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A local structure result for line graphs of strong quasi-graphical matrices

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Abstract

Let A be a binary matrix which generates a regular matroid with no $M^*(K_{3,3})$ minor and let $L(A)$ be the line graph of A, namely, the graph whose vertices correspond to the columns of A and where there is an edge between two vertices if the supports of the corresponding columns intersect. In this paper we prove that every vertex of $L(A)$ has at most four neighbors on every hole of $L(A)$. This result generalizes a local structure result proved by Golumbic and Jamison [Golumbic and Jamison, J. Comb. Theory Ser. B 38, 1985] for Edge-Path-Tree graphs, that is line graphs of network matrices reduced modulo 2.

Keywords: Line graphs, regular matrices, wheels.

1 Introduction

An *Edge-Path-Tree* (EPT) matrix is a binary matrix whose columns are incidence vectors of edge-sets of paths in a given tree referred to as the *underlying tree* of the matrix. EPT matrices are fundamental combinatorial objects. After a deep result of Seymour [17], EPT matrices are (essentially) the building blocks of regular matrices that is those binary matrices that can be *signed* to become *Totally Unimodular*. Recall that a *signing* of a matrix A consists of multiplying some of the entries of A by -1. The image of the signing is thus a $\{-1,0,1\}$ matrix. A $\{-1,0,1\}$ matrix is *totally unimodular* if the absolute value of the determinant of each of its nonsingular submatrices is 1. EPT matrices have a rich combinatorial structure and can be tough of as the unsigned patterns (i.e., reduced modulo 2) of the well-known *network matrices*. A $\{-1,0,1\}^{M\times N}$ matrix is a *network matrix* if there exists a connected directed graph $D=(V,M\cup N)$ admitting a spanning tree $T=(V,M)$ such that if P_j denotes the unique path connecting the endpoints of edge j in T (for $j\in N$) then

$$a_{i,j}=\begin{cases}-1 & \text{if arc } i \text{ occurs in } P_j \text{ in the } \textit{forward} \text{ direction}\\ 0 & \text{if arc } i \text{ does not occur in } P_j\\ 1 & \text{if arc } i \text{ occurs in } P_j \text{ in the } \textit{backward} \text{ direction}\end{cases}$$

the *forward direction in* P_j being the direction of j in the cycle obtained by adding j to P_j. EPT matrices became even more popular after the seminal paper [8] on *Edge-Path-Tree* (EPT)

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graphs, defined as the *line graphs* of EPT matrices. The *line graph* of matrix $A \in \{0,1\}^{M\times N}$, M and N being finite, is the graph $L(A)$ with vertex set N and where j, $j' \in N$ are adjacent if $a_{i,j} = a_{i,j'} = 1$, for some $i \in M$ (i.e., the supports of columns j and j' intersect).

In [8] Golumbic and Jamison obtained a number of results for the class of EPT graphs. Among them they proved that EPT graphs have polynomially many cliques and gave a characterization of the holes of an EPT graph in terms of certain configurations called *pies* which we now briefly describe.

Let $G = L(A)$ be an EPT graph for some EPT matrix $A \in \{0,1\}^{M\times N}$ underlain by a tree T. A *GJ-pie*—the prefix GJ stands for Golumbic and Jamison–in T with respect to a set $J \subseteq N$ of k elements is a star subgraph of T with k edges $v_0v_1 \dots, v_0v_k$, such that, for $j = 1, \dots, k$, each *slice* $\{v_0v_j, v_0v_{j+1}\}$ is contained in a different member of $\{\widetilde{A^j} \mid j \in J\}$ where $\widetilde{A^j}$ is the *support* of column A^j of A and addition over indices is modulo k. Recall that the *support* of a vector $u \in \{0,1\}^M$ is the set $\widetilde{u} = \{i \in M \mid a_{i,j} = 1\} \subseteq M$ and that a *hole* in a graph is a chordless cycle of length at least four.

By relying on the notion of pie, Golumbic and Jamison proved:

Theorem 1 *An EPT graph contains a hole if and only if it contains a pie*

Corollary 1 *Let C be a hole in an EPT graph G and let $v \in V(G) \setminus V(C)$. Then v has at most four neighbors on C.*

Corollary 2 *An EPT graph does not contain $\bar{C}_k$, $k \geq 7$ as induced subgraph.*

The main aim of this paper is to extend the above results to proper superclasses of EPT graphs. After having defined a slightly more general notion of pie we show that Theorem 1 holds in the class of column-odd-wheel-matrices. The reader is referred to the next section for the definition of the various classes of matrices occurring in the paper. A simple applications of the result also clarifies the containment relationships among subclasses of column-odd-wheel-free perfect matrices. The result is then specialized (Section 4) to a sharper superclass of EPT graphs, namely, line graphs of strong quasi-graphical-families, within which the above two corollaries hold as well. The proof of the latter result contains ideas from [3].

1.1 Regular and related matrices

The *bipartite graph* of a matrix $A \in \{0,1\}^{M\times N}$, is the bipartite graph $B(A)$ whose color classes are M and N and where two vertices i and j are adjacent if $a_{i,j} = 1$. Conversely, with every bipartite graph B with color classes M and N one can associate the binary matrix $A(B)$ whose rows and columns are indexed by M and N, respectively, and where $a_{i,j} = 1$, $i \in M$, $j \in N$, if i and j are adjacent in B. In a graph, a *hole* in a induced cycle of length at least four. In bipartite graphs holes have even length thus we call a hole *even* if its length is divisible by four and *odd* otherwise. A *hole submatrix* of A is a square submatrix A' of order $k \geq 3$ such that $B(A')$ is a hole in $B(A)$. The length of the hole is twice the order of A'. If the order of A is six then A' is said to be a *triangle submatrix* of A'.

Two binary matrices are $GF(2)$-*equivalent* if one arises from the other by a sequence of the following operations: permuting rows, permuting columns and $GF(2)$-*pivoting* on a nonzero entry.

