \in D$.

A lot of general results on stability of hybrid systems can be found in literature. Generalizations of global Lyapunov-like conditions for stability, $\mathcal{KLL}$-stability, invariance principle, robustness of stability and converse Lyapunov theorems, can be found in [27, 30, 31, 32, 45, 62, 63, 64, 99, 125, 126].

Further analysis of stability theory is out of the scope of this introduction to hybrid systems. Stability topics will be considered again in the next chapter, for a class of homogeneneous hybrid systems. In particular, we will present a set of local Lyapunov-like conditions for inferring global pre-asymptotic stability of systems. Then, we will prove that these conditions are mild, namely, that each globally pre-asymptotically stable system must satisfy them, and we will propose a sum of squares algorithm for deciding whether or not a given hybrid system is stable. Following a similar approach, we will define several conditions for studying instability and overshoots of solutions of such a class of hybrid system.

Chapter 2

Stability for a Class of Homogeneous Hybrid Systems by Local Lyapunov Analysis

Consider a class of hybrid systems $\mathcal{H} = (\mathbb{R}^n, C, D, F, G)$, $n \in \mathbb{Z}_{\geq 0}$, whose data satisfy the following conditions. For any given $\lambda \in \mathbb{R}_{>0}$

- for each $x \in C$, $\lambda x \in C$;
- for each $x \in D$, $\lambda x \in D$;
- for each $x \in C$ and each $f \in F(x)$, $\lambda f \in F(\lambda x)$;
- for each $x \in D$ and each $g \in G(x)$, $\lambda g \in G(\lambda x)$.

These systems can be considered as a particular subset of the class of *homogeneous hybrid systems* [153], namely hybrid systems that are homogeneous with respect to a specific generalized notion of *dilation* [151, 152]. We do not introduce the notions of dilation and of homogeneity of a hybrid system with respect to a given dilation, but interesting results on stability, robust stability, and converse results are summarized in [153].

In what follows, we present a method to study properties of *stability* of the point $x_e = 0$ for systems that satisfy the conditions above. We present also results on *overshoots* of solutions for such kinds of systems, namely when the norm of the solution ξ at some time instant exceeds the norm of the initial state of ξ. The chapter is organized as follows: in Section 2.1 we define the class of hybrid systems considered. Main theoretical results on stability, a sum of squares algorithm and an example are developed in Section 2.2. Theoretical results on overshoots of solutions and on instability properties are developed in Section 2.3. Further comments on sum of squares implementation are presented in Section 2.4.

In Section 2.2 we propose a *local Lyapunov-like approach* to the study of stability properties of such a class of hybrid systems. We define a set of Lyapunov-like conditions whose satisfaction, in a suitable subset of the state space, guarantees global pre-asymptotic stability of the point $x_e = 0$ of the system. We also show that these conditions are mild, that is, they are verified by any given hybrid system whose point x_e is pre-asymptotically stable. Based on that results, we define a sum of squares algorithm [113], that constructs a suitable function to automatically satisfies such conditions.

The use of sum of squares algorithms in control and, in particular, the use of sum of squares algorithms to construct Lyapunov functions, is well developed. See for example [111, 120, 141, 143]. Sum of squares formulations have been used in [12, 84, 112, 119] on arbitrary switching systems, on switched systems and on hybrid automata. In that works, the system dynamics is usually defined by polynomial functions $\dot{x} = f_i(x)$, or by affine functions $\dot{x} = A_i x + a_i$, for $i \in \{1, \dots, N\}$, each of them enabled in a subset of the state-space [84, 112, 119] or enabled by a particular updating rule based on the state value [12]. The stability analysis in [12, 84, 119] is developed for a part of the state that never jumps. Systems in [112] allow resets of the state, provided that no solutions are Zeno or discrete. Stability is characterized by constructing continuous and piecewise continuous Lyapunov functions.

The functions constructed by our algorithm satisfy usual Lyapunov conditions for pre-asymptotic stability [62] but only in a suitable subset of the state space. Then, for the class of systems considered, it is possible to generalize those "local" Lyapunov functions to the whole space, guaranteeing global pre-asymptotic stability. Our approach can be used on systems with Zeno and discrete solutions and produces smooth functions, that may exhibit non-convex level sets in the subset of the state space in which they resemble to classical Lyapunov functions. It follows that our method can be applied on global pre-asymptotically stable systems for which a convex Lyapunov function does not exist, see [23]. Finally, based on recent results on homogeneous approximations of hybrid systems [64], our method can be used on a properly defined homogeneization of general hybrid systems to infer local pre-asymptotic stability of the point $x_e = 0$ of that systems.

Within the considered class of hybrid system, in Section 2.3 we study the behavior of hybrid solutions in the neighboorhod of the point $x_e = 0$. We analyze the following cases.

1. Solutions that do not satisfy the classical (δ, ε) argument of stability concepts, that is, solutions ξ for which there exists an $\varepsilon \in \mathbb{R}_{>0}$ and a set $\mathcal{U} \subset \mathbb{R}^n$, $x_e \in \mathcal{U}$, such that for each $\delta \in \mathbb{R}_{>0}$, if $\xi(0,0) \in \mathcal{U} \cap \delta\mathbb{B}$ then, for some $(T, J) \in \operatorname{dom} \xi$, $\xi(T, J) \notin \varepsilon\mathbb{B}$, no matter how small δ is.

2. Solutions that grow unbounded from a suitable subset of the state-space, that is, solutions ξ such that for any given $\varepsilon \in \mathbb{R}_{>0}$, there exists a set $\mathcal{U} \subset \mathbb{R}^n$, $x_e \in \mathcal{U}$, such that for each $\delta \in \mathbb{R}_{>0}$ if $\xi(0,0) \in \mathcal{U} \cap \delta\mathbb{B}$ then, for some $(T, J) \in \operatorname{dom} \xi$, $\xi(T, J) \notin \varepsilon\mathbb{B}$, no matter how big ε is.

3. Solutions that grow by a factor ρ, that is, solutions ξ for which there exists a set $\mathcal{U} \subset \mathbb{R}^n$ and a $\rho \in \mathbb{R}_{>1}$, such that if $\xi(0,0)$ in $\mathcal{U}$ then then $|\xi(T, J)| > \rho|\xi(0,0)|$. Such behavior is denoted as *overshoot*.

Point 1 is analyzed by proposing a Chetaev-like theorem [87, Theorem 4.3] generalized to the hybrid systems framework. Points 2 and 3 are addressed following a Lyapunov-like approach, by defining a set of conditions whose satisfaction, in a suitable subset of the state-space, guarantees 2 or 3. Based on these results, we propose two sum of squares algorithms that construct a suitable function to automatically satisfy those conditions. Note that the point $x_e = 0$ of a given hybrid system is unstable if some solution either satisfies point 1 or satisfies point 2. Point 3 can be related to the shape of a possible Lyapunov function.

A study of solutions behavior with sum of squares, not related to stability problems, can be found in [118], where safety problems are taken into account (namely problems in which solutions *must not enter* a given subset of the state space or they *must reach* some particular subset of the state space). A similar approach based on approximations of solutions with polyhedra is proposed in [36] Here we propose an approach to study the solutions to a hybrid system in the neighborhood of the point $x_e = 0$. Based on this analysis, if some solution either satisfies 1 or satisfies 2 then the point $x_e = 0$ is unstable. Intuitively, 3 is related to the properties of convergence of solutions to the point x_e..

Finally, a correlation between the results of Sections 2.2 and 2.3 is proposed at the end of Section 2.3 and some remarks on possible problems of sum of square implementation is in Section 2.4.

2.1 The Class of Hybrid Systems

We consider a particular class of hybrid systems in which flow set and jump set are defined as the union of closed polyhedral cones, and flow map and jump map are defined, respectively, as the convex hull and the union of several linear vector fields. Indeed, let i be an index number in $\mathbb{Z}_{\geq 0}$, and let $R^{(i)}$ be a closed set defined as follows

$$R^{(i)} = \left\{ x \,\middle|\, \begin{bmatrix} m_1^{(i)} \\ \dots \\ m_{r^{(i)}}^{(i)} \end{bmatrix} x \geq 0 \right\} \tag{2.1}$$

where $r^{(i)}$ belongs to $\mathbb{Z}_{\geq 0}$ and $m_j^{(i)} \in \mathbb{R}^{1\times n}$ is a row vector, for each $j = 1, \dots, r^{(i)}$. Then, C and D can be defined as

$$C = \bigcup_{i\in I_C} R^{(i)} \qquad D = \bigcup_{i\in I_D} R^{(i)} \tag{2.2}$$

where I_C, I_D are disjoint and finite index sets. Note that C and D can overlap. Note also that it is possible to have $C \cup D \neq \mathbb{R}^n$.

In a similar way, consider set-valued mappings $F_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, for $i \in I_C$, and $G_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, for $i \in I_D$, defined as follows. For each $i \in I_C$, $F_i(x)$ is a convex and closed set defined by

$$F_i(x) = \begin{cases} \overline{\text{co}}\{f \,|\, f = F_{ik}x \text{ for } k = 1 \dots r_F\} & \text{if } x \in R^{(i)} \\ \emptyset & \text{otherwise} \end{cases} \tag{2.3}$$

where $F_{ik} \in \mathbb{R}^{n\times n}$ and $r_F \in \mathbb{Z}_{\geq 0}$. For each $i \in I_D$, $G_i(x)$ is a set defined by

$$G_i(x) = \begin{cases} \{g \,|\, g = G_{ik}x \text{ for } k = 1 \dots r_G\} & \text{if } x \in R^{(i)} \\ \emptyset & \text{otherwise} \end{cases} \tag{2.4}$$

where $G_{ik} \in \mathbb{R}^{n\times n}$ and $r_G \in \mathbb{Z}_{\geq 0}$. Then, flow and jump mappings, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, can be defined as

$$F(x) = \overline{\text{co}} \bigcup_{i\in I_C} F_i(x) \qquad G(x) = \bigcup_{i\in I_D} G_i(x) \tag{2.5}$$

Note that $F(x)$ reduces to $F_i(x)$ when x belongs only to one cone $R^{(i)}$, for some $i \in I_C$. The same holds for $G(x)$.

Hybrid systems of the form (1.5),(2.1)-(2.5) satisfy the basic conditions, as stated in the following Claim.

Claim 2.1 *A hybrid system $\mathcal{H}$ of Equations* (1.5),(2.1)-(2.5) *satisfies the* basic conditions *(Definition 1.9).*

Proof. See Appendix 6.2.1 □

Remark 2.1 Switched linear systems with state dependent switching policies, [94, Sections 3.3 and 3.4], can be characterized within the family of hybrid systems considered above. For example, consider the system

$$\dot{x} = A_i x \;\text{ if }\; x \in C_i, \qquad i = 1, \dots, N.$$

where $N \in \mathbb{Z}_{\geq 0}$ and, for each $i = 1, \dots, N$, $A_i \in \mathbb{R}^{n \times n}$ and C_i is a conic subset of $\mathbb{R}^n$. Such systems can be easily defined within the class of hybrid systems considered above, by defining $F_i(x) = A_i x$ if $x \in C_i$ and $F_i(x) = \emptyset$ otherwise, for each $i = 1, \dots, N$. In such a case, $D = \emptyset$. Moreover, switched linear systems under arbitrary switching policies, [94, Section 2.1.4], can be written as hybrid systems (1.5),(2.1)-(2.5), based on a single differential inclusion of the form (2.3), defined by the convex hull of the linear vector fields of the switched linear system, and $C = \mathbb{R}^n$.

2.2 Stability

2.2.1 Main Results

In what follows we show some results on stability of the point $x_e = 0$ of hybrid systems of Equations (1.5),(2.1)-(2.5). Indeed, we study the stability of the set $\mathcal{A} = \{x_e\}$ by following a Lyapunov-like approach, namely, by using a suitable selected function V that satisfies a defined set of conditions. Then, we present an algorithm to effectively construct that function V.

Global pre-asymptotic stability of a hybrid system of Equations (1.5),(2.1)-(2.5) can be inferred from a "local" analysis of the system by finding a function $V : \mathbb{R}^n \to \mathbb{R}$ that respects some specific properties on the set $\{x \mid c \leq |x| \leq \rho c\}$, where $c \in \mathbb{R}_{\geq 0}$ and $\rho \in \mathbb{R}_{>1}$. The "local" satisfaction of those properties will guarantee global pre-asymptotic stability of the system, as stated in the following theorem.

Definition 2.1 A function $\vartheta : \mathbb{R}^n \to \mathbb{R}$ is said to be *homogeneous of degree* k if for some $k \in \mathbb{Z}_{\geq 0}$ and for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}_{\geq 0}$, $\vartheta(\lambda x) = \lambda^k \vartheta(x)$,

Theorem 2.1 *For a hybrid system* $\mathcal{H}$ *of Equations* (1.5),(2.1)-(2.5), *suppose that there exist a function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and constants* $c \in \mathbb{R}_{\geq 0}$ *and* $\rho \in \mathbb{R}_{>1}$ *such that,*

– *for each* x *in* $\{x \mid c \leq |x| \leq \rho c\}$,

1. $V(x) \geq 0 \qquad x \in C \cup D$;
2. $\langle \nabla V(x), f \rangle < 0 \qquad x \in C, \forall f \in F(x)$;
3. $V(g) - V(x) < 0 \qquad x \in D, \forall g \in G(x)$;

– *there exist* $\ell_1, \ell_2 \in \mathbb{R}_{>0}$, $\ell_1 < \ell_2$,

(4) $\max_{|x|=c} V(x) \leq \ell_1$ *and* $\min_{|x|=\rho c} V(x) \geq \ell_2$;

(5) if $x \in D \cap \{x \mid c \leq |x| \leq \rho c\} \cap \{x \mid V(x) \leq \ell_2\}$ *and* $g \in G(x)$ *then* $|g| \leq \rho c$;

(6) if $x \in D \cap \{x \mid |x| \leq c\}$ *and* $\forall g \in G(x)$ *then* $g \notin \{x \mid V(x) > \ell_1, c \leq |x| \leq \rho c\}$;

– *the restriction of* V *to* $\{x \mid c \leq |x| \leq \rho c\} \cap (C \cup D)$ *is a smooth function.*

Then, for any given constant $k \in \mathbb{R}_{>0}$, *there exists a function* $\overline{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and some constants* $a_2 \geq a_1 > 0$, $\mu > 0$, $0 < \nu < 1$ *in* $\mathbb{R}$ *such that* $\overline{V}$ *is a homogeneous function, smooth in* $\mathbb{R}^n \setminus \{0\}$, *and*

$$
\begin{array}{cc}
a_1|x|^k \leq \overline{V}(x) \leq a_2|x|^k & \forall x \in C \cup D \\
\left\langle \nabla \overline{V}(x), f \right\rangle \leq -\mu \overline{V}(x) & \forall x \in C, \forall f \in F(x) \\
\overline{V}(g) \leq \nu \overline{V}(x) & \forall x \in D, \forall g \in G(x)
\end{array}
\tag{2.6}
$$

Proof. The proof follows the line of reasoning of the proof of [153, Theorem 2]. A function $\overline{V}$ is constructed by integration as suggested in [124], and we prove that $\overline{V}$ is a smooth and homogeneous Lyapunov function that satisfies (2.6). See Appendix 6.2.2, for details. □

The meaning of Conditions (1)-(6) of the theorem can be explained by considering Figure 2.1, in which we summarize the case of a planar hybrid system for which conditions (1)-(6) are satisfied. In general, Conditions (1)-(3) can be interpreted as usual Lyapunov conditions for pre-asymptotic stability but, in this case, each inequality must be satisfied only for $c \leq |x| \leq \rho c$. For extending those Lyapunov-like but "local" properties in Conditions (1)-(3), to global properties in (2.6), we need some extra-conditions on function $V(x)$ and on the dynamics of the system. With this aim, $\ell_1 < \ell_2$ and Condition (4) force $V(x)$ to be smaller at $|x| = c$ than at $|x| = \rho c$, guaranteeing the existence of two level sets of $V(x)$ that surround the origin, both contained in the annulus $\{x \,|\, c \leq |x| \leq \rho c\}$. This is represented in Figure 2.1 by the closed curves with labels ℓ_1 and ℓ_2. By looking at the figure, Condition (5) ensures that no jumps from a state within the set enclosed by ℓ_2 can bring the state out of ρc. In a similar way, Condition (6) establishes that no jumps are allowed from a point inside the circle of radius c, say x, to a point, say g, such that $V(g) > \ell_1$.

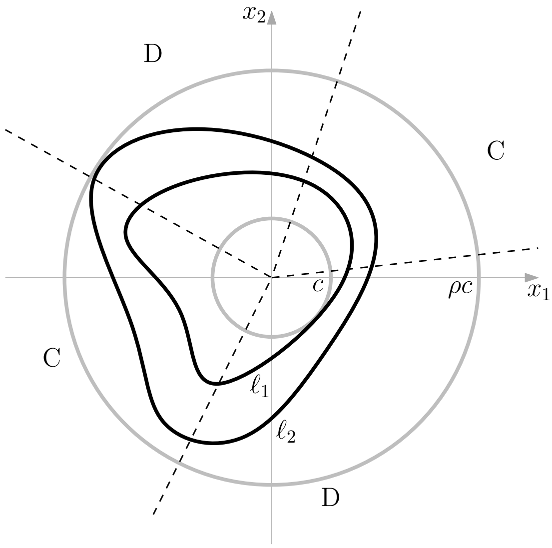


Figure 2.1: A function V that satisfies the conditions of Theorem 2.1 for a planar hybrid system.

Note that the conditions in Theorem 2.1 are quite mild, as stated in the following theorem.

Theorem 2.2 *For a a hybrid system $\mathcal{H}$ of Equations* (1.5),(2.1)-(2.5)*, if the point $x_e = 0$ is globally pre-asymptotically stable, then there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies Conditions (1)-(6) of Theorem 2.1, for some $c \in \mathbb{R}_{\geq 0}$ and $\rho \in \mathbb{R}_{>1}$.*

Proof. By the converse result in [32, Theorem 3.14], for any global pre-asymptotically stable hybrid system $\mathcal{H}$, there exists a smooth Lyapunov function V that satisfies Theorem 1.5. V satisfies also the conditions of Theorem 2.1, for some given $c \in \mathbb{R}_{\geq}0$ and $\rho \in \mathbb{R}_{>1}$. See the appendix, section 6.2.2, for details. □

By Theorem 2.1, the global stability of a hybrid system of Equations (1.5),(2.1)-(2.5) can be deduced by a local analysis of the system, i.e. by finding a suitable function that works as a Lyapunov function in $c \leq |x| \leq \rho c$ and satisfies some other requirements. In what follows we propose a sum of squares algorithm, [113, 143], for constructing a function V that satisfies the conditions of Theorem 2.1. Therefore, by looking at (2.6) in Theorem 2.1, if this algorithm succeeds in the construction of V for a hybrid system $\mathcal{H}$, then the point x_e of $\mathcal{H}$ is global pre-asymptotically stable.

Remark 2.2 Following [64], for a general hybrid system $\mathcal{H}$, (1.5), that satisfies the basic conditions, *local pre-asymptotic stability* of the point $x_e = 0$ can be deduced from the pre-asymptotic stability of the point $x_e = 0$ of a suitable approximation $\mathcal{H}_L$ of $\mathcal{H}$. Therefore, Theorem 2.1 can be used to infer local pre-asymptotic stability of the point x_e of general hybrid systems $\mathcal{H}$ whose approximations $\mathcal{H}_L$ is definable within the class of hybrid systems of Equations (1.5),(2.1)-(2.5). Indeed, following [64, Theorem 3.16], Theorem 2.1 can be applied to homogeneous approximations $\mathcal{H}_L$ of hybrid systems $\mathcal{H}$ based on a dilation $M(\lambda) = \lambda I$ ([64, Definition 3.7]), such that (i) the tangent cones $T_C(x_e)$ and $T_D(x_e)$ are polyhedral cones ([64, Definition 3.9]) and (ii) the set-valued mappings $\overline{\overline{\text{co}}}F^{M,0}$ and $G^{M,0}$ are definable as combinations of a finite number of linear vectors field ([64, Definition 3.13]).

2.2.2 Sum of Squares Algorithm

In this section we present an algorithm for finding a function V that satisfies the conditions of Theorem 2.1 for hybrid systems of Equations (1.5), (2.1)-(2.5). The general idea is to construct a set of polynomial inequalities that imply the conditions of the theorem. Then, a solution to such a set of inequalities is computed by (i) relaxing each inequality to a sum of squares decomposition, and by (ii) using a semidefinite program solver for seeking a solution to the whole sum of squares decomposition problem. Of course, some conservativeness is introduced [120, 113]. Algorithm 1 works as follows.

(i) The input of the algorithm is filled by the data of the hybrid system $\mathcal{H}$ and by the parameters $\varepsilon, c, \rho, d_1$ and d_2, as stated in `INPUT`.

(ii) A set of inequalities is then constructed, parameterized on $\mathcal{H}$ and on $\varepsilon, c, \rho, d_1$ and d_2, as stated in `CONSTRAINTS`. It should be noted that

(iii) A solution is computed by relaxing the satisfiability problem of the whole set of inequalities to a sum of squares decomposition problem. A semidefinite program solver runs over this problem.

(iv) If the solver finds a solution, the set of inequalities is feasible and the algorithm ends positively, as stated in `OUTPUT`.

For the description of the algorithm the following definitions are needed.

Definition 2.2 Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{\geq 0}$ be two constants and let $\Delta_1(\varepsilon_1, \varepsilon_2, \cdot) : \mathbb{R}^n \to \mathbb{R}$ be a map defined with respect to ε_1 and ε_2 as follows

$$\Delta_1(\varepsilon_1, \varepsilon_2, x) = -(|x|^2 - \varepsilon_1^2)(|x|^2 - \varepsilon_2^2) \tag{2.7}$$

Definition 2.3 For any given $i \in I_C \cup I_D$, the function $\Delta_2^{(i)}(x) : \mathbb{R}^n \to \mathbb{R}$ is defined as follows

$$\begin{aligned}
&\Delta_2^{(i)}(x) = \\
&= \sum_{j=1}^{r^{(i)}} p_j(x)\, m_j^{(i)} x + \sum_{j=1}^{r^{(i)}} \sum_{k=j+1}^{r^{(i)}} p_{jk}(x)\, m_j^{(i)} x\, m_k^{(i)} x + \\
&+ \sum_{j=1}^{r^{(i)}} \sum_{k=j+1}^{r^{(i)}} \sum_{h=k+1}^{r^{(i)}} p_{jkh}(x)\, m_j^{(i)} x\, m_k^{(i)} x\, m_h^{(i)} x + \ldots + \\
&+ p_{1,2,\ldots,r}(x) m_1 x\, m_2 x \cdot \ldots \cdot m_{r^{(i)}} x
\end{aligned} \tag{2.8}$$

where, for any given combination of indices j,k,$\ldots$, $p_j, p_{jk}, \ldots$ denote functions in $\mathbb{R}^n \to \mathbb{R}_{\geq 0}$, defined by polynomials of a given degree. We refer to the whole set of polynomials $p_j, p_{jk}, \ldots$ by using the name of *slack polynomials.*

$\Delta_1(\varepsilon_1, \varepsilon_2, x)$ is positive for $\varepsilon_1 \leq |x| \leq \varepsilon_2$, and is strictly negative otherwise. For each $i \in I_C \cup I_D$, $\Delta_2^{(i)}(x)$ is positive for each x in $R^{(i)}$ while it is possibly negative if $x \notin R^{(i)}$, based on the particular configuration of slack polynomials. In the following algorithm, Δ_1 and Δ_2 are used for relaxing the conditions on V to hold only in a subset of $\mathbb{R}^n$. A planar example of subset of $\mathbb{R}^n$ with positive Δ_1 and Δ_2 is in Figure 2.2.

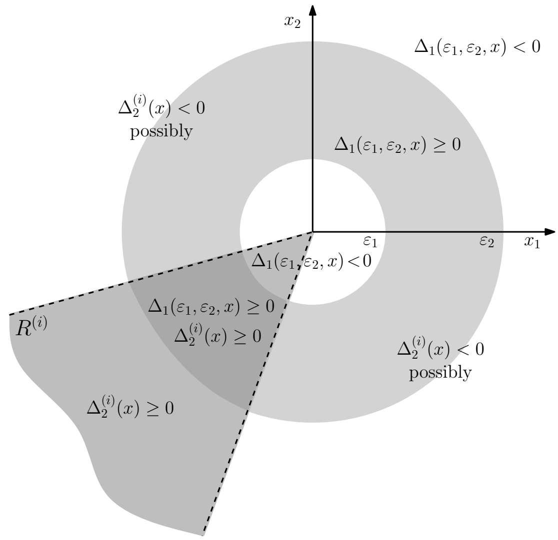


Figure 2.2: Subsets of the state-space related to the sign of Δ_1 and Δ_2.

Algorithm 1

`INPUT:`
Data $\langle F, G, C, D\rangle$ of the hybrid system $\mathcal{H}$;
constants $\varepsilon, c, \rho \in \mathbb{R}_{>0}$, satisfying $\varepsilon \ll c$ and $\rho > 1$;
constants $d_1, d_2 \in \mathbb{Z}_{\geq 0}$, satisfying $d_1 \geq d_2$.

`OUTPUT:`
Feasibility of the sum of squares problem.

`VARIABLES:`
Scalar variables ϵ, ℓ_1, ℓ_2;
polynomials $V(x)$, $s_4(x)$, $s_5(x)$, $s_1^{(i)}(x)$, for each $i \in I_C \cup I_D$, $s_2^{(ik)}(x)$, for each $i \in I_C$ and each $k = 1, \ldots, r_F$, $s_3^{(ik)}(x)$, $s_6^{(ik)}(x)$, $s_7^{(ik)}(x)$, $s_8^{(ik)}(x)$, $s_9^{(ik)}(x)$, for each i in I_D and each $k = 1, \ldots, r_G$, and all the slack polynomials.

`CONSTRAINTS:`
$V(x)$ is a polynomial of degree d_2. ϵ is a scalar variable.

• $\forall i \in I_C \cup I_D$, $s_1^{(i)}(x)$ is a polynomial of degree d_1 and

$$\begin{aligned} V(x) - \Delta_2^{(i)}(x) - s_1^{(i)}(x)\Delta_1(c, \rho c, x) \geq 0 \\ s_1^{(i)}(x) \geq 0 \end{aligned} \tag{2.9}$$

• $\forall i \in I_C$, $\forall k \in \{1 \ldots, r_F\}$, $s_2^{(ik)}(x)$ is a polynomial of degree d_1 and

$$\begin{aligned} -\nabla V(x)^T F_{ik} x - \Delta_2^{(i)}(x) - s_2^{(ik)}(x)\Delta_1(c, \rho c, x) > 0 \\ s_2^{(ik)}(x) \geq 0 \end{aligned} \tag{2.10}$$

• $\forall i \in I_D$, $\forall k \in \{1 \ldots, r_G\}$, $s_3^{(ik)}(x)$ is a polynomial of degree d_1 and

$$\begin{aligned} V(x) - V(G_{ik}x) - \Delta_2^{(i)}(x) - s_3^{(ik)}(x)\Delta_1(c, \rho c, x) > 0 \\ s_3^{(ik)}(x) \geq 0 \end{aligned} \tag{2.11}$$

• $s_4(x)$ and $s_5(x)$ are polynomials of degree d_1, ℓ_1 and ℓ_2 are scalar variables and

$$\begin{aligned} \ell_1 - V(x) - s_4(x)\Delta_1(c, c + \varepsilon, x) \geq 0 \\ V(x) - \ell_2 - s_5(x)\Delta_1(\rho c - \varepsilon, \rho c, x) \geq 0 \\ \ell_2 - \ell_1 > 0 \\ s_4(x), s_5(x), \ell_1, \ell_2 \geq 0 \end{aligned} \tag{2.12}$$

• $\forall i \in I_D$, $\forall k \in \{1 \ldots, r_G\}$, $s_6^{(ik)}(x)$, $s_7^{(ik)}(x)$, $s_8^{(ik)}(x)$ and $s_9^{(ik)}(x)$ are polynomials of degree d_1, ℓ_1, ℓ_2 are scalar variables and

$$\begin{aligned} V(x) - \ell_2 - s_6^{(ik)}(x)(x' G_{ik}^T G_{ik} x - \rho^2 c^2) + & \\ -\Delta_2^{(i)}(x) - s_7^{(ik)}(x)\mathit{\Delta}_1(c, \rho c, x) \geq 0 & \\ \ell_1 - V(G_{ik}x) - s_8^{(ik)}(x)(c^2 - x^T x) - \Delta_2^{(i)}(x) + & \\ -s_9^{(ik)}(x)\Delta_1(c, \rho c, G_{ik}x) \geq 0 & \\ s_6^{(ik)}(x), s_7^{(ik)}(x), s_8^{(ik)}(x), s_9^{(ik)}(x) \geq 0 & \end{aligned} \tag{2.13}$$

• For each use of $\Delta_2^{(i)}(x)$ in (2.9)-(2.11) and (2.13) a *new fresh* set of slack polynomials must be used. Moreover for each slack polynomial, say $p(x)$, a new inequality $p(x) \geq 0$ is added.

Remark 2.3 The last bullet of Algorithm 1 requires a new set of slack polynomials for each use of $\Delta_2^{(i)}(x)$. For example, the slack polynomials of $\Delta_2^{(i)}(x)$ used in an inequality that involves G_{ik_1} in (2.11) must not be confused with slack polynomials of $\Delta_2^{(i)}(x)$ used in an inequality that involves G_{ik_2} in (2.11), with $k_1 \neq k_2$.

Each inequality of Algorithm 1 is constructed by considering two goals: the first part of the left-hand side of each inequality is used to enforce some constraint on V so that V satisfies the conditions of Theorem 2.1; the second part of the left-hand side of each inequality uses Δ_1 and $\Delta_2^{(i)}$, for $i \in I_C \cup I_D$, to guarantee that V satisfies some constraints only in a subset of $\mathbb{R}^n$, leaving V basically unconstrained in the rest of the space.

Consider now to run Algorithm 1 for some given hybrid system $\mathcal{H}$, and to find a feasible solution.

- By (2.9), $V(x)$ is a non-negative function in $(C \cup D) \cap \{x \,|\, c \leq |x| \leq \rho c\}$.
- Inequalities (2.10) and (2.11) guarantee that (i) the directional derivative of $V(x)$, $\langle \nabla V(x), f\rangle$, is negative for each x in the set $C \cap \{x \,|\, c \leq |x| \leq \rho c\}$ and each $f \in F(x)$, and (ii) the increment of $V(x)$, $V(g) - V(x)$, is negative for each x in the set $D \cap \{x \,|\, c \leq |x| \leq \rho c\}$ and each $g \in G(x)$.
- The first inequality of (2.12) implies $\max_{c \leq |x| \leq c+\varepsilon} V(x) \leq \ell_1$. The second inequality of (2.12) implies $\min_{\rho c-\varepsilon \leq |x| \leq \rho c} V(x) \geq \ell_2$. Note that $\ell_1 < \ell_2$ by the third inequality.
- The first inequality of (2.13) guarantees that a hybrid arc of $\mathcal{H}$ cannot escape the set $\{x \,|\, |x| \leq \rho c\}$ by a jump from $D \cap \{x \,|\, c \leq |x| \leq \rho c\} \cap \{x \,|\, V(x) \leq \ell_2\}$. The second inequality of (2.13) guarantees that a hybrid arc of $\mathcal{H}$ cannot jump to $\{x \,|\, V(x) > \ell_1\} \cap \{x \,|\, c \leq |x| \leq \rho c\}$ from the set $\{x \,|\, |x| \leq c\}$.

It follows that a feasible solution to the set of constraints produces a function V that satisfies the conditions of Theorem 2.1, as stated in the following proposition.

Proposition 2.1 *For any given hybrid system $\mathcal{H}$ defined by Equations* (1.5),(2.1)-(2.5)*, if the set of inequalities of Algorithm 1 has a feasible solution for some parameters $c \in \mathbb{R}_{\geq 0}$ and $\rho \in \mathbb{R}$, $\rho > 1$, then the function V constructed by Algorithm 1 satisfies the conditions of Theorem 2.1 with the same c and ρ.*

Proof. See the appendix, section 6.2.2. □

Remark 2.4 Despite the number of indices i, j, k used during the description of Algorithm 1, in practical cases the algorithm is much more simple. For example, the switched systems in [94, Sections 3.3 and 3.4] require a single matrix F_i for each cone $R^{(i)}$. Therefore $k = 1$ in (2.10). Note also that for a hybrid system with $C \cup D = \mathbb{R}^n$, we can replace the conditions of the form (2.9), where $i \in I_C \cup I_D$, with a single condition $V(x) - s_1(x)\Delta_1(c, \rho c, x) \geq 0$, $s_1(x) \geq 0$.

Remark 2.5 $V(x)$ must be positive only in the set $(C \cup D) \cap \{x, |\, c \leq |x| \leq \rho c\}$, for some c and some ρ given as input of Algorithm 1. In the rest of the space, $V(x)$ is basically unconstrained. This kind of "local" requirement introduces some extra-degree of freedom during the construction of $V(x)$, due to the fact that $V(x)$ must be positive only in a subset of $\mathbb{R}^n$. This allows the solver to find a polynomial $V(x)$ that can be non-positive near the origin and non-positive far from the origin, i.e. with low order terms and high order terms not necessarily positive. The effect of this "local" requirement is to allow for the construction of functions $V(x)$ with not necessarily convex level sets. In fact, it could be the case that hybrid systems characterized by (1.5),(2.1)-(2.5) do not admit convex Lyapunov functions, as shown in [23]. Therefore the possibility of constructing a "local" Lyapunov functions with non-convex level-set is important.

2.2.3 Example

We use Algorithm 1 for studying the stability of a hybrid system $\mathcal{H}$ of Equation (1.5),(2.1)-(2.5) defined by the following quantities:

$C = C_1 \cup C_2$ where $C_1 = \{x \,|\, M_1 x \geq 0\}$, $C_2 = \{x \,|\, M_2 x \geq 0\}$, and $D = \{x \,|\, M_3 x \geq 0\}$.

$$\begin{array}{rlll} F(x) = & \left\{ \begin{array}{l} F_1 x \\ \overline{\text{co}}\{F_1 x, F_2 x\} \\ F_2 x \end{array} \right. & \begin{array}{l} \textit{if} \\ \textit{if} \\ \textit{if} \end{array} & \begin{array}{l} x \in C_1 \setminus C_2 \\ x \in C_1 \cap C_2 \\ x \in C_2 \setminus C_1 \end{array} \\ G(x) = & \quad Gx & \textit{if} & x \in D. \end{array} \tag{2.14}$$

where $F_1 = \begin{bmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$, $F_2 = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} \end{bmatrix}$, $G = \begin{bmatrix} e^{\frac{1}{10}} & 0 \\ 0 & e^{-\frac{1}{10}} \end{bmatrix}$,
$M_1 = \begin{bmatrix} 1 & \frac{1}{200} \\ 1 & 2 \end{bmatrix}$, $M_2 = \begin{bmatrix} -1 - 2 \end{bmatrix}$, $M_3 = \begin{bmatrix} -1 & -\frac{1}{200} \\ 1 & 2 \end{bmatrix}$.

Some level sets of the function $V(x)$ constructed by Algorithm 1 for $c = 0.5$ and $\rho = 4.4$ are summarized in Figure 2.3. Note that part of the dynamics of $\mathcal{H}$ can be considered as an adaptation to the hybrid system framework of [23, Section 3].

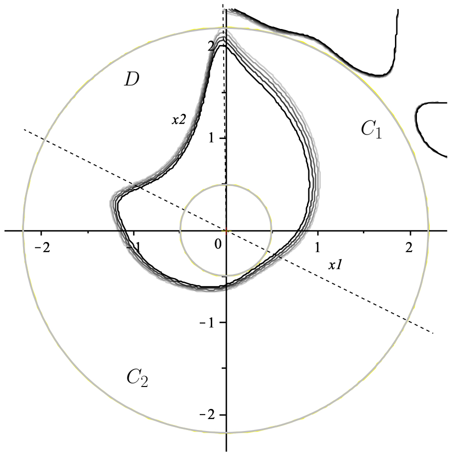


Figure 2.3: Some level sets of $V(x)$, Algorithm 1, Example 2.2.3.

2.3 Overshoots and Instability

2.3.1 Main Results

Both Theorem 2.1 and Algorithm 1 are parameterized by the constants c and ρ. Such constants are used to define a particular subset of $\mathbb{R}^n$ and are connected to the solutions of the hybrid system. In fact, by Theorem 2.1, if the point x_e of the system is unstable then c and ρ cannot be found. Moreover, consider a hybrid system $\mathcal{H}$ with an asymptotically stable point x_e and suppose that, for some c and ρ, there exists a solution x to $\mathcal{H}$ such that $|x(0,0)| = c$ and $|x(t,j)| > \rho c$, for some $(t,j) \in \operatorname{dom} x$. For that case, we cannot find a function V that satisfies the conditions of Theorem 2.1, i.e. the set of inequalities of Algorithm 1 would be infeasible (intuitively, such a hybrid arc would cross any level set contained in $c \leq |x| \leq \rho c$).
We take into account these cases by showing results on instability of the point $x_e = 0$ of $\mathcal{H}$ and results on "overshoots" of solutions to $\mathcal{H}$, namely, the behavior of a solution ξ to move from $|\xi(0,0)| = c$ to $|\xi(t,j)| < \rho c$, with $c \in \mathbb{R}_{>0}$ and $\rho \in \mathbb{R}_{>1}$, and for some $(t,j) \in \operatorname{dom} \xi$. Theorem 2.3 and Corollary 2.1 below summarize the results on instability while the overshoots problem is addressed in Theorem 2.4.

The following theorem is a generalization of Chetaev Theorem [87, Theorem

4.3] to hybrid systems of Equations (1.5),(2.1)-(2.5). Then, it can be used to characterize *unstable* points and it is related to *Case 1* of the introduction.

Theorem 2.3 *(Chetaev-like theorem) Consider a hybrid system* $\mathcal{H}$ *of Equations* (1.5),(2.1)-(2.5) *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function in* $C \cup D$ *such that* $V(0) = 0$ *and* $V(x) > 0$ *for some* $x \in C \cup D$ *with arbitrarily small* $|x|$*. Choose* $r \in \mathbb{R}_{>0}$ *and define* $U = \{x \in C \cup D \,|\, V(x) > 0,\, |x| \leq r\}$*. Suppose*

1. $\langle \nabla V(x), f \rangle > 0 \qquad \forall x \in C \cap U, \forall f \in F(x)$;
2. $V(g) - V(x) > 0 \qquad \forall x \in D \cap U, \forall g \in G(x)$;
3. *Each maximal solution* ξ *to* $\mathcal{H}$ *with initial state* $\xi(0,0) \in U$ *is complete.*

Then $x_e = 0$ *is unstable.*

Proof. By using completeness of solutions, the proof of Theorem 2.3 can be developed by following the proof of [87, Theorem 4.3]. See Appendix 6.2.3 for details. □

Remark 2.6 If $C \cup D = \mathbb{R}^n$ then each maximal solution to $\mathcal{H}$ is a complete solution to $\mathcal{H}$ or it escapes in finite time from any compact set. In such a case, condition (3) of Theorem 2.3 is not needed. Note also that Theorem 2.3 can be used on general hybrid systems of Equation (1.5) that satisfy the conditions of Claim 2.1.

The following theorem can be used to study "overshoots" of solutions to hybrid systems of Equations (1.5),(2.1)-(2.5). The theorem is parameterized with respect to c and ρ and it guarantees the existence of at least one solution ξ to $\mathcal{H}$ such that $c \leq |\xi(0,0)| \leq c + \varepsilon$, with ε small, and $|\xi(T,J)| \geq \rho c$, for some $(T,J) \in \operatorname{dom} \xi$. Theorem 2.4 is related to *Case 3* of the introduction of this chapter.

Theorem 2.4 *Consider a hybrid system* $\mathcal{H}$ *of Equations* (1.5),(2.1)-(2.5)*. Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function such that for some* $\ell \in \mathbb{R}_{>0}$ *and* $c \in \mathbb{R}_{\geq 0}$*,*

1. $\max_{|x|=c} V(x) \leq \ell$;
2. *there exist* $\varepsilon \in \mathbb{R}_{>0}$ *and* $x \in C \cup D$ *such that* $|x| = c + \varepsilon$ *and* $V(x) > \ell$

Choose $\rho \in \mathbb{R}_{>1}$ *and define* $U = \{x \in C \cup D \,|\, V(x) > \ell,\, c \leq |x| \leq \rho c\}$*. Suppose*

(3) $\langle \nabla V(x), f \rangle > 0 \qquad \forall x \in C \cap U, \forall f \in F(x);$

(4) $V(g) - V(x) > 0 \qquad \forall x \in D \cap U, \forall g \in G(x);$

(5) $|g| > c \qquad \forall x \in D \cap U, \forall g \in G(x);$

(6) Each maximal solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) \in U$ is complete.

Then, for each $\lambda \in \mathbb{R}_{>0}$, there exists a solution ξ to $\mathcal{H}$ such that if $|\xi(0,0)| = \lambda(c+\varepsilon)$ then $|\xi(T,J)| \geq \lambda\rho c$, for some $(T,J) \in \operatorname{dom}\xi$.

Proof. See Appendix 6.2.3 □

The meaning of the conditions of the theorem above can be explained by looking at Figure 2.4. In Figure 2.4 we consider the case of a planar hybrid system for which the conditions of Theorem 2.4 are satisfied. Conditions (1) and (2) guarantee that the level set ℓ of V is close to the circle of radius c, while conditions (3)-(6) guarantee that no solution can stay forever in the intersection of the grey shaded set of Figure 2.4 with $c \leq |x| \leq \rho c$.

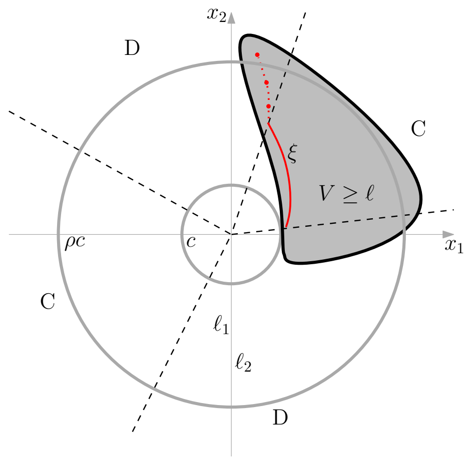


Figure 2.4: A function V that satisfies the conditions of Theorem 2.4, for a planar hybrid system.

By adding a simple condition to Theorem 2.4 it is possible to characterize the instability of the point $x_e = 0$, as stated in the following corollary. The key point of that condition is that it guarantees that $\{x \,|\, |x| = \rho c\} \subseteq \{x \,|\, V(x) > \ell\}$.

Thus, it is possible to show that there exists a solution that grows unbounded. Corollary 2.1 is related to *Case 2* of the introduction.

Corollary 2.1 *Under the hypothesis of Theorem 2.4, if the conditions (1)-(5) hold and the following condition is satisfied*

(7) $\min_{|x|=\rho c} V(x) > \ell$,

then $x_e = 0$ *is unstable.*

Proof. See section 6.2.3 □

Remark 2.7 It is important to mention that Theorems 2.3 and 2.4 are conservative. In fact, both overshoots of solutions to a hybrid system $\mathcal{H}$ and instability properties of the point $x_e = 0$ of $\mathcal{H}$ are the results of the "behavior" of *one* solution to $\mathcal{H}$ only, while Theorems 2.3 and Theorem 2.4 requires a particular "behavior" for an *entire set* of solutions.

Remark 2.8 Theorem 2.4 and Corollary 2.1 still work when Condition (1) is replaced by $\max_{|x|\leq c} V(x) \leq \ell$ and Condition (5) is deleted. The proof of this fact can be developed by following an argument similar to the one in Section 6.2.3. In fact, by $\max_{|x|\leq c} V(x) \leq \ell$, each jump from $x \in U$ to some $g \in G(x)$ with $|g| < c$ would fall in $\{x \,|\, V(x) \leq \ell\}$, that is forbidden by Condition (3) of Theorem 2.4.

The properties on overshoots and instability are clearly related to the stability results of Section 2.2. A straightforward example is given by Theorem 2.3 and Theorem 2.1, whose conditions cannot be satisfied at the same time. The following theorem characterizes some relationships among Theorems 2.1, 2.3, 2.4 and Corollary 2.1. We add a subscript to c and ρ to avoid confusion among constants of different theorems and corollaries. For example, $c_{(T2.1)}$ denotes the value of the constant c of Theorem 2.1 while $\rho_{(C2.1)}$ denotes the value of ρ used in Corollary 2.1.

Theorem 2.5 *Consider a hybrid system* $\mathcal{H}$ *of Equations* (1.5),(2.1)-(2.5) *and suppose* $C \cup D = \mathbb{R}^n$. *Then,*

1. *if Theorem 2.3 holds then* $\forall c_{(T2.1)} > 0, \forall \rho_{(T2.1)} > 1$ *Theorem 2.1 cannot be satisfied;*

2. *if* $\exists c_{(T2.1)} > 0, \exists \rho_{(T2.1)} > 1$ *such that Theorem 2.1 holds then Theorem 2.3 cannot be satisfied;*

3. *if* $\exists c_{(C2.1)} > 0, \exists \rho_{(C2.1)} > 1$ *such that Corollary 2.1 holds then* $\forall c_{(T2.1)} > 0, \forall \rho_{(T2.1)} > 1$ *Theorem 2.1 cannot be satisfied;*

4. *if* $\exists c_{(T2.1)} > 0, \exists \rho_{(T2.1)} > 1$ *such that Theorem 2.1 holds then* $\forall c_{(C2.1)} > 0, \forall \rho_{(C2.1)} > 1$ *Corollary 2.1 cannot be satisfied;*

5. *if* $\exists c_{(T2.4)} > 0, \exists \rho_{(T2.4)} > 1$ *such that Theorem 2.4 holds then* $\forall c_{(T2.1)} > 0, \forall \rho_{(T2.1)} < \rho_{(T2.4)}$*, Theorem 2.1 cannot be satisfied;*

6. *if* $\exists c_{(T2.1)} > 0, \exists \rho_{(T2.1)} > 1$ *such that Theorem 2.1 holds and each maximal solution* ξ *to* $\mathcal{H}$ *is complete, then* $\forall c_{(T2.4)} > 0, \forall \rho_{(T2.4)} > \rho_{(T2.1)}$ *Theorem 2.4 cannot be satisfied.*

Proof. See Section 6.2.3. □

2.3.2 Sum of Squares Algorithms

Under the assumption $C \cup D = \mathbb{R}^n$, we can use the following algorithms to find functions V that satisfy the conditions of Theorem 2.4. Then, we will use one of those algorithms to construct functions V that satisfy also the conditions of Corollary 2.1.

Algorithm 2 is defined by a set of inequalities parameterized by the parameters: $(case, k_1, k_2)$. A solution to the set of inequalities is then computed by relaxing the satisfaction problem of such inequalities to a sum of squares decomposition problem. Then, if a *solution is found*, the algorithm ends. Otherwise, the algorithm runs on a new set of inequalities, constructed on a different selection of $(case, k_1, k_2)$, until each possible case of $(case, k_1, k_2)$ has been considered. In fact, by using a parameterization with $(case, k_1, k_2)$, a non-convex search problem is reduced to several convex problems, suitable for sum-of-squares implementation. Therefore, by running Algorithm 2 several times, each time on a different set of parameters, we explore a non-convex search-space, searching for a function V that fullfills the conditions of Theorem 2.4. At each run:

(i) the input of the algorithm is filled by the data of the hybrid system $\mathcal{H}$, by some parameters $\varepsilon, c, \rho, d_1$ and d_2, and by a selection of $(case, k_1, k_2)$, as stated in section `INPUT`.

(ii) A set of inequalities is then constructed, as stated in section `CONSTRAINTS`. Each inequality uses the variables defined in `VARIABLES`.

(iii) A semidefinite program solver runs over the set of inequalities. A solution is computed by relaxing the feasibility problem of the whole set of inequalities to a sum of squares decomposition problem. The sum of squares decomposition problem is then solved by using a semidefinite program solver.

(iv) If the solver finds a solution, then the set of constraints is feasible and algorithm 2 has a positive output, as stated in `OUTPUT`.

The description of Algorithm 2 is based on the quantities in Definitions 2.2 and 2.3 and on a polynomial $q(x)$ defined as follows.

Definition 2.4 Let Q be a *symmetric* matrix in $\mathbb{R}^{n\times n}$, defined as follows.

$$Q = \begin{bmatrix} q_{11} & \dots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \dots & q_{nn} \end{bmatrix} \tag{2.15}$$

Let $q(x)$ be a polynomial defined as follows.

$$q(x) = \begin{bmatrix} q_{01} & \dots & q_{0n} \end{bmatrix} x + x'Qx \tag{2.16}$$

where q_{0i} belongs to $\mathbb{R}$, for each $i \in \{1, \dots, n\}$.

By suitable conditions on the elements of Q and on the elements of $[q_{01} \ \dots \ q_{0n}]$, $q(x)$ can be used as a function of x that is positive in some subset of $\mathbb{R}^n$. In particular, the parameterization $(case, k_1, k_2)$ defines some specific conditions on Q and on $[q_{01} \ \dots \ q_{0n}]$ so that $q(x)$ is necessarily greater than zero in some subset of $\mathbb{R}^n$. For example, consider a planar space and assume that $q_{11}+q_{22} > 0$ and $q_{12} = 0$. Then, $q(x) = q_{01}x_1 + q_{02}x_2 + q_{11}x_1^2 + q_{22}x_2^2$ is positive in a conic subset of $\mathbb{R}^2$. A numerical example is summarized in Figure 2.5.

Algorithm 2

`INPUT:`
Data $\langle F, G, C, D\rangle$ of the hybrid system $\mathcal{H}$;
constants $\varepsilon, c, \rho \in \mathbb{R}_{>0}$, with $\varepsilon \ll c$ and $\rho > 1$;
constants $d_1, d_2 \in \mathbb{Z}_{\geq 0}$, *case* $\in \{1, 2, 3\}$ and, if *case* $\neq 1$, $k_1 \in \{1, \dots n\}$, $k_2 \in \{k_1{+}1, \dots, n\}$.

`OUTPUT:`
Feasibility of the sum of squares problem.

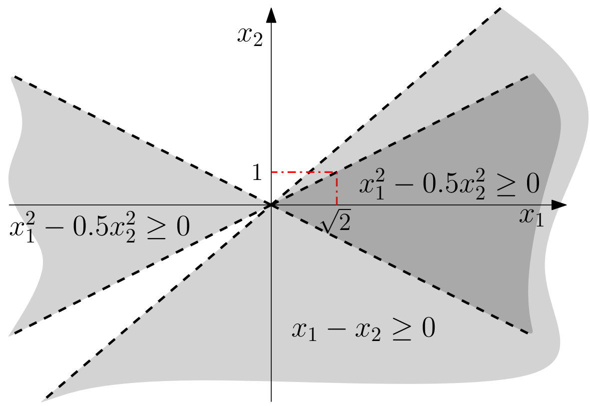


Figure 2.5: Suppose that $q(x) = x_1 - x_2 + x_1^2 - 0.5x_2^2$, then the intersection of $\{x \,|\, x_1 - x_2 \geq 0\}$ with $\{x \,|\, x_1^2 - 0.5x_2^2 \geq 0\}$ is a conic subset of $\{x \,|\, q(x) \geq 0\}$.

`VARIABLES:`
Scalars ℓ;
polynomials $V(x)$, $s_1^{(ik)}(x)$, for each i in I_C and $k \in \{1, \ldots, r_F\}$, $s_2^{(ik)}(x)$ for each i in I_D and $k \in \{1, \ldots, r_G\}$, $s_3(x)$, $s_4(x)$, and $s_5^{(i)}(x)$, $s_6^{(i)}(x)$, for each i in I_D, and all the slack polynomials.

`CONSTRAINTS:`
Let $V(x)$ be a polynomial of degree d_2. Let ϵ be a scalar variable.

- $\forall i \in I_C$, $\forall k \in \{1, \ldots, r_F\}$, let $s_1^{(ik)}(x)$ be a polynomial of degree d_1

$$\begin{aligned} \frac{\partial V}{\partial x}(x) F_{ik} x - \Delta_2^{(i)}(x) - s_1^{(ik)}(x)\Delta_1(c, \rho c, x) > 0 \\ s_1^{(ik)}(x) \geq 0 \end{aligned} \tag{2.17}$$

- $\forall i \in I_D$, $\forall k \in \{1, \ldots, r_G\}$, let $s_2^{(ik)}(x)$ be a polynomial of degree d_1

$$\begin{aligned} V(G_{ik}x) - V(x) - \Delta_2^{(i)}(x) - s_2^{(ik)}(x)\Delta_1(c, \rho c, x) > 0 \\ s_2^{(ik)}(x) \geq 0 \end{aligned} \tag{2.18}$$

- Assume $c + 3\varepsilon < \rho c$. Let $s_3(x)$, $s_4(x)$, $s_5^{(ik)}(x)$ and $s_6^{(ik)}(x)$ be polynomials of degree d_1

$$\begin{aligned}
\ell - V(x) - s_3(x)\Delta_1(c, c+\varepsilon, x) &\geq 0 \\
V(x) - \ell - s_4(x)\Delta_1(c+2\varepsilon, c+3\varepsilon, x) - q(x) &\geq 0 \\
\ell &\geq 0 \\
s_3(x), s_4(x) &\geq 0 \\
\forall i \in I_D, \forall k \in \{1, \ldots, r_G\}, & \\
\ell - V(x) - s_5^{(ik)}(x)(c^2 - x' G'_{ik} G_{ik} x) - \Delta_2^{(i)}(x) + & \\
-s_6^{(ik)}(x)\Delta_1(c, \rho c, x) &\geq 0 \\
s_5^{(ik)}(x), s_6^{(ik)}(x) &\geq 0
\end{aligned} \tag{2.19}$$

- $q(x)$ satisfies the following inequalities:

$$\begin{array}{ll}
\text{if } \mathit{case} = 1 & \sum_{i=1}^{n} q_{ii} > 0 \\
\text{if } \mathit{case} = 2 & \left\{ \begin{array}{c} \forall i \in \{1 \ldots, n\}, \; q_{ii} \leq 0 \\ 2q_{k_1k_2} + q_{k_1k_1} + q_{k_2k_2} > 0 \end{array} \right. \\
\text{if } \mathit{case} = 3 & \left\{ \begin{array}{c} \forall i \in \{1 \ldots, n\}, \; q_{ii} \leq 0 \\ -2q_{k_1k_2} + q_{k_1k_1} + q_{k_2k_2} > 0 \end{array} \right.
\end{array} \tag{2.20}$$

- Each use of $cone^{(i)}(x)$ in inequalities (2.17),(2.18) and (2.19) requires a *new fresh* set of slack polynomials. Moreover for each slack polynomial, say $p(x)$, a new constraint $p(x) \geq 0$ is added.

Analogously to Algorithm 1, each inequality of Algorithm 2 can be divided into two parts: the first part defines some constraints on V while the second part uses Δ_1, Δ_2 and q to guarantee the satisfaction of such constraints only in a specific subset of $\mathbb{R}^n$. Suppose now to run Algorithm 2 and to find a feasible solution to the set of inequalities constructed by Algorithm 2, for some hybrid system $\mathcal{H}$ and for some selection of parameters *case*, k_1 and k_2. The set of inequalities of Algorithm 2 guarantees the following properties.

- By (2.17) and (2.18), the derivative of $V(x)$ is positive, for each $x \in C$ and each $f \in F(x)$ such that $c \leq |x| \leq \rho c$. The difference $V(g) - V(x)$ is positive, for each $x \in D$ and each $g \in G(x)$ such that $c \leq |x| \leq \rho c$.
- By (2.20), $q(x)$ is not a non-positive function. To see this, note that if Q is not negative semi-definite, then there exists a conic subset of $\mathbb{R}^n$ such that $q(x) > 0$. And so, each inequality in (2.20) breaks a necessary condition for negative semi-definiteness of Q.

• The first inequality of (2.19) guarantees that $V(x) \leq \ell$ for each $c \leq |x| \leq c+\varepsilon$. The second inequality of (2.19) guarantees that $V(x) > \ell$ for some $c+2\varepsilon \leq |x| \leq c+3\varepsilon$. Then, $V(x) = \ell$ in at least one point of $c+\varepsilon \leq |x| \leq c+2\varepsilon$.
• If the system $\mathcal{H}$ jumps from a state x in $\{x \,|\, c \leq |x| \leq \rho c\}$ to a state g in $\{x \,|\, |x| \leq c\}$, then the next to the last inequality of (2.19) guarantees that $V(x) \leq \ell$. Therefore, the system cannot jump from the set $\{x \,|\, V(x) > \ell\} \cap \{c \leq |x| \leq \rho c\}$ to the set $\{x \,|\, |x| < c\}$.
It follows that a feasible solution to the set of constraints above produces a function V that satisfies the conditions of Theorem 2.4.

Proposition 2.2 *For any given hybrid system $\mathcal{H}$ defined by Equations* (1.5),(2.1)-(2.5)*, with $C \cup D = \mathbb{R}^n$, if the set of inequalities of Algorithm 2 has a feasible solution for some parameters $c \in \mathbb{R}_{>0}$, $\rho \in \mathbb{R}$, $\rho > 1$ and $(case, k_1, k_2)$, then the function V constructed by Algorithm 2 satisfies the conditions of Theorem 2.4, with the same c and ρ.*

Proof. See Appendix 6.2.3. □

The following modification to Algorithm 2 guarantees that the function $V(x)$ is greater than a costant $\overline{\ell} > \ell$ for each point x such that $|x| = \rho c$, as required by Corollary 2.1. Replace the second inequality of (2.19) with

$$\begin{aligned} V(x) - \overline{\ell} - s_7(x)\Delta_1(\rho c - \varepsilon, \rho c, x) &\geq 0 \\ \overline{\ell} &> \ell \\ s_7(x) &\geq 0 \end{aligned} \tag{2.21}$$

and delete (2.20). Then, the following proposition hold.

Proposition 2.3 *For any given hybrid system $\mathcal{H}$ defined by Equations* (1.5),(2.1)-(2.5)*, with $C \cup D = \mathbb{R}^n$ if the modified set of inequalities of Algorithm 2 has a feasible solution for some parameters $c \in \mathbb{R}_{\geq 0}$, $\rho \in \mathbb{R}$, $\rho > 1$ and $(case, k_1, k_2)$, then the function V constructed by the modified version of Algorithm 2 satisfies Corollary 2.1, with the same c and ρ.*

Proof. See Appendix 6.2.3. □

Remark 2.9 According to Remark 2.8, Algorithm 2 still works if we replace $\ell - V(x) - s_3(x)\Delta_1(c, c+\varepsilon, x) \geq 0$ in (2.19) with $\ell - V(x) - s_3(x)((c+\varepsilon)^2 - x^T x) \geq 0$ and we delete the fifth inequality in (2.19). Note that this approach forces the function V to be lower than ℓ for $|x| \leq c$, while Algorithm 2 leaves V basically unconstrained near the origin. In fact, the fifth inequality in (2.19) enforces a condition on V only if some jump $g \in G(x)$, $|g| \leq c$ from $c \leq |x| \leq \rho c$ occurs.

By (2.17), (2.18), Algorithm 2 searches for a function V whose directional derivative and increment are both positive in $c \leq |x| \leq \rho c$. According to Theorem 2.4, these conditions on V can be relaxed by requiring that both the directional derivative and the increment of V are positive only in a suitable subset of $\mathbb{R}^n$. Algorithm 3 takes into account this problem by defining a set of inequalities parameterized by two triples of parameters: $(case_a, k_1, k_2)$ and $(case_b, k_3, k_4)$. Analogously to Algorithm 2, the problem of finding a solution to the set of inequalities is relaxed to a sum of squares decomposition problem, so that, a semidefinite program solver can be used to find a solution. Also Algorithm 3 runs on each possible selection of $(case_a, k_1, k_2)$ and $(case_b, k_3, k_4)$ until a solution is found.

The description of Algorithm 3 is based on Definitions 2.2 and 2.3 and on polynomials $q_a(x)$ and $q_b(x)$ defined as follows. Note that $q_a(x)$ and $q_b(x)$ have the same structure of $q(x)$ in Definition 2.4.

Definition 2.5 Let Q_a, Q_b be two *symmetric* matrices in $\mathbb{R}^{n\times n}$, defined as follows.

$$Q_i = \begin{bmatrix} q_{i_{11}} & \cdots & q_{i_{1n}} \\ \vdots & \ddots & \vdots \\ q_{i_{n1}} & \cdots & q_{i_{nn}} \end{bmatrix} \tag{2.22}$$

where $i \in \{a, b\}$. The polynomials $q_a(x)$ and $q_b(x)$ are defined as follows.

$$\begin{aligned} q_a(x) &= \begin{bmatrix} q_{a_{01}} & \ldots & q_{a_{0n}} \end{bmatrix} x + x' Q_a x \\ q_b(x) &= \begin{bmatrix} q_{b_{01}} & \ldots & q_{b_{0n}} \end{bmatrix} x + x' Q_b x \end{aligned} \tag{2.23}$$

where $q_{a_{0i}}, q_{b_{0i}}$ belong to $\mathbb{R}$, for each $i \in \{1, \ldots, n\}$.

Algorithm 3

`INPUT:`

Data $\langle F, G, C, D\rangle$ of the hybrid system $\mathcal{H}$;
constants $\varepsilon, c, \rho \in \mathbb{R}_{>0}$, with $\varepsilon \ll c$ and $\rho > 1$;
constants $d_1, d_2 \in \mathbb{Z}_{\geq 0}$, $case_a$, $case_b \in \{1, 2, 3\}$ and, if $case_a \neq 1$, $k_1 \in \{1, \ldots n\}$, $k_2 \in \{k_1+1, \ldots, n\}$, and if $case_b \neq 1$, $k_3 \in \{1, \ldots n\}$, $k_4 \in \{k_3+1, \ldots, n\}$.

`OUTPUT:`

Feasibility of the sum of squares problem.

`VARIABLES:`

Scalars ℓ_1, ℓ_2;

polynomials $V(x)$, $s_1^{(ik)}(x)$, for each $i \in I_C$ and $k \in \{1,\dots,r_F\}$, $s_2^{(ik)}(x)$ for each $i \in I_D$ and $k \in \{1,\dots,r_G\}$, $s_3(x)$, $s_4(x)$, $s_5(x)$, $s_6^{(ik)}(x)$ and $s_7^{(ik)}(x)$ for each $i \in I_D$ and $k \in \{1,\dots,r_G\}$, and all the slack polynomials.

`CONSTRAINTS:`
Let $V(x)$ be a polynomial of degree d_2. Let ϵ be a scalar variable.

- $\forall i \in I_C$, $\forall k \in \{1,\dots,r_F\}$, let $s_1^{(ik)}(x)$ be a polynomial of degree d_1

$$\begin{aligned} \frac{\partial V}{\partial x}(x)F_{ik}x - q_a(x) - \Delta_2^{(i)}(x) - s_1^{(ik)}(x)\Delta_1(c,\rho c,x) > 0 \\ s_1^{(ik)}(x) \geq 0 \end{aligned} \tag{2.24}$$

- $\forall i \in I_D$, $\forall k \in \{1,\dots,r_G\}$, let $s_2^{(ik)}(x)$ be a polynomial of degree d_1

$$\begin{aligned} V(G_{ik}x) - V(x) - q_a(x) - \Delta_2^{(i)}(x) + & \\ -s_2^{(ik)}(x)\Delta_1(c,\rho c,x) > 0 & \\ s_2^{(ik)}(x) \geq 0 & \end{aligned} \tag{2.25}$$

- Assume $c+3\varepsilon < \rho c$. Let $s_3(x)$, $s_4(x)$, $s_5(x)$, $s_6^{(ik)}(x)$ and $s_7^{(ik)}(x)$ be polynomials of degree d_1

$$\begin{aligned} \ell_1 - V(x) - s_3(x)\Delta_1(c,c+\varepsilon,x) \geq 0 \\ V(x) - \ell_1 - s_4(x)\Delta_1(c+2\varepsilon,c+3\varepsilon,x) - q_b(x) \geq 0 \\ \ell_2 - V(x) - s_5(x)\Delta_1(c,\rho c,x) + q_a(x) \geq 0 \\ \ell_1 > \ell_2 \geq 0 \\ s_3(x), s_4(x), s_5(x) \geq 0 \\ \forall i \in I_D, \forall k \in \{1,\dots,r_G\}, \qquad\qquad\qquad\qquad \\ \ell_1 - V(x) - s_6^{(ik)}(x)(c^2 - x'G_{ik}'G_{ik}x) - \Delta_2^{(i)}(x) + \\ -s_7^{(ik)}(x)\Delta_1(c,\rho c,x) \geq 0 \\ s_6^{(ik)}(x), s_7^{(ik)}(x) \geq 0 \end{aligned} \tag{2.26}$$

- $q_a(x)$ satisfies the following inequalities:

$$\begin{array}{ll} \text{if } case_a = 1 & \sum_{i=1}^n q_{a_{ii}} > 0 \\ \text{if } case_a = 2 & \left\{ \begin{array}{c} \forall i \in \{1\dots,n\}, \, q_{a_{ii}} \leq 0 \\ 2q_{a_{k_1k_2}} + q_{a_{k_1k_1}} + q_{a_{k_2k_2}} > 0 \end{array} \right. \\ \text{if } case_a = 3 & \left\{ \begin{array}{c} \forall i \in \{1\dots,n\}, \, q_{a_{ii}} \leq 0 \\ -2q_{a_{k_1k_2}} + q_{a_{k_1k_1}} + q_{a_{k_2k_2}} > 0 \end{array} \right. \end{array} \tag{2.27}$$

- $q_b(x)$ satisfies the following inequalities:

$$\begin{array}{ll} \text{if } case_b = 1 & \sum_{i=1}^{n} q_{b_{ii}} > 0 \\ \text{if } case_b = 2 & \begin{cases} \forall i \in \{1 \ldots, n\}, \; q_{b_{ii}} \leq 0 \\ 2q_{b_{k_3 k_4}} + q_{b_{k_3 k_3}} + q_{b_{k_4 k_4}} > 0 \end{cases} \\ \text{if } case_b = 3 & \begin{cases} \forall i \in \{1 \ldots, n\}, \; q_{b_{ii}} \leq 0 \\ -2q_{b_{k_3 k_4}} + q_{b_{k_3 k_3}} + q_{b_{k_4 k_4}} > 0 \end{cases} \end{array} \tag{2.28}$$

- Each use of $cone^{(i)}(x)$ in inequalities (2.24),(2.25) and (2.26) requires a *new fresh* set of slack polynomials. Moreover for each slack polynomial, say $p(x)$, a new constraint $p(x) \geq 0$ is added.

