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Abstract

The topic of this thesis are compactifications in string theory and supergravity. We study dimensional reductions of type II theories on backgrounds with fluxes, using the techniques of Hitchin's generalized geometry.

We start with an introduction of the needed mathematical tools, focusing on SU(3)×SU(3) structures on the generalized tangent bundle $T \oplus T^*$, and analyzing their deformations. Next we study the four dimensional $N = 2$ gauged supergravity which can be defined reducing type II theories on SU(3)×SU(3) structure backgrounds with general NSNS and RR fluxes: we establish the complete bosonic action, and we show how its data are related to the generalized geometry formalism on $T \oplus T^*$. In particular, we derive a geometric expression for the full $N = 2$ scalar potential. Then we focus on the relations between the 10d and 4d descriptions of supersymmetric flux backgrounds: we spell out the $N = 1$ vacuum conditions within the 4d $N = 2$ theory, as well as from its $N = 1$ truncation, and we establish a precise matching with the equations characterizing the $N = 1$ backgrounds at the ten dimensional level. We conclude by presenting some concrete examples, based on coset spaces with SU(3) structure. We establish for these spaces the consistency of the truncation based on left-invariance, and we explore the landscape of vacua of the corresponding theory, taking string loop corrections into account.

Résumé

Cette thèse porte sur les compactifications en théorie des cordes et supergravité. Nous étudions les réductions dimensionnelles des théories de type II sur des fonds avec flux, en utilisant les techniques de la géométrie généralisée de Hitchin.

Nous commençons en introduisant les outils mathématiques nécessaires : nous nous concentrons sur les structures SU(3)×SU(3) sur le fibré tangent généralisé $T \oplus T^*$, en analysant leurs déformations. Ensuite nous étudions la théorie de supergravité $N = 2$ quadri-dimensionnelle définie par réduction des théories de type II sur des fonds à structure SU(3)×SU(3) avec flux généraux de NSNS et RR: nous établissons l'action bosonique complète, et nous montrons comment ses donées sont reliées au formalisme de la géométrie généralisée sur $T \oplus T^*$. En particulier, nous trouvons une expression géométrique pour le potentiel scalaire $N = 2$. Puis nous nous concentrons sur les relations entre les descriptions à 10d et à 4d des fonds supersymétriques avec flux: nous dérivons les conditions de vide $N = 1$ dans la théorie $N = 2$ à 4d, ainsi que dans sa troncation $N = 1$, et nous prouvons une correspondance précise avec les équations qui caractérisent les vides $N = 1$ au niveau dix-dimensionnel. Nous terminons en présentant des exemples concrets, basés sur des espaces quotients avec structure SU(3). Nous établissons pour ces espaces la cohérence de la troncation basée sur l'invariance gauche, et nous explorons les vides de la théorie associée, en prenant en compte les corrections des boucles des cordes.

Riassunto

Argomento di questa tesi sono le compattificazioni in teoria di stringa e supergravità. Studiamo le riduzioni dimensionali delle teorie di tipo II su dei backgrounds con flussi, utilizzando le tecniche della geometria generalizzata di Hitchin.
La tesi si apre con un'introduzione agli strumenti matematici necessari. Ci interessiamo in particolare alle strutture SU(3)×SU(3) sul fibrato tangente generalizzato $T \oplus T^*$, ed all'analisi delle loro deformazioni. In seguito studiamo la supergravità $N = 2$ gauged definita tramite riduzione delle teorie di tipo II su dei backgrounds a struttura SU(3)×SU(3) in presenza di un sistema generale di flussi NSNS e RR: stabiliamo l'azione bosonica completa, e mostriamo in che modo i suoi elementi costitutivi sono in relazione con il formalismo della geometria generalizzata su $T \oplus T^*$. In particolare, troviamo un'espressione geometrica per il potenziale scalare $N = 2$. Ci volgiamo poi allo studio delle relazioni tra le descrizioni 10d e 4d dei backgrounds supersimmetrici con flussi: scriviamo le condizioni di vuoto $N = 1$ che discendono dalla teoria $N = 2$ a 4d, così come dalla sua troncazione $N = 1$, e dimostriamo una precisa corrispondenza con le equazioni caratterizzanti i backgrounds $N = 1$ a livello dieci-dimensionale. Concludiamo presentando alcuni esempi concreti, basati su degli spazi quoziente con struttura SU(3). Dimostriamo per questi spazi la consistenza della troncazione basata sulla left-invariance, ed esploriamo i vuoti della teoria corrispondente, includendo le correzioni di loop di stringa.

Résumé détaillé

Compactifications avec flux en théorie des cordes, et géométrie généralisée

Cette thèse porte sur les théories des cordes et de supergravité. Ces théories jouent un rôle fondamental dans la physique théorique contemporaine. La théorie des cordes, qui interprète les constituants élementaires de la matière comme des petits objets unidimensionnels (les cordes) et non ponctuels, est considérée comme la meilleure candidate pour fournir une description unifiée de toutes les forces connues dans la nature. En particulier, elle comprend en un seul cadre cohérent à la fois la gravité et la mécanique quantique. Strictement reliées à la théorie des cordes sont les théories de supergravité, qui sont des théories de la gravité incorporant la relativité générale mais ayant une symétrie supplémentaire: la supersymétrie.

Pour des raisons de cohérence mathématique les théories des cordes, ainsi que les théories de supergravité associées, sont normalement formulées en dix dimensions d'espace-temps. Une théorie qui décrit la physique dans les quatre dimensions d'espace-temps usuelles peut être obtenue grâce à la procédure de compactification, dans laquelle six dimensions spatiales sont "enroulées" en un volume d'espace extrêmement petit, et ainsi invisibles.

Ce travail de thèse s'insère dans les développements les plus récents de ce sujet: dans les dernières années, des nouveaux outils mathématiques sophistiqués ont permis d'étudier le problème des compactifications de façon plus systématique, et le pouvoir prédictif de la théorie a été ainsi perfectionné. Notamment, la géométrie généralisée de Hitchin permet d'étudier les reductions dimensionnelles des théories de type II sur des fonds avec flux: nous nous intéressons aux conditions qui permettent de définir une théorie de supergravité $N = 2$ en quatre dimensions. Nous nous concentrons sur les espaces six-dimensionnels dits à structure SU(3)×SU(3), en étudiant les relations entre leurs proprietés géométriques et la théorie de supergravité à quatre dimensions issue de la compactification.

La thèse est ainsi structurée.

Nous commençons dans le *chapitre 2* par une introduction aux outils mathématiques nécessaires pour l'analyse des compatifications. L'ansatz spinoriel lié aux compactifications qui présèrvent $N = 2$ à quatre dimensions nous amène à traiter les G-structures sur la variété six-dimensionnelle compacte. Ensuite, en étendant la notion de G-structure à la somme $T \oplus T^*$ des fibrés tangent et cotangent, on introduit la géométrie généralisée de Hitchin. Notamment, nous nous concentrons sur la notion clé de structure SU(3)×SU(3), qui décrit tous les degrés de liberté NSNS sur la variété compacte. Dans la perspective d'une étude des termes cinétiques scalaires en quatre dimensions, nous considérons la métrique sur l'espace des paramètres des champs NSNS internes. Cette dernière peut être reformulée en termes des déformations des spineurs purs du groupe $O(6,6)$ qui caractérisent la structure SU(3)×SU(3) . Cela permet de faire émerger une structure spéciale Kähler, définie localement sur la variété compacte. Ce résultat, auquel nous avons contribué, se développe en parallèle avec la structure de l'espace des modules des variétés Calabi-Yau, que l'on présente aussi.

Dans le *chapitre 3* nous considérons l'étude des compactifications des théories de type II sur des fonds avec flux qui admettent une structure SU(3)×SU(3). On commence par une revue rapide de la formulation 'démocratique' de la supergravité de type II [48], qui est adaptée aux applications en géométrie généralisée. Ensuite nous présentons les réductions dimensionnelles sur les variétés Calabi-Yau, qui représentent le modèle de référence pour les développements successifs. Nous discutons les relations entre flux et jaugeages. La troncation de la supergravité de type II est effectuée sur des fonds à structure SU(3)×SU(3) à l'aide d'un développement des champs dix-dimensionnels sur une base finie de formes differentielles sur la varieté compacte. La compatibilité avec la supergravité $N = 2$ en quatre dimensions impose sur cette base une série de contraintes géométriques assez restrictives. Celles-ci ont été identifiée dans [45, 46, 49], et nous les réécrivons d'une façon differente. Puis on déduit l'action bosonique complète à quatre dimensions. En particulier, nous nous concentrons sur la façon dont ses donées sont déterminés par la géométrie généralisée.

Premièrement nous établissons la réduction du secteur NSNS de la supergravité de type II. On fait le lien avec l'espace des déformations étudié dans le chapitre précédent. Ensuite, nous étudions le rôle de l'opérateur Hodge-$*$ deformé par le champ B. En particulier, nous montrons comment son action sur la base des formes differentielles généralise au contexte des structures SU(3)×SU(3) l'expression pour l'action de l'opérateur Hodge-$*$ usuel sur les 3–formes harmoniques d'une variété Calabi-Yau. Ceci nous permet d'obtenir une formule pour les matrices des périodes de la géométrie spéciale Kähler $N = 2$. Ensuite, nous nous focalisons sur le potentiel scalaire en quatre dimensions: nous demontrons une formule qui exprime la courbure de Ricci de la variété compacte en termes des données de la géométrie généralisée. Puis nous utilisons cette formule afin d'obtenir une expression géométrique pour le potentiel scalaire. Une fois écrite en termes des variables quadri-dimensionnelles, celle-ci donne l'expression précédemment trouvée en [50] en utilisant exclusivement des méthodes de supergravité $N = 2$ quadri-dimensionnelle.

Dans la dernière partie du chapitre nous passons au secteur RR, en nous concentrant sur la supergravité du type IIA. Plutot que de réduire directement l'action, nous choisissons de réduire les équations du mouvement. En vertu d'une contrainte d'auto-dualité, celles-ci peuvent aussi être vues comme des indentités de Bianchi. Le développement du champ de RR sur la base de formes internes introduit automatiquement des formes de tous degrés dans l'espace-temps quadri-dimensionnel. On interprète un sous-ensemble des équations de RR réduites comme des identités de Bianchi; leur solution donne un système de champs fondamentaux à quatre dimensions. Les équations restantes sont interprétées comme équations du mouvement, à partir desquelles on reconstruit l'action réduite.

Dans le *chapitre 4* nous discutons plus en détail la cohérence entre le résultat de la réduction dimensionnelle et le formalisme de la supergravité $N = 2$ jaugée. Une formulation cohérente en présence d'un système général de flux exige l'introduction de multiplets tensoriels. Nous analysons en particulier les quantités qui déterminent le jaugeage. Ensuite, en considerant les expressions des prépotentiels de Killing, et en utilisant des résultats généraux concernant la supergravité $N = 2$ avec multiplets tensoriels, nous dérivons les shifts fermioniques associés aux variations de supersymétrie à quatre dimensions.

Dans la deuxième partie du chapitre nous faisons une comparaison entre l'approche quadri-dimensionnelle et celle dix-dimensionnelle des vides supersymétriques avec flux. Au niveau dix-dimensionnel, les équations pour le vide $N = 1$ ont été reformulées dans le langage de la géométrie généralisée en [26, 27]. Afin d'effectuer une comparaison avec les contraintes définies en quatre dimensions, nous réécrivons ces équations en termes de variables quadri-dimensionnelles, et nous intégrons sur la variété interne. Successivement, nous trouvons les conditions du vide $N = 1$ définies dans la théorie $N = 2$ à quatre dimensions, en imposant l'annulation des shifts fermioniques sous une seule transformation de supersymétrie. En utilisant les propriétés de la géométrie spéciale Kähler, nous établissons une correspondance précise avec la version intégrée des équations de [26, 27]. Nous effectuons aussi une analyse similaire en considérant la supergravité quadri-dimensionnelle définie par une troncation de la théorie $N = 2$ précédemment analysée. Pour cela, on trouve les expressions pour le superpotentiel et les D-termes et on impose les conditions de F-flatness et de D-flatness.

Enfin, dans le *chapitre 5*, on présente des exemples concrets de compactifications $N = 2$, basés sur des espaces quotients avec structure SU(3). Cela peut être considéré comme une application de l'étude générale présentée dans les chapitres précédents. Le contrôle précis sur la géométrie que la structure de l'espace quotient comporte, nous permet une analyse explicite. En particulier, on établit la cohérence de la réduction dimensionnelle basée sur un ansatz invariant à gauche. On explore les vides supersymétriques et non-supersymétriques associés à la compactification, en paramétrant la solution en termes des flux. En utilisant les contraintes imposées par la supersymétrie $N = 2$, on étudie les corrections des boucles des cordes au potentiel scalaire quadri-dimensionnel, et on esquisse une recherche préliminaire de solutions de Sitter.

Dans le *chapitre 6* on tire nos dernières considérations.

On destine quelques discussions techniques aux *appendices.* L'appendice A résume nos conventions. L'appendice B donne quelques détails sur le 'Mukai pairing' et la 'Clifford map' utilisés dans les calculs en géométrie généralisée. L'appendice C présente la relation entre la formulation démocratique et celle standard de la supergravité de type II. Nous discutons aussi quelques subtilités dues à la présence des flux. Les appendices D et E donnent la définition et certaines propriétés respectivement des variétés 'spéciale Kähler' et 'quaternionic-Kähler', qui jouent en rôle central dans la supergravité $N = 2$. Enfin, les appendices F et G réunissent du matériel liés aux réductions dimensionnelles des espaces quotients du chapitre 5.

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Chapter 1

Introduction

The Standard Model of elementary particles represents our current paradigm for understanding the weak, electromagnetic and strong interactions. Successfully tested to impressively high precision up to an energy scale of the order of 100 GeV [1], it has conferred to quantum field theory based on the gauge principle a central role in the description of the fundamental laws of nature.

However, the Standard Model cannot be regarded as the ultimate theory of fundamental interactions. Indeed, as a first thing it has the drawback of being a rather ad hoc construction, involving a relevant number of parameters which are not fixed by the model itself: from a theoretical viewpoint, it would be much more appealing to dispose of a derivation from first principles, in which the various parameters are dynamically determined. Secondly, it does not at all include gravity, which is instead described by Einstein's general relativity. Though it shares with the Standard Model the fact of being a field theory, general relativity is based on completely independent conceptual grounds, dealing with the intimate dynamics of spacetime, and the interaction of its geometry with the distribution of matter. Furthermore, it is a classical theory, i.e. the principles of quantum mechanics do not enter in its formulation.

Now, for the purpose of describing the interactions of microscopic particles at relatively low energy scales, ignoring gravity is a perfectly sensible approximation, since at such scales this force is much weaker than the remaining three. However, as one approaches the scale of 10^{19} GeV, known as the Planck scale, the strength of gravity becomes comparable to the one of the other forces, so that it cannot be neglected anymore. Also general relativity has its own regime of application: it works very well at long distances, where gravity, being always attractive, dominates over the other forces; for this reason, it is the relevant theory for the explication of large-scale astrophysical phenomena, where indeed it has proved very successful. However, general relativity breaks down at short distances, where quantum effects become important. In fact, from a field theory perspective gravity is non-renormalizable, and all attempts to quantize it in this scheme have failed. The scale at which general relativity breaks down is again the Planck scale. We can conclude that the picture of two distinct theories – the Standard Model and general relativity – being based on independent conceptual schemes, and describing different kinds of phenomena, seems itself an effective description, valid at the energy scales we are used to, but breaking down at higher energies.

Even if 10^{19} GeV is an extremely high energy scale, far beyond the reach of any conceivable experiment, still the question of a possible unification of the quantum theory of elementary particles with general relativity is not just of aesthetical value: indeed, understanding the quantum nature of gravity could provide the missing tools for describing the very early universe and other extreme regimes like the physics of black holes, where such scale of energies is relevant. Furthermore, since the time of Newton's intuition that the gravitational force governing the fall of an apple is the same force that ties the planets to their orbits, the unification of different conceptual schemes has often led to deep breakthroughs in physics.

For these and other reasons, while the Standard Model was so accurately tested in experiments, theoreticians pushed forward their imagination, speculating on its possible extensions at higher energies, and elaborating new, bold ideas. Part of these ideas is now coming to be probed at the Large Hadron Collider (LHC) experiment, whose (re-)start is scheduled for the next few months. It is believed that new physics will appear at the LHC scale of $1-10$ TeV, and it is likely that this will discriminate among the plethora of existing possible extensions of the Standard Model, or even stimulate totally new elaborations (see e.g. [2]). There is also the hope that LHC might provide indirect hints of the physics at even higher scales.

One of the most urgent questions that theoreticians would like to be answered by LHC is whether nature is supersymmetric. Supersymmetry is a powerful and very constraining symmetry which extends the Poincaré spacetime symmetry of relativistic quantum field theories via fermionic generators, so that it transforms any bosonic particle into a fermionic partner, and vice-versa. Actually, it has to transform any observed (bosonic or fermionic) particle into a never-observed one. Indeed, as a first thing one notices that exact supersymmetry implies that the superpartners be mass degenerate, and this is in obvious contradiction with the observations. Hence, if supersymmetry exists, then it has to be realized in a broken phase. However, even assuming broken supersymmetry, it turns out that due to a mismatch of quantum numbers no known particles can be mutual superpartners. It remains the possibility that the superpartners be hidden at higher energies, and it is here that LHC enters into the game. We conclude that supersymmetry basically doubles the number of elementary particles.

Despite these awkward features, there are several strong reasons to consider attractive the supersymmetric extension of the Standard Model. For instance, as a consequence of a non-renormalization theorem, supersymmetry protects the Higgs mass against large quantum corrections, stabilizing the weak scale even within a theory having a higher scale cutoff. Furthermore, supersymmetry provides a natural candidate for explaining the dark matter problem. Finally, it determines a highly non-trivial fact: with supersymmetry, the running coupling constants of the electromagnetic, weak and strong interactions strikingly meet at at the 'Grand Unification' scale of 10^{16} GeV. This is interpreted as a hint of the important role that supersymmetry may play in embedding the Standard Model into a more fundamental theory. More in general, and independently of the phenomenological purposes, it has been learned over the years that supersymmetry nicely improves the tractability of quantum field theories.

Even though the supersymmetric extension of the Standard Model does not describe gravity either, supersymmetry and general relativity are not unrelated. Indeed, by applying the gauge principle, supersymmetry can be made local. Since it contains the Poincaré symmetry, and since invariance under local translations leads to general relativity, we have that the latter is naturally incorporated in a theory of local supersymmetry, which has therefore been named *supergravity*.

Despite the initial hope that the addition of supersymmetry might cure the problem of divergencies in general relativity and make it renormalizable as a quantum field theory, this turns out not to be the case (at least not for the realistic models with a minimal amount of supersymmetry – for the maximal $N = 8$ supergravity recent developements suggest that it might actually be finite). Hence the mere fact of making supersymmetry local does not solve the problem of quantizing gravity, but can nevertheless be interpreted as a first step towards a unified theory. Indeed, if supersymmetry is relevant for the Standard Model, then it is very likely that the associated theory of gravity also be supersymmetric. Now, an ultraviolet completion of supergravity exists and is *string theory*.

String theory [3, 4] arises from the simple but radical idea of replacing the point particles of ordinary field theory with one-dimensional objects, the *strings*. These can be open strings, with the topology of a segment, or closed strings, with the topology of a circle. For each of the these topologies, the quantized string has a discrete spectrum of vibrating modes which, at distances much larger than the characteristic string length $\ell_s = \sqrt{\alpha'}$ (which is usually assumed tiny, with $1/\sqrt{\alpha'}$ not too far from the Planck scale), can effectively be interpreted as different point particles. The spectrum of oscillations is formed by an infinite tower, whose lowest-lying states are massless, while the upper levels are massive, with a spacing of $1/\sqrt{\alpha'}$. Among the massless states of a closed string there is one that has spin 2; by identifying it with the graviton, it follows that string theory automatically incorporates gravity. Actually, it proposes itself as a consistent theory of *quantum* gravity: indeed, the minimal length of the string provides an ultraviolet regularization of the graviton scattering amplitudes, whose badly divergent behaviour in quantum field theory is due to the point-like nature of the interaction.