Recall that *pivoting* A over $GF(2)$ on a nonzero entry (the *pivot element*) means replacing

$$A = \begin{pmatrix} 1 & a \\ b & D \end{pmatrix} \quad \text{by} \quad \widetilde{A} = \begin{pmatrix} 1 & a \\ b & D + ba \end{pmatrix}$$

where the rows and columns of A have been permutated so that the pivot element is $a_{1,1}$ ([6], p. 69, [15], p. 280). The operation of $GF(2)$-pivoting a matrix $A \in \{0,1\}^{M \times N}$ can be described in terms of the bipartite graph $B(A)$ as follows: if A' is the result of $GF(2)$-pivoting on a nonzero entry $a_{i,j}$ of A, then $B(A')$ results from $B(A)$ by complementing the edges between $N(i) \setminus \{j\}$ and $N(j) \setminus \{i\}$, where, for a vertex h of $B(A)$, $N(h)$ denotes the set of neighbors of h (see [5, 6]).

The following bipartite graphs (and the associated matrices) play an important role throughout the rest of the paper:

—***wheels***: A *wheel* is a graph (C, l) consisting of a hole C (the *rim*) and a vertex $l \notin V(C)$ (the *center*) that has at least three neighbors on C; each edge of the wheel incident to l is called a *spoke*. A wheel *odd* if it has an odd number of spokes; otherwise it is *even*. In this paper we are mostly concerned with wheels that are bipartite graphs and, unless otherwise stated, when we say wheel we mean a bipartite wheel. For any such wheel the rim has always an even number of vertices and we say that it is *complete* if it has $|C|/2$ spokes, $|C|$ being the order of C. A k-wheel is a wheel with k-spokes. A k-wheel whose center is in the color class N is denoted by $W_k(N)$. Analogously, $W_k(M)$ will denote a k-wheel whose center is in M, while W_k will always denote a k-wheel regardless of the color class its center belongs to.

—***3-path-configurations***: an *uv-3-path configuration* (3PC(u,v)) is a bipartite graph consisting of three internally vertex-disjoint uv-paths P_1, P_2 and P_3 such that $V(P_i) \cup V(P_j)$, $i \neq j$, induces a chordless cycle and u and v are not adjacent. A 3-*path configuration* (3PC) is a 3PC(u,v) for some u and v. Since 3PC(u,v) is a bipartite graph, the length of each of the three uv-paths is odd or even accordingly to whether u and v belong to different color classes or to the same color class, respectively. In the former case each path has length at least three and the 3PC is said to be *odd*. In the latter case, if each path has length at least four, we say that the 3PC is *even*[1]. For even 3PC's we write 3PC(M) and 3PC(N) for those even 3PC(u,v) with $u, v \in M$ and $u, v \in N$, respectively.

The two special matrices in (1) have odd wheels with three spokes as their bipartite representation. They are the *Fano* and the *dual Fano* matrix, respectively.

$$F_7 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \qquad F_7^* = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \tag{1}$$

A graph G is H-free if G does not contain any induced subgraph isomorphic to H. A matrix $A \in \{0,1\}^{M \times N}$ is *column-odd-wheel-free* if $B(A)$ does not contain any $W_k(N)$ as induced subgraph for k odd. It is *row-odd-wheel-free* if $B(A)$ does not contain any $W_k(M)$ as induced subgraph for k odd; it is *odd-wheel-free* if it does not contain any odd wheel as induced subgraph. Throughout the rest of the paper we use the following concrete bicoloring for the graph $B(A)$ of a matrix $A \in \{0,1\}^{M \times N}$: vertices in M are represented by empty circles while those in N by solid circles. The class of *column-odd-wheel-free* matrices contains several interesting classes of matrices which

[1]We stress here that if H is a 3PC(u,v), with u and v belonging to the same color class, but u and v are linked by a path of length two, then H must not be considered an even wheel.

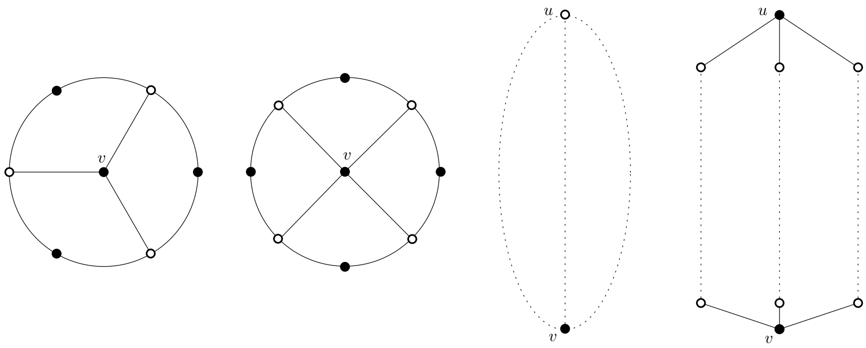

Figure 1: W_3, complete $W_4(N)$, odd 3PC of the form 3PC(u, v) and even 3PC(N) of the form 3PC(u, v). Solid lines represent edges and dotted lines represent paths.

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we are going to list. To this end recall first a $\{-1, 0, 1\}$ matrix A is *balanced* if the sum of the entries of each *hole* submatrix is divisible by 4. This notion, due to Truemper [18], generalizes Berge's notion of binary balanced matrix (see e.g [4]). It is well known that totally unimodular matrices are balanced.

The following classes are proper subclasses of the class of column-odd-wheel-free matrices:

- *balanceable matrices*, i.e., those matrices admitting a balanced signing; these matrices have a $W_{2h+1}(N)$-free bipartite representation, $h \in \mathbb{N}$, by a result of Truemper [18] asserting that a binary matrix A is balanceable if and only if $B(A)$ contains neither odd-wheels nor odd 3PC's as induced subgraphs;
- *regular matrices*, i.e., those matrices admitting an unimodular signing; these matrices have a $W_{2h+1}(N)$-free bipartite representation, $h \in \mathbb{N}$, because of Tutte's characterization of regular matrices (see e.g., [15]) asserting that a matrix is regular if and only if it cannot be transformed into the matrix F_7^* in (1) by permuting rows and columns, taking submatrices, taking transpose and pivoting over $GF(2)$; as matrices of the form $A(W_{2h+1}(N))$ can be $GF(2)$-pivoted into matrices containing F_7^* as submatrix [5] (see also Lemma 2) and since $B(F_7^*) \cong W_3(N)$ (and $B(F_7) \cong W_3(M)$), it follows that regular matrices are odd-wheel-free for k odd. In particular, regular matrices are balanceable [5];
- EPT matrices; these matrices are regular because they can be signed to become network matrices and such matrices are totally unimodular matrices (see e.g., [16] Vol. A, p. 213).