To satisfy the conditions of Theorem 2.4, the directional derivative and the increment of V must be positive in the set $(C \cup D) \cap U$, namely the set $\{x \,|\, V(x) > \ell, x \in C \cup D\}$. These goals are achieved by using the inequalities in (2.24) and in (2.25), together with the third inequality of (2.26). Intuitively, for each $c \leq |x| \leq \rho c$, we have that the directional derivative and the increment of V are positive in the set $\{x \,|\, q_a(x) \geq 0\}$. At the same time, we have that $q_a(x) \leq 0$ implies $V < \ell$. It follows that, if $V(x) \geq \ell$ then $q_a(x) > 0$, that is, the directional derivative and the increment of V are positive in a subset of $\mathbb{R}^n$ that includes $\{x \,|\, V(x) \geq \ell, c \leq |x| \leq \rho c\}$, as required by Theorem 2.4. See Figure 2.6 for a graphical interpretation of inequalities of Algorithm 3, in a planar case.

A formal argument on (2.24),(2.25), on the third inequality of (2.26), and on the use of the remaining inequalities is developed in the proof of the following proposition.

Proposition 2.4 *For any given hybrid system $\mathcal{H}$ defined by Equations* (1.5),(2.1)-(2.5)*, with $C \cup D = \mathbb{R}^n$, if the set of inequalities of Algorithm 3 has a feasible solution for some parameters $c \in \mathbb{R}_{>0}$, $\rho \in \mathbb{R}$, $\rho > 1$ and some $(case_a, k_1, k_2)$, $(case_b, k_3, k_4)$, then the function V constructed by Algorithm 3 satisfies the conditions of Theorem 2.4, with the same c and ρ.*

Proof. See Appendix 6.2.3. □

2.3.3 Example

Let us consider the following hybrid system

$$\mathcal{H} = \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & \lambda_r \end{bmatrix} x & x \in C \\ x^+ = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} x & x \in D \end{cases} \tag{2.29}$$

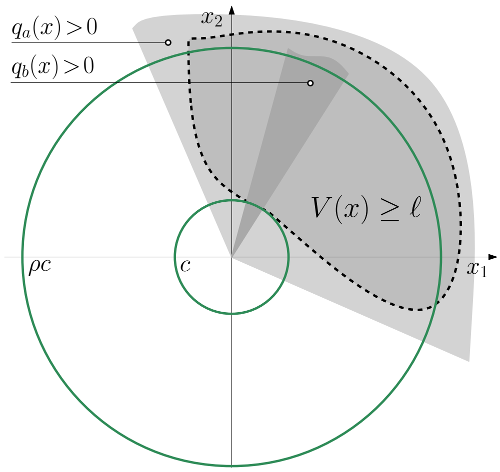


Figure 2.6: An interpretation of inequalities of Algorithm 3. The directional derivative and the increment of V are both positive in the subset $\{x \mid q_a(x) \geq 0\}$
.

where $\lambda_r \in \mathbb{R}$ is a parameter and C and D are defined as follows

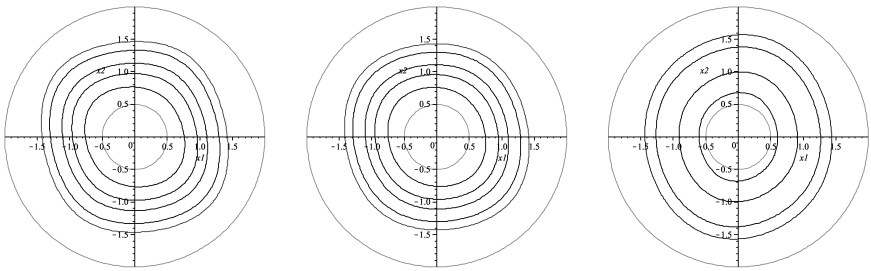
$$\begin{aligned} C &= \left\{ x \mid \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x \geq 0 \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x \geq 0 \right\} \\ D &= \left\{ x \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq 0 \text{ or } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x \geq 0 \right\}. \end{aligned} \tag{2.30}$$

We increase λ_r progressively so that the continuous dynamics of the hybrid system is characterized (i) by an asymptotic stable system, (ii) by a stable system and (iii) by an unstable system. For $c = 0.5$ and $\rho = 4$, the set of inequalities of Algorithm 1 has a feasible solution. Some level sets of the functions $V(x)$ constructed by Algorithm 1 are shown in Figure 2.7. A possible solution to $\mathcal{H}$ is illustrated in Figure 2.7(f).

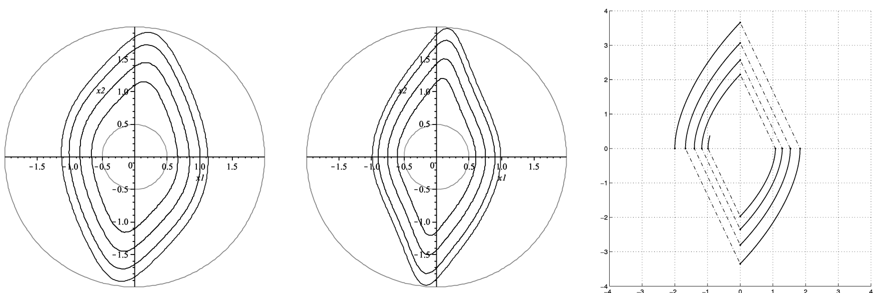
On the same system, We use Algorithm 2 to estimate the overshoot of $\mathcal{H}$. Indeed, we study $\mathcal{H}$ for increasing values of λ_r and for each λ_r we run several times Algorithm 2 (for $c = 0.5$ and $d_2 = 10$) looking for the greatest values of ρ for which the set of constraints are still feasible. Some level sets of the function $V(x)$ constructed by Algorithm 2 are shown in Figure 2.8.

The same study is repeated with Algorithm 3. Results are shown in Figure

2.9.



(a) $\lambda_r = -1$, degree of $V = 6$ (b) $\lambda_r = -0.5$, degree of $V = 6$ (c) $\lambda_r = 0$, degree of $V = 6$



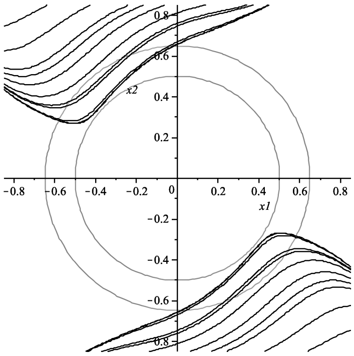
(d) $\lambda_r = 0.5$, degree of $V = 6$ (e) $\lambda_r = 1$, degree of $V = 6$ (f) $\lambda_r = 1$, a hybrid arc of the system.

Figure 2.7: Functions V constructed by Algorithm 1, Example 2.3.3

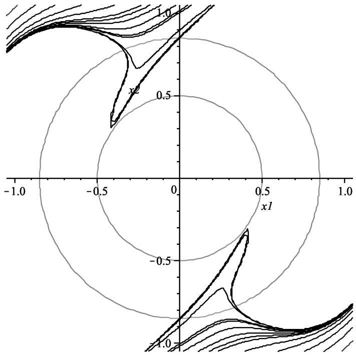
2.4 Notes on Sum of Squares Implementation

The results in Theorem 2.5 cannot be directly extended to the algorithms. In fact, each algorithm satisfies a set of conditions that is more conservative than the set of conditions of the theorem that the algorithm is based on.

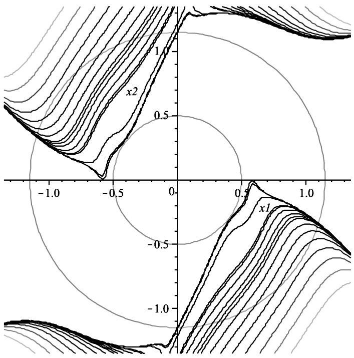
The problem of finding a solution to the set of inequalities of each algorithm is addressed by replacing each inequality with a sum-of-squares decomposition. In fact, the left-hand side of each inequality involving polynomials is a polynomial, say $p(x)$. It follows that the inequalities $p(x) \geq 0$ can be replaced by $p(x)$ *is a sum-of-squares* . Note that each strict inequality of the form $p(x) > 0$ can



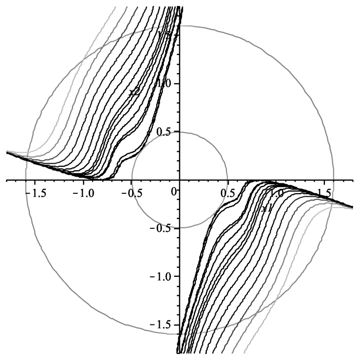
(a) $\lambda_r = 0.5$, $\rho = 1.3$



(b) $\lambda_r = 1$, $\rho = 1.7$



(c) $\lambda_r = 2$, $\rho = 2.3$

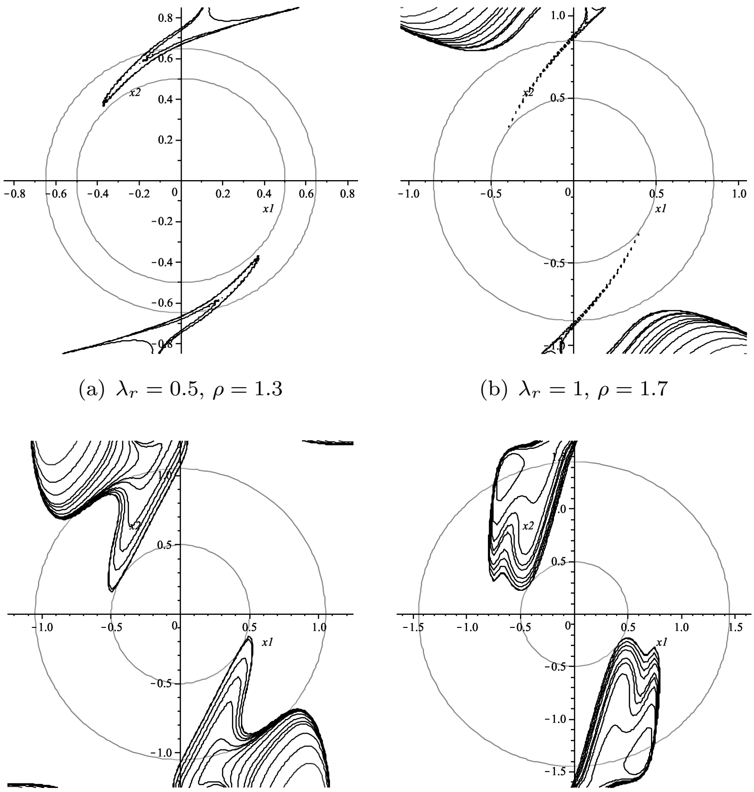


(d) $\lambda_r = 4$, $\rho = 3.2$

Figure 2.8: Level sets greater or equal than ℓ of the function $V(x)$ constructed by algorithm 2, Example 2.3.3. Note that the system is globally pre-asymptotically stable for $\lambda_r = 0.5$ and for $\lambda_r = 1$ while it is unstable for $\lambda_r = 2$ and for $\lambda_r = 4$.

be transformed to a non-strict inequality of the form $p(x) - \epsilon x^T x \geq 0$, with $\epsilon > 0$ variable of the problem and this step is needed to write the set of inequality as a sum of squares decomposition problem.

From a computational point of view, finding a sum-of-squares decomposition is much easier than using a general algorithm for finding a solution to the inequality constraints. At the same time, it could be the case that a solution



(a) $\lambda_r = 0.5$, $\rho = 1.3$

(b) $\lambda_r = 1$, $\rho = 1.7$

(c) $\lambda_r = 2$, $\rho = 2.1$

(d) $\lambda_r = 4$, $\rho = 2.9$

Figure 2.9: Level sets greater or equal than ℓ_1 of the function $V(x)$ constructed by Algorithm 3, Example 2.3.3. The system is globally pre-asymptotically stable for $\lambda_r = 0.5$ and for $\lambda_r = 1$ while it is unstable for $\lambda_r = 2$ and for $\lambda_r = 4$.

to the inequality constraints exists while the sum of squares decomposition fails to exist. Moreover, even though (i) polynomial inequalities constructed by each algorithm are linear with respect to the set of variables and (ii) a sum of squares decomposition problem can be solved in polynomial time, the computational complexity of finding a solution to the set of inequalities grows rapidly with the dimension of the state-space of $\mathcal{H}$, with the degree of the free polynomials used in the set of inequalities, with the number of disjoint cones of $C \cup D$, and with the number of matrices F_{ik}, G_{ik}.

It is important to underline that a sum-of-squares decomposition is satisfied within the limits of the numerical computation, therefore it cannot be exact. Fortunately, we are not interested in an exact decomposition. What we really need is that, despite the numerical approximation errors of the construction of the sum of square decomposition, the constructed polynomials are still a feasible solution to the set of inequalities. By following [98], this goal can be achieved by considering a perturbed polynomial with a perturbation magnitude that depends on the numerical approximations errors of the decomposition (residuals). By the fact that a sum-of-squares problem can be formulated as an equivalent SDP problem and then solved by a SDP solver, we can use [98, Theorem 4] to guarantee that the approximate solution to the sum of squares decomposition problem is a feasible solution for the set of inequalities. For instance, suppose that we want to find a polynomial $p(x)$ such that $p(x) \geq 0$. Then,

– we relax the problem to *find* $p(x)$ *such that* $p(x)$ *is a sum-of-squares* ;

– the data of the SDP formulation are the matrices A and b;

– the solution is $P \in \mathbb{R}^{M \times M}$, for some $M \in \mathbb{Z}_{\geq 0}$;

– $p(x)$ can be written as $v(x)' P v(x)$, where $v(x)$ is a base of monomials.

By [98, Theorem 4], if

$$\lambda_{\min}(P) \geq M \| \, A(P) - b \, \|_{\infty} \tag{2.31}$$

then $v(x)' P v(x)$ is non-negative, i.e. the satisfaction of each inequality is certified. Note that (2.31) can be used as a termination condition for the algorithms.

Finally, each algorithm and the test condition in (2.31) can be implemented and solved by using packages like YALMIP [97], and SeDuMi [138].

Chapter 3

Formal Verification of Hybrid Systems

In Chapters 2 we presented some results on stability, overshoot, and instability of a particular family of hybrid systems. Then, we constructed some sum of squares algorithms that effectively use that results to characterize the stability of that particular family of hybrid systems. In this chapter we continue to work on the analysis of hybrid systems by defining (i) a formalized language to express classes of properties on these systems and (ii) a method for deciding whether or not the system satisfies the properties of interest, in case they are decidable. We will consider the following *verification problem*: given a hybrid system $\mathcal{H}$, a state x of $\mathcal{H}$, and a formula φ, establish whether or not the state x of $\mathcal{H}$ satisfies the property φ, denoted $\mathcal{H}, x \vDash \varphi$.

In Chapter 2 we studied a particular verification problem: the stability problem. We considered a system $\mathcal{H}$ and a set $\mathcal{A}$ and we proposed some procedures (the algorithms) to answer to the question: *is the set $\mathcal{A}$ asymptotically stable for $\mathcal{H}$?* Here we define a specific *temporal logic* [22, 39], to express properties like "if A happens then B happens within 5 units of time", or "A holds for every time instant of the solutions", and we propose a *verification method* to decide, if at all possible, whether or not a hybrid system and a state satisfy the property expressed by a specific temporal logic formula.

There are two main approaches to the verification problem on hybrid systems. One is the *model checking* approach in which the set of states that satisfy a given formula is computed by successive approximations [39]. Examples of the model checking approach on particular classes of hybrid systems can be found in [2, 54, 66]. The other approach is the *deductive verification* approach in which a set of transformation rules is applied to the formula that express the property of interest. The result is, in general, a proof tree whose leaves establishes whether or not a given state satisfies the property [25, 26]. This approach has been applied to real-time systems in [72] and to general hybrid systems in [44]. Notable works are [115, 116].

Formal verification is usually based on the following abstract components.

- *A language to describe a class of processes of interest.* In our case the language is the one of hybrid systems, where we take take into account the interaction of continuous and discrete processes.

- *A language to express a class of properties of interest.* In our case we use a slightly modified version of *Timed Computation Tree Logic* TCTL [13], denoted as $\mathcal{H}$TCTL, *Hybrid Time Computation Tree Logic.* Although $\mathcal{H}$TCTL has the same syntax of TCTL, its semantics is based on solutions to hybrid systems.

- *A verification procedure to decide, if at all possible, whether or not a given state of a process satisfies a given property.* In this chapter we show a method to reduce the verification problem of $\mathcal{H}$TCTL to the membership problem on a set denoted by a fixpoint expression. Such a method follows the approach of [73], and generalizes that approach to a broader class of systems and to a broader class of solutions. Then, either a model checking approach [73], or a deductive approach [26], can be used on the fixpoint expression to decide whether or not a given state of a process belongs to the set denoted by the fixpoint expression.

3.1 A Model for Hybrid Systems

By following [1, 5, 73], we define a branching time logic similar to CTL for untimed systems and similar to TCTL for timed systems [13]. The $\mathcal{H}$TCTL has the same sintax of TCTL, but a different semantics. In particular, the semantics of $\mathcal{H}$TCTL is defined by using the notion of a solution to a hybrid system (Definition 1.4), instead of the usual approach based on transition systems or Kripke structures [13] and [39, Chapters 2 and 3]. Therefore, we need to characterize some properties of the solutions to a hybrid system before presenting the logic and defining its semantics

In the following definition we consider two important properties on sets of solutions to a hybrid system $\mathcal{H}$: *suffix-closure* and *fusion-closure*. We follow the approach of [73, Definition 2.3], by considering the solutions to a hybrid system instead of the *real-time trajectories* of [73, Definition 2.2].

Definition 3.1 A set Π of solutions to a hybrid system $\mathcal{H}$ of Equation 1.5 is defined as $\Pi \subseteq \{\xi \mid \xi \text{ is a solution to } \mathcal{H}\}$.

Definition 3.2 A set Π of solutions to a given hybrid system $\mathcal{H}$ is

1. *suffix-closed* if for all solutions $x \in \Pi$ and for each state $x(t, j)$, there exists a solution $y \in \Pi$ such that if $(\tau, i) \in \operatorname{dom} x$ and $\tau + i \geq t + j$ then $y(\tau - t, i - j) = x(\tau, i)$.

2. *fusion-closed* if for all solutions $x, y \in \Pi$ and all states $x(t, j)$ and $y(\tau, i)$ if $x(t, j) = y(\tau, i)$ then the hybrid arc $z(t, j)$ constructed as

$$\begin{cases} z(t,j) = x(t,j) & \text{if } t + j \leq t_x + j_x \\ z(t,j) = y(t_y + t - t_x, j_y + j - j_x) & \text{if } t + j \geq t_x + j_x \end{cases} \tag{3.1}$$

is in Π.

Closure properties of sets of solutions of Definition 3.2 are related to the well-known fact that solutions to hybrid systems of Equation (1.5) are completely determined by the current state of the system, i.e. no informations on previous states is needed. Indeed, consider a hybrid system $\mathcal{H}$ and a solution x to $\mathcal{H}$. Define a hybrid arc y so that $y(0, 0) = x(t_x, j_x)$ for some given $(t_x, j_x) \in \operatorname{dom} x$ and $y(t_y, j_y) = x(t_x + t_y, j_x + j_y)$ for each $(t_x + t_y, j_x + j_y) \in \operatorname{dom} x$ and $t_y \geq 0$, $j_y \geq 0$. Then, y is a solution to $\mathcal{H}$ from the initial state $x(t_x, j_x)$. It follows that the set of *all solutions* to a hybrid system $\mathcal{H}$, $\{\xi \mid \xi \text{ is a solution to } \mathcal{H}\}$, is necessarily a suffix-closed set of solutions. Of course, a subset Π of the whole set of solutions can be not suffix-closed. By following a similar argument, consider two solutions x and y to $\mathcal{H}$ and suppose $x(t_x, j_x) = y(t_y, j_y)$ for some $(t_x, j_x) \in \operatorname{dom} x$ and $(t_y, j_y) \in \operatorname{dom} y$. Then, because of the dependence on the initial state, we can construct a solution $\overline{z}$ to $\mathcal{H}$ with initial state $z(0, 0) = x(t_x, j_x) = y(t_y, j_y)$ that follows either x or y from this initial state. Then, the set of all solutions to a hybrid system $\mathcal{H}$ is necessarily fusion-closed. As before, a subset Π of the whole set of solutions can be not fusion-closed.

Lemma 3.1 *For any given hybrid system $\mathcal{H}$, the set of all solutions to $\mathcal{H}$ is suffix-closed and fusion-closed.*

The following definition take into account [73, Definition 2.4] together with the notions of hybrid time and of solution to a hybrid system. The notions of *model* and *premodel* of Definition 3.3 will be used in Section 3.2 to define the semantics of the temporal logic.

Definition 3.3 For any given hybrid system $\mathcal{H}$, a *premodel* is a set of solutions to $\mathcal{H}$ that is suffix-closed and fusion-closed. A *model* is a premodel whose solutions are complete (Definition 1.5).

It is worth mentioning that our definition of a model takes into account Zeno and discrete solutions, and it is different from the notion proposed in [73], in which sets of solutions with Zeno and discrete solutions are not models (by the fact that they contains solutions that are not *divergent*, see [73, Definition 2.2]). The notion of a model proposed in Definition 3.3 preserves the symmetry between continuous solutions and discrete solutions to a hybrid system. Indeed, the hybrid time in Definition 1.2 uses a variable for taking into account the number of jumps within a solution, and it uses a variable to take into account the elapsing of time, during flow intervals. Therefore, it seemed natural to consider discrete and continuous solutions symmetrically and include both of them in a model for hybrid systems. (In [73] continuous solutions lead to divergent paths and a set of solutions that contains continuous solutions can be a model for the hybrid system, while discrete solutions produce non-divergent paths and a set of solutions that contains discrete solutions cannot be a model for the hybrid system). Note that a Zeno solution has an unbounded domain, that is, it is a complete solution, thus a set that contains Zeno solutions can be a model.

Remark 3.1 The notion of complete solutions in Definition 1.5 is related to [73, Definition 2.2] and Definitions 3.2 and 3.3 are related to [73, Definitions 2.3 and 2.4]. Our definitions are based on the notion of hybrid time in Definition 1.2, that differs from the dense-time model used in [1, 5, 73]. This leads to the following differences between our definitions and the relative definitions in [73]:

- the notion of a model of a hybrid system in Definition 3.3 generalizes [73, Definition 2.4], by considering also Zeno solutions and discrete solutions;

- complete solutions in Definition 1.5 can be compared to *divergent paths* of [73, Definition 2.2]. Then, each divergent path of [73, Definition 2.2] can be considered as a complete solution to $\mathcal{H}$ while the contrary does not hold. In fact, Zeno solutions and discrete solutions to $\mathcal{H}$ are complete solutions but not divergent paths.

These differences between divergent paths and complete solutions lead to a different notion of a model of a hybrid system. Thus, sets of solutions that contains Zeno and discrete solutions can be considered as models for a hybrid system only for the notion of model in Definition 3.3. Note that, although in [73] the authors call *real-time system* what we call here a model, Definition 3.3 and [73, Definition 2.4] are similar.

3.2 $\mathcal{H}$TCTL

In this section we present the $\mathcal{H}$TCTL, a branching time logic quite similar to TCTL of [1, 13, 73]. The $\mathcal{H}$ at the beginning of the $\mathcal{H}$TCTL tells the reader that $\mathcal{H}$TCTL is a version of TCTL that uses the notion of hybrid time (see Definition 1.2). We consider a syntax of the $\mathcal{H}$TCTL that coincides with the syntax of TCTL, defined in [1, 13], and we give a semantics for the $\mathcal{H}$TCTL based on the notions of hybrid time (Definition 1.2) and model of a hybrid system (Definition 3.3). Then, we compare the semantics of TCTL [13, 73] to that of $\mathcal{H}$TCTL.

Path operators of TCTL in [1, 13] are parameterized with time-intervals denoted by $rop\, c$, where $rop \in \{<, \leq, =, \geq, >\}$ and $c \in \mathbb{Z}_{\geq 0}$. We follow a similar approach, using *hybrid time intervals* on path operators. We denote these intervals as $rop\,(c_t, c_j)$, where $rop \in \{<, \leq, =, \geq, >\}$ and $(c_t, c_j) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and (t, j) satisfies $rop\,(c_t, c_j)$ iff $t\; rop\; c_t$ and $j\; rop\; c_j$.

Definition 3.4 [Syntax of $\mathcal{H}$TCTL]
Let P be a set of *atomic propositions*. A formula φ of Hybrid Time Computation Tree Logic is inductively defined from the set of *atomic propositions* as follows:

$$\varphi \equiv p \,|\, \neg\varphi \,|\, \varphi_1 \vee \varphi_2 \,|\, \exists\varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)}\varphi_2 \,|\, \forall\varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)}\varphi_2 \tag{3.2}$$

where $p \in P$ and $rop\,(c_t, c_j)$ is a hybrid time interval.

In what follows we will denote the symbols $\exists$ and $\forall$ as *quantifiers on solutions* or, for short, *quantifiers*, and the symbol $\mathcal{U}$ as *path operator*. For semplicity, we will sometimes use the name of *time interval* for a *hybrid time interval* $rop\,(c_t, c_j)$.

To define the semantics of $\mathcal{H}$TCTL we need an interpretation for the atomic propositions, that is, for a hybrid system $\mathcal{H} = (O, C, D, F, G)$ with state-space of dimension n, a function that maps each atomic proposition to a subset of $\mathbb{R}^n$.

Definition 3.5 Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ with state-space of dimension $n \in \mathbb{Z}_{\geq 0}$. Consider a set P of atomic propositions and a premodel

M of $\mathcal{H}$. The function $[\![\cdot]\!]_M : P \to 2^{\mathbb{R}^n}$ maps each atomic propositions $p \in P$ to the *characteristic set* $[\![p]\!]_M \subseteq \mathbb{R}^n$ of p of states in which p holds.

Note that, in general, $[\![p]\!]_M$ is an infinite set of states and a language is needed to symbolically denote that set of states. For example, using the language of linear inequalities to denote subset of $\mathbb{R}^n$, $n \in \mathbb{Z}_{\geq 0}$, consider a hybrid system $\mathcal{H}$ with state vector $x = [\, x_1 \; x_2 \; \ldots \; x_n]^T$ and consider an atomic proposition p. Then, an atomic proposition p can be mapped to $[\![p]\!]_M = \{x \,|\, x_1 + x_2 + \cdots + x_n \geq 0\}$.

By using the saturation function on atomic propositions, we can now give the definition of the $\mathcal{H}$TCTL semantics, as follows. It is worth mentioning that the $\mathcal{H}$TCTL semantics is defined similarly to [13, 73], but it is based on the model notion given in Definition 3.3.

Definition 3.6 [Semantics of $\mathcal{H}$TCTL]
For any given $n \in \mathbb{Z}_{\geq 0}$, consider a premodel M, a set P of atomic propositions, a function $[\![\cdot]\!]_M : P \to 2^{\mathbb{R}^n}$, that maps each $p \in P$ to its characteristic set $[\![p]\!]_M$, and a state $x \in \mathbb{R}^n$. The meaning of a formula φ of $\mathcal{H}TCTL$ is inductively defined as follows

$M, x \vDash p$ iff $x \in [\![p]\!]_M$

$M, x \vDash \neg\varphi$ iff $M, x \nvDash \varphi$

$M, x \vDash \varphi_1 \vee \varphi_2$ iff $M, x \vDash \varphi_1$ or $M, x \vDash \varphi_2$;

$M, x \vDash \exists\varphi_1 \mathcal{U}^{\,rop\,(c_t, c_j)} \varphi_2$ iff $\exists \xi \in M$, with $\xi(0,0) = x$, such that

- $\exists (T, J) \in \operatorname{dom} \xi$, such that $(T, J)\, rop\, (c_t, c_j)$ and $M, \xi(T, J) \vDash \varphi_2$, and $\forall (t, j) \in \operatorname{dom} \xi$, if $(t, j) \leq (T, J)$ then $M, \xi(t, j) \vDash \varphi_1 \vee \varphi_2$.

$M, x \vDash \forall\varphi_1 \mathcal{U}^{\,rop\,(c_t, c_j)} \varphi_2$ iff $\forall \xi \in M$, with $\xi(0,0) = x$,

- $\exists (T, J) \in \operatorname{dom} \xi$ such that $(T, J)\, rop\, (c_t, c_j)$ and $M, \xi(T, J) \vDash \varphi_2$, and $\forall (t, j) \in \operatorname{dom} \xi$, if $(t, j) \leq (T, J)$ then $M, \xi(t, j) \vDash \varphi_1 \vee \varphi_2$.

Intuitively, a state x and a premodel M satisfy $\exists\varphi_1 \mathcal{U}^{\,rop\,(c_t, c_j)} \varphi_2$, that is, $M, x \vDash \exists\varphi_1 \mathcal{U}^{\,rop\,(c_t, c_j)} \varphi_2$, if for at least one solution ξ from x there is a time (T, J) such that $M, \xi(T, J) \vDash \varphi_2$ and for all $(t, j) \leq (T, J)$, $M, \xi(t, j) \vDash \varphi_1 \vee \varphi_2$. The meaning of $\forall\varphi_1 \mathcal{U}^{\,rop\,(c_t, c_j)} \varphi_2$ is similar, but it requires that the relation between φ_1 and φ_2 is enforced on each solution ξ from x. An example is given in Figure 5.6.

Note that $\exists\varphi_1 \mathcal{U}^{\,rop\,(c_t, c_j)} \varphi_2$ and $\forall\varphi_1 \mathcal{U}^{\,rop\,(c_t, c_j)} \varphi_2$ can be divided in two parts: a path operator and a quantifier on solutions. Indeed,

(i) the path operator can be interpreted as a *quantification on time instants of the solution*. Without the quantifier on solutions, $\varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2$ defines a particular relation between φ_1 and φ_2, on a given solution $\xi \in M$. When a solution ξ satisfies this particular relation between φ_1 and φ_2, that is,

- for some $(T,J)\; rop\,(c_t,c_j),\;\; M,\xi(T,J) \vDash \varphi_2$ and
- for all $(t,j) \leq (T,J),\;\; M,\xi(t,j) \vDash \varphi_1 \vee \varphi_2$,

we say that ξ *satisfies the semantics of* $\varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2$;

(ii) the quantifier on solutions defines when either *some* solution in M, or *each* solution in M, must satisfy the semantics of $\varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)}$.

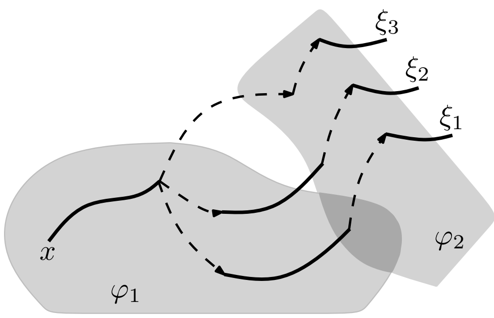


Figure 3.1: We represent three solutions from the initial state x. Flow intervals are represented by a continuous line. Jumps are represented by a dashed line. The characteristic sets of the $\mathcal{H}$TCTL formulas φ_1 and φ_2 are represented by the grey shaded shapes of the figure. x satisfies $\exists \varphi_1 \mathcal{U}^{\geq(0,0)} \varphi_2$ by the fact that ξ_1 and ξ_3 satisfies the semantics of $\varphi_1 \mathcal{U}^{\geq(0,0)} \varphi_2$, therefore at least one solution from x satisfies the semantics of $\varphi_1 \mathcal{U}^{\geq(0,0)} \varphi_2$. x does not satisfy $\forall \varphi_1 \mathcal{U}^{\geq(0,0)} \varphi_2$ by the fact that ξ_2 does not satisfies the semantics of $\varphi_1 \mathcal{U}^{\geq(0,0)} \varphi_2$. Therefore, not each solution from x satisfies the semantics of $\varphi_1 \mathcal{U}^{\geq(0,0)} \varphi_2$.

The definition of the $\mathcal{H}$TCTL semantics is based on a set P of atomic propositions and on a particular premodel M. In general, the meaning of a $\mathcal{H}$TCTL formulas is defined on each set of solutions to $\mathcal{H}$ which is suffix and fusion closed. Note that the set of all solutions to a hybrid system is a premodel and we can replace M with $\mathcal{H}$ whenever the set of all solutions to $\mathcal{H}$ is considered. Note also that if each solution to $\mathcal{H}$ is complete, the set of all solutions to $\mathcal{H}$ is a model.

Denote by Ψ the set of all $\mathcal{H}$TCTL formulas and consider a hybrid system $\mathcal{H}$ with state dimension $n \in \mathbb{Z}_{\geq 0}$. Following the semantics defined above, we can define the semantic function $[\![\cdot]\!]_M : \Psi \to 2^{\mathbb{R}^n}$, that maps each $\mathcal{H}$TCTL formula to a subset of $\mathbb{R}^n$. Note that the semantic function below overload the function that maps atomic propositions to their characteristic sets.

Definition 3.7 Let M be a model and let Ψ be the set of all $\mathcal{H}$TCTL formulas. The *semantic function* $[\![\cdot]\!]_M : \Psi \to 2^{\mathbb{R}^n}$ maps each formula $\varphi \in \Psi$ to $[\![\varphi]\!]_M = \{x \,|\, M, x \vDash \varphi\}$.

Example 3.1 Consider a hybrid system $\mathcal{H} = (\mathbb{R}^2, C, D, F, G)$ defined by the following equation. We use x to denote the state.

$$\begin{cases} \dot{x} = [\,1 \;\; 1\,]^T & x \in \{x \,|\, |x| \leq 1\} \\ x^+ = \frac{1}{2}x & x \in \{x \,|\, |x| \geq 1\} \end{cases} \tag{3.3}$$

Consider a premodel M defined by the set of all complete solutions ξ to $\mathcal{H}$ and define a set of atomic propositions $P = \{p_0, p_1, \mathit{true}\}$ such that $[\![p_0]\!]_M = \{x \,|\, 0 \leq |x| \leq 1\}$, $[\![p_1]\!]_M = \{x \,|\, |x| = 1\}$ and $[\![\mathit{true}]\!]_M = \mathbb{R}^2$. For that premodel M, a state x satisfies the formula $\varphi \equiv p_0 \vee \forall \mathit{true}\, \mathcal{U}^{\geq (0,0)} p_1$, that is, $M, x \vDash \varphi$, if $x \in [\![p_0]\!]_M$ or for each solutions ξ to $\mathcal{H}$ with $\xi(0,0) = x$, there exists a $(T, J) \in \operatorname{dom} \xi$, $(T, J) \geq (0, 0)$, such that $\xi(T, J) \in [\![p_1]\!]_M$. Therefore, φ is satisfied by states x such that $0 \leq |x| \leq 1$ and such that there exists a solution ξ with $\xi(0,0) = x$ and $|\xi(T, J)| = 1$, for some $(T, J) \in \operatorname{dom} \xi$.

Now, consider a solution ξ to $\mathcal{H}$ such that $|\xi(0,0)| \leq 1$. ξ has an initial state within the flow set, therefore it satisfies $\xi(t, 0) = t + \xi(0, 0)$ for all $0 \leq t \leq T$, where $T \in \mathbb{R}_{\geq 0}$ is such that $T + \xi(0,0) \in \{x \,|\, |x| = 1\}$. It follows that each state x within the flow set satisfies $\forall \mathit{true}\, \mathcal{U}^{\geq (0,0)} p_1$. Now consider a solution ξ with initial state $|\xi(0,0)| > 1$. ξ satisfies $\xi(0, j+1) = \frac{1}{2}\xi(0, j)$, for each $j \in \mathbb{Z}_{\geq 0}$ such that $0 \leq j \leq J$, where J is the time instant at which $|\xi(0, J)| < 1$ (ξ enters the flow set in a finite number of jumps). From there, ξ flows and, for some $T \in \mathbb{R}_{\geq 0}$, $\xi(T, J) \in [\![p_1]\!]_M$. It follows that each state of $\mathcal{H}$ satisfies φ. In Figure 3.2 we have represented two possible solutions of the hybrid system (3.3).

Note that if we replace φ by $\varphi' \equiv p_0 \vee \forall \mathit{true}\, \mathcal{U}^{\leq (10,2)} p_1$, then the a state x such that $x = |5|$ does not satisfy φ'. Indeed, each solution ξ from x needs three jumps to reaches the ball of radius 1, that is, there exists a $J \geq 3$ such that $|\xi(0, J)| < 1$ and does not exist a $J \leq 2$ for which $|\xi(0, J)| < 1$. From there, ξ flows to $|x| = 1$, that is, $\xi(T, J) \in [\![p_1]\!]_M$ for some $T \geq 0$. Constraints on the $\mathcal{U}$ operator, however, requires at most two jumps.

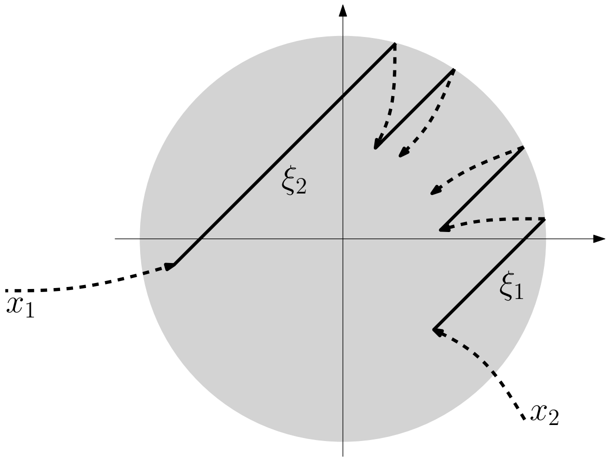


Figure 3.2: We represent two possible solutions ξ_1 and ξ_2 of the hybrid system (3.3), respectively from x_1 and x_2. Both solutions satisfy $\exists\varphi_1\mathcal{U}^{\geq(0,0)}\varphi_2$. Note that the jumps are represented by a dashed line. Each jump moves the state from a point $v \in \mathbb{R}^2$ to $\frac{1}{2}v$. The flows are represented by continuous lines. The gray shaded circle represent $[\![p_0]\!]_M$.

Remark 3.2 We could consider different kinds of hybrid time intervals. For example, a possible interval I subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ can be defined as $[\underline{t}, \overline{t}] \times \{j, j+1, \ldots, j+n\}$, for some $\underline{t}, \overline{t} \in \mathbb{R}_{\geq 0}$ and $j, n \in \mathbb{Z}_{\geq 0}$. The semantics of $\mathcal{H}$TCTL given in Definition 3.6 can be restated on these hybrid intervals. We decided to consider simpler hybrid time intervals for reasons of simplicity.

Note that the operator $\mathcal{U}$ of $\mathcal{H}$TCTL has a semantics that differs from the usual semantics of the operator $\mathcal{U}$ of CTL. For instance, consider a solution ξ to $\mathcal{H}$, and a formula $\varphi_1\mathcal{U}^{\,rop\,(c_t,c_j)}\varphi_2$. In $\mathcal{H}$TCTL, this formula requires that $\xi(T,J)$ satisfies φ_2, for some $(T,J) \in \operatorname{dom}\xi$, such that $(T,J)\; rop\,(c_t,c_j)$, and it requires that for each $(t,j) \in \operatorname{dom}\xi$, $(t,j) \leq (T,J)$, $\xi(t,j)$ satisfies $\varphi_1 \vee \varphi_2$. Instead, the CTL semantics for $\mathcal{U}$ requires that $\xi(T,J)$ satisfies φ_2, for some $(T,J) \in \operatorname{dom}\xi$, such that $(T,J)\; rop\,(c_t,c_j)$, but, for each $(t,j) \in \operatorname{dom}\xi$, it requires that $\xi(t,j)$ satisfies φ_1 only. The semantics of the operator $\mathcal{U}$ in $\mathcal{H}$TCTL follows [13, 73] and can be justified by the following example.

Example 3.2 Consider a solution $\xi : \operatorname{dom}\xi \to \mathbb{R}$ to a hybrid system $\mathcal{H}$ that, for all $t \in \mathbb{R}_{\geq 0}$, satisfies $\xi(t,0) = 0 + t$. Consider two atomic proposition p_1 and

p_2 such that $[\![p_1]\!]_M = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$ and $[\![p_2]\!]_M = \{x \in \mathbb{R} \,|\, 1 \leq x \leq 2\}$. ξ satisfies the $\mathcal{H}$TCTL semantics of $p_1\mathcal{U}^{\geq(1.5,0)}p_2$ if $\xi(T,0) \in [\![p_2]\!]_M$ for some $T \geq 1.5$ and $\xi(t,0) \in [\![p_1]\!]_M \cup [\![p_2]\!]_M$ for each $t \leq T$. With $T = 1.5$, we have that ξ satisfies the semantics of $p_1\mathcal{U}^{\geq(1.5,0)}p_2$. Now consider the CTL semantics of the operator $\mathcal{U}$. It requires that $\xi(t,0) \in [\![p_1]\!]_M$ for each $t \leq T$. However, $\xi(t,0) \notin [\![p_1]\!]_M$ for each $t \in (1,1.5]$ therefore ξ does not satisfy the CTL semantics of $p_1\mathcal{U}^{\geq(1.5,0)}p_2$.

It is worth mentioning that ξ satisfies both the $\mathcal{H}$TCTL and the CTL semantics of $p_1\mathcal{U}^{\geq(0,0)}p_2$ in Example 3.2. This is a general result and is related to the fact that in CTL the semantics of $p_1\mathcal{U}^{\geq(0,0)}p_2$ is equivalent to the semantics of $(p_1 \vee p_2)\mathcal{U}^{\geq(0,0)}p_2$.

By following [13], we can define some induced $\mathcal{H}$TCLT operators as follows.

Definition 3.8 The propositional logic operators $\wedge$, $\rightarrow$, *true* can be constructed in the usual way, that is, for any given formulas φ_1, φ_2, $\varphi_1 \wedge \varphi_2 \equiv \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $\varphi_1 \rightarrow \varphi_2 \equiv \neg\varphi_1 \vee \varphi_2$, and $true \equiv \varphi_1 \rightarrow \varphi_1$. Then,

$$\begin{aligned}
\exists \bigcirc \varphi &\equiv \exists\, true\, \mathcal{U}^{=(0,1)}\varphi \\
\forall \bigcirc \varphi &\equiv \forall\, true\, \mathcal{U}^{=(0,1)}\varphi \\
\exists F^{rop\,(c_t,c_j)}\varphi &\equiv \exists\, true\, \mathcal{U}^{rop\,(c_t,c_j)}\varphi \\
\forall F^{rop\,(c_t,c_j)}\varphi &\equiv \forall\, true\, \mathcal{U}^{rop\,(c_t,c_j)}\varphi \\
\exists G^{rop\,(c_t,c_j)}\varphi &\equiv \neg\forall F^{rop\,(c_t,c_j)}\neg\varphi \\
\forall G^{rop\,(c_t,c_j)}\varphi &\equiv \neg\exists F^{rop\,(c_t,c_j)}\neg\varphi
\end{aligned}$$

Finally, we avoid to mention explicitely $rop\,(c_t, c_j)$ on operators $\mathcal{U}$, F and G whenever it coincides with $\geq (0,0)$.

3.2.1 $\mathcal{H}$TCTL and CTL

In this section we study the relationship between $\mathcal{H}$TCTL and CTL. We use a hybrid system to model a discrete process and we compare the semantics of $\mathcal{H}$TCTL formulas, interpreted on this specific hybrid system, with the semantics of CTL formulas, interpreted with the usual semantics given by the Kripke structure [39] that abstract the discrete process. It turns out that, for this specific case, a suitable subset of $\mathcal{H}$TCTL formulas has the same *expressivity* of

CTL formulas, that is, for each state x and for each $\mathcal{H}$TCTL formula $\varphi_{\mathcal{H}TCTL}$, if x satisfies $\varphi_{\mathcal{H}TCTL}$, then there is a CTL formula φ_{CTL} such that x satisfies φ_{CTL}, and viceversa.

Note that several discrete processes can be modeled as hybrid systems. An example is the case of finite automata [81, 114] that can be easily rewritten to hybrid systems.

Example 3.3 Let $A = (Q, \Sigma, q_0, F, \delta)$ be a finite automaton, where:

- Q is a finite set of *states*,
- Σ is a finite alphabet,
- q_0 is an element of Q, called the *initial state*,
- $F \subseteq Q$ is the set of *final states*, and
- δ is a total function, called the *transition function* from $Q \times \Sigma$ to Q.

Usually, labeled multigraph are used to represent finite automaton. The states of the automaton are represented as nodes of the multigraph, while the transition function of the automaton is represented as labeled edges of the multigraph. Indeed, for every states $q_1, q_2 \in Q$ and for every symbol $v \in \Sigma$, if $\delta(q_1, v) = q_2$ then there exists an edge from node q_1 to node q_2 with label v. An example is given in Figure 3.3.

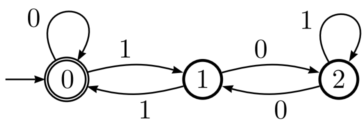


Figure 3.3: A finite automaton which accepts the empty string ε and the binary numerals denoting natural numbers divisible by 3. For example, the string 110 that denotes the number 6 is accepted, [114, Page 28].

For any given word w whose symbols are in Σ, a finite automaton starts from the initial state q_0 and reads the word from left to right, one character in *Sigma* at time. The reading oa character determines the change of the state according to an updating rule defined by the transition function δ. Indeed, for each $i \in \mathbb{Z}_{\geq 0}$, $q_{i+1} = \delta(q_i, s_i)$, where $s_i \in \Sigma$ is the ith symbol of the word w. A word is accepted by a finite automaton if the last state q_n reached when reading the last character is a final state.

It is possible to model the a finite automaton $\mathcal{A}$ as a hybrid system $\mathcal{H}$ of the form $(\mathbb{R}^2, \emptyset, \mathbb{R}^2, \emptyset, G)$, as follows. Consider a state vector $x \in \mathbb{R}^2$ whose components are $x = [\, x_p \;\; x_n \,]^T$, and define an injection *code* : $Q \to \mathbb{Z}_{\geq 0}$ that maps each state $q \in Q$ to an integer number $N \in \mathbb{Z}_{\geq 0}$. Without lost of generality, we assume $\mathit{code}(q_0) = 0$. Then,

$$\begin{bmatrix} x_p \\ x_n \end{bmatrix}^+ \in \bigcup_{s \in \Sigma} \{x_n\} \times \{\mathit{code}(\delta(s, \mathit{code}^{-1}(x_n)))\} \tag{3.4}$$

We consider also an output function $y = h(x)$ that does not modify the dynamics of the system, and it is used to produce a state based output. The output function is defined by

$$\begin{cases} s & \text{if } \mathit{code}^{-1}(x_n) = \delta(\mathit{code}^{-1}(x_p), s) \\ \varepsilon & \text{otherwise} \end{cases} \tag{3.5}$$

Consider now a solution $\xi(0,0) = [\,0 \;\; 0\,]^T$. For some $s \in \Sigma$, $\xi(0,1)$ is the vector $[\,\mathit{code}(\delta(q_0, s)) \;\; 0\,]^T$. $h(\xi(0,1)) = s$ while $h(\xi(0,0)) = \varepsilon$. From (3.4) and (3.5), each solution to $\mathcal{H}$ from $[\,0 \;\; 0\,]^T$ can be associated to a sequence of states of the automaton, and each sequence of states of the automaton can be associated to a solution to $\mathcal{H}$ starting from $[\,0 \;\; 0\,]^T$. It follows that, by h, each word generated by $\mathcal{H}$ is a word accepted by $\mathcal{A}$. Note that the *emptiness* problem on automata, that is, the problem of deciding whether or not an automaton accepts at leastone word, is equivalent to the problem of deciding whether or not $\mathit{Reach}([\,0 \;\; 0\,]^T)$ is empty.

The procedure to rewrite an automaton to a hybrid system is general and it can be applied to any discrete process defined by a transition relation that depends on the state. In general, those discrete processes can be modeled as hybrid systems of the form $\mathcal{H} = (O, \emptyset, D, \emptyset, G)$ where the flow set C is empty, so that no flow intervals occur. In that case, solutions to $\mathcal{H}$ are sequences of jumps and we are interested in comparing the semantics of $\mathcal{H}$TCTL formulas, when models with only discrete solutions are considered, and the semantics of CTL formulas.

Considering the semantics of CTL given in [13, 39] and based on infinite paths, we can define the following notion of *path from a solution ξ to a hybrid system $\mathcal{H}$*.

Definition 3.9 Consider a model M that contains only discrete solutions. An *infinite path* π *from a solution* $\xi \in M$ is a ω-sequence of states

$$\pi = \langle \xi(0,0), \xi(0,1), \dots \xi(0,n), \dots \rangle \tag{3.6}$$

A *set* Π_M *of paths from a model* M is the set

$$\Pi_M = \{\pi \,|\, \pi \text{ is an infinite path from a solution } \xi \in M\} \tag{3.7}$$

For any $i \in \mathbb{Z}_{\geq 0}$, we denote by $\pi(i)$ the ith element of a path π, that is, $\pi(i) = \xi(0,i)$[1].

With the notion of a path, we can briefly recall syntax and semantics of CTL.

Definition 3.10 Let P be a set of *atomic propositions*. A formula φ of Computation Tree Logic is inductively defined from the set of *atomic propositions* as follows:

$$\varphi \equiv p \,|\, \neg\varphi \,|\, \varphi_1 \vee \varphi_2 \,|\, \exists\varphi_1\mathcal{U}\varphi_2 \,|\, \forall\varphi_1\mathcal{U}\varphi_2 \tag{3.8}$$

For a given hybrid system $\mathcal{H}$ with state denoted by x, and for a given model M, define a set Π_M of paths from M. Then, for a given state x of $\mathcal{H}$, the semantics of a CTL formula φ can be defined as follows

$\Pi_M, x \vDash p$ iff $x \in [\![p]\!]_M$

$\Pi_M, x \vDash \neg\varphi$ iff $\Pi_M, x \nvDash \varphi$

$\Pi_M, x \vDash \varphi_1 \vee \varphi_2$ iff $\Pi_M, x \vDash \varphi_1$ or $\Pi_M, x \vDash \varphi_2$;

$\Pi_M, x \vDash \exists\varphi_1\mathcal{U}\varphi_2$ iff there exists a path $\pi \in \Pi_M$ with $\pi(0) = x$, such that

- $\exists J \in \mathbb{Z}_{\geq 0}$ such that $\Pi_M, \pi(J) \vDash \varphi_2$, and $\forall j \leq J$, $\Pi_M, \pi(j) \vDash \varphi_1$.

$\Pi_M, x \vDash \forall\varphi_1\mathcal{U}\varphi_2$ iff for all paths $\pi \in \Pi_M$ with $\pi(0) = x$,

- $\exists J \in \mathbb{Z}_{\geq 0}$ such that $\Pi_M, \pi(J) \vDash \varphi_2$, and $\forall j \leq J$, $\Pi_M, \pi(j) \vDash \varphi_1$.

We can now state the main result of this section. To avoid confusion, for a model M, a state x, and a $\mathcal{H}$TCTL formula φ, we denote the satisfiability relation of $\mathcal{H}$TCTL as $M, x \vDash^{\mathcal{H}TCTL} \varphi$. For a set Π_M of paths, a state x and a CTL formula φ, we denote the satisfiability relation of CTL as $\Pi_M, x \vDash^{CTL} \psi$.

[1]Note that each solution ξ in a model M has an unbounded domain. Therefore, each solution ξ to M allows for the construction of an infinite path.

Theorem 3.1 *Consider a hybrid system $\mathcal{H}$, a model M of discrete solutions to $\mathcal{H}$, and a set Π_M of paths from M. Consider a set of atomic proposition P and the subset of $\mathcal{H}TCTL$ formulas defined inductively by the following syntax:*

$$\varphi \equiv p \,|\, \neg\varphi \,|\, \varphi_1 \vee \varphi_2 \,|\, \exists\varphi_1 \mathcal{U} \varphi_2 \,|\, \forall\varphi_1 \mathcal{U} \varphi_2 \tag{3.9}$$

where p belongs to P. Then, for each formula φ expressed using the syntax in (3.9),

$$M, x \vDash^{\mathcal{H}TCTL} \varphi \qquad \textit{iff} \qquad \Pi_M, x \vDash^{CTL} \varphi \tag{3.10}$$

Proof. Note that the subset of formulas of $\mathcal{H}$TCTL defined by (3.9) and the set of formulas of CTL have the same syntax. M is a model, therefore each solution is complete, that is, from each solution ξ to $\mathcal{H}$ we can construct an infinite path from ξ. Then, the result follows by induction on the syntax rules, comparing the CTL semantics interpreted over paths from solutions $\xi \in M$, and the $\mathcal{H}$TCTL semantics interpreted over solutions $\xi \in M$. Note that, in CTL, $\exists\varphi_1\mathcal{U}\varphi_2$ is equivalent to $\exists(\varphi_1 \vee \varphi_2)\mathcal{U}\varphi_2$, and $\forall\varphi_1\mathcal{U}\varphi_2$ is equivalent to $\forall(\varphi_1 \vee \varphi_2)\mathcal{U}\varphi_2$. Note also that usual CTL operators $\exists F$, $\forall F$, $\exists G$, $\forall G$, $\exists\bigcirc$, $\forall\bigcirc$ can be defined by using operators in (3.9) as specified in Definition 3.8. □

3.3 Abstractions

By looking at the inductive definition of syntax and semantics of $\mathcal{H}$TCTL, it is possible to see one of the main advantage of using formalized languages to express properties of hybrid systems: the possibility of inferring the truth of a formula by a compositional evaluation of the truth of its subformulas. For example, consider a hybrid system $\mathcal{H}$ and a given state x of $\mathcal{H}$, and suppose that φ is the formula $\varphi \equiv \varphi_1 \wedge \exists F\varphi_2$, based on subformulas φ_1 and $\exists F\varphi_2$. The problem of deciding whether or not x satisfies φ in a given model M, that is $M, x \vDash \varphi$, can be addressed by deciding the truth of each subformula φ_1 and $\exists F\varphi_2$ and composing those *truth values* according to the semantics of $\wedge$. Thus, *iterative evaluation* with base case on atomic propositions, and *composition* of such evaluations, characterize a general approach to the solution of the decision problem $M, x \vDash \varphi$.

$\mathcal{H}$TCTL formulas involve two different kinds of quantifiers: one which quantifies on the time instants of a solution and which quantifies on the solution of a given set of solutions to a hybrid system. Therefore, an iterative procedure to evaluate the truth of a formula must deal with them. We do so asl follows.

We define two functions between states of $\mathcal{H}$, whose iterated application capture the motion of the state given by the solutions to $\mathcal{H}$. Then, we use that functions to evaluate the truth of formulas that involve operators like $\mathcal{U}$, G and F. Indeed, for any given hybrid system $\mathcal{H} = (O, C, D, F, G)$, we consider two functions $\delta_f : 2^O \times O \to 2^O$ and $\delta_b : 2^O \times O \to 2^O$ such that

(i) δ_f maps each state x to the set X of states that are reachable either by a continuous flow from x or by a jump from x (*forward*), that is, intuitively, δ_f maps x to the set of points y such that there exists a solution ξ to $\mathcal{H}$ and a time $(t, j) \in \operatorname{dom} \xi$ such that $x = \xi(0, 0)$ and $y = \xi(t, j)$,with $(t, j) \in \mathbb{R}_{\geq 0} \times 0$ or $(t, j) = (0, 1)$;

(ii) δ_b maps each state x to the set X of states from which it is possible to reach x by a continuous flow or by a jump (*backward*), that is, intuitively, δ_b maps x to the set of points y such that there exists a solution ξ to $\mathcal{H}$ and a time $(t, j) \in \operatorname{dom} \xi$ such that $y = \xi(0, 0)$ and $x = \xi(t, j)$, with $(t, j) \in \mathbb{R}_{\geq 0} \times 0$ or $(t, j) = (0, 1)$.

Definition 3.11 Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$. The functions $\delta_f : 2^O \times O \to 2^O$ and $\delta_b : 2^O \times O \to 2^O$, respectively map each set $S \subseteq O$ and each $x \in O$ to the set $\delta_f(S, x)$ and to the set $\delta_b(S, x)$ defined as follows:

$$
\begin{aligned}
\delta_f(S, x) = \{y \,|\, \exists \xi \text{ solution to } \mathcal{H} \text{ with } \xi(0,0) = x \in S \text{ and,}\\
&\text{either } (0,1) \in \operatorname{dom} \xi,\ \xi(0,1) = y \in S\\
&\text{or } \exists (t,0) \in \operatorname{dom} \xi,\ \xi(t,0) = y \text{ and } \forall 0 \leq \tau \leq t, \xi(\tau, 0) \in S\}\\
\delta_b(S, x) = \{y \,|\, \exists \xi \text{ solution to } \mathcal{H} \text{ with } \xi(0,0) = y \in S \text{ and,}\\
&\text{either } (0,1) \in \operatorname{dom} \xi,\ \xi(0,1) = x \in S\\
&\text{or } \exists (t,0) \in \operatorname{dom} \xi,\ \xi(t,0) = x \text{ and } \forall 0 \leq \tau \leq t, \xi(\tau, 0) \in S\}
\end{aligned}
$$

The first argument S of δ_f and δ_b can be interpreted as a parameter used to enforce some conditions on the set of solutions ξ considered during the evaluation of δ_f and δ_b. Indeed, from the definition of δ_f and δ_b, a solution ξ to $\mathcal{H}$ must *stay* within the set S, that is, for some $(t, j) \in \mathbb{R}_{\geq 0} \times \{0\}$ or $(t, j) = (0, 1)$, if $x = \xi(0, 0)$ and $y = \xi(t, j)$ then, for all $(\tau, i) \in \operatorname{dom} \xi$ and $(\tau, i) \leq (t, j)$, $\xi(\tau, i) \in S$. An example is represented in Figure 3.4.
We can generlize $\delta_f(S, x)$ and $\delta_b(S, x)$ to set domains as follows.

Definition 3.12 Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and two sets $S, X \subseteq O$. The functions $\delta_f^* : 2^O \times 2^O \to 2^O$ and $\delta_b^* : 2^O \times 2^O \to 2^O$ respectively

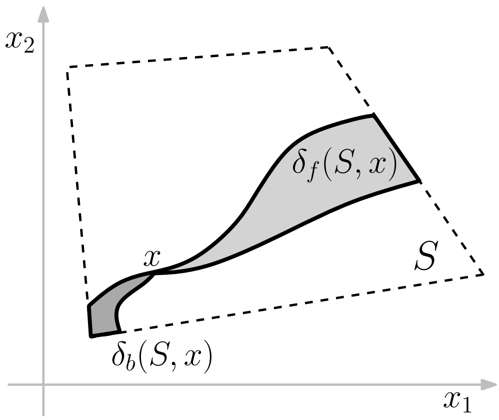


Figure 3.4: A possible representation for $\delta_f(S,x)$ and $\delta_b(S,x)$ for a system with state $x \in \mathbb{R}^2$. The set S is represented by a dashed line. $\delta_f(S,x)$ and $\delta_b(S,x)$ are represented by gray shaded shapes.

map each $S, X \subseteq O$ to the set $\delta_f^*(S,X)$ and to a set $\delta_b^*(S,X)$, defined as follows.

$$\delta_f^*(S,X) = \bigcup_{x \in X} \delta_f(S,x)$$
$$\delta_b^*(S,X) = \bigcup_{x \in X} \delta_b(S,x)$$

We denote the argument S of δ_f^* and δ_b^* the *parameterization* of δ_f^* and δ_b^*.

Consider a parameterization $S \subseteq O$, then δ_f^* maps a set $X \subseteq O$ to a set $Y \subseteq O$ defined by the set of states y reachable, either by a continuous flow or by a jump, by some solution ξ to $\mathcal{H}$ with $\xi(0,0) = x \in X$, provided that if $y = \xi(t,j)$, for some $(t,j) \in \operatorname{dom}\xi$, then for each $(\tau,i) \in \operatorname{dom}\xi$ and $(t,i) \leq (t,j)$, $\xi(t,i)$ stays within the set S. Similarily for δ_b^*.

From a theoretical point of view, δ_f^* and δ_b^* define an important connection between solutions to a hybrid system $\mathcal{H}$ and states of $\mathcal{H}$: a set of solutions can be studied by iterated application of those functions on sets of states. Therefore, the approach to the analysis of a hybrid system based on the study of its solutions can be replaced by an approach that works by iterated applications of functions δ_f^* and δ_b^* to subsets of $\mathcal{O}$. For example, by fusion and suffix closure of solutions to $\mathcal{H}$, in Definition 3.2, for a given hybrid system $\mathcal{H}$ and a given set X

of states, it is possible to use δ_f^* to generate the set of reachable states from X (Definition 1.6). To see this, (i) we characterize some important features of δ_f^* and δ_b^*, (ii) we define a specific iterated application of these functions on X, and (iii) we show that the iterated application of δ_f^* constructs the set $\mathit{Reach}(X)$, of Definition 1.6. Note that the characterization of $\mathit{Reach}(X)$ by functions on sets will involve only δ_f^*. δ_b^* will be extensively used in the next section to give a fixpoint characterization of $\mathcal{H}$TCTL formulas.

With the following proposition, we show that δ_f^* and δ_b^* are *monotonic* and $\cup$-*continuous*. Note that $\cap$-continuity does not hold. See Section 6.3.1 for the proofs.

Proposition 3.1 *Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$, a parameterization set $S \subseteq O$, and two sets $X_0, X_1 \subseteq O$,*

$$\text{if } X_0 \subseteq X_1 \text{ then } \delta_f^*(S, X_0) \subseteq \delta_f^*(S, X_1) \text{ and } \delta_b^*(S, X_0) \subseteq \delta_b^*(S, X_1). \tag{3.11}$$

Moreover, for any given ω-chain $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq \dots$ of subsets of O,

$$\delta_f^*(S, \cup_i X_i) = \cup_i \delta_f^*(S, X_i) \text{ and } \delta_b^*(S, \cup_i X_i) = \cup_i \delta_b^*(S, X_i). \tag{3.12}$$

The iterated application of δ_f^* to a given set X_0, computes the set of states reachable by solutions to $\mathcal{H}$ that start from X_0. This is a general result, summarized in the following proposition, whose proof is in Section 6.3.2.

Proposition 3.2 *Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and a set $X_0 \subseteq C \cup D$.*

$$\mathit{Reach}(X_0) = \mu X. X_0 \cup \delta_f^*(O, X) \tag{3.13}$$

By monotonicity of δ_f^* in Proposition 3.1, the least fixpoint exists, [7, Theorem 1.2.8], and can be computed by iterated application of δ_f^*, [7, Theorem 1.2.11]. See also [144]. Moreover, as stated in the following propostion, the iteration for computing the least fixpoint is not greater than ω, [7, Theorem 1.2.14]. The proof of Proposition 3.3 is in Section 6.3.3[2].

Proposition 3.3 *Consider a hybrid system $\mathcal{H}$ and a set $X_0 \subseteq O$.*

$$\mathit{Reach}(X_0) = \bigcup_{i \in \omega} \left(\lambda X. \left(\delta_f^*(O, X)\right)\right)^i X_0 \tag{3.14}$$

[2]We recall that $\lambda X. \delta_f^*(O, X)$ denotes the function in $2^O \to 2^O$ that maps each $X \subseteq O$ to the set $\delta_f^*(O, X) \subseteq O$.

The problem of Turing-computability of δ_f^* and δ_b^*, has been addressed for several classes of hybrid systems in recent years, [5, 6, 10, 54, 69, 71, 91, 93, 103, 117, 150]. In this thesis we assume δ_f^* and δ_b^* as Turing-computable. The existence of an algorithm to compute $\delta_f^*(S, X)$ and $\delta_b^*(S, X)$, for a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and for given $S, X \in O$, depends on the particular class of hybrid systems considered and on the shape of S and X. In fact, the definition of δ_f^* and δ_b^* is based on solutions ξ to $\mathcal{H}$. Therefore, two main issues may occur:

– A solution ξ to $\mathcal{H}$ cannot be defined by a mathematical expression (i.e. as a composition of polynomials, exponentials, trigonometric functions, and so on).

– A solution ξ to $\mathcal{H}$ has a mathematical expression but the existentially quantified formula in the definition of δ_f and of δ_b cannot be transformed to a formula without quantifiers, that is, it belongs to a theory that does not admit quantifier elimination.