The dimensionful parameter α' is the only free parameter entering in the definition of string theory. Consistency conditions at the quantum level impose further, strong contraints. For instance, certain tachyonic instabilities can be avoided by including supersymmetry; the resulting formulation is known as superstring theory. Furthermore, a Weyl anomaly cancellation condition implies that the spacetime in which superstrings propagate has to be ten-dimensional. The existence of extra-dimensions is no doubt one of the most striking predictions of string theory. Taking all the consistency conditions into account, it turns out that there are only five possible superstring theories; these are termed type IIA, type IIB, type I, heterotic SO(32) and heterotic $E_8 \times E_8$. Over the years, it has been understood that these different formulations are related to each other by various dualities, which, beside the strings, involve further dynamical multidimensional objects, called the branes. These interconnections among the different superstring theories might be the signal of an underlying more fundamental theory, which is conjectured to live in eleven spacetime dimensions, and has been given the name of M-theory.

As mentioned above, the limit $\alpha' \to 0$ is a low energy limit where string theory is approximated by a theory of massless point particles. Not too surprisingly, the latter turns out to be supergravity: specifically, the low energy limit of type IIA and type IIB superstring theories corresponds to type IIA and type IIB supergravity, while taking the low energy limit of type I and of heterotic strings one obtains type I supergravity, coupled to a super Yang-Mills theory with gauge group SO(32) or $E_8 \times E_8$. These are the only ten-dimensional supergravity theories which are also anomaly-free. Furthermore, the low energy limit of M-theory is defined to be the unique supergravity theory existing in eleven dimensions. In this limit, the branes contained in string theory are identified with certain solitonic solutions of supergravity.

Superstring theory not only offers a way to quantize gravity, but, since its massless spectrum also contains gauge bosons, it is also a promising candidate for a unified theory of all the fundamental interactions. Taking seriously this hypothesis, an immediate question which arises is how to pass from a fundamental theory formulated in ten spacetime dimensions to the four-dimensional world we have experience of. This has been one of the major areas of research in string theory since about twenty-five years. The most studied way to achieve this reduction from ten to four dimensions is by a *compactification* procedure. In this approach, one assumes that the ten-dimensional spacetime has only four extended spacetime dimensions, which are identified with our world; the remaining six spatial dimensions are instead wrapped to form a compact space, which is chosen very small in order to explain why we don't have access to it (actually, there are some hopes to gain indications of these extra dimensions from LHC; see e.g. [5] for an account). In other words, the background spacetime is chosen with a topology of the type $M_{10} = M_4 \times M_6$, where M_4 can for instance be Minkowski$_4$, while M_6 is a compact, 'internal' manifold. The subsequent step is to 'integrate' the ten dimensional theory over the internal space M_6, in order to be left with an effectively four dimensional theory.

The properties of the lower-dimensional theory crucially depend on the geometry chosen for the compact space. On the one hand, this is an attractive feature, because it provides a very elegant and controllable way to build lower-dimensional theories. On the other hand, it is a drawback: even if the higher-dimensional theory is unique, the compactification reintroduces a large amount of arbitrariness in the determination of the four-dimensional physics.

One of the main features that are studied is the amount of supersymmetry preserved by the compactification. For phenomenological purposes, the most desiderable situation is a compactification preserving just a minimal fraction of the supersymmetry of string theory: indeed, the supersymmetric extension of the Standard Model, to which one would like to make contact (the scale of supersymmetry breaking is usually assumed well below the compactification scale), requires just a minimal amount of supersymmetry. The non-complete breaking of supersymmetry is also technically convenient, since the constraints imposed by supersymmetry allow a good control on the compactified theory. Indeed, the conditions dictated by supersymmetry of the background translate into differential conditions for the spinors existing on the compact manifold, and this strongly constrains the geometry. In a very popular case, the supersymmetry condition requires the compact space to be Calabi-Yau, namely a 6d compact manifold with SU(3) holonomy [6]. Now, it turns out that there is an enormous number of allowed Calabi-Yau manifolds, so that the choice of the supersymmetric background is far from being unique.

In spite of this freedom inherent to the compactification, and though many progresses have been made, it has to be said that a consistent embedding of the (supersymmetric extension of the) Standard Model into string theory has not been achieved in a fully satisfactory way yet. A further physical quantity that one would hope to derive from a theory combining gravity and particle physics is a realistic cosmological constant (realistic means very small and positive, the observed value being $\Lambda \simeq 10^{-120} M_P^4$, where M_P is the Planck scale). This involves one of the main unsolved hierachy problems in contemporary theoretical physics, and its solution presumably requires a better understanding of quantum effects in gravity.

Initially, the efforts to build realistic compactification models where concentrated on the heterotic and type I string theories (see e.g. [3, vol. 2]), because these come already endowed with a large, non-abelian gauge group, while this is not the case for type II theories. Dramatically new perspectives have been opened by the discovery of D-branes in type II theories [7], which also support non-abelian gauge groups, even in a very flexible way. Furthermore, gauge groups can also arise directly from the geometry of the compact manifold.

We see that the modern approach to model building via compactifications involves several ingredients. These include fairly complicated, curved compact spaces, as well as localized objects like D-branes, which typically are taken intersecting among them. For various reasons, to these one is also led to add other tools like orientifold planes, fluxes of the higher-dimensional fields and non-perturbative quantum effects. Clearly, it is very hard to have full control on all these ingredients at once, so that often one prefers to concentrate on certain aspects of the whole picture. In this thesis, we will focus on the interrelations between geometry and supersymmetry in compactifications of type II theories.

1.1 Compactifications, fluxes and generalized geometry

The idea that the unification of the fundamental interactions may be related with the existence of supplementary spacetime dimensions is old, and dates back to the works of Kaluza and Klein [8], who derived Einstein's general relativity together with Maxwell's electromagnetism in four dimensions from a pure gravity action defined in five dimensions. Furthermore, many supergravity theories in diverse dimensions are related by compactification. In fact, since the early ages of supergravity, dimensional reductions have also served as a powerful tool to construct lower-dimensional supergravity theories starting from higher-dimensional ones, whose field content is simpler. A first, prominent example is the Cremmer-Julia derivation of the complete 4d maximal $N = 8$ supergravity [9], starting from eleven-dimensional supergravity [10].

In order to illustrate some general features of compactifications, we now briefly review the Kaluza-Klein model in arbitrary dimension, following [11].

We wish to reduce a $(d + 1)$-dimensional theory of pure gravity to d dimensions, by performing a compactification on a circle S^1, of radius L. Calling y the coordinate along the circle, the $(d+1)$-dimensional coordinates z^M split as $z^M = (x^\mu, y)$, where $M = 0, \ldots, d$ and $\mu = 0, \ldots, d-1$. We start from the usual Einstein-Hilbert action for the metric field

$\hat{g}_{MN}(x,y)$:

$$\hat{S} = \frac{1}{2\hat{\kappa}^2}\int d^{d+1}z\,\sqrt{-\hat{g}}\,\hat{R}\,,$$

where the hat symbol denotes quantities in $d+1$ dimensions. Since the metric tensor has to be periodic along the circle, we can expand it in Fourier series as

$$\hat{g}_{MN}(x,y) \;=\; \sum_n g^{(n)}_{MN}(x)\,e^{iny/L}\,.$$

In principle we could simply substitute this into the action $\hat{S}$, and integrate over the compact space S^1 in order to define a d-dimensional action. However, in doing this we would obtain a theory containing an infinite number of fields, labeled by the Fourier mode number n. In order to define a lower-dimensional theory with a finite number of fields, we need to perform a truncation of the spectrum of $\hat{g}_{MN}$. The criterium to define the truncation in this case is readily available. Indeed, it turns out that the modes of $\hat{g}_{MN}$ with $n=0$ are massless from the d-dimensional viewpoint, while those with $n \neq 0$ are massive, with a mass of the order of $|n|/L$. This can be seen by linearizing the Einstein equation $\hat{R}_{MN}=0$ for small fluctuations around the flat vacuum solution given by Minkowski$_d \times S^1$:

$$\langle \hat{g}_{MN}\rangle dz^M dz^N = \eta_{\mu\nu}dx^\mu dx^\nu + (dy)^2\,,$$

where here we are choosing $\langle g_{dd}\rangle = 1$. If we choose a circle with a very small radius L, the states with $n \neq 0$ will be very massive, so that they can be neglected in a low energy approximation. We conclude that one can define a truncation to d-dimensional massless modes by taking all the fields independent of the compact coordinate y. Now, the most convenient way to identify the field content of the d-dimensional theory is to parameterize the generic higher-dimensional metric (independent of y) as

$$\hat{g}_{MN}(x)dz^M dz^N = e^{2\alpha\phi(x)}g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\beta\phi(x)}(dy+A)^2\,,$$

where $A = A_\mu(x)dx^\mu$ is a 1–form, and α and $\beta \neq 0$ are arbitrary constants. The vacuum solution written above is obviously recovered setting $\langle g_{\mu\nu}\rangle = \eta_{\mu\nu}\,,\ \langle A\rangle = 0$ and $\langle e^\phi\rangle = 1\,.$ Substitution in the higher-dimensional action $\hat{S}$ and integration over S^1 yield the d-dimensional action

$$S = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}\left(R - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{4}e^{-2(d-1)\alpha\phi}F_{\mu\nu}F^{\mu\nu}\right)\,,$$

where the d-dimensional gravitational coupling constant is $\kappa^2 = \hat{\kappa}^2/2\pi L$, and we defined the field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. Furthermore, in order to get canonically normalized kinetic terms the constants α and β have been chosen appropriately in terms of d [11]. We conclude that by reducing a theory of pure gravity on S^1 one obtains a Maxwell-Einstein theory, also involving a scalar field ϕ. From the metric ansatz above we see that the gauge symmetry of the Maxwell field $A \to A + d\lambda(x)$ is inherited from the invariance under x-dependent reparameterizations of the circle coordinate y.

Nowadays, the example illustrated here should be regarded just as a toy model, and typically the techniques required in supergravity and string theory compactifications are much more involved (both because the higher-dimensional action is not just pure gravity, and because the compactification manifold is not simply S^1). However, in the following we highlight a couple of points, which have general validity and will be important in the remainder of this thesis.

As a first thing, we observe that if we wish that the higher-dimensional theory be compactified to a lower-dimensional theory which has a finite number of fields (as it is always the case), a truncation of the modes of the higher-dimensional fields on the internal space is always required. We will call the prescription selecting the degrees of freedom to be kept the *truncation ansatz*. In the example above, the appropriate truncation ansatz was not hard to identify. However, in general, things are not that easy, and different approaches can be adopted.

A physically well motivated prescription, relevant if one is interested in describing the low-energy physics around a given ground state, is the so called Kaluza-Klein ansatz (see e.g. [12] for a review), whose identification proceeds through the following steps:

i) choose a ground state of the higher-dimensional theory displaying a 'spontaneous compactification', namely a solution which is a direct product of two independent spaces.

ii) linearize the higher-dimensional equations of motion by considering small field fluctuations around the chosen vacuum, and identify the contributions to be interpreted as mass terms from the lower-dimensional viewpoint. Generically, these mass terms will include wave operators on the compact manifold, arising from the splitting of the higher-dimensional kinetic terms in a 4d spacetime part and an internal part.

iii) expand the higher-dimensional fields in a basis of eigenmodes of the identified mass operators; then truncate the spectrum, keeping just the lightest modes (typically, the massless ones).

In our example, the truncation ansatz valid at linear order around the Minkowski$_d \times S^1$ ground state has been easily extended to non-linear order simply by asking independence of the circle coordinates, but already considering slightly more complicated spaces, like e.g. higher-dimensional spheres, this step would become highly non-trivial, if not impossible. As a consequence, in general the outcome of a Kaluza-Klein analysis is limited to a chosen vacuum, and describes the physics of small field fluctuations around it.

A different approach to dimensional reductions would be to define a truncation ansatz by requiring the preserved degrees of freedom to be invariant under some given symmetry. For instance, one can demand invariance under the action of (a subgroup of) the isometry group of the internal manifold. A typical case in which this alternative approach can be pursued is a dimensional reduction on group manifolds or coset spaces. Notice that, since $S^1 \cong U(1)$, actually the simple dimensional reduction described above is an example of this second approach as well: indeed, demanding independence of the internal coordinates corresponds to keep only the singlets under the action of U(1). This kind of truncation ansatz is not always physically motivated, in that the obtained lower-dimensional theory does not necessarily capture the complete low energy physics. On the other hand, it is independent of the choice of a vacuum, is perfectly well-defined from the mathematical viewpoint, and has the advantage of yielding consistent reductions, where by definition a reduction is called consistent if all solutions (not only the vacua) of the lower-dimensional theory lift to solutions of the original, higher-dimensional theory. In this thesis we will discuss an explicit example of this second approach in chapter 5.

As a second point concerning our example above, let us consider the role of the scalar field ϕ. If we restrict to linear order in the field fluctuations, this is a free, massless field, whose propagation describes the variation of the size of the circle S^1 along the d-dimensional space. Its presence in the massless spectrum is not accidental, but is a consequence of the fact that its vev, describing the size of the circle in the Minkowski$_d \times S^1$ solution, is arbitrary. In fact, this is a characteristic feature of compactifications: it often happens that the compactification background displays a continuous degeneracy, whose parameters are called *moduli*; these include the allowed variations in shape and size of the compact manifold. In particular, typically the supersymmetric Calabi-Yau string backgrounds mentioned above come with a large number of moduli. In the lower-dimensional theory, the moduli always appear as massless scalar fields. Now, from a phenomelogical perspective these are unwelcome, and determine one of the major problems of compactifications, known as the *moduli problem*. Indeed, these massless scalar fields would carry long range interactions, which are not observed in the real world. Furthermore, the various couplings in the low energy 4d effective action depend on the vevs of the moduli; since these are undetermined, the theory loses most of its predictive power. It follows that string compactifications, and specifically Calabi-Yau compactifications, will not be fully satisfactory until this problem is solved.

A way out to the moduli problem would be to generate a non-trivial scalar potential in the 4d action, having the twofold effect of stabilizing the vevs and of providing mass terms for the moduli. In the last years, it has been realized that a promising mechanism to generate a potential for the moduli can be obtained by considering string theory compactifications with *fluxes* [13] (see [14, 15] for previous work). Since then, flux compactifications have been the object of an intense research activity. Some very nice recent reviews on the subject are [16, 17, 18].

Fluxes are associated with a nonvanishing background value of the p-form field-strengths which are contained in the higher-dimensional supergravity theories (notice that, as far as one takes the typical length of the compactification manifold well below the string scale ℓ_s, it is justified to work in the supergravity approximation to string theory). More precisely, let F_p be a p-form field strength, satisfying the Bianchi identity $dF_p = 0$, and let Σ_p be a non-trivial p-cycle of the compact manifold. Then one has a flux of F_p threading Σ_p if

$$\int_{\Sigma_p} F_p = n \neq 0\,.$$

As for Dirac's magnetic monopole, fluxes are subject to quantization conditions, so that in a quantum mechanical picture n can take just discrete values.

The reason why fluxes generate a 4d potential for the parameters controlling the compact geometry is apparent by considering the kinetic term for the internal field strength F_p in the higher-dimensional action. This reads

$$S = \int_{M_4} \ldots \underbrace{\int_{M_6} F \wedge * F}_{V}\,,$$

Figure 1.1: A schematic representation of a compactification with fluxes.

where M_6 is the compact manifold, and V is a function of the geometric moduli, since F_p couples with the metric on the compact manifold through the Hodge-$*$. In particular, the potential V will depend on the parameters controlling the size of the cycles threaded by the flux.

Moduli stabilization is not the only motivation for studying flux compactifications. Indeed, fluxes are also naturally sourced by the spacetime-filling D-branes which are considered in the modern approach to realistic compactifications. Furthermore, non-vanishing background values of the supergravity field strengths open new perspectives in the study of the geometry of string theory vacua. Indeed, they yield a nonvanishing constribution to the energy-momentum tensor of the higher-dimensional Einstein equation; it follows that the empty-space Ricci-flatness condition is removed. For instance, Calabi-Yau manifolds are no more available solutions (indeed, SU(3) holonomy implies Ricci-flatness). In some cases the backreaction due to the fluxes is mild, and one can still work with an underlying Calabi-Yau geometry [13], while in other situations it can be more drastic, and strongly deform the compact geometry (see e.g. [19] for an early example). In order to achieve a deeper understanding of the structure of string theory, and its relations with the lower-dimensional world, it becomes therefore very interesting to explore, and eventually classify, the possible compactifications with fluxes.

A systematic study of compactifications with fluxes is a challenging goal which requires new mathematical techniques. For this task, supersymmetry is again a powerful ally. Indeed, even though in the presence of fluxes the differential conditions for supersymmetric backgrounds become more involved, still one finds interesting underlying structures. In particular, one is led [20] to consider six-dimensional manifolds whose structure group lies in SU(3) [21]. Manifolds with *SU(3) structure* share with Calabi-Yau manifolds the existence of a globally defined and nowhere vanishing spinor, but are more general since the latter needs not being covariantly constant in the Levi-Civita connection. This global spinor is required in order to properly decompose the spinorial generators of the higher-dimensional supersymmetry transformations.

Actually, type II theories – on which this thesis focuses – involve *two* spinorial parameters, so that there is also the possibility to employ a pair of internal spinors in the decomposition associated with the compactification. This yields an enhanced freedom in the study of compactifications preserving a minimal fraction of supersymmetry. As we will discuss in detail in chapter 2, a suitable formalism for studying these compactifications is provided by *generalized geometry*, introduced in the mathematical literature by Hitchin in

2002 [22], and further developed in [23, 24, 25]. Generalized geometry deals with structures defined on $T \oplus T^*$, the sum of the tangent and cotangent bundle of the compact manifold.

One of the main applications of the generalized geometry formalism in string theory has been an elegant reformulation of the supersymmetry conditions for type II flux vacua [26, 27]. These take the form of differential equations for the differential forms which characterize the generalized geometry, and can in part be understood as an integrability condition for structures on $T \oplus T^*$. Subsequent related work, employing generalized geometry for the study of flux backgrounds, can be found in [28]–[40]. In physics, generalized geometry has also been applied to the study of supersymmetric worldsheet σ-models, starting with [42, 43]; see e.g. [44] for a review of this topic.

In compactifications, the study of the ground state is usually understood as a first (essential) step towards the determination of the lower-dimensional theory. At the level of the action, the minimal amount of supersymmetry preserved by type II pure supergravity compactifications is $N = 2$ in four dimensions. Reductions preserving just $N = 1$ are possible if one adds further ingredients, like certain projections induced by localized sources, relating the two ten-dimensional supersymmetry parameters. On the same footing as for the background, the best studied case of compactification leading to an $N = 2$ supergravity action in 4d again involves Calabi-Yau manifolds. As we will see, possible deformations of the Calabi-Yau dimensional reductions can be studied using the tools of generalized geometry. A program in this direction was started in [45] and pursued in [46], as a natural consequence of the previous studies of reductions on SU(3) structure manifolds, pioneered in [47].

In this thesis, we will build on this line of research to further study a general procedure for truncating type II theories to $N = 2$ supergravity in four dimensions. As it is generically the case when considering compactifications with fluxes, we will be led to study *gauged* $N = 2$ supergravities, which as an essential feature involve a scalar potential.

1.2 Outline of the thesis

The main aim of this thesis is to study dimensional reductions of type II theories leading to $N = 2$ supergravity in four dimensions. In doing this, we allow for a general set of background fluxes. The principal tools that are employed throughout the work are generalized geometry and gauged $N = 2$ supergravity.

The thesis is structured as follows.

We start in *chapter 2* with an introduction to the mathematical notions necessary for the analysis of the compactification. The spinor ansatz for dimensional reductions preserving $N = 2$ in 4d leads us to discuss G-structures on the 6d compact manifold. Then, by extending the notion of G-structure to the generalized tangent bundle $T \oplus T^*$, we introduce Hitchin's generalized geometry. In particular, we focus on the key notion of SU(3)$\times$SU(3) structure, which encodes all the NSNS degrees of freedom on the compact manifold. In view of the study of the scalar kinetic terms in 4d, we consider the metric on the parameter space of the internal NSNS fields. The latter can be reformulated in terms of deformations of the O(6,6) pure spinors which characterize the SU(3)$\times$SU(3) structure, and this allows to highlight an underlying special Kähler geometry, defined locally on the internal manifold. This result, to which we give original contributions, parallels to some extent the structure of the Calabi-Yau moduli space, which we also review.