There is another interesting class of matrices which is properly sandwiched between the class of EPT matrices and the class of regular matrices. Such a class, which is our main concern, is defined as the class of those regular matrices that cannot be transformed into the following matrix

$$H_{3,3} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

by permuting rows and columns, taking submatrices and pivoting over $GF(2)$. Any such matrix will be referred to as *strongly-quasi-graphical*. It can be seen [2, 3] that A is strongly quasi-graphical if and only if matrix $[I_m|A]$, I_m being the identity matrix of order m, generates a binary regular matroid with no $M^*(K_{3,3})$ minor (see Sections 3.3, 4 and the references cited therein for some background). Moreover, strongly quasi-graphical matrices form a proper subclass of the class of *quasi-graphical* matrices defined as those balanceable matrices whose bipartite representation contains neither $W_k(N)$ for $k \geq 4$, k even, nor even 3PC(N) [3]. This explains why strongly graphical families are so called: quasi-graphical because they generalize EPT matrices, namely, graphical objects, strongly because they are more than quasi-graphical in that the class is closed under $GF(2)$-pivoting.

2 Holes in column-odd-wheel-free matrices

Our results strongly rely upon an appropriate extension of the notion of *pie* introduced in [8] for EPT graphs. *Pies* can be viewed natural generalization to families of sets of the notion of circuit in a graph. For $j \in N$, let A^j denote the j-th column of $A \in \{0,1\}^{M \times N}$. For $J \subseteq N$, A^J is the submatrix of A obtained by deleting the column whose index is not in J. A *pie* in A is a submatrix of the form A^J for some $J \subseteq N$, $|J| = k \geq 3$, such that

(a) for some permutation $(j_1, \ldots, j_k)$ of J one has $\widetilde{A^{j_l}} \cap \widetilde{A^{j_{l+1}}} \neq \emptyset$ and $\widetilde{A^{j_h}} \cap \widetilde{A^{j_l}} = \emptyset$ if $|l-h| \notin \{1, k-1\}$, (addition over indices is modulo k);

(b) if $k = 3$ then $\cap_{j \in J} \widetilde{A^j} = \emptyset$.

Two columns (members) A^h and A^l ($\widetilde{A^h}$ and $\widetilde{A^l}$) of a pie are *consecutive* if $\widetilde{A^h} \cap \widetilde{A^l} \neq \emptyset$. The number k is the *size* of the pie; The pie is *odd* if k is odd and *even* otherwise. Remark that if A^J is a pie in A then each $i \in \cup_J \widetilde{A^j}$ occurs in at most two members of the pie. Also notice that if A contains a pie then $B(A)$ contains a hole. On the other hand $B(A)$ might contain hole while A does not contain any pie: for instance F_7^* is such. A set is called *odd* if it has odd cardinality.

Theorem 2 *Let A be a column-odd-wheel-free matrix and let C be an odd hole in $B(A)$. Then, for some odd $K \subseteq V(C) \cap J$, A^K in an odd-pie in A.*

Proof. Let C be a minimum size counterexample with $V(C) = I \cup J$, $I \subseteq M$, $J \subseteq N$. Possibly after renumbering, we may suppose that $J = \{1, 2, \ldots, k\}$ and that A^h and A^j are consecutive if and only if $|h - j| \in \{1, k-1\}$. Since A^J is not a pie in A there is some $i \in M$ such that the corresponding row intersects more than two columns. Thus $J^* = \{j \in J \mid i \in \widetilde{A^j}\}$ is nonempty and contains at least three indices. Therefore, $|J^*| \geq 3$ and $J^* = J_1 \cup \ldots \cup J_t$, for some disjoint intervals of J^*. Without loss of generality $1 \in J_1$ and $k \notin J_t$. Let $K_1, \ldots K_t$, be the intervals factorizing $J \setminus J^*$. We claim that K_l is odd for at least one $l \in \{1, \ldots, t\}$. Suppose that all of the K_l's are even. Thus $|J^*|$ as the same parity as k and hence it is odd, k being odd. Therefore i has an odd number of neighbors on C and $V(C) \cup \{i\}$ induces an odd wheel in $B(A)$. A contradiction. Consequently K_l is odd for some $l \in \{1, \ldots, t\}$. Let E_l be the set of the neighbors of the vertices in K_l on C, and let i' and i'' be the two vertices of E_l having exactly one neighbor in K_l. Moreover, Let j' and j'' be the unique neighbors of i' and i'', respectively, outside K_l. Thus $E_l \cup K_l \cup \{j', j''\}$ induces a subpath P of C with $j', j'' \in J^*$. The neighbors of i on P are j' and j''. Hence $V(P) \cup \{i\}$

induces an odd hole C' in $B(A)$ of size $2(|K_l|+2) \leq 2(|J \setminus J^*|+2) < 2k$, ($|J^*|$ being odd and greater than 1). Since $K_l = V(C') \cap N$ it follows that $A^{J'}$ is not a pie for any odd $J' \subseteq K_l$ (because C is a counterexample). Hence C' is still a counterexample contradicting the minimality of C. □

Let A' be a hole submatrix of $A \in \{0,1\}^{M \times N}$ and let $J' \subseteq N$ be the index set of the columns of A'. We say that A' is *minimal* if $A^{J'}$ contains no hole matrix of smaller order.

Remark 1 *Let A be a column-odd-wheel-free matrix and let $C = B(A')$ be the bipartite graph of a minimal hole submatrix and let J' be the index set of the columns of A'. By the proof of Theorem 2, either $A^{J \cap V(C)}$ is a pie or $B(A^{J \cap V(C)})$ contains a complete even wheel as subgraph whose* rim *is C.*

Corollary 3 *Let A be a column-odd-wheel-free matrix and let $G = L(A)$. If H is a hole in G then $A^{V(H)}$ is a pie in A.*

Proof. If H is a hole in $L(A)$ then A contains a hole submatrix A'. Moreover, A' is minimal because H is chordless. Clearly the set of columns of A' is $V(H)$. Let $C = B(A')$. Thus $V(C) \cap N = V(H)$, N being the index set of the columns of A. In view of Remark 1, either $A^{V(H)}$ is a pie in A or, for some $i \in I$, $B(A^{V(H)})$ contains a complete even wheel (C, i). In the latter case since $i \in \cap_{j \in V(H)} \widetilde{A^j}$, H cannot be a hole in G, $V(H)$ being actually a clique in G. □

3 Some Application

In this section we exploit the preceding observations to give extensions a new proofs of known results for EPT matrices and EPT graphs.