In both cases, the definition of δ_f^* and of δ_b^* cannot be used to compute those functions. Then, approximated computation of δ_f^* and of δ_b^*, namely the computation of a set $Y \subseteq O$ such that, for any given $S, X \in O$,

– Y *over-approximates* $\delta_f^*(S, X)$, that is, $Y \supseteq \delta_f^*(S, X)$, and

– Y *under-approximates* $\delta_f^*(S, X)$, that is, $Y \subseteq \delta_f^*(S, X)$,

and analogously for δ_b^*, are usually considered when issues on Turing-computability of δ_f^* and δ_b^* occur. However, when approximated computation are introduced, the verification problem must be restricted to, a subset of the formulas of the logic, [8, 9, 37, 38, 47, 48, 70, 92].

3.4 From $\mathcal{H}$TCTL Formulas to Fixpoints

In the previous section we presented two monotonic functions, δ_f^* and δ_b^*, whose iterated application can be used to study the solutions to a hybrid system. In this section we use that functions to evaluate when a state x satisfies a given $\mathcal{H}$TCTL formula φ. The main idea is to *reduce* a formula to a fixpoint expression based on δ_b^*. Indeed, we propose a procedure that, for a given hybrid system $\mathcal{H}$, inductively reduces a $\mathcal{H}$TCTL formula φ to a fixpoint expression E such that the following equivalence holds: let M be the model defined by the set of all complete solutions to $\mathcal{H}$, let x be a state of $\mathcal{H}$ and let $|E|$ be the subset of O denoted by the fixpoint expression E, then

$$M, x \vDash \varphi \qquad \textit{iff} \qquad x \in |E| \tag{3.15}$$

We present that procedure in three steps:

- We propose the notion of *extended hybrid system* whose solutions are defined in a way so that that they take into account explictely the hybrid time. The *extended hybrid system* is defined in Section 3.4.1.
- We propose a procedure for reducing $\mathcal{H}$TCTL formulas based on general hybrid time intervals, to $\mathcal{H}$TCTL formulas based on a *normalized form* of hybrid time intervals. This procedure uses the notion of extended hybrid system and is presented in Section 3.4.2.
- Finally, we propose a procedure for reducing $\mathcal{H}$TCTL formulas based on normalized hybrid time intervals to fixpoint expressions. This procedure is presented in Section 3.4.3.

It is worth mentioning that δ_f^* and δ_b^* are defined with respect to the set of all solutions to a given hybrid system $\mathcal{H}$. Then, when a $\mathcal{H}$TCTL formula is reduced to a fixpoint expression, based on δ_b^*, we consider a semantics for that formula that is based on a premodel that coincides with the whole set of solutions to $\mathcal{H}$. In order to consider a different model or premodel M of solutions to $\mathcal{H}$, the definitions of δ_b and δ_f must be restated as follows: the quantification $\exists\xi$ *solution to* $\mathcal{H}$, in their definitions must be replaced by $\exists\xi$ *solution to* M. Then, with that new definitions, the whole construction presented below for a model defined by the set of all solutions to $\mathcal{H}$ can be extended to the case of a generic model M of solutions to $\mathcal{H}$.

In what follows, for any given hybrid system $\mathcal{H}$, we write $\mathcal{H}, x \vDash \varphi$ instead of $M, x \vDash \varphi$, to underline the fact that we are considering premodels M that coincide with the whole set of solutions to $\mathcal{H}$.

3.4.1 The Extended Hybrid System

Let us consider the $\mathcal{H}$TCTL formula $\varphi \equiv \exists\varphi_1 \mathcal{U}^{rop\,(c_t,c_j)} \varphi_2$. Let us also consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and suppose that the state x of $\mathcal{H}$ is a vector of dimension $n \in \mathbb{Z}_{\geq 0}$. Then, following Definition 3.6, for a given hybrid system $\mathcal{H}$, a state x satisfies the formula φ, that is, $\mathcal{H}, x \vDash \varphi$, if there exists a solution ξ to $\mathcal{H}$ such that

$$\begin{aligned} &\xi(0,0) = x,\ (T,J)\, \mathit{rop}\,(c_t, c_j),\ \xi(T,J) \vDash \varphi_2 \text{ and} \\ &\forall (t,j) \in \operatorname{dom}\xi,\ \text{if } (t,j) \leq (T,J) \text{ then } \xi(t,j) \vDash \varphi_1 \vee \varphi_2. \end{aligned} \tag{3.16}$$

The definition of $\mathcal{H}, x \vDash \varphi$ exhibits conditions on solutions ξ to $\mathcal{H}$ that involve (i) *state constraints*, namely conditions on *states* reached by a solution ξ at some specific time instants, and (ii) *time constraints*, namely conditions on *time instants* at which some specific state must be reached. Note that the definition of of $\mathcal{H}, x \vDash \forall \varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2$ can be decomposed in a similar way.

Following [73], by a suitable construction of a new hybrid system from $\mathcal{H}$, denoted *extended hybrid system*, we can define an equivalent definition of $\mathcal{H}, x \vDash \varphi$ that uses only state constraints. This is the first step to reduce φ to a fixpoint expression.

Definition 3.13 Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ where the state x is vector of dimension $n \in \mathbb{Z}_{\geq 0}$. The *extended hybrid system* $\mathcal{H}_{ext} = (O_{ext}, C_{ext}, D_{ext}, F_{ext}, G_{ext})$ *from* $\mathcal{H}$ can be constructed as follows. Sets O_{ext}, C_{ext} and D_{ext} are defined as follows:

- $O_{ext} = O \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$
- $C_{ext} = C \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$
- $D_{ext} = D \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$

For each $x_{ext} = [\,x^T\ t\ \ j\,]^T \in O_{ext}$, where $x \in O$, $t \in \mathbb{R}_{\geq 0}$ and $j \in \mathbb{R}_{\geq 0}$, the set-valued mappings $F_{ext} : O_{ext} \rightrightarrows \mathbb{R}^{n+2}$ and $G_{ext} : O_{ext} \rightrightarrows O_{ext}$ are defined as follows:

- $F_{ext}(x_{ext}) = \begin{bmatrix} F(x) \\ 1 \\ 0 \end{bmatrix}$
- $G_{ext}(x_{ext}) = \begin{bmatrix} G(x) \\ t \\ j+1 \end{bmatrix}$

Then, $\mathcal{H}_{ext}$ can be represented as follows:

$$\mathcal{H}_{ext}:\begin{cases}\begin{bmatrix}\dot{x}\\ t\\ j\end{bmatrix}\in\begin{bmatrix}F(x)\\ 1\\ 0\end{bmatrix} & \begin{bmatrix}x\\ t\\ j\end{bmatrix}\in(C\cap O)\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\\ \begin{bmatrix}x\\ t\\ j\end{bmatrix}^{+}\in\begin{bmatrix}G(x)\\ t\\ j+1\end{bmatrix} & \begin{bmatrix}x\\ t\\ j\end{bmatrix}\in(D\cap O)\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\end{cases}\tag{3.17}$$

Consider a solution η to $\mathcal{H}_{ext}$ from the initial state $[\,x^T\ 0\ \ 0\,]^T$, $x\in O$, and write η as $[\,\eta_x^T\ \eta_t\ \eta_j\,]^T$ where η_x, η_t and η_j take into account the elements of the solution η relative to x, t and j, respectively. Then, the components η_t and η_j of η satisfy the following equations

– $\forall(t,j)\in\operatorname{dom}\eta,\ \eta_t(t,j)=t,$

– $\forall(t,j)\in\operatorname{dom}\eta,\ \eta_j(t,j)=j.$

From a different initial state than $[\,x^T\ 0\ \ 0\,]^T$, $x\in O$, those relationships between η_t,η_j and the hybrid time (t,j) is lost. Nevertheless, a solution η to $\mathcal{H}_{ext}$ from an initial state $[\,x^T\ \alpha_t\ \ \alpha_j\,]^T$, with $x\in O$, $\alpha_t\in\mathbb{R}_{\geq 0}$ and $\alpha_j\in\mathbb{R}_{\geq 0}$, satisfies the following relations:

– $\forall(t,j)\in\operatorname{dom}\eta,\ \eta_t(t,j)=\alpha_t+t,$

– $\forall(t,j)\in\operatorname{dom}\eta,\ \eta_j(t,j)=\alpha_j+j.$

We also have the following proposition on the relationship between the η_x component of a solution η to $\mathcal{H}_{ext}$ and the solutions ξ to $\mathcal{H}$. From (3.17), the η_x component of the solutions η to $\mathcal{H}_{ext}$ from the initial state $[\,x^T\ 0\ \ 0\,]$, $x\in O$ coincides with some solution ξ to $\mathcal{H}$ from x, and viceversa, as stated in the following

Proposition 3.4 *Consider a hybrid system $\mathcal{H}$, with state vector x, and consider the extended hybrid system $\mathcal{H}_{ext}$ from $\mathcal{H}$, with state vector $x_{ext}=[\,x^T\ t\ \ j\,]^T$. Then,*

– *for any $x\in O$ and any solution $\eta=[\,\eta_x^T\ \eta_t\ \ \eta_j\,]^T$ to $\mathcal{H}_{ext}$ from the initial state $x_{ext}=[\,x^T\ 0\ \ 0\,]^T$, there exists a solution ξ to $\mathcal{H}$ from the initial state x such that*

$$\begin{aligned}&\operatorname{dom}\eta=\operatorname{dom}\xi\ \textit{and}\\ &\forall(t,j)\in\operatorname{dom}\eta,\ \eta_x(t,j)=\xi(t,j),\eta_t(t,j)=t,\eta_j(t,j)=j;\end{aligned}\tag{3.18}$$

– *for any* $x \in O$ *and any solution* ξ *to* $\mathcal{H}$ *from the initial state* x*, there exists a solution* $\eta = [\,\eta_x^T \;\; \eta_t \;\; \eta_j\,]^T$ *to* $\mathcal{H}_{ext}$ *from the initial state* $x_{ext} = [\,x^T \; 0 \; 0\,]^T$*, such that* (3.18) *holds.*

Proof. This proposition is a direct consequence of the following facts: (i) both flow and jump dynamics of the component x of x_{ext} in $\mathcal{H}_{ext}$ are defined by the same set-valued mappings of the flow and jump dynamics of $\mathcal{H}$, (ii) flow and jump sets of $\mathcal{H}_{ext}$ restricted to the component x of x_{ext} are the flow and jump sets of $\mathcal{H}$, and (iii) the dynamics of the t and j components of x_{ext} in $\mathcal{H}_{ext}$ do not have effects on the dynamics of the x component of x_{ext} in $\mathcal{H}_{ext}$ and, by the definition of C_{ext} and D_{ext}, $\mathcal{H}_{ext}$ jumps if and only if $\mathcal{H}$ jumps and $\mathcal{H}_{ext}$ flows if and only if $\mathcal{H}$ flows. $\square$

From Proposition 3.4, we have that the first component η_x of a solution $\eta = [\,\eta_x^T \;\; \eta_t \;\; \eta_j\,]^T$ to the extended hybrid system $\mathcal{H}_{ext}$ from $\mathcal{H}$ is a hybrid arc that coincides with one solution ξ to $\mathcal{H}$, that is, $\operatorname{dom}\eta = \operatorname{dom}\xi$ and $forall(t,j) \in \operatorname{dom}\eta$, $\eta_x(t,j) = \xi(t,j)$. (and viceversa). Moreover, the second and the third component η_t and η_j store the hybrid time. In this sense, we say that the hybrid time is *embedded* in the solution η.

3.4.2 From Time Intervals $rop\,(c_t, c_j)$ to Time Intervals $\geq (0{,}0)$

In this section we propose a procedure to normalize the hybrid times intervals of $\mathcal{H}$TCTL formulas. We claim that, for any given hybrid system $\mathcal{H}$, and any state x of $\mathcal{H}$, the definition of $\mathcal{H}, x \vDash \varphi$, based on *state* and *time* constraints on solutions ξ to $\mathcal{H}$, can be stated by using only *state* constraints on solutions η to $\mathcal{H}_{ext}$.

Proposition 3.5 *Consider a hybrid system* $\mathcal{H}$ *with state vector* x *and consider the extended hybrid system* $\mathcal{H}_{ext}$ *from* $\mathcal{H}$ *with state vector* $x_{ext} = [x^T \;\; t \;\; j\,]^T$*. Consider also the* $\mathcal{H}$*TCTL formulas* $\exists \varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2$ *and* $\forall \varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2$*. Then,*

$$
\begin{aligned}
\mathcal{H}, x \vDash \exists \varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2 \quad &\textit{iff} \quad \mathcal{H}_{ext}, \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \vDash \exists \varphi_3 \mathcal{U} \varphi_4 \\
\mathcal{H}, x \vDash \forall \varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2 \quad &\textit{iff} \quad \mathcal{H}_{ext}, \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \vDash \forall \varphi_3 \mathcal{U} \varphi_4
\end{aligned}
\tag{3.19}
$$

where $\varphi_3 = \varphi_1 \vee \varphi_2$ *and* $\varphi_4 = \varphi_2 \wedge t\, rop\, c_t \wedge j\, rop\, c_j$

Proof. See Section 6.3.4. □

Proposition 3.5 can be justified by looking to a single solution ξ to $\mathcal{H}$ that satisfies the semantics of $\varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2$. Let ξ be a solution to $\mathcal{H}$ such that $\xi(0,0) = x$, $\mathcal{H}, \xi(T,J) \vDash \varphi_2$ for $(T,J)\; rop\; (c_t, c_j)$, $(T,J) \in \operatorname{dom}\xi$, and $\mathcal{H}, \xi(t,j) \vDash \varphi_1 \vee \varphi_2$ for each $(t,j) \leq (T,J)$. Then, from Proposition 3.4, there exists a solution $\eta = \begin{bmatrix} \eta_x^T & \eta_t & \eta_j \end{bmatrix}^T$ to $\mathcal{H}_{ext}$ from $\begin{bmatrix} x^T & 0 & 0 \end{bmatrix}^T$ such that η_x coincides with ξ, that is, $\operatorname{dom}\eta = \operatorname{dom}\xi$ and $\forall (t,j) \in \operatorname{dom}\eta$, $\eta_x(t,j) = \xi(t,j)$. It follows that $\mathcal{H}, \eta_x(T,J)$ satisfies φ_2, $\eta_t(T,J) = T\, rop\, c_t$ and $\eta_j = J\, rop\, c_j$, that is, $\mathcal{H}_{ext}, \eta(T,J) \vDash \varphi_4$. Moreover, for each $(t,j) \leq (T,J)$, there are no conditions on $\eta_t(t,j)$ and $\eta_j(t,j)$ while $\eta_x(t,j) = \xi(t,j)$. Therefore, for each $(t,j) \leq (T,J)$, $\mathcal{H}_{ext}, \eta \vDash \varphi_3$.

Remark 3.3 Note that the results in Proposition 3.4 can be restricted to subsets of solutions ξ to $\mathcal{H}$ that define a premodel $M_{\mathcal{H}}$ to $\mathcal{H}$. In that case, the premodel $M_{\mathcal{H}_{ext}}$ would define the set of solution η to $\mathcal{H}_{ext}$ such that, for each solution ξ of $M_{\mathcal{H}}$, there exist a solution $\eta \in \mathcal{H}_{ext}$, such that $\operatorname{dom}\eta = \operatorname{dom}\xi$ and $\forall (t,j) \in \operatorname{dom}\eta$, $\eta_x(t,j) = \xi(t,j)$ and $\eta_t(0,0) = 0$ and $\eta_j(0,0) = 0$, and viceversa.

From Proposition 3.5, conditions on time instants and conditions on states in a given $\mathcal{H}$TCTL formula can be re-casted to a $\mathcal{H}$TCTL formula that uses only state conditions. Indeed, from a theoretical point of view, the differences between time conditions and state conditions are just syntactical, based on the fact that conditions on the time parameterization of solutions can be stated as conditions on the state of solutions to an extended hybrid system that embed the hybrid time.

Proposition 3.5 has also a practical application for the reduction of a $\mathcal{H}$TCTL formula to a fixpoint expression. Fixpoint expressions will be defined by considering variables ranging over subsets of the state-space, therefore a necessarily step for reducing a $\mathcal{H}$TCTL formula φ to a fixpoint expression is to rewrite φ to an equivalent formula that involves only state constraints. In this sense, Proposition 3.5 define a procedure to reduce a $\mathcal{H}$TCTL formula to an equivalent $\mathcal{H}$TCTL formula (based on the notion of extended hybrid system) that involves only state constraints. Then, following [73, Section 5], the reduction of any given $\mathcal{H}$TCTL formula φ to a fixpoint expression can be finally performed by defining a procedure that reduce $\mathcal{H}$TCTL formulas $\exists \varphi_3 \mathcal{U} \varphi_4$ and $\forall \varphi_3 \mathcal{U} \varphi_4$ to a fixpoint expression.

3.4.3 From $\exists\varphi_1\mathcal{U}\varphi_2$ and $\forall\varphi_1\mathcal{U}\varphi_2$ to Fixpoints

For any given hybrid system $\mathcal{H}$, a procedure to reduce a $\mathcal{H}$TCTL formula to a fixpoint expression relies on a suitable use of extended hybrid systems from $\mathcal{H}$ and on the reduction of $\mathcal{H}$TCTL formulas $\exists\varphi_1\mathcal{U}\varphi_2$ and $\forall\varphi_1\mathcal{U}\varphi_2$, to fixpoint expressions.

In this section, for any given hybrid system $\mathcal{H} = (O, C, D, F, G)$, we propose a procedure for reducing a $\mathcal{H}$TCTL formula of the form $\exists\varphi_1\mathcal{U}\varphi_2$ to a fixpoint expression E and we claim that the verification problem $\mathcal{H}, x \vDash \exists\varphi_1\mathcal{U}\varphi_2$ is equivalent to the membership problem $x \in |E|$, where $|E|$ is the subset of O denoted by E. We follow a similar approach for $\mathcal{H}, x \vDash \forall\varphi_1\mathcal{U}\varphi_2$.

Note that we do not distinguish here between a hybrid system $\mathcal{H}$ and an extended hybrid systems $\mathcal{H}_{ext}$ from $\mathcal{H}$, by the fact that the reduction procedure and the equivalence between verification problem and membership problem apply on both $\mathcal{H}, x \vDash \exists\varphi_1\mathcal{U}\varphi_2$ and $\mathcal{H}_{ext}, x_{ext} \vDash \exists\varphi_1\mathcal{U}\varphi_2$, where x is the state vector of $\mathcal{H}$ and x_{ext} is the state vector of $\mathcal{H}_{ext}$. Analogously for $\forall\varphi_1\mathcal{U}\varphi_2$

For simplicity of the notation, in this section we use the following definitions.

- a solution ξ *stays* in a set S if $\forall(t, j) \in \operatorname{dom}\xi$, $\xi(t, j) \in S$;
- a solution ξ *stays initially* in a set S if $\forall(t, j) \in \operatorname{dom}\xi$ such that $(t, j) \in \mathbb{R}_{\geq 0} \times \{0\}$ or $(t, j) \in \{(0, 0), (0, 1)\}$, $\xi(t, j) \in S$;
- a solution ξ *reaches* the set S, if for some $(t, j) \in \operatorname{dom}\xi$, $\xi(t, j) \in S$[3];
- a solution ξ *reaches initially* the set S, if for some $(t, j) \in \operatorname{dom}\xi$ such that $(t, j) \in \mathbb{R}_{\geq 0} \times \{0\}$ or $(t, j) \in \{(0, 0), (0, 1)\}$, $\xi(t, j) \in S$.

Following [39, Chapter 6] and [73, Section 5] we can state the following result.

Proposition 3.6 *Consider a hybrid system $\mathcal{H}$ and two $\mathcal{H}$TCTL formulas φ_1 and φ_2. Then,*

$$\mathcal{H}, x \vDash \exists\varphi_1\mathcal{U}\varphi_2 \quad \textit{iff} \quad x \in \mu X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X) \tag{3.20}$$

Proof. See Section 6.3.5 □

Proposition 3.6 can be explained by considering the function $f = \lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)$, At each application, f explores backward the solutions ξ that stay in $[\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$ and that reach $[\![\varphi_2]\!]_{\mathcal{H}}$ in a given number of flow intervals and

[3]This notion is recalled from Definition 1.6.

jumps. For instance, $f(\emptyset) = [\![\varphi_2]\!]_{\mathcal{H}}$; $f^2(\emptyset)$ computes the set of states x from which there is a solution ξ that stays initially in $[\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$ and reaches initially $[\![\varphi_2]\!]_{\mathcal{H}}$. Generalizing this approach, for any $n \in \omega$, $f^n(\emptyset)$ computes the set of states from which a solution ξ stays in $[\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$ and reaches $[\![\varphi_2]\!]_{\mathcal{H}}$ in at most $n \in \mathbb{Z}_{\geq 0}$ flow intervals or jumps. Therefore, at each application, f finds new initial states of solutions that satisfy the semantics of $[\![\varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$.

Reducing $\varphi \equiv \forall \varphi_1 \mathcal{U} \varphi_2$ to a fixpoint expression is much more complicated. To see this, consider a state x of a given hybrid system $\mathcal{H}$ and suppose that it satisfies φ, that is, each solution ξ to $\mathcal{H}$ from x satisfies the semantics of $\varphi_1 \mathcal{U} \varphi_2$. For simplicity, we suppose that for some $x \notin [\![\varphi_2]\!]_{\mathcal{H}}$ there exists only one solution ξ to $\mathcal{H}$ with $\xi(0,0) = x$ such that

- $\exists (T, 0) \in \operatorname{dom} \xi$ such that $\xi(T, 0) \in [\![\varphi_2]\!]_{\mathcal{H}}$ and
- $\forall (t, 0) \in \operatorname{dom} \xi$, $(t, 0) \leq (T, 0)$, $\xi(t, 0) \in [\![\varphi_1]\!]_{\mathcal{H}}$

However, using δ_b^*, we have that $x \in \delta_b^*(\{x\})$, moreover $x \in \delta_b^*(\{x\})^i$, for each $i \in \omega$. Therefore, despite $\mathcal{H}, x \vDash \varphi$, δ_b^* generates, for example, a sequence of states of the following form

$$x = \xi(0,0) \to \xi(0,0) \to \xi(0,0) \to \xi(0,0) \to \xi(0,0) \to \dots \tag{3.21}$$

That sequence of states denotes a hybrid arc that (i) it is not a solution to $\mathcal{H}$, and (ii) it does not satisfy the semantics of $\varphi_1 \mathcal{U} \varphi_2$. Unfortunately, by using δ_b^*, that sequence of states cannot be distinguished from a solution to $\mathcal{H}$. It follows that δ_b^* may introduce new solutions/hybrid arcs from x that possibly do not satisfy the semantics of $\varphi_1 \mathcal{U} \varphi_2$.

This is a side effect of using iteratively δ_b^* instead of performing a direct analysis of solutions to $\mathcal{H}$, and it is at the base of the problems of computing $\forall \varphi_1 \mathcal{U} \varphi_2$ as a fixpoint. These problems do not occur for $\exists \varphi_1 \mathcal{U} \varphi_2$ for which we need that *at least one solution* satisfies the semantics of $\varphi_1 \mathcal{U} \varphi_2$.

Following [73, Section 5.3], we reduce $\varphi \equiv \forall \varphi_1 \mathcal{U} \varphi_2$ to a fixpoint expression, by the following steps:

- A new function $\overline{\delta}_b^*$ is defined (Definition 3.14). It is quite similar to δ_b^* but it is based on a universal quantification on solutions ξ to $\mathcal{H}$.
- In Lemma 3.2, φ is reduced to a fixpoint expression based on $\overline{\delta}_b^*$.
- The notion of *finite variability* of a set is defined (Definition 3.15). Finite variability plays a fundamental role in the proof of the equivalence between the set computed by $\overline{\delta}_b^*$ and the set denoted by a suitable fixpoint expression that involves δ_b^*.

- Lemma 3.3 presents a method to reduce the definition of $\overline{\delta}_b^*$ to a fixpoint expression based on δ_b^*.
- Proposition 3.7 finally defines a procedure to reduce $\forall\varphi_1\mathcal{U}\varphi_2$ to a fixpoint expression.

Definition 3.14 Consider a hybrid system $\mathcal{H}$. The function $\overline{\delta}_b^* : (\mathbb{R}_{\geq 0} \cup \infty) \times 2^O \times 2^O \to 2^O$, *parameterized* with respect to $c \in \mathbb{R}_{\geq 0}$ and $S \in O$, maps each set $X \in O$ to the set $\overline{\delta}_b^*(c, S, X)$, defined as follows

$$\begin{aligned}\overline{\delta}_b^*(c, S, X) = \{x \mid \forall\xi \text{ solution to } \mathcal{H} \text{ with } \xi(0,0) = x,\\ \text{either } (0,1) \in \operatorname{dom}\xi,\ \xi(0,1) \in X \text{ and } x \in S \cup X,\\ \text{or } \exists(t,0) \in \operatorname{dom}\xi,\ t \leq c,\ \xi(t,0) \in X \text{ and } \forall 0 \leq \tau \leq t,\ \xi(\tau,0) \in S \cup X\}\end{aligned}$$

$\overline{\delta}_b^*(\infty, S, X)$ and $\delta_b^*(S, X)$ differ for the quantification on solutions to $\mathcal{H}$. While $x \in \delta_b^*(S, X)$ if there exists *at least* one solution ξ to $\mathcal{H}$ from x that satisfies a given property, say P, $x \in \overline{\delta}_b^*(\infty, S, X)$ if *each* solution to $\mathcal{H}$ from x satisfies P. It follows that $x \in \overline{\delta}_b^*(\infty, S, X)$ implies $x \in \delta_b^*(S, X)$, provided that x is the initial state of at least one solution to $\mathcal{H}$. The role of c in $\overline{\delta}_b^*(c, S, X)$ is to give a bound on the length of flows intervals during the computation of $\overline{\delta}_b^*(c, S, X)$, that is, each solution ξ to $\mathcal{H}$ must reach x by a bounded interval of flow, whose time length is shorter than c, or by one jump to x.

$\overline{\delta}_b^*$ is used in the following lemma to reduce the $\mathcal{H}$TCTL formula $\forall\varphi_1\mathcal{U}\varphi_2$ to a fixpoint expression. The role of $\overline{\delta}_b^*$ can be shown by the following example. Consider a constant $c \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and consider a state $x \in \overline{\delta}_b^*(c, [\![\varphi_1]\!]_{\mathcal{H}}, [\![\varphi_2]\!]_{\mathcal{H}})$. Then, each solution ξ to $\mathcal{H}$ starting from x reaches φ_2 by one jump or within an interval of continuous flows shorter than c, and it remains inside the set $[\![\varphi_1]\!]_{\mathcal{H}} \cup [\![\varphi_2]\!]_{\mathcal{H}} = [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$. Therefore, x would belong to $[\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$.

Note that, for any given costant c and set $S \subseteq O$, $\lambda X.\overline{\delta}_b^*(c, S, X)$ is monotonic and $\cup$-continuous (the proof of this fact is analogous to that of Proposition 3.1, for δ_b^*).

Lemma 3.2 *Consider a hybrid system $\mathcal{H}$ and two $\mathcal{H}$TCTL formulas φ_1 and φ_2. Suppose that each solution to $\mathcal{H}$ is complete. Then, for any given $c \in \mathbb{R}_{\geq 0} \cup \{\infty\}$,*

$$\mathcal{H}, x \vDash \forall\varphi_1\mathcal{U}\varphi_2 \quad \textit{iff} \quad x \in \mu X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \overline{\delta}_b^*(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X) \tag{3.22}$$

Proof. See Section 6.3.6 □

By virtue of the boundedness of the flows intervals, it will be possible to replace $\overline{\delta}_b^*$ by a suitable iterative application of δ_b^*, provided that X is *finitely variable* for the solutions to $\mathcal{H}$.

Definition 3.15 Consider a hybrid system $\mathcal{H}$, a solution ξ to $\mathcal{H}$, a set M of solutions to $\mathcal{H}$ and a set $X \subseteq O$. Denote with I_j the maximal subset of $\mathbb{R}_{\geq 0}$ such that $I_j \times \{j\} \in \operatorname{dom} \xi$. Then,

- X is *finitely variable* for ξ if $\exists \tau \in \mathbb{R}_{>0}$, $\forall j \in \{j \,|\, (t,j) \in \operatorname{dom} \xi\}$, I_j can be divided in subintervals of length τ, say $\{I_j^{\tau}, I_j^{2\tau}, \ldots, I_j^{n\tau}\}$, $n \in \omega$, $I_j \subseteq \bigcup_k I_j^{k\tau}$, such that $\forall 1 \leq k \leq n$,
  (i) either $(\operatorname{gph} \xi \cap I_j^{k\tau} \times \{j\} \times O) \subseteq X$,
  (ii) or $\operatorname{gph} \xi \cap I_j^{k\tau} \times \{j\} \cap X = \emptyset$
- X is *finitely variable* for a set M of solutions to $\mathcal{H}$ if X is finitely variable for each solution ξ that belongs to M.

Finite variability is an important property for the correctness of the results of the next Lemma. Intuitively, it enforces regularity on the flow dynamics of solutions ξ to $\mathcal{H}$, so that, for a small quantity τ of time, the solution ξ remains inside or outside of X for at least τ units of time. For example, by looking at Figure 3.5, X is finitely variable for the solution ξ_2. In fact, ξ_2 crosses the border of X infinitely many times by jumping, and the flow intervals decreases in length after each jump. But each flow interval is entirely confined either inside or outside X, therefore we can find a τ for which ξ considered on subintervals of length at most τ is either completely inside of X or completely outside of X. Nevertheless, the solution ξ_1 flows across the border at X for each flow interval, and each flow interval shrinks after each jump. Then, we cannot find a small τ for which the solution ξ considered on a piece of length at most τ of each flow interval, is entirely confined either inside or outside of X, that is, X is not finitely variable for ξ_1.

Example 3.4 Consider a hybrid system $\mathcal{H}$ and a continuous solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) = 0$ and such that, for each $t \in \mathbb{R}_{\geq 0}$, it satisfies the following equation.

$$\begin{cases} \xi(t,0) \in X & \text{if } t = \frac{1}{n}, \text{ for some } n \in \omega, \\ \xi(t,0) \notin X & \text{otherwise} \end{cases} \tag{3.23}$$

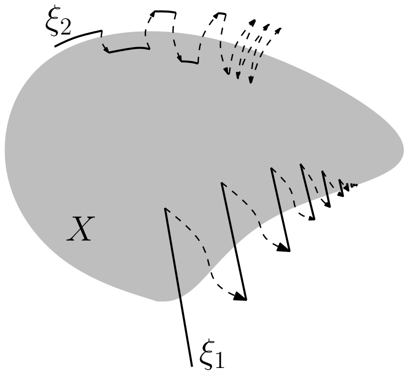


Figure 3.5: A representation of two solutions to a given hybrid system $\mathcal{H}$. The set X is represented by the gray shaded shape of the figure. X is not finitely variable for ξ_1 but it is finitely variable for ξ_2.

Then, X is not finitely variable for ξ. In fact, for any given $\tau \in \mathbb{R}_{>0}$, and any given partition of pieces $I_0^{k\tau}$, $k \in \omega$, of length τ, points (i) and (ii) of Definition 3.15 are not satisfied. To see this, take the first piece I_0^{τ}, then there exist $t_1, t_2 \in I_0^{\tau}$ for which $\xi(t_1, 0) \notin X$ and $\xi(t_2, 0) \in X$, no matter how small τ is.

Using the finite variability property, we can reduce the definition of $\overline{\delta}_b^*$ to a fixpoint expression on δ_b^*, as stated in the following lemma.

Lemma 3.3 *Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ with state dimension n. Suppose that each solution to $\mathcal{H}$ is complete (Definition 1.5). For any given sets $S, X \subseteq O$ and constant $c \in \mathbb{R}_{>0}$, if X is finitely variable for each solution ξ to $\mathcal{H}$ then the set of states computed by $\overline{\delta}_b^*(c, S, X)$ coincides with the set of states computed by the following fixpoint equation on states of the extended hybrid system $\mathcal{H}_{ext} = (O_{ext}, C_{ext}, D_{ext}, F_{ext}, G_{ext})$ from $\mathcal{H}$:*

$$\overline{\delta}_b^*(c, S, X) = \mathbb{R}^n \setminus \{x \mid \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \in \mu Z. q(c, S, X) \cup \delta_b^*((\mathbb{R}^n \setminus X) \times \mathbb{R}^2, Z)\} \quad (3.24)$$

where

$$q(c, S, X) = \Big((\mathbb{R}^n \setminus (S \cup X)) \times \mathbb{R}^2 \cap [\![(t \leq c \wedge j = 0) \vee (t = 0 \wedge j \leq 1)]\!]_{\mathcal{H}_{ext}}\Big) \cup \\ \cup [\![(t > c \wedge j = 0) \vee (t = 0 \wedge j > 1)]\!]_{\mathcal{H}_{ext}}. \quad (3.25)$$

Proof. See Section 6.3.7 □

The following proposition collects all the results in previous lemmas. Therefore, Proposition 3.7 presents a procedure to reduce the formula $\forall\varphi_1\mathcal{U}\varphi_2$ to a fixpoint expression.

Proposition 3.7 *Consider a hybrid system $\mathcal{H} = (O, C, D, F, G)$ and suppose that each solution to $\mathcal{H}$ is complete. Consider two $\mathcal{H}$TCTL formulas φ_1 and φ_2 Then,*

$$\mathcal{H}, x \models \forall\varphi_1\mathcal{U}\varphi_2 \quad \textit{iff} \quad x \in \mu X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \overline{\delta}_b^*(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X) \tag{3.26}$$

where, under the hypothesis that the fixpoint in (3.26) *is finitely variable for each solution ξ to $\mathcal{H}$, for any given $c > 0$, $\overline{\delta}_b^*(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)$ is*

$$\mathbb{R}^n \setminus \{x \,|\, \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \in \mu Z.q(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X) \cup \delta_b^*((\mathbb{R}^n \setminus X) \times \mathbb{R}^2, Z)\} \tag{3.27}$$

where the variable Z ranges over the state-space O_{ext} of the extended hybrid system $\mathcal{H}_{ext} = (O_{ext}, C_{ext}, D_{ext}, F_{ext}, G_{ext})$ from $\mathcal{H}$, and $q(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)$ is

$$\begin{aligned}\big((\mathbb{R}^n \setminus ([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}} \cup X)) \times \mathbb{R}^2 \cap [\![(t \le c \wedge j = 0) \vee (t = 0 \wedge j \le 1)]\!]_{\mathcal{H}_{ext}}\big) \cup \\ \cup\, [\![(t > c \wedge j = 0) \vee (t = 0 \wedge j > 1)]\!]_{\mathcal{H}_{ext}}.\end{aligned} \tag{3.28}$$

Note that the intricacies of reducing a formula to a fixpoint expression occur only for formulas involving $\forall\varphi_1\mathcal{U}\varphi_2$. Usual *safety* properties like reachability can be expressed by formulas that involve only $\exists\varphi_1\mathcal{U}\varphi_2$. In those cases, following Proposition 3.6, this properties can be easily written as fixpoint equations.

3.5 The Verification Procedure

Definitions 3.5 and 3.8, Propositions 3.5, 3.6 suggest a possible verification procedure for $\mathcal{H}$TCTL formulas on hybrid systems whose solutions are complete. Such a procedure is based on the progressive reduction of formulas to fixpoint expressions, and it requires that for each equation constructed by following Proposition 3.7, the external fixpoint (namely, the fixpoint that is the one most on the left of a fixpoint expression) is finitely variable for any solution to $\mathcal{H}$.

It is worth mentioning that such a fixpoint characterization of $\mathcal{H}$TCTL formulas relies on the δ_b^* function and, in general, this function is not computable on hybrid systems. Nevertheless, there are classes of systems for which δ_b^* is computable. Timed automata [4] and linear hybrid automata [74] are examples of simple classes of hybrid systems for which δ_b^* is computable. In this case, we can use the procedure above to rewrite each formula to a fixpoint expression. Then, the set of states denoted by fixpoint expression can be computed by iterative computation of the fixpoint approximants, until the fixpoint is found. This approach has been studied in [3, 5, 73] and, for hybrid systems with simple dynamics, several tools are available for computing δ_b^* (see [135]). Note that, in that a case, finite variability is also required on each approximant of the external fixpoint of the fixpoint expression constructed by Proposition 3.7. Note also that negation and union of $\mathcal{H}$TCTL can be reduced to union and complementation of sets. Therefore, each $\mathcal{H}$TCTL formula is reduced to the union and the complemenetation of several fixpoint, each of them denoted by a fixpoint expressions. Then, each fixpoint expression is defined by at most two nested least fixpoints. That structure guarantees that the iterative computation of fixpoint approximants can be simplified, as shown in [7, Sections 1.2.3 and 1.3].

A different approach based on the fixpoint characterization of formulas is the *local* model checking of [25, 26]. Instead of computing the fixpoint by successive approximations, that method applies several rewriting rules on fixpoint expressions, based on the system dynamics. The result is a proof tree whose configuration says whether or not a given state x satisfies the formula. Note that, within the hybrid systems framework, tableaux methods in [25, 26] can be considered as a structured approach to the construction of a proof for properties of hybrid systems.

Finally, it is important to underline that the verification problem on hybrid systems is in general not decidable. In fact, for any given two counter machine M (which is powerful as a Turing machine), it is possible to construct a simple hybrid system $\mathcal{H}$, that encodes the computation of M. Therefore, reachability on hybrid systems is not decidable [71, Section 4]. Following [71, Theorem 4.1], we have the following

Theorem 3.2 *Let L_1, L_2 be two constants in $\mathbb{R}$, with $L_1 \neq L_2$. Then, there exists a hybrid system $\mathcal{H}$ with a state vector $x = [\, x_1\ x_2\,]^T \in \mathbb{R}^2$, whose continuous solutions satisfy the dynamics $\dot{x}_1 = L_1$ and $\dot{x}_2 = L_2$, and such that the reachability problem for $\mathcal{H}$ is undecidable.*

Chapter 4

Some Synthesis Problems on Constrained Systems

In the previous chapters we proposed some results on analysis of hybrid systems. We considered stability problems and we proposed some sum of squares algorithms to decide whether or not a system satisfies some given stability property. Then, we considered a temporal logic similar to CTL to express properties of interest, and we generalized the approach of [73] to deal with Zeno and discrete solutions. Here and in Chapter 5 we consider problems related to the synthesis of controllers in the classical framework of dynamical control systems.

In this chapter we consider the case of continuous processes with bounds on the inputs, and we propose controllers based on anti-windup [58] and controllers that, on a specific planar case, blend together linear and optimal control laws [53]. It is important to point out that those approaches produce non-hybrid controllers (they do not have any discrete transition) that blend together two control devices, one for global performance, like global asymptotic stability or optimal convergence, and another one for local performance, like exponential decay rate and robustness of the stabilization. The control authority is continuously moved from a control device to the other, based on a suitable relation that depends on the state of the system and on the particular features of the input signals. Thus, by relaxing the requirement on continuity of the management of control authority, it is possible to introduce switching policies and resets of the controllers state, so that hybrid phenomena occur. This approaches are not pursued here but they could be of interest for future works. Some cases of hybrid control systems, namely when hybrid controllers are used to control continuous processes, will be presented in Chapter 5.

4.1 Globally Stabilizing Quasi-Optimal Control of Planar Saturated Linear Systems

Planar systems with input saturation have been long studied in the control literature, perhaps because they are often used as a starting point for more general theories, or because of the evident advantages arising from constraining trajectories on a plane, where several results can be employed (see, e.g., [87, Chapter 2]). Moreover, for experimental purposes, planar models are already sufficient to characterize the main dynamic behavior of a wide family of plants, so that high performance control laws arising from studies on planar systems might become very effective in several applications (see, e.g., the case studies mentioned in [122] or the application in [60]). Clearly, in light of saturation, when addressing the design of high performance controllers for these systems it becomes evident that one seeks for solutions of the time-optimal (or bang-bang) type, so that the control input authority is fully exploited most of the time.

While there are several valuable studies on time-optimal control of nonlinear planar systems (see the pioneering paper [140] as well as the later work in [28] and references therein), we focus here the attention on linear saturated systems. For this class of systems, as well as for linear saturated systems in general, global asymptotic stabilization can only be achieved if the plant poles are in the closed left half plane [131, 136]. The corresponding class is called ANCBI, asymptotically null-controllable with bounded inputs. For non ANCBI planar linear systems the null controllability region is bounded in the exponentially unstable direction [52, 139].

Taking a closer look at the class of ANCBI systems is quite interesting. While linear systems with poles in the open left half plane can be globally *exponentially* stabilized by bounded inputs (this is evident, because they *already* are exponentially stable with zero input), linear ANCBI systems with at least one pole on the imaginary axis can only be globally *asymptotically* stabilized and performance becomes a key issue when designing (linear or nonlinear) global stabilizers. Note that a triple integrator cannot be globally asymptotically stabilized by a linear saturated feedback law [55].

Clearly, when looking at planar systems, this limitation does not apply. (indeed, as an example, linear saturated feedback is always stabilizing for a double integrator as long as it stabilizes the plant without saturation [49, Example 4.4]). Most of the existing literature for these systems actually applies to the double integrator case, which is probably the most critical selection among all the planar ANCBI systems. A complete overview of several recent approaches and a careful study of how they deal with practical implementation problems can be found in [122]. External stability properties of the double integrator with saturated linear feedback (in particular, $\mathcal{L}_p$, $\mathcal{L}_2$ and ISS stability, respectively) have also been addressed in [82, 85, 133]. Finally, some general approaches can be applied, as a special case, to the control of the saturated double integrator (see, e.g., [105, 101]).

We propose a family of static nonlinear controllers for ANCBI planar linear systems. The novelty of the approach stands in recognizing that when looking at plants with poles close to the imaginary axis (a perfect example being the double integrator) linear saturated state feedback typically leads to poor performance if the parameters are not suitably adjusted for different signal levels. However, these systems are always found in practical control design (think about DC motors controlling a position via the action of a torque or any system where the acceleration is a control input and the position is the measurement output) and the literature seems to lack of a high performance design methodology that overcomes the evident limitations of a linear saturated feedback by way of a nonlinear design. The key idea in the selection of our nonlinear control laws is to guarantee a closed-loop response which is extremely close to being (time or fuel) optimal, while preserving the nice robustness properties of a Lipschitz state-feedback law. Moreover, whenever useful, the control laws can be designed in such a way that locally they behave like a prescribed linear state feedback law.

In Section 4.1.1 we introduce a family of state feedback stabilizers parameterized by a nonlinear function which needs to satisfy a simple gradient condition. In Section 4.1.2 we discuss useful selections of this parameter leading to quasi-optimal responses for certain classes of systems. In Section 4.2.4 we provide several examples illustrating the advantages of the proposed stabilizers. The proof of the main theorem is in Section 6.4.

4.1.1 A Family of State Feedback Stabilizers

Consider the following linear planar saturated system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_1 x_1 - a_2 x_2 + \mathrm{sat}_M(u) \end{aligned} \tag{4.1}$$

where $\mathrm{sat}_M(\cdot)$ is the symmetric scalar saturation function with saturation limits $\pm M$ and u is the control input. For the plant (4.1), we will make the following standing assumption.

Assumption 4.1 *The linear plant (4.1) is globally stabilizable from u.*

Remark 4.1 Based on results on global stabilizability of linear systems from bounded inputs [131, 136], Assumption 4.1 holds if and only if plant (4.1) is not exponentially unstable. This, in turns, is equivalent to requiring that $a_1 \geq 0$ and $a_2 \geq 0$ because of the companion form of the state-space representation.

Given the linear saturated plant (4.1) we will study the design of a (nonlinear in general) static state feedback stabilizer:

$$u = -k\beta(x), \tag{4.2}$$

where k is a positive constant and $\beta(\cdot)$ is a suitable nonlinear function.

We will give several recipes for the static controller (4.2), geared toward the achievement of almost time-optimal and (possibly) fuel-optimal responses. Moreover, we will allow in several cases to enforce an arbitrary local linear behavior on the tail of the closed-loop responses (namely in a suitable neighborhood of the origin). To this aim, it is useful to formally define here the set of functions $\beta(\cdot)$ which are guaranteed to induce desirable stability and convergence properties on the closed-loop, as formally stated in the following assumption and theorem.

Assumption 4.2 *The function $\beta(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ in (4.2) is a locally Lipschitz function satisfying $\beta(0) = 0$ and the following*

1. *there exists a class $\mathcal{K}$ function $\eta(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that* [1]
   - *if $x_1 \geq 0$ and $x_2 \geq 0$, then $\beta(x) \geq \eta(|x|)$;*
   - *if $x_1 \leq 0$ and $x_2 \leq 0$, then $\beta(x) \leq -\eta(|x|)$;*
2. *$\frac{\partial}{\partial x_2}\beta(x) \geq 0$ a.e in $\mathbb{R}^2$ and there exists an open set $\mathcal{A}$ such that $0 \in \overline{\mathcal{A}}$ and such that $\frac{\partial}{\partial x_2}\beta(x) > 0$ a.e. in $\mathcal{A}$.*

[1] A class $\mathcal{K}$ function is zero at zero and strictly increasing.

Theorem 4.1 *Given the plant (4.1), if the control law* (4.2) *satisfies Assumption 4.2 and is such that*

$$k > \inf_{s>0} \frac{M}{\eta(s)} = \lim_{s\to+\infty} \frac{M}{\eta(s)}, \tag{4.3}$$

then the following holds:

1. *all trajectories of the closed-loop (4.1),* (4.2) *converge to the origin.*
2. *moreover if* $\beta(\cdot)$ *is differentiable at the origin and*

$$\frac{\partial}{\partial x_1}\beta(0) > -a_1, \qquad \frac{\partial}{\partial x_2}\beta(0) > -a_2, \tag{4.4}$$

then the origin is a locally exponentially stable and globally asymptotically stable equilibrium point.

Proof. See Section 6.4 □

Remark 4.2 (Interpretations of Assumption 4.2) The intuitive meaning of Assumption 4.2 is that the state feedback (4.2) should preserve the equilibrium at the origin (namely $\beta(0) = 0$) and that β is strictly positive on the first and third closed quadrants take away the origin (item 1). Note that this last property is stated in terms of a class $\mathcal{K}$ function to simplify the statement of the fact that when β grows unbounded in those quadrants (namely this class $\mathcal{K}$ function is also $\mathcal{K}_\infty$) any positive k guarantees stability because in (4.3) the right hand side becomes zero (see the following Remark 4.4). Finally, the constraint on the derivative of β with respect to x_2 at item 2 provides a sufficient condition to guarantee that the β does not induce new equilibria nor limit cycles. The peculiar requirement on the set $\mathcal{A}$ (namely that it is open and its closure contains the origin) is motivated by the fact that for any (arbitrarily small) neighborhood of the origin, we need the strict inequality to hold in a set of positive measure, contained in that neighborhood (see the proof of the theorem for details).

As a last observation, the requirements on β in Assumption 4.2 are fairly general and allow to incorporate in β quasi optimal properties in a wide range of situations (see the cases discussed in the next Section 4.1.2).

Remark 4.3 (On global exponential stability) If the plant (4.1) is exponentially stable (i.e., both a_1 and a_2 are strictly positive), then under the conditions at item 2 of Theorem 4.1 (which in this case reduce to $\frac{\partial}{\partial x_1}\beta(0) > -a_1$

because of item 2 of Assumption 4.2), global exponential stability of the closed-loop can actually be proven. This is because each trajectory can be bounded by a uniform exponential bound in a first compact time interval outside a large enough ball (this comes from the boundedness of the control input and the GES property of the plant), a uniform exponential bound on the tail of the trajectory inside a small enough ball (this comes from LES) and a uniform bound in the remaining donut applying to a compact time interval. These three bounds can be combined to construct a uniform exponential bound that holds globally.

Remark 4.4 (On the lower bound on the gain k) Whenever the function $\eta(\cdot)$ belongs to $\mathcal{K}_\infty$ (namely it is of class $\mathcal{K}$ and $\lim_{s\to+\infty}\beta(s) = +\infty$), the lower bound on k enforced by (4.3) corresponds to zero. This means that if the function $\beta(\cdot)$ is unbounded in the first and third quadrant (namely it goes to infinity as x goes to infinity along any direction with $x_1x_2 \geq 0$), then the corresponding stabilizer can be scaled by any constant (arbitrarily small) gain. In the opposite case (if $\beta(\cdot)$ is bounded in some unbounded direction in the first and third quadrant), then (4.3) enforces a lower bound on k corresponding to requiring that for large enough values of x in those quadrants, the control law (4.2) exceeds the saturation limits.

Remark 4.5 (Robustness properties from Lipschitz continuity) Note that ensuring that the proposed controller is locally Lipschitz guarantees useful robustness properties on the nonlinear closed-loop. As a matter of fact, if the conditions of item 2 of Theorem 4.1 are satisfied, so that GAS holds, then the results in [148, Theorem 2] guarantee that 1) there exists a smooth converse Lyapunov function for the closed-loop system and, as a consequence, 2) the global asymptotic stability property is robust in the sense that the system can tolerate a suitable perturbation of the dynamics via inner (namely, measurement errors) and outer (namely, actuation errors) inflations (see, [148, Definition 8] for details).

It seems important to explain why this type of robustness holds here, even though it is well known that, in general, for ANCBI systems the closed-loop cannot tolerate arbitrarily small perturbations of the system matrix A in (4.1) that make a_1 or a_2 strictly positive (as a matter of fact in that case the plant becomes exponentially unstable which cannot be globally asymptotically stabilized from a bounded input). The reason for this is that when selecting the inflation function in [148, Definition 8], this function is indeed constrained to be nonzero everywhere except for the origin. However, it is allowed to decrease and become arbitrarily small for arbitrarily large values of x, so that the bounded input still has enough authority to compensate for this perturbation. This fact is quite interesting because (thanks to the results in [148]) it allows us to use robustness properties even for ANCBI systems by ruling out the problems at infinity (typically one is interested in robustness properties for signals of reasonable size and is not so worried about arbitrarily large ones).

Remark 4.6 (Linear saturated feedback stabilizers) Note that a trivial selection of the function $\beta(\cdot)$ which satisfies Assumption 4.2 is linear state feedback of the form $\beta(x) = -\alpha_1 x_1 - \alpha_2 x_2$, with $\alpha_1 > 0$ and $\alpha_2 > 0$. This type of feedback has been studied in many papers (e.g. [52] and references therein [49, 50, 85, 122, 133]) where several stability (both internal and external) properties are established. For our case, Theorem 4.1 ensures global asymptotic (and local exponential) stability of the origin using such feedbacks. However, it is well known that the performance obtained with these linear feedback laws cannot be desirable for all ranges of signals (namely parameter selections that lead to good small signal performance, also lead to undesirable large signal transients, and vice-versa). Therefore, the nonlinear selections that we'll propose next for β will provide improved closed-loop performance.

4.1.2 Parameter Selections for Quasi Time-Optimal Responses

We briefly recall here some facts about time-optimal control laws for linear systems [11, Chapters 6 and 7]. For simplicity of the exposition, all the statements are formalized for the planar ANCBI case, therefore conditions must be added to generalize them to state space of higher dimension.

Time-Optimal Feedback Laws

Time-optimal inputs for linear plants with bounded inputs are bang-bang, that is, they can only assume the maximum and minimum allowed values, and can be expressed as a state feedback defined in terms of a suitable switching surface. The switching surface can usually be expressed either in the form $\{x : x_1 + \alpha(x_2) = 0\}$, in which case the corresponding feedback law is

$$u = -M \operatorname{sgn}(x_1 + \alpha(x_2)), \tag{4.5}$$

or in the form $\{x : x_2 + \bar{\alpha}(x_1) = 0\}$, in which case the corresponding feedback law is

$$u = -M \operatorname{sgn}(x_2 + \bar{\alpha}(x_1)). \tag{4.6}$$

For example, for a double integrator with control input bounded between $\pm M$ the optimal feedback is given by

$$u = -M\mathrm{sgn}\left(x_1 + \frac{1}{2M}x_2|x_2|\right), \tag{4.7}$$

and in this case the function $\alpha(\cdot)$ in (4.5) is locally Lipschitz, strictly increasing and such that

$$s\alpha(s) > 0, \quad \forall s \neq 0, \tag{4.8}$$

i.e. $\alpha(\cdot)$ lies strictly in the first and third quadrant (note that this implies $\alpha(0) = 0$ so that the equilibrium at the origin is preserved). For a harmonic oscillator ($a_1 = \omega^2$, $a_2 = 0$) with control input bounded between $\pm M$ the optimal feedback is given by

$$u = -M\mathrm{sgn}\left(x_2 + \bar{\alpha}(x_1)\right), \tag{4.9a}$$

$$\bar{\alpha}(x_1) = \frac{M}{\omega}\mathrm{sgn}\left(x_1\right)\sqrt{1 - \left(\frac{\omega^2}{M}x_1 - 2\left\lfloor\frac{\omega^2}{2M}x_1\right\rfloor - 1\right)^2}, \tag{4.9b}$$

where $\lfloor s \rfloor$ denotes the integer part of s (i.e. the integer h which is closest to s and such that $|h| \leq |s|$), and in this case the function $\bar{\alpha}(\cdot)$ in (4.6) is such that

$$s\bar{\alpha}(s) \geq 0, \quad \forall s, \tag{4.10}$$

i.e. $\bar{\alpha}(\cdot)$ lies in the closed first and third quadrant (see Figure 4.1). Note that $\bar{\alpha}(\cdot)$ is neither monotonically increasing nor locally Lipschitz (the Lipschitz property does not hold at $x_2 = 0$, $x_1 = \frac{2M}{\omega^2}h$, $h \in \mathbb{Z}$).

Quasi Time-Optimal, Locally Linear Lipschitz Feedback

As pointed out before, the advantage of having a locally Lipschitz (instead of a discontinuous, bang-bang) feedback consists in better robustness to noise and disturbances (see Remark 4.5); moreover, in a neighborhood of the origin it is desirable to have a linear control law in order to have at least local exponential stability.

Notice that, if the optimal feedback law has the form (4.5), and $\alpha(\cdot)$ is a strictly increasing Lipschitz function, then the function $\beta(x_1, x_2) = x_1 + \alpha(x_2)$ satisfies Assumption 4.2 and then using this $\beta(x_1, x_2)$ in (4.2) yields a Lipschitz controller ensuring global convergence according to Theorem 4.1 (this is

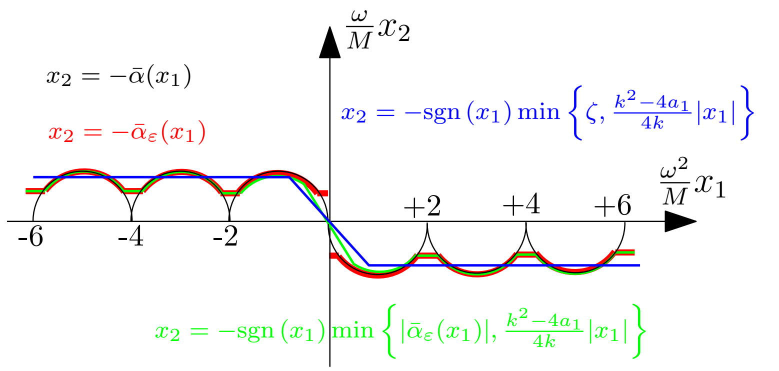


Figure 4.1: The normalized time-optimal switching curve for the harmonic oscillator and its approximations: $x_2 = -\bar{\alpha}(x_1)$ (black), $x_2 = -\bar{\alpha}_\varepsilon(x_1)$ (red).

the case for the time-optimal control of the double integrator, see (4.7) and the subsequent comments). However, in a neighborhood of the origin, the above selection of $\beta(x_1, x_2)$ can lead to a nonlinear feedback inducing a highly oscillatory behavior (this can be shown in a double integrator example); hence, it is advisable to introduce a local linear feedback inducing a critically damped local response (i.e. in the neighborhood of the origin where the control input arising from this linear feedback is inside the saturation limits, the closed loop has two coincident negative eigenvalues). The Lipschitz nonlinear control law and the local linear one can be blended by choosing $\beta(x_1, x_2)$ as[2]

$$\beta(x_1, x_2) = x_1 + \mathrm{sgn}\,(x_2)\max\left\{|\alpha(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}, \tag{4.11}$$

so that (4.2) becomes:

$$u = -k\left(x_1 + \mathrm{sgn}\,(x_2)\max\left\{|\alpha(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}\right). \tag{4.12}$$

Note that $\mathrm{sat}_M(u)$ with u given by (4.12) and the time-optimal feedback (4.5) coincide everywhere except for a stripe around the curve $x_1 + \alpha(x_2) = 0$ having width $\frac{2}{k}$ in the x_1 direction; hence, by increasing k, the above feedback

[2]For simplicity, we omit the dependence on k of $\beta(x_1, x_2)$ in (4.11); the role of this dependence is clear from (4.12).

can be made arbitrarily close to the optimal one (at the price of monotonically increasing the locally Lipschitz constants around the switching region – indeed increasing k one becomes closer and closer to the discontinuous law). In particular, for the case of the double integrator (4.12) becomes

$$u = -k\left(x_1 + x_2 \max\left\{\frac{|x_2|}{2}, \frac{2}{\sqrt{k}}\right\}\right). \tag{4.13}$$

Wholly similar comments hold if the optimal feedback law has the form (4.6), and the selection $\beta(x_1, x_2) = x_2 + \bar{\alpha}(x_1)$ is made, provided that $\alpha(\cdot)$ is a Lipschitz function contained in the first and third quadrant and such that $\liminf_{|s|\to+\infty} |\bar{\alpha}(s)| > 0$ (this condition is weaker than requiring $\bar{\alpha}(\cdot)$ to be strictly increasing). Now, in order to highlight an additional possible obstruction and its solution, consider again the optimal feedback (4.9) for the harmonic oscillator. Although the just stated condition on $\bar{\alpha}(\cdot)$ is weaker than the one required before, it is clear that the function $\bar{\alpha}(x_1)$ considered in the time-optimal feedback law (4.9) for the harmonic oscillator does not respect this condition: in fact, $\bar{\alpha}(x_1)$ is zero for $x_1 = \dfrac{2M}{\omega^2}h$, $h \in \mathbb{Z}$ (hence $\liminf_{|x_1|\to+\infty} |\bar{\alpha}(x_1)| = 0$) and is not Lipschitz at the same points (see Figure 4.1). However, both problems can be overcome by a blending which is slightly more general than the one in (4.12), where in addition to introduce a linear behavior in a neighborhood of the origin, the general time optimal switching curve $x_2 + \bar{\alpha}(x_1) = 0$ is modified as $x_2 + \bar{\alpha}_\varepsilon(x_1) = 0$, where

$$\bar{\alpha}_\varepsilon(x_1) = \operatorname{sgn}(x_1) \max\{|\bar{\alpha}(x_1)|, \varepsilon\}, \tag{4.14}$$

in order to exclude a neighborhood of each point where the Lipschitz property is violated (see Figure 4.1). The overall arising formula for the harmonic oscillator is

$$u = -k\left(x_2 + \operatorname{sgn}(x_1) \min\left\{|\bar{\alpha}_\varepsilon(x_1)|, \frac{k^2 - 4a_1}{4k}|x_1|\right\}\right), \tag{4.15}$$

where ε is a sufficiently small positive constant (in particular, $\varepsilon \in \left(0, \frac{M}{\omega}\right)$) and $\bar{\alpha}(x_1)$ is given by (4.9b). It is easy to see that, as was the case with (4.12), also (4.15) is quasi-optimal, in the sense that $\mathrm{sat}_M(u)$ with u given by (4.15) and the time-optimal feedback (4.9) coincide except on a stripe around the curve $x_2 + \bar{\alpha}(x_1) = 0$ having width proportional to k^{-1} in the x_2 direction; hence, by increasing k, the above feedback can be made arbitrarily close to the optimal one (at the price of monotonically increasing the associated local Lipschitz constants, as also commented above for the double integrator case).

Comparing the feedback law (4.15) for the harmonic oscillator and the law (4.13) for the double integrator, it is evident that the implementation of (4.15) is more complex than the implementation of (4.13); however, replacing (4.15) by the following expression

$$u = -k\left(x_2 + \operatorname{sgn}(x_1)\min\left\{\zeta, \frac{k^2 - 4a_1}{4k}|x_1|\right\}\right), \tag{4.16}$$

(where for the harmonic oscillator a typical choice of the parameters would be $k > 2\omega$ and $\zeta = M\left(\frac{1}{\omega} - \frac{1}{k}\right)$) leads to a much simpler law which also guarantees global attractiveness of the origin (due to Theorem 4.1), and better global performance than any linear stabilizing law (this is easily seen by comparing the regions where the two kinds of feedbacks differ from the time-optimal feedback law).

Remark 4.7 (Quasi-optimality, and performance-simplicity trade-off) The above discussion has highlighted that the proposed approach leads to *quasi-optimal control laws*, since it allows to recover the optimal control feedback on all $\mathbb{R}^2$ apart from a small stripe around the curve $\beta(x) = 0$ whose width is a decreasing function of k, converging to a set of measure zero when k goes to infinity.

Another useful feature of the approach is that it allows for a *trade-off between optimality and simple implementation.* In fact (again, cfr classic books as [11]), the exact switching surfaces can be rather complex, and for ease of implementation it can be desirable to choose a simpler curve as the set where $\beta(x) = 0$; as long as the $\beta(x)$ associated to this simpler curve satisfies the requirements in Assumption 4.2, the above approach guarantees global asymptotic and local exponential stability of the origin.

Of course, if an approximate simpler switching surface is used, the subset of $\mathbb{R}^2$ where the control law is really optimal will depend both on the approximation and on k, and in general will not converge to a set of measure zero when k goes to infinity (so that quasi optimality is lost, although only at the degree specified by the choice of the simpler suboptimal switching curve); at the same time, the approach is flexible enough to allow for rather general curves (e.g. far beyond globally stabilizing linear ones that in general lead to very poor performance for large enough initial states).

Remark 4.8 (Robust convergence and nominal performance) It is perhaps useful to stress that the proposed approach leads to *robust global asymptotic stabilization*, in the sense that, as far as the considered system is in the form (4.1) (namely, it has relative degree 2 from u to x_1, or, stated otherwise, the first equation preserves the kinematic nature $\dot{x}_1 = x_2$, so that x_1 can be interpreted as position and x_2 as velocity), the proposed control law will still guarantee global asymptotic stabilization, even if the values of $a_1 \geq 0$, $a_2 \geq 0$ are not the nominal values considered during the design stage; moreover, quasi-optimality will be achieved if those parameters have (or if they are very close to) their nominal values, and the function $\beta(x_1, x_2)$ and the parameter k have been chosen in order to recover the optimal feedback law.

Remark 4.9 (Step reference tracking and regulation) When at least one of the two eigenvalues of the plant (4.1) is zero (i.e. whenever $a_1 = 0$), the optimal feedback law for regulating the state to zero also provides the optimal feedback law for tracking the step reference $r(t) = \bar{r}$, $\forall t \geq 0$, provided that $\beta(x_1, x_2)$ is replaced by $\beta(x_1 - \bar{r}, x_2)$. In fact, via the change of variables $\bar{x}_1 = x_1 - \bar{r}$, $\bar{x}_2 = x_2$, the above tracking problem is easily seen to be equivalent to the problem of regulating to zero the new state $\bar{x}_1$, $\bar{x}_2$, since $\dot{\bar{x}}_1 = \bar{x}_2$ and $\dot{\bar{x}}_2 = -a_2\bar{x}_2 - \mathrm{sat}_M(k\beta(\bar{x}_1, \bar{x}_2))$ (which has exactly the same form encountered in the regulation problem). On the other hand, for $a_1 \neq 0$ an additional constant term $a_1\bar{r}$ would appear in the $\dot{\bar{x}}_2$ equation, thus preventing the regulating control law from being optimal for the tracking problem.

4.1.3 Parameter Selections for Quasi Fuel-Optimal Responses

We briefly recall here some facts about fuel-optimal control laws for linear systems [11, Chapters 6 and 8]. As before, for simplicity of the exposition, all the statements are formalized for the planar ANCBI case, therefore conditions must be added to generalize them to state space of higher dimension.

Fuel-Optimal Feedback Laws

When the objective is to minimize fuel consumption, it can be shown that optimal solutions do not exist in relevant cases (there exist open regions in the state space such that given an initial state in such a region and any control input ensuring convergence to the origin from that state, it is possible to find a different control input achieving convergence with less fuel, although in a longer time). In order to avoid such situations, it is necessary to bound the maximum allowed transfer time: hence, if $T_m(x_0)$ is the transfer time for x_0 when the time-optimal input is used,

– *fixed response time* fuel-optimal inputs guarantee that, for a given a positive value $\bar{T}$, the transfer from x_0 is achieved with minimum fuel expenditure in at most $\bar{T}$ time units if $T_m(x_0) \leq \bar{T}$ (i.e. if x_0 is close to the origin) or in $T_m(x_0)$ time units otherwise;

– *bounded response time* fuel-optimal inputs guarantee that, for a given a positive value $\gamma > 1$, the transfer from x_0 is achieved with minimum fuel expenditure in at most $\gamma T_m(x_0)$ time units.