In *chapter 3* we turn to the study of type II compactifications on flux backgrounds admitting SU(3)×SU(3) structure. We start with a brief review of the 'democratic' version of type II supergravities [48], which is particularly suitable for generalized geometry applications. We also discuss dimensional reductions on Calabi-Yau manifolds, which represent the model of reference for the subsequent developements. We illustrate the relation between fluxes and gaugings.

The truncation of type II supergravity is implemented on general SU(3)×SU(3) structure backgrounds via the expansion of the higher-dimensional fields in a finite basis of differential forms on the compact manifold. Compatibility with $N = 2$ supergravity in 4d requires this basis to respect a restrictive set of geometrical constraints, which have been identified in [45, 46, 49], and which we revisit. Then we fill a gap existing in the literature by deriving via dimensional reduction the complete four-dimensional bosonic action. In particular, we focus on the way its data are determined by generalized geometry, and we establish various results.

First we deal with the reduction of the NSNS sector of type II supergravity. We make the link with the space of deformations studied in the previous chapter. Then we study the role of a B-twisted Hodge star operator, and in particular we show how its action on the basis of forms generalizes to the SU(3)×SU(3) context the well-known expression for the usual Hodge-$*$ acting on the harmonic three-forms of a Calabi-Yau manifold. This allows to derive a formula for the period matrices of the $N = 2$ special Kähler geometry. Next we focus on the 4d scalar potential: we prove a formula expressing the internal Ricci curvature in terms of the generalized geometry data, and we apply it to deduce a geometric expression for the scalar potential. Once restated in terms of 4d variables, this gives back the symplectically invariant and mirror-symmetric expression found in [50] by means of purely 4d gauged supergravity methods.

In the last part of the chapter we move to the RR sector, with a focus on type IIA. Instead of directly reducing the action, we choose to reduce the equations of motion. As a consequence of a self-duality constraint, these can also be read as Bianchi identities. The expansion of the democratic RR field on the internal basis automatically introduces forms of all possible degrees in the 4d spacetime. We interpret a subset of the reduced RR equations as 4d Bianchi identities; by their solution we define the 4d fundamental fields. The remaining equations are seen as 4d equations of motion, from which we reconstruct the reduced action.

In *chapter 4* we illustrate further the consistency between the outcome of the dimensional reduction and the formalism of gauged $N = 2$ supergravity. A consistent formulation in the presence of a complete set of fluxes requires the introduction of tensor multiplets. We focus on the quantities determining the gauging, whose associated charges are generated by the NSNS, RR and geometric fluxes. Then, starting from the expression of the $N = 2$ Killing prepotentials, and using some general results about $N = 2$ supergravity with tensor multiplets, we deduce the fermionic shifts in the 4d supersymmetry variations.

In the second part of the chapter we confront the 4d and 10d approaches to the $N = 1$ flux backgrounds. At the 10d level, we adopt the generalized geometry reformulation of the equations for an $N = 1$ vacuum found in [26, 27]. In order to perform a comparison with the 4d supersymmetry conditions, we rephrase these equations in a 4d framework performing the integral over the internal manifold. Next we derive the $N = 1$ vacuum conditions within the 4d $N = 2$ theory, by imposing the vanishing of the fermionic shifts under a single susy transformation. Exploiting the properties of special Kähler geometry, we establish a precise matching with the integrated version of the pure spinor equations. We also perform a similar study by considering the 4d $N = 1$ supergravity which arises as a truncation of the previously analyzed $N = 2$ theory: we derive the expressions for the superpotential and D-terms, and we impose the F-flatness and D-flatness conditions; again we find precise correspondence with the 10d equations.

Finally, in *chapter 5* we present some concrete examples of $N = 2$ compactifications, based on coset spaces with SU(3) structure. To some extent, these can be seen as an application of the general study done in the previous chapters. Thanks to the full control on the geometry allowed by the coset structure, we can perform an explicit analysis. In particular, we establish the consistency of the dimensional reduction based on a left-invariant truncation ansatz. This gives a solid justification to the choice of the expansion forms. Then we explore the supersymmetric and non-supersymmetric backgrounds associated with the compactification, parameterizing the solutions in terms of the fluxes. Exploiting the constraints imposed by $N = 2$ supersymmetry, we study the string loop corrections to the 4d scalar potential, and we perform a preliminary search of de Sitter extrema.

In *chapter 6* we draw our final considerations.

We relegate some technical discussions to the *appendix*. Appendix A summarizes our conventions. Appendix B gives some details about the Mukai pairing and the Clifford map used in the generalized geometry computations. Appendix C discusses the relation between the democratic and the standard formulation of type IIA supergravity, clarifying some subtleties related to the presence of fluxes. Appendices D and E give the definition and some properties respectively of special Kähler manifolds and of quaternionic-Kähler manifolds, which play a central role in $N = 2$ supergravity. Finally, appendix F collects some details of the coset space dimensional reductions discussed in chapter 5, and appendix G derives the string loop corrections to the associated 4d, $N = 1$ vacua.

This thesis is based on the following papers:

[P1] D. Cassani and A. Bilal, *Effective actions and N=1 vacuum conditions from SU(3)×SU(3) compactifications*, JHEP **0709** (2007) 076 [arXiv:0707.3125 [hep-th]].

[P2] D. Cassani, *Reducing democratic type II supergravity on SU(3) × SU(3) structures*, JHEP **0806** (2008) 027 [arXiv:0804.0595 [hep-th]].

[P3] D. Cassani and A. K. Kashani-Poor, *Exploiting N=2 in consistent coset reductions of type IIA*, Nucl. Phys. B **817** (2009) 25 [arXiv:0901.4251 [hep-th]].

Chapter 2

Generalized structures in type II supergravity

In this chapter the needed mathematical notions are introduced. The spinor ansatz for dimensional reductions preserving $N = 2$ in 4d leads us to discuss SU(3)×SU(3) structures on the generalized tangent bundle. These conveniently encode all the NSNS degrees of freedom on the internal manifold. In view of the derivation of the lower dimensional scalar kinetic terms, we study deformations of SU(3)×SU(3) structures, and discuss their special Kähler geometry.

2.1 Motivation

We are interested in dimensional reductions preserving a fraction of supersymmetry. As a first consequence, we restrict our attention to internal spaces supporting spinors, namely spin manifolds.[1] Next, in order to determine the amount of supersymmetry preserved by the compactification, we have to exhibit a truncation ansatz for the spinorial parameters appearing in the higher dimensional supersymmetry transformations. Indeed, assuming a spacetime topology of the form $M_D = M_d \times M_{D-d}$ (where M_d is a Lorentzian spacetime and M_{D-d} is a Riemannian compact manifold), each of these parameters decomposes under $\mathrm{Spin}(D-1,1) \to \mathrm{Spin}(d-1,1) \times \mathrm{Spin}(D-d)$, and its expansion in a given number of internal spinors defines the possible susy parameters of the would-be lower dimensional theory. Calling ϵ the higher dimensional susy parameter, its decomposition generically reads

$$\epsilon = \sum_{k=1}^{N} \varepsilon^k \otimes \eta^k \ ,$$

where ε^k are (anticommuting) $\mathrm{Spin}(d-1,1)$ spinors on the d dimensional spacetime, and η^k are (commuting) $\mathrm{Spin}(D-d)$ spinors on the compact manifold. The latter are usually required to be globally defined, and have to satisfy some specified condition. The number N of independent spinors fulfilling such condition determines the number of independent lower

[1] A manifold M is spin if it admits a spin structure. This is possible if and only if M is orientable and $w_2(T(M)) = 0$, where $w_2 \in H^2(M, \mathbb{Z}_2)$ is the second Stiefel-Whitney class. Examples of manifolds which are orientable but not spin are provided by the even complex projective spaces $\mathbb{CP}^{2n}$.

dimensional supersymmetry transformations, with parameter ε^k. For instance, consider the dimensional reduction of 11d supergravity on the torus T^7, leading to the maximal 4d $N = 8$ supergravity [9]. The Kaluza-Klein ansatz, guaranteeing a truncation to the massless 4d fields only, prescribes a decomposition of the Majorana 11d susy parameter ϵ in which the η^k are chosen to be constant. Since on T^7 there is the maximal number - i.e. eight - of independent constant spinors, we see that precisely the eight 4d supersymmetry parameters of $N = 8$ supergravity are defined.

Since now on we will focus on dimensional reductions of type II theory leading to four dimensional supergravity. The higher dimensional spacetime M_{10} is chosen to be topologically a product $M_{10} = M_4 \times M_6$, where M_4 is the 4d spacetime and M_6 is an 'internal' 6d compact manifold. This implies a breaking of the local Lorentz SO(9,1) invariance to SO(3,1)×SO(6). It follows that at each point of M_{10} the Spin(9,1) spinor representation decomposes according to Spin(9,1)→Spin(3,1)× Spin(6).

Our task is to study general type II compactifications preserving the minimal $N = 2$ off-shell supersymmetry in 4d. Let us call ϵ^1 and ϵ^2 the two supersymmetry parameters associated with the left-moving and the right-moving sector of the type II superstring. Their expansion in a given number of internal Spin(6) spinors defines the 4d susy parameters. The general spinor ansatz possibly preserving $N = 2$ in 4d is [45]

$$\begin{aligned} \epsilon^1(x,y) &= \varepsilon_1(x) \otimes \eta^1_{\mp}(x,y) + c.c. \\ \epsilon^2(x,y) &= \varepsilon_2(x) \otimes \eta^2_{+}(x,y) + c.c.\,, \end{aligned} \tag{2.1}$$

where x and y are coordinates on M_4 and M_6 respectively, and the upper/lower sign in the first line refer to type IIA/IIB. The Spin(3,1) Weyl spinors $\varepsilon_1, \varepsilon_2$ have positive-chirality, and parameterize the eight supercharges of 4d $N = 2$ local supersymmetry. In our choice of the spinor algebra conventions (see appendix A.3 for a detailed account), if ε is a Spin(3,1) Weyl spinor with positive chirality ($\gamma_5\varepsilon = \varepsilon$), then its complex conjugate ε^* is again a Weyl spinor, with negative chirality. The Spin(6) spinors in eq. (2.1) are Weyl spinors; η^1_+ has positive chirality, while $\eta^1_- \equiv (\eta^1_+)^*$ has negative chirality, and analogously for $\eta^2_\pm$ (again, we refer to appendix A.3 for details). It follows that the Spin(9,1) spinors ϵ^1, ϵ^2 are Majorana-Weyl, as required by type II supergravity. Note that ϵ^1 has negative/positive chirality respectively for type IIA/IIB, while ϵ^2 has always positive chirality.

Ansatz (2.1) includes as a particular case the spinor decomposition associated with compactifications on Calabi-Yau 3–folds, which is obtained by taking $\eta^1_+ = \eta^2_+ \equiv \eta_+$, where η_+ is the unique (up to constant rescalings) covariantly constant spinor with positive chirality existing on any Calabi-Yau manifold. See section 2.3 for an introduction to Calabi-Yau geometry. However, ansatz (2.1) is more general than the the Calabi-Yau one, since η^1_+ and η^2_+ are not necessarily proportional; moreover, at this stage we don't require the internal spinors to satisfy any differential condition. The only condition we impose is that

η^1_+, η^2_+ be globally defined and nowhere vanishing on the compact manifold M_6.

This ensures that the reduction ansatz is everywhere well-defined. As we will discuss below, this is a non-trivial topological requirement, constraining the structure group of the compact manifold M_6. It allows to employ the formalism of *G-structures*, reviewed in section 2.2 below, for a systematic study of flux compactifications.

Clearly, the existence of globally defined nowhere vanishing spinors alone does not automatically imply the dimensional reduction to yield an $N = 2$ supergravity. For this to be possible, further conditions need to be imposed. While these further constraints will be discussed in chapter 3, in this chapter we study the general consequences arising by the mere existence of the globally defined nowhere vanishing spinors η^1 and η^2. We will see in subsection 2.2.2 that any globally defined nowhere vanishing chiral spinor η_+ on M_6 identifies a subgroup SU(3) of the generic structure group SU(4) $\cong$ Spin(6) of the principal spin bundle, and this determines an SU(3) structure on M_6, meaning that the transition functions of the frame bundle are constrained to lie in SU(3) $\subset$ SU(4). One therefore deduces that the decomposition (2.1) implies the existence of a pair of SU(3) structures (namely, two different embeddings of SU(3) in SU(4)), one for each of the two globally defined spinors η^1_+, η^2_+. Now, different situations can occur, corresponding to the different possible intersections of the two SU(3) subgroups of Spin(6). If the spinors η^1_+ and η^2_+ are everywhere parallel, then the two SU(3) subgroups actually coincide, determining what we will call a *strict* SU(3) structure on M_6. However, generically we will consider the two spinors being independent almost everywhere, and becoming parallel at some points: in this situation locally η^1_+ and η^2_+ identify an SU(2) subgroup of SU(4), but this fails at the points where the spinors become linearly dependent. Nowhere parallel η^1_+ and η^2_+ select an SU(2) everywhere on M_6, defining in this way an SU(2) structure.[2]

In section 2.4 we will see how the aforementioned different situations concerning the internal spinors can be conveniently described in a unified fashion by considering *generalized structures* on $TM_6 \oplus T^*M_6$, the sum of the tangent and the cotangent bundles of M_6. After having introduced and reviewed the relevant notions of the generalized structure formalism, in the second part of section 2.4 we will discuss how the spinor ansatz (2.1) can be associated with the existence of an SU(3)$\times$SU(3) structure on $TM_6 \oplus T^*M_6$. As we will illustrate, SU(3)$\times$SU(3) structures also provide an useful description of the full NSNS sector of type II supergravity on the compact manifold. In section 2.5 we turn to study deformations of SU(3)$\times$SU(3) structures, investigating their special Kähler geometry properties. In particular, having in mind the application to the problem of compactifications, to be implemented in chapter 3, we establish the relation between the deformation space of SU(3)$\times$SU(3) structures, and the space of parameters of a metric g and a 2–form b on M_6. Section 2.6 closes the chapter with an overview and further comments.

2.2 G-structures

Motivated by the above arguments, we now start introducing the mathematical notions that we'll need in the next chapters to study compactifications of type II theories. While we save the study of generalized structures for the second part of this chapter, in this section we focus on a prerequisite, namely the notion of G-structure.

Starting from [20, 47, 52], over the last few years the G-structure formalism has been fruitfully applied to flux compactifications. In particular, [47] pioneered the use of this technology to approach the problem of deriving the 4d effective actions associated with

[2] In this last case the spinor ansatz (2.1) might also be extended to possibly define $N = 4$ compactifications, since each of the 10d susy parameters can be decomposed on either η^1_+ *and* η^2_+, see e.g. [51].

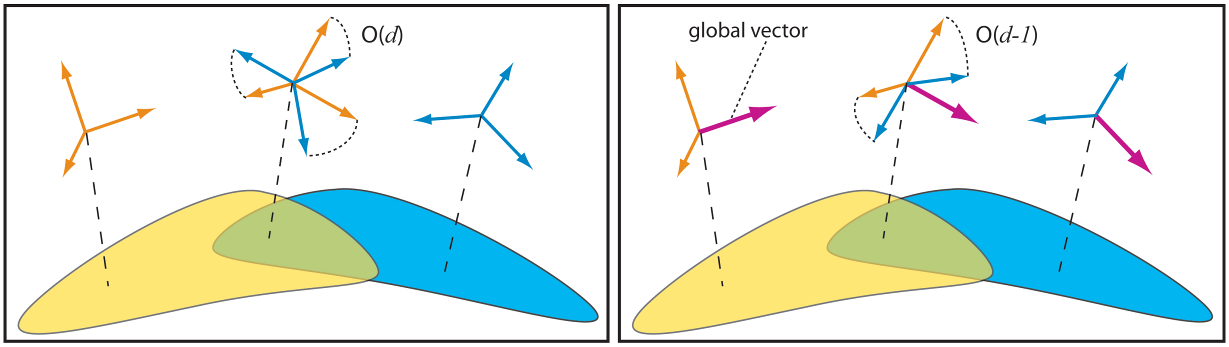

Figure 2.1: On the left: The generic structure group of the frame bundle of a Riemannian manifold M_d is $\mathrm{O}(d)$. Its action sends the orthonormal frame at a point of a patch into the corresponding orthonormal frame on an overlapping patch. On the right: a simple example of reduction of the structure group, induced by the existence of a globally defined nowhere vanishing vector. Frames can be chosen in such a way that the vector takes the same form everywhere; this constrains the transition functions on patch overlaps to lie in $\mathrm{O}(d-1)$.

certain non-Calabi-Yau compactifications. In the context of flux compactifications, the main advantage of the G-structure formalism is that it allows for a systematic analysis. This is made possible by the fact that the various higher dimensional fields, including the fluxes, decompose in irreducible representations of the structure group G characterizing the G-structure on the internal manifold. It follows that the terms in the equations of motion and/or in the supersymmetry conditions transforming differently under G decouple, and can be considered separately. This clearly facilitates the analysis and provides a classification of the possible solutions.[3]

It is time to come to the proper definition of a G-structure. Recall first that the *frame bundle* of a d-dimensional manifold M is the principal bundle over M whose fibre at any point $p \in M$ is made of all ordered bases - the frames - of the tangent space T_pM. Generically the fibre can be identified with $\mathrm{GL}(d, \mathbb{R})$.

Now, let G be a Lie subgroup of $\mathrm{GL}(d, \mathbb{R})$. Then a *$G$-structure* on M is a principal subbundle of the frame bundle, with fibre G. In other words, we have a G-structure when the generic $\mathrm{GL}(d, \mathbb{R})$ structure group of the frame bundle is reduced to a subgroup G. We see that the notion of G-structure is topological in nature.

Several well-known geometric structures on M_d can be seen as G-structures. This is related to the fact that reductions of the generic $\mathrm{GL}(d, \mathbb{R})$ structure group are characterized by the existence of tensors on M_d which are globally defined and nowhere degenerate. Indeed, given a globally defined non-degenerate tensor, frames can be chosen in such a way that this takes the same fixed form everywhere on the manifold. This amounts to state that the tensor is invariant under the action of the transition functions gluing the frames on two overlapping patches of M_d. The transition functions are then constrained to be in a subgroup G of $\mathrm{GL}(d, \mathbb{R})$, and a G-structure is determined.

[3]See e.g. [16] for a review of these applications of the G-structure formalism. A more mathematically oriented, but still pedagogical, discussion of G-structures can be found in [58]. Both these reviews discuss generalized geometry as well.

Riemannian manifold. For instance, if M_d admits a Riemannian metric g, on each patch frames can be chosen in such a way that in frame indices g reads as the identity matrix. This condition is preserved just by the transition functions of the frame bundle living in $\mathrm{O}(d) \subset \mathrm{GL}(d,\mathbb{R})$, hence an $\mathrm{O}(d)$ structure is determined. The converse is also true: a reduction of the $\mathrm{GL}(d,\mathbb{R})$ structure to $\mathrm{O}(d)$ amounts to fix a Riemannian metric.

In the following we discuss further examples of the correspondence between G-structures and interesting global non-degenerate tensors on M_d. All our examples are summarized in table 2.1. Figure 2.1 depicts the notion of $\mathrm{O}(d)$ structure, and its further reduction due to a globally defined nowhere zero object (in the case of the figure, a vector).

Oriented manifold. A globally defined and nowhere zero d–form on M_d (i.e. a volume form) is the same as an $\mathrm{SL}(d,\mathbb{R})$ structure. This fixes an orientation for M_d. If the volume form is associated with a Riemannian metric, then we have a further reduction to an $\mathrm{SO}(d)$ structure.

Parallelizable manifold. The maximal possible reduction, i.e. a trivial $\{e\}$-structure, corresponds to the trivialization of the frame bundle induced by a global frame. A manifold with trivial frame bundle is called *parallelizable*. Beside all the Lie groups, an example of parallelizable manifold which is particularly relevant for supergravity compactifications is the 7–sphere S^7.

Almost symplectic manifold. A non-degenerate real 2–form J determines an $\mathrm{Sp}(d,\mathbb{R})$ structure. Non-degeneracy requires d even.[4] Setting $n = d/2$, a volume form is then determined by

$$vol_d = \frac{1}{n!} J^n \,, \tag{2.2}$$

where $J^n = \underbrace{J \wedge \ldots \wedge J}_{n \text{ times}}$.