3.1 Perfection of column-odd-wheel-free matrices

For $\mathbf{w} \in \mathbb{Z}_+^N$ let us associate with $A \in \{0,1\}^{M \times N}$ the following pair of dual linear programs:

$$\max\{\mathbf{wx} \mid A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\} = \min\{\mathbf{1y} \mid \mathbf{y}A \geq \mathbf{w}, \mathbf{y} \geq \mathbf{0}\} \tag{2}$$

The polytope $P(A)$ on the left hand side of (2) is referred to as the *fractional packing polyhedron* of A. A binary matrix A is *perfect* if its *fractional packing polyhedron* is integral, i.e., has integral vertices only. By a Theorem of Lovász (see e.g., [16]), A is perfect if and only the minimization problem on the right hand of (2) has an integral optimal solution for any choice of $w \in \mathbb{Z}_+^N$ (i.e, the defining system of $P(A)$ is *Totally Dual Integral*).

A $\{0,1\}$-matrix is *totally balanced* if A does not contain any hole matrix. Let **N**, **B**, **TB** and **U** denote the classes of perfect, balanced, totally balanced and totally unimodular binary matrices, respectively. It is well known (see e.g., [16]) that **U** ⊆ **B** ⊆ **N** and that **TB** ⊆ **B** ⊆ **N**. In general all inclusions are strict; moreover, **TB** and **U** are in general inclusionwise incomparable classes. Remark that the class of perfect matrices is denoted by **N** (rather than by **P**) because the hypergraph defined by identifying the columns of a perfect binary matrix with their supports is a *Normal* hypergraph (see e.g., [4], [16]).

Corollary 4 *Let A be a column-odd-wheel-free binary matrix. Then A is perfect if and only if it is balanced.*

Proof. The *if* part is trivial. Let us prove that a perfect column-odd-wheel-free matrix is balanced. No perfect column-odd-wheel-free matrix A can contain an odd pie A^J for some $J \subseteq N$ where, possibly after renumbering, A^h and A^j are consecutive if and only if $|h-j| \in \{1, k-1\}$. Indeed, by letting $x_j = 1/2$, for $j \in \{1, \ldots, k\}$ and $x_j = 0$ for $N \setminus J$, one defines a feasible fractional solution in the polyhedron of (2) whose value is $k/2$. By choosing $i_j \in \widetilde{A^j} \cap \widetilde{A^{j+1}}$, (addition over indices is modulo k) and by letting $y_{i_j} = 1/2$ if $j \in \{1, \ldots k\}$ and $y_i = 0$ for $i \in M - \{i_1, \ldots, i_k\}$, one defines a feasible fractional dual solution whose value is $k/2$. Hence A cannot be perfect, k being odd. Thus, in view of Theorem 2, if A is perfect column-odd-wheel-free matrix then A cannot contain odd hole submatrices, otherwise $B(A)$ would contain odd holes and thus A odd pies. Hence A is balanced and the *only if* part is established. □

Balancedeness is a self-dual property. Therefore as the transpose of balanced matrix is balanced as well and the transpose of a column-odd-wheel-free matrix is a row-odd-wheel-free matrix, Corollary 4 implies:

Corollary 5 *Let A be a row-odd-wheel-free binary matrix. Then A is perfect family if and only if it is balanced.*

Corollary 6 *Within the class of regular matrices one has* $\mathbf{N} = \mathbf{B} = \mathbf{U} \subseteq \mathbf{TB}$. *In particular a regular matrix is perfect if and only if it does not contain odd pies, while if it is totally balanced then it does not contain any pie.*

Proof. Every balanced signing of a regular matrix is totally unimodular ([5, 6]). In particular, if a regular matrix is balanced then it is totally unimodular. Hence the thesis follows by Corollary 4 and from the inclusion $\mathbf{TB} \subseteq \mathbf{B}$. □

Remark 2 *Observe that being pie-free is only a necessary condition for a matrix A being totally balanced. The complete bipartite wheel with four spokes correspond to a matrix whose transpose contains no pies. However it contains a hole submatrix.*

3.2 Hellyness

A family of subsets of a given ground set is *Helly* (or has the *Helly Property*) if pairwise intersecting members have nonempty intersection. Let $A \in \{0,1\}^{M \times N}$ be a binary matrix and $\widetilde{A} = (\widetilde{A^j} \,|\, j \in N)$ be the family of the supports of the columns of A (these are subsets of M). We say that A is Helly if so is $\widetilde{A}$. The transpose of A is denoted by A^* and the family $\widetilde{A^*}$ is called the dual family (of A). Matrices whose dual is Helly are called *conformal.* A matrix A is strong Helly (or has the strong Helly property) if every of its submatrices is Helly. Strong Helly matrices are characterized by the following result of Ryser [14] (see also [10, 13] for more results).

Theorem 3 (Ryser [14]) *A is strong Helly if and only if it does not contain any triangle submatrix.*

As a further corollary of Theorem 2 we collect below some straightforward consequences for the Helly Property (more results can be found in [2]). After Corollary 9 and subsequent Remark 4 (see Section 4) these imply hellyness of families of arc sets of directed paths in a directed tree as proved by Monma and Wey [11] and the conformality of their dual as proved by Gutierrez and Meidanis [9].

Corollary 7 *Let A be a column-odd-wheel-free matrix. Then the following statement are equivalent:*

(1) *A is Helly;*

(2) *A does not contain 3-pies;*

(3) *A is strong Helly.*

If in addition A is regular then

(4) *A is Helly if and only if so is A^*.*

Proof. ((1)⇒ (2)). 3-pies are non-Helly families. ((3)⇒ (1)). trivial. ((2)⇒ (3)) By Theorem 2 A does not contain any triangle submatrix hence the result follows by Theorem 3. If A is regular then both A and A^* are column-odd-wheel-free matrices. Hence (4) follows from the first part. □

Remark 3 *Matrix F_7 is a column-odd-wheel-free matrix free which is conformal (its transpose is F_7^* which is Helly) but not Helly.*