Similarly to time-optimal inputs, (fixed or bounded response time) fuel-optimal inputs for linear time invariant systems are bang-off-bang (i.e. can only assume the maximum and minimum allowed values, plus the zero value), and can be expressed as a state feedback defined in terms of a suitable switching surface; for example, for a double integrator with control input bounded between $\pm M$ the switching surfaces in the bounded response time case are given by $\left\{x : x_1 + \frac{1}{2M} x_2 |x_2| = 0\right\}$ and $\left\{x : x_1 + \frac{m_\gamma}{M} x_2 |x_2| = 0\right\}$, where m_γ is given by (8-213) in [11, Sec. 8.7], namely

$$m_\gamma = \frac{\gamma}{2\gamma - 2\sqrt{\gamma(\gamma-1)} - 1} - \frac{1}{2}$$

and the optimal feedback is given by

$$u = \begin{cases} -\mathrm{sgn}\left(x_1 + \frac{m_\gamma}{M} x_2 |x_2|\right) & \text{if } x_1(x_1 + \frac{m_\gamma}{M} x_2 |x_2|) \geq 0 \\ -\mathrm{sgn}\left(x_1 + \frac{1}{2M} x_2 |x_2|\right) & \text{if } x_1(x_1 + \frac{1}{2M} x_2 |x_2|) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Quasi Fuel-Optimal, Locally Linear Lipschitz Feedback

The key ideas are similar to those expressed in Section 4.1.2; for brevity, the discussion will be limited to the case when the switching surfaces of interest can be expressed as $\{x : x_1 + \alpha(x_2) = 0\}$, $\{x : x_1 + \widetilde{\alpha}(x_2) = 0\}$ with $\alpha(s)$ and $\widetilde{\alpha}(s)$ two strictly increasing Lipschitz functions lying in the first and third quadrant and such that

$$s\widetilde{\alpha}(s) \geq s\alpha(s) > 0, \quad \forall s \neq 0;$$

this is the case for the fuel optimal (with either bounded or fixed response time) solution for the double integrator or plants having one null eigenvalue and one negative eigenvalue.

Defining the following functions:

$$\hat{\alpha}(s) := \max\{\widetilde{\alpha}(s), \alpha(s)\}, \quad \check{\alpha}(s) := \min\{\widetilde{\alpha}(s), \alpha(s)\},$$

and using the same blending approach already used and motivated in the first part of Section 4.1.2, the proposed control law is

$$u = \begin{cases} -k\left(x_1 + \operatorname{sgn}(x_2)\max\left\{|\check{\alpha}(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}\right) & \text{if } x \in \mathcal{R}_1, \\ -k\left(x_1 + \operatorname{sgn}(x_2)\max\left\{|\hat{\alpha}(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}\right) & \text{if } x \in \mathcal{R}_2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.17}$$

where

$$\mathcal{R}_1 = \left\{x : x_1 + \operatorname{sgn}(x_2)\max\left\{|\check{\alpha}(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\} \geq 0\right\}, \tag{4.18a}$$

$$\mathcal{R}_2 = \left\{x : x_1 + \operatorname{sgn}(x_2)\max\left\{|\hat{\alpha}(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\} \leq 0\right\}. \tag{4.18b}$$

Notice that (4.17) can also be written as

$$\begin{aligned} u = -k\min\Bigg\{& x_1 + \operatorname{sgn}(x_2)\max\left\{|\check{\alpha}(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}, \\ & \max\left\{x_1 + \operatorname{sgn}(x_2)\max\left\{|\hat{\alpha}(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}, 0\right\}\Bigg\}, \end{aligned}$$

With the obvious substitutions, remarks analogous to those in Section 4.1.2 apply here too. In particular for the double integrator, (4.17) becomes

$$u = \begin{cases} -k\left(x_1 + x_2\max\left\{\left|\min\left\{\frac{m_\gamma}{M}x_2, \frac{x_2}{2M}\right\}\right|, \frac{2}{\sqrt{k}}\right\}\right), \\ \qquad\qquad \text{if } x \in \mathcal{R}_1 \\ -k\left(x_1 + x_2\max\left\{\left|\max\left\{\frac{m_\gamma}{M}x_2, \frac{x_2}{2M}\right\}\right|, \frac{2}{\sqrt{k}}\right\}\right), \\ \qquad\qquad \text{if } x \in \mathcal{R}_2, \\ 0, \qquad \text{otherwise,} \end{cases} \tag{4.19}$$

and (4.18) in turn become

$$\mathcal{R}_1 = \left\{x : x_1 + x_2\max\left\{\left|\min\left\{\frac{m_\gamma}{M}x_2, \frac{x_2}{2M}\right\}\right|, \frac{2}{\sqrt{k}}\right\} \geq 0\right\}, \tag{4.20a}$$

$$\mathcal{R}_2 = \left\{x : x_1 + x_2\max\left\{\left|\max\left\{\frac{m_\gamma}{M}x_2, \frac{x_2}{2M}\right\}\right|, \frac{2}{\sqrt{k}}\right\} \leq 0\right\}. \tag{4.20b}$$

4.1.4 Simulation Examples

Double Integrator

The constant reference tracking problem for a double integrator plant with saturated input can be solved by a bang-bang time-optimal strategy. By blending the bang-bang feedback control strategy with a stabilizing linear feedback, it is possible to almost recover the time-optimal behavior for large error signals and to avoid problems near the equilibrium due to noise and disturbances, at the same time. Assuming a symmetric saturation with limit $M = 1$, the quasi-optimal control proposed is (4.13) and the simulations results are in fig. 4.2 where different curves corresponding to different selections of k are reported and compared to the time-optimal one to show how the response becomes closer and closer to the optimal one.

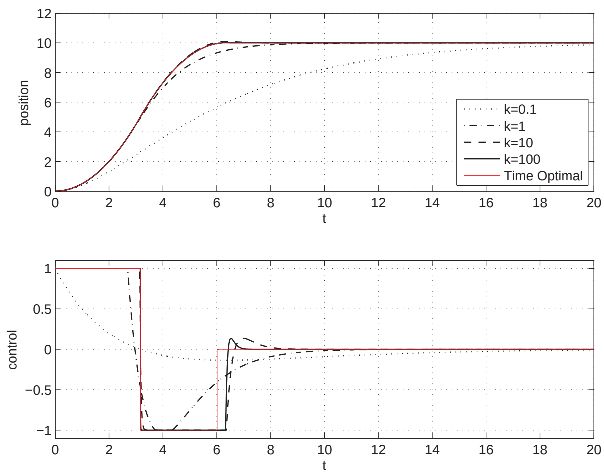


Figure 4.2: $M = 1$, quasi time-optimal strategy.

Note that the parameter k can be used to select the level of approximation of the optimal response: increasing k induces a better recovery of the optimal response and the use of a more aggressive linear control law. The selection of the constant $\frac{2}{\sqrt{k}}$ inside the max function imposes a critically damped (two coincident poles) linear closed-loop near the equilibrium, i.e. where the linear feedback law is used and the input signal is not saturated. Note that the proposed control satisfies Assumption 4.2, therefore by our main theorem it guarantees global asymptotic stability, with some loss of optimality, for any parameter variation that preserves the kinematic relation $\dot{x}_1 = x_2$ (see Remark 4.8).

A similar approach can be used to perform a quasi fuel-optimal control strategy. The control law (4.19), (4.20) blends the fuel-optimal strategy for high error levels, with a linear feedback law near the origin. Choosing $m_\gamma \approx 11.66$ forces the closed loop system with fuel-optimal strategy to converges to the equilibrium in a time no longer than two times the optimal one ($\gamma = 2$). The resulting responses are in fig. 4.3 where different curves corresponding to different selections of k are reported and compared to the fuel optimal one, similar to the previous case.

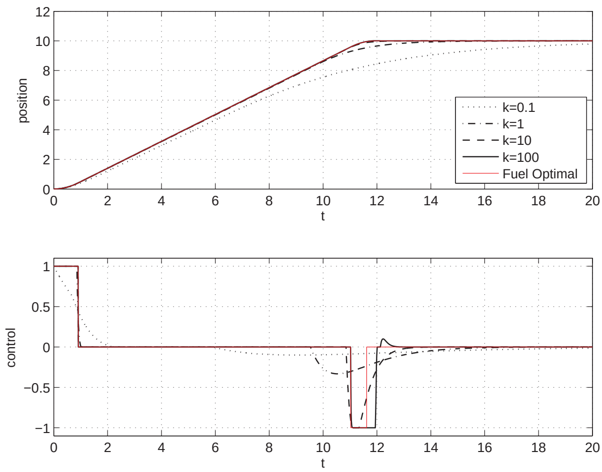


Figure 4.3: $M = 1$, $\gamma = 2$, quasi fuel-optimal strategy.

Pure Oscillator

As a last example we consider the stabilization problem for a pure oscillator of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega^2 x_1 - \mathrm{sat}(u), \end{aligned}$$

which can be solved by the quasi time-optimal control law (4.15), (4.9b).

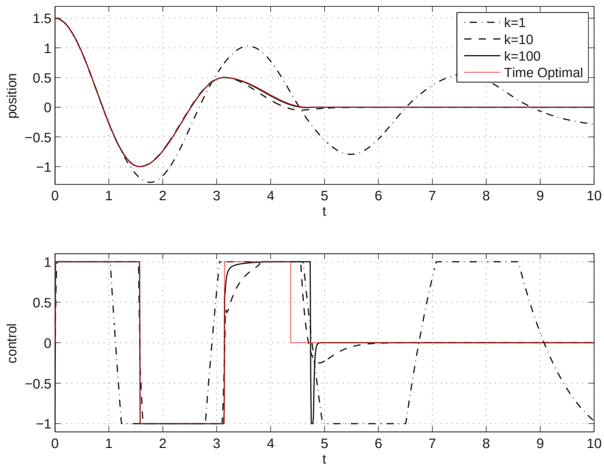


Figure 4.4: $M = 1$, $\omega = 2$, quasi time-optimal strategy.

Note that when the first argument inside min works, it guarantees (locally) a critically damped feedback linear system. The optimal $\bar{\alpha}(\cdot)$ in (4.9b) cannot be used as second argument of the min function because it doesn't satisfy the condition required at item 1 of Assumption 4.2 (it is zero for arbitrary large values of x_1 on the $x_2 = 0$ axis, and it is not Lipschitz at the same points). By selecting ε as a very small constant, the time-optimal response can be almost recovered with large values of k. Fig. 4.4 shows simulation results with increasing values of k and compares them to the time-optimal response.

4.2 Model Recovery Anti-Windup for Rate and Magnitude Saturated Plants

Input saturation is a relevant problem in any high performance control system where lightweight structures and/or full exploitance of the available input power is required. Indeed, these phenomena can be neglected whenever it is possible to oversize the actuators so that during normal operation the saturation limits are never reached by the controller command. Much research has been carried out in the past years to characterize and address the problem of magnitude and rate saturation. This arises whenever the actuator under consideration imposes constraints not only on the size of the requested input effort, but also on the variation of that request. This type of problem has been most studied in the aerospace context where it has been shown to be relevant [20, 21, 29, 134], but it also arises in plasma control systems in Tokamaks [132]. As with magnitude-only saturation, rate and magnitude saturation can be addressed by designing a controller which directly accounts for the limitations (see, e.g., the approaches in [16, 17, 65, 79, 80, 86] and references therein), or by adding some modifications to an existing small signal controller, which achieves a desirable performance as long as the saturation limits are not exceeded.

Anti-windup compensation schemes have been historically addressed in the magnitude-only saturation context, where two main approaches have been proposed to solve the problem: Direct Linear Anti-Windup for linear control systems [78] and Model Recovery Anti-Windup (MRAW), also called "$\mathcal{L}_2$ anti-windup", see [147] and also [59, 162]). In the recent literature, a number of anti-windup results considered the case of rate and magnitude saturation and, for the MRAW approach, some of them use a LMI-based design. These approaches mostly arise from selecting a suitable characterization of the rate+magnitude saturation nonlinearity, for which the LMI-based approach can be extended and successfully applied. For example, in the anti-windup solution of [21, 121], a dynamic actuator model suitable for flight control applications has been used, which incorporates both magnitude and rate saturation.

Here we propose two constructive solutions to the rate+magnitude anti-windup problem when cast into the Model Recovery Anti-Windup framework of [162, 147]. The first solution arising from treating the magnitude and rate saturation altogether as a single nonlinearity, establishing key input/output properties of this nonlinearity and performing the anti-windup action by way of a possibly linear solutions whose design can be carried out by solving a set of Bilinear Matrix Inequalities (BMIs) depending on the plant parameters. This solution leads to global results with exponentially stable plants, semiglobal results with ANCBI plants (namely plants globally stabilizable by nonlinear feedbacks) and local results in all other cases (namely regional results with guaranteed regions of attraction). In the second solution that we propose, which was preliminarily sketched in [160], the saturation is separated out into the two (magnitude and rate) components and a scheme accounting for the two phenomena separately is proposed, with extra states added to the anti-windup compensator. Also in this second case, the design method hinges upon the selection of a possibly nonlinear stabilizer for which we propose a linear design leading to global results with exponentially stable plants and local results in all other cases. As in the previous solution, we give an optimality-based design to maximize the domain of attraction. In this case, the design uses convex Linear Matrix Inequalities (LMIs), thus becoming more numerically attractive then the previous BMI solution. On the other hand, semiglobal results with ANCBI plants cannot be guaranteed here, thus making this second solution less appealing for polynomially unstable plants. In both approaches, a generalized sector condition [43, 77] is used to characterize the saturation, in order to reduce the conservativeness of the design.

We comparatively discuss the two techniques on a simulation example, which highlights that the first approach leads to tighter bounds on the achievable performance, in spite of the fact that it relies on non-convex design tools.

Notation used in this section
The placeholder $\mathrm{sat}_{MR}(\cdot)$ denotes the rate and magnitude saturation phenomenon; in particular, for any $s(\cdot) : \mathbb{R} \to \mathbb{R}^m$, the expression $\mu = \mathrm{sat}_{MR}(s)$ is a shortcut to say that μ is the unique solution of the following discontinuous dynamics:

$$\dot{\mu}(t) = \mathrm{diag}(R)\mathrm{sign}(\mathrm{sat}_M(s(t)) - \mu(t)), \tag{4.21}$$

where $\mathrm{sat}_M(\cdot)$ is the decentralized magnitude saturation with saturation limits $M := [M_1, \ldots, M_m]$, $\mathrm{sign}(\cdot)$ is the decentralized sign function [3] and $R := [R_1, \ldots, R_m]$ is the vector containing the rate saturation limits. Moreover, $\mathrm{dz}_M(s) := s - \mathrm{sat}_M(s)$ denotes the decentralized deadzone. The symbols $\mathrm{sat}_{\eta MR}(\cdot)$ or $\mathrm{sat}_{MR\eta}(\cdot)$ will be used to denote a rate and magnitude saturation rescaled by the factor $\eta \in [0, 1]$, namely such that its levels are given by ηM and ηR.

4.2.1 Problem Definition

Consider the following linear plant

$$\dot{x} = Ax + B_u u + B_d d, \tag{4.22a}$$
$$y = C_y x + D_{yu} u + D_{yd} d, \tag{4.22b}$$
$$z = C_z x + D_{zu} u + D_{zd} d, \tag{4.22c}$$

where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^m$ is the plant control input, y is the measurement output, z is the performance output and d is a disturbance input. We assume that plant (4.22) is stabilizable from u.

[3] An exact description of the discontinuous dynamics (4.21) would require the use of set valued maps and differential inclusions. However to keep the discussion simple we will abuse notation here and assume that the sign$(\cdot)$ function in (4.21), when evaluated at zero, returns the correct value to guarantee existence and uniqueness of solutions (see [158, Lemma B.1, p. 145]).

Assumption 4.3 *The pair (A, B_u) is stabilizable.*

Following the standard anti-windup approach, we assume that a controller has been already designed for plant (4.22). We make very few assumptions on the structure of the controller that can be described by the following nonlinear dynamic equations:

$$\dot{x}_c = f(x_c, u_c, r), \qquad y_c = g(x_c, u_c, r) \tag{4.23}$$

where x_c is the controller state, u_c is its measurement input and r is an external reference signal. To guarantee existence and uniqueness of solutions, we assume that both f and g are locally Lipschitz functions.

The following assumption entails the necessary property that the closed-loop between plant (4.22) and controller (4.23) is well behaved in the absence of saturation, namely with the following "unconstrained" interconnection:

$$u_c = y, \quad u = y_c. \tag{4.24}$$

Assumption 4.4 *The closed-loop between plant (4.22) and controller (4.23) via the interconnection (4.24) is well posed and forward complete.*

We address the so-called anti-windup augmentation problem for the interconnection (4.22), (4.23), (4.24) when rate and magnitude saturation affects the control input of the plant. In particular, we address and solve the problems arising when interconnecting plant (4.22) and controller (4.23) via the following saturated interconnection

$$u_c = y, \quad u = \mathrm{sat}_{MR}(y_c). \tag{4.25}$$

In light of the negative effects that can be often experienced in the saturated closed-loop (4.22), (4.23), (4.21), we address the following anti-windup augmentation problem. *For compact notation*, given external signals $r(\cdot)$ and $d(\cdot)$ and initial states for the plant (4.22) and the controller (4.23) the response of the unconstrained closed loop (given by (4.22), (4.23) and (4.24)) will be denoted by a hat $\hat{\cdot}$ over the variable of interest (*e.g.* $\hat{u}_c$ and $\hat{x}$, respectively, for the input to the controller and the state of the plant in the unconstrained closed loop), whereas the response of the anti-windup closed loop (given by (4.22), (4.23) and the anti-windup dynamics with suitable interconnection conditions) to the same external signals and initial states (and suitable initial states for the anti-windup compensator) will be denoted by a bar $\bar{\cdot}$ over the variable of interest (*e.g.* $\bar{u}_c$ and $\bar{x}$, respectively, for the input to the controller and the state of the plant in the anti-windup closed loop).

Problem 4.1 *Given the plant-controller pair (4.22), (4.23) and the magnitude and rate saturation in (4.21), design a dynamic compensator which only uses measurements from the controller signals and injects modifications at the controller input and output and whose interconnection to the plant-controller pair (4.22), (4.23) guarantees (at least one of) the following properties:*

1. *(global anti-windup) for any scalar* $\varepsilon \in (0, 1)$ *and for any pair* $(r(\cdot), d(\cdot))$ *such that* $\mathrm{dz}_{M(1-\varepsilon)}(\hat{u}) \in \mathcal{L}_2$ *and* $\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}}) \in \mathcal{L}_2$*, there exists a class* $\mathcal{K}_\infty$ *function* $\gamma(\cdot)$ *such that*
$$\|\bar{z} - \hat{z}\|_2 < \gamma\left(\left\| \begin{bmatrix} \mathrm{dz}_{M(1-\varepsilon)}(\hat{u}) \\ \mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}}) \end{bmatrix} \right\|_2\right); \qquad (4.26)$$

2. *(local anti-windup) there exists* $\rho > 0$ *and a class* $\mathcal{K}$ *function* $\gamma(\cdot)$ *such that inequality* (4.26) *holds for any scalar* $\varepsilon \in (0, 1)$*, for any initial state such that* $\|(\hat{x}(0), \hat{x}_c(0))\| < \rho$ *and for any pair* $(r(\cdot), d(\cdot))$ *such that* $\|\mathrm{dz}_{M(1-\varepsilon)}(\hat{u})\|_2 < \rho$ *and* $\|\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}})\|_2 < \rho$*;*

3. *(regional anti-windup with exponential recovery) for any scalar* $\varepsilon \in (0, 1)$ *and for any pair* $(r(\cdot), d(\cdot))$ *such that* $\mathrm{dz}_{M(1-\varepsilon)}(\hat{u})$ *and* $\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}})$ *have compact support*[4] $[0, T]$*, if the state of the anti-windup compensator for* $t = T$ *belongs to a suitable region* $\mathcal{R}$ *(possibly depending on* $\hat{y}_c(T)$*), then* $\bar{z}(t) - \hat{z}(t)$ *converges to zero exponentially fast with convergence rate* γ

Remark 4.10 Note that we don't ask for any stability or convergence property in Assumption 4.4. Although guaranteed by essentially any reasonable choice of the given controller (4.23), these properties are actually not required to state our main results that are expressed in terms of the deviation of the actual response from the unconstrained one. Clearly, saturation will impose some limits on the trackable responses, *i.e.* on the unconstrained responses that can actually be recovered by the anti-windup closed loop system; in particular, in order to be trackable a response must leave enough room (quantified by ε in Problem 4.1) for the action of the anti-windup compensator, and this is the reason why the only responses considered in Problem 4.1 are the responses such that, for some $\varepsilon \in (0, 1)$, it holds that $\mathrm{dz}_{M(1-\varepsilon)}(\hat{u}) \in \mathcal{L}_2$, $\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}}) \in \mathcal{L}_2$, and not simply the responses such that $\mathrm{dz}_M(\hat{u}) \in \mathcal{L}_2$, $\mathrm{dz}_R(\dot{\hat{u}}) \in \mathcal{L}_2$.

The difference between the requirements in item 1 and 2 of Problem 4.1 lies in the fact that the performance degradation in the anti-windup closed loop

[4] A function $f(\cdot)$ has support $[0, T]$ if $f(t) = 0$ for $t \notin [0, T]$.

with respect to the unconstrained closed loop (as measured by the $\mathcal{L}_2$ norm of the difference between the corresponding performance outputs) is guaranteed to be bounded for any response such that $\|\mathrm{dz}_{M(1-\varepsilon)}(\hat{u})\|_2$ and $\|\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}})\|_2$ are finite in item 1, whereas in item 2 it is guaranteed to be bounded only if $\|\mathrm{dz}_{M(1-\varepsilon)}(\hat{u})\|_2$, $\|\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}})\|_2$ and the initial states are sufficiently small. It will be shown in this section that it is possible to come up with solutions of either form 1) or 2) of the problem, at the price of sacrificing the performance achievable on responses such that $\|\mathrm{dz}_{M(1-\varepsilon)}(\hat{u})\|_2$ and $\|\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}})\|_2$ are small (in the global case), or at the price of not being able to guarantee the recovery of the unconstrained responses such that $\|\mathrm{dz}_{M(1-\varepsilon)}(\hat{u})\|_2$ and $\|\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}})\|_2$ are not sufficiently small (in the local case). Hence, in order to identify more desirable solutions (achieving a trade-off between the merits and the pitfalls of the "easy" solutions cited above) item 3 of Problem 4.1 is introduced, where an explicit quantification is given of both the guaranteed (exponential) convergence rate and of the size of the region of the state space of the anti-windup compensator where such a convergence rate is achieved. The reason for giving this third formulation in terms of a region in state space instead of in terms of a bound ρ on $\|\mathrm{dz}_{M(1-\varepsilon)}(\hat{u})\|_2$ and $\|\mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}})\|_2$ (as in the second formulation) is that in this way unnecessary restrictions are avoided and then the definition applies to a larger set of unconstrained responses.

4.2.2 Plant-Order Anti-Windup Solution

A first solution that we propose to Problem 4.1 is carried out along the same lines as those in [146, 15, 14], where a dynamical system reproducing the dynamics of the plant (4.22) from the control input u to the measurement output y is inserted in the closed-loop to generate the mismatch between the actual plant behavior and the virtual behavior in the absence of saturation. In particular, this system, called "anti-windup compensator" corresponds to the following dynamics:

$$\dot{x}_{aw} = Ax_{aw} + B_u(u - y_c) \tag{4.27a}$$

$$y_{aw} = C_y x_{aw} + D_{yu}(u - y_c) \tag{4.27b}$$

$$z_{aw} = C_z x_{aw} + D_{zu}(u - y_c), \tag{4.27c}$$

The anti-windup compensator (4.27) is to be connected to the plant (4.22) and the controller (4.23) via the following anti-windup interconnection equations:

$$u_c = y - y_{aw}, \quad u = \mathrm{sat}_{MR}(\mathrm{sat}_{MR(1-\varepsilon)}(y_c) + v_1), \tag{4.28}$$

where the signal v_1 is a degree of freedom left by the compensation scheme to guarantee that the actual plant response converges to the virtual response corresponding to the absence of the saturation effects. A block diagram of the overall anti-windup scheme is represented in Figure 4.5.

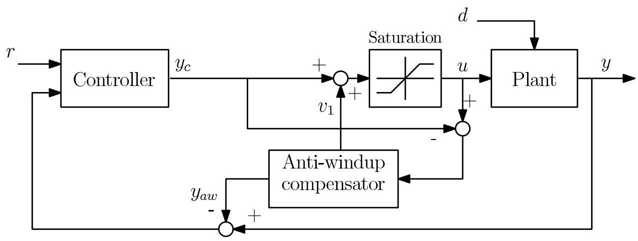


Figure 4.5: Block diagram of the plant-order anti-windup solution.

The following theorem, whose proof is reported in Section 6.4.2, establishes a few results on suitable designs for the stabilizing signal v_1 in (4.28), each of them having certain advantages and disadvantages as illustrated after the proof of the theorem.

Theorem 4.2 *Consider the anti-windup closed-loop (4.22), (4.23), (4.27), (4.28). Under Assumptions 4.3 and 4.4, the following holds:*

1. *If A is Hurwitz, then the selection $v_1 = 0$ solves the global anti-windup problem in Problem 4.1.*

2. *Selecting v_1 as any stabilizing linear state feedback for $\dot{x}_{aw} = Ax_{aw} + B_u v_1$ solves the local anti-windup problem in Problem 4.1.*

3. *Selecting $v_1 = Kx_{aw}$, where K is any feasible solution to the following BMI problem in the variables $P = P^T > 0$, K, H, $\gamma > 0$, $\frac{1}{\alpha} > 0$, $U_M > 0$ diagonal*

$$\frac{1}{\alpha} I > P \tag{4.29a}$$

$$0 > \mathrm{He} \begin{bmatrix} P(A + \gamma I + B_u K) & -PB_u \\ U_M(K - H) & -U_M \end{bmatrix} \tag{4.29b}$$

$$0 \leq \begin{bmatrix} \varepsilon^2 R_i^2 P & \star \\ [K(A + B_u K)]_i & 1 \end{bmatrix}, i = 1, \dots m, \tag{4.29c}$$

$$0 \leq \begin{bmatrix} \varepsilon^2 M_i^2 P & \star \\ [H]_i & 1 \end{bmatrix}, i = 1, \dots m, \tag{4.29d}$$

(where $\star$ denotes symmetrical terms and $[Z]_i$ denotes the i-th row of the matrix Z) solves the local anti-windup *and the* regional anti-windup with exponential recovery *problems in Problem 4.1 with exponential bound γ in the guaranteed region $\mathcal{B}(\alpha)$, namely a ball of size α.*

Each of the solutions proposed in Theorem 4.2 deserves some discussion. The solution at item 1 corresponds to a generalization of the so-called IMC anti-windup solution (which is a very well known anti-windup solution applying to the magnitude saturation case – see e.g., [89]). In particular, this solution corresponds to relying on the exponentially decaying modes of the plant (whose matrix A is Hurwitz) and is also well known for its global exponential stability guarantees in spite of poor performance when used for lightly damped plants (this is because the solution itself relies on the intrinsic exponential decay given by the plant).

The solution at item 2 corresponds to a very practical approach to the problem, wherein the saturation effects are completely disregarded in the design of v_1. Since the saturation acts like an identity for small enough signals, any such a solution will guarantee the local statement in Problem 4.1, however there's no guarantee on the size of the region from which the unconstrained performance can be recovered by the compensated system. The advantages of this solution are simplicity and local performance. The main disadvantage is the lack of stability guarantees for large signals. This solution was adopted in the application study carried out in [146].

The last solution at item 3 tries to overcome the limitations of the previous two approaches by enforcing a guaranteed exponential decay of the performance output mismatch $\bar{z} - \hat{z}$ while ensuring that this bound holds in a guaranteed region. The trade off in the BMIs (4.29) is between γ (associated with the decaying exponential bound) and α (associated with the size of the guaranteed region). A last comment pertains to the BMI nature of the conditions (4.29), which makes them not straightforward to solve in general. In Section 4.2.4 we discuss a case study where using the branch-and-bound solver `bmibnb` in YALMIP [97] and the commercial package PENBMI [88] it is possible to derive a solution. Regarding the solution at item 3 of Theorem 4.2, the constraints (4.29) are typically solved in one of the following two ways: either a desired decay rate $\bar{\gamma}$ is fixed and the BMIs are solved with $\gamma = \bar{\gamma}$ with the goal of maximizing α, so that the associated guaranteed region is maximized, or a desired guaranteed region size $\bar{\alpha}$ is fixed and the BMIs are solved with $\alpha = \bar{\alpha}$ with the goal of maximizing γ, so that the associated decay rate is maximized. In this last case, an appealing feature is that as long as the plant is not exponentially unstable (thus also including the polynomially unstable case), the BMI constraints are semiglobal, namely they are feasible for any arbitrarily large $\bar{\alpha}$ thereby giving constructive solutions for any arbitrarily large guaranteed region. This fact is formalized in the next statement (the proof is in Section 6.4.3). Note that this is as good as one can get because exponentially unstable plants are known to have bounded controllability regions [136] and linear stabilizers are known to be insufficient to globally stabilize certain polynomially unstable plants with bounded inputs [55].

Proposition 4.1 *If the matrix A only has eigenvalues in the closed left half plane, then given any fixed $\alpha > 0$, the LMIs (4.29) in the variables $P = P^T > 0$, K, $\gamma > 0$, $U_M > 0$ diagonal are feasible.*

4.2.3 Extended Anti-Windup Solution

In this section we propose an alternative solution to Problem 4.1 which uses, within the MRAW framework, a recently proposed alternative method to represent rate and magnitude saturation in anti-windup schemes (see the recent paper [57]). The core idea behind this approach is to assume that it is possible [5] to compute the derivative of the controller output y_c in (4.23) and impose the rate saturation directly on this signal, so that the arising dynamics is not discontinuous and the magnitude and rate saturation limits are still satisfied. To this aim, we define an extended anti-windup compensator (extended because as compared to the previous solution in (4.27), this compensator has a larger number of states) having the following form:

$$\dot{x}_{aw} = Ax_{aw} + B_u(u - y_c) \tag{4.30a}$$

$$\dot{\delta} = \mathrm{sat}_R(y_{c,dot} + v_1) \tag{4.30b}$$

$$y_{aw} = C_y x_{aw} + D_{yu}(u - y_c) \tag{4.30c}$$

$$z_{aw} = C_z x_{aw} + D_{zu}(u - y_c), \tag{4.30d}$$

where v_1 is a stabilizing signal to be designed, similar to the previous section and $y_{c,dot}$ is a signal reproducing as accurately as possible the derivative of the controller output y_c. The extended anti-windup compensator (4.30) should be interconnected to the plant-controller pair (4.22), (4.23) via the following

[5] This assumption is always satisfied if the controller is strictly proper. Nevertheless, if the controller is not strictly proper then approximate implementations are possible (see the following Remark 4.11).

anti-windup interconnection:

$$u_c = y - y_{aw}, \quad u = \mathrm{sat}_M(\delta). \tag{4.31}$$

A block diagram of the overall anti-windup scheme is represented in Figure 4.6.

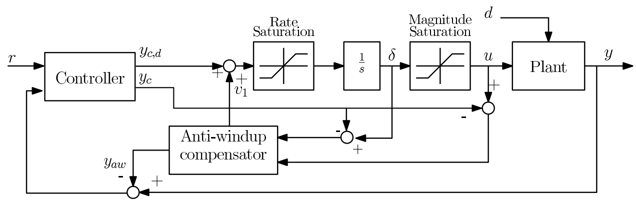


Figure 4.6: Block diagram of the extended anti-windup solution.

The following theorem, whose proof is reported in Section 6.4.4, establishes a few results on suitable designs for the stabilizing signal v_1 in (4.30), each of them having certain advantages and disadvantages as illustrated after the proof of the theorem.

Theorem 4.3 *Consider the anti-windup closed-loop (4.22), (4.23), (4.30), (4.31). Under Assumptions 4.3 and 4.4, the following holds:*

1. *If A is Hurwitz, then for any diagonal $K_\delta > 0$, the selection $v_1 = -K_\delta(\delta - y_c)$ solves the global anti-windup problem in Problem 4.1.*

2. *Selecting $v_1 = K_{aw}\left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]$, where K_{aw} is any stabilizing linear state feedback for*

$$\dot{x}_{aw} = Ax_{aw} + B_u\delta_{aw} \tag{4.32a}$$

$$\dot{\delta}_{aw} = K_{aw}\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right] \tag{4.32b}$$

solves the local anti-windup problem in Problem 4.1.

3. *Consider any feasible solution from the following generalized eigenvalue problem in the variables $Q = Q^T > 0$, X, $\gamma > 0$, $\alpha > 0$, $W_M > 0$*

diagonal:

$$\alpha I < Q \tag{4.33a}$$

$$0 > \mathrm{He}\left[\begin{array}{c|c} \begin{bmatrix} A+\gamma I_n & B_u \end{bmatrix} Q & -B_u W_M \\ \hline \begin{bmatrix} 0 & \gamma I_m \\ 0 & I_m \end{bmatrix} Q + \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} X & \begin{matrix} 0 \\ -W_M \end{matrix} \end{array}\right] \tag{4.33b}$$

$$0 \leq \begin{bmatrix} \varepsilon^2 S_i Q & [X]_i^T \\ [X]_i & 1 \end{bmatrix}, i = 1, \ldots, 2m, \tag{4.33c}$$

where $S_i = M_i$ *and* $S_{m+i} = R_i$ *for all* $i = 1, \ldots, m$. *Then, the following LMIs in the variables* K_x, K_δ, k_{max}, $W_R > 0$ *diagonal are feasible:*

$$0 > \mathrm{He}\left[\begin{array}{c|c} \begin{bmatrix} A+\gamma I_n & B_u \\ K_x & K_\delta + \gamma I_m \end{bmatrix} Q & \begin{matrix} -B_u W_M & 0 \\ 0 & -W_R \end{matrix} \\ \hline \begin{bmatrix} 0 & I_m \\ K_x & K_\delta \end{bmatrix} Q + \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} X & \begin{matrix} -W_M & 0 \\ 0 & -W_R \end{matrix} \end{array}\right] \tag{4.34a}$$

$$0 \leq \begin{bmatrix} k_{max} I & [K_x\ K_\delta] \\ \star & k_{max} I \end{bmatrix}. \tag{4.34b}$$

Moreover, selecting $v_1 = [K_x\ K_\delta] \left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]$, *where* K_x *and* K_δ *arise from any solution to (4.34), solves the local and the exponential anti-windup problems in Problem 4.1 with exponential bound* γ *in the guaranteed region* $\mathcal{B}(\alpha)$.

4. *If* A *is Hurwitz, consider any solution to the generalized eigenvalue problem (4.33b) with* $X = 0$, *in the variables* $Q = Q^T > 0$, $\gamma > 0$, $W_M > 0$ *diagonal. Then, with that solution, the LMIs (4.34) with* $X = 0$ *in the variables* K_x, K_δ, k_{max}, $W_R > 0$ *diagonal are feasible. Moreover, selecting* $v_1 = [K_x\ K_\delta] \left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]$, *where* K_x *and* K_δ *arise from any solution to (4.34), solves the global and the exponential anti-windup problems in Problem 4.1 with exponential bound* γ *with guaranteed region corresponding to the whole space.*

The solution at item 1 parallels the solution at item 1 of Theorem 4.2 as some generalization of the IMC anti-windup scheme (see [89]). Indeed, in this solution only the δ subsystem in the dynamics (4.32) is stabilized by the feedback function and the rest of the state (namely, x_{aw}) will converge to zero following the decay rate of A (see the proof of the theorem for details). Due to this reason, this solution only applies to exponentially stable plants and behaves in unacceptable ways when the plant dynamics is lightly damped.

The solution at item 2 parallels the solution at item 2 of Theorem 4.2 and has the same advantages/disadvantages discussed after the proof of Theorem 4.2. This solution was adopted for its simplicity in [159].

Similarly, the solution at item 3 parallels the solution at item 3 of Theorem 4.2 even though the trade off between α and γ is carried out here by way of convex (or, quasi convex) constraints, so that global optima can always be determined. This is a strong advantage of this second approach versus the one of Section 4.2.2.

Finally, the solution at item 4 arises from the possibility of transforming the regional constraints at the previous item into global ones, so that anti-windup with global exponential performance can be determined. Unfortunately, this solution only applies to exponentially stable plants, as imposing $X = 0$ in the constraints (4.33) makes them infeasible whenever A is not Hurwitz.

Regarding the solution at item 3 of Theorem 4.3, the constraints (4.33) are typically solved in one of the following two ways: either a desired decay rate $\bar{\gamma}$ is fixed and the LMIs arising from fixing $\gamma = \bar{\gamma}$ are solved with the goal of maximizing α, so that the associated guaranteed region is maximized, or a desired guaranteed region size $\bar{\alpha}$ is fixed and the constraints arising from fixing $\alpha = \bar{\alpha}$ are solved maximizing γ via a generalized eigenvalue problem, so that the associated decay rate is maximized.

As compared to the constraints given in Theorem 4.2 in section 4.2.2, the advantage of the formulation in Theorem 4.3 is that the constraints are convex (or quasi-convex because of the gevp) and can be efficiently solved by determining the globally optimal solution using commercial solvers such as the Matlab LMI Toolbox [56] (there wasn't such a guarantee with the BMIs of Theorem 4.2). Another advantage of this approach is that when the plant is exponentially stable the results in item 4 provide a global solution to the problem of maximizing the exponential convergence rate. On the other hand, a drawback of the approach proposed here is that semiglobal results cannot be established for plants having poles in the closed left half plane. In other words no parallel statement to that in Proposition 4.1 can be proven[6].

Remark 4.11 One of the main difficulties in implementing the anti-windup scheme proposed in this section is that the signal $y_{c,dot}$, *i.e.* the derivative of the controller output y_c, must be generated to be used in (4.30). If a strictly proper controller is used, such a derivative can be explicitly and easily computed;

[6]The main reason for this limitation stands in the nature of the dynamics (6.59), which shows internal saturations unlike the parallel dynamics (6.50) which only have input saturations.

otherwise, a viable alternative route, provided that $D_{yu} = 0$ in (4.22b), consists in filtering y_c by $F(s) = \frac{1}{1+\tau_d s}[1 \quad s]^T$ in order to produce $[\bar{y}_c \quad \dot{\bar{y}}_c]^T$ and replacing y_c by $\bar{y}_c$ in the anti-windup scheme above (with the advantage that $\dot{\bar{y}}_c$ is explicitly available). If this approach is taken, the proposed anti-windup scheme will recover the response of the modified unconstrained closed loop (with y_c replaced by $\bar{y}_c$) instead of the response of the original unconstrained closed loop; but this also guarantees that the original response is essentially recovered, since it is possible to show that the modified and the original unconstrained closed loop responses can be made arbitrarily close if a sufficiently small τ_d is chosen. Note that the condition $D_{yu} = 0$ in (4.22b) (which is sufficient to ensure that the closed loop including the small time constant τ_d is stable provided that the closed loop before the introduction of τ_d is such) can always be enforced by a preliminary loop transformations, redefining $y = C_y x + D_{yd} d$ in (4.22b) and $u_c = y + D_{yu} y_c$ in (4.24) (the arising algebraic loop is well-posed by Assumption 4.4). This strategy is adopted in our example in Section 4.2.4, where the controller is not strictly proper.

Remark 4.12 To improve the transient performance induced by the anti-windup closed-loop, it is possible to modify the anti-windup scheme by using saturated versions of y_c and $y_{c,dot}$, namely replacing (4.30b) by

$$\dot{\delta} = \mathrm{sat}_R(\mathrm{sat}_{R(1-\varepsilon)}(y_{c,dot}) + v_1) \tag{4.35}$$

and by choosing signal v_1 as a feedback signal from $\begin{bmatrix} x_{aw} \\ \delta - \mathrm{sat}_{M(1-\varepsilon)}(y_c) \end{bmatrix}$, rather than $\begin{bmatrix} x_{aw} \\ \delta - y_c \end{bmatrix}$ (a similar idea was adopted in [105]). Then it can be proved (details are omitted for brevity) that all the stated closed-loop properties are preserved and the transient performance of the anti-windup law is improved because the peaks in y_c and $y_{c,dot}$ are trimmed out.

4.2.4 Simulation Example

Consider the short-period longitudinal dynamics of the VISTA/MATV F-16 at Mach 0.2 and altitude 10000 feet (corresponding to a dynamic pressure value of $40.8\,psf$) at a trim angle of attack of 28 degrees, described locally by a second order plant as in (4.22) with two states corresponding to the angle of attack and the pitch rate, respectively, and two inputs corresponding to the deviations of the elevator deflection and of the pitch thrust vectoring from the trim condition (see [146] for details). As in [146], the controller (4.23) is nonlinear and corresponds to a daisy chained allocation of the inputs, driven by a reference signal for the angle of attack.

We design two anti-windup compensators for this example. The first one corresponds to the construction at item 3 of Theorem 4.2, where we select $\alpha = 0.001$ and obtain a guaranteed regional performance of $\gamma = 3.7849$ by solving the BMIs with the software YALMIP [97] and the commercial package PENBMI [88]. The corresponding simulations are represented by bold curves in Figure 4.7, where they are compared to the unconstrained response (solid), to the saturated response (dotted) and to the response using the construction in [146] (dashed), which can be seen as using the approach at item 2 of Theorem 4.2 (see [146] for details). The anti-windup design guarantees fast and desirable recovery of the unconstrained trajectory.

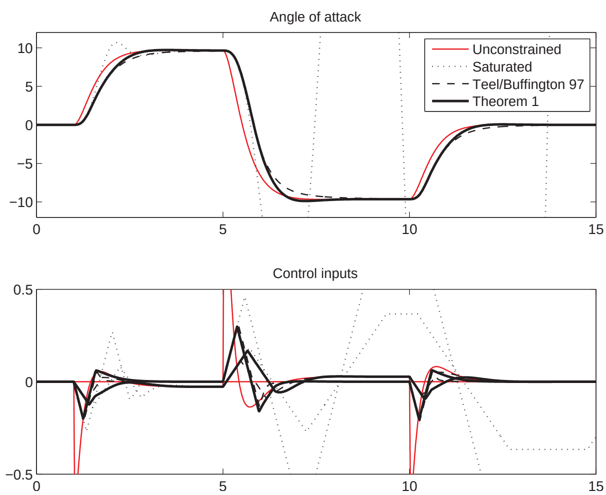


Figure 4.7: Comparison between different responses when referring to Theorem 4.2.

The second construction is the one at item 3 of Theorem 4.3, where we select $\alpha = 0.001$ so that the same guaranteed region is obtained. The arising guaranteed regional performance is $\gamma = 2.4537$ obtained by solving the LMIs with the software YALMIP [97] and the commercial package Matlab LMI Toolbox [56]. Note that this performance is worse than the one obtained with the previous approach. To construct the signal $y_{c,dot}$, we rely on Remark 4.11 by selecting $\tau_d = 0.1$ because the controller isn't strictly proper. Moreover, to improve the transient performance we insert the extra saturation suggested in Remark 4.12. The corresponding simulations are represented by bold curves in Figure 4.8, where they are compared to the unconstrained response (solid), to the saturated response (dotted) and to the response using the construction in [146] (dashed). Once again, the anti-windup design guarantees fast and desirable recovery of the unconstrained trajectory. Note however that, in spite of the worse guaranteed performance level, the unconstrained response recovery is slightly more desirable than the previous case. This probably derives from the conservativeness of the performance bounds.

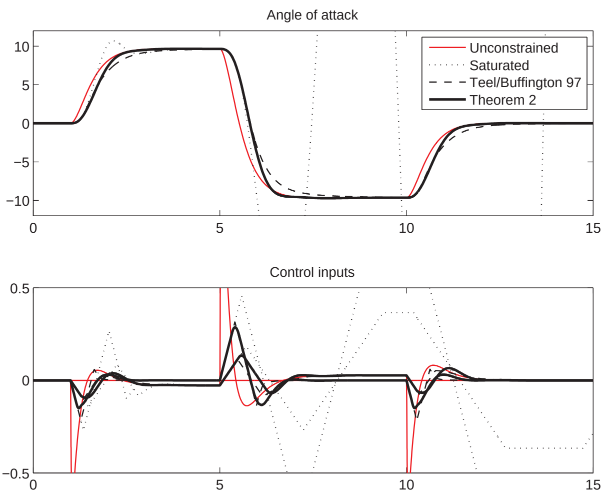


Figure 4.8: Comparison between different responses when referring to Theorem 4.3.

Finally, in Figure 4.9 we show the two curves arising from the trade off between α and γ for our two constructions with guaranteed exponential decay rate. In the figure, the region on the bottom left of the curve is the feasibility region and the region on the top right is the infeasibility region. Note that, quite interestingly, for this example the first construction appears to achieve a better performance than the second one. This fact is compensated by the converse properties in terms of computational burden, indeed the BMIs associated with the first approach are solved suboptimally by the PENBMI solver [88] which requires significant computational effort and is extremely sensitive to numerical problems. The LMIs associated with the second approach, instead, are efficiently solved using MATLAB's LMI toolbox [56] and are guaranteed to always provide the optimal solution.

Acknowledgement. The authors would like to thank Andy Teel for his suggestions with the proof of Lemma 6.4.

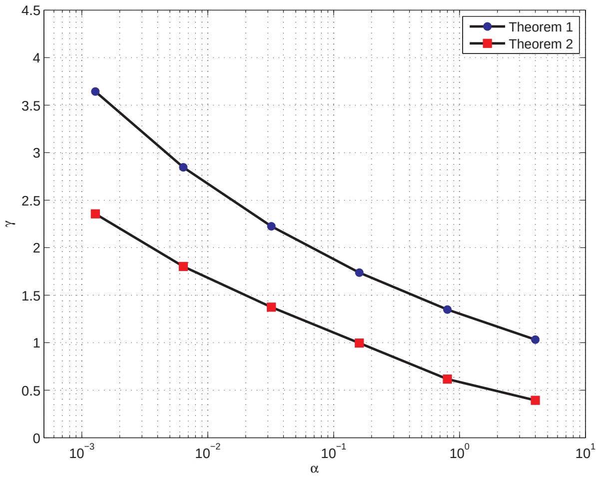


Figure 4.9: Trade-off between achievable stability region size (α) and guaranteed exponential convergence rate (γ) using the two methods at item 3 of Theorems 4.2 and 4.3.

Chapter 5

Case Studies of Hybrid Control Systems

In recent years, much attention has been given to the design problem of control systems in the hybrid context [30, 62, 95, 107, 108, 128], namely when the closed-loop dynamics obeys either a continuous law imposing a constraint on the derivative of the solution, when it belongs to the so-called flow set, and/or a discrete law imposing a constraint on the jumps that the solution undertakes when it belongs to the so-called jump set.

A theoretical motivation for considering hybrid controllers is related to the existence of continuous/discontinuous control laws for steering the state of a system to zero. In the survey work [137], such a problem is considered for systems of the form $\dot{x} = f(x, u)$. Following [137], the open-loop property of *null-asymptotic controllability* requires that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each $|x_0| < \delta$ there is some measurable, locally essentially bounded control $u : [0, \infty) \to \mathbb{R}^m$, where m is the dimension of the input, such that the trajectory $x(\cdot)$ of the system, resulting from the initial state x_0 and the input u satisfies

$$x(t), u(t) \to 0 \text{ as } t \to \infty \text{ and } \forall t > 0,\ |x(t)| + |u(t)| < \varepsilon. \tag{5.1}$$

The first condition is a classical convergence property while the second condition is a classical boundedness property, closely related to the concept of stability. When such conditions is verified for each $x_0 \in \mathbb{R}^n$, we have *global asymptotic controllability.*

For scalar cases, global asymptotic controllability means that for each x we can find an input u such that $xf(x,u) < 0$ (that can be restricted to small x for local asymptotic controllability). Consider the set $\mathcal{O}$ of pairs (x,u) that satisfies that condition:

$$\mathcal{O} = \{(x,u) \mid xf(x,u) < 0\}. \tag{5.2}$$

Then, following [137], global asymptotic controllability implies that the projection of $\mathcal{O}$ on the x variable, that is, $O_x = \{x \mid \exists u,\, xf(x,u) < 0\}$, is equal to $\mathbb{R} \setminus \{0\}$. Intuitively, $O_x = \mathbb{R} \setminus \{0\}$ guarantees that for each initial state there exists some u to steer the state to 0.

Consider now the system

$$\dot{x} = x\left[(u-1)^2 + 1 - x\right]\left[x - 2 + (u+1)^2\right] \tag{5.3}$$

Then, the set $\mathcal{O}$ depends on $[(u-1)^2 + 1 - x][x - 2 + (u+1)^2] < 0$, represented by the blank part of Figure 5.1. The shaded part refers to $[(u-1)^2 + 1 - x][x - 2 + (u+1)^2] \geq 0$. For the system (5.3), $O_x = \mathbb{R} \setminus \{0\}$, therefore the system is globally asymptotically controllable, but a continuous feedback control law $u = k(x)$, $k : \mathbb{R} \to \mathbb{R}$, does not exist. Nevertheless, a discontinuous feedback law may exists. Therefore, hybrid systems are a suitable framework to model the effects of a discontinuous control law on the closed loop dynamics. Moreover, such discontinuous law can be the result of a complex decision process that may involve, for example, the use of logic variables, resets phenomena or reactions to some events from the environment. Then, the decision process can be conveniently modeled within the hybrid systems framework, leading to the design of *hybrid controllers*, namely controllers whose behavior is defined by a suitable hybrid system. This is why the mathematical tools developed for hybrid systems, and partially showed in previous chapters, can be used for the analysis of closed loop systems.

In what follows we propose two hybrid approaches within the classical framework of dynamical control systems. In Section 5.1 we propose a class of passive controllers whose passivity is induced by a suitable reset strategy. The approach that we follow is to rewrite both the continuous dynamics of the controller and the reset strategy as a hybrid system with inputs. Then, by studying the solutions to such a hybrid system, it turns out that continuous systems possibly not passive, or possibly not stable, can be transformed to a passive system by resets. Note that hybrid systems with inputs are closely related to the autonomous hybrid systems of previous chapters, but a slightly different notion of solution must be developed to take into account input signals. In Section 5.1, we do not develop a complete theory of hybrid systems with inputs (see e.g. [30]) and we

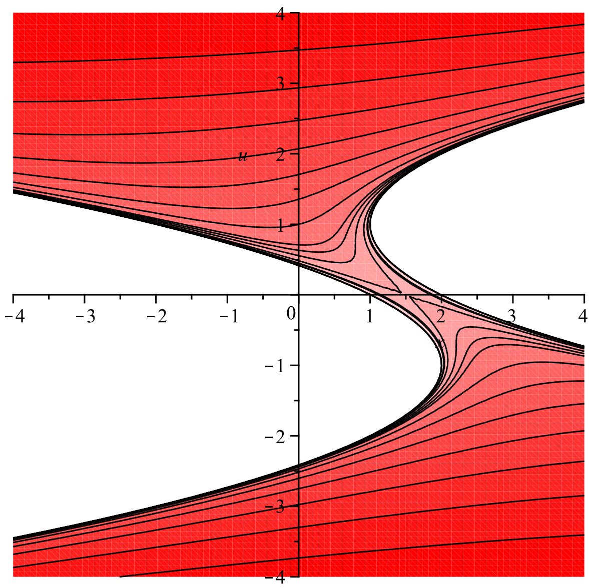


Figure 5.1: For some initial states, there is no continuous feedback control law that steers the state of the system (5.3) to zero. x is on the horizontal axis, u is on the vertical axis.

write each definition in a form that works specifically for the problem that we are considering. In Section 5.2, we propose a technique to break the signals continuity in a feedback loop, by defining specific policies that decide when samples of such signals must be transmitted to the controller. We propose a centralized policy and a decentralized policy, each of them based on the state of the control system. The first one forces a synchronized update of the measurements vector while the second one allows for an asynchronous update of the measurements vector.

5.1 Passification of Controllers via Time-Regular Reset Map

In this section we consider a class of hybrid control systems characterized by continuous-time plants controlled by a hybrid controller, namely a hybrid closed-loop where the jumps only affect the controller states. Within this class of systems, a relevant example consists in the reset control systems first introduced in [41], where a jump linear system (the "Clegg integrator") generalizing a linear integrator was proposed. This generalization was then further developed in [76] where it was extended to first order linear filters, and therein called First Order Reset Elements (FORE). FORE received much attention in recent years and have been proven to overcome some intrinsic limitations of linear controller [18]. Moreover, by relying on Lyapunov approaches, suitable analysis and synthesis tools for the stability of a class of reset systems generalizing control systems with FORE have been proposed in [19, 109] and references therein. Moreover, in the recent paper [34] the $\mathcal{L}_2$ stability of reset control systems has been addressed in the passivity context, by showing interesting properties of the reset system under the assumption that the continuous-time part of the reset controller is passive before resets and that a suitable non-increase condition is satisfied by the storage function at jumps. In [34] it was also shown by a simulation example that resets do help closed-loop performance in passivity-based closed-loops.

In this section we further develop over the ideas of [34] by using a specific temporally regularized reset strategy for the reset controller. The reset strategy generalizes the new interpretation of FOREs and Clegg integrators proposed in [109, 161] and references therein. We show that, with the proposed reset strategy, passification is possible for any continuous-time underlying dynamics under some sector growth assumption on the right hand side of the continuous-time dynamics of the controller. The obtained passivity property is characterized by an excess of output passivity and a lack of input passivity whose size can be made arbitrarily small by suitably adjusting the reset rule. As an example, the proposed reset strategy allows to establish a passivity property for any FORE, including those characterized by an exponentially unstable pole, while the results in [34] only allow to establish passivity of FOREs with stable poles. This increased potential of the reset rule proposed here is illustrated on a nonlinear simulation example.

5.1.1 A Class of Nonlinear Reset Controllers

Consider the following nonlinear controller mapping the input v to the output u,

$$\begin{array}{rcl} \dot{x}_c & = & f(x_c) + g(x_c, v) \\ u & = & h(x_c), \end{array} \tag{5.4}$$

where $u \in \mathbb{R}^q$, $v \in \mathbb{R}^q$, so that the controller is square and where the following regularity assumption is satisfied by the right hand side.

Assumption 5.1 *The functions $f(\cdot)$ and $h(\cdot)$ are continuous and sector bounded, namely there exist two constants L_f and L_h such that for all x_c, $|f(x_c)| \leq L_f|x|$ and $|h(x_c)| \leq L_h|x_c|$.*

Moreover, $g(\cdot, \cdot)$ is continuous in both its arguments and uniformly sector bounded in the second argument, namely there exists a constant L_g such that for all x_c and all v, $|g(x_c, v)| \leq L_g|v|$.

We propose a hybrid modification of the controller (5.4) aimed at making it passive from v to u, regardless of the properties of the original dynamics in (5.4). In particular, the modified controller follows the continuous-time dynamics of (5.4) at times when the input/output pair belongs to a certain subset of the input/output space. When the input/output pair exits that subset, the state of the controller is reset to zero, intuitively re-initializing the controller within the set where it is allowed to flow.

To avoid Zeno solutions, we also embed the hybrid modification with a temporal regularization clock, imposing that the controller cannot be reset to zero before ρ times after the previous reset (see also [83, 109]).

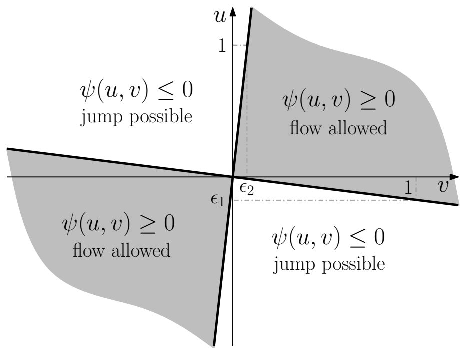


Figure 5.2: Input/output space of the controller (5.5) and subsets where $\psi(u,v) \gtreqless 0$. The blank area defines the set of pairs (u,v) for which the occurrence of a jump is *possible* provided that the condition $\tau \geq \rho$ is satisfied (time regularization).

The proposed hybrid controller is given by

$$\begin{cases} \dot{x_c} = f(x_c) + g(x_c, v) \\ \dot{\tau} = 1 \end{cases} \quad \textit{if } \tau \leq \rho \textit{ or } \psi(u,v) \geq 0$$
$$\begin{cases} x_c^+ = 0 \\ \tau^+ = 0 \end{cases} \quad \textit{if} \, \tau \geq \rho \textit{ and } \psi(u,v) \leq 0 \tag{5.5a}$$
$$u = h(x_c)$$

where $\psi(u,v)$ is defined as

$$\psi(u,v) = (u + \epsilon_1 v)^T (v - \epsilon_2 u) \tag{5.5b}$$

and ϵ_1 and ϵ_2 are some (typically small) non-negative scalars. As usual in the hybrid system framework, we call C the set $\{(x_c, \tau, v) : \tau \leq \rho \textit{ or } \psi(h(x_c), v) \geq 0\}$ and D the set $\{(x_c, \tau, v) : \tau \geq \rho \textit{ and } \psi(h(x_c), v) \leq 0\}$.

The rationale behind the reset controller (5.4) is illustrated in Figure 5.2 where the input/output space of (5.5) is represented for the case $q = 1$. In the figure, the shaded region corresponds to the set $\psi(u,v) \geq 0$ where the system always flows, regardless of the value of τ. Instead, in the remaining region, where $\psi(u,v) \leq 0$, the system will jump provided that $\tau \geq \rho$. Note also that when $\epsilon_1 = \epsilon_2 = 0$, the shaded region reduces to the first and third quadrant, resembling the resetting rule characterized for the first order reset element (FORE) in [161, 109]. When the reset occurs, since $h(0) = 0$, the u component of the input/output pair will jump at zero thus resulting in a vertical jump to the horizontal axis. Moreover, ϵ_1 and ϵ_2 allow to have extra degrees of freedom in the resetting rule. In particular, the goal of ϵ_1 is to guarantee that the reset rule maps the new input/output pair in the interior of the shaded set whenever $v \neq 0$. Instead, as it will be clear next, the goal of ϵ_2 is to modify the resetting rule to obtain some strict output passivity for the reset controller (5.5).

Controller (5.5) will be dealt with in this section following the framework of [63, 62, 30]. In particular, by Assumption 5.1, controller (5.5) satisfies the hybrid basic conditions (see, e.g., [30]), which ensure desirable regularity properties of the solutions, such as existence, and robustness to arbitrarily small perturbations (see [62] for details).

The motion of the state ξ of the hybrid system (5.5), depends on the input signal v, so that both ξ and v must be defined on hybrid time domain. By following [30], we call *hybrid signal* each function defined on a hybrid time domain. A hybrid signal $v : \operatorname{dom} v \to \mathcal{V}$ is a *hybrid input* if $v(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid signal ξ : $\operatorname{dom} \xi \to \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ is a *hybrid arc* if $\xi(\cdot, j)$ is locally absolutely continuous, for each j. With the basic conditions satisfied, a hybrid arc $\xi = (\xi_x, \xi_\tau)$ and a hybrid input v is a *solution pair* (ξ, v) to the hybrid system (5.5) if $\operatorname{dom} \xi = \operatorname{dom} v$, $(\xi(0,0), v(0,0)) \in C \cup D$, and

s.1 for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \operatorname{dom} \xi$,

$$\begin{aligned} &(\xi(t,j), v(t,j)) \in C \\ &\dot{\xi}_x(t,j) = f(\xi_x(t,j)) + g(\xi_x(t,j), v(t,j)); \\ &\dot{\xi}_\tau(t,j) = 1; \end{aligned} \tag{5.6}$$

s.2 for all $(t, j) \in \operatorname{dom} \xi$ such that $(t, j+1) \in \operatorname{dom} \xi$,

$$\begin{aligned} &(\xi(t,j), v(t,j)) \in D \\ &\xi_x(t,j+1) = 0; \\ &\xi_\tau(t,j+1) = 0; \end{aligned} \tag{5.7}$$

We say that a set of solutions pairs (ξ, v) is *uniformly non-Zeno* if there exists $T \in \mathbb{R}_{>0}$ and $J \in \mathbb{Z}_{>0}$ such that, for any given $(t, j), (t', j') \in \operatorname{dom} \xi$, if $|t - t'| \leq T$ then $|j - j'| \leq J$, that is, in any time period of length T, no more than J jumps can occur. Note that multiple instantaneous jumps are still possible, [63].

Note that any continuous-time signal $\overline{v} : \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ can be rewritten as hybrid signal with domain E, for any given hybrid domain E. In fact, suppose that $E = \bigcup [t_j, t_{j+1}] \times \{j\}$ is an hybrid time domain. Then, we can define a hybrid signal v *lifted* from $\overline{v}$ on E as follows: $v(t,j) = \overline{v}(t)$ for each $(t,j) \in E$. Conversely, suppose that (ξ, v) is a solution pair to the hybrid system (5.5). Then, the output signal $u = h(\xi_x)$ is a hybrid signal and $\operatorname{dom} u = \operatorname{dom} \xi$. From u we can construct an continuous-time signal $\overline{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ *projected* from u on $\mathbb{R}_{\geq 0}$ as follows: $\overline{u}(t) = u(t,j)$ for each $(t,j) \in \operatorname{dom} u$ such that $(t, j+1) \notin \operatorname{dom} u$, and $\overline{u}(t) = u(t, j+1)$ otherwise.

We denote by $\|\overline{v}\|_p$ the $\mathcal{L}_p$ gain of a continuous-time signal $\overline{v}$. The $\mathcal{L}_p$ gain of a hybrid signal v, related to the continuous part of its domain, will be denoted by $\|v\|_{c,p} = \left(\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} |v(t,j)|^p dt \right)^{1/p}$. Note that for any continuous-time signal $\overline{v}$ projected from a hybrid signal $\overline{v}$ on $\mathbb{R}_{\geq 0}$, we have that $\|\overline{v}\|_q = \|v\|_{c,p}$. Conversely, for any hybrid signal v lifted from a continuous-time signal $\overline{v}$ on a given hybrid time domain E, we have that $\|v\|_{c,p} = \|\overline{v}\|_p$.

Finally, the following lemma characterizes regularity of the solutions to (5.5).

Lemma 5.1 *Under Assumption 5.1, all the solutions to (5.5) are uniformly non-Zeno. Moreover, for each $\mathcal{L}_p$ integrable input signal $\overline{v}$, a solution pair (ξ, v) where v is the hybrid input signal lifted from $\overline{v}$ on $\operatorname{dom} \xi$, is a complete solution pair.*

Proof. For a solution pair (ξ, v), define $t_j = \min\{t \,|\, (t,j) \in \operatorname{dom} \xi\}$. By the definition of C and D given after (5.5), given any solution pair $(\xi, v) = ((\xi_x, \xi_\tau), v)$ of (5.5), $t_j - t_{j-1} \geq \rho$ for all $(t,j) \in \operatorname{dom}(x)$, $j \geq 2$. This implies that the uniformly non-Zeno definition in [63] (see also [42]) is satisfied with $T = \rho$ and $J = 2$.

By $C \cup D = \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathcal{V}$, $\operatorname{dom} \xi$ is bounded only if ξ blows up in finite time. Looking at the dynamics of the system in (5.5a), by Assumption 5.1, $|\dot{x_c}| \leq |f(x_c) + g(x_c, v)| \leq L_f |x_c| + L_g |v|$ and $|\dot{\tau}| = 1$. Therefore, if $|v|$ is $\mathcal{L}_p$ integrable, $|\xi|$ is bounded in any given compact subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. $\square$

5.1.2 Passivity of the Reset Controller

The following theorem shows that the hybrid controller (5.5) is almost passive with a shortage of input passivity proportional to the temporal regularization constant ρ plus ϵ_1. Moreover, the slight modification of the function $\psi(\cdot, \cdot)$ enforced by ϵ_2 induces some excess of output passivity.

Theorem 5.1 *Consider the hybrid controller* (5.5) *satisfying Assumption 5.1. Define*

$$\varepsilon_1 := \frac{\epsilon_1}{1-\epsilon_1\epsilon_2}\,, \qquad \varepsilon_2 := \frac{\epsilon_2}{1-\epsilon_1\epsilon_2}$$

$$k(\rho) = \rho L_h L_g \max\{1, \rho e^{L_f \rho}\} \tag{5.8}$$

$$\overline{k}(\rho) = k(\rho)(1+\varepsilon_2 k(\rho))$$

Given a $\mathcal{L}_2$ *integrable input signal* $\overline{v} \in \mathbb{R}_{\geq 0} \to \mathcal{V}$ *and a solution pair* (ξ, v) *to* (5.5)*, with* v *lifted from* $\overline{v}$ *on* dom ξ*, then*

$$\int_0^\infty \overline{u}(t)^T \overline{v}(t)dt \geq -\left(\varepsilon_1 + \overline{k}(\rho)\right) \|\overline{v}(\cdot)\|_2^2 + \varepsilon_2 \|\overline{u}(\cdot)\|_2^2 \tag{5.9}$$

where the output signal $\overline{u} \in \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ *is projected from the hybrid output signal* $u : \operatorname{dom} u \to \mathbb{R}^q$ *corresponding to the solution pair* (ξ, v)*.*

Proof. See Section 6.5.1. □

Remark 5.1 Note that Theorem 5.1 establishes the passivity of (5.5) based on the norm $\|\cdot\|_{c,2}$, namely only taking into account the continuous-time nature of the hybrid solutions. This type of passivity concept is relevant because of Lemma 5.1 and, moreover, allows to rely on standard passivity results [130] to conclude properties of the closed loop between (5.5) and a plant, as detailed in Section 5.1.3.

Remark 5.2 It is important to underline that the passive behavior of the hybrid controller (5.5) is strongly related to the definition of the jump and flow sets D and C, more than to the dynamic equations of the controller. Roughly speaking, the passive behavior of the controller can be considered as an effect of the definition of $\psi(u, v)$, that forces a particular shape of the sets C and D. Following this intuition, while $\psi(u, v)$ constrains C and D to induce passivity, time regularization adds some extra constraint on C and D possibly destroying part of this passivity property. This results in a shortage of passivity parameterized with ρ.

5.1.3 Application to Feedback Systems

In this section we use the *passivity theorem* [130] to establish useful stability properties of the reset controller (5.5) interconnected to any passive nonlinear plant: [1]

$$\begin{aligned} \dot{x}_p &= f_p(x_p, u + d) \\ y &= h_p(x, u + d), \end{aligned} \tag{5.10}$$

via the negative feedback interconnection $v = w - y$, where w is an external signal. In (5.10), d is an additive disturbance acting at the plant input. The following statement directly follows from the properties of (5.5) established in Theorem 5.1.