Almost complex manifold. An almost complex structure on M_d is a globally defined map

$$I \,:\, TM_d \to TM_d \qquad \text{such that} \qquad I^2 = -\mathbb{1} \,. \tag{2.3}$$

A necessary condition for its existence is that the dimension of the manifold be even: $d = 2n$. I splits the complexified tangent bundle $TM \otimes \mathbb{C}$ into $+i$ and $-i$ eigenbundles, of equal complex dimension n; vector fields of these subbundles are called of type $(1,0)$ and type $(0,1)$ respectively. This decomposition implies that the structure group of the frame bundle is reduced to $\mathrm{GL}(n,\mathbb{C})$. Indeed, given an $n \times n$ complex matrix transforming the $(1,0)$–frames on patch overlaps, the transformation of the $(0,1)$–frames is determined by the complex conjugate matrix.

From the $(1,0)$– and $(0,1)$–vectors, by duality one can define the $(1,0)$– and $(0,1)$–forms on $T^*M \otimes \mathbb{C}$, and then build (p,q)–forms by wedging. Calling $\Omega^k(M,\mathbb{C})$ the space of complex k–forms on M, and $\Omega^{(p,q)}(M)$ the space of (p,q)–forms, one has the decomposition

$$\Omega^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{(p,q)}(M) \,.$$

[4] The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is the Lie group of real $2n \times 2n$ matrices A preserving the matrix $S = \left(\begin{smallmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{smallmatrix}\right)$, i.e. satisfying $A^T S A = S$. Its dimension is $n(2n+1)$.

Global, non-degenerate tensors	G-structure
Riemannian metric g	$\mathrm{O}(d)$
volume form	$\mathrm{SL}(d,\mathbb{R})$
volume form associated with g	$\mathrm{SO}(d)$
global frame	$\{e\}$
real 2–form J $(\Rightarrow d$ even$)$	$\mathrm{Sp}(d,\mathbb{R})$
almost complex structure I $(\Rightarrow d$ even$)$	$\mathrm{GL}(d/2,\mathbb{C})$
decomposable $d/2$–form Ω $(\Rightarrow d$ even$)$	$\mathrm{SL}(d/2,\mathbb{C})$
$\left.\begin{array}{c} J \text{ and } I,\ \text{ with } I^T J I = J \text{ and } g := JI > 0 \\ I \text{ and } g,\ \text{ with } I^T g I = g \quad (\Rightarrow J = -gI) \\ g \text{ and } J,\ \text{ with } J^T g^{-1} J = g \quad (\Rightarrow I = -g^{-1}J) \end{array}\right\}$	$\mathrm{U}(d/2)$
J and Ω, with $J\wedge\Omega = 0$ and $g := JI > 0$	$\mathrm{SU}(d/2)$

Table 2.1: G-structures on a d-dimensional manifold M_d induced by globally defined non-degenerate tensors. One can also think in the other way around: given a G-structure, one or more invariant tensors are determined.

In particular, it follows that locally one can always define an $(n,0)$–form, but in general this won't be a global section of $\wedge^{(n,0)}T^*M$. If this is the case, the $\mathrm{GL}(n,\mathbb{C})$ structure is further reduced to $\mathrm{SL}(n,\mathbb{C})$. Indeed, an $(n,0)$–form is left invariant precisely by the $\mathrm{GL}(n,\mathbb{C})$ transformations having unit determinant.

Actually, the existence of a global, non-degenerate, complex decomposable[5] n–form Ω_n already determines an $\mathrm{SL}(n,\mathbb{C})$ structure. It follows that Ω_n has to define an almost complex structure I on M_d, with respect to which it is of type $(n,0)$. For $d=6$ (the case of interest for us) the explicit formula for I is [53]

$$I^m{}_n = 4\,\frac{\mathrm{Re}\Omega_3\wedge dy^m\wedge \iota_{\partial_n}\mathrm{Re}\Omega_3}{i\Omega_3\wedge\bar\Omega_3}\ , \tag{2.4}$$

where at each point of M_d we define the ratio of two top–forms by the ratio of their unique elements.

Almost hermitian manifold. This corresponds to a $\mathrm{U}(n)$ structure, where again the dimension of M is $d=2n$. Since $\mathrm{U}(n)$ is the intersection of the groups $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{O}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$, this structure incorporates a Riemannian metric g, an almost complex structure I and an almost symplectic structure J. Actually,

$$\mathrm{U}(n) = \mathrm{GL}(n,\mathbb{C})\cap\mathrm{O}(2n) = \mathrm{O}(2n)\cap\mathrm{Sp}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R})\cap\mathrm{GL}(n,\mathbb{C})$$

(see figure 2.2 for a pictorial representation). It follows that two out of the three tensors I, g and J determine the remaining one, provided a certain compatibility condition is satisfied.

[5]A p–form is decomposable if locally it can be written as the wedging of p complex 1–forms.

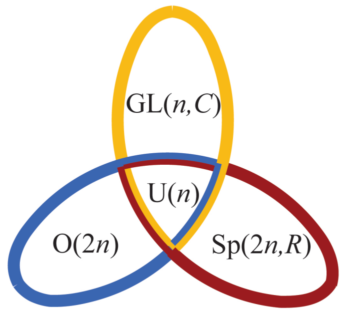

Figure 2.2: The intersection of two of the three groups $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{O}(2n)$, $\mathrm{Sp}(2n,\mathbb{R})$ is equal to the intersection of all the three, and coincides with $\mathrm{U}(n)$.

Hence we have three possible ways to define a $\mathrm{U}(n)$ structure via invariant tensors, see table 2.1. Here we discuss the case in which I and J are given. The compatibility relation they have to satisfy is

$$I^T J I = J\,. \tag{2.5}$$

This says that J is $(1,1)$ with respect to I and, together with (2.3), implies that the tensor

$$g := JI \tag{2.6}$$

is symmetric. In order to have a $\mathrm{U}(n)$ structure, and not a $\mathrm{U}(p,n-p)$ one, we also need to require that g be positive, i.e. that it is a Riemannian metric.

Note that g is automatically hermitian:

$$I^T g I = g\,.$$

SU(n) structure. If the almost complex structure I of an almost hermitian $2n$-dimensional manifold is determined by the globally definite, non-degenerate, decomposable n–form Ω_n introduced above, then the $\mathrm{U}(n)$ structure is further reduced to $\mathrm{SU}(n)$. The compatibility relation (2.5) can now be rephrased as

$$J \wedge \Omega_n = 0\,. \tag{2.7}$$

Indeed, since Ω_n is $(n,0)$ with respect to I, eq. (2.7) is equivalent to say that the real 2–form J is $(1,1)$. One also imposes the top-form normalization condition

$$(-1)^{n(n-1)/2}(i/2)^n \Omega_n \wedge \overline{\Omega}_n \;=\; J^n/n! \;\equiv\; vol_{2n}\,, \tag{2.8}$$

which is also another way of stating the non-degeneracy of J and Ω_n.

$\mathrm{SU}(n)$ structures (or subgroups thereof) are particularly interesting for supergravity applications, because they imply the existence of globally defined and nowhere vanishing spinors. In the forthcoming subsection 2.2.2 we discuss in detail the 6-dimensional case. Before however we introduce the concept of integrability.

2.2.1 Integrability

The geometric structures discussed above are all topological notions. To the ones denoted with 'almost' it is possible to associate a differential geometric notion, holding if a certain *integrability* condition is satisfied. The latter is a differential condition on the tensors defining the structure, and can be seen as a vanishing torsion requirement for the corresponding G-structure (see e.g. [54] for details). If the integrability equation is satisfied, then one can drop the 'almost' in the denomination of the geometric structure. In particular:

- a *symplectic manifold* is an almost symplectic manifold with $dJ = 0$.
- A *complex manifold* is an almost complex manifold whose associated Nijenhuis tensor, defined as
$$N_{mn}{}^{p} \;:=\; I^{p}{}_{q}\partial_{[m}I^{q}{}_{n]} + \partial_{q}I^{p}{}_{[m}I^{q}{}_{n]}\,,$$
vanishes.
- Further, a *hermitian manifold* is an almost hermitian manifold with integrable almost complex structure.
- If also the almost symplectic structure J of a hermitian manifold is integrable (namely, closed), then we have a *Kähler manifold*. In this case, J is called the *Kähler form*. A Kähler manifold has holonomy $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$. An important property characterizing Kähler manifolds is their metric can locally be derived from a *Kähler potential* K via the relation $g_{i\bar{\jmath}} = \partial_i\partial_{\bar{\jmath}}K$ (here $i, \bar{\jmath}$ label complex coordinates on the manifold).
- Finally, a compact manifold admitting a torsionless $\mathrm{SU}(n)$ structure is a *Calabi–Yau n-fold*; as it will be discussed further for $n = 3$ in the next subsection, the no-torsion requirement is expressed in terms of J and Ω_n by
$$dJ = 0 \quad , \quad d\Omega_n = 0\,,$$
and coincides with the usual $\mathrm{SU}(n)$ holonomy condition. We will give more details on Calabi-Yau geometry in the forthcoming subsection 2.3.

2.2.2 SU(3) structures and spinors

In view of the applications to supergravity, we are interested in manifolds supporting spinors. A first result in this sense, involving G-structures, is the following [54, prop. 3.6.2]: suppose that a manifold M of dimension $d \geq 3$ admits a G-structure, where G is a connected, simply-connected subgroup of $\mathrm{SO}(d)$. Then M is spin.

In the following we will consider just manifolds satisfying these requirements, so that the spin property doesn't need to be imposed separately.

Actually, as discussed in section 2.1, we wish to study compact six dimensional manifolds admitting at least one globally defined nowhere vanishing spinor. It is at this point that G-structures, in particular SU(3) structures, enter into the game in an essential way. Indeed, the existence of a globally defined and nowhere vanishing spinor is equivalent to a reduction of the structure group to SU(3). To see this, assume we have a global non-zero spinor η_+, with positive chirality. In order to discuss spinors, we have to preliminarly require an SO(6) structure – i.e. a metric and an orientation – to be defined on M_d; then the structure group of the spin bundle is the double cover of SO(6), i.e. Spin(6) $\cong$ SU(4), and positive chirality spinors transform in its fundamental representation on patch overlaps. By a suitable choice of frame, we can set $\eta_+ = (0,0,0,z)^T$ everywhere. Then the SU(4) elements leaving η_+ invariant are of the form $\left(\begin{smallmatrix} U & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the 3×3 matrix U being in SU(3). Conversely, suppose an SU(3) structure is defined on M_6. The fundamental representation of SU(4) decomposes under SU(3) as $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$. The fact that the decomposition contains a singlet means that there is an invariant spinor η_+. Because η_+ is invariant, it has to be globally defined and nowhere vanishing. Of course the same argument applies to the conjugate spinor $\eta_- = \eta_+^*$, this time starting from the $\bar{\mathbf{4}}$ of SU(4).

The method of decomposing a given SO(6) representation in irreps of the structure group $G \subset$ SO(6) in order to identify the invariant objects is general, the latter always corresponding to the singlets of the decomposition. For instance, the forms J and Ω_3 introduced above (in the following Ω_3 will be called simply Ω) correspond to the singlets appearing in the decomposition under SU(3) respectively of the **15** and of the **20** antisymmetric tensor representations of SO(6).

Since the spinor η_+, together with a metric g, determines an SU(3) structure, then it has to define J and Ω too. Indeed, assuming $\eta_+^\dagger \eta_+ = 1$, these are given by the bilinears

$$J_{mn} = -i\eta_+^\dagger \gamma_{mn} \eta_+ \qquad , \qquad \Omega_{mnp} = -i\eta_-^\dagger \gamma_{mnp} \eta_+ \,. \tag{2.9}$$

Notice that the metric g plays a role here, since it enters in the definition of the gamma-matrices with curved indices. In section A.4 of the appendix we collect some further relations satisfied by the SU(3)-structure invariant objects.

The decomposition in irreducible representations of SU(3) can also be used to classify the different possible SU(3) structures. This is done evaluating the *intrinsic torsion* of the SU(3) structure, defined as the measure of the failure of J and Ω to be closed.[6] The decomposition of dJ and $d\Omega$ in SU(3) irreps yields five *torsion classes* W_i [21]:

$$\begin{aligned} dJ &= \frac{3}{2}\mathrm{Im}(\bar{W}_1 \Omega) + W_4 \wedge J + W_3 \\ d\Omega &= W_1 \wedge J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega \,, \end{aligned} \tag{2.10}$$

where W_1 is a complex scalar, W_2 is a complex primitive (1,1)–form (primitive means $W_2 \wedge J \wedge J = 0$), W_3 is a real primitive (1,2) + (2,1)–form (primitive $\Leftrightarrow W_3 \wedge J = 0$), W_4 is a real 1–form, and W_5 is a complex (1,0)–form. The W_i transform differently under SU(3) (see e.g. [16] for more details), and classify the SU(3) structure. For instance, recalling the definitions of subsection 2.2.1, we see that an SU(3) structure manifold is

- symplectic if $W_1 = W_3 = W_4 = 0$;

- complex when $W_1 = W_2 = 0$. Indeed, if the manifold is complex then $d\Omega$ has to be a (3,1)–form; it follows that in eq. (2.10) only the term containing W_5 is allowed, the

[6] The intrinsic torsion can equivalently be defined as the failure of the spinor η_+ to be covariantly constant in the Levi-Civita connection.

other two being (2,2). The converse is also true, i.e. the vanishing of the (2,2) terms in $d\Omega$ sets to zero the Nijenhuis tensor associated with the almost complex structure I.

- Kähler if $W_1 = W_2 = W_3 = W_4 = 0$.
- Finally, a Calabi-Yau 3–fold is a compact manifold admitting an SU(3) structure with vanishing torsion: $W_i = 0\ \forall i$.

In chapter 5 we will discuss explicit constructions of SU(3) structures on coset spaces, having $W_3 = W_4 = W_5 = 0$.

2.3 Calabi-Yau manifolds and their moduli space

Before moving to the study of generalized structures, we consider in more detail the geometry of Calabi-Yau n-folds. Compactification on Calabi-Yau 3-folds represents by far the best studied case of dimensional reduction of type II theory leading to $N = 2$ supergravity. For this reason, in the next chapter we will use it as a guiding example for developing dimensional reductions on more general spaces.

2.3.1 Definition and properties

Above we introduced Calabi-Yau 3–folds as compact 6-dimensional manifolds admitting an SU(3) structure with vanishing torsion. This characterization can be generalized to arbitrary even dimension, and is equivalent to the following definition, close to the one given in [54, 55]: let M be a compact Kähler manifold of complex dimension n, with complex structure I and Kähler metric g. Then (M, I, g) is a Calabi-Yau n-fold if g has SU(n) holonomy.

In the following it will always be understood that the holonomy coincides with the full group SU(3), and not a subgroup thereof.

The SU(n) holonomy condition implies that the Kähler metric is Ricci-flat. If the manifold is simply connected, the converse is also true.

Moreover, Ricci-flatness implies the vanishing of the first Chern class $c_1(M)$, which on compact Kähler manifolds (M, I, g) with Ricci tensor $R_{mp}(g)$ can be seen as the cohomology class of the Ricci form, divided by 2π:

$$c_1(M) = \left[\frac{1}{2\pi}\mathcal{R}\right], \quad \text{where} \quad \mathcal{R} = \frac{1}{2}\mathcal{R}_{mn} dy^m \wedge dy^n, \quad \text{with} \quad \mathcal{R}_{mn} = R_{mp} I^p{}_n \,.$$

The converse statement follows from a conjecture formulated by Calabi in 1954, and eventually proved by Yau in 1976 in a celebrated theorem. The latter implies that if M is a compact complex manifold with $c_1(M) = [0] \in H^2(M, \mathbb{R})$, then every Kähler class on M contains a unique Ricci-flat Kähler metric.[7] Beside explaining the naming 'Calabi-Yau',

[7] The Kähler class $[J] \in H^{1,1}(M, \mathbb{R})$ is the cohomology class of the Kähler form J. In complex coordinates μ, ν, the Kähler form and the Kähler metric are related by $J_{\mu\bar{\nu}} = i g_{\mu\bar{\nu}}$. Concerning complex coordinates, see the remark below eq. (2.12).

this property also justifies an alternative definition which is often adopted, by which a Calabi-Yau n-fold is a compact Kähler manifold of vanishing first Chern class.

Below we collect some further well-known facts about Calabi-Yau manifolds.

A Calabi-Yau n-fold admits a globally defined, nowhere vanishing, covariantly constant holomorphic $(n,0)$-form Ω_n, that is unique up to multiplication by a non-zero complex number. Imposing relation (2.8) fixes Ω_n up to a constant overall phase.

We see therefore that Calabi-Yau n-folds admit an SU(n) structure (J,Ω_n) such that $dJ=0$ and $d\Omega_n=0$. The converse is also true: for $n=3$, we saw above that an SU(3) structure manifold with $dJ=0$ and $d\Omega=\bar{W}_5\wedge\Omega$ is Kähler, hence it has U(n) holonomy. If on top of this $W_5=0$, i.e. Ω is closed, then one has a reduction of the holonomy to SU(3) (or a subgroup thereof). Analogous arguments hold for $n\neq 3$.

The SU(3) invariant spinor η_+ associated with the torsionless SU(3) structure is covariantly constant, i.e. it satisfies

$$D_m\eta_+=0\,.$$

Finally, we briefly recall the cohomology properties of Calabi-Yau 3–folds. As we will review in section 3.2, these play a crucial role in determining the lower dimensional effective theory arising from string compactification on Calabi-Yau manifolds.

On any Kähler manifold M, the de Rahm cohomology groups decompose in Dolbeault cohomologies $H^{p,q}(M)$. Their dimensions are the Hodge numbers $h^{p,q}$, arranging in the Hodge diamond (here presented for a manifold of complex dimension $n=3$)

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}\,.$$

By complex conjugation, one has $h^{p,q}=h^{q,p}$, and by Poincaré duality $h^{p,q}=h^{3-p,3-q}$. Furthermore, one always has $h^{0,0}=h^{3,3}=1$. For a Calabi-Yau 3-fold the Hodge diamond is restricted even further. Provided the 3–fold has strict SU(3) holonomy, it takes the form

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \qquad (2.11)$$

The fact that $h^{3,0}=1$ means that there exists a unique (up to constant rescalings) holomorphic (3,0)–form.

2.3.2 The moduli space of Calabi-Yau 3-folds

Knowledge of the possible metric deformations of a given class of manifolds is crucial in compactifications of higher dimensional gravity theories. Indeed, in this context the parameters describing the space of possible internal metrics are promoted to dynamical scalar fields propagating along the uncompactified directions. Hence, the moduli space of the metric on the compact manifold is interpreted as a target space in which the scalars take value. The metric on the moduli space provides then the kinetic terms for these scalars.

In the following we give a brief account of the results presented in the classic paper [56], where the moduli space of Calabi-Yau 3–folds M is shown to locally split in a product of two factors, respectively describing deformations of the complex structure and of the Kähler structure on M. Both these factors display a special Kähler structure, meaning that the the geometry is governed by a Kähler potential which is itself determined by a certain holomorphic function, named the prepotential. As we will see, here differential forms, and in particular the cohomology representatives, play a crucial role in describing the metric deformations of a Calabi-Yau manifold.

We start observing that, once a given topology is assigned, the local structure of the moduli space of Calabi-Yau 3-folds can be explored by deforming the Ricci-flat Calabi-Yau metric g, and requiring that Ricci-flatness be preserved, i.e.

$$R_{mn}(g+\delta g) = 0\,.$$

At first order in the deformation, this condition takes the form

$$\nabla^p\nabla_p\delta g_{mn} + 2R_{m\ n}^{\ p\ q}\,\delta g_{pq} = 0, \tag{2.12}$$

known as the *Lichnerowicz equation*. This is best seen in complex coordinates. In this section, and only here, we will label the complex coordinates on the Calabi-Yau manifold M_6 with greek letters $\mu, \nu, \ldots$ (while in the remainder of the thesis, these label real coordinates on the 4d spacetime). In complex indices, (2.12) splits in two independent equations for the 'pure type' metric deformations $\delta g_{\mu\nu}$ and the 'mixed type' deformations $\delta g_{\mu\bar{\nu}}$. One finds that

- $\delta g_{\mu\nu}$ satisfies (2.12) if and only if the complex (2,1)–form

$$\delta\chi := -\frac{1}{4}\delta g_{\bar{\mu}\bar{\nu}}\,\Omega^{\bar{\nu}}{}_{\kappa\lambda} d\bar{y}^{\bar{\mu}}\wedge dy^{\kappa}\wedge dy^{\lambda} \tag{2.13}$$

is harmonic.