3.3 Holes in EPT graphs and Pies in matrices

We shall prove that notion of pie and GJ-pie are equivalent. To be consistent with the aim of the paper we give a "vertex-free" proof of this fact namely, a proof which uses only the graphic matroid associated with an EPT matrix (the reader is referred to [12, 19, 20] for the elementary background in matroid theory we need in this paper. We follow Section **7** and Section **8** of [6] and Chapter 6 of [20] which contain all what we need here). Recall that with every binary matrix $A \in \{0,1\}^{M\times N}$ with $m = |M|$ rows one can associate the binary matroid $M(A)$ generated on $M \cup N$ by the columns of $[I_m, A]$, I_m being the identity matrix of order m. Such a matroid is defined as the matroid whose circuits are the minimal supports of the vectors in the nullspace of $[I_m, A]$, $[I_m, A]$ being a viewed as a matrix over $GF(2)$. If A and A' are $GF(2)$-equivalent matrices then they generate *isomorphic* binary matroids and, conversely, if A and A' have the same order and $M(A)$ and $M(A')$ are *isomorphic* (written $M(A) \cong M(A')$) then A and A' are $GF(2)$-equivalent. *Isomorphism* between two binary matroids is meant as the isomorphism between the binary spaces generated by them. A graphic matroid is the binary matroid $M(A)$, where A is some EPT matrix [7]. Given a graph G and one of its spanning forest T, the EPT matrix of G with respect to T is the binary matrix $A_{G,T}$ whose columns are indexed by $N = E(G) - E(T)$ and for any $j \in E(G) - E(T)$, $A^j_{G,T}$ is incidence vector of the unique path in T connecting the endpoints of j. The cycle matroid of a graph G is the graphic matroid generated by $M(A_{G,T})$ for some spanning forest T of G. While isomorphic graphs (in the graphical theoretical sense) have isomorphic graphic matroids (in the matroid theoretical sense) the converse is not true in general. Whitney's 2-isomorphism Theorem describes when such a converse statement is true. To state it precisely we shall describe the operations of vertex identification and twisting. We follow chapter 6.1 of [20]. Recall first that if U is a subsets of vertices of a graph G the graph $G - U$ is the graph induced by $V(G) \setminus U$. If U is the singleton $\{u\}$ we write as customary $G - u$ in place of $G - \{u\}$. Let G_1 and G_2 be two disjoint graphs and let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. The graph G' obtained by identifying v_1 and v_2 as a new vertex $v \notin V(G_1) \cup V(G_2))$ is the graph with vertex set $(V(G_1) \setminus \{v_1\}) \cup (V(G_1) \setminus \{v_1\}) \cup \{v\}$ and edge set $E(G_1 - v_1) \cup E(G_2 - v_2) \cup \{uv \mid uv_1 \in$

$E(G_1)$ for some $u \in V(G_1)\} \cup \{uv \mid uv_2 \in E(G_2)$ for some $u \in V(G_2)\}$. A loopless 2-connected graph is *block*. Let G be a block and u, $v \in V(G)$ be such that $\{u, v\}$ is a vertex cut and let G_1 and G_2 be the connected components arising after the removal of u and v. The graph obtained by twisting at u and v is the graph defined as follows.

– Let G'_1 be the graph induced in G by $V(G_1) \cup \{u, v\}$, with vertex u and v renamed by u_1 and v_1, respectively;

– if u and v induce an edge of G let G'_2 be the graph induced in G by $V(G_1) \cup \{u, v\}$ with edge uv removed and u and v renamed by u_2 and v_2, respectively;

– if u and v are independent let G'_2 be the graph induced in G by $V(G_1) \cup \{u, v\}$ with u and v renamed by u_2 and v_2, respectively;

– finally let G' be the graph obtained by identifying u_1 with v_2 as a new vertex u and v_1 with u_2 as a new vertex v.

Theorem 4 (Whitney's 2-isomorphism Theorem, 1933) *If H and G are blocks and $M(H) \cong M(G)$ then H can be obtained from G by a succession of twisting.*

A graph G is *homeomorphic from* a graph H, written $G \prec H$, if either $H \cong G$ or G is isomorphic to a graph obtained from H by edge subdivisions. We also say that G is (isomorphic to) a *subdivision* of H. A graphic matroid M *uniquely determines* a graph G if M is isomorphic to the cycle matroid of G and any other graph whose cycle matroid is isomorphic to M is isomorphic to G. Throughout the rest of the section (but not longer) we reserve the term *wheel* to those non-bipartite wheels whose center is adjacent to every vertex of the rim.

Lemma 1 *Let H be a graph homeomorphic from a wheel. Then the cycle matroid $M(H)$ of H uniquely determines H (up to isomorphisms).*

Proof. Call each edge on the rim of a wheel a *tourniquet*. Since $H \prec W_k$ for some $k \in \mathbb{N}$, $k \geq 3$, each tourniquet as well as each spoke of W_k is replaced by a path of positive length (possibly one). We refer to such paths as the *tourniquets* and the *spokes* of H. Now H is 2-connected but is not 3-connected (unless $H \cong W_k$). Therefore the vertex cuts of H are pairs of vertices. Such vertex cuts consist of nonadjacent vertices of H. Moreover, it is easily checked that if $\{u, v\}$ is any such 2-vertex cut the u, v both belong either to the same tourniquet or to the same spoke. It follows that one of the two components of $H - \{u, v\}$, is a path (possibly with zero length). Therefore twisting at $\{u, v\}$ amounts to rename u by v and v by u, that is taking an automorphism of H. Thus twisting at u and v leaves H unchanged up to isomorphism. To complete the proof it suffices to invoke Whitney's 2-isomorphism Theorem. □

Theorem 5 *Let $A \in \{0, 1\}^{M \times N}$ be an EPT matrix and let T be any of its underlying trees. Then A contains a pie of size k if and only if T contains CJ-pie of size k with respect to some $J \subseteq N$ of k elements.*

Proof. If T contains a GJ-pie of size k with respect to some $J \subseteq N$ of k elements then clearly A^J fulfils conditions (a) and (b) in the definition of pie given in Section 1.