Proposition 5.1 *Consider the hybrid controller* (5.5) *satisfying Assumption 5.1 in feedback interconnection $v = w - y$ with the plant (5.10).*

For any $\epsilon_1 \geq 0$, $\epsilon_2 > 0$ and $\rho > 0$, given ε_1 and $\overline{k}(\rho)$ as in (5.9), if the plant is output strictly passive with excess of output passivity $\delta_P > \varepsilon_1 + \overline{k}(\rho)$, then the closed-loop system (5.5), (5.10) with $v = w - y$ is finite-gain $\mathcal{L}_2$ stable from (w, d) to (u, v).

In Proposition 5.1 we require a specific excess of output passivity from the plant because we assume that the controller requires implementation with certain prescribed selections of ϵ_1 and ρ. In the case where it is possible to reduce arbitrarily these two parameters, it is possible to relax the requirements of Proposition 5.1 as follows:

Proposition 5.2 *Consider the hybrid controller* (5.5) *satisfying Assumption 5.1 in feedback interconnection $v = -y$ with the plant (5.10).*

If the plant (5.10) is output strictly passive, then for any $\epsilon_2 > 0$, there exist small enough positive numbers ϵ_1^* and ρ^* such that for all $\epsilon_1 \leq \epsilon_1^*$ and all $\rho \leq \rho^*$, the closed-loop system (5.5), (5.10) with $v = w - y$ is finite-gain $\mathcal{L}_2$ stable from (w, d) to (u, v).

Proof. The proposition is a straightforward consequence of Proposition 5.1 noting that for a fixed ϵ_2, the lack of output passivity established in Theorem 5.1 decreases monotonically to zero as ϵ_1 and ρ go to zero. Then it is always possible to reduce the two parameters to match the passivity condition in [130]. $\square$

Both Propositions 5.1 and 5.2 either require an explicit bound on the excess of output passivity of the plant or constrain the controller parameters ϵ_1 and ρ to be small enough. An alternative solution to this is to add an extra feedforward loop to the reset controller (5.5), following the derivations in [87, page 233], to guarantee that the arising reset system is very strictly passive, namely it is both input strictly passive and output strictly passive. To this aim, we modify the

[1] See also [34] for a similar application of the passivity theorem to reset controllers.

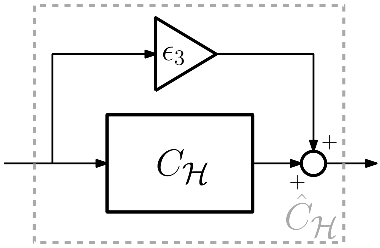


Figure 5.3: The very strictly passive version (5.11) of the reset controller ($C_{\mathcal{H}}$ corresponds to (5.5)).

output equation of (5.5) by adding the feedforward term $\epsilon_3 v$, as represented in Figure 5.3. The corresponding reset controller can then be written as:

$$\begin{aligned} &\begin{cases} \dot{x_c} = f(x_c) + g(x_c, v) \\ \dot{\tau} = 1 \end{cases} && \textit{if } \tau \le \rho \textit{ or } \hat{\psi}(\hat{u}, v) \ge 0 \\ &\begin{cases} x_c^+ = 0 \\ \tau^+ = 0 \end{cases} && \textit{if } \tau \ge \rho \textit{ and } \hat{\psi}(\hat{u}, v) \le 0 \\ &\ \hat{u} = h(x_c) + \epsilon_3 v \end{aligned} \tag{5.11a}$$

where $\hat{\psi}(\hat{u}, v)$ is defined as

$$\hat{\psi}(\hat{u}, v) = ((\hat{u} + (\epsilon_1 - \epsilon_3)v)^T((1 + \epsilon_2\epsilon_3)v - \epsilon_2\hat{u}) \tag{5.11b}$$

and $\epsilon_3 > 0$ is suitably selected as specified below. When using the modified reset controller (5.11), the following result holds.

Proposition 5.3 *Consider the hybrid controller* (5.11) *satisfying Assumption 5.1 in feedback interconnection $v = w - y$ with a passive plant (5.10).*

For any $\epsilon_1 \ge 0$, $\epsilon_2 > 0$ and $\rho > 0$, given ε_1 and $\overline{k}(\rho)$ as in (5.9), if $\epsilon_3 > \varepsilon_1 + \overline{k}(\rho)$, then the closed-loop system (5.11), (5.10) with $v = w - y$ is finite-gain $\mathcal{L}_2$ stable from (w, d) to (u, v).

Proof. Define a new output $\hat{u} = u + \epsilon_3 v$ and denote by $\overline{\hat{u}}$ the output signal projected from $\hat{u}$ on $\mathbb{R}_{\ge 0}$. Then, from (5.9), we have that

$$\begin{aligned} \int_0^\infty \overline{\hat{u}}(t)^T \overline{v}(t)\, dt &\ge \epsilon_2 \int_0^\infty \overline{u}^T \overline{u}\, dt + \left(\epsilon_3 - \varepsilon_1 - \overline{k}(\rho)\right) \int_0^\infty \overline{v}^T \overline{v}\, dt \\ &\ge \frac{1}{1 + 2\epsilon_2\epsilon_3} \left(\epsilon_2 \int_0^\infty \overline{\hat{u}}^T \overline{\hat{u}}\, dt + \left(\epsilon_3 - \varepsilon_1 - \overline{k}(\rho)\right) \int_0^\infty \overline{v}^T \overline{v}\, dt \right). \end{aligned}$$

It follows that

$$\int_0^\infty \overline{\hat{u}}(t)^T \overline{v}(t)\, dt \geq \eta_1 \|\overline{\hat{u}}\|_2^2 + \eta_2 \|\overline{v}\|_2^2 \tag{5.12}$$

with $\eta_1 = \frac{\epsilon_2}{1+2\epsilon_2\epsilon_3} > 0$ and $\eta_2 = \frac{\epsilon_3 - \epsilon_1 - \overline{k}(\rho)}{1+2\epsilon_2\epsilon_3} > 0$.

Replace now the output u of the controller (5.5) with $\hat{u} = u + \epsilon_3 v = h(x_c) + \epsilon_3 v$. Then, $\hat{\psi}(\hat{u}, v)$ is obtained by substituting $u = \hat{u} - \epsilon_3 v$ in the expression of $\psi(u, v)$ of Equation (5.5b). By the *passivity theorem* in [130], Proposition 5.3 follows. □

Remark 5.3 The results in this section can be seen as a generalization of the results on full reset compensators in [34], where passivity techniques are used to establish finite gain $\mathcal{L}_2$ stability of the closed-loop between passive nonlinear plants and reset controllers. When focusing on linear reset controllers such as Clegg integrators [41] and First Order Reset Elements (FORE) [76, 19], the novelty of Theorem 5.1 as compared to the results in [34] is that those results establish passivity of FORE whose underlying linear dynamics is already passive (namely FORE with stable poles). Conversely, our results of Theorem 5.1 apply regardless of what the underlying dynamics of the controller is. Therefore, for example, any FORE with arbitrarily large unstable poles would still become passive using the flow and jump sets characterized in (5.5). Note however that, as compared to the approach in [34], we are using a different selection of the flow and jump sets. In the example section we illustrate the use of unstable FOREs within (5.5).

5.1.4 Simulation Example

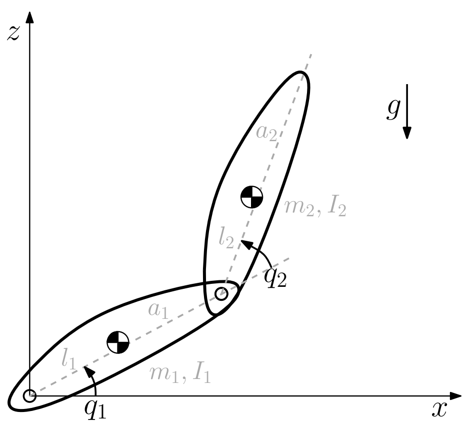
We consider a planar two-link rigid robot manipulator in Figure 5.4, as modeled in [104]. Denoting by $q \in \mathbb{R}^2$ the two joint positions and by $\dot{q} \in \mathbb{R}^2$ the corresponding velocities, the manipulator is modeled as

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + h(q) = u_p \tag{5.13}$$

where $D(q)$ is the inertia matrix, $C(q, \dot{q})\dot{q}$ comprises the centrifugal and Coriolis terms, $h(q)$ is the gravitational vector, and u_p represents the external torques applied to the two rotational joints of the robot. In Figure 5.4, m_1 and m_2 represent the links masses, a_1 and a_2 represent the links lengths, l_1 and l_2 represent the distances of the center of mass of each link from the preceding joint, and I_1 and I_2 represent the rotational inertias at the two joints. The numerical values of the parameters are listed in the table of Figure 5.4. Denoting

$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}$, $C(q,\dot{q}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{bmatrix}$, and $h(q) = \begin{bmatrix} h_1 & h_2 \end{bmatrix}^T$, we get:

$$\begin{aligned}
&d_{11} = I_1 + m_1 l_2^2 + I_2 + m_2(a_1^2 + l_2^2 + 2a_1 l_2 \cos(q_2)),\\
&d_{12} = I_2 + m_2(l_2^2 + a_1 l_2 \cos(q_2)),\\
&d_{22} = I_2 + m_2 l_2^2,\\
&c_{11} = -m_2 a_1 l_2 \sin(q_2)\dot{q}_2,\\
&c_{12} = -m_2 a_1 l_2 \sin(q_2)(\dot{q}_1 + \dot{q}_2),\\
&c_{21} = m_2 a_1 l_2 \sin(q_2)\dot{q}_1,\\
&h_1 = g(m_1 l_1 + m_2 a_1)\cos(q_1) + g m_2 l_2 \cos(q_1 + q_2),\ h_2 = g m_2 l_2 \cos(q_1 + q_2).
\end{aligned}$$



Link	l_i $[m]$	m_i $[kg]$	I_i $[kgm^2]$	a_i $[m]$
1	0.5	6	0.2	1
2	0.25	5	0.1	0.5

Figure 5.4: The robot example and its parameters.

Given a reference signal $r \in \mathbb{R}^2$ representing the desired joint position, following a standard passivity based approach, it is possible to close a first control loop around the robot (5.13) to induce the equilibrium point $(q,\dot{q}) = (r, 0)$ while guaranteeing passivity from a suitable input u to the joint velocity output $\dot{q}$, as shown in Figure 5.5. In particular, define $V(q,r) = \frac{k_p}{2}(q-r)^T(q-r)$, where

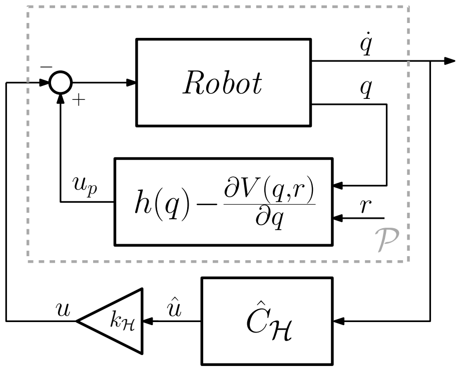


Figure 5.5: Control loop of the two-links robot.

the scalar $k_p > 0$ is a weight parameter on the position error, and choose

$$u_p = -\frac{\partial V(q,r)}{\partial q} + h(q) + u. \tag{5.14}$$

Then, the interconnection (5.13), (5.14) corresponds to

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V(q,r)}{\partial q} = u \tag{5.15}$$

and, following similar steps to those in [51], it can be shown to be passive from u to $\dot{q}$. In particular, use the storage function $E = \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q,r)$ to conclude

$$\begin{aligned} \dot{E} =& \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + k_p(q-r)^T\dot{q} \\ =& \dot{q}^T u + \dot{q}^T\left(\frac{1}{2}\dot{D}(q) - C(q,\dot{q})\right)\dot{q} \\ =& \dot{q}^T u \end{aligned} \tag{5.16}$$

where the second equality follows from (5.15) and the third equality follows from the well known fact that $z^T(\dot{D}(q) - 2C(q,\dot{q}))z = 0$, for all $z \in \mathbb{R}^2$.

For the outer loop, we rely on the very strictly passive controller (5.11) where the dynamics in (5.11a) is selected as a pair of decentralized First Order Reset Elements, namely denoting $x_c = [x_{c1}\ x_{c2}]^T$, we select $f(x_c) = [\lambda_1 x_{c1}\ \lambda_2 x_{c2}]^T$

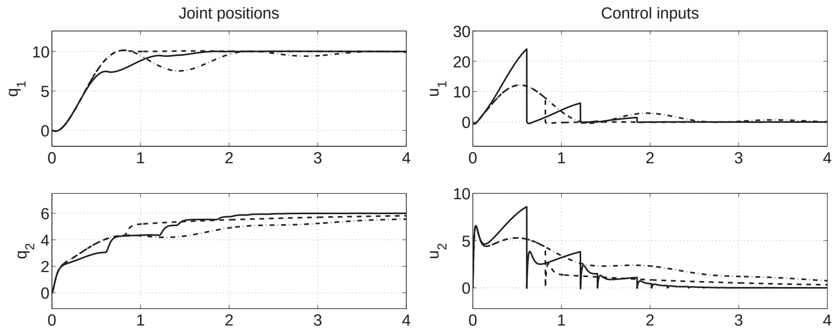


Figure 5.6: Simulations results. Stable FORE and no resets (dash-dotted), stable FORE with resets (dashed) and unstable FORE with resets (solid).

and $g(x_c, \dot{q}) = \dot{q}$. Moreover, as shown in Figure 5.5, we choose $u = k_{\mathcal{H}}\hat{u}$, where $k_{\mathcal{H}}$ is a positive constant.

By Proposition 5.3, the closed loop system (5.13), (5.14), (5.11a) with $u = k_{\mathcal{H}}\hat{u}$ is finite-gain $\mathcal{L}_2$ stable. Figure 5.6 compares several simulation results for this closed-loop using the constant reference signal $r = [10\,6]^T$ and the following values of the parameters: $k_p = 100$, $k_{\mathcal{H}} = 100$ and $\rho = 0.1$. First, we select stable FORE poles $(\lambda_1, \lambda_2) = (-2, -1)$ so that the closed-loop stability can be concluded also using the results in [34]. For this case, when no resets occur, the position output (namely q) and plant input (namely u) responses correspond to the dash-dotted curves in Figure 5.6. That response is converging because the system without resets is passive due to the stability of the FORE poles. When introducing resets, the response becomes the dashed curves in the figure, where it can be appreciated that a single reset occurring around $t = 0.8$ s significantly improves the closed-loop response. A last simulation is carried out by selecting an unstable FORE with $(\lambda_1, \lambda_2) = (2, 1)$. In this case the speed of convergence of the second joint is faster at the price of a reduction of the speed of convergence of the first joint. Note also that the dwell time imposed by the temporal regularization is never active for this specific simulation, as each jump occurs after more than $\rho = 0.1$ seconds from the previous jump. We don't include a simulation with the unstable FORE without resets because this leads to diverging trajectories.

5.2 Control over Network: Lazy Sensors

In recent years, much attention has been devoted to the study of networked control systems. The interest in this class of control systems is motivated by the increased computational capability required by control and estimation algorithms in addition to the presence of emerging control applications wherein the systems to be controlled are spread over a wide territory or are technologically built in such a way that several subcomponents of the control system communicate over a shared and low capacity network (see, e.g., the recent surveys [157, 75] and references therein). While networked control systems denote many different situations where a network is in some sense involved in the transmission of the control signals, a case of interest is that when the network is used as a communication channel between the plant with its sensing/actuating devices and the device hosting the control algorithm. This specific context is studied, e.g., in [33, 35, 102, 106, 142, 155, 156].

A typical way to represent and suitably write the dynamics of systems acting on networks is to use the hybrid systems notation. For example in [33, 106], Lyapunov-like tools are used to model ISS properties of network control systems and the MATI - maximum allowable transfer interval, to preserve asymptotic stability.

In this section we consider a linear control system that consists of a controller that uses the output of a given plant and produces a suitable input to asymptotically stabilize the whole closed-loop system. Usually, the measured output y of the plant is connected to the input of the controller u, so that the signal y is continuously transmitted to the controller. Here we break this continuity by considering a not necessarily periodic sample and hold approach. In particular, we suppose that the wire from the *measured output* y to the *controller input* u is replaced by a network, that is, each measurement is sampled and routed to the controller input. Then, we define an updating policy for sending such measurements samples, based on the current state of the system and on the error between the value of the output y and the value of the samples sent to the controller. The devices performing this scheduling policy are called "lazy sensors" to resemble the fact that their goal is to avoid transmitting too often, so to keep low the network load. The structure of the considered system is represented in Figure 5.7.

Each lazy sensor is able to perform some computation on the measured plant output and, possibly, on extra input signals. Then, each sensor decides whether or not to send a sampled measurement to the input of the controller. The contribution of this section consists in casting the above problem using the

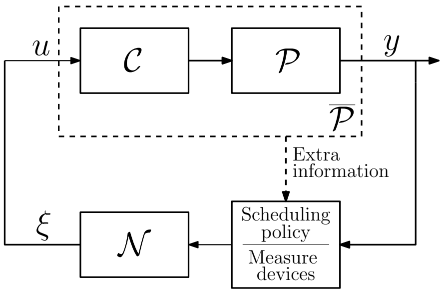


Figure 5.7: A closed-loop $\mathcal{S}_{\mathcal{N}}$ over a network using lazy sensors.

framework in [62] and proposing a number of measurement transmission (or update) policies which depend on the state of the plant and on the measurement error through a suitable Lyapunov-like function. Then, we show that the proposed transmission policies preserve the closed-loop stability. In particular, we propose the following three solutions, suitable for different practical contexts:

– a synchronous updating policy where each sensor is aware of the conditions of the other sensors so that the samples update is a global decision. Specifically, the sensors send a new sample all together when some suitable condition occur;

– an asynchronous updating policy where each sensor knows its own measurement error and the state of the plant. Then, it decides autonomously whether or not to send a new sample to the controller;

– a synchronous updating policy based on the measurement errors and the output signal of the plant, by using an observer to reconstruct the state, assuming that it is not available for measurement.

Since the ultimate goal of the above policies is to use the network as little as possible, we call *lazy* these intelligent sensors, to resemble the fact that they are reluctant to transmit and that they do so only when it is strictly necessary, w.r.t. the satisfaction of a suitable Lyapunov-like condition, to preserve closed-loop stability. A possible implementation context could be that of a CAN network where the shared information is broadcast on the network by the controller node, which has highest priority over the other nodes. Then, the other nodes could correspond to the lazy sensors, each of them equipped with an onboard intelligence deciding whether or not to transmit over the network.

The approach developed in this section can be seen as a constructive solution along the general lines of [33, 106], where Lyapunov tools and the hybrid framework of [62] are used as well to address networked control systems. Our work can also be associated with the many interesting results in [35, 102, 142, 155] and references therein. Here, differently from [142, 155], we only take into account linear systems by proposing updating rules that do not necessarily force each sample to be updated to the current measure of the output. Moreover, asynchronous updating policies and output based updating policies studied here are not taken into account in [142, 155].

This section is structured as follows. In Section 5.2.1 we introduce the problem data and in the following Sections 5.2.2, 5.2.3, 5.2.4 we discuss the three approaches outlined above. In Section 5.2.5 we give a simulation example and proofs are given in the appendix.

5.2.1 Problem statement

Consider a *nominal closed-loop system*, $\mathcal{S}$, composed by a *linear controller* $\mathcal{C}$, with input u_c and output y_c, and by a *linear plant* $\mathcal{P}$, with input u_p and output y_p. The controller drives the plant by the connection $u_p = y_c$ and the output of the plant, y_p, is connected to the input u_c of the controller (feedback signal). In what follows we denote with $\overline{\mathcal{P}}$ the *cascade* of the the controller $\mathcal{C}$ and of the plant $\mathcal{P}$, through the connection $u_p = y_c$. $\overline{\mathcal{P}}$ can be represented as follows

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Hx \end{array} \right. \tag{5.17}$$

where we assume $u = u_c$ and $y = y_p$. Thus, the nominal closed loop system $\mathcal{S}$ is constructed by connecting (5.17) through

$$u = y. \tag{5.18}$$

Then, the closed-loop system $\mathcal{S}$ of Equations (5.17),(5.18) can be characterized as follows.

$$\left\{ \begin{array}{rcl} \dot{x} & = & (A + BH)\, x \\ y & = & Hx \end{array} \right. \tag{5.19}$$

and we consider the following standing assumption

Assumption 5.2 *The nominal closed-loop system $\mathcal{S}$ is exponentially stable.*

Consider now to replace the direct feedback interconnection (5.18) with a non-continuous communication policy $u = \mathcal{N}(y)$ between the output y and the input u. $\mathcal{N}$ can be considered a sample and hold network of digital sensor devices, that brings each sensor measurement of y to the input u of the controller. The *networked closed-loop system* $\mathcal{S}_\mathcal{N}$, namely the closed-loop system of (5.17) through the interconnection $u = \mathcal{N}(y)$, combines together the continuous dynamics of the plant-controller cascade $\overline{\mathcal{P}}$ and the discrete behavior of the network of digital sensor devices $\mathcal{N}$. Thus, it can be conveniently characterized within the *hybrid systems framework*.

In particular, we can write a hybrid model for the networked closed-loop system $\mathcal{S}_\mathcal{N}$. It is characterized by three main components: (i) the continuous dynamics of the cascade $\overline{\mathcal{P}}$ of controllers and plant; (ii) the dynamics of the sample-and-hold device, namely the mechanism that holds a sensors sample until a new one occur; (iii) the updating policy, that decides when a new sample of the measured output y must be submitted to the controllers. Then, a possible model for $\mathcal{S}_\mathcal{N}$ is

$$\begin{cases} \dot{x} &= Ax + B\xi \\ \dot{\xi} &= 0 \end{cases} \qquad (x,\xi) \in C \tag{5.20a}$$

$$\begin{cases} x^+ &= x \\ \xi^+ &= g(x,\xi) \end{cases} \qquad (x,\xi) \in D \tag{5.20b}$$

$$y = Hx \tag{5.20c}$$

Consider the continuous dynamics in (5.20a): x takes into account the dynamics of the plant-controller cascade, while ξ is the value that is currently enforced at the input of the controller. The dynamics of ξ takes into account the sample-and-hold behavior of the network, whose derivative must be zero (it "holds"). Moreover, the dynamics of $\overline{\mathcal{P}}$ is now driven by ξ, replacing the connection $u = y$ with $u = \mathcal{N}(y)$, that is, with $u = \xi$. The set C in which the system may flow is a design parameter, that is, it will be used to define the updating policy of the measurement samples. Consider the discrete dynamics in (5.20b): We model the updating mechanism of a measurement sample to the controller input as a *jump*. Therefore, during a jump, the state x of $\overline{\mathcal{P}}$ is not modified, while the state ξ of the network is modified in accordance with a suitable updating policy, whose behavior depends on the function g and on sets C and D. Intuitively, a measurement is updated to a new value given by g when some suitable condition on x and ξ is satisfied, that is, when $(x, \xi) \in D$. For now, we do not make any assumption on g. The structure of g, as well as its values, depend on the particular feedback that we consider and will be defined in the next sections.

Remark 5.4 In this work, we consider a very simple model for the network $\mathcal{N}$. In fact, we consider $\mathcal{N}$ as a general discrete process that routes each sensor measurement to an output point. Usually, this operation introduces time-delays and quantizations of signals. Moreover, the amount of data routed by the network is limited by the physical data-rate bounds of the network. In our model we do not take into account time-delays and quantization problems, assuming that each measurement is instantaneously routed to the controller. Instead, we consider an updating policy that guarantees a low data-rate on the network.

5.2.2 State feedback: synchronous approach

Consider the networked closed-loop system $\mathcal{S}_{\mathcal{N}}$ in Equation (5.20) and the coordinate transformation $e = \xi - y$, related to the *error* between the measured output and its samples, induced by the sample and hold mechanism of the network. The system can be written as follows.

$$\left\{ \begin{array}{lcl} \dot{x} & = & \overline{A}_{11}x + \overline{A}_{12}e \\ \dot{e} & = & \overline{A}_{21}x + \overline{A}_{22}e \end{array} \right. \qquad (x,e) \in \overline{C} \tag{5.21a}$$

$$\left\{ \begin{array}{lcl} x^{+} & = & x \\ e^{+} & = & \overline{g}(x,e) \end{array} \right. \qquad (x,e) \in \overline{D} \tag{5.21b}$$

$$y = Hx \tag{5.21c}$$

where $\overline{A}_{11} = (A + BH)$, $\overline{A}_{12} = B$, $\overline{A}_{21} = -H(A + BH)$ and $\overline{A}_{22} = -HB$. $\overline{g}(x,e)$, $\overline{C}$ and $\overline{D}$ characterize the updating policy and their definition is the goal of this work. They will be defined by the design method proposed below. In general, $\overline{g}$ is a function in $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ satisfying $\overline{g}(0,0) = 0$.

Remark 5.5 Suppose that $\overline{g}(x,e)$, $\overline{C}$ and $\overline{D}$ have been constructed by a suitable design method. Then, $g(x,\xi)$, C and D of (5.20) can be defined from $\overline{g}(x,e)$, $\overline{C}$ and $\overline{D}$ as follows.

- Suppose $\overline{C} = \{(x,e) \,|\, r(x,e)\}$ where r is a given relation on x and e. Then, $C = \{(x,\xi) \,|\, r(x, \xi - Hx)\}$, which is equivalent to defining a set C parameterized by the current output y, namely $\{(x,\xi) \,|\, r(x, \xi - y)\}$. Analogously for D.

- $g(x,\xi) = Hx + \overline{g}(x, \xi - Hx)$. An equivalent characterization for g, based on the current output y, is $y + \overline{g}(x, \xi - y)$.

The first transformation is straightforward. To see the second one, note that $e^+ = \xi^+ - y^+ = g(x, e+y) - y^+ = g(x, e+y) - Hx^+ = -Hx + g(x, e+y)$. Then, the result follows by solving $\overline{g}(x, \xi - Hx) = -Hx + g(x, \xi)$.

Remark 5.6 It is worth to mention that (5.21a) and (5.21c) can now be compared to the dynamics of (5.19), by adding the effect of the error $e = \xi - y$ to the right-hand side of (5.19). Moreover, from Assumption 5.2, there exists a symmetric and positive definite matrix P_{11} such that the function $V_{11}(x) = \frac{1}{2}x^T P_{11} x$ decreases along the trajectories of (5.19), that is $\langle \nabla V_{11}(x), \overline{A}_{11} x\rangle \leq -x^T Q x$, for any given symmetric and positive definite matrix Q.

In what follows we work with the model (5.21), and we define a possible updating policy for the lazy sensors that decides when the measurements samples must be routed to the controller input, so that the stability of the networked closed -loop systems is preserved. Indeed, we propose a Lyapunov-like characterization of the updating policy, that is, we find a policy whose routing events are defined with respect to a suitable Lyapunov function, so that the point $(x, e) = (0, 0)$ is asymptotically stable. Consider the following *Lyapunov-function candidate*

$$V(x, e) = \frac{1}{2}\begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{5.22}$$

where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ is symmetric and positive definite.

Then, by denoting by $F(x, e)$ the right-hand side of (5.21a), the directional derivative $\langle \nabla V(x,e), F(x,e)\rangle$ of V is less than or equal to

$$-x^T Q x + x^T R_{11} x + x^T R_{12} e + e^T R_{22} e \tag{5.23}$$

where Q is a symmetric and positive definite matrix, still to be selected, and

$$\begin{aligned} R_{11} &= P_{12}\overline{A}_{21} \\ R_{12} &= P_{11}\overline{A}_{12} + P_{12}\overline{A}_{22} + \overline{A}_{11}^T P_{12} + \overline{A}_{21}^T P_{22} \\ R_{22} &= P_{12}^T \overline{A}_{12} + P_{22}\overline{A}_{22} \end{aligned} \tag{5.24}$$

Note that the existence of Q is guaranteed by Assumption 5.2 (See Remark 5.6). By denoting by $G(x, e)$ the right-hand side of (5.21b), the increment $V(G(x, e)) - V(x, e)$ of V is

$$xP_{12}(\overline{g}(x, e) - e) + \frac{1}{2}\overline{g}(x, e)^T P_{22}\overline{g}(x, e) - \frac{1}{2}e^T P_{22} e. \tag{5.25}$$

Define now

$$\overline{C} = \left\{(x,e)\,|\langle\nabla V(x,e), F(x,e)\rangle \leq -\varepsilon|x|^2\,\right\} \tag{5.26a}$$
$$\overline{D} = \left\{(x,e)\,|\langle\nabla V(x,e), F(x,e)\rangle \geq -\varepsilon|x|^2\,\right\} \tag{5.26b}$$

where ε and Q are chosen so that

$$Q - R_{11} - \varepsilon I > 0. \tag{5.27}$$

Then, the following theorems hold (the proofs are in Appendix 6.5.2).

Theorem 5.2 *Let $\overline{C}$ and $\overline{D}$ be defined as in* (5.26). *Under Assumption 5.2, for each continuous function $\overline{g}$ such that*

(1) $V(G(x,e)) - V(x,e) \leq 0$ *for all* $(x,e) \in \overline{D}$,

(2) $(x, \overline{g}(x,e)) \notin \overline{D} \setminus \{(0,0)\}$ *for all* $(x,e) \in \overline{D}$,

the origin of the system $\mathcal{S}_{\mathcal{N}}$ of equations (5.21) *is globally pre-asymptotically stable (GpAS).*

Theorem 5.3 *Let $\overline{C}$ and $\overline{D}$ be defined as in* (5.26) *and α be a class $\mathcal{K}$ function. Under Assumption 5.2, for each continuous function $\overline{g}$ such that*

(1) $V(G(x,e)) - V(x,e) \leq -\alpha(|e|)$ *for all* $(x,e) \in \overline{D}$,

the origin of the system $\mathcal{S}_{\mathcal{N}}$ of equations (5.21) *is globally pre-asymptotically stable.*

Remark 5.7 Note that the existence of an updating policy for the lazy sensors is guaranteed by Assumption 5.2. In fact, the closed-loop system (5.17), (5.18) is exponentially stable, therefore it is robust with respect to small error signals e that vanish with x. Consider now (5.21). The dynamics of e is linear, therefore there exists a sufficiently small τ such that an updating policy that routes a new measurement sample ($e = 0$) with an intersample time not greater than τ would preserve the stability of the closed-loop system.

Remark 5.8 Note that the asymptotic stabilization of the point $(x,e) = (0,0)$ can be relaxed to the asymptotic stabilization of the set $\mathcal{A} = \{(x,e)\,|\, x = 0, -\underline{c} \leq |e|_\infty \leq \overline{c}\}$, for some given $\underline{c}, \overline{c} \in \mathbb{R}_{\geq 0}$. In fact, if $\mathcal{A}$ is globally pre-asymptotically stable, then the state of $\overline{\mathcal{P}}$ is driven to zero as in (5.19). In such a case, we are relaxing the stabilization problem by requiring only a bounded error (e.g. a periodic non zero error). Note that stabilizing $\mathcal{A}$ instead of 0 would not affect the output of the system.

A possible construction for $\overline{C}$ and $\overline{D}$

By using the exponential stability property of the nominal closed-loop system, a solution to the stabilization problem of the networked closed-loop system can be constructed as follows. A candidate Lyapunov function V can be defined as

$$V = \frac{1}{2}\begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{5.28}$$

where P_{11} and P_{22} are positive definite matrices and P_{11} satisfies

$$\overline{A}_{11}^T P_{11} + P_{11}\overline{A}_{11} \leq -Q \tag{5.29}$$

with Q symmetric positive definite matrix. Therefore, $\overline{C}$ and $\overline{D}$ can be defined as in equations (5.26), with $R_{11} = 0$, $R_{12} = P_{11}\overline{A}_{12} + \overline{A}_{21}^T P_{22}$ and $R_{22} = P_{22}\overline{A}_{22}$.

By resetting the error to zero whenever a jump occurs, that is, by defining $\overline{g}(x, e) = 0$ for all x and all e, we fullfill the requirements of both Theorems 5.2 and 5.3. Indeed,

$$V(G(x, e)) - V(x, e) = -\frac{1}{2}e^T P_{22} e \tag{5.30}$$

which satisfies condition (1) of both Theorem 5.2 and of Theorem 5.3. Moreover, by resetting the error to zero we have that

$$\begin{aligned} -x^T Q x + x^T R_{11} x + x^T R_{12} e + e^T R_{22} e &= x^T (R_{11} - Q) x \\ &< -\varepsilon x^T x, \end{aligned} \tag{5.31}$$

which, by (5.27), brings the state to the interior of $\overline{C}$, fullfilling condition (2) of Theorem 5.2.

It is important to note that this possible construction can be an effective model of the updating policy only if the state of the plant $\overline{\mathcal{P}}$ is known. In fact, a data is updated only if the state of the plant $\overline{\mathcal{P}}$ and the error $e = \xi - y$ characterize a configuration that do not belongs to $\overline{C}$. From a constructive point of view, we need sensors that evaluate the inequality in (5.26) and, based on such an evaluation, decide whether or not to update the data. Such a configuration is illustrated in Figure 5.8. Note that resetting e to zero is equivalent to reseting ξ to y.

Remark 5.9 The data-rate in the network is related to the definition of $\overline{C}$ and $\overline{D}$. In fact, longer flow intervals for $\mathcal{S}_{\mathcal{N}}$ guarantee lower data-rate on the network. For example, by choosing P_{22} so that $\overline{\sigma}(P_{22})$ is small, we are giving less consideration to the error e. This *naive* selection of P_{22} increases the length of the flow interval, therefore the jump rate decreases.

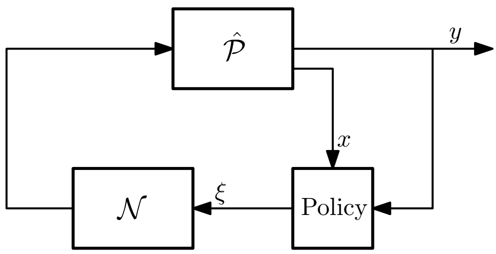


Figure 5.8: A possible configuration of the networked closed loop system $\mathcal{S}_{\mathcal{N}}$.

5.2.3 State feedback, asynchronous approach

The characterization of $\overline{C}$, $\overline{D}$ and $\overline{g}(x, e)$ of the previous section is based on the knowledge of the full state vector of the plant $\overline{\mathcal{P}}$ and of the complete error $e = \xi - y$. Such architecture needs that the sensors take into account the state x and the error e and decide whether or not to update the whole vector of (measured) output to the input vector of $\overline{\mathcal{P}}$.

In this section we propose an asynchronous updating policy in which each sensor decides autonomously its own update time. For instance, the knowledge of each sensor is limited to the state x of $\overline{\mathcal{P}}$ and to its own error, say e_i, given by $e_i = \xi_i - y_i$, where ξ_i and y_i are the ith components of ξ and y, respectively. Each sensor i decides to update ξ_i by taking into account the state vector x and the error e_i only. No shared knowledge of the state of others sensors, say $j \neq i$, is allowed.

Consider the hybrid system $\mathcal{S}_{\mathcal{N}}$ in (5.21). The *asynchronous behavior* of each sensor and the effect of such a behavior on the dynamics of the whole system can be modeled by the following definition of $\overline{C}$, $\overline{D}$ and $\overline{g}(x, e)$:

- $\overline{C}$ and $\overline{D}$ as the intersection and union of sets $\overline{C}_i$ and $\overline{D}_i$. For any given $i \in \{1, \ldots, q\}$, $\overline{C}_i$ or $\overline{D}_i$ are subsets of $\mathbb{R}^n \times \mathbb{R}$ whose elements are the pairs (x, e_i), where e_i is the ith component of e;

- define $\overline{g} \in \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ as the vector $[\,\overline{g}_1(x, e_1), \ldots, \overline{g}_q(x, e_q)\,]^T$ where each $\overline{g}_i$, $i \in \{1, \ldots, q\}$, is a function in $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

Let us consider a candidate Lyapunov function V as in (5.22). Equations (5.23) and (5.25) characterize the directional derivative and the increment of V. For each $i \in \{1, \ldots, q\}$, define now

$$\begin{aligned}\overline{C_i} = \{(x, e_i) \,|\, &- \alpha_i x^T Q x + \alpha_i x^T R_{11} x \\ &+ K_1 |x||e_i| + K_2 e_i^2 \le -\alpha_i \varepsilon |x|^2 \,\} \\ \overline{D_i} = \{(x, e_i) \,|\, &- \alpha_i x^T Q x + \alpha_i x^T R_{11} x \\ &+ K_1 |x||e_i| + K_2 e_i^2 \ge -\alpha_i \varepsilon |x|^2 \,\}\end{aligned} \tag{5.32}$$

where

- for each $i \in \{1, \ldots, q\}$, $\alpha_i \in \mathbb{R}_{>0}$ and $\sum_{i=1}^{q} \alpha_i = 1$,
- $K_1 = \max_{|x|=1, |e|=1} |x^T R_{12} e|$,
- $K_2 = \max_{|e|=1} |e^T R_{22} e|$,
- Q and ε satisfy (5.27).

Then, we can define $\overline{C}$ and $\overline{D}$ as follows.

$$\overline{C} = \{(x, e) \,|\, \text{for each } 1 \le i \le q,\ (x, e_i) \in \overline{C_i}\} \tag{5.33a}$$

$$\overline{D} = \{(x, e) \,|\, \text{there exists } 1 \le i \le q,\ (x, e_i) \in \overline{D_i}\} \tag{5.33b}$$

Remark 5.10 Note that $\overline{D}$ in (5.33b) is the closed complement of $\overline{C}$ in (5.33a). This fact and the definition of $\overline{D}$ imply that a jump occurs when at least one combination of e_i and x, $i \in \{1, \ldots, q\}$, satisfies the condition in $\overline{D}_i$.

The asynchronous behavior of the sensors is then guaranteed by assuming that each function $\overline{g}_i$, $i \in \{1, \ldots, q\}$, coincides with the identity function that maps e_i to e_i, for $(x, e_i) \notin \overline{D}_i$. In fact, suppose that a jump is enabled by the ith sensor only, that is, $(x, e_i) \in \overline{D}_i$. Then the ith sensor sends a new sample based on the value given by $\overline{g}_i(x, e_i)$, while the behavior of all the other sensors, say $j \ne i$, is given by $\overline{g}_j(x, e_j) = e_j$, that is, their value is not modified.

To state the main result of this section, in Theorem 5.4, we need the following technical definition.

Definition 5.1 For each $i \in \{1, \ldots, q\}$, $\overline{g}_i \in \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ is a function such that

- $\overline{g}_i(x, e_i) = e_i$ if $(x, e_i) \notin \overline{D}_i$;

– the restriction of g_i on $\overline{D}_i$ is a continuous function.

Then, we say that $\overline{g} \in \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ defined by $\overline{g}(x,e) = [\,\overline{g}_1(x,e_1),\ldots,\overline{g}_q(x,e_q)\,]^T$ is *asynchronous.*

Theorem 5.4 *Let $\overline{C}$ and $\overline{D}$ be defined as in* (5.33) *and α a $\mathcal{K}$ function. Under Assumption 5.2, for each* asynchronous *function $\overline{g}$, if for each $(x,e) \in \overline{D}$*

(1) $V(G(x,e)) - V(x,e) < 0 \quad$ if $e \neq 0$,

then the origin of system $\mathcal{S}_{\mathcal{N}}$ (5.21) *is globally pre-asymptotically stable.*

Proof. See Appendix 6.5.3. □

A possible construction for $\overline{C}$ and $\overline{D}$

A solution to the stabilization problem of the networked closed loop system can be constructed as follows. Consider a candidate Lyapunov function V defined as

$$V = \frac{1}{2}\begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} \tag{5.34}$$

where P_{11} and P_{22} are positive definite matrices, P_{11} satisfies

$$\overline{A}_{11}^T P_{11} + P_{11}\overline{A}_{11} \leq -x^T Q x \tag{5.35}$$

for some given positive definite and symmetric Q, and

$$P_{22} = \mathrm{diag}\{\, P_{22}^{(1)},\ldots,P_{22}^{(q)}\,\}. \tag{5.36}$$

The sets $\overline{C}$ and $\overline{D}$ can be defined as in equation (5.33) with

$$\begin{aligned} \overline{C_i} &= \big\{(x,e_i)\,|\,-\alpha_i x^T Q x + K_1|x||e_i| + K_2 e_i^2 \leq -\alpha_i\varepsilon|x|^2\big\} \\ \overline{D_i} &= \big\{(x,e_i)\,|\,-\alpha_i x^T Q x + K_1|x||e_i| + K_2 e_i^2 \geq -\alpha_i\varepsilon|x|^2\big\} \end{aligned} \tag{5.37}$$

where K_1 and K_2 satisfy

$$\begin{aligned} K_1 &= \max_{|x|=1,|e|=1} |x^T(P_{11}\overline{A}_{12} + \overline{A}_{21}^T P_{22})e| \\ K_2 &= \max_{|e|=1} |e^T P_{22}\overline{A}_{22}e| \end{aligned} \tag{5.38}$$

By defining $\overline{g}(x,e)$ as follows

$$\overline{g}(x,e) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_q \end{bmatrix} \quad \text{where} \quad \begin{cases} v_i = 0 & \text{if } (x,e_i) \in \overline{D}_i \\ v_i = e_i & \text{otherwise} \end{cases} \tag{5.39}$$

we fullfill the requirements of Theorem 5.4. Indeed, $\overline{g}$ has an asynchronous structure because the reset of v_i to zero depends on x and e_i only, for each $i = 1, \ldots, q$. Moreover,

$$\begin{aligned} V(G(x,e)) - V(x,e) &= \frac{1}{2}(\overline{g}(x,e)^T P_{22} \overline{g}(x,e) - e^T P_{22} e) \\ &= \frac{1}{2} \sum_{i=1}^{q} P_{22}^{(i)} (v_i^2 - e_i^2). \end{aligned} \tag{5.40}$$

Since $\overline{g}$ is applied only if the state (x,e) is in $\overline{D}$, it follows that there exists at least one $j \in \{1 \ldots, q\}$ such that $v_j = 0$. Therefore,

$$V(G(x,e)) - V(x,e) \leq -\frac{1}{2} P_{22}^{(j)} e_j^2 \tag{5.41}$$

for some $j \in \{1, \ldots, q\}$. This satisfies condition (1) of Theorem 5.4.

From a constructive point of view we need q sensors. Each sensor, say i, evaluates the inequality in (5.32), that depends only on the measured output y_i and on the state x. Based on such an evaluation, the sensor decides whether or not to update the sample ξ_i, namely whether or not to transmit its measurement. Such a configuration is illustrated in Figure 5.9. Note that resetting e_i to zero is equivalent to reset ξ_i to y_i.

Remark 5.11 Note that α_i can be used to increase the update-rate of a sensor with respect to the others. Indeed, a greater α_i allows for a larger error bound, therefore the update-rate decreases. Note also that each α_i can be modified at runtime. As long as $\sum_{i=0}^{q} \alpha_i = 1$, the stability is preserved.

Remark 5.12 In general, if $\overline{g}(x,e)$ does not depend on the state, that is, its definition does not use x to define the value of $\overline{g}(x,e)$, then we can reduce the quantity of information sent to the sensors. In fact, for each $i \in \{1, \ldots, q\}$, both $\overline{C}_i$ and $\overline{D}_i$ can be redefined by using only e_i (i.e. ξ_i) and the following two signals $s_1 = x^T(-Q + R_{11})x$ and $s_2 = |x|$.

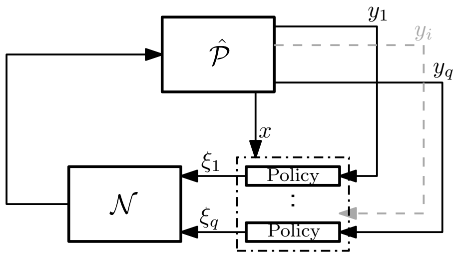


Figure 5.9: A possible asynchronous configuration of the networked closed loop system $\mathcal{S}_{\mathcal{N}}$

5.2.4 Output feedback approach

Consider the nominal closed-loop system of equations (5.17) and (5.18) and assume now that the state of controller x of the $\mathcal{C}$ and of the plant $\mathcal{P}$ can only be reconstructed from the output measurements. Despite the lack of information on the state, the approach of Section 5.2.2 can still be used by considering a suitable estimate of the state. We need the following assumption.

Assumption 5.3 *The pair* (A, H) *in* (5.17) *is detectable.*

The introduction of a classical continuous-time observer of the state in the networked closed-loop system $\mathcal{S}_{\mathcal{N}}$ leads to the following model

$$\left\{\begin{array}{l} \dot{x} = Ax + B\xi \\ \dot{\xi} = 0 \\ \dot{\hat{x}} = A\hat{x} + B\xi + L(y - H\hat{x}) \end{array}\right. \qquad (\hat{x},\xi) \in C \text{ or } \left\| \begin{bmatrix} \hat{x} \\ \xi - H\hat{x} \end{bmatrix} \right\| \le \rho \tag{5.42a}$$

$$\left\{\begin{array}{lcl} x^+ & = & x \\ \xi^+ & = & g(\hat{x},\xi) \\ \hat{x}^+ & = & \hat{x} \end{array}\right. \qquad (\hat{x},\xi) \in D \text{ and } \left\| \begin{bmatrix} \hat{x} \\ \xi - H\hat{x} \end{bmatrix} \right\| \ge \rho \tag{5.42b}$$

$$y = Hx \tag{5.42c}$$

where L is the observer matrix in $\mathbb{R}^{n\times q}$ and g is a function in $\mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ with q dimension of the output y of $\overline{\mathcal{P}}$. C and D are subsets of $\mathbb{R}^n$ and $\rho \in \mathbb{R}_{\ge} 0$. Note that the flow and jump sets of (5.42) can be considered as the combination of the flow and jump sets of (5.20) with a new condition $\left\| \begin{bmatrix} \hat{x} \\ \xi - H\hat{x} \end{bmatrix} \right\| \ge \rho$. This condition guarantees that if the estimate $\hat{x}$ and the sampling error $\xi - H\hat{x}$ are small enough (than ρ), then the system continues to flow without updating the value of the samples.

We can use the coordinate transformation $\hat{x}$, $e = \xi - H\hat{x}$ and $\eta = x - \hat{x}$ to rewrite (5.42) as follows:

$$\left\{\begin{array}{rcl} \dot{\hat{x}} & = & \overline{A}_{11}\hat{x} + \overline{A}_{12}e + LH\eta \\ \dot{e} & = & \overline{A}_{21}\hat{x} + \overline{A}_{22}e - HLH\eta \\ \dot{\eta} & = & (A - LH)\eta \end{array}\right. \qquad \begin{array}{c} (\hat{x}, e) \in \overline{C} \text{ or} \\ \left\| \begin{bmatrix} \hat{x} \\ e \end{bmatrix} \right\| \le \rho \end{array} \tag{5.43a}$$

$$\left\{\begin{array}{rcl} \hat{x}^+ & = & \hat{x} \\ e^+ & = & \overline{g}(\hat{x}, e) \\ \eta^+ & = & \eta \end{array}\right. \qquad \begin{array}{c} (\hat{x}, e) \in \overline{D} \text{ and} \\ \left\| \begin{bmatrix} \hat{x} \\ e \end{bmatrix} \right\| \ge \rho \end{array} \tag{5.43b}$$

$$y = H\hat{x} + H\eta \tag{5.43c}$$

where $\overline{A}_{11}$, $\overline{A}_{12}$, $\overline{A}_{21}$, $\overline{A}_{22}$ are defined as in Section 5.2.2. To extend the results of Section 5.2.2, $\overline{C}$ and $\overline{D}$ are defined as in (5.26), and $\overline{g}(\hat{x}, e) = M[\,\hat{x}^T\ e^T\,]^T$, where M is a matrix of dimensions $q \times (n + q)$. Then, the following theorem holds.

Theorem 5.5 (Global practical asymptotic stability) *Suppose that the conditions of Theorem 5.2 or of Theorem 5.3 are satisfied with the state x replaced by the estimation $\hat{x}$ and with $\overline{g}(\hat{x}, e) = M[\,\hat{x}^T\ e^T\,]^T$, where M is a matrix of dimension $q \times (n + q)$. Suppose that the gain-matrix L of the observer guarantees that* $\mathrm{eig}(A - LH)$ *is hurwitz.*

Then, there exists a $\overline{\gamma} \in \mathbb{R}_{>0}$ such that for any given ρ in (5.43)*, there exists a set $\mathcal{A} \subseteq \overline{\gamma}\rho\mathbb{B} \subset \mathbb{R}^{n+q}$, such that $\mathcal{A} \times \{0\} \subset \mathbb{R}^{n+q} \times \mathbb{R}^n$ is globally pre-asymptotically stable.*

Proof. See Appendix 6.5.4. □

Corollary 5.1 *If the conditions of Theorem 5.5 are satisfied, then each solution $(x, \xi, \hat{x})$ to* (5.42) *is such that $\hat{x}$ converges to x and $(x, \xi - y)$ converges to a ball of radius $\overline{\gamma}\rho\mathbb{B}$.*

Proof. Note that the union of the flow set and of the jump set of (5.43) coincides with the whole state-space $\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^n$. Thus, from any given initial state, each maximal solution is a complete solution. The coordinate transformation $(x, \xi, \hat{x}) \to (\hat{x}, e, \eta)$ is invertible, therefore the convergence of η of (5.43) to 0 implies that $\hat{x}$ converges to x. Thus, the convergence of $(\hat{x}, e)$ to $\overline{\gamma}\rho\mathbb{B}$ implies $(x, \xi - Hx)$ converges to $\overline{\gamma}\rho\mathbb{B}$. □

Remark 5.13 g, C and D of (5.42) can be constructed from $\overline{g}$, $\overline{C}$ and $\overline{D}$ as follows.

- Suppose $\overline{C} = \{(\hat{x}, e) \,|\, r(\hat{x}, e)\}$ where r is a given relation on $\hat{x}$ and e. Then, $C = \{(\hat{x}, \xi) \,|\, r(\hat{x}, \xi - H\hat{x})\}$. For D is the same.
- $g(\hat{x}, \xi) = H\hat{x} + \overline{g}(\hat{x}, \xi - H\hat{x})$. In fact, $e^+ = \xi^+ - H\hat{x}^+ = g(\hat{x}, \xi) - H\hat{x} = \overline{g}(\hat{x}, \xi - H\hat{x})$, where the last equality follows from (5.43b).

Remark 5.14 The result of Theorem 5.5 extends to the asynchronous case in Section 5.2.3 but it requires that the output $\hat{x}$ of the observer is shared among the sensors, thus breaking the decentralized structure of that approach.

5.2.5 Simulation example

We consider the following exponentially unstable linear plant $\mathcal{P}$ defined as follows

$$\mathcal{P} = \begin{cases} \dot{x_p} & = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_p + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_p \\ y_p & = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_p. \end{cases} \tag{5.44}$$

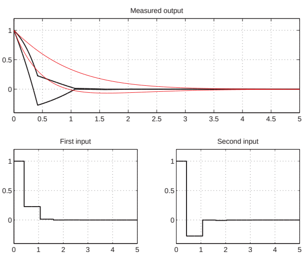
The nominal closed-loop system is constructed by connecting the plant $\mathcal{P}$ to the following LQR static controller $\mathcal{C}$.

$$y_c = \begin{bmatrix} -2.1961 & -0.7545 \\ -0.7545 & -2.7146 \end{bmatrix} u_c. \tag{5.45}$$

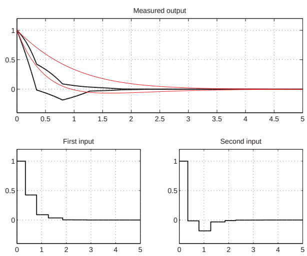
through the interconnection $u_p = y_c$ and $u_c = y_p$. With this controller the nominal closed-loop system is exponentially stable.

In the networked closed-loop system, the interconnection $u_c = y_p$ is replaced by $u_c = \xi$ where ξ is the vector of samples of the measured output that the controller is currently using. The vector of samples ξ is updated by following the policy defined in section 5.2.2. Four simulations tests with different values of the parameters are reported in Figure 5.10.

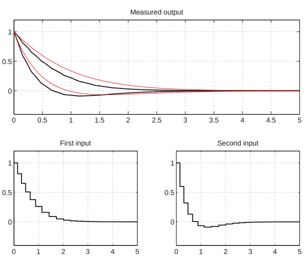
Consider now the closed-loop system given by equation (5.44) and (5.45). In this example we use the asynchronous policy of section 5.2.3, The results are illustrated in Figure 5.11, where six different parameters values are used. Note that each sensor i updates the measured output sample ξ_i without any kind of synchronization with the other sensors. Moreover, by choosing different α_1 and α_2, we force one sensor to allow for a larger error bound on $e_i = \xi_i - y_i$ before forcing an update. Therefore, one sensor will reset its state ξ_i more frequently than the other.



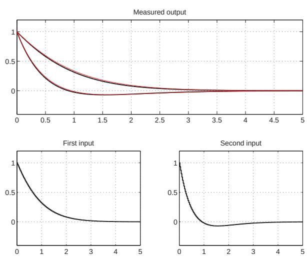
(a) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 0.1I$



(b) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 1I$

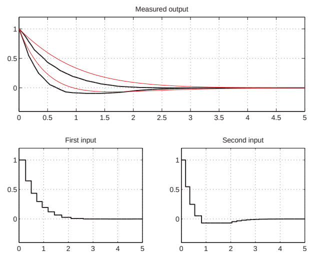


(c) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 10I$

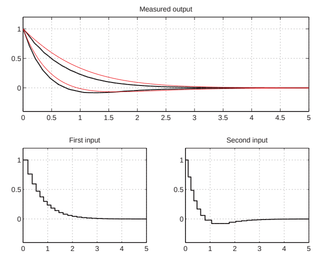


(d) $Q = 10I$, $\varepsilon = 0.01$, $P_{22} = 100I$

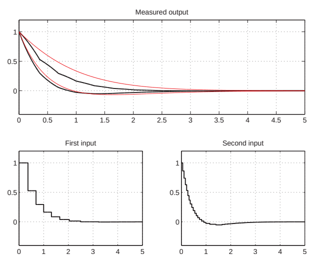
Figure 5.10: Input and output of $\mathcal{S}_{\mathcal{N}}$ in the synchronous case, for different choices of P_{22}. The thin line in each figure is the output of the nominal closed loop.



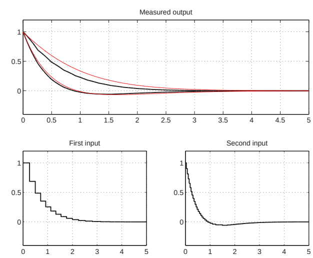
(a) $P_{22} = I$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$



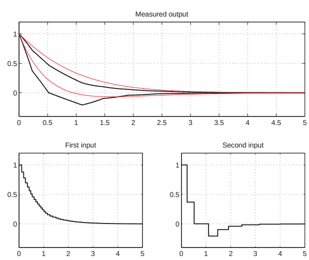
(b) $P_{22} = 10I$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$



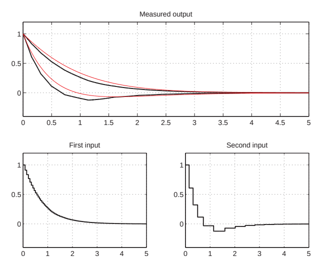
(c) $P_{22} = I$, $\alpha_1 = 0.9$, $\alpha_2 = 0.1$



(d) $P_{22} = 10I$, $\alpha_1 = 0.9$, $\alpha_2 = 0.1$



(e) $P_{22} = I$, $\alpha_1 = 0.1$, $\alpha_2 = 0.9$



(f) $P_{22} = 10I$, $\alpha_1 = 0.1$, $\alpha_2 = 0.9$

Figure 5.11: Input and output of $\mathcal{S}_{\mathcal{N}}$ in the asynchronous case, for different choices of P_{22}. The thin line in each figure is the output of the nominal closed-loop system. Extra parameters common to all cases are parameters are $Q = 10I$ and $\varepsilon = 0.01$.

Chapter 6

Proofs

6.1 Proof of the Results in Chapter 1

6.1.1 Proof of Theorem 1.5.

Proof. Under the assumption of the Theorem 1.5, there exists ϵ small enough such that $(\mathcal{A} + 2\epsilon\mathbb{B}) \subseteq \mathcal{U}$. Then, there exists a r_ϵ such that,

$$\begin{aligned}&\text{if } V(x) < r_\epsilon \text{ and } x \in (\mathcal{A} + 2\epsilon\mathbb{B}) \cap (C \cup D) \text{ then}\\ &x \in (\mathcal{A} + \epsilon\mathbb{B}) \cap (C \cup D) \text{ and } G(x) \subseteq (\mathcal{A} + 2\epsilon\mathbb{B}) \cap (C \cup D).\end{aligned} \tag{6.1}$$

In fact, by continuity of V, there exists $r'_\varepsilon > 0$ such that $V(x) < r''_\epsilon$ and $x \in (\mathcal{A} + 2\epsilon\mathbb{B}) \cap (C \cup D)$ imply $x \in (\mathcal{A} + \epsilon\mathbb{B}) \cap (C \cup D)$. Note now that $u_D(x) \leq 0$ for each $x \in \mathcal{A} \cap D$ and V is positive definite in $C \cup D$. It follows that $G(\mathcal{A} \cap (C \cup D)) \subseteq \mathcal{A} \cap (C \cup D)$. Moreover, G is outer semicontinuous and locally bounded, therefore, there is a $\gamma > 0$ such that $G(\mathcal{A} + \gamma\mathbb{B}) \subseteq \mathcal{A} + \epsilon\mathbb{B}$, [123, Proposition 5.12]. As before, V is positive definite, then there exists $r''_\epsilon > 0$ such that $V(x) < r''_\epsilon$ and $x \in (\mathcal{A} + 2\epsilon\mathbb{B}) \cap (C \cup D)$ implies $x \in (\mathcal{A} + \gamma\mathbb{B}) \cap (C \cup D)$, that is, $G(x) \subseteq (\mathcal{A} + \epsilon\mathbb{B}) \cap (C \cup D)$. It follows that implication (6.1) is true for $r_\epsilon = \min\{r'_\epsilon, r''_\epsilon\}$.

Consider now the set

$$\mathcal{N} = \{x \in (\mathcal{A} + \epsilon\mathbb{B}) \cap (C \cup D) \,|\, V(x) \leq r_\epsilon\} \tag{6.2}$$

Then, $\mathcal{N}$ is strictly forward invariant for $\mathcal{H}$, that is, for each solution ξ to $\mathcal{H}$ from a point $x \in \mathcal{N}$, $\operatorname{rng} x \subseteq \mathcal{N}$. In fact, for each $x \in \mathcal{N}$, consider a solution

ξ to $\mathcal{H}$ from x. If $(0,1) \in \operatorname{dom} \xi$ then, by (6.1), $\xi(0,0) = x$ and $\xi(1,0) \in \mathcal{N}$. If $[0,T] \times \{0\} \subseteq \operatorname{dom} \xi$, by $u_C(x) \leq 0$ for $x \in C \cap \mathcal{U}$ and (6.1), for each $t \in [0,T]$, $V(\xi(t,0)) \leq V(x)$. Then, by (6.1), for each $t \in [0,T]$ we have $\xi(t,0) \in (A + \varepsilon\mathbb{B}) \cap (C \cup D)$ and $V(\xi(t,0))$. Therefore, $\xi(t,0) \in \mathcal{N}$, that is, $\mathcal{N}$ is forward invariant.

Finally, by continuity of V, given any small $\epsilon > 0$, pick r_ϵ so that (6.1) holds. Then, we can find $\delta \in (0, \varepsilon)$ such that if $x \in \mathcal{A} + \delta\mathbb{B}$, then $V(x) \leq r_\epsilon$. Then, by forward invariance of $\mathcal{N}$, each solution ξ to $\mathcal{H}$ from some point in $\mathcal{A} + \delta\mathbb{B}$ is so that $\operatorname{rng} \xi \subseteq \mathcal{A} + \epsilon\mathbb{B}$. Thus, A is stable.

If $u_c(x) < 0$ for each $x \in (C \setminus \mathcal{A}) \in \mathcal{U}$ and $u_c(x) < 0$ for each $x \in (D \setminus \mathcal{A}) \in \mathcal{U}$ then, by [126, Theorem 4.7] complete solutions must converge to $\mathcal{A}$.

□

6.2 Proof of the Results in Chapter 2

6.2.1 Proof of Claim 2.1.

By (2.1), $R^{(i)}$ is a closed cone for each $i \in I_C$ and each $i \in I_D$. Point (i) follows from (2.2), by the fact that C and D are defined as finite union of closed sets.

The outer semicontinuity of F in (ii) can be shown by developing a proof for the outer semicontinuity of F_i, for each $i \in I_C$. The proof is developed by showing that for all $x \in R^{(i)}$, and for all sequences $x_j \to x$, $y_j \in F_i(x_j)$ such that $y_j \to y$, we have $y \in F_i(x)$. Then, outer semicontinuity of F can be developed with a similar approach.

Consider $i \in I_C$ and a mapping F_i defined in (2.3). Such a definition is equivalent to

$$F_i(x) = \left\{ f \mid f = \sum_{k=1}^{r_F} \lambda_k F_{ik} x \text{ and } \sum_{k=1}^{r_F} \lambda_k = 1 \right\} \tag{6.3}$$

where $\lambda_k \geq 0$, for each $k = 1 \ldots, r_F$. Suppose that a sequence x_j converges to x, that is $x_j \to x$, and suppose that there exist $y_j \in F_i(x_j)$ such that $y_j \to y$. Then, by (2.3), there exist $\lambda_1^j, \ldots, \lambda_{r_F}^j$, $\sum_{k=1}^{r_F} \lambda_k^j = 1$, such that $y_j = \sum_{k=1}^{r_F} \lambda_k^j F_{ik} x_j$, from which $\sum_{k=1}^{r_F} \lambda_k^j F_{ik} x_j \to y$.

Such a relation, the fact that $x_j \to x$, and the fact that $R^{(i)}$ is closed, imply that there exists $\lambda_1, \ldots, \lambda_{r_F}$, $\sum_{k=1}^{r_F} \lambda_k = 1$, such that $\lambda_k^j \to \lambda_k$, for each $k = 1, \ldots, r_F$, and $\sum_{k=1}^{r_F} \lambda_k F_{ik} x = y$.

In fact, suppose that $\sum_{k=1}^{r_F} \lambda_k F_{ik} x = y$ but $y \notin F_i(x)$. $R^{(i)}$ is a closed set, so $x \in R^{(i)}$ and F_i is defined on x. Therefore, by (2.3), we have that $y \notin F_i(x)$ implies

$\sum_{k=1}^{r_F} \lambda_k \neq 1$. This means that there exists a neighborhood of $\lambda_1, \ldots, \lambda_{r_F}$ such that, for some J, $\forall j \geq J$, $\sum_{k=1}^{r_F} \lambda_k^j \neq 1$. that contradicts the assumption of $y_j \in F_i(x_j)$, for each j.
Consider now F. The analysis of outer semicontinuity of F differs from the analysis of F_i only for sequences $x_j \to x$ whose tail periodically visits two or more cones C_i. In such a case, $x_j \in \bigcup_{i \in I} C_i$, where $I \subseteq I_C$. By the fact that the finite union of closed set is closed, $x \in \bigcup_{i \in I} C_i$ and, precisely, $x \in \bigcap_{i \in I} C_i$. It follows that, for each j, we can write y_j as the convex combination of vectors $F_{ik} x_j$, where i belongs to I and $k = 1, \ldots, r_F$. By the fact $x \in \bigcap_{i \in I} C_i$ is a closed set, it follows that $F(x)$ is defined on x and coincides with the convex combination of matrices F_{ik}, for $i \in I$ and $k = 1, \ldots, r_F$. Therefore, by an argument similar to the one used for F_i, we have that $y \in F(x)$.

$F_i(x)$ is defined by the convex hull of bounded vectors $F_{ik} x$ therefore it is a compact set. $F(x)$ is defined by the convex hull of $F_i(x)$, from which it is a compact set too. For each compact set $K \subset C$ and each $x \in K$, it follows that $F(x)$ is locally bounded.

From (2.3) and (2.5), $F(x)$ is nonempty in C. Convexity follows from the application of the convex hull operator in 2.5.

Outer semicontinuity of G in (iii) follows from the outer semicontinuity of G_i. For a given $i \in I_D$, consider a sequence $x_i \to x$ and $y_j \in G_i(x_j)$ so that $y_j \to y$. From (2.4) $y_j = G_{ik_j} x_j$ for some $k_j = 1, \ldots, r_G$, therefore $G_{ik_j} x_j \to y$, for a suitable selection of indices k_j. It follows that $x_j \to x$ and $y_j \to y$ imply that $G_{ik_j} x$ converges to some $G_{ik} x$, for some $k = 1 \ldots, r_G$.
In fact, without loss of generality, suppose that $x_j \to x$ and $y_j \to y$ and, for x_j sufficiently close to x, G_{ik_j} alternates between two matrices: G_{ik_a} if j is even and G_{ik_b} if j is odd. Then, for x_j sufficiently close to x, we have that $G_{ik_j} x_j \to y$ only if both $G_{ik_a} x_j \to y$ and $G_{ik_b} x_j \to y$, respectively for j even and j odd. This implies that, for $j \in \mathbb{Z}_{\geq 0}$, both $G_{ik_a} x_j \to y$ and $G_{ik_b} x_j \to y$ as $j \to \infty$. From which, $G_{ik_a} x = G_{ik_b} x = y$.
For $i \in I_D$, the outer semicontinuity of G_i follows from the fact that $R^{(i)}$ is a closed set and from the fact that $G_i(x)$ is defined for each $x \in R^{(i)}$. Therefore, for $x_j \to x$, $y = G_{ik} x \in G_i(x)$.
Note now that G differs from G_i only for sequences $x_j \to x$ whose tail periodically visits two or more cones D_i. In such a case, $x_j \in \bigcup_{i \in I} D_i$, where $I \subseteq I_D$. By the fact that the finite union of closed set is closed, $x \in \bigcup_{i \in I} D_i$ and, precisely, $x \in \bigcap_{i \in I} D_i$. It follows that, for each j, we can choose a matrix $G_{i_j k_j}$ with $i_j \in I$ and $k_j = 1, \ldots, r_G$ so that $y_j = G_{i_j k_j} x_j$. By the fact that $x \in \bigcap_{i \in I} D_i$ is a closed set, it follows that $G(x)$ is defined on x and, by an argument similar

to the one used above for G_i, it coincides with $y_j = G_{ik}x_j \in G(x)$, for some i that belongs to I and $k = 1, \ldots, r_G$.