- $\delta g_{\mu\bar{\nu}}$ satisfies (2.12) if and only if the real (1,1)–form

$$\delta J = \, i\delta g_{\mu\bar{\nu}} dy^{\mu}\wedge d\bar{y}^{\bar{\nu}} \tag{2.14}$$

is harmonic.

While $\delta g_{\mu\bar{\nu}}$ clearly corresponds to a deformation of the Kähler class,[8] $\delta g_{\mu\nu}$ is associated with a variation of the complex structure of the Calabi-Yau. Indeed, the only way to recast the new metric $g_{\mu\bar{\nu}} + \delta g_{\mu\nu}$ in hermitian form is by a non-holomorphic coordinate transformation, which is allowed only if the complex structure is modified.

We deduce that the moduli space locally splits in a tensor product of two subspaces, respectively associated with complex structure deformations and with Kähler structure deformations. Since the solutions of the Lichnerowicz equation are in one-to-one correspondence with the complex elements of $H^{2,1}(M)$ and the real elements of $H^{1,1}(M)$, we conclude that these two independent parameter spaces have real dimensions $2h^{2,1}$ and $h^{1,1}$ respectively.

Next we wish to study the metric on the moduli space of Calabi-Yau 3–folds. As we will discuss in detail in the next chapter (see section 3.2 and the beginning of section 3.4), in compactifications of 10d supergravity the fluctuations of the metric g come in pair with the fluctuations of the NS 2–form b, and the kinetic terms for the associated 4d scalars are defined by the σ-model metric

$$ds^2 \;=\; \frac{1}{8\int vol_6 e^{-2\phi}}\int_{M_6} vol_6 e^{-2\phi} g^{mp}g^{nq}(\delta g_{mn}\delta g_{pq} + \delta b_{mn}\delta b_{pq})\;. \tag{2.15}$$

This expression can be understood as a metric on the space of metrics and b-fields living on the compact manifold M_6. When compactifying on a Calabi-Yau manifold (see section 3.2 for details), the dilaton ϕ is taken constant along M_6, and δb is required to be harmonic, necessarily of type (1,1) since $h^{2,0} = 0$. Hence (2.15) can be rewritten in complex coordinates as

$$ds^2 = \frac{1}{4Vol}\int_M vol_6\, g^{\kappa\bar{\mu}}g^{\lambda\bar{\nu}}\left[\delta g_{\kappa\lambda}\delta g_{\bar{\mu}\bar{\nu}} + (\delta g_{\kappa\bar{\nu}}\delta g_{\lambda\bar{\mu}} + \delta b_{\kappa\bar{\nu}}\delta b_{\bar{\mu}\lambda})\right]\,, \tag{2.16}$$

where $Vol = \int vol_6$. Neglecting the term in δb, this is a metric on the moduli space of Calabi-Yau manifolds. Notice that it is block diagonal, the two blocks corresponding respectively to a metric on the complex structure and on the Kähler class deformation spaces. The term involving δb can be incorporated by considering variations of a complexified version of the Kähler form, namely $b+iJ$. The associated parameter space has real dimension $2h^{1,1}$. In the following we look deeper into the geometric structure of each of these two independent deformation spaces, which we call $\mathscr{M}_{cs}$ and $\mathscr{M}_{ks}$ respectively.

The complex structure deformations

Let us first consider the piece in (2.16) associated with the complex structure deformations, i.e. the metric on $\mathscr{M}_{cs}$. One can show that this can be written as

$$\frac{1}{4Vol}\int vol_6\, g^{\kappa\bar{\nu}}g^{\mu\bar{\lambda}}\,\delta g_{\kappa\mu}\delta g_{\bar{\lambda}\bar{\nu}} \;=\; -\frac{\int \delta\chi\wedge\delta\bar{\chi}}{\int \Omega\wedge\bar{\Omega}}, \tag{2.17}$$

where $\delta\chi$ was defined in (2.13). Calling $z^i, i = 1,\ldots,h^{2,1}$ the complex coordinates on $\mathscr{M}_{cs}$, we can write $\delta\chi = \chi_i\delta z^i$, where the $\chi_i = -\frac{1}{4}\frac{\partial g_{\bar{\mu}\bar{\nu}}}{\partial z^i}\,\Omega^{\bar{\nu}}{}_{\kappa\lambda}d\bar{y}^{\bar{\mu}}\wedge dy^{\kappa}\wedge dy^{\lambda}$ form a basis for $H^{2,1}$.

[8] Indeed (recalling that $J_{\mu\bar{\nu}} = ig_{\mu\bar{\nu}}$) a harmonic deformation $\delta J = i\delta g_{\mu\bar{\nu}}\, dy^{\mu}\wedge dy^{\bar{\nu}}$ obviously changes the cohomology class of the Kähler form J.

Therefore, in coordinates the metric on $\mathscr{M}_{cs}$ – call it $\mathscr{G}^{cs}_{i\bar{\jmath}}$ – reads

$$\mathscr{G}^{cs}_{i\bar{\jmath}} \;=\; -\frac{\int \chi_i \wedge \bar{\chi}_{\bar{\jmath}}}{\int \Omega \wedge \bar{\Omega}}\,. \tag{2.18}$$

In order to proceed, we need to introduce the Kodaira formula, stating that the variation of the holomorphic (3,0)–form $\Omega(z)$ with respect to the complex structure moduli z^i reads

$$\frac{\partial \Omega}{\partial z^i} = \kappa_i \Omega + \chi_i \qquad\qquad i = 1, \dots, h^{2,1}, \tag{2.19}$$

where κ_i are coefficients which can depend on z but not on the coordinates of M_6, and the χ_i coincide with the ones introduced here above. Using (2.19), and recalling that the χ_i are of type (2,1), one finds for (2.18):

$$\mathscr{G}^{cs}_{i\bar{\jmath}} \;=\; -\frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{\jmath}}} \log i \int \Omega \wedge \bar{\Omega}\,. \tag{2.20}$$

This relation states that $\mathscr{M}_{cs}$ is Kähler, with Kähler potential

$$K_{cs} \;=\; -\log i \int \Omega \wedge \bar{\Omega}\,. \tag{2.21}$$

It can be shown that actually $\mathscr{M}_{cs}$ is a Kähler manifold of *special* type. To see this, it is convenient to parameterize $\mathscr{M}_{cs}$ via the *periods* of the holomorphic (3,0) form Ω. Let (A^I, B_J), $I, J = 0, \dots, h^{2,1}$ a canonical basis for the homology $H_3(M, \mathbb{Z})$, and (α_I, β^J) the dual cohomology basis for $H^3(M)$, such that

$$\int_{A^J} \alpha_I = \int_M \alpha_I \wedge \beta^J = \delta_I^J \qquad , \qquad \int_{B_I} \beta^J = \int_M \beta^J \wedge \alpha_I = -\delta_I^J\,. \tag{2.22}$$

Notice that α_I and β^J are real forms. Then the periods of Ω are defined as

$$Z^I \;:=\; \int_{A^I} \Omega = \int_M \Omega \wedge \beta^J \qquad , \qquad \mathcal{G}_I \;:=\; \int_{B_I} \Omega = \int_M \Omega \wedge \alpha_I, \tag{2.23}$$

and it turns out that $\mathcal{G}_I = \mathcal{G}_I(Z)$. Hence we dispose of $h^{2,1} + 1$ parameters Z^I to describe a space of dimension $h^{2,1}$. We notice that a rescaling $Z^I \to \lambda Z^I$ by a non-zero λ just corresponds to a rescaling $\Omega \to \lambda\Omega$, and that this does not modify the complex structure. It follows that the Z^I are projective coordinates for $\mathscr{M}_{cs}$, and that Ω is homogeneous of degree 1 in these coordinates. A set of coordinates z^i on $\mathscr{M}_{cs}$ is defined via $z^i = Z^I/Z^0$, for a non-vanishing Z^0. The dependence of Ω on the Z^I can be read from the expansion

$$\Omega = Z^I \alpha_I - \mathcal{G}_I(Z)\beta^I, \tag{2.24}$$

inferred from (2.23). Notice that, since the basis 3–form (α_I, β^I) are defined only up to a symplectic rotation, the periods $(Z^I, \mathcal{G}_I)^T$ form a symplectic vector.

From (2.19) it follows that

$$\int \Omega \wedge \frac{\partial \Omega}{\partial Z^I} = 0\,.$$

Substituting (2.24), this yields

$$\mathcal{G}_I = Z^J \frac{\partial}{\partial Z^I}\mathcal{G}_J = \frac{1}{2}\frac{\partial}{\partial Z^I}(Z^J \mathcal{G}_J) = \frac{\partial}{\partial Z^I}\mathcal{G}\,,$$

where

$$\mathcal{G} \equiv \frac{1}{2} Z^I \mathcal{G}_I\,.$$

Thus the $\mathcal{G}_I$ are derivatives of a function $\mathcal{G}(Z)$, which is homogeneous of degree 2 and goes under the name of the *prepotential*. This terminology is due to the fact that $\mathcal{G}$ determines the Kähler potential. Indeed, substituting the expansion (2.24) in (2.21), one finds

$$K_{cs} = -\log i\big(\bar{Z}^I \mathcal{G}_I - Z^I \bar{\mathcal{G}}_I\big)\,, \qquad \text{where } \; \mathcal{G}_I = \frac{\partial \mathcal{G}}{\partial Z^I}. \tag{2.25}$$

An n-dimensional Kähler manifold admitting projective coordinates $Z^I(z)$, $I = 0, 1, \dots, n$, and for which there exists a holomorphic function $\mathcal{G}(Z^I)$ of homogeneity degree 2, determining the Kähler potential via a relation of the form (2.25), is called *special Kähler*. Sometimes the specification *local* is added, in order to distinguish this geometry from the *rigid* special Kähler geometry which governs the scalar manifolds in rigid supersymmetry. In appendix D we provide a more intrinsic definition of a local special Kähler manifold, as well as several notable relations that will be useful in the next chapters. Some of these relations are expressed in terms of the *period matrix* of special Kähler geometry, relating the upper and lower components of the symplectic vector $(Z^I, \mathcal{G}_I)^T$. Naming it $\mathcal{M}$, its definition is

$$\mathcal{G}_I = \mathcal{M}_{IJ} Z^J \qquad , \qquad D_k \mathcal{G}_I = \overline{\mathcal{M}}_{IJ} D_k Z^J, \tag{2.26}$$

where the Kähler covariant derivative D_k acts on the periods as $D_k = \partial_{z^k} + \partial_{z^k} K_{cs}$. $\mathcal{M}$ can be computed from the prepotential, using formula (D.5). As we will see in next chapter, $\mathcal{M}$ is also an important ingredient of the compactification, and it appears explicitly in the 4d effective action.

The Kähler structure deformations

We have just seen that the space $\mathscr{M}_{cs}$ of complex structures on a Calabi-Yau manifold is special Kähler. As we now briefly recall, again following [56], the same geometric structure exists on the parameter space $\mathscr{M}_{ks}$ of cohomology classes of the complexified Kähler form

$$t := b + iJ\,.$$

Let ω_a , $a = 1, \dots, h^{1,1}$ be a *real* basis for $H^{1,1}(M)$, and expand

$$b = b^a \omega_a \quad , \quad v = v^a \omega_a \quad \Rightarrow \quad t = t^a \omega_a \equiv (b^a + i v^a)\omega_a\,. \tag{2.27}$$

Typically for the ω_a one chooses the harmonic representatives of the $H^{1,1}$ cohomology classes. The $h^{1,1}$ complex parameters t^a can be seen as complex coordinates on $\mathscr{M}_{ks}$. Now, the block of the metric (2.16) corresponding to the metric on $\mathscr{M}_{ks}$ can be rewritten as

$$\frac{1}{4Vol}\int_M vol_6\, g^{\kappa\bar{\mu}} g^{\lambda\bar{\nu}}(\delta g_{\kappa\bar{\nu}}\delta g_{\lambda\bar{\mu}} + \delta b_{\kappa\bar{\nu}}\delta b_{\bar{\mu}\lambda}) \;=\; \frac{1}{4Vol}\int_M \delta t \wedge *\delta\bar{t}\,, \tag{2.28}$$

depending just on the cohomology class of δt [71]. Hence, with $\delta t = \delta t^a \omega_a$, the coordinate form of the metric on $\mathscr{M}_{ks}$ – call it $\mathscr{G}^{ks}_{ab}$ – is

$$\mathscr{G}^{ks}_{ab} \;=\; \frac{1}{4Vol}\int \omega_a \wedge *\omega_b \,. \tag{2.29}$$

This is a Kähler metric, as it can be derived from the Kähler potential

$$K_{ks} := -\log \frac{4}{3}\int J\wedge J\wedge J \tag{2.30}$$

via

$$\mathscr{G}^{ks}_{ab} \;=\; \frac{\partial}{\partial t^a}\frac{\partial}{\partial \bar{t}^b}K_{ks}\,. \tag{2.31}$$

The 4/3 factor in K_{ks} doesn't play any role in the derivation of $\mathscr{G}^{ks}_{ab}$, and in chosen just for consistency with the rest of the discussion below.

The manifold $\mathscr{M}_{ks}$ is actually special Kähler. In order to see this, one defines a set of projective coordinates $X^A = (X^0, X^a)$ (hence $A = 0, 1, \ldots, h^{1,1}$), related to the t^a via $t^a = X^a/X^0$. As we will see in the next chapter, in the context of string compactifications it is natural to identify X^0 with $e^{-\phi}$, where ϕ is the dilaton. However, at this stage X^0 can be considered just as a redundant coordinate, useful to exhibit the special Kähler structure of $\mathscr{M}_{ks}$. Next one introduces the following function $\mathcal{F}$, holomorphic and homogeneous of degree 2 in the X^A:

$$\mathcal{F}(X) := -\frac{1}{3!}\mathcal{K}_{abc}\frac{X^aX^bX^c}{X^0}\,, \tag{2.32}$$

where the $\mathcal{K}_{abc}$, known as the triple intersection numbers, are defined as

$$\mathcal{K}_{abc} := \int \omega_a\wedge\omega_b\wedge\omega_c\,. \tag{2.33}$$

It is now easy to see that K_{ks} given in (2.30) is precisely reproduced by

$$K_{ks} = -\log i\big(\bar{X}^A\mathcal{F}_A - X^A\bar{\mathcal{F}}_A\big)\,, \qquad \text{where } \mathcal{F}_A = \frac{\partial\mathcal{F}}{\partial X^A}\,, \tag{2.34}$$

and with X^0 set to one after differentiation. We conclude therefore that – by the same token of the complex structure moduli space $\mathscr{M}_{cs}$ – the complexified Kähler structure moduli space $\mathscr{M}_{ks}$ is a special Kähler manifold; its prepotential is the function $\mathcal{F}$, whose explicit form is given in (2.32).

Again one can introduce a period matrix, which this time we denote by $\mathcal{N}$, defined as

$$\mathcal{F}_A = \mathcal{N}_{AB}X^B \qquad , \qquad D_a\mathcal{F}_B = \overline{\mathcal{N}}_{BC}D_aX^C, \tag{2.35}$$

where here the Kähler covariant derivative is $D_a = \partial_{t^a} + \partial_{t^a}K_{ks}$. The matrix $\mathcal{N}$ can be computed from the prepotential $\mathcal{F}$ via eq. (D.5) (translated in the notation for $\mathscr{M}_{ks}$).

This very symmetric structure between the two factors composing the moduli space of Calabi-Yau manifolds was the main hint suggesting *mirror symmetry* [57], a duality stating that for every Calabi-Yau manifold M there exists a 'mirror' Calabi-Yau W having the complex structure and the Kähler structure moduli spaces interchanged:

$$\mathscr{M}_{cs}(M) = \mathscr{M}_{ks}(W) \qquad , \qquad \mathscr{M}_{ks}(M) = \mathscr{M}_{cs}(W)\,. \tag{2.36}$$

This implies that if M has Hodge diamond (2.11), with assigned $h^{1,1}$ and $h^{2,1}$ numbers, then the Hodge diamond of W has the $h^{1,1}$ and $h^{2,1}$ numbers interchanged. In other words, mirror symmetry acts on the Hodge diamond as a reflection along its diagonal.

2.4 Generalized structures

As discussed in section 2.1, the spinor ansatz (2.1) for $N = 2$ reductions requires the existence of a pair of SU(3) structures on M_6, one for each of the two globally defined spinors η^1, η^2. In particular, we have two pairs I_1, J_1 and I_2, J_2 of almost hermitian structures, related to the same Riemannian metric g by (2.6). It turns out that these data can be conveniently repackaged in certain objects defined on $TM_6 \oplus T^*M_6$, the sum of the tangent and the cotangent bundles of M_6 (*generalized tangent bundle* in the following). These objects can be seen as arising from the extension of the standard concept of G-structure to the generalized tangent bundle, and for this reason we will speak of *generalized structures*.

Generalized geometry, studying structures on $T \oplus T^*$, has recently been introduced in a mathematical context by Hitchin [22], and further developed in [23, 24, 25]. Its formalism was first employed in the derivation of 4d $N = 2$ supergravities from type II compactifications in ref. [45]. A physicists' review of generalized geometry can be found in [33] (see also [16, 58]). Further related references have been given in section 1.1.

One of the main purposes of this thesis is to study the problem of compactifications taking the point of view of generalized geometry. In the following of this section we provide a basic introduction to generalized structures, focusing on the aspects that are more relevant for this task. Most of the mathematical results we present can be found in the original works [22, 23]. While they are generically valid for any even dimensional manifold, here we will restrict to the six-dimensional case.

The bundle $TM_6 \oplus T^*M_6$ is automatically endowed with an O(6,6) structure. Indeed, take two vector fields v, w and two 1–forms ζ, ξ on the manifold M, and build the *generalized vector fields* $X = v + \zeta$ and $Y = w + \xi$ on $TM \oplus T^*M$. Then a natural symmetric pairing $\mathcal{I}$ between X and Y is

$$\mathcal{I}(X, Y) \;:=\; \xi(v) + \zeta(w) \;=\; \xi_m v^m + \zeta_m w^m\,. \tag{2.37}$$

Representing a generalized vector in coordinates as $(v^m, \xi_m)^T$, the corresponding matrix form of $\mathcal{I}$ is

$$\mathcal{I} = \begin{pmatrix} 0 & \mathbb{1}_6 \\ \mathbb{1}_6 & 0 \end{pmatrix}. \tag{2.38}$$

The metric $\mathcal{I}$ is globally defined and non-degenerate. Having $(6,6)$-signature, it determines an $\mathrm{O}(6,6)$ structure on $TM_6 \oplus T^*M_6$.

A key observation is that the group of $O(6,6)$ transformations preserving the metric $\mathcal{I}$ contains, beside the diffeomorphisms, a transformation involving a 2–form b, acting on $TM \oplus T^*M$ as

$$v + \zeta \;\mapsto\; v + \zeta + \iota_v b\,. \tag{2.39}$$

This natural incorporation of a 2–form in the generalized geometry formalism is one of the reasons which make it particularly suitable for string theory applications, where b is identified with the NS 2–form (sometimes called the B-field in the following).

Analogously to the case of G-structures, reductions of the generic $O(6,6)$ structure on the generalized tangent bundle can be induced by globally defined and non-degenerate objects acting on tensor products of $TM_6 \oplus T^*M_6$. We call *generalized structures* such reductions.