Let us show conversely that, no matter how T is chosen, if A^J is a pie of size k in A, for some $J \subseteq N$, then T contains CJ-pie of size k with respect to some $J \subseteq N$. Let $K = \cup_{j \in J} \widetilde{A^j}$ and let A^J_K be the submatrix of A obtained by deleting the rows and the columns whose index is not in K and J, respectively. Clearly A^J_K is an EPT matrix as well and it is underlain by T^J_K, the subgraph spanned by K in T—observe that, since A is an EPT matrix and T is a tree, then T^J_K is a subtree of T–. Indeed, T underlies any submatrix A' of A obtained by deleting columns. Thus T^J_K underlies any submatrix of A' obtained by deleting zero rows, i.e., rows corresponding to elements not occurring in any member of $\widetilde{A'}$—remark in passing that these rows correspond to co-loops of $M(A')$ and, accordingly, each of the corresponding elements does not belong to any circuit of any graph whose cycle matroid is $M(A')$–. It follows that T contains CJ-pie of size k with respect to J (and A) if and only so does T^J_K with respect to J (and A^J_K). By a known construction (see e.g., [7]) if (D, F) is the pair formed by an EPT matrix $D \in \{0,1\}^{R\times S}$ underlain by a tree F then $M(D)$ is isomorphic to the cycle matroid of the graph obtained from F by adding an edge s between the endpoints of $\widetilde{d_s}$, $s \in S$. Let G^J_K be the graph obtained by applying the preceding construction to (A^J_K, T^J_K). It is readily seen that $M(A^J_K)$ is isomorphic to the cycle matroid of a subdivision of a wheel: just observe that, up to a permutation of rows and columns, A^J_K is the EPT matrix of a subdivision of W_k taken with respect to a spanning tree obtained by deleting one edge from each tourniquet. Since graphs homeomorphic from a wheel are blocks, Lemma 1 applies and thus $G^J_K \prec W_k$. In particular, A^J_K is the EPT matrix of G^J_K taken with respect to T^K_J. Therefore, no matter how T is chosen to underly A, the edges of T^K_J incident to the center of G^K_J form a CJ-pie of size k with respect to J. □

Corollary 8 (Theorem 2 in [8]) *Let G be an EPT graph with underlying tree T. If G contains a hole H then T contains a GJ pie on $V(H)$.*

Proof. EPT matrices are regular and hence column-odd-wheel-free matrices (see the Section 1.1). The result thus follows by Corollary 3 after Theorem 5. □

For EPT matrices Corollary 6 immediately implies the following consequence proved directly in [1].

Corollary 9 ([1]) *Let A be an EPT matrix which does not contain any odd pie. Then A is a $\{0,1\}$ network matrix.*

Remark 4 *$\{0,1\}$ network matrices are better known as Directed-Path-Tree (DPT) Matrices. These matrices are defined as the incidence matrices of families of arc-sets of directed paths in a directed tree.*

4 An extension of the local structure result

Here we show that the local structure result can be extended in a sharper way from the class of line graphs of EPT matrices to the class of line graphs of strong quasi-graphical matrices. To this end we need to list some subgraph that is forbidden in the bipartite representation of a strong quasi-graphical matrix.

Lemma 2 *None of the following graphs is an induced subgraph in the bipartite representation of a strong quasi-graphical matrix $A \in \{0,1\}^{M\times N}$.*

(i) *odd 3PC's and odd wheels;*

(ii) *even 3PC*(N) *and even* $W_k(N)$.

Proof. Recall from Section 1.1 that if A' is the result of pivoting on a nonzero entry $a_{i,j}$ of A, then $B(A')$ results from $B(A)$ by complementing the edges between $N(i) \setminus \{j\}$ and $N(j) \setminus \{i\}$, where, for $l \in I \cup J$, $N(l)$ denote the set of neighbors of l (see [5, 6]). Thus the subgraphs in (i) are forbidden because they can be pivoted into graphs containing either $B(F_7) = W_3(M)$ or $B(F_7^*) = W_3(N)$ [5, 6], contradicting the regularity of strongly quasi-graphical matrices. To prove that those in (ii) are forbidden as well we shall prove the following claim.

(3) Let G be a bipartite graph such that G is either an even 3-path configuration $3PC(u, v)$ or an even wheel (C, v). Then G can be pivoted into a bipartite graph containing a complete 4-wheel whose center is in the same color class of u and v if $G = 3PC(u, v)$ and in the same color class of v if $G = (C, v)$.

Proof of (3): It suffices to show that by pivoting and taking subgraphs even 3PC's and even wheels can be transformed into a complete 4-wheel. In the first place observe that any chordless uv-path of odd length can be transformed into the single edge uv by repeatedly pivoting on a inner edge $u'v'$, say, and deleting u' and v'. Thus any even 3PC(u, v) can be transformed into an even 3PC(u, v), G say, where the three paths have length four. Now let ux and yv be the end edges of any of the three paths of G and let z be the middle vertex on such a path. Notice that z has the same color as u and v. By pivoting on ux and yv and deleting x and y results in a complete 4-wheel centered at z. Analogously, any even wheel (C, v) can be transformed into a complete even wheel (C', v) with the same center and the same number of spokes. If C' has eight vertices we are done; otherwise let u_1, v_1, u_2, v_2, u_3 and v_3 induce a subpath on C' where u_i is adjacent to v, $i = 1, 2, 3$. The following sequence of operations preformed on (C', v) takes (C', v) into an even complete wheel (C'', v) with two spokes less: pivot on u_2v_2 (hence v_1 and u_3 become adjacent) and delete u_2 and v_2; pivot on v_1u_3 (hence u_1 and v_3 become adjacent) and delete v_1 and u_3. A repeated application of these procedures eventually yields a complete 4-wheel. ■

By (3), the subgraphs in (ii) are forbidden because they can be pivoted into subgraphs containing the bipartite representation of the matrix $H_{3,3}$, namely, a complete 4-wheel centered at a vertex representing a column. □

Theorem 6 *Let G be the line graph of a strong quasi-graphical matrix. Then each vertex of G has at most four neighbors on every hole which does not contain it.*

Proof. Let $G = L(A)$ for some strong quasi-graphical matrix $A \in \{0, 1\}^{M \times N}$. Let $J \subseteq V(G)$ be a set of $k \geq 5$ vertices that induces a hole in G (if $k = 4$ the result is trivial). Possibly after renumbering $J = \{1, 2, \ldots, k\}$ and, for $i, j \in J$, A^i and A^j are consecutive if and only if $|i - j| \in \{1, k - 1\}$. Hence J is endowed by a cyclic order and for each $j \in J$ we denote by j^+ the successor of j in such an order. Thus $1^+ = 2, \ldots (k-1)^+ = k$ and $k^+ = 1$. For $r \in N \setminus J$. By Corollary 3 A^J is a pie in A. Let $B_j = \widetilde{A^j} \cap \widetilde{A^{j+1}}$, $j \in J$ (addition over the indices is taken modulo k) be the j-th *branch* of the pie. Observe that by the definition of pie one has $B_i \cap B_j = \emptyset$, for $i \neq j$, $i, j \in J$. Furthermore let Z_j be the set of elements of $\cup_{j=1}^{k} \widetilde{A^j}$ occurring only in $\widetilde{A^j}$. We say that B_h, B_i (Z_h, Z_i) are *consecutive* in A^J, if $|i - h| \in \{1, k - 1\}$. Let $S(r) = \{j \in J \mid \widetilde{A^r} \cap B_j \neq \emptyset\}$ and $T(r) = \{j \in J \mid \widetilde{A^r} \cap Z_j \neq \emptyset\}$ and denote by $s(r)$ and $t(r)$ the respective cardinalities. Moreover, for $j \in J$ let $b(j, r)$ be an element of $B_j \cap \widetilde{A^r}$ if $B_j \cap \widetilde{A^r} \neq \emptyset$ else let $b(j, r)$ be any element of B_j.