Local boundedness of G follows from the fact that, for each $x \in D$, $G(x)$ is the union of bounded vectors. From (2.4), for each $i \in I_D$ and each $x \in R^{(i)}$, $G_i(x)$ is defined. Therefore, $G(x)$ is nonempty for each $x \in D$. □

6.2.2 Stability Proofs

Proof of Theorem 2.1.

Consider a smooth non-decreasing function $\sigma : \mathbb{R} \to [0, 1]$ defined as follows

$$\sigma(s) = \begin{cases} 0 & \text{if } s \leq \ell_1 \\ \frac{s-\ell_1}{\ell_2-\ell_1} & \text{if } l_1 \leq s \leq l_2 \\ 1 & \text{if } \ell_2 \leq s \end{cases} \tag{6.4}$$

where $l_1, l_2 \in \mathbb{R}_{>0}$ and $\ell_1 < l_1 < l_2 < \ell_2$, and consider a smooth function $q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as follows

$$q(x) = \begin{cases} q_{\ell_1}(x) & |x| \leq c \text{ or } (c \leq |x| \leq \rho c, V(x) \leq \ell_1) \\ V(x) & c \leq |x| \leq \rho c, \ell_1 \leq V(x) \leq \ell_2 \\ q_{\ell_2}(x) & (c \leq |x| \leq \rho c, V(x) \geq \ell_2) \text{ or } |x| \geq \rho c \end{cases} \tag{6.5}$$

where
(i) $q_{\ell_1} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a $\mathcal{C}^\infty$ function with image in $[0, \ell_1]$ that coincides with V on the set $\{x \in \mathbb{R}^n \,|\, c \leq |x| \leq \rho c \text{ and } V(x) = \ell_1\}$ and on the set $\{x \in \mathbb{R}^n \,|\, c \leq |x| \leq \rho c \text{ and } q_{\ell_1}(x) = \ell_1\}$; (ii) $q_{\ell_2} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a $\mathcal{C}^\infty$ function with image in $[\ell_2, \infty]$ that coincides with V on the set $\{x \in \mathbb{R}^n \,|\, c \leq |x| \leq \rho c \text{ and } V(x) = \ell_2\}$ and on the set $\{x \in \mathbb{R}^n \,|\, c \leq |x| \leq \rho c \text{ and } q_{\ell_2}(x) = \ell_2\}$; (iii) the junctions of q_{ℓ_1} with V and of q_{ℓ_2} with V are smooth.

Note that it is always possible to define q_{ℓ_1} and q_{ℓ_2} so that points (i)-(iii) above are satisfied and

$$\alpha_1(|x|) \leq q(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n \tag{6.6}$$

holds, where $\alpha_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\alpha_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are $\mathcal{K}_\infty$ functions. In fact, $q(x)$ coincides with $V(x)$ for $c \leq |x| \leq \rho c$ and for positive values of $V(x)$ between ℓ_1 and ℓ_2. Because $V(x)$ has a minimum and a maximum value in the compact set $c \leq |x| \leq \rho c$, it is always possible to find a lower bound function α_1 of V and a upper bound function α_2 of V in such a compact set. For the rest of the space, q coincides with q_{ℓ_1} or q_{ℓ_2}, therefore we can define such a function for satisfying (6.6).

With the definitions given above, the function $\overline{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ can be constructed as

$$\overline{V}(x) = \begin{cases} \int_0^\infty \frac{1}{t^{k+1}}\sigma(q(tx))dt & \text{if } x \in \mathbb{R}^n \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases} \tag{6.7}$$

where k is a given constant greater than 0. We prove that such a function is a smooth and homogeneous Lyapunov function for $\mathcal{H}$. Note that with $k > 0$ the integral in (6.7) converges, therefore $\overline{V}(x)$ is well defined.

Part 1: $a_1|x|^k \leq \overline{V}(x) \leq a_2|x|^k$.
From the definition of $\overline{V}$ and from (6.6), it follows that

$$\begin{aligned} \overline{V}(x) &= \int_0^\infty \frac{1}{t^{k+1}}\sigma(q(tx))dt \leq \int_0^\infty \frac{1}{t^{k+1}}\sigma(\alpha_2(|tx|))dt \\ &= \int_0^\infty \frac{1}{t^{k+1}}\sigma(\alpha_2(t|x|))dt = \int_0^\infty \frac{|x|^k}{s^{k+1}}\sigma(\alpha_2(s))ds \\ &= |x|^k \int_0^\infty \frac{1}{s^{k+1}}\sigma(\alpha_2(s))ds = a_2|x|^k \end{aligned} \tag{6.8}$$

where the first inequality holds because σ is a non-decreasing function, the third equality holds by $s = t|x|$, and $a_2 = \int_0^\infty \frac{1}{s^{k+1}}\sigma(\alpha_2(s))ds$ is well-defined because the integral converges for any $\mathcal{K}_\infty$ function α_2. In a similar way

$$\begin{aligned} \overline{V}(x) &= \int_0^\infty \frac{1}{t^{k+1}}\sigma(q(tx))dt \geq \int_0^\infty \frac{1}{t^{k+1}}\sigma(\alpha_1(|tx|))dt \\ &= \int_0^\infty \frac{|x|^k}{s^{k+1}}\sigma(\alpha_1(s))ds = a_1|x|^k \end{aligned} \tag{6.9}$$

where $a_1 = \int_0^\infty \frac{1}{s^{k+1}}\sigma(\alpha_1(s))ds$.

Part 2: $\overline{V}$ is smooth.
Continuity can be proved as follows. Define $\underline{t}(x) = \inf\{t \,|\, \sigma(q(tx)) > 0\}$ and $\overline{t}(x) = \sup\{t \,|\, \sigma(q(tx)) < 1\}$. For any given $\xi \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$,

$$\begin{aligned} \overline{V}(\xi) - \overline{V}(\xi+d) = &\int_{\underline{t}(\xi)}^{\overline{t}(\xi)} \frac{1}{t^{k+1}}\sigma(q(t\xi))dt + \int_{\overline{t}(\xi)}^\infty \frac{1}{t^{k+1}}dt + \\ &- \int_{\underline{t}(\xi+d)}^{\overline{t}(\xi+d)} \frac{1}{t^{k+1}}\sigma(q(t(\xi+d)))dt - \int_{\overline{t}(\xi+d)}^\infty \frac{1}{t^{k+1}}dt \end{aligned} \tag{6.10}$$

Note that $q(t\xi)$ can be replaced with $V(t\xi)$ in the first integral and $q(t(\xi+d))$ can be replaced with $V(t(\xi+d))$ in the third integral. In fact, by the definition of $\underline{t}(x)$, $\sigma(q(tx))>0$ only if tx satisfies both $c\le|tx|\le\rho c$ and $\ell_1<V(tx)<\ell_2$. It follows that $\underline{t}(x)x$ satisfies $c\le|\underline{t}(x)x|\le\rho c$ and $\ell_1\le V(\underline{t}(x)x)\le\ell_2$. By using the same argument, we can say that $\overline{t}(x)x$ satisfies $c\le|\overline{t}(x)x|\le\rho c$ and $\ell_1\le V(\overline{t}(x)x)\le\ell_2$. It follows that $c\le|tx|\le\rho c$ for each $t\in[\underline{t}(x),\overline{t}(x)]$. Finally, note that $\sigma(q(tx))=\sigma(V(tx))$ for $c\le|tx|\le\rho c$.

Case A: $\underline{t}(\xi)\le\underline{t}(\xi+d)\le\overline{t}(\xi)\le\overline{t}(\xi+d)$

$$\overline{V}(\xi)-\overline{V}(\xi+d)=\int_{\underline{t}(\xi)}^{\underline{t}(\xi+d)}h_1+\int_{\underline{t}(\xi+d)}^{\overline{t}(\xi)}h_2+\int_{\overline{t}(\xi)}^{\overline{t}(\xi+d)}h_3 \tag{6.11}$$

where

$$\begin{aligned} h_1&=\frac{1}{t^{k+1}}\sigma(V(t\xi))dt\\ h_2&=\frac{1}{t^{k+1}}[\sigma(V(t\xi))-\sigma(V(t(\xi+d)))]dt\\ h_3&=\frac{1}{t^{k+1}}[1-\sigma(V(t(\xi+d)))]dt \end{aligned} \tag{6.12}$$

By considering that (i) $\sigma(V(t\xi))\le1$, (ii) $1-\sigma(V(t(\xi+d)))\le1$, (iii) for any $t\in[\underline{t}(\xi+d),\overline{t}(\xi)]$, there exists a function $K_\xi(|d|):\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ such that $|\sigma(V(t\xi))-\sigma(V(t(\xi+d)))|\le K_\xi(|d|)$ and such that $K_\xi(|d|)\to0$ as $|d|\to0$ (by continuity of $\sigma(V(t\xi))-\sigma(V(t(\xi+d)))$ in the interval $[\underline{t}(\xi+d),\overline{t}(\xi)]$), it follows that

$$\begin{aligned}\overline{V}(\xi)-\overline{V}(\xi+d)\le M_\xi|\underline{t}(\xi+d)-\underline{t}(\xi)|+M_\xi K_\xi(|d|)\cdot\\ \cdot|\overline{t}(\xi)-\underline{t}(\xi+d)|+M_\xi|\overline{t}(\xi+d)-\overline{t}(\xi)|\end{aligned} \tag{6.13}$$

where $M_\xi=\frac{1}{(\underline{t}(\xi))^{k+1}}$.

The analysis of this case can be concluded by showing that $\underline{t}(\xi+d)\to\underline{t}(\xi)$ and $\overline{t}(\xi+d)\to\overline{t}(\xi)$ as $|d|\to0$. Indeed, $|\underline{t}(\xi+d)-\underline{t}(\xi)|=|\inf\{t\,|\,\sigma(q(t(\xi+d)))>0\}-\inf\{t\,|\,\sigma(q(t\xi))>0\}|$, and $|\overline{t}(\xi+d)-\overline{t}(\xi)|=|\sup\{t\,|\,\sigma(q(t(\xi+d)))<1\}-\sup\{t\,|\,\sigma(q(t\xi))<1\}|$, that both shrink to zero as $|d|\to0$, by the continuity of functions q and σ.

Case B: $\underline{t}(\xi)\le\overline{t}(\xi)<\underline{t}(\xi+d)\le\overline{t}(\xi+d)$.

For $|d|$ sufficiently small this case is impossible. For instance, suppose that $|d|\to0$ and $\underline{t}(\xi+d)\to\underline{t}(\xi)$ but $\underline{t}(\xi)\le\overline{t}(\xi)\le\underline{t}(\xi+d)$. It follows $|\underline{t}(\xi)-\overline{t}(\xi)|\to0$. This contradicts the fact that, by $\ell_1<\ell_2$, $|\underline{t}(\xi)-\overline{t}(\xi)|\ge\varepsilon$, for some $\varepsilon>0$.

Case C: $\underline{t}(\xi) \le \underline{t}(\xi+d) \le \overline{t}(\xi+d) \le \overline{t}(\xi)$.

$$\overline{V}(\xi) - \overline{V}(\xi+d) = \int_{\underline{t}(\xi)}^{\underline{t}(\xi+d)} h_1 + \int_{\underline{t}(\xi+d)}^{\overline{t}(\xi+d)} h_2 + \int_{\overline{t}(\xi+d)}^{\overline{t}(\xi)} h_3 \tag{6.14}$$

where h_1 and h_2 have been defined in (6.12)

$$h_4 = \frac{1}{t^{k+1}}[\sigma(V(t\xi)) - 1]dt \tag{6.15}$$

We can write

$$\begin{aligned}\overline{V}(\xi) - \overline{V}(\xi{+}d) \le M_\xi|\underline{t}(\xi{+}d) - \underline{t}(\xi)| + M_\xi K_\xi(|d|)\cdot \\ \cdot|\overline{t}(\xi{+}d) - \underline{t}(\xi{+}d)| + M_\xi|\overline{t}(\xi) - \overline{t}(\xi{+}d)|\end{aligned} \tag{6.16}$$

where $M(\xi)$ and $K_\xi(|d|)$ can be defined as in case A, but $K_\xi(|d|)$ is now a bound for $[\sigma(V(t\xi)) - \sigma(V(t(\xi+d)))]$ in the interval $[\underline{t}(\xi+d), \overline{t}(\xi+d)]$. From here, the argument coincides with case A.

Remaining cases can be analyzed as in A, B or C.

A similar path can be used for proving that $\overline{V}$ is differentiable for each degree k of differentiation. In fact, $\sigma(q(\cdot))$ is a smooth function and the integral can be compute by subdividing the domain of integration in several parts, say $I_1, I_2, \ldots, I_\nu$, for some $\nu \in \mathbb{Z}_{\ge 0}$, following the approach used for continuity. Then, we can prove that for each $i \in \{1,\ldots,\nu\}$, the sub-domain of integration I_i (a) converges to zero as $|d| \to 0$ or (b) converges to a domain for which the argument of the integral shrinks to 0 as $|d| \to 0$.

Part 3: $\forall x \in C, \forall f \in F(x)$, $\left\langle \nabla\overline{V}(x), f \right\rangle \le \mu\overline{V}(x)$.

By Condition (2) of Theorem 2.1 we have that for each $x \in C \cap \{x \,|\, c \le |x| \le \rho c\}$, and for each $f \in F(x)$, $\langle \nabla V(x), f\rangle$ is strictly negative. It follows that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $\forall x \in C \cap \{x \,|\, c \le |x| \le \rho c\}$ and $\forall f \in F(x)$,

$$\langle \nabla V(x), f\rangle \le -\varepsilon \max_{c\le|x|\le\rho c} V(x) \le -\varepsilon V(x). \tag{6.17}$$

Note that the existence of ε is guaranteed by the fact that V and ∇V are bounded in $C \cap \{x \,|\, c \le |x| \le \rho c\}$ and for each x in such a set $F(x)$ is a closed and bounded set. By the fact that $\sigma(q(\cdot))$ is a smooth function we can write

$$\begin{aligned} \langle \nabla \overline{V}(x), f \rangle &= \nabla \overline{V}(x)^T f \\ &= \int_0^\infty \frac{1}{t^{k+1}} \sigma'(q(tx)) \nabla V(tx)^T t f \, dt \\ &= \int_0^\infty \frac{1}{t^{k+1}} \sigma'(q(tx)) \langle \nabla V(tx), t f \rangle \, dt \\ &\leq -\varepsilon \int_0^\infty \frac{1}{t^{k+1}} \sigma'(q(tx)) V(tx) dt \end{aligned} \tag{6.18}$$

where $\sigma'(s) = \frac{\partial}{\partial s}\sigma(s)$. $\nabla V(tx)$ replaces $\nabla q(tx)$ because $\sigma'(q(x))$ is zero for x in $\{x \mid q(x) \neq V(x)\}$. The inequality can be explained as follows. (i) Suppose that x belongs to the intersection of some cone, say $\bigcap_{i \in I_x} C_i$, where $I_x \subseteq I_C$. The intersection of cones is a cone, therefore if $x \in \bigcap_{i \in I_x} C_i$ then $tx \in \bigcap_{i \in I_x} C_i$, for each $t \geq 0$. Moreover $I_x = I_{tx}$. (ii) $F(x)$ is the convex hull of matrices F_{ik} where $k = 1, \ldots, r_F$ and $i \in I_x$. Therefore, $f \in F(x)$ can be written as convex combination of vectors $F_{ik}x$ where $k = 1, \ldots, r_F$ and $i \in I_x$. It follows that tf can be written as the convex combination of vectors $tF_{ik}x$, that is equal to the convex combination of vectors $F_{ik}tx$, for $k = 1, \ldots, r_F$ and $i \in I_{tx}$. By points (i) and (ii), it follows that $tf \in F(tx)$, for each $t \geq 0$. Finally, $tf \in F(tx)$ and the fact that $\sigma'(q(tx))$ is zero for $tx \notin \{x \mid c \leq |tx| \leq \rho c\}$, allow to replace $\langle \nabla V(tx), tf \rangle$ with $-\varepsilon V(tx)$.

From (6.18), we can write

$$\begin{aligned} \langle \nabla \overline{V}(x), f \rangle &\leq -\varepsilon \int_0^\infty \frac{1}{t^{k+1}} \sigma'(q(tx)) q(tx) dt \\ &\leq -\varepsilon \int_0^\infty \frac{1}{t^{k+1}} \sigma'(\alpha_1(|tx|)) \alpha_1(|tx|) dt \\ &= -\varepsilon |x|^k \int_0^\infty \frac{1}{s^{k+1}} \sigma'(\alpha_1(s)) \alpha_1(s) ds \end{aligned} \tag{6.19}$$

where the second inequality follows from $\alpha_1(|x|)) \leq q(x)$, for each x, and the last expression is the results of the substitution $s = t|x|$. From (6.8), $\overline{V}(x) \leq a_2 |x|^k$, that is $\frac{1}{a_2}\overline{V}(x) \leq |x|^k$. It follows that

$$\begin{aligned} \langle \nabla \overline{V}(x), f \rangle &\leq -\frac{\varepsilon}{a_2} \overline{V}(x) \int_0^\infty \frac{1}{s^{k+1}} \sigma'(\alpha_1(s)) \alpha_1(s) ds \\ &\leq -\mu \overline{V}(x) \end{aligned} \tag{6.20}$$

where $\mu = \frac{\varepsilon}{a_2}\int_0^\infty \frac{1}{s^{k+1}}\sigma'(\alpha_1(s))\alpha_1(s)ds$. Note that such an integral converges by the fact that σ' is non zero only for $\ell_1 \leq \alpha_1(s) \leq \ell_2$.

Part 4: $\forall x \in D, \forall g \in G(x),\ \overline{V}(g) \leq \nu\overline{V}(x)$.

Let us consider $x \in \bigcap_{i\in I_x} D_i$ where $I_x \subseteq I_D$. Analogously to the previous case, $x \in \bigcap_{i\in I_x} D_i$ implies $tx \in \bigcap_{i\in I_x} D_i$, for each $t \geq 0$. Note that $I_x = I_{tx}$ and note also that, for $x \in \bigcap_{i\in I_x} D_i$, $i \in I_x$, we have that $g \in G(x)$ if and only if $g = G_{ik}x$ for some $k = 1, \ldots, r_G$ and some $i \in I_x$. Moreover, $tg = tG_{ik}x = G_{ik}tx$, for any $t \geq 0$, for some $k = 1, \ldots, r_G$ and some $i \in I_x$. It follows that $g \in G(x)$ implies $tg \in G(tx)$.

$$\begin{aligned}\overline{V}(g) - \overline{V}(x) &= \int_0^\infty \frac{1}{t^{k+1}}\sigma(q(tg))dt - \int_0^\infty \frac{1}{t^{k+1}}\sigma(q(tx))dt \\ &= \int_0^\infty \frac{1}{t^{k+1}}\big(\sigma(q(tg)) - \sigma(q(tx))\big)dt \leq 0\end{aligned} \tag{6.21}$$

Last inequality can be explained by considering that, for $x \in D$ and $t \geq 0$, $\sigma(q(tg)) - \sigma(q(tx)) \leq 0$. In fact,

(i) by Condition (3) of the theorem, if $c \leq |tx| \leq \rho c$ and $c \leq |tg| \leq \rho c$ then $V(tg) - V(tx) < 0$. Therefore $\sigma(q(tg)) - \sigma(q(tx)) \leq 0$;

(ii) if $c \leq |tx| \leq \rho c$ and $|tg| \leq c$ then $\sigma(q(tg)) = 0$. Then $\sigma(q(tg)) - \sigma(q(tx)) = -\sigma(q(tx)) \leq 0$;

(iii) if $c \leq |tx| \leq \rho c$ and $|tg| \geq \rho c$ then $\sigma(q(tg)) = 1$. By Condition (5) of the theorem, $q(tx) \geq \ell_2$ then $\sigma(q(tx)) = 1$. It follows that $\sigma(q(tg)) - \sigma(q(tx) = 0$;

(iv) if $|tx| \geq \rho c$ then $\sigma(q(tx)) = 1$. Because $\sigma(q(tg)) \leq 1$, it follows that $\sigma(q(tg)) - \sigma(q(tx)) \leq 0$;

(v) if $|tx| \leq c$ and $|tg| \leq \rho c$ then $\sigma(q(tx)) = 0$. By Condition (6), $q(tg) \leq \ell_1$, therefore $\sigma(q(tg)) = 0$. Then $\sigma(q(tg)) - \sigma(q(tx)) = 0$;

(vi) suppose $|tx| \leq c$ and $|tg| > \rho c$. Then, there exists $\bar{t} < t$ such that $|\bar{t}x| \leq c$ and $|\bar{t}g| = \rho c$. In such a case $V(\bar{t}g) \geq \ell_2 > \ell_1$. It follows that we found a point $\overline{x} = \bar{t}x$ such that $|\overline{x}| \leq c$ and $\overline{g} \in G(\overline{x})$ satisfies $\overline{g} \in \{x \,|\, V(x) > \ell_1 \text{ and } c \leq |x| \leq \rho c\}$. This contradicts Condition (6) of the theorem.

Note also that, for any given $x \in D \cap \{x \,|\, c \leq |x| \leq \rho c\}$, $V(g) - V(x)$ is strictly negative. Then, by (6.4) and (6.5), for each $x \in D$ such that $q(x) \in [l_1, l_2]$, we have that $q(x) = V(x) \geq l_1$ and either $q(g) = V(g) < q(x)$ or $q(g) \leq \ell_1$. It follows that $q(g) < q(x)$. Moreover, there exists a constant $\varepsilon \in [0,1) \subseteq \mathbb{R}$ such that

$$q(g) \leq \varepsilon q(x) \qquad \forall x \in D, q(x) \in [l_1, l_2], \forall g \in G(x) \tag{6.22}$$

In fact, $\{x \,|\, x \in D, q(x) \in [l_1, l_2]\}$ is a compact set and g is defined as $g = G_{ik}x$ where i takes values in a finite index set I_x and $k = 1, \ldots, r_G$, therefore we can find a small $\varepsilon > 0$ such that $q(g) - \varepsilon q(x) \leq 0$, for each $x \in D$, $q(x) \in [l_1, l_2]$, and each $g \in G(x)$.

From equation (6.4), for any given constant $s \in [l_1, l_2]$ and any given constant $\varepsilon \in [0,1)$ we have that $\sigma(\varepsilon s) < \sigma(s)$. This is straightforward for each $s \in (l_1, l_2]$ while, for $s = l_1$, it follows from the fact that the function is smooth, therefore the right-derivative of $\sigma(s) = \frac{1}{\ell_2 - \ell_1}$ on $s = l_1$ must be equal to the left-derivative, and from the fact that σ is non-decreasing. Finally, we can say that for any given $\varepsilon \in [0,1)$, there exists $\overline{\varepsilon} \in [0,1) \subseteq \mathbb{R}$ such that

$$\sigma(\varepsilon s) \leq \overline{\varepsilon} \sigma(s) \qquad \forall s \in [l_1, l_2] \tag{6.23}$$

By using (6.22) and (6.23), for any given $x \in D$, and for any given $t_1, t_2 \in \mathbb{R}_{\geq 0}$ so that $q(tx) \in [l_1, l_2]$ for each $t \in [t_1, t_2]$, we have

$$\begin{aligned} \overline{V}(g) &= \int_0^\infty \frac{1}{t^{k+1}} \sigma(q(tg))dt = \int_0^{t_1} h_g + \int_{t_1}^{t_2} h_g + \int_{t_2}^\infty h_g \\ &= \int_0^{t_1} h_g + \int_{t_1}^{t_2} \frac{1}{t^{k+1}} \sigma(\varepsilon q(tx))dt + \int_{t_2}^\infty h_g \\ &\leq \int_0^{t_1} h_g + \overline{\varepsilon} \int_{t_1}^{t_2} \frac{1}{t^{k+1}} \sigma(q(tx))dt + \int_{t_2}^\infty h_g \\ &\leq \int_0^{t_1} h_x + \overline{\varepsilon} \int_{t_1}^{t_2} h_x + \int_{t_2}^\infty h_x \end{aligned} \tag{6.24}$$

where $h_g = \frac{1}{t^{k+1}} \sigma(q(tg))dt$ and $h_x = \frac{1}{t^{k+1}} \sigma(q(tx))dt$. Note that last inequality follows from $\sigma(q(tg)) - \sigma(q(tx)) \leq 0$.

Consider now vectors $\eta \in D$ such that $|\eta| = 1$. Define two constant $s_{1\eta}, s_{2\eta} \in \mathbb{R}_{\geq 0}$ as follows: $s_{1\eta}$ is the smallest value such that $q(s_{1\eta}\eta) = l_1$ and $s_{2\eta}$ is the greatest value such that $q(s_{2\eta}\eta) = l_2$, and define

$$\begin{aligned}
I_{1\eta} &= \int_0^{s_{1\eta}} \frac{1}{s^{k+1}}\sigma(\alpha_2(s))ds \\
I_{2\eta} &= \int_{s_{1\eta}}^{s_{2\eta}} \frac{1}{s^{k+1}}\sigma(\alpha_1(s))ds \qquad (6.25)\\
I_{3\eta} &= \int_{s_{2\eta}}^{\infty} \frac{1}{s^{k+1}}\sigma(\alpha_2(s))ds.
\end{aligned}$$

Choose $\nu_\eta \in [\overline{\varepsilon}, 1)$ so that

$$(1-\nu_\eta)(I_{1\eta}+I_{3\eta}) - (\nu_\eta - \overline{\varepsilon})I_{2\eta} \le 0 \qquad (6.26)$$

For any given $\lambda > 0$, consider $x = \lambda\eta$ and take $t_{1\eta} = \frac{s_{1\eta}}{|x|}$ and $t_{2\eta} = \frac{s_{2\eta}}{|x|}$. From the definition of $s_{1\eta}$ and $s_{2\eta}$ we have that $q(t_{1\eta}x) = l_1$ and $q(t_{2\eta}x) = l_2$. Then

$$\begin{aligned}
\overline{V}(g) \le \int_0^{t_{1\eta}} + \overline{\varepsilon}\int_{t_{1\eta}}^{t_{2\eta}} + \int_{t_{2\eta}}^{\infty} &= \nu_\eta\left(\int_0^{t_{1\eta}} + \int_{t_{1\eta}}^{t_{2\eta}} + \int_{t_{2\eta}}^{\infty}\right) + \\
&+ (1-\nu_\eta)\left(\int_0^{t_{1\eta}} + \int_{t_{2\eta}}^{\infty}\right) - (\nu_\eta - \overline{\varepsilon})\int_{t_{1\eta}}^{t_{2\eta}} = \\
&= \nu_\eta\overline{V}(x) + (1-\nu_\eta)\left(\int_0^{t_{1\eta}} + \int_{t_{2\eta}}^{\infty}\right) - (\nu_\eta - \overline{\varepsilon})\int_{t_{1\eta}}^{t_{2\eta}} \qquad (6.27)\\
&\le \nu_\eta\overline{V}(x) + |x|^k((1-\nu_\eta)(I_{1\eta}+I_{3\eta}) - (\nu_\eta - \overline{\varepsilon})I_{2\eta}) \\
&\le \nu_\eta\overline{V}(x)
\end{aligned}$$

where the argument of each integral is $\frac{1}{t^{k+1}}\sigma(q(tx))dt$. Note that the second inequality can be obtained by using $s = t|x|$ as variable of integration and by using α_2 instead of q inside the integrals $\int_0^{t_1}$ and $\int_{t_2}^{\infty}$, and by using α_1 instead of q inside $\int_{t_1}^{t_2}$.

Finally, we need to show that there exists a $\nu \in [\overline{\varepsilon}, 1)$ such that for each $|\eta| = 1$ we have $\nu_\eta \le \nu$. With this aim, define $S = \{\eta \in \mathbb{R}^n \mid |\eta| = 1\}$ and define three functions $p_i : S \to \mathbb{R}_{\ge 0}$ that map $\eta \in S$ to $I_{i\eta}$, for $i \in \{1, 2, 3\}$. By the continuity of q, α_1 and α_2, p_i is continuous, for each $i \in \{1, 2, 3\}$. By the fact that the inequality in (6.26) is satisfied by $\max\{\overline{\varepsilon}, \frac{I_{1\eta}+\overline{\varepsilon}I_{2\eta}+I_{3\eta}}{I_{1\eta}+I_{2\eta}+I_{3\eta}}\} \le \nu_\eta < 1$, we can construct a function $p_4 : S \to \mathbb{R}_{\ge 0}$ defined as

$$p_4(\eta) = \max\left\{\overline{\varepsilon}, \frac{p_1(\eta) + \overline{\varepsilon}p_2(\eta) + p_3(\eta)}{p_1(\eta) + p_2(\eta) + p_3(\eta)}\right\} \qquad (6.28)$$

where η belongs to S, so that $\nu_\eta = p_4(\eta)$ satisfies (6.26) for each $\eta \in S$. Note that $p_4(\eta) < 1$ for each η in S. Moreover, p_4 is a continuous on the compact set S, therefore it has a maximum value $v = \max_{|\eta|=1} p_4(\eta) < 1$. Note that, for each $\eta \in S$, $\nu_\eta = \nu$ satisfies (6.26)

Part 5: Homogeneity.
Define $s = \lambda t$

$$\begin{aligned}\overline{V}(\lambda x) &= \int_0^\infty \frac{1}{t^{k+1}} \sigma(q(t\lambda x))dt \\ &= \int_0^\infty \frac{\lambda^k}{s^{k+1}} \sigma(q(sx))ds \\ &= \lambda^k \overline{V}(x)\end{aligned} \tag{6.29}$$

□

Proof of Theorem 2.2.

By the converse result in [32, Theorem 3.14], for a a hybrid system $\mathcal{H}$ of Equations (1.5),(2.1)-(2.5), if the point $x_e = 0$ is globally pre-asymptotically stable, then there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\begin{array}{cl} \alpha_1(|x|) \le V(x) \le \alpha_2(|x|) & \forall x \in \mathbb{R}^n \\ \langle \nabla V(x), f\rangle \le -V(x) & \forall x \in C, \forall f \in F(x) \\ V(g) \le e^{-1}V(x) & \forall x \in D, \forall g \in G(x) \end{array} \tag{6.30}$$

(1,2,3) Conditions (1)-(3) of Theorem 2.1 are immediately satisfied.
(4) Choose a constant $\ell_1 > 0$ and define $c = \alpha_2^{-1}(\ell_1)$. Choose a constant $\ell_2 > \ell_1$ so that $\rho = \alpha_2^{-1}(\ell_2)/c$ is strictly greater then 1. It follows that $\max_{|x|=c} V(x) \le \alpha_2(c) = \ell_1$ and $\min_{|x|=\rho c} V(x) \ge \alpha_1(\rho c) = \ell_2$. Condition (4) of Theorem 2.1 holds.
(5) Suppose now that there exists a $x \in D \cap \{x \,|\, c \le |x| \le \rho c\} \cap \{x \,|\, V(x) \le \ell_2\}$ and a $g \in G(x)$ such that $|g| > \rho c$. Then, $\ell_2 = \alpha_1(\rho c) < \alpha_1(|g|) \le V(g)$. Therefore $V(g) > V(x)$, that contradicts (6.30). It follows that Condition (5) of Theorem 2.1 holds.
(6) In a similar way, suppose that there exists a $x \in D \cap \{x \,|\, |x| \le c\}$ and a $g \in G(x)$ so that $c \le |g| \le \rho c$ and $V(g) > \ell_1$. Then, $V(x) \le \alpha_2(c) = \ell_1$. Therefore $V(g) - V(x) > 0$, that contradicts (6.30). Condition (6) of Theorem 2.1 is satisfied. □

Proof of Proposition 2.1.

$V(x)$ is a polynomial function, so it is smooth.
(1) For each $i \in I_C \cup I_D$, (2.9) can be written as $V(x) \geq \Delta_2^{(i)}(x) + s_1^{(i)}(x)\Delta_1(c, \rho c, x)$. $\Delta_1(c, \rho c, x) \geq 0$ for $c \leq |x| \leq \rho c$, $\Delta_2^{(i)}(x) \geq 0$ for $x \in R^{(i)}$ and $s_1^{(i)}(x) \geq 0$ for each $x \in \mathbb{R}^n$. It follows that $V(x) \geq 0$ in $\{x \,|\, c \leq |x| \leq \rho c, x \in R^{(i)}\}$. By $i \in I_C \cup I_D$, $V(x) \geq 0$ in $\{x \,|\, c \leq |x| \leq \rho c\} \cap (C \cup D)$, i.e. Condition (1) of Theorem 2.1 holds.
(2) For each $i \in I_C$ and each $k = 1, \ldots, r_F$, (2.10) can be written as $\nabla V(x) F_{ik} x < -\Delta_2^{(i)}(x) - s_2^{(ik)}(x)\Delta_1(c, \rho c, x)$. Therefore $\nabla V(x) F_{ik} x < 0$ in $\{x \,|\, c \leq |x| \leq \rho c, x \in R^{(i)}\}$, for each $i \in I_C$ and $k = 1, \ldots, r_F$. Suppose now that x belongs to the intersection of some sets $R^{(i)}$, for $i \in I \subseteq I_C$. Then, for each $f \in F(x)$ we can write

$$\begin{aligned} \langle \nabla V(x), f \rangle &= \langle \nabla V(x), \sum_{i \in I, k=1,\ldots,r_F} \lambda_{ik} F_{ik} x \rangle \\ &= \sum_{i \in I, k=1,\ldots,r_F} \lambda_{ik} \langle \nabla V(x), F_{ik} x \rangle \end{aligned} \tag{6.31}$$

for some $\sum_{i \in I, k=1,\ldots,r_F} \lambda_{ik} = 1$. It follows that $\langle \nabla V(x), f \rangle < 0$ in $\{x \,|\, c \leq |x| \leq \rho c\} \cap C$, i.e. Condition (2) of Theorem 2.1 holds.
(3) (2.11) implies Condition (3). To see this, an argument similar to the one above on (2.10) can be used. No convex combination of vectors $G_{ik} x$ is needed in this case, in fact $g \in G(x)$ if and only if $g \in G_{ik} x$ for some $i \in I \subseteq I_D$ and $k = 1, \ldots, r_G$.
(4) First inequality in (2.12) can be written as $V(x) \leq \ell_1 - s_4(x)\Delta_1(c, c+\varepsilon, x)$, that implies $V(x) \leq \ell_1$ for $x \in [c, c+\varepsilon]$. It follows that $\max_{|x|=c} V(x) \leq \ell_1$. A similar argument can be used to show that the second inequality in (2.12) guarantees $\min_{|x|=\rho c} V(x) \geq \ell_2$. Therefore, Condition (4) of Theorem 2.1 is satisfied.
(5) For each $i \in I_D$ and each $k = 1, \ldots, r_G$, first inequality in (2.13) implies $V(x) - \ell_2 - s_6^{(ik)}(x)(x' G_{ik}^T G_{ik} x - \rho^2 c^2) \geq 0$ for x in $\{x \,|\, x \in R^{(i)}, c \leq |x| \leq \rho c\}$. We can write such an inequality as $s_6^{(ik)}(x)(\rho^2 c^2 - x' G_{ik}^T G_{ik} x) \geq \ell_2 - V(x)$, and from $s_6^{(ik)}(x) \geq 0$, it follows that $\rho^2 c^2 - x' G_{ik}^T G_{ik} x \geq 0$ for x in $\{x \,|\, x \in R^{(i)}, c \leq |x| \leq \rho c, V(x) \leq \ell_2\}$. Note that $\rho^2 c^2 - x' G_{ik}^T G_{ik} x \geq 0$ is equivalent to $|G_{ik} x| \leq \rho c$, and such a relation hold for each $i \in I_D$ and each $k = 1, \ldots, r_G$, whenever x belongs to $\{x \,|\, x \in R^{(i)}, c \leq |x| \leq \rho c, V(x) \leq \ell_2\}$. It follows that $|g| \leq \rho c$ for each x in $\{x \,|\, x \in D, c \leq |x| \leq \rho c, V(x) \leq \ell_2\}$ and each $g \in G(x)$, i.e. Condition (5) of Theorem 2.1 is satisfied.
(6) For each $i \in I_D$ and each $k = 1, \ldots, r_G$, second inequality in (2.13) implies

$s_8^{(ik)}(x)(c^2 - x^T x) \leq \ell_1 - V(G_{ik}x) - s_9^{(ik)}(x)\Delta_1(c, \rho c, G_{ik}x)$ for each $x \in R^{(i)}$. By $s_8^{(ik)}(x) \geq 0$, we have that if $V(G_{ik}x) > \ell_1$ and $c \leq |G_{ik}| \leq \rho c$ then $(c^2 - x^T x) < 0$, for each $x \in R^{(i)}$. Such implication holds for each $i \in I_D$ and each $k = 1, \ldots, r_G$, therefore we can say that if $V(g) > \ell_1$ and $c \leq |g| \leq \rho c$ then $|c| < |x|$, for each $x \in D$ and each $g \in G(x)$. By negation, we have that for each $x \in D$ and each $g \in G(x)$, if $|x| \leq |c|$ then $g \notin \{g \,|\, V(g) > \ell_1, c \leq |g| \leq \rho c\}$. Condition (6) of Theorem 2.1 holds. □

6.2.3 Overshoots and Instability Proofs.

Proof of Theorem 2.3.

From the assumptions of Theorem 2.3, we have that $V(x) > 0$ for some point x arbitrarily close to 0. It follows that U is not empty and, by continuity of $V(x)$ in $C \cup D$, x_e belongs to the border of U.
Let ξ be a solution to $\mathcal{H}$ with initial state $\xi(0,0) \in U$. By conditions (1), (2) and (3) of the theorem, ξ must leave U. In fact, suppose that $V(\xi(0,0)) = a$, for some $a \in \mathbb{R}_{>0}$. Then, by conditions (1) and (2) of the theorem, $V(\xi(t,j)) \geq a$ for each $(t,j) \in \operatorname{dom}\xi$, and the set $\{x \in U \,|\, V(x) \geq a\}$ is a compact subset of U. By the compactness of such a set and Claim 2.1, we can say that there exist $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ such that $\langle \nabla V(x), f\rangle > \gamma_1$, for each $x \in C \cap \{x \in U \,|\, V(x) \geq a\}$ and each $f \in F(x)$, and $V(g) - V(x) > \gamma_2$, for each $x \in D \cap \{x \in U \,|\, V(x) \geq a\}$ and each $g \in G(x)$. It follows that $V(\xi(t,j)) \geq a + \gamma_1 t + \gamma_2 j$, for each $(t,j) \in \operatorname{dom}\xi$. Then, by condition (3) of the theorem and by the fact that V has a maximum on $\{x \in U \,|\, V(x) \geq a\}$, ξ cannot stay forever in such a compact set.
By (1) and (2) of the theorem, ξ cannot leave U by flowing across $\{x \in \mathbb{R}^n \,|\, V(x) = 0\}$ or by jumping to $\{x \in \mathbb{R}^n \,|\, V(x) \leq 0, |x| \leq r\}$, therefore it leaves U by flowing across $\{x \in U \,|\, |x| = r\}$ or by jumping to $\{x \in C \cup D \,|\, |x| > r\}$. Because this happens for points x arbitrarily close to 0, the point $x_e = 0$ is unstable. □

The following lemma will be used in the proof of Theorem 2.4 and of Corollary 2.1

Lemma 6.1 *Consider a hybrid system $\mathcal{H}$ of Equations* (1.5),(2.1)-(2.5) *and suppose ξ is a solution to $\mathcal{H}$. Then, for each $\lambda \in \mathbb{R}_{>0}$, $\lambda\xi$ is a solution to $\mathcal{H}$.*

Proof. By (2.1), (2.2), for each $(t,j) \in \operatorname{dom}\xi$, if $\xi(t,j) \in \bigcap_{i \in I} R^{(i)}$, for some $I \subseteq I_D$ or some $I \subseteq I_C$, then $\lambda\xi(t,j) \in \bigcap_{i \in I} R^{(i)}$. By this fact and by Equations (2.3), (2.4) and (2.5), we can say that
- for each $(t,j) \in \operatorname{dom}\xi$ such that $(t, j+1) \in \operatorname{dom}\xi$, suppose $\xi(t, j+1) \in$

$G(\xi(t,j))$. Then, $\lambda\xi(t,j+1) \in G(\lambda\xi(t,j))$;
- for each $\underline{t}, \overline{t} \in \mathbb{R}_{\geq 0}$ such that $[\underline{t}, \overline{t}] \times \{j\} \subseteq \operatorname{dom}\xi$. if $\dot{\xi}(t,j) \in F(\xi(t,j))$, for almost all $t \in [\underline{t}, \overline{t}]$, then $\lambda\dot{\xi}(t,j) \in F(\lambda\xi(t,j))$, for almost all $t \in [\underline{t}, \overline{t}]$.
It follows that $\lambda\xi$ is a solution to $\mathcal{H}$. □

Proof of Theorem 2.4.

By the continuity of V and from assumptions (1) and (2) of Theorem 2.4, we have that U is not empty and $V(x) = \ell$ Let ξ be a solution to $\mathcal{H}$ with initial state $\xi(0,0) \in U$. Conditions (3)-(6) of Theorem 2.4 guarantee that ξ must leave U in finite time (this can be shown by following the argument of the proof of Theorem 2.3). Consider a solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) \in U$. By Condition (3), such a solution cannot leave U by flowing across $\{x \in \mathbb{R}^n \,|\, V(x) = \ell\}$, by Condition (4), it cannot leave U by jumping to $\{x \in \mathbb{R}^n \,|\, V(x) \leq \ell, c \leq |x| \leq \rho c\}$ and, by Condition (6), ξ cannot jump to $\{x \in \mathbb{R}^n \,|\, |x| \leq c\}$. It follows that ξ leaves U by flowing across $\{x \in \mathbb{R}^n \,|\, |x| = \rho c\}$ or by jumping to $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$.
Consider now a solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) \in U$ and, by (2), $|\xi(0,0)| = c + \varepsilon$. Such a solution leaves U in finite time, that is, $|\xi(T,J)| \geq \rho c$, for some $(T,J) \in \operatorname{dom}\xi$. Therefore, by Lemma 6.1, the result of the theorem follows. □

Proof of Corollary 2.1.

As stated in Theorem 2.4, each solution ξ to $\mathcal{H}$ leaves U in finite time. Note that, by Condition (7), each point $x \in C \cup D$ with $|x| = \rho c$ belongs to U. This implies that the set U surrounds the origin, therefore if a solution ξ leaves U, it cannot go back to the set $\{x \,|\, x \in (C \cup D), x \leq \rho c\}$ any more.
By the continuity of V and by Conditions (1) and (7), we can find two constants $\ell_1, \ell_2 \in \mathbb{R}$ such that $\min_{|x|=\rho c} V(x) = \ell_2$, $\ell < \ell_1 < \ell_2$ and the set $U_1 = \{x \in C \cup D \,|\, V(x) = \ell_1, \, |x| \leq \rho c\}$ surrounds th origin. It follows that (i) by continuity of V and by Conditions (1) and (7), U_1 is a subset of U, so that solutions ξ to $\mathcal{H}$ with $\xi(0,0) \in U_1$ escapes U in finite time by flowing or jumping to $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$, and (ii) by Conditions (3)-(6) and by Lemma 6.1 we can use pieces of solutions to $\mathcal{H}$ from U_1 to $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ to construct a solution ξ to $\mathcal{H}$ that grows unbounded. Indeed, inductively, consider a solution ξ_i to $\mathcal{H}$ with initial state $\xi_i(0,0) \in U_1$, where i is a positive integer (an index). Such a solution enters the set $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ in finite time, say $(t_i, j_i) \in \operatorname{dom}\xi_i$. The point $\xi_i(t_i, j_i) \in \{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ can be scaled so that $\lambda_i \xi_i(t_i, j_i) \in U_1$,

for some $\lambda_i \in \mathbb{R}_{>0}$. Then, consider a solution ξ_{i+1} to $\mathcal{H}$ with initial state $\xi_{i+1}(t_{i+1}, j_{i+1}) = \lambda_i \xi_i(t_i, j_i)$. Also such a solution enters the set $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ in finite time. Therefore, by using solutions ξ_i with $i \geq 0$ we can inductively define an unbounded solution ξ as follows.
Base case:

$$\begin{aligned} \xi(0,0) &= \xi_0(0,0) \\ \xi(t,j) &= \xi_0(t,j) \quad \forall (t,j) \in \operatorname{dom} \xi_0, (t,j) \leq (t_0, j_0); \end{aligned}$$

Inductive case: for each $i > 0$

$$\xi(t + \sum_{k=0}^{(i-1)} t_k, j + \sum_{k=0}^{(i-1)} j_k) = \frac{1}{\lambda_i} \xi_i(t,j),$$
$$\forall (t,j) \in \operatorname{dom} \xi_i,\ (t + \sum_{k=0}^{(i-1)} t_k, j + \sum_{k=0}^{(i-1)} j_k) \leq (t_i, j_i).$$

ξ grows unbounded by the fact that each solution ξ_i begins from a U_1 that is a proper subset of $\{x \in \mathbb{R}^n \,|\, |x| < \rho c\}$ and enters $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ in finite time. Instability of x_e follows from Lemma 6.1. □

Proof of Theorem 2.5

Point (1)-(4) of Theorem 2.5 follow from the fact that Theorem 2.1 is a sufficient condition to the asymptotic stability of the point $x_e = 0$ and Theorem 2.3 and Corollary 2.1 are sufficient conditions to the instability of $x_e = 0$.

Point (5) of Theorem 2.5 can be proved as follows. By Theorem 2.4, there exists at least one solution ξ to $\mathcal{H}$ such that (i) $|\xi(0,0)| = c_{(T2.4)} + \varepsilon$ and (ii) ξ escapes the set $U = \{x \in C \cup D \,|\, |x| \leq \rho_{(T2.4)} c_{(T2.4)}\}$ in finite time. Suppose now that conditions Theorem 2.1 are satisfied with $c_{(T2.1)} = c_{(T2.4)} + \varepsilon$ and $\rho_{(T2.1)} > 1$ such that $\rho_{(T2.1)} c_{(T2.1)} \leq \rho_{(T2.4)} c_{(T2.4)}$. Define ℓ_1 of Theorem 2.1 as $\ell_1 = \max_{|x| = c_{(T2.1)}} V_{(T2.1)}(x)$, for some function $V_{(T2.1)}$ used in Theorem 2.1. By conditions (2) and (3) of Theorem 2.1, $V_{(T2.1)}(\xi(t,j)) < V_{(T2.1)}(\xi(0,0))$ for each $(t,j) \in \operatorname{dom} \xi$ such that $c_{(T2.4)} + \varepsilon \leq |\xi(t,j)| \leq \rho_{(T2.4)} c_{(T2.4)}$. Two cases are possible:
(i) ξ flows through $\{x \in \mathbb{R}^n \,|\, |x| = \rho_{(T2.4)} c_{(T2.4)}\}$, then $\min_{|x| = \rho c} V(x) < \ell_1$ that contradicts Condition (4) of Theorem 2.1, i.e. $\min_{|x| = \rho c} V(x) \geq \ell_2 > \ell_1$, or
(ii) ξ jumps to $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho_{(T2.4)} c_{(T2.4)}\}$ that contradicts Condition (5) Theorem 2.1.
By Lemma 6.1, we can use the argument above with $c_{(T2.1)} > 0$, $\rho_{(T2.1)} > 1$

such that: $c_{(T2.1)} = \lambda(c_{(T2.4)} + \varepsilon)$, and $\rho_{(T2.1)}c_{(T2.1)} \leq \lambda c_{(T2.4)}\rho_{(T2.4)}$, for any given $\lambda > 0$ and any $\varepsilon > 0$ sufficiently small. Therefore $\rho_{(T2.1)} \leq \frac{\lambda c_{(T2.4)}\rho_{(T2.4)}}{\lambda(c_{(T2.4)}+\varepsilon)} = \frac{c_{(T2.4)}\rho_{(T2.4)}}{c_{(T2.4)}+\varepsilon}$. Finally, ε can be taken sufficiently small but strictly greater then zero, therefore $1 < \rho_{(T2.1)} < \rho_{(T2.4)}$.

Point (6) of Theorem 2.5 can be proved as follows. By Condition (4) of Theorem 2.1 we have $\max_{|x|=c_{(T2.1)}} V_{(T2.1)}(x) \leq \ell_1$ and $\min_{|x|=\rho_{(T2.1)}c_{(T2.1)}} V_{(T2.1)}(x) \geq \ell_2$, for some function $V_{(T2.1)}$ that satisfies Theorem 2.1, and some constants $c_{(T2.1)}$, $\rho_{(T2.1)}$, $\ell_1 < \ell_2$.
By Conditions (2),(3) and (6) of Theorem 2.1, each solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) \in S_1 \triangleq \{x \in C \cup D \,|\, c_{(T2.1)} \leq |x| \leq \rho_{(T2.1)}c_{(T2.1)}, V(|x|) \leq \ell_2\}$ does not enter the set $S_2\{x \in C \cup D \,|\, |x| > \rho_{(T2.1)}c_{(T2.1)}\}$. Because $\max_{|x|=c_{(T2.1)}} V_{(T2.1)}(x) \leq \ell_1 < \ell_2$, we have that each point $|x| = c_{(T2.1)}$ belongs to S_1, therefore each solution ξ to $\mathcal{H}$ with $|\xi(0,0)| = c_{(T2.1)}$ does not enter S_2.
For some $\varepsilon > 0$ sufficiently small, define $c_{(T2.4)}+\varepsilon = c_{(T2.1)}$ and $\rho_{(T2.4)} > 1$ such that $\rho_{(T2.4)}c_{(T2.4)} > \rho_{(T2.1)}c_{(T2.1)}$. By the result above on solutions from S_1 to S_2, we can say that no solutions ξ to $\mathcal{H}$ with initial state $|\xi(0,0)| = c_{(T2.4)} + \varepsilon$ enter the set $\{x \in C \cup D \,|\, |x| > \rho_{(T2.4)}c_{(T2.4)}\}$, that is, Theorem 2.4 cannot be applied with this $c_{(T2.4)}$ and $\rho_{(T2.4)}$.
By Lemma 6.1, for any given $\lambda > 0$ and any $\varepsilon > 0$ sufficiently small, we can use the argument above with $c_{(T2.4)} > 0$ and $\rho_{(T2.4)} > 1$ such that: $c_{(T2.4)} + \varepsilon = \lambda c_{(T2.1)}$ and $\rho_{(T2.4)}c_{(T2.4)} > \lambda c_{(T2.1)}\rho_{(T2.1)}$. Therefore, $\rho_{(T2.4)} > \frac{\lambda c_{(T2.1)}\rho_{(T2.1)}}{c_{(T2.4)}} > \frac{\lambda c_{(T2.1)}\rho_{(T2.1)}}{\lambda c_{(T2.1)}-\varepsilon}$. Finally, ε can be taken sufficiently small but strictly greater then zero, therefore $\rho_{(T2.4)} > \rho_{(T2.1)}$. $\square$

Proof of Proposition 2.2

V is a polynomial, therefore it is continuously differentiable.
(1) First inequality of (2.19) can be rewritten as $\ell - V(x) \geq s_3(x)\Delta_1(c, c+\varepsilon, x)$. Therefore, $\ell - V(x) \geq 0$ for each $c \leq |x| \leq c + \varepsilon$, that implies Condition (1) of Theorem 2.4.
(2) Rewrite the second inequality of (2.19) as $V(x) - \ell \geq s_4(x)\Delta_1(c + 2\varepsilon, c + 3\varepsilon, x) + q(x)$, then $V(x) - \ell \geq 0$ for $c + 2\varepsilon \leq |x| \leq c + 3\varepsilon$ and $q(x) \geq 0$. By (2.20), $q(x)$ is non-negative in a conic subset of $\mathbb{R}^n$, therefore $V(x) - \ell \geq 0$ in a subset of $c + 2\varepsilon \leq |x| \leq c + 3\varepsilon$, as required by Condition (2) of Theorem 2.4. In fact, denote $\varepsilon_{(Alg.2)}$ and $\varepsilon_{(Thm.2.4)}$ respectively the constants ε of Algorithm 2 and of Theorem 2.4, then V satisfies Condition (2) of Theorem 2.4 with $\varepsilon_{(Thm.2.4)} \geq 2\varepsilon_{(Alg.2)}$

(3,4) (2.17) and (2.24) imply conditions (3) and (4) of Theorem 2.4, respectively.
(5) The fifth inequality of (2.19) can be interpreted as $\ell - V(x) \geq s_5^{(ik)}(x)(c^2 - x'G'_{ik}G_{ik}x)$ for each $x \in R^{(i)}$ and $c \leq |x| \leq \rho c$, where $i \in I_D$. Therefore $\ell - V(x) \geq 0$ if $c - g \geq 0$, for each $x \in R^{(i)}$, $c \leq |x| \leq \rho c$ and each $g \in G(x)$. By negation, if $V(x) > \ell$ then $c - g < 0$, for each $x \in R^{(i)}$, $c \leq |x| \leq \rho c$ and each $g \in G(x)$, as required by Condition (5) of Theorem 2.4.
(6) Condition (6) is implied by $C \cup D = \mathbb{R}^n$ and the fact that no solutions ξ to $\mathcal{H}$ of Equations (1.5),(2.1)-(2.5) can blow up in finite time. □

Proof of Proposition 2.3

Inequality (2.21) can be written as $V(x) \geq \overline{\ell} + s_7(x)\Delta_1(\rho c - \varepsilon, \rho c, x)$ from which $V(x) \geq \overline{\ell} > \ell$, for each $\rho c - \varepsilon \leq |x| \leq \rho c$. Therefore, $\min_{|x|=\rho c} V(x) \geq \overline{\ell} > \ell$, that satisfies Condition (6) of Corollary 2.1. □

Proof of Proposition 2.4

Define $\ell = \ell_1$ and note that, by (2.27) and (2.28), $q_a(x)$ and $q_b(x)$ are positive in some conic subset of $\mathbb{R}^n$.
(1,2) First inequality of (2.26) can be written as $\ell_1 - V(x) \geq 0$, for $c \leq |x| \leq c + \varepsilon$, that implies Condition (1) of Theorem 2.4. Second inequality of (2.26) guarantees that $V(x) - \ell_1 \geq q_b(x)$, for $c + 2\varepsilon \leq |x| \leq c + 3\varepsilon$, that implies $V(x) \geq \ell_1$ for some x such that $q_b(x) \geq 0$ and $c + 2\varepsilon \leq |x| \leq c + 3\varepsilon$. By continuity of V, there exists a point x $c \leq |x| \leq c + 2\varepsilon$ with $V(x) = \ell_1 = \ell$, as required by Condition (2) of Theorem 2.4.
(3,4) (2.24) and (2.25) guarantee $\langle \nabla V(x), f \rangle > q_a(x)$ for each $x \in C$ such that $c \leq |x| \leq \rho c$ and each $f \in F(x)$, and $V(g) - V(x) > q_a(x)$ for each $x \in D$ such that $c \leq |x| \leq \rho c$ and each $g \in G(x)$. Third inequality in (2.26) enforces the constraint $V(x) - \ell_2 \leq q_a(x)$ for each $c \leq |x| \leq \rho c$, that is, if $q_a(x) \leq 0$ then $V(x) \leq \ell_2 < \ell$, for $c \leq |x| \leq \rho c$. It follows that, for each $c \leq |x| \leq \rho c$, (i) $V(x) > \ell$ implies $q_a(x) > 0$, that is, (ii) $\langle \nabla V(x), f \rangle > 0$ for each x such that $V(x) > \ell$ and each $f \in F(x)$, and (iii) $V(g) - V(x) > 0$ for each x such that $V(x) > \ell$ and each $g \in g(x)$. It follows that Conditions (3) and (4) of Theorem 2.4 are satisfied.
(5,6) By an argument similar to points (5) and (6) of section 6.2.3, conditions (5) and (6) of Theorem 2.4 respectively follow from the next to the last inequality of (2.26) and from the fact $C \cup D = \mathbb{R}^n$. □

6.3 Proof of the Results in Chapter 3

6.3.1 Proof of Proposition 3.1

Monotonicity:
Consider a state $y \in \delta_f^*(S, X_0)$. From the definition of δ_f^*, $y \in \delta_f(S, x)$ for some $x \in X_0 \subseteq X_1$. Then, $y \in \delta_f^*(S, X_1)$. A similar argument can be repeated for δ_b^*.

$\cup$-*continuity*:

$$\begin{aligned} y \in \delta_f^*(S, \cup_i X_i) &\equiv y \in \delta_f^*(S, \{x \,|\, \exists i, x \in X_i\}) \\ &\equiv y \in \bigcup_{\exists i,\, x \in X_i} \delta_f(S, x) \\ &\equiv \exists i, \exists x \in X_i, y \in \delta_f(S, x) \\ &\equiv \exists i, y \in \delta_f^*(S, X_i) \\ &\equiv y \in \cup_i \delta_f^*(S, X_i). \end{aligned} \tag{6.32}$$

A similar argument can be repeated for δ_b^*. □

It is worth mentioning that $\cap$-continuity holds only for ω-chains of closed subset of O.

Lemma 6.2 *For any given ω-chain $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \ldots$ of closed subsets of O,*

$$\delta_f^*(S, \cap_i X_i) = \cap_i \delta_f^*(S, X_i) \text{ and } \delta_b^*(S, \cap_i X_i) = \cap_i \delta_b^*(S, X_i). \tag{6.33}$$

Proof. (6.33) *can be proved as follows.*

$$\begin{aligned} y \in \delta_f^*(S, \cap_i X_i) &\equiv y \in \delta_f^*(S, \{x \,|\, \forall i, x \in X_i\}) \\ &\equiv y \in \bigcup_{\forall i,\, x \in X_i} \delta_f(S, x) \\ &\equiv y \in \{\delta_f(S, x) \,|\, \exists x, \forall i, x \in X_i\} \\ &\equiv y \in \{\delta_f(S, x) \,|\, \forall i, \exists x, x \in X_i\} \\ &\equiv \forall i, y \in \delta_f^*(S, X_i) \\ &\equiv y \in \cap_i \delta_f^*(S, X_i). \end{aligned} \tag{6.34}$$

where the fourth equivalence holds by the fact that for each convergent sequence $\{x_i\} \to x$, x belongs to each set X_i. In fact, each X_i is closed and, for each

i, $X_i \supseteq X_{i+1}$, therefore, x can be seen as the limit point of the sequence $\{x_i\}$. Consider now the set X_i, and take the subsequence $\{x_j\}_{j\geq i, j\in\omega} \subseteq \{x_i\}_{i\in\omega}$. $\{x_j\}_{j\geq i}$ converges to x and for each $j \geq i$, x_j belongs to X_i. Then $x \in X_i$ by the fact that X_i is closed. It follows that each sequence $\{x_i\}$ whose elements satisfies $\forall i$, $x_i \in X_i$, that is, $\{x_i\}$ is a witness of $\forall i, \exists x, x \in X_i$, converges to a point x that belongs to each set X_i, that is, x is a witness of the predicate. The converse is straightforward. A similar argument can be repeated for δ_b^*. $\square$

Note that, by monotonicity of δ_f^*, $\cup_i \delta_f^*(S, X_i) \subseteq \delta_f^*(S, \cup_i X_i)$ and $\cap_i \delta_f^*(S, X_i) \supseteq \delta_f^*(S, \cap_i X_i)$, [7, Proposition 1.2.5]. The same holds for δ_b^*.

6.3.2 Proof of Proposition 3.2

$\mathit{Reach}(X_0) \subseteq X_0 \cup \delta_f^*(O, \mathit{Reach}(X_0))$:
Suppose $x \in \mathit{Reach}(X_0)$ then, either $x \in X_0$ or $x \notin X_0$. In this second case, there exists a solution ξ to $\mathcal{H}$ such that $\xi(0,0) \in X_0$ and $\xi(T,J) = x$ for some $(T,J) \in \operatorname{dom}\xi$. Moreover, $\forall (t,j) \in \operatorname{dom}\xi$, $\xi(t,j)$ belongs to $\mathit{Reach}(X_0)$. The, the solution ξ reaches x either by jumping or by flowing. In the first case, $(T,J), (T,J-1) \in \operatorname{dom}\xi$. Define $\overline{\xi}$ such that $\overline{\xi}(0,0) = \xi(T,J-1)$ and $\overline{\xi}(0,1) = \xi(T,J) = x$. Then, $\overline{\xi}$ is a solution to $\mathcal{H}$ and $\overline{\xi}(0,0) \in \mathit{Reach}(X_0)$. From the definition of δ_f^*, it follows that $x \in \delta_f^*(O, \mathit{Reach}(X_0))$. In the second case, consider an interval $[T_0, T] \times \{j\} \subseteq \operatorname{dom}\xi$ and define $\overline{\xi}(t,0) = \xi(T_0 + t, j)$ for $t \in [0, T - T_0]$. Then, ξ is a solution to $\mathcal{H}$, $\overline{\xi}(0,0)$ belongs to $\mathit{Reach}(X_0)$ and $\overline{\xi}(T - T_0, 0) = x$. From the definition of δ_f^*, $x \in \delta_f^*(O, \mathit{Reach}(X_0))$.

$\mathit{Reach}(X_0) \supseteq X_0 \cup \delta_f^*(O, \mathit{Reach}(X_0))$:
Suppose $x \in X_0 \cup \delta_f^*(O, \mathit{Reach}(X_0))$. If $x \in X_0$ then $x \in \mathit{Reach}(X_0)$. If $x \in \delta_f^*(O, \mathit{Reach}(X_0))$, from the definition of δ_f^*, there exists a solution ξ_1 to $\mathcal{H}$ such that $\xi_1(0,0) \in \mathit{Reach}(X_0)$ and either (iA) $\xi_1(t,0) = x$ for some $(t,0) \in \operatorname{dom}\xi_1$, or (iiA) $\xi_1(0,1) = x$ for $(0,1) \in \operatorname{dom}\xi_1$. From the definition of $\mathit{Reach}(X_0)$, there exists a solution ξ_2 to $\mathcal{H}$ such that $\xi_2(0,0) \in X_0$ and $\xi_2(T,J) = \xi_1(0,0)$ for some (T,J) in $\operatorname{dom}\xi_2$. Define now a hybrid arc ξ as follows: $\xi(t,j) = \xi_2(t,j)$, for $(t,j) \in \operatorname{dom}\xi_2$ and $t + j \leq T + J$, and either (iB) $\xi(T+t, J) = \xi_1(t,0)$ or (iiB) $\xi(T, J+1) = \xi_1(0,1)$, where we use (iB) if (iA) holds and (iiB) if (iiA) holds. Then, ξ is a solution to $\mathcal{H}$ and either (iC) $\xi(T+t, J) = x$, for some $(T+t, J) \in \operatorname{dom}\xi$ or (iiC) $\xi(T, J+1) = x$. It follows that $x \in \mathit{Reach}(X_0)$.

$\mathit{Reach}(X_0)$ is the *least* fixpoint:
Consider the function $\lambda X.X_0 \cup \delta_f^*(O,X) : 2^O \to 2^O$, that maps each $X \subseteq O$ to $X_0 \cup \delta_f^*(O,X)$. By monotonicity of $\delta_f^*(O,X)$, $\lambda X.X_0 \cup \delta_f^*(O,X)$ is monotonic, then the least fixpoint of the equation $X = X_0 \cup \delta_f^*(O,X)$ exists and

is unique, [7, Theorem 1.2.8], and can be computed by iterative application of $\lambda X.X_0 \cup \delta_f^*(O, X)$ from the emptyset, [7, Theorem 1.2.11]. See also [144]. Indeed, $\mu X.X_0 \cup \delta_f^*(X) = \bigcup_{i\in\lambda} \big(\lambda X.(X_0 \cup \delta_f^*(O, X))\big)^{(i)}\emptyset$, where λ is a sufficiently large ordinal. We prove that *Reach*(X_0) is the least fixpoint of $X = X_0 \cup \delta_f^*(O, X)$, by induction on the time-length of solutions ξ to $\mathcal{H}$.
For a given solution ξ to $\mathcal{H}$ define $t_i = \inf\{t \,|\, (t, i) \in \operatorname{dom}\xi\}$. Then,

- *Base:* $\big(\lambda X.(X_0 \cup \delta_f^*(O, X))\big)\emptyset = X_0$ and $\xi(0, 0) \in X_0$.
- *Induction:* suppose that $\forall j \leq i, \forall (t, j) \in \operatorname{dom}\xi$, if $t + j \leq t_i + i$ then $\xi(t, j) \in \big(\lambda X.(X_0 \cup \delta_f^*(O, X))\big)^{(j+1)}\emptyset$. It follows that,
  1. if $(t_i, i{+}1) \in \operatorname{dom}\xi$, then $\xi(t_i, i{+}1) \in \delta_f(O, \xi(t_i, i)) \subseteq \delta_f^*(O, \{\xi(t_i, i)\}) \subseteq \delta_f^*((\lambda X.(X_0 \cup \delta_f^*(O, X)))^{(i+1)}\emptyset) \subseteq (\lambda X.(X_0 \cup \delta_f^*(O, X)))^{(i+2)}\emptyset$;
  2. if the interval $[t_i, t_i + \tau] \times \{i\} \subseteq \operatorname{dom}\xi$, then $\forall 0 \leq \tau \leq t_{i+1} - t_i$, $\xi(t_i + \tau, i) \in \delta_f(O, \xi(t_i, i)) \subseteq \delta_f^*(O, \{\xi(t_i, i)\}) \subseteq \delta_f^*(O, (\lambda X.(X_0 \cup \delta_f^*(O, X)))^{(i+1)}\emptyset) \subseteq (\lambda X.(X_0 \cup \delta_f^*(O, X)))^{(i+2)}\emptyset$.