The first reduction we consider is associated with the notion of *generalized almost complex structure*. By definition, this is a map

$$\mathcal{J} \;:\; TM \oplus T^*M \;\to\; TM \oplus T^*M$$

satisfying

$$\mathcal{J}^2 = -\mathbb{1}_{T\oplus T^*}\,, \tag{2.40}$$

as well as the orthogonality condition with respect to $\mathcal{I}$:

$$\mathcal{J}^T \mathcal{I} \mathcal{J} = \mathcal{I}\,. \tag{2.41}$$

The analogy with a standard almost complex structure is apparent, and indeed $\mathcal{J}$ shares several features with the latter. For instance, a generalized almost complex structure has $\pm i$ eigenvalues, and splits $(TM_6 \oplus T^*M_6)\otimes\mathbb{C}$ in $+i$ and $-i$ eigenbundles. Furthermore, obstructions for finding a generalized almost complex structure $\mathcal{J}$ are the same as the ones for finding a standard almost complex structure I [23, prop. 4.15]. A generalized almost complex structure reduces the structure group on $TM_6 \oplus T^*M_6$ to $U(3,3) = O(6,6) \cap GL(6,\mathbb{C})$. There also exists an integrability condition for $\mathcal{J}$, which, if satisfied, makes the manifold M *generalized complex*. This condition can be expressed in terms of the Courant bracket, replacing the usual Lie bracket for vector fields. Since we will not really need this last notion in the following, we refer e.g. to [23, 33] for details on this point.

2.4.1 Making up generalized structures

Now, the data contained in the almost hermitian structures I_1, J_1 and I_2, J_2, as well as – this is the bonus of generalized geometry – in the internal NS 2-form b, can all be encoded in *a pair* of generalized almost complex structures. Indeed, it is readily checked that each of the two matrices

$$\mathcal{J}_{\pm\,\Sigma}^{\Lambda} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}\begin{pmatrix} -I_1 \pm I_2 & -J_1^{-1} \mp J_2^{-1} \\ J_1 \pm J_2 & I_1^T \mp I_2^T \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \tag{2.42}$$

satisfies (2.40), (2.41). The writing J^{-1} denotes matrix inversion, and does not involve the metric on M_6. Here and in the following, the indices $\Lambda, \Sigma = 1, \ldots, 12$ label the tangent and cotangent space coordinates; they are raised and lowered with the O(6,6) metric $\mathcal{I}_{\Lambda\Sigma}$.

In the case in which $J_1 = J_2 \equiv J$, $I_1 = I_2 \equiv I$, i.e. when M_6 has strict SU(3) structure, $\mathcal{J}_\pm$ in (2.42) reduce to the simpler form (here we also choose $b = 0$)

$$\mathcal{J}_+ = \begin{pmatrix} 0 & -J^{-1} \\ J & 0 \end{pmatrix} \quad , \quad \mathcal{J}_- = \begin{pmatrix} -I & 0 \\ 0 & I^T \end{pmatrix} . \tag{2.43}$$

We see therefore that the notion of generalized almost complex structure includes as special cases the concept of standard almost complex structure as well as the one of almost symplectic structure. It can also be shown [23] that integrability of $\mathcal{J}_+$ and $\mathcal{J}_-$ in (2.43) is equivalent to the integrability respectively of J and I discussed in subsection 2.2.1. We conclude that generalized complex geometry incorporates, and treats on the same footing, both complex and symplectic geometry. Actually, it can also smoothly interpolate between the two, as shown in [23, sect. 4.6] via a 4d example based on the K3 surface.

Let's come back to the $\mathcal{J}_\pm$ in (2.42). These define a metric on $T \oplus T^*$: indeed, observing that

$$[\mathcal{J}_+, \mathcal{J}_-] = 0 \,, \tag{2.44}$$

and recalling (2.40), (2.41), we see that the generalized tensor

$$\mathcal{G} := -\mathcal{I}\mathcal{J}_+\mathcal{J}_- \tag{2.45}$$

is symmetric. Recalling (2.6), from (2.42) we find

$$\mathcal{G} = -\mathcal{I}\mathcal{J}_+\mathcal{J}_- = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} g - bg^{-1}b & -bg^{-1} \\ g^{-1}b & g^{-1} \end{pmatrix}, \tag{2.46}$$

which is positive definite thanks to the positive-definiteness of g.

2.4.2 Extracting data from generalized structures

We have thus seen how the two U(3) structures associated with the spinor ansatz (2.1), together with the NS 2–form b, can be repackaged in a pair of generalized almost complex structures, defining a positive definite metric on $T \oplus T^*$. A logically converse approach would be to *define* the two U(3) structures, as well as b, *via* the introduction of generalized tensors determining the appropriate reduction of the generic O(6,6) structure group on $T \oplus T^*$. In the following we briefly review how this is achieved. Table 2.2 summarizes all the various progressive reductions we mention; while in the text we discuss just the six-dimensional case, there we assume a generic even dimension d for M_d.

A positive definite metric $\mathcal{G}$ on $TM_6 \oplus T^*M_6$ satisfying the compatibility condition

$$\mathcal{G}\mathcal{I}^{-1}\mathcal{G} = \mathcal{I} \tag{2.47}$$

defines a reduction of the O(6,6) structure on $TM_6 \oplus T^*M_6$ to its maximal compact subgroup O(6)×O(6). Indeed, (2.47) says that $\mathcal{I}^{-1}\mathcal{G}$ squares to $\mathbb{1}$, and therefore splits the generalized tangent bundle into the two 6d eigenbundles $C_\pm$ associated with its ± 1 eigenvalues. Since $\mathcal{G}$ is positive definite, again from (2.47) we see that $\mathcal{I}$ is positive on C_+ and negative on C_-, and this fixes the O(6)×O(6) structure. It can be shown that $\mathcal{G}$ always

Non-degenerate tensors	Generalized structure
natural maximally indefinite metric $\mathcal{I}$	$\mathrm{O}(d,d)$
generalized almost complex structure $\mathcal{J}$	$\mathrm{U}(d/2,d/2)$
globally defined pure spinor Φ	$\mathrm{SU}(d/2,d/2)$
positive generalized metric $\mathcal{G}$, with $\mathcal{G}\mathcal{I}^{-1}\mathcal{G}=\mathcal{I}$	$\mathrm{O}(d)\times\mathrm{O}(d)$
$\mathcal{J}_+,\ \mathcal{J}_-,$ with $[\mathcal{J}_+,\mathcal{J}_-]=0$ and $\mathcal{G}:=-\mathcal{I}\mathcal{J}_+\mathcal{J}_->0$	$\mathrm{U}(d/2)\times\mathrm{U}(d/2)$
compatible pure spinors Φ_+ , Φ_-	$\mathrm{SU}(d/2)\times\mathrm{SU}(d/2)$

Table 2.2: Generalized structures on $TM_d \oplus T^*M_d$, for d even, and the associated globally defined non-degenerate objects. In the main text, we discuss the case $d=6$.

takes the form (2.46), where again b is a 2–form, while g is a metric on M_6, positive definite thanks to the assumed positive-definiteness of $\mathcal{G}$. Hence, an O(6)×O(6) structure on $TM_6 \oplus T^*M_6$ is equivalent to a metric on M_6, together with a B-field.

On the other hand, we saw above that a generalized almost complex structure $\mathcal{J}$ induces a reduction of the generic O(6,6) structure on $TM \oplus T^*M$ to $\mathrm{U}(3,3)$. The simultaneous presence of a positive generalized metric $\mathcal{G}$ and of $\mathcal{J}$, satisfying $\mathcal{J}^T\mathcal{G}\mathcal{J}=\mathcal{G}$, yields a reduction to the maximal compact subgroup $\mathrm{U}(3)\times\mathrm{U}(3)\subset\mathrm{U}(3,3)$. This can equally well be described by a *compatible* pair of generalized almost complex structures $\mathcal{J}_+,\mathcal{J}_-$, where compatibility means that

i) $\mathcal{J}_+,\mathcal{J}_-$ satisfy (2.44), and

ii) the generalized metric $\mathcal{G}$ constructed via (2.45) is positive definite.

Now, an U(3)×U(3) structure is equivalent to a pair of U(3) structures on M_6, together with a B-field. Indeed, firstly g and b can be read from (2.45). Secondly, it was shown in [23] that $\mathcal{J}_\pm$ can always be put in the form (2.42), where (I_k, J_k), $k=1,2$, define two almost hermitian structures, satisfying $g=J_1I_1=J_2I_2$. These two almost hermitian structures degenerate into a single one when $\mathcal{J}_+,\mathcal{J}_-$ are of the form (2.43).

2.4.3 Description via pure Spin(6,6) spinors: the polyform picture

Generalized almost complex structures can alternatively be described in terms of pure Spin(6,6) spinors. This alternative representation will lead us to discuss the eventual reduction of the $T\oplus T^*$ structure group we are interested in, namely the SU(3)×SU(3) structure. This is the most relevant for our purposes, since it determines two SU(3) structures on M_6, and therefore a pair of globally defined nowhere vanishing Spin(6) spinors η^1,η^2. It also encode the full NSNS sector of type II supergravity.

As a first thing, we recall that the Spin(6,6) spinors are isomorphically mapped to sections of $\wedge^\bullet T^*M_6$, the bundle of forms of mixed degree (*polyforms* since now on) on M_6. Let $X=v+\zeta\in\Gamma(TM\oplus T^*M)$ be a generalized vector field, where, in a standard notation, by $\Gamma(E)$ we denote the space of sections of any given bundle E. Then a Clifford action on any $C \in \wedge^{\bullet} T^* M$ is realized by

$$X \cdot C = (\iota_v + \zeta \wedge) C \ . \tag{2.48}$$

This is a Clifford action in that it satisfies

$$(X \cdot Y + Y \cdot X) \cdot C \ = \ \mathcal{I}(X, Y)\, C \, . \tag{2.49}$$

As a consequence, the Cliff(6,6) gamma matrices Γ^{Λ} locally can be identified with the coordinate basis of $T_p M \oplus T_p^* M$:

$$\Gamma^{\Lambda} \ = \ \begin{pmatrix} dy^m \wedge \\ \iota_{\partial_m} \end{pmatrix} \qquad , \qquad \{\Gamma^{\Lambda}, \Gamma^{\Sigma}\} = \mathcal{I}^{\Lambda\Sigma} \ . \tag{2.50}$$

The Spin(6,6) spinor representation decomposes in two irreducible Weyl representations, and this is reflected in the splitting $\wedge^{\bullet} T^* = \wedge^{\text{even}} T^* \oplus \wedge^{\text{odd}} T^*$. In this way, an even/odd form of mixed degree can be regarded as a Weyl spinor of Spin(6,6) with positive/negative chirality.

A bilinear product between polyforms $A, C \in \wedge^{\bullet} T^* M_6$ can be defined through the Mukai pairing $\langle \, , \rangle$:

$$\langle A \, , C \rangle \ := \ \big[\lambda(A) \wedge C \big]_{\text{top}} \ , \tag{2.51}$$

where $[\ ]_{\text{top}}$ picks the 6-form component, while the involution λ acts on a k–form A_k as

$$\lambda(A_k) = (-)^{[\frac{k+1}{2}]} \, A_k \ . \tag{2.52}$$

In six dimensions $\langle \, , \rangle$ is antisymmetric. Further properties are collected in appendix B.

Since it yields a top form, and not a scalar, the Mukai pairing does not exactly correspond to a bilinear product between two Spin(6,6) spinors. The reason can be traced back to the fact that the isomorphism between the Spin(6,6)-bundle and the bundle of forms is not canonical, in that it requires the choice of a volume form on M_6 (see for instance [23, 45] for more details). In order to map the result of the Mukai pairing to a scalar, one has to mod out by a chosen volume form (which exists since we assume the manifold M_6 is orientable).

As already mentioned, a prominent role in relation with the generalized almost complex structures is played by *pure* Spin(6,6) spinors. In order to define pure spinors, we need to introduce the notions of isotropic subbundle and of annihilator space. A subbundle $L \subset T \oplus T^*$ is *isotropic* if for any pair of sections X, Y of L one has $\mathcal{I}(X, Y) = 0$. L is said maximally isotropic if its fibre has the maximal possible dimension, which is 6 since $\mathcal{I}$ has (6,6) signature. The first isotropic subbundle we are interested in is the *annihilator space* L_{Φ} of a complex Spin(6,6) spinor Φ, defined by

$$L_{\Phi} := \{ \, X \in (T \oplus T^*) \otimes \mathbb{C} \, : \, X \cdot \Phi = 0 \, \} \ , \tag{2.53}$$

where isotropy follows directly from (2.49). Now, by definition we say that Φ is a *pure spinor* if its annihilator space L_{Φ} is maximally isotropic.

We can now state the relation between complex pure spinors Φ and generalized almost complex structures $\mathcal{J}$. The key feature to be remarked is that any $\mathcal{J}$ is uniquely determined by its $+i$ eigenbundle, and that this is a maximally isotropic subbundle of $(T \oplus T^*) \otimes \mathbb{C}$. Isotropy is showed recalling (2.41): indeed, for any pair X, Y of generalized vectors in the $+i$ eigenbundle of $\mathcal{J}$, one has

$$\mathcal{I}(X,Y) = \mathcal{I}(\mathcal{J}X, \mathcal{J}Y) = \mathcal{I}(iX, iY) = -\mathcal{I}(X,Y)\,.$$

The isotropic subbundle is maximal since its rank (namely, the dimension of its fibre) is half the one of $(T \oplus T^*) \otimes \mathbb{C}$.

A correspondence between Φ and $\mathcal{J}$ is then established by identifying the annihilator L_Φ of Φ with the $+i$ eigenbundle of $\mathcal{J}$. Since the annihilator L_Φ determines Φ modulo rescalings, one actually has a one-to-one correspondence between generalized almost complex structures and complex *line bundles* of pure spinors; in other words, $\mathcal{J}$ defines a pure spinor modulo rescalings. Moreover, at each point of M_6, the pure spinor generating the complex line has to satisfy the condition $\langle \Phi, \bar{\Phi} \rangle \neq 0$.

An explicit formula for $\mathcal{J}$ in terms of Φ that will be useful in the following is [22, 45, 33]

$$\mathcal{J}^\Lambda{}_\Sigma = -4\frac{\langle \mathrm{Re}\Phi, \Gamma^\Lambda{}_\Sigma \mathrm{Re}\Phi\rangle}{i\langle \Phi, \bar{\Phi}\rangle}\,, \tag{2.54}$$

where $\Gamma^{\Lambda\Sigma}$ denotes the antisymmetrized product of two Cliff(6,6) gamma matrices, and again the $T \oplus T^*$ indices Λ, Σ are raised and lowered with the metric $\mathcal{I}$. In order to see that $\mathcal{J}$ does not depend on complex rescalings of Φ, one should remark that $\langle \Phi, \Gamma^{\Lambda\Sigma}\Phi\rangle = 0$ [22], which implies for the numerator of (2.54):

$$2\langle \mathrm{Re}\Phi, \Gamma^{\Lambda\Sigma}\mathrm{Re}\Phi\rangle = \langle \Phi, \Gamma^{\Lambda\Sigma}\bar{\Phi}\rangle\,.$$

Beside playing the role of a normalization factor offsetting the real rescalings of Φ, the denominator of (2.54) also ensures that $\mathcal{J}$ doesn't depend on the choice of the volume form for M_6.

If the line bundle of pure spinors has a global section, i.e. if there is a globally defined pure spinor, then the $\mathrm{U}(3,3)$ structure group on $T \oplus T^*$ determined by the corresponding $\mathcal{J}$ is reduced to $\mathrm{SU}(3,3)$. This is analogous to the $\mathrm{SL}(3,\mathbb{C}) \subset \mathrm{GL}(3,\mathbb{C})$ reduction we discussed in section 2.2. Indeed, there we saw that an almost complex structure I reduces the generic $\mathrm{GL}(6,\mathbb{R})$ structure group of M_6 to $\mathrm{GL}(3,\mathbb{C})$, and also determines a (3,0) form Ω, in general only modulo a complex function and hence locally (since on patch overlaps we are allowed to perform a complex rescaling of Ω). If Ω is instead globally defined, then we have a further reduction, to $\mathrm{SL}(3,\mathbb{C})$. The analogy with the present situation is clear: $\mathcal{J}$ reduces the generic $T \oplus T^*$ structure group $\mathrm{O}(6,6)$ to $\mathrm{U}(3,3) = \mathrm{O}(6,6) \cap \mathrm{GL}(6,\mathbb{C})$, and also determines Φ, modulo rescalings. If Φ is globally defined, then one has the reduction to $\mathrm{SU}(3,3)$. The example involving I and Ω is actually a particular case of this more general construction, see $\mathcal{J}_-$ in (2.43) and Φ_- in (2.65).

Above we characterized an $\mathrm{U}(3)\times\mathrm{U}(3)$ structure on $TM_6 \oplus T^*M_6$ via a compatible pair of generalized almost complex structures $\mathcal{J}_+, \mathcal{J}_-$. Thanks to the correspondence between generalized almost complex structures and line bundles of pure spinors, an $\mathrm{U}(3)\times\mathrm{U}(3)$

structure is equivalently specified by a pair of pure spinors Φ_+ and Φ_-. Seen as polyforms, these are sections of $(\wedge^{\text{even}}T^*)\otimes\mathbb{C}$ and $(\wedge^{\text{odd}}T^*)\otimes\mathbb{C}$ respectively (so that Φ_+ has positive Spin(6, 6) chirality, while Φ_- has negative chirality), satisfying the compatibility relation [23, 45]

$$\langle\Phi_+, X\cdot\Phi_-\rangle \;=\; 0 \;=\; \langle\bar{\Phi}_+, X\cdot\Phi_-\rangle \qquad\quad \forall X\in\Gamma(T\oplus T^*)\,, \tag{2.55}$$

which is the equivalent of (2.44) – see [35] for a proof. As above, we also require positive-definiteness of the generalized metric $\mathcal{G}$ defined via (2.45). Finally, at each point of M_6 the generators of the pure spinor lines need to satisfy $\langle\Phi_\pm,\bar{\Phi}_\pm\rangle\neq 0$. If these conditions are satisfied, then we say that Φ_+,Φ_- are *compatible* (since so are the generalized almost complex structures $\mathcal{J}_+,\mathcal{J}_-$ they determine).

We eventually come to the last generalized structure we wish to introduce, namely the SU(3)×SU(3) structure, which will be the one of main interest for our purposes. If both the pure spinor line bundles defining the U(3)×U(3) structure admit a global section, i.e. if both $\Phi_\pm$ are globally defined, then the structure group of $TM_6\oplus T^*M_6$ is further reduced to SU(3)×SU(3). In our definition of SU(3)×SU(3) structure we also include the requirement

$$\langle\Phi_+,\bar{\Phi}_+\rangle = \langle\Phi_-,\bar{\Phi}_-\rangle\;. \tag{2.56}$$

The pure spinors $\Phi_\pm$ are invariant under the action of the SU(3)×SU(3)$\,\subset\,$Spin(6,6) structure group they identify, much as a globally defined Spin(6) chiral spinor $\eta_\pm$ is invariant under the action of SU(3)$\,\subset\,$ Spin(6).[9]

An SU×SU(3) structure specifies an SU(3) structure inside each of the two U(3) structures on M_6, and therefore provides two globally defined nowhere vanishing Spin(6) chiral spinors η^1_+,η^2_+, which we employ in our spinor ansatz (2.1) for type II compactifications. We will discuss further the relation between Spin(6,6) pure spinors and standard Spin(6) spinors in the next subsection.

Concerning the bosonic degrees of freedom, if an U(3)×U(3) structure already provided a metric g and a B-field b, the fact that now the pure spinors are globally defined fixes the missing datum to complete the NSNS sector of type II supergravity, namely the dilaton. Recalling the comment below eq. (2.52), we define the norm $||\Phi||$ of a pure spinor Φ by

$$||\Phi||^2\, vol_6 \;:=\; i\langle\Phi,\bar{\Phi}\rangle\,, \tag{2.57}$$

where the volume form vol_6 we choose in order to extract a scalar from the Mukai pairing is the natural one associated with the metric g determined by $\mathcal{J}_\pm$. Then the non-zero equal norms (recall (2.56)) of $\Phi_\pm$ define a nonvanishing real scalar function all over M_6. We relate this with the dilaton ϕ, as follows[10]

$$||\Phi_\pm||^2 = 8e^{-2\phi}\,. \tag{2.58}$$

We remark that the pure spinor norm is completely independent of $\mathcal{J}_\pm$, and hence of $\mathcal{G}$.

[9] In the case of η we didn't need to explicitly ask for purity, since in 6d any chiral spinor is pure.

[10] More generally, this function can be associated with a combination of the dilaton and the warp factor, see e.g. [37, 41]. In this thesis, however, the warp factor is always assumed to be trivial. This also justified by the fact that a non-trivial warp factor appears to explicitly break the $N=2$ description to $N=1$ [37].