Observe in the first place that

(4) $N_G(r) \cap J = S(r) \cup T(r) \cup S^+(r), \ \forall r \in N \setminus J,$

where we have set $S^+(r) = \{j^+ \mid j \in S(r)\}$. To prove the theorem thus it suffices to show that $|S(r) \cup T(r) \cup S^+(r)| \leq 4$. First we claim that

(5) $s(r) \leq 2, \ \forall r \in N \setminus J.$

Proof of (5): consider the graph W induced in $B(A)$ by $\{b(1,r) \dots, b(k,r)\} \cup J \cup \{r\}$ and observe that if $s(r)$ were greater than two then $W \cong W_{s(r)}(N)$ contradicting Lemma 2. ■

Next we claim that

(6) $s(r) = 0 \Rightarrow t(r) \leq 2, \ \forall r \in N \setminus J$; moreover, $t(r) = 2$ if and only if $\widetilde{A^r}$ intersects consecutive Z_j's.

Proof of (6): suppose not. Hence we can find elements $a_{j_h} \in \widetilde{A^{j_h}} \cap Z_{j_h}$, $h = 1, \dots, p$, for some $p \in \mathbb{N}$ such that $3 \leq p \leq k$. Since $k \geq 5$ among $Z_{j_1}, \dots Z_{j_p}$ we can find two members, Z_{j_h} and Z_{j_l} say, that are nonconsecutive in A^J. The graph induced by $\{b(1,r), \dots, b(k,r)\} \cup J \cup \{r, a_{j_h}, a_{j_k}\}$ is thus isomorphic to an even $3\text{PC}(a_{j_h}, a_{j_k})$ still contradicting Lemma 2. We conclude that $s(r) = 0 \Rightarrow t(r) \leq 2$ with $t(r) = 2$ if and only if $\widetilde{A^r}$ intersects consecutive Z_j's. ■

We claim also

(7) $S(r) = \{j\} \Rightarrow |T(r) \cap \{j, j+1\}| \leq 1, \ \forall r \in N \setminus J.$

Proof of (7): without loss of generality $j = 1$. If $T(r)$ contained both 1 and 2 then we could find $a_1 \in Z_1$ and $a_2 \in Z_2$ such that $\{b(2,r), \dots, b(k,r)\} \cup J \cup \{r\} \cup \{a_1, a_2\}$ induces a hole C' in $B(A)$. Now $b(1,r) \notin V(C')$ but $b(1,r)$ has exactly three neighbors on C', namely, 1, 2 and r. Hence $V(C') \cup \{b(1,r)\}$ would induce a wheel with three spokes contradicting the regularity of A.■

Finally we claim that

(8) if, for some $r \in N \setminus J$, $S(r)$ and $T(r)$ are both nonempty then for each $j \in T(r)$ either j or $j-1$ (addition is modulo k) belongs to $S(r)$.

Proof of (8): to prove the claim we assume for the sake of contradiction that it is false. Since $T(r) \neq \emptyset$ there is $j \in T(r)$. Let $a \in \widetilde{A^r} \cap Z_j$ and consider the graph W induced by $\{b(1,r), \dots, b(k,r)\} \cup J \cup \{a, j\}$. Let us distinguish two cases: $s(r) = 1$ (referred to as case 1) and $s(r) = 2$ (referred to as case 2). In case 1 we may suppose that $S(r) = \{1\}$; thus $T(r) \cap \{1, 2\} = \emptyset$ and $j \in T(r) \setminus \{1, 2\}$; it follows that W is an odd $3\text{PC}(a, j)$ in $B(A)$. In case 2 we may suppose that $S(r) = \{1, h\}$, with $h \neq 1, k$; thus $T(r) \cap \{1, 2, h, h+1\} = \emptyset$ and $j \in T(r) \setminus \{1, 2, h, h+1\}$ (remark that h might coincide with 2); now, in W, $b(1,r)$, $b(h,r)$ and a are adjacent to r while a is adjacent to j; by pivoting on edge ja and deleting vertex a results in a wheel with four spokes. In either cases Lemma 2 is contradicted and the claim is thus proved. ■

After (7) and (8) we see that if $s(r) > 0$ then $T(r) \subseteq S(r) \cup S^+(r)$. Therefore $|N_G(r) \cap J| \leq |S(r) \cup S^+(r)| \leq 2|S(r)| \leq 4$ the latter inequality being due to (5) (notice that the inequality can be strict: if $S(r) = \{i, j\}$ then $N_G(r) \cap J = \{i, i+1\} \cup \{j, j+1\}$; hence if the branches intersected by r are consecutive then $|N_G(r) \cap J| = 3$). The only case left is $s(r) = 0$. In this case $|N_G(r) \cap J| = t(r) \leq 2$ by (6). The theorem is thus completely proved. □

The following consequence of Theorem 6 extends Corollary 3 in [8] from the class of EPT graphs to the class of intersection graphs of quasi-graphical families.

Corollary 10 *If $G = L(A)$ for some quasi-graphical matrix A, then for every chordless cycle H on vertices $j_1, \ldots j_k$, $k \geq 4$ and every $r \notin V(H)$ exactly one of the following holds:*

(1) $N_G(r) \cap V(H) = \emptyset$,

(2) $N_G(r) \cap V(H) = \{j_h\}$ *for some* h,

(3) $N_G(r) \cap V(H) = \{j_h, j_{h+1}\}$ *for some* h,

(4) $N_G(r) \cap V(H) = \{j_h, j_{h+1}, j_{h+2}\}$ *for some* h,

(5) $N_G(r) \cap V(H) = \{j_h, j_{h+1}, j_i, j_{i+1}\}$ *for some* h *and* i,

where addition over indices is modulo k.