By induction, for each solution ξ to $\mathcal{H}$ with initial state in X_0 and for any given $(t, i) \in \operatorname{dom}\xi$, $t + i \leq t_i + i$, $\xi(t, i) \in (\lambda X.(X_0 \cup \delta_f^*(O, X)))^{(i+1)}\emptyset$. It follows that *Reach*$(X_0) \subseteq \bigcup_{i\in\lambda}(\lambda X.(X_0 \cup \delta_f^*(O, X)))^{(i)}\emptyset$. But *Reach*$(X_0)$ is a fixpoint, therefore it must be the least fixpoint. □

6.3.3 Proof of Proposition 3.3

Consider the function $\lambda X.\delta_f^*(O, X) : 2^O \to 2^O$ that maps each set $X \subseteq O$ to $\delta_f^*(O, X)$. Monotonicity of δ_f^* in Proposition 3.1 guarantees that $\lambda X.\delta_f^*(O, X)$ is a monotonic function. It is also $\cup$-continuous, that is,

$$\lambda X.\delta_f^*(O, X)\,(\cup_i X_i) = \cup_i \big(\lambda X.\delta_f^*(O, X) X_i\big) \tag{6.35}$$

for any given ω-chain $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \ldots$ of subsets of O. For instance,

$$\begin{aligned} \lambda X.\delta_f^*(O, X)\,(\cup_i X_i) &= \lambda X.\delta_f^*(O, X)(\cup_i X_i) \\ &= \delta_f^*(O, \cup_i X_i) \\ &= \cup_i \delta_f^*(O, X_i) \\ &= \cup_i \big(\lambda X.\delta_f^*(O, X) X_i\big) \end{aligned} \tag{6.36}$$

Finally note that $X_0 \cup \delta_f^*(O, \emptyset) = \big(\lambda X.\delta_f^*(O, X)\big)^{(0)} X_0 = X_0$.

Therefore, by [7, Proposition 1.2.13] the fixpoint is *Reach*(X_0) and by [7, Theorem 1.2.14], the induction ends at ω. □

6.3.4 Proof of Proposition 3.5.

Consider a solution ξ to $\mathcal{H}$ with initial condition $\xi(0,0) = x$ that satisfies the semantics of $\mathcal{U}^{\,rop\,(c_t,c_j)}\varphi_2$, that is,

(1) $\exists (T,J) \in \operatorname{dom}\xi$, $(T,J)\, rop\,(c_t,c_j)$, $\mathcal{H}, \xi(T,J) \vDash \varphi_2$, and

(2) $\forall (t,j) \in \operatorname{dom}\xi$, $(t,j) \leq (T,J)$, $\mathcal{H}, \xi(t,j) \vDash \varphi_1 \vee \varphi_2$.

By Proposition 3.4, there exists a solution $\eta = \begin{bmatrix} \eta_x^T & \eta_t & \eta_j \end{bmatrix}^T$ to $\mathcal{H}_{ext}$ from $x_{ext} = \begin{bmatrix} x^T & 0 & 0 \end{bmatrix}^T$, such that $\operatorname{dom}\eta = \operatorname{dom}\xi$ and for each $(t,j) \in \operatorname{dom}\eta$, $\eta_x(t,j) = \xi(t,j)$. Then,

(1) $\exists (T,J) \in \operatorname{dom}\eta$, $(T,J)\, rop\,(c_t,c_j)$, $\mathcal{H}_{ext}, \eta(T,J) \vDash \varphi_4$. In fact, $\eta_x = \xi$ therefore exists a $(T,J) \in \operatorname{dom}\eta$, $(T,J)\, rop\,(c_t,c_j)$, such that $\eta_x(T,J)$ satisfies φ_2. By Proposition 3.4, $\eta_t(T,J) = T$ therefore $\eta_t(T,J)\, rop\, c_t$, $\eta_j(T,J) = J$ therefore $\eta_j(T,J)\, rop\, c_j$. Moreover,

(2) $\forall (t,j) \in \operatorname{dom}\eta$, $(t,j) \leq (T,J)$, $\mathcal{H}_{ext}, \eta(t,j) \vDash \varphi_3$. In fact, for such (t,j), φ_3 does not define any conditions on $\eta_t(t,j)$ and on $\eta_j(t,j)$, while $\eta_x(t,j) = \xi(t,j)$ therefore it satisfies $\varphi_1 \vee \varphi_2$.

It follows that a solution η to $\mathcal{H}_{ext}$ with initial state $\eta(0,0) = \begin{bmatrix} x^T & 0 & 0 \end{bmatrix}^T$ satisfies the semantics of $\varphi_3 \mathcal{U} \varphi_4$.

Consider now a solution $\eta = \begin{bmatrix} \eta_x^T & \eta_t & \eta_j \end{bmatrix}^T$ to $\mathcal{H}_{ext}$ with initial state $x_{ext} = \begin{bmatrix} x^T & 0 & 0 \end{bmatrix}^T$ that satisfies the semantics of $\varphi_3 \mathcal{U} \varphi_4$, that is,

(1) $\exists (T,J) \in \operatorname{dom}\eta$, $(T,J) \geq (0,0)$, $\mathcal{H}_{ext}, \eta(T,J) \vDash \varphi_4$, and

(2) $\forall (t,j) \in \operatorname{dom}\eta$, $(t,j) \leq (T,J)$, $\mathcal{H}_{ext}, \eta(t,j) \vDash \varphi_3$.

By Proposition 3.4, there exists a solution ξ to $\mathcal{H}$ from x such that $\operatorname{dom}\xi = \operatorname{dom}\eta$ and for each $(t,j) \in \operatorname{dom}\xi$, $\xi(t,j) = \eta_x(t,j)$. Then

(1) $\exists (T,J) \in \operatorname{dom}\xi$ such that $(T,J)\, rop\,(c_t,c_j)$ and $\mathcal{H}, \xi(T,J) \vDash \varphi_2$. In fact, $\xi = \eta_x$ therefore there exists a $(T,J) \in \operatorname{dom}\xi$ such that $\xi(T,J) = \eta_x(T,J)$ satisfies φ_2, $T = \eta_t(T,J)\, rop\, c_t$ and $J = \eta_j(T,J)\, rop\, c_j$. Moreover,

(2) $\forall (t,j) \in \operatorname{dom}\xi$, $(t,j) \leq (T,J)$, $\mathcal{H}, \xi \vDash \varphi_1 \vee \varphi_2$. In fact, for such (t,j), $\xi(t,j) = \eta_x(t,j)$ and $\eta_x(t,j)$ satisfies $\varphi_1 \vee \varphi_2$.

It follows that a solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) = x$ satisfies the semantics of $\varphi_1 \mathcal{U}^{\,rop\,(c_t,c_j)} \varphi_2$.

The equivalences in (3.19) follow from the one to one correspondence between solutions to $\mathcal{H}$ from initial state $x \in O$ and solutions to $\mathcal{H}_{ext}$ from initial state $\begin{bmatrix} x^T & 0 & 0 \end{bmatrix}^T \in O_{ext}$, in Proposition 3.4. □

6.3.5 Proof of Proposition 3.6

$[\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}} \subseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}})$.
Suppose that $x \in [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$ then, either $x \in [\![\varphi_2]\!]_{\mathcal{H}}$ or $x \notin [\![\varphi_2]\!]_{\mathcal{H}}$. In this second case, there exists a solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) = x \in [\![\varphi_1]\!]_{\mathcal{H}}$ that satisfies the semantics of $\varphi_1 \mathcal{U} \varphi_2$. Then $\xi(0,0) = x \in [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$ and, from the definition of δ_b^*, $x \in \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}})$.

$[\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}} \supseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}})$.
If $x \in [\![\varphi_2]\!]_{\mathcal{H}}$, then $x \in [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$. Suppose $x \in \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}})$. Then, from the semantics of δ_b^*, there exists a solution ξ_1 to $\mathcal{H}$ such that $\xi_1(0,0) = x \in [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$ and

- either $(0,1) \in \operatorname{dom} \xi_1$, $\xi_1(0,1) \in [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$,
- or, for some value $t \in \mathbb{R}_{>0}$ such that $[0,t] \times \{0\} \in \operatorname{dom} \xi_1$, $\xi_1(t,0) \in [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$, and $\forall 0 \leq \tau \leq t$, $\xi_1(\tau,0) \in [\![\exists \varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$.

Moreover, from the semantics of $\varphi_1 \mathcal{U} \varphi_2$, there exists a solution ξ_2 to $\mathcal{H}$ that begins either from the point $\xi_1(0,1)$ or from the point $\xi_1(t,0)$, that remains in $[\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$ until it enters $[\![\varphi_2]\!]_{\mathcal{H}}$.

Consider now the hybrid arc ξ that coincides with solution ξ_1, until ξ_1 reaches either the state $\xi_1(0,1)$ or the state $\xi_1(t,0)$ and, from that point, ξ coincides with ξ_2. Then, ξ is a solution to $\mathcal{H}$ with initial state $\xi(0,0) = x$ that satisfies the semantics of $\varphi_1 \mathcal{U} \varphi_2$. It follows that $x \in [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$.

$[\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$ is the *least* fixpoint.

For any given $x \in [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$, suppose $x \in \left(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\right)^i \emptyset$ for some $i \in \omega$. Then $[\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}} \subseteq \bigcup_{i \in \omega} \left(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\right)^i \emptyset$. where the induction ends at ω by the fact that $\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)$ is $\cup$-continuous. But $[\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$ is a fixpoint, then it would be the least fixpoint.

Consider now $x \in [\![\exists \varphi_1 \mathcal{U} \varphi_2]\!]_{\mathcal{H}}$ and consider the solution ξ to $\mathcal{H}$ from $\xi(0,0) = x$ that satisfies the dynamics of $\varphi_1 \mathcal{U} \varphi_2$, that is,

– $\xi(T,J) \in [\![\varphi_2]\!]_{\mathcal{H}}$ for some $(T,J) \in \operatorname{dom}\xi$,

– $\xi(t,j) \in [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$ for all $\forall(t,j) \in \operatorname{dom}\xi$, $(t,j) \leq (T,J)$.

Define also $t_j = \inf\{t \mid (t,j) \in \operatorname{dom}\xi\}$.

Let us call $y_n = \xi(T,J)$, where n is an index in ω. From the semantics of $\varphi_1\mathcal{U}\varphi_2$,

$$\begin{aligned} y_n \in [\![\varphi_2]\!]_{\mathcal{H}} &\subseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \emptyset) \\ &\subseteq \big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)\emptyset. \end{aligned} \tag{6.37}$$

By computing $\delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_n\})$, one of the following two cases occurs:

(i) either $\xi(T, J-1) \in \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_n\})$,

(ii) or $\xi(t_J, J) \in \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_n\})$.

Let us call $y_{n-1} = \xi(T, J-1)$ if case (i) occurs and $y_{n-1} = \xi(t_J, J)$ otherwise. Then,

$$\begin{aligned} y_{n-1} \in \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_n\}) &\subseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\varphi_2]\!]_{\mathcal{H}}) \\ &\subseteq \big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)[\![\varphi_2]\!]_{\mathcal{H}} \\ &\subseteq \big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)^2\emptyset \end{aligned} \tag{6.38}$$

By computing $\delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_{n-1}\})$, one of the following two cases occurs:

(i) either $\xi(t_{J-1}, J-1) \in \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_n\})$,

(ii) or $\xi(t_J, J-1) \in \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_n\})$.

By defining $y_{n-2} = \xi(t_{J-1}, J-1)$ if case (i) occurs and $y_{n-2} = \xi(t_J, J-1)$ otherwise, we can use the same argument above to show that y_{n-2} belongs to $\big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)^3\emptyset$.

In general, consider $y_{n-m} = \xi(t,j)$ for some $(t,j) \in \operatorname{dom}\xi$, where $m < n$, and suppose $y_{n-m} \in \big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)^{m+1}\emptyset$. Then, define $y_{n-(m+1)}$ either as $\xi(t_j, j)$, if $(t_j, j) \in \operatorname{dom}\xi$, or as $\xi(t, j-1)\}$, otherwise. From the definition of δ_b^*, we have that

$$\begin{aligned} \{y_{n-(m+1)}\} &\subseteq \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_{n-m}\}) \\ &\subseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, \{y_{n-m}\}) \\ &\subseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}\big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)^{m+1}\emptyset) \\ &\subseteq \big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)^{m+2}\emptyset. \end{aligned} \tag{6.39}$$

Note that we explore backward the solution ξ by taking into account, at each step, either the beginning (t_j, j) of a flow interval $[t_j, t] \times \{j\} \in \operatorname{dom}\xi$, or the initial time $j-1$ of a jump whose time domain is $(t_j, j-1), (t_j, j) \in \operatorname{dom}\xi$. By the fact that $\xi(T, J)$ belongs to $[\![\varphi_2]\!]_{\mathcal{H}}$ for some finite $(T, J) \in \operatorname{dom}\xi$, the alternation of jumps and flows on ξ is finite and the number of jumps and of flows intervals is lower than $2J+1$. Therefore, the backward exploration of ξ is well-founded, requires no more that $2J+1$ steps and ends when $\xi(0,0) = x$ is reached. It follows that, $x \in \big(\lambda X.[\![\varphi_2]\!]_{\mathcal{H}} \cup \delta_b^*([\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, X)\big)^{n+1}$, for some $n \in \omega$. $\square$

6.3.6 Proof of Lemma 3.2

Each solution to $\mathcal{H}$ is complete therefore each solution to $\mathcal{H}$ as an unbounded time domain, that is, each maximal solution to $\mathcal{H}$ is complete and the set of complete solutions to $\mathcal{H}$ is a model.

[$[\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}} \subseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \overline{\delta}_b^*(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}})$]
Suppose $x \vDash \forall\varphi_1\mathcal{U}\varphi_2$ Then either $x \in [\![\varphi_2]\!]_{\mathcal{H}}$ or $x \notin [\![\varphi_2]\!]_{\mathcal{H}}$. For this second case, Consider the whole set, say Ξ, of solutions ξ with initial condition in $\xi(0,0) = x$: Then,
$\forall\xi \in \Xi$, $\exists(t, j) \in \xi$, $\xi(t, j) \in [\![\varphi_2]\!]_{\mathcal{H}}$, and
$\forall(\tau, k) \le (t, j)$, if $(\tau, k) \in \operatorname{dom}\xi$ then $\xi(\tau, k) \in [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$
It follows that each of such a solution $\xi \in \Xi$ can be divided in two parts and each of such a part satisfies a well defined set of properties, as shown below. Consider $\xi \in \Xi$ and define two hybrid arc ξ_1 and ξ_2 as follows. Let $\overline{t} = \max\{t \mid (t, 0) \in \operatorname{dom}\xi,\ t \le c\}$.

(i) If $\overline{t} \neq 0$, then
$\forall\tau \le \overline{t}$, $\xi_1(\tau, 0) = \xi(\tau, 0)$;
$\forall(\tau, k) \ge (t, 0)$ $\xi_2(\tau - \overline{t}, k) = \xi(\tau, k)$

(ii) If $\overline{t} = 0$ then
$\xi_1(0,0) = \xi(0,0)$ and $\xi_1(0,1) = \xi(0,1)$;
$\forall(\tau, k) \ge (0, 1)$ $\xi_2(\tau, k-1) = \xi(\tau, k)$

In both cases ξ_1 and ξ_2 are solutions to $\mathcal{H}$. Moreover, consider case (i). From the assumption on $x \in [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$, it follows that

(ia) either $\exists t \le \overline{t}$, $y_1 = \xi_1(\tau, 0) \in [\![\varphi_2]\!]_{\mathcal{H}} \subseteq [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$, and $\forall\tau \le t$, $\xi_1(\tau, 0) \in [\![\varphi_2 \vee \varphi_2]\!]_{\mathcal{H}}$;

(ib) or $y_2 = \xi_2(0,0) \in [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$, and $\forall(t,0) \in \operatorname{dom}\xi_1$, $\xi_1(t,0) \in [\![\varphi_2 \vee \varphi_2]\!]_{\mathcal{H}}$.

For case (ii), it follows that

(iia) either $y_1 = \xi_1(0,1) \in [\![\varphi_2]\!]_{\mathcal{H}} \subseteq [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$, and $\xi_1(0,0) \in [\![\varphi_2 \vee \varphi_2]\!]_{\mathcal{H}}$;

(iib) or $y_2 = \xi_2(0,0) \in [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$, and $\xi_1(0,0), \xi_1(0,1) \in [\![\varphi_2 \vee \varphi_2]\!]_{\mathcal{H}}$.

Note that $y_2 \in [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$ can be justified by considering that for both (ib) $y_2 = \xi(\bar{t},0)$ and (iib) $y_2 = \xi(0,1)$, the solution ξ did not reach $[\![\varphi_2]\!]_{\mathcal{H}}$ yet. From the assumption on x, each solution $\xi \in \Xi$ from x satisfies the semantics of $\varphi_1\mathcal{U}\varphi_2$. Therefore, the solution ξ_2 from y_2 will necessarily reach $[\![\varphi_2]\!]_{\mathcal{H}}$, staying within $[\![\varphi_2 \vee \varphi_2]\!]_{\mathcal{H}}$.

The analysis shown above can be successfully repeated on each solution ξ in Ξ. Therefore, from x each solution ξ can be divided in two parts ξ_1 and ξ_2, and the first part ξ_1 is a solution to $\mathcal{H}$ from x that either flows for at most a c-bounded interval of time, or jumps. In both cases ξ_1 reaches $[\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$ staying within the set $[\![\varphi_2 \vee \varphi_2]\!]_{\mathcal{H}}$. It follows that $x \in \overline{\delta}_b^*(c, [\![\varphi_2 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}})$

[$[\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}} \supseteq [\![\varphi_2]\!]_{\mathcal{H}} \cup \overline{\delta}_b^*(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}})$]

Take a state $x \in [\![\varphi_2]\!]_{\mathcal{H}} \cup \overline{\delta}_b^*(c, [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}, [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}})$. Each solutions ξ to $\mathcal{H}$ from x satisfies one of the following points.

- $\xi(0,0) \in [\![\varphi_2]\!]_{\mathcal{H}}$;
- $\exists(t,0) \in \operatorname{dom}\xi$, $t \leq c$, $\xi(t,0) \in [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$, and $\forall(\tau,0) \leq (t,0)$, $\xi(t,0) \in [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$;
- $(0,1) \in \operatorname{dom}\xi$, $\xi(0,1) \in [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$ and $\xi(0,0) \in [\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{H}}$

Since this is true for each solution ξ to $\mathcal{H}$ from x, then $x \in [\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$.

The proof that $[\![\forall\varphi_1\mathcal{U}\varphi_2]\!]_{\mathcal{H}}$ is the least fix point can developed by an analysis similar to the one in Proof 6.3.5 (of Proposition 3.6).

6.3.7 Proof of Lemma 3.3

Each solution to $\mathcal{H}$ is complete therefore each solution to $\mathcal{H}$ as an unbounded time domain, that is, each maximal solution to $\mathcal{H}$ is complete and the set of complete solutions to $\mathcal{H}$ is a model.

For any given set $S, X \subseteq O$ and any constant $c \in \mathbb{R}_{>0}$, consider the definition of $\overline{\delta}_b^*(c, S, X)$, that is,

$$\begin{aligned}\overline{\delta}_b^*(c, S, X) = \{x \,|\, \forall \xi \text{ solution to } \mathcal{H} \text{ with } \xi(0,0) = x,\\ \text{either } (0,1) \in \operatorname{dom} \xi,\ \xi(0,1) \in X \text{ and } x \in S \cup X,\\ \text{or } \exists (t,0) \in \operatorname{dom} \xi,\ t \leq c,\ \xi(t,0) \in X \text{ and } \forall 0 \leq \tau \leq t,\ \xi(\tau,0) \in S \cup X\}\end{aligned}$$

Assume X finitely variable for each solution ξ to $\mathcal{H}$. We claim that $x \notin \overline{\delta}_b^*(c, S, X)$ if and only if $x \in Q$, where Q is defined as the union of the following four sets:

$$\begin{aligned}Q &= A_1 \cup A_2 \cup B_1 \cup B_2\\ A_1 &= \{x \,|\, \exists \xi \text{ solution to } \mathcal{H},\ \xi(0,0) = x,\ \exists (0,j) \in \operatorname{dom} \xi,\ j \leq 1,\\ &\qquad \xi(0,j) \notin S \cup X \text{ and } \forall k \leq j,\ \xi(0,k) \notin X\}\\ A_2 &= \{x \,|\, \exists \xi \text{ solution to } \mathcal{H},\ \xi(0,0) = x,\ \forall (0,j) \in \operatorname{dom} \xi,\ j \leq 1,\ \xi(0,j) \notin X\}\\ B_1 &= \{x \,|\, \exists \xi \text{ solution to } \mathcal{H},\ \xi(0,0) = x,\ \exists (t,0) \in \operatorname{dom} \xi, t \leq c, \text{ such that}\\ &\qquad \xi(t,0) \notin S \cup X \text{ and } \forall 0 \leq \tau \leq t, \xi(\tau,0) \notin X\}\\ B_2 &= \{x \,|\, \exists \xi \text{ solution to } \mathcal{H},\ \xi(0,0) = x,\ \forall (t,0) \in \operatorname{dom} \xi,\ \text{if } t \leq c \text{ then } \xi(t,0) \notin X\}\end{aligned} \tag{6.40}$$

[$x \in Q$ *then* $x \notin \overline{\delta}_b^*(c, S, X)$]
Suppose $x \in A_1$. Then, there exists a solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) = x \notin X$ and $\xi(0,1) \notin S \cup X$. Therefore, $\xi(0,1) \notin X$ that implies $x \notin \overline{\delta}_b^*(c, S, X)$. Suppose $x \in A_2$. Then, there exists a solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) = x \notin X$ and $\xi(0,1) \notin X$. Again, $x \notin \overline{\delta}_b^*(c, S, X)$.
Suppose $x \in B_1$, then there exists some $(a,0) \in \operatorname{dom} \xi$, $(a,0) \leq (c,0)$ such that $\xi(a,0) \notin S \cup X$ and $\forall 0 \leq b \leq a, \xi(b,0) \notin X$. Then,

- suppose that for some $t \in (a,c]$ $(t,0) \in \operatorname{dom} \xi$ and $\xi(t,0) \in X$. Then, the definition of $\overline{\delta}_b^*(c, S, X)$ requires that $\forall 0 \leq \tau \leq t$, $\xi(\tau,0) \in S \cup X$. But for $a < t$, $\xi(a,0) \notin S \cup X$.
- for any $0 \leq t \leq a$, $\xi(t,0) \notin X$;

It follows that x does not satisfy the requirements of the definition of $\overline{\delta}_b^*$, therefore x does not belong to $\overline{\delta}_b^*(c, S, X)$. Finally, consider the case $x \in B_2$. Then, there exists a solution ξ to $\mathcal{H}$ such that for each $(t,0) \in \operatorname{dom}\xi$ if $t \leq c$ then

$\xi(t,0) \notin X$ ($x \in B_2$). Therefore, from the definition of $\overline{\delta}_b^*(c,S,X)$, it cannot be the case that x belongs to $\overline{\delta}_b^*(c,S,X)$.

[$x \notin \overline{\delta}_b^*(c,S,X)$ *then* $x \in Q$]

Suppose $x \notin \overline{\delta}_b^*(c,S,X)$. Then, there exists a solution ξ to $\mathcal{H}$ with initial state $\xi(0,0) = x$ such that one of the following cases occurs:

(i) if $\xi(0,1) \in \mathrm{dom}\,\xi$, $\xi(0,1) \in X$ then $x \notin S \cup X$;

(ii) $\forall (t,0) \in \mathrm{dom}\,\xi$, if $(t,0) \leq (c,0)$ and $\xi(t,0) \in X$ then $\exists (\tau,0) \leq (t,0)$, $\xi(\tau,0) \notin S \cup X$;

(iii) $\forall (t,0) \in \mathrm{dom}\,\xi$, if $(t,0) \leq (c,0)$ then $\xi(t,0) \notin X$ and $\xi(t,0) \in S$

Case (i) is directly captured by A_1 and A_2. For instance, A_1 captures the solutions that begins with $\xi(0,0) \notin S \cup X$, regardless to $\xi(0,1)$. A_2 captures the solutions $\xi(0,1) \notin X$ and $\xi(0,0) \in\notin$ but $\xi(0,0) \in S$, that A_1 cannot catch. For case (ii), define $a = \inf\{t \mid \xi(t,0) \in X\}$. Then, for each $0 \leq t < a$ $\xi(t,0) \notin X$ and either $\xi(a,0) \in X$ or, by finite variability assumption, there exists a sufficiently small $\varepsilon > 0$ such that for each $0 < \tau \leq \varepsilon$, $\xi(a+\tau,0) \in X$.

- Consider the case $\xi(a,0) \in X$. From (ii), there is $(b,0) < (a,0)$ such that $\xi(b,0) \notin X \cup S$.
- Consider the case $\xi(a+\tau,0) \in X$, $0 < \tau \leq \varepsilon$. From (ii), there is $(b,0) \leq (a,0)$ such that $\xi(b,0) \notin X \cup S$.
- For both cases above, for each $0 \leq t < a$, $\xi(t,0) \notin X$.

It follows that ξ satisfies the conditions of the definition of B_1.
Finally, case (iii) is captured by B_2. Suppose that for each $(t,0) \in \mathrm{dom}\,\xi$, $t \leq c$, $\xi(t,0) \notin X$ but $\xi(t,0) \in S$. Then, ξ satisfies the the definition of B_2.

[*From* Q *to fixpoint*]

In what follows, we replace *and* and *or* connectives with $\wedge$ and $\vee$.

We have shown above the equivalence between $\overline{\delta}_b^*(c,S,X)$ and $\mathbb{R}^n \setminus Q$, where Q is defined in Equation (6.40). Now, we use such a relation to rewrite $\overline{\delta}_b^*(c,S,X)$ as a set of fixpoint expressions based on $\delta_b^*(S,X)$. The definition of Q uses several conditions on time variables. Therefore, we will use the the extended hybrid system $\mathcal{H}_{ext}$ from $\mathcal{H}$ (Section 3.4.2), to transform that *time constraints* to *state constraints*.

$$
\begin{aligned}
A_1 = \{x \mid \exists\xi \text{ solution to } \mathcal{H},\ \xi(0{,}0){=}x,\ \exists(0,j) \in \operatorname{dom}\xi,\ j \le 1,& \\
\xi(0,j) \notin S{\cup}X \text{ and } \forall k \le j,\ \xi(0{,}k){\notin}X\}& \\
= \{x \mid \exists\eta \text{ solution to } \mathcal{H},\ \eta(0{,}0){=}\left[\begin{array}{ccc} x^T & 0 & 0 \end{array}\right]^T,& \\
\exists(T{,}J){\in}\operatorname{dom}\eta,\ \eta(T{,}J){\in}(\mathbb{R}^n \setminus (S \cup X)) \times \mathbb{R}^2 \cap [\![t = 0 \wedge j \le 1]\!]_{\mathcal{H}_{ext}}& \\
\text{and } \forall(\tau,k) \le (T,J),\ \eta(\tau{,}k){\in}(\mathbb{R}^n \setminus X) \times \mathbb{R}^2\}& \\
= \{x \mid \left[\begin{array}{ccc} x^T & 0 & 0 \end{array}\right]^T{\in}\ \mu Z.\Big((\mathbb{R}^n \setminus (S \cup X)) \times \mathbb{R}^2 \cap [\![t = 0 \wedge j \le 1]\!]_{\mathcal{H}_{ext}}\Big)\cup& \\
\cup\ \delta_b^*((\mathbb{R}^n \setminus X) \times \mathbb{R}^2, Z)\}&
\end{aligned}
$$

$$
\begin{aligned}
A_2 = \{x \mid \exists\xi \text{ solution to } \mathcal{H},\ \xi(0{,}0){=}x,\ \forall(0,j) \in \operatorname{dom}\xi,\ j \le 1,\ \xi(0,j) \notin X\}& \\
= \{x \mid \exists\xi \text{ solution to } \mathcal{H},\ \xi(0{,}0){=}x,\ \exists(0,j) \in \operatorname{dom}\xi,\ j > 1,& \\
\xi(0,j) \in O \text{ and } \forall k \le j,\ \xi(0{,}k){\notin}X\}& \\
= \{x \mid \exists\eta \text{ solution to } \mathcal{H},\ \eta(0{,}0){=}\left[\begin{array}{ccc} x^T & 0 & 0 \end{array}\right]^T,& \\
\exists(T{,}J){\in}\operatorname{dom}\eta,\ \eta(T{,}J){\in}O_{ext} \cap [\![t = 0 \wedge j > 1]\!]_{\mathcal{H}_{ext}}& \\
\text{and } \forall(\tau,k) \le (T,J),\ \eta(\tau{,}k){\in}(\mathbb{R}^n \setminus X) \times \mathbb{R}^2\}& \\
= \{x \mid \left[\begin{array}{ccc} x^T & 0 & 0 \end{array}\right]^T \in \mu Z.\Big(O_{ext} \cap [\![t = 0 \wedge j > 1]\!]_{\mathcal{H}_{ext}}\Big)\cup& \\
\cup\ \delta_b^*((\mathbb{R}^n \setminus X) \times \mathbb{R}^2, Z)\}&
\end{aligned}
$$

$$
\begin{aligned}
B_1 = \{x \mid \exists\xi \text{ solution to } \mathcal{H},\ \xi(0,0) = x,\ \exists(t,0) \in \operatorname{dom}\xi, t \le c,& \\
\xi(t,0) \notin S \cup X \text{ and } \forall 0 \le \tau \le t, \xi(\tau,0) \notin X\}& \\
= \{x \mid \exists\eta \text{ solution to } \mathcal{H}_{ext},\ \eta(0,0) = \left[\begin{array}{ccc} x^T & 0 & 0 \end{array}\right]^T,& \\
\exists(T,J) \in \operatorname{dom}\eta,\ \eta(T{,}J){\in}(\mathbb{R}^n \setminus (S \cup X)) \times \mathbb{R}^2 \cap [\![t \le c \wedge j{=}0]\!]_{\mathcal{H}_{ext}}& \\
\text{and } \forall(\tau,k) \le (T,J), \eta(\tau{,}k){\in}(\mathbb{R}^n \setminus X) \times \mathbb{R}^2\}& \\
= \{x \mid \left[\begin{array}{ccc} x^T & 0 & 0 \end{array}\right]^T \in \mu Z.\Big((\mathbb{R}^n \setminus (S \cup X)) \times \mathbb{R}^2 \cap [\![t \le c \wedge j = 0]\!]_{\mathcal{H}_{ext}}\Big)\cup& \\
\cup\ \delta_b^*((\mathbb{R}^n \setminus X) \times \mathbb{R}^2, Z)\}&
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= \{x \mid \exists \xi \text{ solution to } \mathcal{H},\ \xi(0,0)=x,\ \forall (t,0)\in \operatorname{dom}\xi,\ \text{ if } t\le c \text{ then } \xi(t,0)\notin X\}\\
&= \{x \mid \exists \xi \text{ solution to } \mathcal{H},\ \xi(0{,}0)=x,\ \exists (t,0)\in \operatorname{dom}\xi,\ t>c,\\
&\qquad\qquad \xi(t,0)\in O \text{ and } \forall \tau\le t,\ \xi(\tau{,}0)\notin X\}\\
&= \{x \mid \exists \eta \text{ solution to } \mathcal{H}_{ext},\ \eta(0,0)=\left[\begin{array}{ccc} x^T & 0 & 0\end{array}\right]^T,\\
&\qquad\qquad \exists (T,J)\in \operatorname{dom}\eta, \eta(T{,}J)\in O_{ext}\cap [\![t>c\wedge j{=}0]\!]_{\mathcal{H}_{ext}}\\
&\qquad\qquad \text{and } \forall (\tau,k)\le (T,J), \eta(\tau{,}k)\in (\mathbb{R}^n\setminus(X))\times\mathbb{R}^2\}\\
&= \{x \mid \left[\begin{array}{ccc} x^T & 0 & 0\end{array}\right]^T \in \mu Z.\Big(O_{ext}\cap [\![t>c\wedge j=0]\!]_{\mathcal{H}_{ext}}\Big)\cup\\
&\qquad\qquad \cup\, \delta_b^*((\mathbb{R}^n\setminus X)\times\mathbb{R}^2, Z)\}
\end{aligned}
$$

Note that, for each set A_1, A_2, B_1 and B_2, the fixpoint in the last equivalence can be justified by considering the definition of the formula $\exists\varphi_1\mathcal{U}\varphi_2$ and by looking at the equivalence in Equation (3.20),

The fixpoint characterization of $\overline{\delta}_b^*$, with respect the extended hybrid system $\mathcal{H}_{ext}$ from $\mathcal{H}$ follows.

$$
\begin{aligned}
\overline{\delta}_b^*(c,S,X) &= \mathbb{R}^n\setminus(A_1\cup A_2\cup B_1\cup B_2)\\
&= \mathbb{R}^n\setminus\{x \mid \left[\begin{array}{ccc} x^T & 0 & 0\end{array}\right]^T \in \mu Z.q(c,S,X)\cup\delta_b^*((\mathbb{R}^n\setminus X)\times\mathbb{R}^2, Z)\}
\end{aligned}
\tag{6.41}
$$

where

$$
\begin{aligned}
q(c,S,X) = &\Big((\mathbb{R}^n\setminus(S\cup X))\times\mathbb{R}^2\cap [\![(t\le c\wedge j=0)\vee(t=0\wedge j\le 1)]\!]_{\mathcal{H}_{ext}}\Big)\cup\\
&\cup [\![(t>c\wedge j=0)\vee(t=0\wedge j>1)]\!]_{\mathcal{H}_{ext}}.
\end{aligned}
\tag{6.42}
$$

Note that each fixpoint characterization of A_1, A_2, B_1 and B_2 can be divided in two parts. The left-part of each fixpoint involves a specific state predicate while the right-part of each fixpoint computes δ_b^* on $((\mathbb{R}^n\setminus X)\times\mathbb{R}^2, Z)\subseteq 2^{O_{ext}}\times 2^{O_{ext}}$. It follows that the union of A_1, A_2, B_1 and B_2 is equivalent to the union of each left-part of their fixpoint characterization, that leads to $q(c,S,X)$. □

Remark 6.1 [73, Page 30]. Finite variability is a key notion for the correctness of the proof. To see this, consider Example 3.4, that is,

$$
\begin{cases} \xi(t,0)\in X & \text{if } t=\frac{1}{n}, n\in\omega,\\ \xi(t,0)\notin X & \text{otherwise}\end{cases}
\tag{6.43}
$$

Take $S = [\![x = 0]\!]$. Then, $0 \notin \overline{\delta}_b^*(S, X)$ but also $0 \notin Q$.

6.4 Proof of the Results in Chapter 4

6.4.1 Proof of Theorem 4.1

The main steps of the proof are the following. We first show in Section 6.4.1 that, for each initial states there exists a compact forward invariant set $\mathcal{I}(x_0)$ containing the origin where the trajectory is confined. Then we show in Section 6.4.1 that the origin is the only equilibrium point in the set $\mathcal{I}(x_0)$. Then, to complete the proof of item 1, in Section 6.4.1 we show that there doesn't exist any periodic orbit in the set $\mathcal{I}(x_0)$ so that, by [67, Theorem 18.1, page 66] (following a Bendixson-like approach), all trajectories necessarily converge to the origin. Finally, in Section 6.4.1 we prove item 2 of the theorem.

Existence of the Forward Invariant Set $\mathcal{I}(x_0)$

Consider the following locally Lipschitz Lyapunov function:

$$V = \begin{cases} a_1 \dfrac{x_1^2}{2} + \dfrac{x_2^2}{2} + M x_1 & \text{if } x_1 \geq 0 \\ a_1 \dfrac{x_1^2}{2} + \dfrac{x_2^2}{2} - M x_1 & \text{if } x_1 \leq 0. \end{cases}$$

Following the nonsmooth analysis in, e.g., [40] (indeed, $V \notin C^1$, so that we cannot use gradients and derivatives in the usual sense), its generalized gradient in $x_1 = 0$ corresponds to the following set:

$$\begin{aligned} \nabla V_0(x_2) \quad &:= \quad \overline{\text{co}} \left\{ \begin{bmatrix} -M \\ x_2 \end{bmatrix}, \begin{bmatrix} M \\ x_2 \end{bmatrix} \right\} \\ &= \quad \left\{ \begin{bmatrix} (1-2\alpha)M \\ x_2 \end{bmatrix}, \ \alpha \in [0,1] \right\}, \end{aligned}$$

and its generalized derivative along the system dynamics results in

$$\dot{V} \in \begin{cases} -a_2 x_2^2 - x_2(\text{sat}(k\beta(x)) - M), & \text{if } x_1 > 0 \\ -a_2 x_2^2 - x_2(\text{sat}(k\beta(x)) + M), & \text{if } x_1 < 0 \\ \nabla V_0(x_2)^T \begin{bmatrix} x_2 \\ -a_2 x_2 - \text{sat}(k\beta(x)) \end{bmatrix}, & \text{if } x_1 = 0, \end{cases} \tag{6.44}$$

where with a slight abuse of notation, $\nabla V_0(x_2)^T w$ corresponds to the set $\{v^T w, \ v \in \nabla V_0(x_2)\}$.

Since k satisfies (4.3), then there exists $\bar{s} > 0$ such that $k = \frac{M}{\eta(\bar{s})} > \inf_{s>0} \frac{M}{\eta(s)}$ Therefore, by item 1 of Assumption 4.2 and since $\eta(\cdot) \in \mathcal{K}$,

$$|k\beta(x)| \geq \left| \frac{M}{\eta(\bar{s})} \eta(|x|) \right| \geq M, \quad \forall |x| \geq \bar{s}, x_1 x_2 \geq 0. \tag{6.45}$$

Therefore, the following bounded set

$$\mathcal{W} := \{x \,:\, |x| \leq \bar{s}\} \cap \{x \,:\, x_1 x_2 > 0\} \subset \mathbb{B}(0, \bar{s})$$

is characterized by the fact that any x outside $\mathcal{W}$ and in the (closed) first and third quadrant will lead to (by (6.45)) $|k\beta(x)| \geq M$, namely will cause the plant input u to saturate.

Consider now the maximum of $V(\cdot)$ in $\mathcal{W}$ and at x_0, namely $\overline{v}(x_0) := \max_{x \in \mathcal{W} \cup \{x_0\}} V(x)$, and choose the set $\mathcal{I}$ as the following sublevel set of V:

$$\mathcal{I}(x_0) := \{x \,:\, V(x) \leq \overline{v}(x_0)\} \supset \mathcal{W}$$

and note that by the radial unboundedness of $V(\cdot)$, the set $\mathcal{I}$ is necessarily compact. Moreover, by definition of $\mathcal{I}(x_0)$ and by item 1 of Assumption 4.2, the input u is saturated for all x belonging to the (closed) first and third quadrants (namely satisfying $x_1 x_2 \geq 0$) intersection with the closed complement of $\mathcal{I}(x_0)$ (denoted in the following by $\overline{\mathcal{I}(x_0)^c}$.

Then the following reasonings prove that $\dot{V} \leq 0$ for all $x \in \overline{\mathcal{I}(x_0)^c}$, i.e. $\mathcal{I}(x_0)$ is a forward invariant set where the trajectory is confined (because $x_0 \in \mathcal{I}(x_0)$ by definition):

- *(2nd and 4th quadrants)* if $x_1 x_2 \leq 0$, then $k\beta(x)$ could be any value, therefore we only know that $\mathrm{sat}(k\beta(x)) \in [-M, M]$. Based on this, the three cases in (6.44) yield:
  1. if $x_1 > 0$, so that necessarily $x_2 \leq 0$, $\dot{V} \leq -x_2(\mathrm{sat}(k\beta(x)) - M) = -|x_2||\mathrm{sat}(k\beta(x)) - M| \leq 0$ (both terms are negative);
  2. if $x_1 < 0$, so that necessarily $x_2 \geq 0$, $\dot{V} \leq -x_2(\mathrm{sat}(k\beta(x)) + M) = -|x_2||\mathrm{sat}(k\beta(x)) + M| \leq 0$ (both terms are positive);
  3. if $x_1 = 0$, then, by definition of $\mathcal{I}(x_0)$, $\mathrm{sat}(k\beta(x)) \in \{-M, M\}$ for all $x \in \overline{\mathcal{I}(x_0)^c} \cap \{x_1 = 0\}$. Moreover, also by item 1 of Assumption 4.2, $\mathrm{sat}(k\beta(x)) = M$ if $x_2 > 0$ and $\mathrm{sat}(k\beta(x)) = -M$ if $x_2 < 0$. Therefore the following bound holds:

$$\begin{aligned} \dot{V} &\leq \max_{\alpha \in [0,1]} x_2[(1 - 2\alpha)M - \mathrm{sat}(k\beta(x))] \\ &= |x_2|M + |x_2|(-M) = 0. \end{aligned}$$

- *(1st and 3rd quadrants)* if $x_1 x_2 > 0$ and $x \in \overline{\mathcal{I}(x_0)^c} \cap \{x_1 = 0\}$, once again $\mathrm{sat}(k\beta(x)) = M$ if $x_2 > 0$ and $\mathrm{sat}(k\beta(x)) = -M$ if $x_2 < 0$, by definition of $\mathcal{I}(x_0)$ and by item 1 of Assumption 4.2. Then the first two cases in (6.44) yield:

  1. if $x_1 > 0$, then $\dot{V} \;\leq\; -x_2(\mathrm{sat}(k\beta(x)) - M) = 0$
  2. if $x_1 < 0$, then $\dot{V} \;\leq\; -x_2(\mathrm{sat}(k\beta(x)) + M) = 0$.

Uniqueness of the Equilibrium Point

Since $\dot{x}_1 = x_2$ any equilibrium has to be on $x_2 = 0$. By item 1 of Assumption 4.2 $\mathrm{sat}(k\beta(x)) = 0$ on the x_1 axis if and only if $x_1 = 0$, and has the same sign as x_1 otherwise. Hence on $\{x_2 = 0\}$, $\dot{x}_2 = -a_1 x_1 - \mathrm{sat}(k\beta(x)) \neq 0$ if $x_1 \neq 0$. This proves that $x = 0$ is the only equilibrium point.

Proof of Convergence Using Bendixson's Criterion

Since the only equilibrium point is the origin, any trajectory not converging to that point must converge to a periodic orbit contained in $\mathcal{I}(x_0)$. Moreover, any such a hypothetical periodic orbit must surround the origin because:

- no such an orbit can happen in a single quadrant, indeed the periodicity of the orbit contradicts the property that in each quadrant $\dot{x}_1 = x_2$ is monotone;
- the trajectory phase is decreasing on the coordinate axes as a matter of fact (see, e.g., [87, §10.5]) the following results from writing the dynamics in polar coordinates $|x|^2\dot{\phi} = x_2^2 - x_1(-a_1 x_1 - a_2 x_2 - \mathrm{sat}(k\beta(x)))$, which clearly implies that the phase ϕ increases on $x_1 = 0$ and, by the property of $\beta(\cdot)$ on the $x_2 = 0$ axis, also implies that ϕ increases on $x_2 = 0$.

Now, reasoning like in the proof of Bendixson criterion (see, e.g., [87, Lemma 2.2]), suppose by contradiction that there exists a periodic orbit γ including the origin and let S be the surface surrounded by γ. Then $\int_\gamma f_2(x)dx_1 - f_1(x)dx_2 = 0$, i.e. by Green's theorem,

$$\int\!\!\int_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 = 0. \tag{6.46}$$

However, this identity cannot be possible under item 2 of Assumption 4.2 which implies that any such an integral is necessarily strictly negative, as shown next (see (6.48)).

To suitably bound the integrand in (6.46), taking into account that sat(·) and $\beta(\cdot)$ are locally Lipschitz functions, the following relations hold almost everywhere:

$$
\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= 0 \\
\frac{\partial f_2}{\partial x_2} &= -a_2 - \frac{\partial}{\partial x_2}\mathrm{sat}(k\beta(x)) \\
&= \begin{cases} -a_2, & \text{if } |k\beta(x)| \geq M \\ -a_2 - k\dfrac{\partial \beta(x)}{\partial x_2}, & \text{if } |k\beta(x)| < M \end{cases}
\end{aligned}
$$

therefore, for almost all x,

$$
\begin{aligned}
|k\beta(x)| \geq M &\Rightarrow \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -a_2 \leq 0 \\
|k\beta(x)| < M &\Rightarrow \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \leq -a_2 - k\frac{\partial \beta(x)}{\partial x_2} \leq 0
\end{aligned}
\tag{6.47}
$$

Moreover, since the intersection of S with the set $\mathcal{A}$ (introduced in item 2 of Assumption 4.2) is necessarily a set with positive measure, [1] then (6.47) and item 2 of Assumption 4.2, together with the fact that $a_2 \geq 0$ by Assumption 4.1 (see also Remark 4.1), imply

$$
\int\int_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 \quad < \quad 0 \tag{6.48}
$$

which contradicts (6.46). Therefore, no periodic orbit γ exists. Since no periodic trajectory exists in $\mathcal{I}(x_0)$ and the origin is the only equilibrium point, by [67, Theorem 18.1, page 66] (following a Bendixson-like approach), the trajectory necessarily converges to the origin.

Proof of Item 2 of the Theorem

The two conditions at item 2 correspond to requiring strict positiveness of the coefficients of the characteristic polynomial of the linearized system around the origin. Therefore it is straightforward to conclude local exponential stability

[1] This is true because there exist points in $\mathcal{A}$ arbitrarily close to the origin and given any point in $\mathcal{A}$, one can find a small enough ball (which evidently has positive measure) around that point completely contained in $\mathcal{A}$.

(LES) of the origin from those conditions. This, together with the global convergence property proven above is sufficient (see, e.g., [149] or [90, pp. 68-72]) to prove global asymptotic and local exponential stability (GAS+LES). $\square$

6.4.2 Proof of Theorem 4.2

The following lemma, which is a reformulation of [43, Lemma 1] (see also [77]), will be useful for the proof of Theorem 4.2 and Theorem 4.3.

Lemma 6.3 *Given any (magnitude) saturation function* $\mathrm{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$, *define the deadzone function as* $\mathrm{dz}(s) := s - \mathrm{sat}(s)$. *Then, for any* $v \in \mathbb{R}^m$ *and any* $w \in \mathbb{R}^m$ *satisfying* $\mathrm{dz}(w) = 0$, *the following bound holds:*

$$\mathrm{dz}(v)^T U(\mathrm{dz}(v) - v + w) \leq 0,$$

where U *is any positive definite diagonal matrix.*

In the following Lemma we show a useful characteristic of the nonlinearity $\mathrm{sat}_{MR}(\cdot)$ described by (4.21)

Lemma 6.4 *Given any pair of functions* $w_1(\cdot) : \mathbb{R} \to \mathbb{R}^m$, $w_2(\cdot) : \mathbb{R} \to \mathbb{R}^m$ *and any* $\epsilon \in (0, 1)$, *the following holds:*

$$\mathrm{sat}_{MR}(\mathrm{sat}_{MR\epsilon}(w_1) + \mathrm{sat}_{MR(1-\epsilon)}(w_2)) = \mathrm{sat}_{MR\epsilon}(w_1) + \mathrm{sat}_{MR(1-\epsilon)}(w_2). \quad (6.49)$$

Moreover, (4.21) *is* $\mathcal{L}_2$ *stable with finite gain not greater than* $\sqrt{2}$, *namely* $s \in \mathcal{L}_2$ *implies that* $\|\mathrm{sat}_{MR}(s)\|_2 \leq \sqrt{2}\|s\|_2$.

Proof. Since decentralized saturations are considered, it is enough to prove the case $m = 1$.
As for (6.49), since $\mathrm{sat}_{MR\epsilon}(w_1)$ has magnitude and rate not exceeding $M\epsilon$ and $R\epsilon$, respectively, and similarly $\mathrm{sat}_{MR(1-\epsilon)}(w_2)$ has magnitude and rate not exceeding $M(1-\epsilon)$ and $R(1-\epsilon)$, respectively, the magnitude and rate of their sum will not exceed M and R, respectively; hence, their sum will not be modified by $\mathrm{sat}_{MR}(\cdot)$ since by [158, Lemma B.1, p. 145] if $s : \mathbb{R} \to \mathbb{R}^m$ is such that $|s(t)| \leq M$ and $|\dot{s}(t)| \leq R$, $\forall t \in \mathbb{R}$ then $\mathrm{sat}_{MR}(s(t)) = s(t)$, $\forall t \in \mathbb{R}$.
As for the bound $\|\mathrm{sat}_{MR}(s)\|_2 \leq \sqrt{2}\|s\|_2$ for $s \in \mathcal{L}_2$, let $\dot{\mu}(t) = R\,\mathrm{sign}(\mathrm{sat}_M(s(t)) - \mu(t))$ and consider the storage function $V(\mu) = \frac{|\mu|^3}{3}$. It will now be shown that $\dot{V}(\mu) < -\mu^2 + 2s^2$, from which the claim easily follows. Taking the time derivative yields $\dot{V}(\mu) = -|\mu|^2 \mathrm{sign}(\mu) R \mathrm{sign}(\mu - \mathrm{sat}_M(s))$. If $|\mu| > |\mathrm{sat}_M(s)|$,

then $\text{sign}(\mu - \text{sat}_M(s)) = \text{sign}(\mu)$ and then $\dot{V}(\mu) = -|\mu|^2 \leq -|\mu|^2 + 2|s|^2$. On the other hand, if $|\mu| \leq |\text{sat}_M(s)|$, then $\dot{V}(\mu) \leq |\mu|^2 \leq -|\mu|^2 + 2|\mu|^2 \leq -|\mu|^2 + 2|\text{sat}_M(s)|^2 \leq -|\mu|^2 + 2|s|^2$. $\square$

The advantage in the interconnection between (4.22), (4.23) and (4.27) via the equation (4.28) is illustrated by the following statement.

Lemma 6.5 *For the closed-loop (4.22), (4.23), (4.27), (4.28) the following holds.*

1. *If $x_{aw}(0) = 0$, then* [2] *the controller state $\bar{x}_c$ and output response $\bar{y}_c$ coincides with the virtual response $\hat{x}_c$ and $\hat{y}_c$ produced by the unconstrained closed-loop (4.22), (4.23), (4.24) from the same initial states and under the action of the same external inputs r and d. Moreover, $\bar{z}_{aw} = \bar{z} - \hat{z}$.*

2. *If there exists a static feedback control law $k(\cdot)$ from x_{aw} such that $|k(x_{aw})|_2 \leq c|x_{aw}|_2$ for some $c > 0$ and the following system*

$$\dot{x}_{aw} = Ax_{aw} + B_u \text{sat}_{\varepsilon MR}(k(x_{aw})) + B_u \sigma \tag{6.50}$$

is locally (respectively, globally) $\mathcal{L}_2$ stable from σ to x_{aw}, then the anti-windup closed-loop (4.22), (4.23), (4.27), (4.28) with

$$v_1 = \text{sat}_{\varepsilon MR}(k(x_{aw})) \tag{6.51}$$

is such that there exists a local (respectively, global) nonlinear $\mathcal{L}_2$ gain from $\begin{bmatrix} \text{dz}_{M(1-\varepsilon)}(\hat{u}) \\ \text{dz}_{R(1-\varepsilon)}(\dot{\hat{u}}) \end{bmatrix}$ to $\bar{z} - \hat{z}$; namely as long as the unconstrained trajectory does not spend infinite energy outside the (restricted) saturation limits, then the actual output response $\bar{z}$ converges in the $\mathcal{L}_2$ sense to the ideal unconstrained output response $\hat{z}$

Proof. Item 1. The proof of this item is carried out along the usual lines with the model recovery schemes (see also [147, 145, 162, 163, 14, 59] for similar reasonings) so it is only sketched here. Writing the anti-windup closed-loop dynamics (4.22), (4.23), (4.27), (4.28) in the following coordinates: $(x_a, x_c, x_{aw}) = (x - x_{aw}, x_c, x_{aw})$, the arising representation is in cascade form, where the first subsystem comprising the states (x_a, x_c) coincides with the unconstrained closed-loop dynamics (4.22), (4.23) and (4.24) and the second subsystem is the anti-windup compensator (4.27), which is driven by the signal y_c produced

[2] Similar to the results in [147], if $x_{aw}(0) \neq 0$ then one experiences an extra transient at startup, but the closed-loop properties remain unchanged.

by the first subsystem. Due to this fact, the controller response coincides with the unconstrained controller response, so that $\bar{y}_c = \hat{y}_c = \hat{u}$. Moreover, $\hat{z} = z_a = \bar{z} - \bar{z}_{aw}$.

Item 2. With the selection for u in (4.28) and with v_1 as in (6.51), by Lemma 6.4 it is easily seen that:

$$\begin{aligned} u - y_c &= \mathrm{sat}_{MR}(\mathrm{sat}_{MR(1-\varepsilon)}(y_c) + \mathrm{sat}_{MR\varepsilon}(v_1)) - y_c \\ &= \mathrm{sat}_{MR(1-\varepsilon)}(y_c) + \mathrm{sat}_{MR\varepsilon}(v_1) - y_c \\ &= \mathrm{dz}_{MR\varepsilon}(y_c) + \mathrm{sat}_{MR\varepsilon}(v_1), \end{aligned}$$

and then (4.27a) becomes (6.50) with $\sigma = \mathrm{dz}_{MR(1-\varepsilon)}(\bar{y}_c)$ which is an $\mathcal{L}_2$ signal if $\begin{bmatrix} \mathrm{dz}_{M(1-\varepsilon)}(\hat{u}) \\ \mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}}) \end{bmatrix} \in \mathcal{L}_2$.

Since $\bar{y}_c = \hat{u}$ and by item 1 of this lemma, $\bar{z} - \hat{z} = \bar{z}_{aw}$, then the result follows from the $\mathcal{L}_2$ stability assumption on (6.50), the fact that

$$\|z_{aw}\|_2 \leq |C_z| \cdot \|x_{aw}\|_2 + |D_{zu}| \cdot \|\mathrm{sat}_{MR\varepsilon}(k(x_{aw}))\|_2$$

the $\mathcal{L}_2$ stability of (4.21) and the fact that $\|k(x_{aw})\|_2 \leq c\|x_{aw}\|_2$ since $|k(x_{aw})| \leq c|x_{aw}|$. □

Based on the preliminary statements in Lemma 6.5, it is possible to prove Theorem 4.2 as follows.

Proof. Theorem 4.2.

Item 1. By Lemma 6.5, it is sufficient to prove that system (6.50) is globally $\mathcal{L}_2$ stable from σ to z_{aw}. Since $v_1 = 0$ and A is Hurwitz, then (6.50) corresponds to a linear exponentially stable system under the action of an $\mathcal{L}_2$ disturbance. Therefore the system has a global finite input/output $\mathcal{L}_2$ gain (see, e.g., [87, Cor. 5.1]) and the result follows.

Item 2. Similar to the previous item, by relying on Lemma 6.5, we address the local $\mathcal{L}_2$ stability of system (6.50). Since for small enough states x_{aw}, the stabilizing signal v_1 remains below the saturation limits, then the origin of (6.50) is locally exponentially stable. This implies local $\mathcal{L}_2$ stability from σ to z_{aw} (see, e.g., [87, Cor. 5.1]).

Item 3. Consider the Lyapunov function $V = x_{aw}^T P x_{aw}$ for system (6.50). Applying a Schur complement (see, [24]) to (4.29c) and to (4.29d), we get, respectively,

$$\begin{aligned} ([K(A + B_u K)]_i)^T([K(A + B_u K)]_i) &\leq \varepsilon^2 R_i^2 P, \\ [H]_i^T [H]_i &\leq \varepsilon^2 M_i^2 P, \end{aligned} \tag{6.52}$$

for $i = 1, \dots m$, which imply that in the set $\mathcal{E}(P, 1) := \{x_{aw} : V(x_{aw}) \le 1\}$ the following bounds hold, respectively:

$$|\varepsilon R^{-1} K(A + B_u K)x_{aw}|_\infty \le 1, \tag{6.53a}$$

$$|\varepsilon M^{-1} H x_{aw}|_\infty \le 1. \tag{6.53b}$$

Using $v_1 = Kx_{aw}$, then for all $x_{aw} \in \mathcal{E}(P, 1)$, the first condition in (6.53) implies that $\text{sat}_{\varepsilon R}(\dot v_1) = \dot v_1$, so that $\text{sat}_{\varepsilon MR}(v_1) = \text{sat}_{\varepsilon M}(v_1)$. Moreover, the second condition in (6.53) implies that $\text{sat}_{\varepsilon M}(Hx_{aw}) = Hx_{aw}$ and, by the generalized sector condition in Lemma 6.3, we get for any positive definite diagonal matrix U_M:

$$\text{dz}_{\varepsilon MR}(Kx_{aw})^T U_M(\text{dz}_{\varepsilon MR}(Kx_{aw}) - (K - H)x_{aw}) \le 0, \tag{6.54}$$

for all $x_{aw} \in \mathcal{E}(P, 1)$.

Consider now the time derivative of V along the dynamics (6.50) with $\sigma = 0$. Using (6.54) and defining $q = \text{dz}_{\varepsilon MR}(Kx_{aw})$, we get for all $x_{aw} \in \mathcal{E}(P, 1)$,

$$\begin{aligned}
\dot V &\le \dot V - 2q^T U_M(q - (K - H)x_{aw}) \\
&= x_{aw}^T \left(\text{He}(P(A + B_u K)x_{aw} - 2PB_u q\right) \\
&\quad - 2q^T U_M(q - (K - H)x_{aw}) \\
&= \begin{bmatrix} x_{aw} \\ q \end{bmatrix}^T \text{He} \begin{bmatrix} P(A + B_u K) & -PB_u \\ U_M(K - H) & -U_M \end{bmatrix} \begin{bmatrix} x_{aw} \\ q \end{bmatrix} \\
&< -2\gamma x_{aw}^T P x_{aw} = -2\gamma V,
\end{aligned}$$

where we have used (4.29b) in the last step. Since by (4.29a), $\mathcal{E}(P, 1) \supseteq \mathcal{B}(\alpha)$, then the bound above on $\dot V$ implies that the origin of (6.50) is locally exponentially stable with region of attraction including $\mathcal{B}(\alpha)$. Similar to the proof of item 2 local $\mathcal{L}_2$ stability of system (6.50) is guaranteed. Moreover, exponential recovery trivially follows from the relation $\dot V \le -2\gamma V$, which holds on the invariant set $\mathcal{E}(P, 1)$ and implies that for $x_{aw}(0)$ in this set $V(x_{aw}(t)) \le e^{-2\gamma t} V(x_{aw}(0))$; denoting by λ_m and λ_M the minimum and maximum eigenvalues of the symmetric matrix $P > 0$, and taking into account that $\lambda_m |x_{aw}|_2^2 \le V(x_{aw}) \le \lambda_M |x_{aw}|_2^2$ since $V(x_{aw}) = x_{aw}^T P x_{aw}$, the exponential decay of V implies that $|x_{aw}(t)|_2 \le \sqrt{\frac{\lambda_M}{\lambda_m}} e^{-\gamma t} |x_{aw}(0)|_2$. □

6.4.3 Proof of Proposition 4.1

Under Assumption 4.3, by [96, Lemma 2.7], for all $\varepsilon_1 > 0$, there exists a unique matrix $P(\varepsilon_1) > 0$ that solves the equation

$$A^T P(\varepsilon_1) + P(\varepsilon_1)A - P(\varepsilon_1)B_u B_u^T P(\varepsilon_1) + \varepsilon_1 I = 0 \tag{6.55}$$

Moreover, $P(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$ and, by choosing $K(\varepsilon_1) = -B_u^T P(\varepsilon_1)$, we have that $A + B_u K(\varepsilon_1)$ is asymptotically stable for all $\varepsilon_1 > 0$.
Consider (4.29a). For any given α, we can find ε_1 small enough that guarantees $P(\varepsilon_1) < \frac{1}{\alpha}$.
Consider (4.29b). Take $H(\varepsilon_1) = K(\varepsilon_1)$ and note that $2x^T P(\varepsilon_1)B_u q \leq 2\varepsilon_2^2 x^T x + 2q^T \frac{B_u^T P(\varepsilon_1)P(\varepsilon_1)B_u}{\epsilon_2^2} q$. Then, consider (4.29b) multiplied on the right and on the left by $[\, x \; q]^T$, it follows that the right-hand side of (4.29b) is less then or equal to

$$2x^T Q_1(\varepsilon_1, \varepsilon_2, \gamma)x + 2q^T Q_2(\varepsilon_1, \varepsilon_2)q \tag{6.56}$$

where $Q_1(\varepsilon_1, \varepsilon_2, \gamma) = P(\varepsilon_1)A + P(\varepsilon_1)B_u K(\varepsilon_1) + \gamma I + \varepsilon_2^2 I$ and $Q_2(\varepsilon_1, \varepsilon_2) = \frac{B_u^T P(\varepsilon_1)P(\varepsilon_1)B_u}{\epsilon_2^2} - U_M$ By choosing γ and ε_2 so that $\gamma + \varepsilon_2^2 < \varepsilon_1$, (6.55) guarantees that $Q_1(\varepsilon_1, \varepsilon_2, \gamma) < 0$. Moreover, by choosing $U_M > \frac{B_u^T P(\varepsilon_1)P(\varepsilon_1)B_u}{\epsilon_2^2}$ we have that $Q_2(\varepsilon_1, \varepsilon_2) < 0$.
Finally, for each i, by applying the Schur complement to (4.29c), we have that $([K(\varepsilon_1)(A + B_u K(\varepsilon_1))]_i)^T([K(\varepsilon_1)(A + B_u K(\varepsilon_1))]_i) \leq \varepsilon^2 R_i^2 P(\varepsilon_1)$ for a small enough ε_1. To see this, note that $K(\varepsilon_1) = -B_u^T P(\varepsilon_1)$ guarantees that the left-hand side shrinks to zero faster than $\varepsilon_1^2 \to 0$, while $P(\varepsilon_1)$ goes to zero as $\varepsilon_1 \to 0$. A similar argument can be used with (4.29d). □

6.4.4 Proof of Theorem 4.3

, The following Lemma will be used to prove Lemma 6.7.

Lemma 6.6 *Given any pair $v, y \in \mathbb{R}$ and any $\varepsilon \in (0, 1)$, there exists $\epsilon \in [\varepsilon, 2-\varepsilon]$ such that the following holds:*

$$\mathrm{sat}_S(y + v) - y = \mathrm{sat}_{S\epsilon}(v) + \sigma, \tag{6.57}$$

where $|\sigma| \leq |2\mathrm{dz}_{S(1-\varepsilon)}(y)|$

Proof. Let $\delta = 1 - \varepsilon$. We have that

$$\mathrm{sat}_S(y + v) - y = \mathrm{sat}_S(\mathrm{sat}_{S\delta}(y) + v) - \mathrm{sat}_{S\delta}(y) + \omega_1 + \omega_2,$$

with $\omega_1 = \mathrm{sat}_S(y+v) - \mathrm{sat}_S(\mathrm{sat}_{S\delta}(y)+v)$ and $\omega_2 = \mathrm{sat}_{S\delta}(y) - y$. Hence $|\omega_1+\omega_2| \le |\omega_1|+|\omega_2| \le |\mathrm{dz}_{S\delta}(y)|$ since $|\omega_1| \le |\mathrm{sat}_S(y+v)-\mathrm{sat}_S(\mathrm{sat}_{S\delta}(y)+v)| \le |y+v-\mathrm{sat}_{S\delta}(y)-v| \le |\mathrm{dz}_{S\delta}(y)|$ and $|\omega_2| = |\mathrm{sat}_{S\delta}(y)-y| \le |\mathrm{dz}_{S\delta}(y)|$. Moreover, there exists $\epsilon \in [\varepsilon, 2-\varepsilon]$ such that

$$\mathrm{sat}_S(\mathrm{sat}_{S\delta}(y)+v) - \mathrm{sat}_{S\delta}(y) = \mathrm{sat}_{S\epsilon}(v) \tag{6.58}$$

In fact, if $|y| \ge \delta$ and $yv \ge 0$ then (6.58) is satisfied by $\epsilon = \varepsilon$ and if $|y| \ge \delta$ and $yv \le 0$ then (6.58) is satisfied by $\epsilon = 2-\varepsilon$. It follows that for $|y| < \delta$ the equation is satisfied by some $\epsilon \in (\varepsilon, 2-\varepsilon)$. □

Similar to the case discussed in the previous section, the key properties of the anti-windup scheme rely on the fact that the signal y_{aw} keeps the controller well behaved, while the action of the stabilizer v_1 enforces the desired unconstrained response recovery. This is formalized in the following lemma.