2.4.4 Spin(6,6) pure spinors and Spin(6) bispinors

In order to make explicit the correspondence between the Spin(6,6) pure spinors Φ_+, Φ_- defining an SU(3)×SU(3) structure and the data $(g, b, \phi, \eta^1_\pm, \eta^2_\pm)$ on M_6, we now review a direct way to build the former in terms of the latter [26, 28, 27]. This exploits the (vector space) isomorphism between the exterior algebra of $\wedge^\bullet T^* M_6$ and the Clifford algebra Cliff(6), implemented by the Clifford map "/":

$$C = \sum_k \frac{1}{k!} C_{m_1 \ldots m_k} dy^{m_1} \wedge \ldots \wedge dy^{m_k} \qquad \longleftrightarrow \qquad \not{C} = \sum_k \frac{1}{k!} C_{m_1 \ldots m_k} \gamma^{m_1 \ldots m_k} \ , \quad (2.59)$$

where the antisymmetrized products of gamma matrices $\gamma^{m_1 \ldots m_k}$ form a basis for Cliff(6). Notice that the metric g already enters here, since it is needed in order to define the gamma matrices with curved indices (actually it was already needed for introducing Spin(6) spinors). Now, out of $\eta^1_+, \eta^2_\pm$ one can build the bispinors

$$\not{\Phi}^0_\pm := 8 e^{-\phi} \eta^1_+ \otimes \eta^{2\dagger}_\pm \ , \quad (2.60)$$

and map them to polyforms $\Phi^0_\pm$ using first the Fierz identity

$$\eta^1_+ \otimes \eta^{2\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} \left(\eta^{2\dagger}_\pm \gamma_{m_k \ldots m_1} \eta^1_+ \right) \gamma^{m_1 \ldots m_k} \ , \quad (2.61)$$

and then (2.59) backwards: $\not{\Phi}^0_\pm \to \Phi^0_\pm$. In (2.60) we assume the spinors are normalized:

$$\eta^{1\dagger}_\pm \eta^1_\pm = \eta^{2\dagger}_\pm \eta^2_\pm = 1 \, , \quad (2.62)$$

and the factor of 8 is introduced just for convenience (it offsets the 1/8 in (2.61)). It is not difficult to see that Φ^0_+ is an even polyform, while Φ^0_- is odd. Furthermore, it turns out that $\Phi^0_\pm$ define a compatible pure spinor pair (we will discuss their annihilators in section 2.5. See [33] for further details). Using the image (B.8) of the Mukai pairing under the Clifford map, one can also see that their norms are equal and satisfy (2.58):

$$i \langle \Phi^0_\pm \, , \, \bar{\Phi}^0_\pm \rangle = 8 e^{-2\phi} \, ||\eta^1_\pm||^2 ||\eta^2_\pm||^2 \, vol_6 = 8 e^{-2\phi} vol_6 \, . \quad (2.63)$$

We deduce that $\Phi^0_\pm$ identify an SU(3)×SU(3) structure on the generalized tangent bundle. Actually, one can define in this way any SU(3)×SU(3) structure with vanishing b–field. The latter can be introduced, without losing any of the previous features, by defining the *B-transformed* spinors

$$\Phi_\pm = e^{-b} \Phi^0_\pm \ , \quad (2.64)$$

where $e^{-b} \equiv 1 - b + \frac{1}{2} b \wedge b - \frac{1}{6} b \wedge b \wedge b$ acts by wedging. In particular, thanks to the property (B.2) of the Mukai pairing, this transformation does not change the pure spinor norm. This exhausts the construction of general compatible pure spinor pairs in terms of $(g, b, \phi, \eta^1_\pm, \eta^2_\pm)$.

One can then work out the form of $\Phi_\pm$ for the cases of interest. The simplest case is when there's just a strict SU(3) structure on M_6, i.e. when η^1 and η^2 are proportional everywhere on M_6. Taking $\eta^1_+ = e^{i\alpha}\eta_+$, $\eta^2_+ = e^{i\beta}\eta_+$ for some (possibly coordinate dependent) phase factors α and β, and recalling relation (2.9) between $\eta_\pm$ and the SU(3) invariant forms J and Ω, one gets

$$\Phi_+ = s_+ e^{-b-iJ} \qquad , \qquad \Phi_- = s_-\Omega\,, \tag{2.65}$$

where $s_+ = e^{-\phi+i(\alpha-\beta)}$ and $s_- = -ie^{-\phi+i(\alpha+\beta)}$ are functions on M_6. The pure spinor compatibility (2.55) reads

$$J\wedge\Omega \,=\, 0 \,=\, b\wedge\Omega\,, \tag{2.66}$$

stating that both J and b have to be of type (1,1), while conditions (2.56)–(2.58) become

$$\frac{i}{8}\Omega\wedge\bar{\Omega} = \frac{1}{6}J\wedge J\wedge J = vol_6\,, \tag{2.67}$$

which is (2.8) with the choice $n = 3$. From (2.43), one can also see that in this case expression (2.54) for $\mathcal{J}_-$ reduces to (2.4). We conclude that when $\eta^1_+ = \eta^2_+$, the compatible pure spinor pair defined by (2.60), (2.64) is equivalent to the SU(3) structure defining forms introduced in section 2.2, together with the additional data of b and ϕ.

In the general situation where η^1_+ and η^2_+ are not proportional, the $\Phi_\pm$ obtained starting from (2.60) can be expressed in terms of the forms characterizing the underlying local SU(2) structure on M_6; we refer e.g. to [33] for details. Further developements on explicit constructions of general compatible pure spinor pairs can be found in [32, 38, 39].

The 'bispinor picture' just described, in which $\Phi^0_\pm$ are seen as in (2.60), has the clear advantage of showing explicitly the way in which the two SU(3) factors of SU(3)×SU(3) act on the pure spinors: one from the left on η^1, the other from the right on η^2. The η's, and therefore $\Phi_\pm$, are invariant under this action. In the next section we will see that the same picture for the action of SU(3)×SU(3) can be extended to the whole space of complex polyforms, which in general transform non-trivially. Thanks to this manageability, the bispinor picture is often convenient in concrete computations. Some technical details on this purpose are reported in appendix B. In particular, eq. (B.8) illustrates how to evaluate the Mukai pairing via bispinors. As an application of this technology, at the end of appendix B we perform an instructive exercise, checking that the generalized almost complex structures defined by the pure spinors (2.64) via formula (2.54) correspond exactly to the matrices $\mathcal{J}_\pm$ provided in (2.42).

A final remark is in order. The 2–form b appearing in (2.64) is supposed to be globally defined. However, this cannot be the case if its field strength $H = db$ is cohomologically non-trivial, a circumstance of obvious interest for our applications to flux compactifications. This situation can be taken into account by replacing the generalized tangent bundle made by the sum of T and T^* by an extended bundle where T^* is non-trivially fibered over T, and where on patch overlaps one allows b-shifts by gauge transformations [22, 23, 46]. However, at least for our purposes, this more complicated construction can be avoided by splitting H as $H = H^{\rm fl} + db$, where $H^{\rm fl}$ is a fixed representative of the cohomology class of H ('fl' stands for 'flux'), and it is understood that both b and $H^{\rm fl}$ are globally defined. The b defined here is identified with the one appearing in (2.64), while the flux piece $H^{\rm fl}$ does not enter in the pure spinors (it will be part of a 'twisted differential' $d - H^{\rm fl}\wedge$ acting on them, see subsection 3.3.5).

2.5 Deformations of SU(3)×SU(3) structures

In this section we study deformations of compatible pure spinors, hence of SU(3)×SU(3) structures. In particular, we are interested in the SU(3)×SU(3) structure deformations affecting the generalized metric $\mathcal{G}$. In the context of compactifications, to be explored in next chapter, the metric on the space of these deformations defines the kinetic terms of the 4d scalars parameterizing the space of both the internal metrics and b-fields. With restriction to the strict SU(3) structure case, a similar analysis has been performed in ref. [45]. Our contribution, developed in [P1, P2], was to extend the results of that paper, working with a general SU(3)×SU(3) structure on $TM_6 \oplus T^*M_6$.[11] The mainstay of our discussion will be Hitchin's result about the special Kähler geometry property of the space of pure spinor deformations at a point of M_6, discussed in subsection 2.5.2 below.

2.5.1 The generalized diamond

The space of complex Spin(6,6) spinors, or equivalently, the space of complex polyforms $(\wedge^\bullet T^*M_6)\otimes\mathbb{C}$, decomposes in irreducible representations of the subgroup SU(3)×SU(3) $\subset$ Spin$(6,6)$ defined by the compatible pair Φ_+, Φ_-. This is analogous to what seen in subsection 2.2.2 when discussing SU(3) structures on M_6, where we had a decomposition of the Spin(6) spinor representation in irreps of the SU(3) subgroup identified by a globally defined nowhere vanishing ordinary spinor η_+. As we will discuss in next subsection, this decomposition of the space of polyforms is an useful tool for classifying the pure spinor deformations we wish to analyze.

Following [46], we call $U_{\mathbf{r},\mathbf{s}}$ the subbundle of $(\wedge^\bullet T^*M_6)\otimes\mathbb{C}$ whose polyforms transform in the $(\mathbf{r},\mathbf{s})$ representation of SU(3)×SU(3), and we organize the different representations in a *generalized diamond* [23, 59][12]

$$\begin{array}{ccccccc}
 & & & U_{\mathbf{1},\bar{\mathbf{1}}} & & & \\
 & & U_{\mathbf{1},\mathbf{3}} & & U_{\bar{\mathbf{3}},\bar{\mathbf{1}}} & & \\
 & U_{\mathbf{1},\bar{\mathbf{3}}} & & U_{\bar{\mathbf{3}},\mathbf{3}} & & U_{\mathbf{3},\bar{\mathbf{1}}} & \\
U_{\mathbf{1},\mathbf{1}} & & U_{\bar{\mathbf{3}},\bar{\mathbf{3}}} & & U_{\mathbf{3},\mathbf{3}} & & U_{\bar{\mathbf{1}},\bar{\mathbf{1}}} \\
 & U_{\bar{\mathbf{3}},\mathbf{1}} & & U_{\mathbf{3},\bar{\mathbf{3}}} & & U_{\bar{\mathbf{1}},\mathbf{3}} & \\
 & & U_{\mathbf{3},\mathbf{1}} & & U_{\bar{\mathbf{1}},\bar{\mathbf{3}}} & & \\
 & & & U_{\bar{\mathbf{1}},\mathbf{1}} & & &
\end{array} \tag{2.68}$$

An important difference with respect to the usual (p,q)-decomposition of complex differential forms is that here the $U_{\mathbf{r},\mathbf{s}}$ are made of forms of mixed degree. It turns out that the even and the odd polyforms transform differently under SU(3)×SU(3), i.e. the polyforms in each $U_{\mathbf{r},\mathbf{s}}$ have definite parity. We have:

$$\begin{aligned}
U_{\mathbf{1},\bar{\mathbf{1}}} \oplus U_{\mathbf{1},\bar{\mathbf{3}}} \oplus U_{\bar{\mathbf{3}},\mathbf{3}} \oplus U_{\mathbf{3},\bar{\mathbf{1}}} \oplus U_{\bar{\mathbf{3}},\mathbf{1}} \oplus U_{\mathbf{3},\bar{\mathbf{3}}} \oplus U_{\bar{\mathbf{1}},\mathbf{3}} \oplus U_{\bar{\mathbf{1}},\mathbf{1}} \;&=\; (\wedge^{\mathrm{even}}T^*M_6)\otimes\mathbb{C} \\
U_{\mathbf{1},\mathbf{3}} \oplus U_{\bar{\mathbf{3}},\bar{\mathbf{1}}} \oplus U_{\mathbf{1},\mathbf{1}} \oplus U_{\bar{\mathbf{3}},\bar{\mathbf{3}}} \oplus U_{\mathbf{3},\mathbf{3}} \oplus U_{\bar{\mathbf{1}},\bar{\mathbf{1}}} \oplus U_{\mathbf{3},\mathbf{1}} \oplus U_{\bar{\mathbf{1}},\bar{\mathbf{3}}} \;&=\; (\wedge^{\mathrm{odd}}T^*M_6)\otimes\mathbb{C}\,.
\end{aligned}$$

[11] See also ref. [37] for an independent analysis, focused on $N=1$ supergravity.

[12] $\bar{\mathbf{1}}$ refers to the singlet coming from the decomposition under SU(3) of the $\bar{\mathbf{4}}$ of Spin(6).

The SU(3)×SU(3) singlets $\Phi_\pm$, $\bar\Phi_\pm$ occupy the vertices of the diamond. More concretely, Φ_+ spans $U_{\mathbf{1},\bar{\mathbf{1}}}$ while Φ_- spans $U_{\mathbf{1},\mathbf{1}}$.

For vanishing b, the SU(3)×SU(3) structure is defined by the $\Phi^0_\pm$ given in (2.60). In this case, an explicit local basis for the whole decomposition (2.68) can be built [27, 33] by exploiting the correspondence between differential forms and bispinors provided by the Clifford map (2.59). Indeed, starting from the lowest/highest weight states $\Phi^0_\pm$ and $\bar\Phi^0_\pm$, and acting in the bispinor picture with holomorphic/antiholomorphic Cliff(6) gamma matrices (to be seen as lowering/raising operators), one can reconstruct the whole decomposition of the space of complex polyforms under SU(3)×SU(3). More specifically, recalling that the two SU(3) factors of SU(3)×SU(3) act respectively from the left and from the right on the bispinors $\not{\Phi}^0_\pm = 8e^{-\phi}\eta^1_+ \otimes \eta^{2\dagger}_\pm$, one defines an action of γ^{i_1}, $\gamma^{\bar\imath_1}$ from the left and of γ^{i_2}, $\gamma^{\bar\imath_2}$ from the right, where γ^{i_1} (respectively, γ^{i_2}) is holomorphic[13] with respect to the almost complex structure I_1 (I_2) associated with η^1_+ (η^2_+). Then the 6 annihilators of the pure spinor $\not{\Phi}^0_+$ are $\overrightarrow{\gamma}^{\,i_1}$ and $\overleftarrow{\gamma}^{\,\bar\imath_2}$, while $\not{\Phi}^0_-$ is annihilated by $\overrightarrow{\gamma}^{\,i_1}$ and $\overleftarrow{\gamma}^{\,i_2}$. The conjugate gamma matrices act as creators. The resulting basis is [33]

$$\begin{array}{ccccccccc}
 & & & & \Phi^0_+ & & & & \\
 & & & \Phi^0_+\gamma^{i_2} & & \gamma^{\bar\imath_1}\Phi^0_+ & & & \\
 & & \Phi^0_-\gamma^{\bar\imath_2} & & \gamma^{\bar\imath_1}\Phi^0_+\gamma^{i_2} & & \gamma^{i_1}\bar\Phi^0_- & & \\
 \Phi^0_- & & & \gamma^{\bar\imath_1}\Phi^0_-\gamma^{\bar\imath_2} & & \gamma^{i_1}\bar\Phi^0_-\gamma^{i_2} & & & \bar\Phi^0_- \\
 & & \gamma^{\bar\imath_1}\Phi^0_- & & \gamma^{i_1}\bar\Phi^0_+\gamma^{\bar\imath_2} & & \bar\Phi^0_-\gamma^{i_2} & & \\
 & & & \gamma^{i_1}\bar\Phi^0_+ & & \bar\Phi^0_+\gamma^{\bar\imath_2} & & & \\
 & & & & \bar\Phi^0_+ & & & &
\end{array} \tag{2.69}$$

where we have dropped the slashes on the pure spinors in order not to clutter the notation. Using the Clifford map backwards, these basis elements can also be seen as polyforms. In this last case, the Cliff(6) gamma matrices are mapped to elements of $T\oplus T^*$, as shown in (B.9).

When the SU(3)×SU(3) structure is defined by pure spinors including also the B-field, namely $\Phi_\pm = e^{-b}\Phi^0_\pm$, a basis for the decomposition (2.68) of the space of polyforms under the action of this different SU(3)×SU(3) is simply obtained by acting with e^{-b} on the basis (2.69) (in the polyform picture). Indeed, this is just the result of performing a B-transformation: for the pure spinors one has

$$\Phi^0_\pm \overset{B-\mathrm{transf}}{\longrightarrow} \Phi_\pm = e^{-b}\Phi^0_\pm\ ,$$

while the raising/lowering operators $\overrightarrow{\gamma}^{\,i_1}$, $\overrightarrow{\gamma}^{\,\bar\imath_1}$, $\overleftarrow{\gamma}^{\,i_2}$, $\overleftarrow{\gamma}^{\,\bar\imath_2}$, viewed as elements of $T\oplus T^*$ (recall (B.9)), are shifted as[14]

$$\overrightarrow{\gamma}^{\,i} = P_1{}^i{}_n(dy^n + iJ_1^{np}\partial_p) \quad \overset{B-\mathrm{transf}}{\longrightarrow} \quad \overrightarrow{\gamma}^{\,i}_{(b)} = P_1{}^i{}_n\big(dy^n + iJ_1^{np}(\partial_p + b_{pq}dy^q)\big) \qquad \text{(analogous for the others).}$$

Here P_1 is the holomorphic projector with respect to the almost complex structure I_1. We deduce that, for instance, $\overrightarrow{\gamma}^{\,i}_{(b)}\bar\Phi_+ = e^{-b}\,\overrightarrow{\gamma}^{\,i}\bar\Phi^0_+$, and similarly for all the other basis elements.

[13]See appendix A.4 for the expression of these gamma matrices in terms of holomorphic projectors.

[14]Recall (2.39) for the B-transformation of a generic element $v+\zeta \in T\oplus T^*$. The sign in front of $\iota_v b$ is positive if the pure spinors transform with e^{-b} [23].

Thanks to the basis for the decomposition in representations $(\mathbf{r},\mathbf{s})$ of SU(3)×SU(3), it's now easy to check that the generalized diamond is orthogonal with respect to the Mukai pairing, i.e. the only nonvanishing pairings are between polyforms transforming in conjugate representations $(\mathbf{r},\mathbf{s})$ and $(\bar{\mathbf{r}},\bar{\mathbf{s}})$ of SU(3)×SU(3). This is best seen in the bispinor picture, using the representation (B.8) of the Mukai pairing under the Clifford map.

2.5.2 The space of deformations

Equipped with the technical tools introduced above, we can now discuss the moduli space of compatible pure spinor pairs, and their relevance for compactifications. Building on previous work [53] dealing with differential forms of pure degree, in ref. [22] Hitchin shows that both the spaces of even and odd pure spinors at a point of M_6 admit a rigid special Kähler structure. This result was first transposed in the context of supergravity in [45], to which we also refer for a review of Hitchin's work. Here we just recall that starting from the rigid special Kähler structure defined by Hitchin, one can obtain a local special Kähler geometry modding out the $\mathbb{C}^*$ action corresponding to a rescaling of the pure spinors. Clearly, it is this local special Kähler structure that is relevant for the supergravity applications. Since modding out the rescalings is the same as working with pure spinor lines, the quotient space corresponds to the deformation space of the generalized almost complex structure $\mathcal{J}$ determined by the pure spinor Φ. The Kähler potentials $K_\pm$ yielding the local special Kähler metrics on these deformation spaces turn out to be the Hitchin functions [22, 45]

$$e^{-K_\pm} = i\langle\Phi_\pm,\bar{\Phi}_\pm\rangle \ . \tag{2.70}$$

We stress that this result is valid *at a point* of the 6d manifold M_6. Correspondingly, in (2.70) no integral is performed over the compact space. Put in the context of type II supergravity compactifications, this means that at this stage we are keeping a full dependence of the higher dimensional fields on both the external spacetime coordinates, on which the moduli depend, and the internal coordinates. We will come back to this issue later on in this chapter, as well as in the first part of the next one.

In the strict SU(3) structure case, substituting the pure spinors (2.65) into (2.70), one gets for $K_\pm$:

$$e^{-K_+} = \frac{4}{3}e^{-2\phi}J\wedge J\wedge J \qquad , \qquad e^{-K_-} = ie^{-2\phi}\Omega\wedge\bar{\Omega} \ , \tag{2.71}$$

expressions that we have already encountered in subsection 2.3.2 when looking at the moduli space of Calabi-Yau 3-folds. In that case, J and Ω are closed, and their allowed deformations are parameterized by cohomology classes in $H^2(M)$ and $H^3(M)$ respectively.[15] This permits to identify the integrated form of expressions (2.71) as the Kähler potentials for the moduli space of the full manifold, not just at a point of it. The dimension of the moduli space is finite since it corresponds to the dimension of the cohomology.