Proof. Let $J = V(H)$ where, possibly after renumbering $J = \{1, \ldots, k\}$. Thus A^J is a pie in A. Suppose that $s(r) = 0$. Then by (6) $t(r) \leq 2$ and $t(r) = 2$ if and only if $\widetilde{A^r}$ intersects consecutive members of $\widetilde{A^J}$. Thus if $s(r) = 0$ then $N_G(r) \cap V(H)$ is either empty or a singleton or it consists of two adjacent vertices, that is, one of (1)÷(3) applies. Suppose now that $s(r) = 1$ and let $S(r) = \{j\}$. By (7) and (7) one has $T(r) \subseteq \{j, j+1\}$ and $|T(r) \cap \{j, j+1\}| \leq 1$. Therefore $\widetilde{A^r}$ intersects $\widetilde{A^j}$ and (possibly) $\widetilde{A^{j+1}}$ and one of (2) and (3) applies. Finally, if $s(r) = 2$ then still by (8) we know that $T(r) \subseteq S(r) \cup S^+(r)$. Hence we are either in case (4) if $S(r) = \{j, j+1\}$ for some $j \in J$ or in case (5) if $S(r) = \{i, j\}$ for some two distinct vertices i, $j \in J$. □

Golumbic and Jamison proved that none of the two graphs G_1 and G_2 of Figure 2 are EPT graphs. Thus none of them can be an induced subgraph in any EPT graph (as the property of being EPT is preserved under taking induced subgraphs). Since G_2 is isomorphic to $\overline{P}_6$ it follows that the complement of any EPT graph cannot contain induced P_6 and hence induced P_k on $k \geq 6$ vertices (Corollary 4 in [8]). ¿From this fact Corollary 2 follows. We are going to show that the same result holds within line graphs of strong quasi-graphical matrices (see Corollary 11). The result implies that G_1 and G_2 are forbidden induced subgraphs in line graphs of strong quasi-graphical matrices as well. Since also the property of being strong quasi-graphical is inherited by deleting columns, this implies that if $G = L(A)$ for some strong quasi-graphical matrix A, then its complement $\overline{G}$ cannot contain an induced P_k on $k \geq 6$ vertices, thus providing a further extension of the local structure result. Corollary 11 relies on the following fact which is proved in [3].

Lemma 3 *Let $A \in \{0,1\}^{M \times N}$ be a quasi-graphical matrix and let A^J be a pie in A of size $k \geq 3$ for some $J \subseteq N$. Suppose without loss of generality that $J = \{1, 2 \ldots, k\}$ and that i and j are consecutive if and only if $|i - j| \in \{1, k-1\}$. For $h \in \{1, \ldots, k\}$ let $K(h, J)$ be the index set of the columns of A whose support intersects the h-th branch of A^J, namely, $K(h, J) = \{r \in N \mid \widetilde{A^r} \cap B_h \neq \emptyset\}$, where $B_{j_h} = \widetilde{A^h} \cap \widetilde{A^{h+1}}$. Then there is $b(h) \in B_h$ such that $b(h) \in \cap_{r \in K(h,J)} \widetilde{A^r}$.*

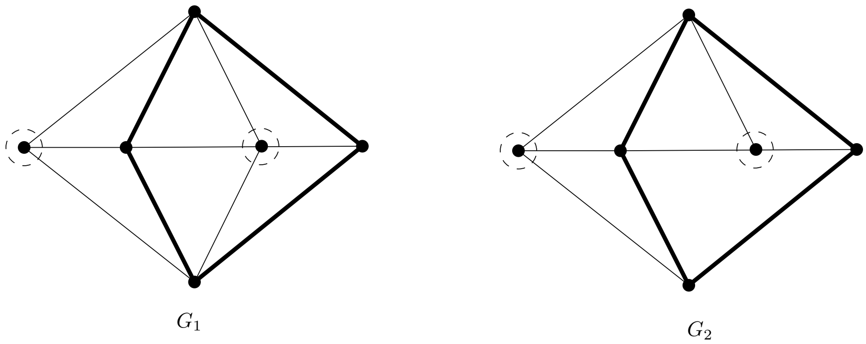

Figure 2: The graphs G_1 and G_2 are forbidden induced subgraphs in any EPT graph. The encircled vertices both have at least three consecutive neighbors on a hole (thick lines) and among these neighbors they have at least two adjacent common neighbors.

Let G be a graph and let H be a hole in G with $V(H) = J = \{1, 2, \ldots, k\}$. For $v \in V(G) \setminus J$ let $D(v, J) = N_G(v) \cap J$ and $d(v, J) = |D(j, J)|$.

Corollary 11 *Let $G = L(A)$ for some strong quasi-graphical matrix $A \in \{0, 1\}^{M \times N}$ and let H be a hole in G with $V(H) = J = \{1, 2, \ldots, k\}$. Let q, $r \in V(G) \setminus J$ be such that either*

$$D(q, J) = \{j-1, j, j+1\} \text{ and } D(r, J) = \{j, j+1, j+2\}$$

or

$$D(q, J) = \{j-1, j, j+1\} \text{ and } D(r, J) = \{j-1, j, j+1, j+2\}.$$

Then q and r are adjacent in G.

Proof. By Corollary 3, A^J is a pie in A. Since $d(q, J)$ and $d(r, J)$ are both not smaller than 3 by Corollary 10 it follows that $D(x, J) = S(x) \cup S^+(x)$ for $x \in \{q, r\}$. In particular, necessarily $S(q) = \{j-1, j\}$ for some $j \in J$, because $D(q, J)$ consists of consecutive elements. By the same reason either $S(r) = \{j, j+1\}$ or $S(r) = \{j-1, j+1\}$. Thus either $j \in S(q) \cap S(r)$ or $j-1 \in S(q) \cap S(r)$. These in turn imply that either q, $r \in K(j, J)$ or q, $r \in K(j-1, J)$. In either case, by Lemma 3, there is $b(h) \in \widetilde{A^q} \cap \widetilde{A^r} \neq \emptyset$ $(h \in \{j-1, j\})$. Consequently, qr is an edge of G. □

Corollary 11 implies that the graph G_1 and G_2 of Figure 2 are not induced in the line graph of any strong quasi-graphical matrix. Such graphs are therefore $\bar{C}_7$-free.

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