Lemma 6.7 *For the closed-loop (4.22), (4.23), (4.30), (4.31) the following holds.*

1. *If $x_{aw}(0) = 0$ and $\delta(0) = y_c(0)$, then [3] the controller state and output response coincides with the virtual response produced by the unconstrained closed-loop (4.22), (4.23), (4.24) from the same initial states and under the action of the same external inputs r and d. Moreover, $\bar{z}_{aw} = \bar{z} - \hat{z}$.*

2. *If there exists a static feedback control law $k(\cdot)$ from $\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw}\end{smallmatrix}\right]$ such that $\left|k(\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw}\end{smallmatrix}\right])\right|_2 \le c\left|\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw}\end{smallmatrix}\right]\right|_2$ for some $c > 0$ and for any function $\epsilon(\cdot) : \mathbb{R} \to \mathbb{R}^m$ such that $\varepsilon \le \epsilon_i(t)$ for all t, $i = 1, \ldots, m$, the following system*

$$\dot{x}_{aw} = Ax_{aw} + B_u \mathrm{sat}_{M_{\epsilon(t)}}(\delta_{aw}) + B_u\sigma_M, \tag{6.59a}$$
$$\dot{\delta}_{aw} = \mathrm{sat}_{R\epsilon(t)}\left(k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw}\end{smallmatrix}\right]\right)\right) + \sigma_R, \tag{6.59b}$$

(with $M_{\epsilon(t)} := \mathrm{diag}(M)\epsilon(t)$) is locally (respectively, globally) $\mathcal{L}_2$ stable from (σ_M, σ_R) to (x_{aw}, δ_{aw}), then the anti-windup closed-loop (4.22), (4.23), (4.30), (4.31) with $v_1 = k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta - y_c\end{smallmatrix}\right]\right)$ is such that there exists a local (respectively, global) $\mathcal{L}_2$ gain from $\begin{bmatrix} \mathrm{dz}_{M(1-\varepsilon)}(\hat{u}) \\ \mathrm{dz}_{R(1-\varepsilon)}(\dot{\hat{u}}) \end{bmatrix}$ to $\bar{z} - \hat{z}$, namely as long as the unconstrained trajectory does not spend infinite energy outside the (restricted) saturation limits, then the actual output response z converges in the $\mathcal{L}_2$ sense to the ideal unconstrained output response $\hat{z}$.

[3] As in [147], if the anti-windup compensation is initialized differently then one experiences an extra transient at startup, but the closed-loop properties remain unchanged.

Proof. Item 1. The proof is a generalization of the proof of item 1 of Lemma 6.5. The closed-loop dynamics in the coordinates $(x_a, x_c, x_{aw}, \delta) = (x - x_{aw}, x_c, x_{aw}, \delta)$ corresponds to a cascade representation where the first subsystem (whose state is (x_a, x_c)) coincides with the unconstrained closed-loop dynamics (4.22), (4.23) and (4.24) and the second subsystem is the anti-windup compensator (4.30), which is driven by the two signals y_c and $y_{c,dot}$. Due to this fact, the controller response coincides with the unconstrained controller response, so that $\bar{y}_c = \hat{y}_c = \hat{u}$. Moreover, $\hat{z} = z_a = \bar{z} - \bar{z}_{aw}$.

Item 2. Consider the dynamics (4.30) with the selection for u in (4.31), in the coordinates $(x_{aw}, \delta_{aw}) := (x_{aw}, \delta - y_c)$ and with $v_1 = k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]\right)$, namely in the new coordinates $v_1 = k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]\right)$:

$$\dot{x}_{aw} = Ax_{aw} + B_u(\mathrm{sat}_M(\bar{y}_c + \delta_{aw}) - \bar{y}_c) \tag{6.60a}$$

$$\dot{\delta}_{aw} = \mathrm{sat}_R(\bar{y}_{c,dot} + v_1) - \bar{y}_{c,dot} \tag{6.60b}$$

$$z_{aw} = C_z x_{aw} + D_{zu}(\mathrm{sat}_M(y_c + \delta_{aw}) - y_c) \tag{6.60c}$$

By Lemma 6.6, (6.60) yields (6.59) with $|\sigma_M| \leq |2\mathrm{dz}_{M(1-\varepsilon)}(\bar{y}_c)|$, $|\sigma_R| \leq |2\mathrm{dz}_{R(1-\varepsilon)}(\bar{y}_{c,dot})|$. Since $\bar{y}_c = \hat{u}$ and $\bar{y}_{c,dot} = \dot{\hat{y}}_c = \dot{\hat{u}}$, and by item 1 of this lemma, $\bar{z} - \hat{z} = \bar{z}_{aw}$, then the result follows from the $\mathcal{L}_2$ stability assumption on (6.59), the fact that $\|z_{aw}\|_2 \leq |C_z| \cdot \|x_{aw}\|_2 + |D_{zu}| \cdot \|\mathrm{sat}_{R\varepsilon}(k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]\right))\|_2$, and $\|k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]\right)\|_2 \leq c\|x_{aw}\|_2$ since $|k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]\right)| \leq c|x_{aw}|$. □

Based on the preliminary statements in Lemmas 6.6 and 6.7, it is possible to prove Theorem 4.3 as follows.

Proof. Theorem 4.3.
Item 1. By Lemma 6.7, it is sufficient to prove that system (6.59) with $k\left(\left[\begin{smallmatrix} x_{aw} \\ \delta - y_c \end{smallmatrix}\right]\right) = -K_\delta \delta_{aw}$ is globally $\mathcal{L}_2$ stable from (σ_M, σ_R) to z_{aw}. With that selection, the second equation in (6.59) becomes $\dot{\delta}_{aw} = -\mathrm{sat}_R(K_\delta \delta_{aw}) + \sigma_R$, which is well known to have a global nonlinear gain from σ_R to δ_{aw} (a direct proof is obtained by a trivial modification of the proof of item (iii) of [57, Lemma 1]). Then the first equation in (6.59) is to an exponentially stable linear system driven by the two $\mathcal{L}_2$ signals $\mathrm{sat}_M(\delta_{aw})$ and σ_M. Then $x_{aw} \in \mathcal{L}_2$ and finally also $z_{aw} \in \mathcal{L}_2$.

Item 2. Similar to the previous item, using Lemma 6.7, we address the local $\mathcal{L}_2$ stability of system (6.59). Since for small enough states (x_{aw}, δ_{aw}), the stabilizing signal v_1 remains below the saturation limits, then the origin of (6.59) is locally exponentially stable. This implies local $\mathcal{L}_2$ stability from (σ_M, σ_R) to z_{aw} [87, Cor. 5.1].

Item 3. We first show that any solution to (4.33a), (4.33c), (4.34), guarantees item 3. Then we show that any solution to (4.33) guarantees feasibility of (4.34).

In the proof we actually disregard constraint (4.34b) because it is always feasible for a large enough k_{max} (it will be used to determine numerically convenient controller gains).

Consider the Lyapunov function $V = \left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]^T P \left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]$ for system (6.59), where $P = Q^{-1}$. Pre- and post- multiplying (4.33c) by the matrix $\left[\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right]$, we get

$$0 \leq \begin{bmatrix} \varepsilon^2 S_i P & [H]_i^T \\ [H]_i & 1 \end{bmatrix}, i = 1, \dots, 2m, \tag{6.61}$$

where $H = XP$. Using a Schur complement [24] on (6.61), we get $[H]_i^T [H]_i \leq \varepsilon^2 S_i^2 P,\ i = 1, \dots 2m$, and then in $\mathcal{E}(P,1) := \left\{ \left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right] : V\left(\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]\right) \leq 1 \right\}$ it holds that $\left|\varepsilon S^{-1} H \left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]\right|_\infty \leq 1$. The last inequality implies that for all $\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right] \in \mathcal{E}(P,1)$, $\mathrm{sat}_{\varepsilon S}\left(H \left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]\right) = H \left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]$ and, by the generalized sector condition in Lemma 6.3, for any positive definite diagonal matrix $U := \left[\begin{smallmatrix} U_M & 0 \\ 0 & U_R \end{smallmatrix}\right]$ and for all $\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right] \in \mathcal{E}(P,1)$,

$$q^T U \left(q - \left(\begin{bmatrix} 0 & I_m \\ K_x & K_\delta \end{bmatrix} - H \right) \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix} \right) \leq 0, \tag{6.62}$$

where $q := \begin{bmatrix} \mathrm{dz}_M^T(\delta_{aw}) & \mathrm{dz}_R^T(K_x x_{aw} + K_\delta \delta_{aw}) \end{bmatrix}^T$.

Consider now the time derivative of V along the dynamics (6.59) with $\sigma_M = 0$ and $\sigma_R = 0$. Using (6.62), we get for all $\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right] \in \mathcal{E}(P,1)$,

$$\begin{aligned}
\dot{V} &\leq \dot{V} - 2q^T U \left(q - \left(\begin{bmatrix} 0 & I_m \\ K_x & K_\delta \end{bmatrix} + H \right) \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix} \right) \\
&= \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix}^T \mathrm{He} \left(P \begin{bmatrix} A & B \\ K_x & K_\delta \end{bmatrix} \right) \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix} \\
&\quad + 2 \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix}^T P \begin{bmatrix} -B & 0 \\ 0 & -I_m \end{bmatrix} q \\
&\quad - 2q^T U \left(q - \left(\begin{bmatrix} 0 & I_m \\ K_x & K_\delta \end{bmatrix} - H \right) \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix} \right) \\
&= w^T \mathrm{He} \begin{bmatrix} P \begin{bmatrix} A & B \\ K_x & K_\delta \end{bmatrix} & P \begin{bmatrix} -B & 0 \\ 0 & -I_m \end{bmatrix} \\ U \left(\begin{bmatrix} 0 & I_m \\ K_x & K_\delta \end{bmatrix} - H \right) & -U \end{bmatrix} w \\
&< -2\gamma \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix}^T P \begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix} = -2\gamma V \left(\begin{bmatrix} x_{aw} \\ \delta_{aw} \end{bmatrix} \right),
\end{aligned}$$

where $w := [x_{aw}^T \;\; \delta_{aw}^T \;\; q^T]^T$ and where the last step follows from (4.34a) after pre- and post-multiplying by the matrix $\left[\begin{smallmatrix} P & 0 \\ 0 & U \end{smallmatrix}\right]$, with $U = \left[\begin{smallmatrix} W_M & 0 \\ 0 & W_R \end{smallmatrix}\right]^{-1}$ (recall that, by definition of H, $XP = H$).

Since by (4.33a), $\mathcal{E}(P,1) \supseteq \mathcal{B}(\alpha)$, then the bound above on $\dot{V}$ implies that the origin of (6.59) is locally exponentially stable with region of attraction including $\mathcal{B}(\alpha)$. Similar to the proof of item 2 local $\mathcal{L}_2$ stability of system (6.59) is guaranteed. Moreover, letting $\xi := \left[\begin{smallmatrix} x_{aw} \\ \delta_a w \end{smallmatrix}\right]$, exponential recovery follows from $\dot{V} \leq -2\gamma V$, which holds on the invariant set $\mathcal{E}(P,1)$ and implies that for $\xi(0)$ in this set $V(\xi(t)) \leq e^{-2\gamma t} V(\xi(0))$; denoting by λ_m and λ_M the minimum and maximum eigenvalues of the symmetric matrix $P > 0$, and taking into account that $\lambda_m |\xi|^2 \leq V(\xi) \leq \lambda_M |\xi|^2$ since $V(\xi) = \xi^T P \xi$, the exponential decay of V implies that $|\xi(t)| \leq \sqrt{\frac{\lambda_M}{\lambda_m}} e^{-\gamma t} |\xi(0)|$.

The proof of item 2 is completed by showing that any solution to (4.33) guarantees feasibility of (4.34). Since (4.34b) is always feasible for a large enough k_{max}, it is enough to show that any solution to (4.33) guarantees feasibility of (4.34a). In particular, only (4.33b) will be necessary to this aim. To this aim, it is useful to write (4.34a) as follows:

$$\mathrm{He}(\Phi_0 + Y[K_x \; K_\delta] Z^T) < 0, \tag{6.63}$$

where the matrices Φ_0, Y and Z are easily derived from (4.34a). By the elimination lemma (see, e.g., [24, Sec. 2.6.2]) there exists a $[K_x \; K_\delta]$ satisfying (6.63) if (and only if):

$$Y_\perp^T \Phi_0 Y_\perp < 0, \qquad Z_\perp^T \Phi_0 Z_\perp < 0, \tag{6.64}$$

where $Y_\perp$ is an orthogonal complement of Y and $Z_\perp$ is an orthogonal complement of Z. By choosing

$$Y_\perp = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \\ 0 & -I_m & 0 \end{bmatrix}, \quad Z_\perp = \begin{bmatrix} 0 \\ I_{2m} \end{bmatrix},$$

after some computations (omitted due to space constraints) the second inequality in (6.64) becomes $\left[\begin{smallmatrix} W_M & 0 \\ 0 & W_R \end{smallmatrix}\right] > 0$, which is always satisfied by assumption, while the first inequality in (6.64) coincides with (4.33b).

Item 4. When $X = 0$ in (4.33), the reasonings of the previous item still apply with the extra feature that the sector condition (6.62) is global (namely it holds for all $\left[\begin{smallmatrix} x_{aw} \\ \delta_{aw} \end{smallmatrix}\right]$). Hence, the results of the previous item hold globally. $\square$

6.5 Proof of the Results in Chapter 5

6.5.1 Proof of Theorem 5.1

Consider an input signal $\overline{v} : \mathbb{R}_{\geq 0} \to \mathcal{V}$ such that $\|\overline{v}\|_2$ is defined, and consider a solution pair $(\xi, v) = ((\xi_x, \xi_\tau), v)$ to the hybrid system (5.5), where v is the hybrid signal lifted from $\overline{v}$ on $\operatorname{dom} \xi$. By Lemma 5.1, $\operatorname{dom} v$ is unbounded.

Define the set $\mathcal{T} = \bigcup_j [t_j, t_j + \rho] \times \{j\}$ where for all j, t_j is such that, for each $\tau \in \mathbb{R}_{>0}$, $(t_j - \tau, j) \notin \operatorname{dom} \xi$. Note that by time regularization, $\mathcal{T} \subseteq \operatorname{dom} \xi$ but $\mathcal{T}$ is not necessarily a hybrid time domain. It follows that $\forall (t, j) \in \operatorname{dom} \xi$ such that $(t, j) \notin \mathcal{T}$ we have $\xi_\tau(t, j) \geq \rho$, therefore

$$u(t,j)v(t,j) + \varepsilon_1 |v(t,j)|^2 - \varepsilon_2 |u(t,j)|^2 \geq 0 \tag{6.65}$$

where $\varepsilon_1 = \frac{\epsilon_1}{1-\epsilon_1\epsilon_2}$ and $\varepsilon_2 = \frac{\epsilon_2}{1-\epsilon_1\epsilon_2}$. Therefore

$$\begin{aligned}
&\int_0^\infty \overline{u}(t)^T \overline{v}(t) dt = \sum_j \int_{t_j}^{t_{j+1}} u(t,j)^T v(t,j) dt = \\
&= \sum_j \left(\int_{t_j}^{t_j+\rho} u(t,j)^T v(t,j) dt + \int_{t_j+\rho}^{t_{j+1}} u(t,j)^T v(t,j) dt \right) \\
&\geq \sum_j \left(\int_{t_j}^{t_j+\rho} u(t,j)^T v(t,j) dt \; + \right. \\
&\qquad\qquad \left. + \int_{t_j+\rho}^{t_{j+1}} -\varepsilon_1 |v(t,j)|^2 dt + \int_{t_j+\rho}^{t_{j+1}} \varepsilon_2 |u(t,j)|^2 dt \right) \\
&\geq \sum_j \left(\int_{t_j}^{t_j+\rho} u(t,j)^T v(t,j) dt - \varepsilon_2 |u(t,j)|^2 dt \; + \right. \\
&\qquad\qquad \left. + \int_{t_j}^{t_{j+1}} -\varepsilon_1 |v(t,j)|^2 dt + \int_{t_j}^{t_{j+1}} \varepsilon_2 |u(t,j)|^2 dt \right).
\end{aligned} \tag{6.66}$$

Consider now the continuous dynamics of x_c in (5.5a). By Assumption 5.1, we have

$$|\dot{x_c}| \leq |f(x_c) + g(x_c, v)| \leq L_f |x_c| + L_g |v| \tag{6.67}$$

Then, for $(t,j) \in [t_j, t_j+\rho] \times \{j\} \subseteq \mathcal{T}$, we have

$$\begin{aligned} |u(t,j)| &\leq L_h \int_{t_j}^{t} e^{L_f(t-s)} L_g |v(s,j)| ds \\ &\leq L_h \int_{t_j}^{t_j+\rho} e^{L_f(t_j+\rho-s)} L_g |v(s,j)| ds \\ &\leq L_h L_g \max\{1, e^{L_f \rho}\} \int_{t_j}^{t_j+\rho} |v(s,j)| ds \end{aligned} \tag{6.68}$$

Note that in (6.68) there is no dependence on the initial condition by the fact that $\xi_x(t_j, j) = 0$. It follows that

$$\begin{aligned} \int_{t_j}^{t_j+\rho} |u(t,j)|^2 dt &\leq \int_{t_j}^{t_j+\rho} L_h^2 L_g^2 \max\{1, e^{2L_f \rho}\} \left(\int_{t_j}^{t_j+\rho} |v(s,j)| ds \right)^2 dt \\ &= \rho L_h^2 L_g^2 \max\{1, e^{2L_f \rho}\} \left(\int_{t_j}^{t_j+\rho} |v(s,j)| ds \right)^2 dt \\ &\leq \rho^2 L_h^2 L_g^2 \max\{1, e^{2L_f \rho}\} \int_{t_j}^{t_j+\rho} |v(s,j)|^2 ds \end{aligned} \tag{6.69}$$

where we used Holder's integral inequality [154, page 274] in the last step of (6.69).

$$\begin{aligned} \int_{t_j}^{t_j+\rho} u(t,j)^T v(t,j) dt &\leq L_h L_g \max\{1, e^{L_f \rho}\} \left(\int_{t_j}^{t_j+\rho} |v(t,j)| dt \right)^2 \\ &\leq \rho L_h L_g \max\{1, e^{L_f \rho}\} \int_{t_j}^{t_j+\rho} |v(t,j)|^2 dt \end{aligned} \tag{6.70}$$

where, as above, the last inequality is obtained by using Holder's integral inequality.

Define $k(\rho) = \rho L_h L_g \max\{1, \rho e^{L_f \rho}\}$. By (6.69), (6.70), we have that

$$\left| \int_{t_j}^{t_j+\rho} u(t,j)^T v(t,j) dt - \varepsilon_2 |u(t,j)|^2 dt \right| \leq k(\rho)(1 + \varepsilon_2 k(\rho)) \int_{t_j}^{t_j+\rho} |v(t,j)|^2 dt \tag{6.71}$$

Define now $\overline{k}(\rho) = k(\rho)(1 + \varepsilon_2 k(\rho))$ then, from (6.66), we can say that

$$\begin{aligned}
\int_0^\infty \overline{u}(t)^T \overline{v}(t)dt \geq & \sum_j \left(-\overline{k}(\rho) \int_{t_j}^{t_j+\rho} |v(t,j)|^2 dt + \right. \\
& \left. + \int_{t_j}^{t_{j+1}} -\varepsilon_1 |v(t,j)|^2 dt + \int_{t_j}^{t_{j+1}} \varepsilon_2 |u(t,j)|^2 dt \right) \\
& \geq - \left(\varepsilon_1 + \overline{k}(\rho)\right) \sum_j \int_{t_j}^{t_{j+1}} |v(t,j)|^2 dt + \sum_j \int_{t_j}^{t_{j+1}} \varepsilon_2 |u(t,j)|^2 dt \\
& = - \left(\varepsilon_1 + \overline{k}(\rho)\right) \|\overline{v}(\cdot)\|_2^2 + \varepsilon_2 \|\overline{u}(\cdot)\|_2^2
\end{aligned} \tag{6.72}$$

6.5.2 Proofs of Theorems 5.2 and 5.3.

System of equations (5.21), with $\overline{C}$ and $\overline{D}$ defined as in Section 5.2.2 or in Section 5.2.3, and the function g continuous or asynchronous (Definition 5.1) satisfies the *basic assumptions* of [62]. Therefore, the following result from [62] can be used to prove Theorems 5.2, 5.3 and 5.4.

We denote with $F(x, e)$ the right-hand side of (5.21a) and we denote with $G(x, e)$ the right-hand side of (5.21b).

Proposition 6.1 *[62, Theorem 23]*
For a system of equations (5.21)*, with C and D closed sets and g continuous or asynchronous (Definition 5.1), if*

$$\begin{aligned}
\langle \nabla V(x,e), F(x,e) \rangle \leq 0 \quad & \textit{for all } x \in C \setminus \{(0,0)\} \\
V(G(x,e)) - V(x,e) \leq 0 \quad & \textit{for all } x \in D \setminus \{(0,0)\}
\end{aligned} \tag{6.73}$$

then $(x, e) = (0, 0)$ is stable. Moreover, if there exists a compact neighborhood K of $(x, e) = (0, 0)$ such that, for each $\mu > 0$, no complete solutions to $\mathcal{S}_\mathcal{N}$ remain in the set $\{(x, e) \,|\, V(x, e) = \mu\} \cap K$, then $(x, e) = (0, 0)$ is pre-asymptotically stable. Finally, $(x, e) = (0, 0)$ is globally pre-asymptotically stable if K can be arbitrarily large and the set $\{(x, e) \,|\, V(x, e) \leq \mu\}$ is compact.

Proof. Theorem 5.2.
From the definition of $\overline{C}$ in (5.26a) and by condition (1) of the theorem,

$$\begin{aligned}
\langle \nabla V(x,e), F(x,e) \rangle \leq 0 \quad & \text{for all } (x,e) \in \overline{C} \\
V(G(x,e)) - V(x,e) \leq 0 \quad & \text{for all } (x,e) \in \overline{D}.
\end{aligned} \tag{6.74}$$

Moreover, (5.26) and condition (2) of the theorem guarantee that for each $(x,e) \in \overline{D}$, $(x, \overline{g}(x,e))$ belongs to the interior of $\overline{C}$ or $(x, \overline{g}(x,e)) = (0,0)$. For the first case, it follows that there exists a compact interval, say $[0, \overline{t}]$, with $\overline{t} \in \mathbb{R}_{>0}$, in which the system can only flow.

Consider now a state (x,e) in $\overline{C}$ and such that $V(x,e) = \mu$, for some $\mu \in \mathbb{R}_{>0}$. By (5.26a), $x \neq 0$, then $\langle \nabla V(x,e), F(x,e) \rangle \leq -\varepsilon |x|^2$. It follows that no complete solutions to $\mathcal{S}_{\mathcal{N}}$, from $x \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$.

Suppose now (x,e) belongs to $\overline{C}$ but $x = 0$ and $e \neq 0$. By (5.21a), the continuous dynamics of x is driven by $\dot{x} = \overline{A}_{12} e$. It follows that $x(t) = 0$ cannot be a solution to $\mathcal{S}_{\mathcal{N}}$ in the interval $t \in [0, \overline{t}]$. Therefore,

- if (x,e) is an interior point of $\overline{C}$ then there exist a time $t \in [0, \overline{t}]$ such that $x(t) \neq 0$, from which $\langle \nabla V(x(t), e(t)), F(x(t), e(t)) \rangle \leq -\varepsilon |x(t)|^2$,
- if (x,e) is on the border of $\overline{C}$ two cases are possible: a jump occurs, that forces the state of the system in the interior of $\overline{C}$, or there exists a compact interval $[0, \overline{t}]$ in which the system can flow. In such case, there exist a time $t \in [0, \overline{t}]$ such that $\langle \nabla V(x(t), e(t)), F(x(t), e(t)) \rangle \leq -\varepsilon |x(t)|^2$.

It follows that, no complete solutions to $\mathcal{S}_{\mathcal{N}}$, from $x = 0$ and $e \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$.

By Proposition 6.1, $(x,e) = (0,0)$ is global pre-asymptotically stable $\square$

Proof. Theorem 5.3.
From (5.26a) and (1) of the theorem, (6.74) hold also for Theorem 5.3.

Suppose that (x,e) belongs to $\overline{D}$ and $V(x,e) = \mu$, for some given $\mu > 0$. By (i), if $e \neq 0$ then $V(G(x,e)) - V(x,e) \leq -\alpha(|e|)$ therefore no complete solutions to $\mathcal{S}_{\mathcal{N}}$, from $e \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$. Moreover, inequality (5.27) guarantees that if $x \neq 0$ and $e = 0$ then (x,e) cannot belong to $\overline{D}$. For instance, let $x \neq 0$ and $e = 0$ then

$$\begin{aligned} -x^T Q x + x^T R_{11} x + x^T R_{12} e + e^T R_{22} e = \\ = -x^T Q x + x^T R_{11} x < -\varepsilon |x|^2 \end{aligned} \tag{6.75}$$

therefore (x,e) belongs to the interior of $\overline{C}$ and the system flows only.

The analysis of the continuous dynamics of $\mathcal{S}_{\mathcal{N}}$ follows the line of the proof of Theorem 5.2. It follows that no complete solutions to $\mathcal{S}_{\mathcal{N}}$ from $(x,e) \in \overline{C}$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$. By Proposition 6.1, it follows that $(x,e) = (0,0)$ is global pre-asymptotically stable. $\square$

6.5.3 Proof of Theorem 5.4.

By (1) of Theorem 5.4, for all $(x,e) \in \overline{D}$

$$V(G(x,e)) - V(x,e) < 0 \quad \text{if } e \neq 0. \tag{6.76}$$

From the definitions of $\overline{C_i}$ and $\overline{C}$ in (5.32) and (5.33a), we have that $\langle \nabla V(x,e), F(x,e)\rangle$ is equal to

$$\begin{aligned}
&= -x^T Q x + x^T R_{11} x + x^T R_{12} e + e^T R_{22} e && (6.77a)\\
&\leq -x^T Q x + x^T R_{11} x + K_1|x||e| + K_2 e^T e && (6.77b)\\
&\leq -x^T Q x + x^T R_{11} x + K_1|x| \sum_{i=1}^{q} |e_i| + K_2 \sum_{i=1}^{q} e_i^2 && (6.77c)\\
&\leq \sum_{i=1}^{q} \big(-\alpha_i x^T Q x + \alpha_i x^T R_{11} x + K_1|x||e_i| + K_2 e_i^2\big) && (6.77d)\\
&\leq \sum_{i=1}^{q} -\alpha_i \varepsilon |x|^2 \leq -\varepsilon |x|^2 \quad \text{for all } (x,e) \in \overline{C}. && (6.77e)
\end{aligned}$$

The inequality between (6.77a) and (6.77b) follows from the definition of K_1 and K_2. The inequality between (6.77b) and (6.77c) follows from the fact that $|e| \leq \sum_{i=1}^{q} |e_i|$, where e_i is the ith component of e, for each $i \in \{1, \ldots, q\}$. (6.77d) follows from (6.77c) by $\sum_{i=1}^{q} \alpha_i = 1$. Finally, from $\overline{C_i}$ and $\overline{C}$, the argument of the sum in (6.77d) can be written as (6.77e). It follows that (6.73) holds.

Suppose now (x,e) belongs to $\overline{D}$ and $V(x,e) = \mu$, for some given $\mu > 0$. If $e \neq 0$ then (6.76) holds and no complete solutions to $\mathcal{S}_{\mathcal{N}}$ from $e \neq 0$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$. If $x \neq 0$ and $e = 0$ then for each $i \in \{1, \ldots, q\}$

$$\begin{aligned}
-\alpha_i x^T Q x + {}&\alpha_i x^T R_{11} x + K_1|x||e_i| + K_2 e_i^2 = \\
&= -\alpha_i x^T Q x + \alpha_i x^T R_{11} x < -\alpha_i \varepsilon |x|^2
\end{aligned} \tag{6.78}$$

where the last inequality follows from (5.27). It follows that (x,e) cannot belong to $\overline{D}$, it belongs to the interior of $\overline{C}$ and the system flows only.

The analysis of the continuous dynamics of $\mathcal{S}_{\mathcal{N}}$ follows the line of the proof of Theorem 5.2. It follows that no complete solutions to $\mathcal{S}_{\mathcal{N}}$ from $(x,e) \in \overline{C}$ remains in the set $\{(x,e) \,|\, V(x,e) = \mu\}$, for any given $\mu > 0$. By Proposition 6.1, it follows that $(x,e) = (0,0)$ is global pre-asymptotically stable. ∎

6.5.4 Proof of Theorem 5.5.

We need the following definition.

Definition 6.1 For each $p \in \mathbb{N}$, a function $\sigma : \mathbb{R}^p \to \mathbb{R}$ is said to be *homogeneous with degree* $\delta \in \mathbb{R}$ if, for all $z \in \mathbb{R}^p$ and $\lambda > 0$, $\sigma(\lambda z) = \lambda^\delta \sigma(z)$.

From Theorem 5.2, 5.3 or 5.4 we know that

$$\left\{ \begin{array}{rcl} \dot{\hat{x}} & = & \overline{A}_{11}\hat{x} + \overline{A}_{12}e \\ \dot{e} & = & \overline{A}_{21}\hat{x} + \overline{A}_{22}e \end{array} \right. \qquad (\hat{x}, e) \in \overline{C} \tag{6.79a}$$

$$\left\{ \begin{array}{rcl} \hat{x}^+ & = & \hat{x} \\ e^+ & = & \overline{g}(\hat{x}, e) \end{array} \right. \qquad (\hat{x}, e) \in \overline{D} \tag{6.79b}$$

$$y = H\hat{x} \tag{6.79c}$$

is GpAS. Then, define $z = \begin{bmatrix} \hat{x} \\ e \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $G(z) = \begin{bmatrix} \hat{x} \\ \overline{g}(\hat{x}, e) \end{bmatrix}$. From the *homogeneity* of (6.79) (e.g. [153], continuous and discrete dynamics are defined by linear vector field and $\overline{C}$ and $\overline{D}$ are cones) and from [32, Theorem 7.9] and [153, Theorem 2], there exists a function $\overline{V} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ that is smooth on $\mathbb{R}^n \times \mathbb{R}^q \setminus \{0\}$ and homogeneous with degree $\delta \in \mathbb{R}$ such that,

$$\alpha_1(|z|) \leq \overline{V}(z) \leq \alpha_2(|z|) \qquad \forall z \in \mathbb{R}^n \times \mathbb{R}^p \tag{6.80a}$$

$$\langle \nabla \overline{V}(z), Az \rangle \leq -\mu \overline{V}(z) \qquad \forall z \in \overline{C} \tag{6.80b}$$

$$\overline{V}(G(z)) \leq \nu \overline{V}(z) \qquad \forall z \in \overline{D} \tag{6.80c}$$

where $\mu > 0$, $\nu \in (0, 1)$ and α_1, $\alpha_2 \in \mathcal{K}_\infty$.

Consider $\delta = 2$, then for each $z, w \in \mathbb{R}^n \times \mathbb{R}^p$,

$$\begin{aligned} \langle \nabla \overline{V}(z), w \rangle &= \lim_{h \to 0} \frac{\overline{V}(z + hw) - \overline{V}(z)}{h} \\ &= \lim_{h \to 0} \frac{\overline{V}(|z|\frac{z}{|z|} + |z|\frac{h}{|z|}w) - \overline{V}(|z|\frac{z}{|z|})}{|z|\frac{h}{|z|}} \\ &= \lim_{h \to 0} \frac{|z|^2}{|z|} \frac{\overline{V}(\frac{z}{|z|} + \frac{h}{|z|}w) - \overline{V}(\frac{z}{|z|})}{\frac{h}{|z|}} \end{aligned} \tag{6.81}$$

where the last equality is the result of the homogeneity of V. Since w is arbitrary, for any $z \neq 0$, $\nabla \overline{V}(z) = |z| \nabla \overline{V}(\frac{z}{|z|})$. Since V is smooth, $|\nabla \overline{V}(z)| \leq \lambda |z|$, where $\lambda = \max_{|z|=1} \overline{V}(z)$. Note that $\alpha_1(1)|z|^2 \leq |z|^2 \overline{V}(\frac{z}{|z|}) \leq \alpha_2(1)|z|^2$.

With these tools we can now prove the global *practical* asymptotic stability of (5.43). Consider $\gamma \in \mathbb{R}_{>0}$, $\gamma \ll 1$ and take $\ell = \alpha_2(\rho + \gamma\rho)$, that implies, $\{z \,|\, |z| \leq \rho\} \subseteq \{z \,|\, \overline{V}(z) < \ell\}$ and consider the compact set $\mathcal{A} = \{z \,|\, \overline{V}(z) \leq \ell\} \times \{0\}$. We prove that $\mathcal{A}$ is globally pre-asymptotically stable for (5.43). Define the candidate Lypunov function $V(z, \eta)$ as follows.

$$V(z,\eta) = \begin{cases} \overline{V}(z) - \ell + \frac{1}{2}\eta^T \overline{P} \eta & \overline{V}(z) \geq \ell \\ \frac{1}{2}\eta^T \overline{P}\eta & \text{otherwise} \end{cases} \tag{6.82}$$

where $\overline{P}$ is a positive definite symmetric matrix of dimension $n \times n$. Note that V is continuous in $\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^n$, 0 in $\mathcal{A}$ and smooth for points $(z, \eta) \in \mathbb{R}^{n+q} \times \mathbb{R}^n$ such that $\overline{V}(z) \neq \ell$. It is locally Lipschitz for points $(z, \eta) \in \mathbb{R}^{n+q} \times \mathbb{R}^n$ such that $\overline{V}(z) = \ell$. For such points, say $(\overline{z}, \overline{\eta})$, we consider the generalized gradient (in the sense of Clarke) of V, that coincides with the convex hull of all limits of sequences $\nabla V(z_i, \eta_i)$ where (z_i, η_i), $i \in \mathbb{N}$, is any sequence converging to $(\overline{z}, \overline{\eta})$ while avoiding an arbitrary set of measurement zero containing all the points at which V is not differentiable [126].

Define $B = \begin{bmatrix} LH \\ -HLH \end{bmatrix}$ and consider (5.43a). The directional derivative of V is less then or equal to

$$\begin{cases} v_1 = \langle \nabla \overline{V}(z), Az + B\eta \rangle + \eta^T \overline{P}(A - LH)\eta & \overline{V}(z) > \ell \\ v_2 = \eta^T \overline{P}(A - LH)\eta & \overline{V}(z) < \ell \\ v_3 \in \overline{\mathrm{co}}\{v_1, v_2\} & \text{otherwise} \end{cases}$$

By Assumption 5.3, $\eta^T \overline{P}(A - LH)\eta \leq \eta^T \overline{Q}\eta$, where $\overline{Q}$ is a negative definite symmetric matrix of dimension $n \times n$. $\overline{Q}$ will be defined below to guarantee negativity of the derivative of V. **(i)** Consider the case $z \in \overline{C}$.

$$\begin{aligned} v_1 &\leq -\mu \overline{V}(z) + \lambda |B||z||\eta| - \eta^T \overline{Q}\eta \\ &\leq \left(-\mu\alpha_1(1) + \varepsilon^2\right)|z|^2 + \left(\frac{\lambda |B|}{\varepsilon^2} - \lambda_{min}(\overline{Q})\right)|\eta|^2 \\ v_2 &\leq -\eta^T \overline{Q}\eta \end{aligned}$$

Therefore v_1 is strictly negative in $\{(z, \eta) \,|\, z \in \overline{C} \text{ or } |z| \leq \rho\} \setminus \mathcal{A}$ for $\varepsilon^2 < \mu\alpha_1$ and $\lambda_{min}(\overline{Q}) > \frac{\lambda |B|}{\varepsilon^2}$. v_2 is strictly negative in $\{(z, \eta) \,|\, z \in \overline{C} \text{ or } |z| \leq \rho\} \setminus \mathcal{A}$ by the fact that, when $\eta = 0$, $z \in \overline{C}$ and $\overline{V}(z) \leq \ell$ imply $z \in \mathcal{A}$. **(ii)** Consider the case $z \notin \overline{C}$. Thus, $|z| \leq \rho$. In this case, $V(z) < \ell$ therefore the directional derivative of V is less then or equal to v_2, that is, it is negative in $\{(z, \eta) \,|\, z \in \overline{C} \text{ or } |z| \leq \rho\} \setminus \mathcal{A}$.

Consider now (5.43b). Then,

$$V(z^+, \eta^+) - V(z, \eta) \leq (\nu - 1)\overline{V}(z) \leq -(1-\nu)\alpha_1(1)|z|^2$$

that is negative in $\{(z, \eta) \,|\, z \in \overline{D} \text{ and } |z| \geq \rho\} \setminus \mathcal{A}$ by the fact that $|z| \geq \rho$. Then, by [126, Theorem 7.6] and [126, Corollary 7.7] the set $\mathcal{A} \times \{0\}$ is globally pre-asymptotically stable.

Note that $\mathcal{A} \times \{0\} \subseteq \alpha_1^{-1}(\alpha_2(\rho + \gamma\rho))\mathbb{B} \times \{0\}$. By the fact that $\alpha_i(s) = |s|^2\alpha_i(1)$, for $i \in \{1, 2\}$, it follows that $\alpha_1^{-1}(s) = \left(\frac{s}{\alpha_1(1)}\right)^{\frac{1}{2}}$. Then, $\alpha_1^{-1}(\alpha_2(\rho + \gamma\rho)) = \left(\frac{\alpha_2(1)}{\alpha_1(1)}\right)^{\frac{1}{2}}(\rho + \gamma\rho)$, that is, $\overline{\gamma} = (1+\gamma)\left(\frac{\alpha_2(1)}{\alpha_1(1)}\right)^{\frac{1}{2}}$. ∎

Index

Bibliography

[1] R. Alur, C. Courcoubetis, and D. Dill. Model-checking in dense real-time. *Information and Computation*, 104:2–34, 1993.

[2] R. Alur, C. Courcoubetis, N. Halbwachs, T.A. Henzinger, P. H. Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138(1):3–34, 1995. Hybrid Systems.

[3] R. Alur, C. Courcoubetis, T.A. Henzinger, and P. Ho. Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems. In R. L. Grossman, A. Nerode, A. P. Ravn, and H. Rischel, editors, *Hybrid Systems*, volume 736 of *Lecture Notes in Computer Science*, pages 209–229. Springer, 1992.

[4] R. Alur and D. L. Dill. A theory of timed automata. *Theor. Comput. Sci.*, 126(2):183–235, 1994.

[5] R. Alur, T.A. Henzinger, and P. Ho. Automatic symbolic verification of embedded systems. *IEEE Transactions on Software Engineering*, 22:181–201, 1996.

[6] H. Anai and V. Weispfenning. Reach set computations using real quantifier elimination. In *HSCC '01: Proceedings of the 4th International Workshop on Hybrid Systems*, pages 63–76, London, UK, 2001. Springer-Verlag.

[7] A. Arnold and D. Niwinski. *Rudiments of μ-calculus.* Elsevier Science B.V., Netherlands, 2001.

[8] E. Asarin, T. Dang, and A. Girard. Reachability analysis of nonlinear systems using conservative approximation. In *In Oded Maler and Amir*

Pnueli, editors, Hybrid Systems: Computation and Control, LNCS 2623, pages 20–35. Springer-Verlag, 2003.

[9] E. Asarin, T. Dang, O. Maler, and O. Bournez. Approximate reachability analysis of piecewise-linear dynamical systems. In *HSCC '00: Proceedings of the Third International Workshop on Hybrid Systems: Computation and Control*, pages 20–31, London, UK, 2000. Springer-Verlag.

[10] E. Asarin, O. Maler, and A. Pnueli. Reachability analysis of dynamical systems having piecewise-constant derivatives. *Theoretical Computer Science*, 138(1):35–65, 1995. Hybrid Systems.

[11] M. Athans and P. L. Falb. *Optimal control: an introduction to the theory and its applications*. McGraw-Hill, 1966.

[12] M. Badamchizadeh, S. Khanmohammadi, G. Alizadeh, A. Aghagolzadeh, and G. Karimian. Using sum of squares decomposition for stability of hybrid systems. *IEICE Transactions*, 90-A(11):2478–2487, 2007.

[13] C. Baier and J.P. Katoen. *Principles of Model Checking (Representation and Mind Series)*. The MIT Press, 2008.

[14] C. Barbu, S. Galeani, A.R. Teel, and L. Zaccarian. Nonlinear anti-windup for manual flight control. *International Journal of Control*, 78(14):1111–1129, September 2005.

[15] C. Barbu, R. Reginatto, A.R. Teel, and L. Zaccarian. Anti-windup for exponentially unstable linear systems with inputs limited in magnitude and rate. In *American Control Conference*, pages 1230–1234, Chicago (IL), USA, June 2000.

[16] C. Barbu, R. Reginatto, A.R. Teel, and L. Zaccarian. Anti-windup for exponentially unstable linear systems with rate and magnitude limits. In V. Kapila and K. Grigoriadis, editors, *Actuator Saturation Control*, chapter 1, pages 1–31. Marcel Dekker, 2002.

[17] A. Bateman and Z. Lin. An analysis and design method for linear systems under nested saturation. *Systems and Control Letters*, 48:41–52, 2003.

[18] O. Beker, C.V. Hollot, and Y. Chait. Plant with an integrator: an example of reset control overcoming limitations of linear feedback. *IEEE Transactions Automatic Control*, 46:1797–1799, 2001.

[19] O. Beker, C.V. Hollot, Y. Chait, and H. Han. Fundamental properties of reset control systems. *Automatica*, 40:905–915, 2004.

[20] J.M. Berg, K.D. Hammett, C.A. Schwartz, and S.S. Banda. An analysis of the destabilizing effect of daisy chained rate-limited actuators. *IEEE Trans. on Control Systems Technology*, 4(2):171–176, March 1996.

[21] J.-M. Biannic and S. Tarbouriech. Optimization and implementation of dynamic anti-windup compensators with multiple saturations in flight control systems. *Control Engineering Practice*, 17(6):703–713, 2009.

[22] P. Blackburn, J.F.A.K. van Benthem, and F. Wolter. *Handbook of Modal Logic, Volume 3 (Studies in Logic and Practical Reasoning).* Elsevier Science Inc., New York, NY, USA, 2006.

[23] F. Blanchini and C. Savorgnan. Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions. *Automatica*, 44(4):1166 – 1170, 2008.

[24] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* Society for Industrial an Applied Mathematics, 1994.

[25] J.C. Bradfield. *Verifying Temporal Properties of Systems.* Birkhauser, 1992.

[26] J.C. Bradfield and C. Stirling. Local model checking for infinite state spaces. *Theoretical Computer Science*, 96(1):157–174, 1992.

[27] M. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Aut. Cont.*, 43(4):475–482, 1998.

[28] A. Bressan and B. Piccoli. A generic classification of time-optimal planar stabilizing feedbacks. *SIAM Journal on Control and Optimization*, 36(1):12–32, 1998.

[29] O. Brieger, M. Kerr, D. Leißling, I. Postlethwaite, J. Sofrony, and M.C. Turner. Flight testing of a rate saturation compensation scheme on the attas aircraft. *Aerospace Science and Technology*, 13(2-3):92–104, 2009.

[30] C. Cai and A.R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[31] C. Cai, A.R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems-Part I: Existence is equivalent to robustness. *IEEE Transactions on Automatic Control*, 52(7):1264–1277, 2007.

[32] C. Cai, A.R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems Part II:(pre) asymptotically stable compact sets. *IEEE Transactions on Automatic Control*, 53(3):734–748, 2008.

[33] D. Carnevale, A.R. Teel, and D. Nešić. Further results on stability of networked control systems: a Lyapunov approach. In *American Control Conference, 2007. ACC '07*, pages 1741 –1746, July 2007.

[34] J. Carrasco, A. Banos, and A. Van der Schaft. A passivity-based approach to reset control systems stability. *Systems & Control Letters*, 59(1):18–24, 2010.

[35] A. Cervin and T. Henningsson. Scheduling of event-triggered controllers on a shared network. In *47th IEEE Conference on Decision and Control, 2008. CDC 2008.*, pages 3601–3606, December 2008.

[36] A. Chutinan and B. H. Krogh. Verification of polyhedral-invariant hybrid automata using polygonal flow pipe approximations. In *HSCC '99: Proceedings of the Second International Workshop on Hybrid Systems*, pages 76–90, London, UK. Springer-Verlag.

[37] A. Chutinan and B.H. Krogh. Verification of infinite-state dynamic systems using approximate quotient transition systems. *IEEE Transactions on Automatic Control*, 46(9):1401 –1410, September 2001.

[38] A. Chutinan and B.H. Krogh. Computational techniques for hybrid system verification. *IEEE Transactions on Automatic Control*, (1):64–75, January 2003.

[39] E. M. Clarke, O. Grumberg, and D. A. Peled. *Model Checking*. The MIT Press, 2000.

[40] F.H. Clarke. *Optimization and Nonsmooth Analysis*. Society for Industrial an Applied Mathematics, 1990.

[41] J.C. Clegg. A nonlinear integrator for servomechanisms. *Trans. A. I. E. E.*, 77 (Part II):41–42, 1958.

[42] P. Collins. A trajectory-space approach to hybrid systems. In *International Symposium on the Mathematical Theory of Networks and Systems*, Leuven, Belgium, 2004.

[43] J.M. Gomes da Silva Jr and S. Tarbouriech. Anti-windup design with guaranteed regions of stability: an LMI-based approach. *IEEE Trans. Aut. Cont.*, 50(1):106–111, 2005.

[44] J.M. Davoren and A. Nerode. Logics for hybrid systems. *Proceedings of the IEEE*, 88(7):985–1010, July 2000.

[45] R.A. Decarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, 88(7):1069 –1082, July 2000.

[46] Schaft A.J. van der and J. M. Schumacher. *Introduction to Hybrid Dynamical Systems.* Springer-Verlag, London, UK, 1999.

[47] A. D'Innocenzo, A.A. Julius, M.D. Di Benedetto, and G.J. Pappas. Approximate timed abstractions of hybrid automata. In *46th IEEE Conference on Decision and Control, 2007*, pages 4045 –4050, December 2007.

[48] A. D'Innocenzo, A.A. Julius, G.J. Pappas, M.D. Di Benedetto, and S. Di Gennaro. Verification of temporal properties on hybrid automata by simulation relations. In *46th IEEE Conference on Decision and Control, 2007*, pages 4039 –4044, December 2007.

[49] D.S. Bernstein F. Tyan. Global stabilization of systems containing a double integrator using a saturated linear controller. *International Journal of Robust and Nonlinear Control*, 9(15):1143–56, 1999.

[50] H. Fang and Z. Lin. Global practical stabilization of planar linear systems in the presence of actuator saturation and input additive disturbance. *IEEE Transactions on Automatic Control*, 51(7):1177–1184, 2006.

[51] I. Fantoni, R. Lozano, and M.W. Spong. Energy based control of the pendubot. *Automatic Control, IEEE Transactions on*, 45(4):725–729, Apr 2000.

[52] J.Y. Favez, Ph. Mullhaupt, B. Srinivasan, and D. Bonvin. Attraction region of planar linear systems with one unstable pole and saturated feedback. *Journal of Dynamical and Control Systems*, 12(3):331–355, 2006.

[53] F. Forni, S. Galeani, and L. Zaccarian. Globally stabilizing quasi time-optimal control of planar saturated linear systems. In *16th IEEE Mediterranean Conference on Control and Automation*, pages 883–888, Ajaccio, Corsica (France), June 2008.

[54] G. Frehse. Phaver: algorithmic verification of hybrid systems past hytech. *Int. J. Softw. Tools Technol. Transf.*, 10(3):263–279, 2008.

[55] A.T. Fuller. In-the-large stability of relay and saturating control systems with linear controllers. *Int. J. Contr.*, 10:457–480, 1969.

[56] P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali. *LMI Control Toolbox.* The MathWorks Inc., 1995.

[57] S. Galeani, S. Onori, A.R. Teel, and L. Zaccarian. A magnitude and rate saturation model and its use in the solution of a static anti-windup problem. *Systems and Control Letters*, 57(1):1–9, 2008.

[58] S. Galeani, S. Tarbouriech, M.C. Turner, and L. Zaccarian. A tutorial on modern anti-windup design. *European Journal of Control*, 15(3-4):418–440, 2009.

[59] S. Galeani, A.R. Teel, and L. Zaccarian. Constructive nonlinear anti-windup design for exponentially unstable linear plants. *Systems and Control Letters*, 56(5):357–365, 2007.

[60] C.E. George and R.T. O'Brien Jr. Vehicle lateral control using a double integrator control strategy. In *Proceedings of the Thirty-Sixth Southeastern Symposium on System Theory*, pages 398–400, 2004.

[61] R. Goebel, J. Hespanha, A.R. Teel, C. Cai, and R.G. Sanfelice. Hybrid systems: Generalized solutions and robust stability. In *NOLCOS*, pages 1–12, Stuttgart, Germany, 2004.

[62] R. Goebel, R. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *Control Systems Magazine, IEEE*, 29(2):28–93, April 2009.

[63] R. Goebel and A.R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573 – 587, 2006.

[64] R. Goebel and A.R. Teel. Preasymptotic stability and homogeneous approximations of hybrid dynamical systems. *SIAM Review*, 52(1):87–109, 2010.

[65] Jr. Gomes da Silva, J.M., S. Tarbouriech, and G. Garcia. Local stabilization of linear systems under amplitude and rate saturating actuators. *IEEE Trans. Aut. Cont.*, 48(5):842–847, May 2003.

[66] H. Guéguen and J. Zaytoon. On the formal verification of hybrid systems. *Control Engineering Practice*, 12(10):1253–1267, 2004. Analysis and Design of Hybrid Systems.

[67] W. Hahn. *Stability of Motion*. Springer-Verlag, 1967.

[68] T.A. Henzinger. The theory of hybrid automata. *Symposium on Logic in Computer Science*, 0:278, 1996.

[69] T.A. Henzinger and P. Ho. Model checking strategies for linear hybrid systems. Technical report, Ithaca, NY, USA, 1994.

[70] T.A. Henzinger, P. Ho, and H. Wong-Toi. Algorithmic analysis of nonlinear hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):540–554, April 1998.

[71] T.A. Henzinger, P. W. Kopke, A. Puri, and P. Varaiya. What's decidable about hybrid automata? *J. Comput. Syst. Sci.*, 57(1):94–124, 1998.

[72] T.A. Henzinger, Z. Manna, and A. Pnueli. Temporal proof methodologies for real-time systems. In *In Proceedings of the 18th Annual Symposium on Principles of Programming Languages*, pages 353–366. ACM Press, 1991.

[73] T.A. Henzinger, X. Nicollin, J. Sifakis, and S. Yovine. Symbolic model checking for real-time systems. *Inf. Comput.*, 111(2):193–244, 1994.

[74] T.A. Henzinger and V. Rusu. Reachability verification for hybrid automata. In *HSCC 98: Hybrid Systems: Computation and Control, Lecture Notes in Computer Science 1386*, pages 190–204. Springer-Verlag, 1998.

[75] JP Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162, 2007.

[76] I. Horowitz and P. Rosenbaum. Non-linear design for cost of feedback reduction in systems with large parameter uncertainty. *Int. J. Contr*, 21:977–1001, 1975.

[77] T. Hu, Z. Lin, and B.M. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38(2):351–359, 2002.

[78] T. Hu, A.R. Teel, and L. Zaccarian. Anti-windup synthesis for linear control systems with input saturation: achieving regional, nonlinear performance. *Automatica (B)*, 44(2):512–519, 2008.

[79] F. Jabbari and İ.E. Kosë. Rate and magnitude-bounded actuators: scheduled output feedback design. *Int. J. Robust and Nonlinear Control*, 14:1169–1184, 2004.

[80] F. Jabbari and T. Nguyen. Output feedback controllers for disturbance attenuation with actuator amplitude and rate saturation. *Automatica*, 36:1339–1346, 2000.

[81] J.D. Ullman J.E. Hopcroft, R. Motwani. *Introduction to Automata Theory, Languages and Computation.* Addison-Wesley, Second edition, 1979.

[82] W.X. Jing and C.R. McInnes. On the l_p-stabilization of the double integrator subject to input saturation. *ESAIM Control, Optimisation and Calculus of Variations*, 6:291–331, 2001.

[83] K.H. Johansson, J. Lygeros, S. Sastry, and M. Egerstedt. Simulation of Zeno hybrid automata. In *Conference on Decision and Control*, pages 3538–3543, Phoenix, Arizona, 1999.

[84] M. Johansson and A. Rantzer. Computation of piecewise quadratic lyapunov functions for hybrid systems. *IEEE Trans. Aut. Cont.*, 43(4):555–559, April 1998.

[85] C.Y. Kao. On the L_2-gain of a double integrator with feedback loop saturation nonlinearity. *IEEE Transactions on Automatic Control*, 46(3):501–504, 2001.

[86] V. Kapila and W.M. Haddad. Fixed structure controller design for systems with actuator amplitude and rate non-linearities. *Int. J. Contr.*, 73:520–530, 2000.

[87] H.K. Khalil. *Nonlinear Systems.* Prentice Hall, USA, 3rd edition, 2002.

[88] Michal Kocvara and Michael Stingl. PENNON - a code for convex nonlinear and semidefinite programming. *Optimization Methods and Software*, 8(3):317–333, 2003.

[89] M.V. Kothare, P.J. Campo, M. Morari, and N. Nett. A unified framework for the study of anti-windup designs. *Automatica*, 30(12):1869–1883, 1994.

[90] J. Kurzweil. On the inversion of Ljapunov's second theorem on stability of motion. *American Mathematical Society Translations, ser. 2*, 24:19–77, 1956.

[91] G. Lafferriere, G. J. Pappas, and S. Yovine. Symbolic reachability computation for families of linear vector fields. *Journal of Symbolic Computation*, 32(3):231–253, 2001.

[92] G. Lafferriere, G.J. Pappas, and S. Shankar Sastry. O-minimal hybrid systems. Technical Report UCB/ERL M98/29, EECS Department, University of California, Berkeley, 1998.

[93] G. Lafferriere, G.J. Pappas, and S. Yovine. A new class of decidable hybrid systems. *Lecture Notes in Computer Science*, 1569:137–151, 1999.

[94] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, 2003.

[95] D. Liberzon and D. Nešić. Input-to-state stabilization of linear systems with quantized state measurements. *IEEE Transactions on Automatic Control*, 52(5):767 –781, May 2007.

[96] Z. Lin, A.A. Stoorvogel, and A. Saberi. Output regulation for linear systems subject to input saturation. *Automatica*, 32(1):29 – 47, 1996.

[97] J. Lofberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the International Symposium on Computer Aided Control System Design*, Taipei, Taiwan, 2004.

[98] J. Lofberg. Pre- and post-processing sum-of-squares programs in practice. *IEEE Transactions on Automatic Control*, 54(5):1007–1011, May 2009.

[99] J. Lygeros, K.H. Johansson, S.N. Simic, J. Zhang, and S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48:2–17, 2003.

[100] Oded Maler, Zohar Manna, and Amir Pnueli. From timed to hybrid systems. In *In Real-Time: Theory in Practice, REX Workshop, LNCS 600*, pages 447–484. Springer-Verlag, 1992.

[101] N. Marchand and A. Hably. Global stabilization of multiple integrators with bounded controls. *Automatica*, 41(12):2147–2152, 2005.

[102] M. Mazo and P. Tabuada. On event-triggered and self-triggered control over sensor/actuator networks. In *47th IEEE Conference on Decision and Control, 2008. CDC 2008.*, pages 435–440, December 2008.

[103] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Review*, 45(1):3–49, 2003.

[104] F. Morabito, S. Nicosia, A.R. Teel, and L. Zaccarian. Measuring and improving performance in anti-windup laws for robot manipulators. In B. Siciliano, A. De Luca, C. Melchiorri, and G. Casalino, editors, *Advances in Control of Articulated and Mobile Robots*, chapter 3, pages 61–85. Springer Tracts in Advanced Robotics, 2004.

[105] F. Morabito, A.R. Teel, and L. Zaccarian. Nonlinear anti-windup applied to Euler-Lagrange systems. *IEEE Trans. Rob. Aut. (A)*, 20(3):526–537, 2004.

[106] D. Nešić and A.R.Teel. Input output stability properties of networked control systems. *IEEE Trans. Automat. Contr.*, 49:1650–1667, 2004.

[107] D. Nešić and D. Liberzon. A small-gain approach to stability analysis of hybrid systems. In *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05.*, pages 5409 – 5414, December 2005.

[108] D. Nešić and A.R. Teel. A Lyapunov-based small-gain theorem for hybrid iss systems. In *47th IEEE Conference on Decision and Control, 2008*, pages 3380 –3385, December 2008.

[109] D. Nešić, L. Zaccarian, and A.R. Teel. Stability properties of reset systems. *Automatica (B)*, 44(8):2019–2026, 2008.

[110] X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. An approach to the description and analysis of hybrid systems. In *Hybrid Systems*, pages 149–178, London, UK, 1993. Springer-Verlag.

[111] A. Papachristodoulou and S. Prajna. On the construction of Lyapunov functions using the sum of squares decomposition. In *Proceedings of the 41st IEEE Conference on Decision and Control, 2002*, volume 3, pages 3482–3487 vol.3, Dec. 2002.

[112] A. Papachristodoulou and S. Prajna. Robust stability analysis of nonlinear hybrid systems. *IEEE Transactions on Automatic Control*, 54(5):1035–1041, May 2009.

[113] P.A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, 2000.

[114] A. Pettorossi. *Automata Theory and Formal Languages*. Aracne, second edition, 2009.

[115] A. Platzer. Differential dynamic logic for verifying parametric hybrid systems. In Nicola Olivetti, editor, *TABLEAUX*, volume 4548 of *LNCS*, pages 216–232. Springer, 2007.

[116] A. Platzer. Differential dynamic logic for hybrid systems. *Journal of Automated Reasoning*, 41(2):143–189, 2008.

[117] A. Platzer and E. M. Clarke. The image computation problem in hybrid systems model checking. In A. Bemporad, A. Bicchi, and G. C. Buttazzo, editors, *HSCC*, volume 4416 of *Lecture Notes in Computer Science*, pages 473–486. Springer, 2007.

[118] S. Prajna and A. Jadbabaie. Safety verification of hybrid systems using barrier certificates. In R. Alur and George J. Pappas, editors, *HSCC*, volume 2993 of *Lecture Notes in Computer Science*, pages 477–492. Springer, 2004.

[119] S. Prajna and A. Papachristodoulou. Analysis of switched and hybrid systems - beyond piecewise quadratic methods. In *Proceedings of the 2003 American Control Conference*, volume 4, pages 2779–2784 vol.4, June 2003.

[120] S. Prajna, A. Papachristodoulou, P. Seiler, and P.A Parrilo. SOSTOOLS and its control applications. In D. Henrion and A. Garulli, editors, *Positive Polynomials in Control*, volume 312, pages 273–292. Springer Verlag, 2005.

[121] I. Queinnec, S. Tarbouriech, and G. Garcia. Anti-windup design for aircraft flight control. In *Conference on Control Applications*, pages 2541–2546, October 2006.

[122] V.G. Rao and D.S. Bernstein. Naive control of the double integrator. *Control Systems Magazine*, 21(5):86–97, 2001.

[123] R.J.-B. Wets. M. Rockafellar, R.T. Wets. *Variational Analysis*. Springer, 2004.

[124] Lionel Rosier. Homogeneous lyapunov function for homogeneous continuous vector field. *Syst. Control Lett.*, 19(6):467–473, 1992.

[125] R.G. Sanfelice. *Robust Hybrid Control Systems*. PhD thesis, University of California, Santa Barbara, 2007.

[126] R.G. Sanfelice, R. Goebel, and A.R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12):2282–2297, 2007.

[127] R.G. Sanfelice, R. Goebel, and A.R. Teel. Generalized solutions to hybrid dynamical systems. *ESAIM: COCV*, 14(4):699–724, October 2008.

[128] R.G. Sanfelice and A.R. Teel. On hybrid controllers that induce input-to-state stability with respect to measurement noise. In *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05.*, pages 4891 – 4896, December 2005.

[129] R.G. Sanfelice and A.R. Teel. Dynamical properties of hybrid systems simulators. *Automatica*, 46(2):239 – 248, 2010.

[130] A.J. Schaft. *L2-Gain and Passivity in Nonlinear Control*. Springer-Verlag New York, Inc., Secaucus, N.J., USA, 1999.

[131] W.E. Schmitendorf and B.R. Barmish. Null controllability of linear systems with constrained controls. *SIAM J. Contr. Opt.*, 18(4):327–345, July 1980.

[132] E. Schuster, M.L. Walker, D.A. Humphreys, and M. Krstic. Plasma vertical stabilization with actuation constraints in the diii-d tokamak. *Automatica*, 41(7):1173 – 1179, 2005.

[133] G. Shi and A. Saberi. On the input-to-state stability (ISS) of a double integrator with saturated linear control laws. In *Proc. of the American Control Conference*, pages 59–61, 2002.

[134] CA. Shifrin. Sweden seeks cause of Gripen crash. *Aviation Week and Space Technology*, 139:78–79, 1993.

[135] B.I. Silva, O. Stursberg, B.H. Krogh, and S. Engell. An assessment of the current status of algorithmic approaches to the verification of hybrid systems. In *Proceedings of the 40th IEEE Conference on Decision and Control, 2001.*, volume 3, pages 2867 –2874 vol.3, 2001.

[136] E.D. Sontag. An algebraic approach to bounded controllability of linear systems. *Int. Journal of Control*, 39(1):181–188, 1984.

[137] E.D. Sontag. Feedback stabilization of nonlinear system. In *Robust control of linear systems and nonlinear control*, volume 4 of *Progr. Systems Control Theory*, pages 61–81. Birkhäuser Boston, MA, 1989.

[138] J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999.

[139] R. Suarez, J. Alvarez-Ramirez, and B. Source. First harmonic analysis of planar linear systems with single saturated feedback. *International Journal of Bifurcation and Chaos*, 6(12):2605–10, 1996.

[140] H. J. Sussmann. The structure of time-optimal trajectories for single-input systems in the plane: The general real analytic case. *SIAM Journal on Control and Optimization*, 25(4):868–904, 1987.

[141] Z. Lin T. Hu, L. Ma. Stabilization of switched systems via composite quadratic functions. *IEEE Trans. Aut. Cont.*, 53(11):2571–2585, December 2008.

[142] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, September 2007.

[143] W. Tan. *Nonlinear Control Analysis and Synthesis using Sum-of-Squares Programming.* PhD thesis, University of California, Berkeley, 2006.

[144] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.

[145] A.R. Teel. Anti-windup for exponentially unstable linear systems. *Int. J. Robust and Nonlinear Control*, 9:701–716, 1999.

[146] A.R. Teel and J.B. Buffington. Anti-windup for an F-16's daisy chain control allocator. In *Proc. of AIAA GNC Conference*, pages 748–754, New Orleans (LA), USA, August 1997.

[147] A.R. Teel and N. Kapoor. The $\mathcal{L}_2$ anti-windup problem: Its definition and solution. In *Proc. 4th ECC*, Brussels, Belgium, July 1997.

[148] A.R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM: Control, Optim. Calc. of Variations*, 5:313–367, 2000.

[149] A.R. Teel and L. Zaccarian. On "uniformity" in definitions of global asymptotic stability for time-varying nonlinear systems. *Automatica*, 42(12):2219–2222, 2006.

[150] A. Tiwari. Abstractions for hybrid systems. *Form. Methods Syst. Des.*, 32(1):57–83, 2008.

[151] S.E. Tuna. *Generalized Dilations and Homogeneity.* PhD thesis, University of California, Santa Barbara, 2005.

[152] S.E. Tuna and A.R. Teel. Generalized dilations and numerically solving discrete-time homogeneous optimization problems. In *43rd IEEE Conference on Decision and Control, 2004.*, volume 5, pages 5403 – 5408 Vol.5, December 2004.

[153] S.E. Tuna and A.R. Teel. Homogeneous hybrid systems and a converse Lyapunov theorem. In *45th IEEE Conference on Decision and Control, 2006*, pages 6235–6240, December 2006.

[154] M. Vidyasagar. *Nonlinear Systems Analysis.* Prentice-Hall, Englewood Cliffs, New Jersey, 2nd edition, 1993.

[155] X. Wang and M.D. Lemmon. Event design in event-triggered feedback control systems. In *47th IEEE Conference on Decision and Control, 2008. CDC 2008.*, pages 2105–2110, December 2008.

[156] X. Wang and M.D. Lemmon. Event-triggered broadcasting across distributed networked control systems. In *American Control Conference, 2008*, pages 3139–3144, june 2008.

[157] TC Yang. Networked control system: a brief survey. *IEE Proceedings-Control Theory and Applications*, 153(4):403–412, 2006.

[158] L. Zaccarian. *Uniting local and global controllers for anti-windup synthesis.* PhD thesis, Università di Roma Tor Vergata, 2000.

[159] L. Zaccarian. On dynamic control allocation for input-redundant control systems. In *Conference on Decision and Control*, pages 1192–1197, New Orleans (LA), USA, December 2007.

[160] L. Zaccarian. Dynamic allocation for input-redundant control systems. *Automatica*, 45:1431–1438, 2009.

[161] L. Zaccarian, D. Nešić, and A.R. Teel. First order reset elements and the Clegg integrator revisited. In *American Control Conference*, pages 563–568, Portland (OR), USA, June 2005.

[162] L. Zaccarian and A.R. Teel. A common framework for anti-windup, bumpless transfer and reliable designs. *Automatica (B)*, 38(10):1735–1744, 2002.

[163] L. Zaccarian and A.R. Teel. Nonlinear scheduled anti-windup design for linear systems. *IEEE Trans. Aut. Cont. (B)*, 49(11):2055–2061, 2004.