In order to work out the general form of the special Kähler metrics derived from (2.70), we now discuss deformations of pure spinors. Following Hitchin [22], we write the generic

[15] In Calabi-Yau compactifications ϕ is constant along M_6, and does not enter in the expression for the moduli space metric derived from the Kähler potentials.

purity-preserving infinitesimal variation $\delta\Phi$ of a pure spinor Φ as

$$\delta\Phi = c\Phi + \sigma\cdot\Phi \qquad , \qquad \sigma\cdot \equiv \sigma_{\Lambda\Sigma}\Gamma^{\Lambda\Sigma} \ ,$$

where $c \in \mathbb{C}$ is a (small) complex function on M_6, while at each point of M_6 $\sigma\cdot$ is an element of the complexified Spin(6,6) algebra $\mathfrak{so}(12,\mathbb{C})$, with infinitesimal parameters $\sigma_{\Lambda\Sigma}$. Recalling (2.50) we can write the $\Gamma^{\Lambda\Sigma}$ as

$$\Gamma^{\Lambda\Sigma} = \left(dy^m\wedge dy^n\wedge \ , \ \frac{1}{2}[dy^m\wedge, \iota_{\partial_n}] \ , \ \frac{1}{2}[\iota_{\partial_m}, dy^n\wedge] \ , \ \iota_{\partial_m}\iota_{\partial_n} \right) .$$

We can also express $\sigma\cdot$ in terms of a basis of creators and annihilators for Φ. The nonzero variations are obtained acting with the antisymmetrized product of two creators, or of a creator and the associated annihilator (in this case the result is proportional to Φ, and we could absorb it in the parameter c).

Consider now the two pure spinors $\Phi_\pm$, together with the SU(3)×SU(3) structure they identify. Decomposing their variations $\delta\Phi_\pm$ in representations of SU(3)×SU(3), and referring to the diamond (2.68), we deduce that

$$\delta\Phi_- \ \in \ \Gamma(U_{\mathbf{1},\mathbf{1}}\oplus U_{\mathbf{1},\mathbf{3}}\oplus U_{\bar{\mathbf{3}},\bar{\mathbf{3}}}\oplus U_{\mathbf{3},\mathbf{1}}) \qquad , \qquad \delta\Phi_+ \ \in \ \Gamma(U_{\mathbf{1},\bar{\mathbf{1}}}\oplus U_{\mathbf{1},\bar{\mathbf{3}}}\oplus U_{\bar{\mathbf{3}},\mathbf{3}}\oplus U_{\mathbf{3},\bar{\mathbf{1}}}) \ .$$

We also need to require that the deformed pure spinors $\Phi_\pm + \delta\Phi_\pm$ again be compatible, and this imposes constraints on the allowed variations. Indeed, varying the compatibility condition (2.55), rewritten as $\langle\Phi_+,\Gamma^\Lambda\Phi_-\rangle = \langle\bar{\Phi}_+,\Gamma^\Lambda\Phi_-\rangle \ = \ 0$, we get

$$\langle\delta\Phi_+,\Gamma^\Lambda\Phi_-\rangle + \langle\Phi_+,\Gamma^\Lambda\delta\Phi_-\rangle = 0 \qquad , \qquad \langle\delta\bar{\Phi}_+,\Gamma^\Lambda\Phi_-\rangle + \langle\bar{\Phi}_+,\Gamma^\Lambda\delta\Phi_-\rangle = 0 \ . \tag{2.72}$$

We eventually rewrite the infinitesimal deformations of $\Phi_\pm$ in a notation reminding the Kodaira formula for the holomorphic 3-form Ω of a Calabi-Yau manifold[16]

$$\delta\Phi_\pm = \delta\kappa_\pm\Phi_\pm + \delta_{\rm tr}\Phi_\pm + \delta\chi_\pm \ . \tag{2.73}$$

Let us discuss the different terms in this formula. The $\delta\kappa_\pm$ are complex scalars describing the local pure spinor rescalings. Because of condition (2.56), their real parts need to be equal (the imaginary parts are instead independent). Anyway, this constraint is ininfluent for the purpose of describing the deformation space of U(3)×U(3) structures, since in this case the pure spinor rescalings are modded out.

The independent deformations $\delta\chi_-$ and $\delta\chi_+$, are sections respectively of the $U_{\bar{\mathbf{3}},\bar{\mathbf{3}}}$ and $U_{\bar{\mathbf{3}},\mathbf{3}}$ subbundles defined in subsection 2.5. Locally on M_6, they can be parameterized using the basis (2.69) as

$$\delta\chi_\pm = e^{-b}\delta\chi^0_\pm \ , \qquad \text{with} \qquad \delta\chi^0_+ = (\delta\chi_+)_{\bar{\imath}_1 j_2}\gamma^{\bar{\imath}_1}\Phi^0_+\gamma^{j_2} \quad , \quad \delta\chi^0_- = (\delta\chi_-)_{\bar{\imath}_1\bar{\jmath}_2}\gamma^{\bar{\imath}_1}\Phi^0_-\gamma^{\bar{\jmath}_2} \ .$$

[16]The Kodaira formula for Ω was introduced in (2.19); in the present notation it reads $\delta\Omega = \delta\kappa\Omega + \delta\chi$. Notice however that here we have an essential limitation. Indeed, the Kodaira formula describes deformations of Ω globally on M_6, and these, being parameterized by cohomology, come in a finite number. On the other hand, here either we work at a point of M_6, and then we lose the global description, or we consider deformations globally on M_6, but then the deformation space is infinite-dimensional since we are not imposing any constraint to the local variations (in particular, here the pure spinors are not even closed).

Here and in the following the indices $\bar{\imath}_1, i_1$ are (anti)holomorphic with respect to the almost complex structure I_1, and analogously for $\bar{\jmath}_2, j_2$ with respect to I_2. The complex tensors $(\delta\chi_+)_{mn}$ and $(\delta\chi_-)_{mn}$ satisfy

$$\bar{P}_{1m}{}^{p} P_{2n}{}^{q} (\delta\chi_+)_{pq} = (\delta\chi_+)_{mn} \qquad , \qquad \bar{P}_{1m}{}^{p} \bar{P}_{2n}{}^{q} (\delta\chi_-)_{pq} = (\delta\chi_-)_{mn} \; , \tag{2.74}$$

where $(P_k)_m{}^n = \frac{1}{2}(\mathbb{1} - iI_k)_m{}^n$ is the holomorphic projector associated with $I_k\,,\ k = 1, 2$.

The remaining deformations $\delta_{\rm tr}\Phi_+$ and $\delta_{\rm tr}\Phi_-$ in (2.73) transform in the SU(3)×SU(3) 'triplets' $(\mathbf{3},\mathbf{1})\oplus(\bar{\mathbf{3}},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\bar{\mathbf{3}})$, hence the denomination '$\delta_{\rm tr}$'. These are precisely the pure spinor deformations constrained by the compatibility condition (2.55), which requires them to be performed simultaneously. More specifically, using the basis (2.69), locally on M_6 a parameterization of these simultaneous variations is

$$\delta_{\rm tr}\Phi_+ = e^{-b}\big(\delta u_{i_1}\gamma^{i_1}\bar{\Phi}^0_- + \delta\bar{v}_{\bar{\imath}_2}\Phi^0_-\gamma^{\bar{\imath}_2}\big) \quad , \quad \delta_{\rm tr}\Phi_- = -e^{-b}\big(\delta u_{i_1}\gamma^{i_1}\bar{\Phi}^0_+ + \delta v_{i_2}\Phi^0_+\gamma^{i_2}\big) \; , \tag{2.75}$$

where

$$\delta u_{i_1} = \frac{1}{2}(\mathbb{1} - iI_1)_{i_1}{}^m \delta u_m \qquad , \qquad \delta v_{i_2} = \frac{1}{2}(\mathbb{1} - iI_2)_{i_2}{}^m \delta v_m \; ,$$

δu_m and δv_m being real and independent small functions on M_6. Via the Clifford map, expression (2.75) can be read either in the bispinor picture, or in the polyform picture (recall that in this case $\overrightarrow{\gamma}^{i_1}$ and $\overleftarrow{\gamma}^{i_2}$ are mapped to elements of $(T\oplus T^*)\otimes\mathbb{C}$, as in (B.9)).

We now evaluate the holomorphic and antiholomorphic variations of the Kähler potentials (2.70), deriving in this way an expression for the special Kähler metrics on the space of compatible pure spinor lines (namely, the space of U(3)×U(3) structures) at a point of M_6. Using (2.73) and the fact that $\Phi_\pm$ depend holomorphically on their parameters, we obtain

$$\begin{aligned} ds^2_\pm \;=\; \delta^{\rm holo}\delta^{\rm anti}K_\pm \;&=\; \frac{\langle\Phi_\pm, \delta\bar{\Phi}_\pm\rangle}{\langle\Phi_\pm, \bar{\Phi}_\pm\rangle}\frac{\langle\delta\Phi_\pm, \bar{\Phi}_\pm\rangle}{\langle\Phi_\pm, \bar{\Phi}_\pm\rangle} - \frac{\langle\delta\Phi_\pm, \delta\bar{\Phi}_\pm\rangle}{\langle\Phi_\pm, \bar{\Phi}_\pm\rangle} \\ &=\; -\frac{\langle\delta\chi_\pm, \delta\bar{\chi}_\pm\rangle}{\langle\Phi_\pm, \bar{\Phi}_\pm\rangle} \;+\; g^{mn}(\delta u_m\delta u_n + \delta v_m\delta v_n)\; . \end{aligned} \tag{2.76}$$

Notice that the rescalings of the pure spinors don't contribute to the metric. The last term arises from the equal contributions (the computation uses (2.75) and (B.8)):

$$-\frac{\langle\delta_{\rm tr}\Phi_\pm, \delta_{\rm tr}\bar{\Phi}_\pm\rangle}{\langle\Phi_\pm, \bar{\Phi}_\pm\rangle} = g^{mn}(\delta u_m\delta u_n + \delta v_m\delta v_n)\; .$$

Since $\delta_{\rm tr}\Phi_-$ and $\delta_{\rm tr}\Phi_+$ are not independent, the space of U(3)×U(3) structures at a point of M_6, with metric

$$ds^2 = ds^2_+ + ds^2_- = -\frac{\langle\delta\chi_-, \delta\bar{\chi}_-\rangle}{\langle\Phi_-, \bar{\Phi}_-\rangle} - \frac{\langle\delta\chi_+, \delta\bar{\chi}_+\rangle}{\langle\Phi_+, \bar{\Phi}_+\rangle} + 2g^{mn}(\delta u_m\delta u_n + \delta v_m\delta v_n) \tag{2.77}$$

is not a direct product of $\mathcal{J}_+$ and $\mathcal{J}_-$ deformation spaces.

2.5.3 Metric deformations

In the following we address the problem of determining which of the SU(3)×SU(3) variations considered above affect the metric g and the 2–form b on M_6, and have therefore a physical meaning.

As we mentioned in subsection 2.3.2 dealing with Calabi-Yau 3–folds – and as we will discuss in next chapter – in dimensional reductions of 10d supergravities the kinetic terms of the 4d scalars describing the fluctuations of the internal metric and B-field are defined by a σ-model whose target space metric can be written as expression (2.15). With restriction to the SU(3) structure case, in [45] it was shown that such kinetic terms are reproduced – with some caveats – by the sum of the special Kähler metrics obtained by variation of the Kähler potentials (2.71). We now extend this result to the more general SU(3)×SU(3) structure environment, analyzing the relation between the pure spinor deformations and the supergravity σ-model (2.15). We will show that the latter can be expressed as the sum of two independent contributions, which arise by varying either one of the generalized almost complex structures $\mathcal{J}_+, \mathcal{J}_-$ while keeping the other fixed. We will also discuss the identification with the two special Kähler metrics (2.76).

Starting from (2.46), we observe that the integrand of (2.15) can be written in terms of fluctuations of the $T \oplus T^*$ metric $\mathcal{G}$ as

$$g^{mp}g^{nq}(\delta g_{mn}\delta g_{pq} + \delta b_{mn}\delta b_{pq}) = -\frac{1}{2}\mathrm{Tr}\big(\delta\mathcal{G}\delta\mathcal{G}\big)\ , \tag{2.78}$$

where the trace is taken over the $T \oplus T^*$ indices. This in turn can be expressed in terms of deformations of the generalized almost complex structures $\mathcal{J}_\pm$. Indeed, recalling (2.46) we have

$$\delta\mathcal{G} = -(\delta\mathcal{J}_+)\mathcal{J}_- - \mathcal{J}_+(\delta\mathcal{J}_-)\,, \tag{2.79}$$

and hence

$$\mathrm{Tr}(\delta\mathcal{G}\delta\mathcal{G}) = \mathrm{Tr}\left[\Big((\delta\mathcal{J}_+)\mathcal{J}_- + \mathcal{J}_+(\delta\mathcal{J}_-)\Big)\Big((\delta\mathcal{J}_+)\mathcal{J}_- + \mathcal{J}_+(\delta\mathcal{J}_-)\Big)\right]\ . \tag{2.80}$$

We wish to evaluate the variations of $\mathcal{J}_\pm$ via the associated pure spinor deformations. From (2.54) we have (omitting the $\pm$ labels for notational clearness):

$$\delta\mathcal{J}_{\Lambda\Sigma} = -\frac{8\langle\mathrm{Re}(\delta\Phi),\Gamma_{\Lambda\Sigma}\mathrm{Re}\Phi\rangle}{i\langle\Phi,\bar{\Phi}\rangle} - \mathcal{J}_{\Lambda\Sigma}\frac{\delta\langle\Phi,\bar{\Phi}\rangle}{\langle\Phi,\bar{\Phi}\rangle}\ , \tag{2.81}$$

where we collected the two terms containing $\mathrm{Re}(\delta\Phi)$ using (B.5). From (2.73), we have

$$\mathrm{Re}(\delta\Phi)\ =\ \mathrm{Re}(\delta\kappa)\mathrm{Re}\Phi - \mathrm{Im}(\delta\kappa)\mathrm{Im}\Phi + \mathrm{Re}(\delta_{\mathrm{tr}}\Phi + \delta\chi)\,.$$

Now, in (2.81) the contribution of $\mathrm{Re}(\delta\kappa)\mathrm{Re}\Phi$ compensates exactly the term containing $\delta\langle\Phi,\bar{\Phi}\rangle$, while it is not difficult to see that $\mathrm{Im}(\delta\kappa)\langle\mathrm{Im}\Phi,\Gamma_{\Lambda\Sigma}\mathrm{Re}\Phi\rangle$ vanishes. Therefore the variation of Φ consisting of a rescaling drops out. This was expected, since generalized almost complex structures are in one-to-one correspondence with complex *lines* of pure spinors. Hence we have

$$\delta\mathcal{J}_{\pm\Lambda\Sigma} = -\frac{8\langle\mathrm{Re}(\delta_{\mathrm{tr}}\Phi_\pm + \delta\chi_\pm)\,,\,\Gamma_{\Lambda\Sigma}\mathrm{Re}\Phi_\pm\rangle}{i\langle\Phi_\pm,\bar{\Phi}_\pm\rangle}\ . \tag{2.82}$$

Let us now consider the variations induced on the generalized almost complex structures $\mathcal{J}_+, \mathcal{J}_-$ by $\delta_{\rm tr}\Phi_\pm$, therefore transforming in the 'triplets' of SU(3)×SU(3). These are precisely the deformations of $\mathcal{J}_\pm$ which leave invariant the generalized metric $\mathcal{G}$ (and so both g and b). Indeed, paraphrasing [45], on the one hand the two compatible generalized almost complex structures are invariant under the action of U(3)×U(3) ⊂ O(6,6), hence the space of $\mathcal{J}_+, \mathcal{J}_-$ at a point of M_6 is the 48-dimensional coset $\frac{\rm O(6,6)}{\rm U(3)\times U(3)}$. On the other hand, since $\mathcal{G}$ is invariant under the larger subgroup O(6)×O(6) ⊂ O(6,6), the space of generalized metrics $\mathcal{G}$ is the 36-dimensional coset $\frac{\rm O(6,6)}{\rm O(6)\times O(6)}$. The $48-36=12$-dimensional space of O(6,6) transformations being in the first and not in the second coset corresponds to the vector representation of O(6,6), decomposing under the SU(3)×SU(3) structure group in the 'triplets' $(\mathbf{3},\mathbf{1})\oplus(\bar{\mathbf{3}},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\bar{\mathbf{3}})$.

The argument just exposed can be made more explicit as follows. Consider the pure spinor variations $\delta_{\rm tr}\Phi_\pm$, parameterized as in (2.75). Starting from (2.82), we evaluate the corresponding deformations of the generalized almost complex structures $\mathcal{J}_+$ and $\mathcal{J}_-$, call them $\delta_{\rm tr}\mathcal{J}_+$ and $\delta_{\rm tr}\mathcal{J}_-$. Performing the computation in the bispinor picture, in particular using (B.8) to evaluate the Mukai pairing and (B.12) for $\Gamma_{\Lambda\Sigma}$, we find that $\mathcal{J}_+(\delta_{\rm tr}\mathcal{J}_-)=-(\delta_{\rm tr}\mathcal{J}_+)\mathcal{J}_-$. More in detail, we obtain

$$\begin{aligned}-(\delta_{\rm tr}\mathcal{J}_+)\mathcal{J}_- \;&=\; +\mathcal{J}_+(\delta_{\rm tr}\mathcal{J}_-)\;=\\ &=\left(\begin{array}{cc}1&0\\-b&1\end{array}\right)\left(\begin{array}{cc}{\rm Im}(\delta u\lrcorner\Omega_1+\delta v\lrcorner\Omega_2)^m{}_n & {\rm Im}(\delta u\lrcorner\Omega_1-\delta v\lrcorner\Omega_2)^{mn}\\ {\rm Im}(\delta u\lrcorner\Omega_1-\delta v\lrcorner\Omega_2)_{mn} & {\rm Im}(\delta u\lrcorner\Omega_1+\delta v\lrcorner\Omega_2)_m{}^n\end{array}\right)\left(\begin{array}{cc}1&0\\b&1\end{array}\right),\end{aligned}$$

where Ω_1 and Ω_2 are the invariant (3,0)–forms of the SU(3) structures associated with η_+^1 and η_+^2 respectively. Recalling (2.79), we conclude that $\mathcal{G}$ is invariant under deformations of the generalized almost complex structures induced by $\delta_{\rm tr}\Phi_\pm$.

The only pure spinor deformations which modify the generalized metric $\mathcal{G}$ are then $\delta\chi_\pm$. We now evaluate (2.80) in terms of these deformations. Since ${\rm Re}(\delta\chi_-)\in\Gamma(U_{\bar{\mathbf{3}},\bar{\mathbf{3}}}\oplus U_{\mathbf{3},\mathbf{3}})$ and ${\rm Re}(\delta\chi_+)\in\Gamma(U_{\bar{\mathbf{3}},\mathbf{3}}\oplus U_{\mathbf{3},\bar{\mathbf{3}}})$, the only nonzero contributions to $\delta\mathcal{J}_\pm$ come from the components of $\Gamma_{\Lambda\Sigma}{\rm Re}\Phi_\pm$ being in the same representations. For $b=0$ these are of the form $\gamma^m{\rm Re}\Phi^0_\pm\gamma^n$ (see (B.12)), while for nonvanishing b there are extra contributions yielding the matrices $\left(\begin{smallmatrix}1&0\\-b&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\b&1\end{smallmatrix}\right)$, as discussed at the end of appendix B. By a quite long but straightforward computation we find that the terms in (2.80) mixing the variations of $\mathcal{J}_+$ and $\mathcal{J}_-$ vanish:

$${\rm Tr}\big[\mathcal{J}_-\mathcal{J}_+(\delta\mathcal{J}_-)(\delta\mathcal{J}_+)\big]\;=\;0\,. \tag{2.83}$$

The computation is performed first substituting $\mathcal{J}_-\mathcal{J}_+=-\mathcal{I}^{-1}\mathcal{G}$, and then going in the bispinor picture; we find cancellation between all the nonzero terms involved in the trace. It follows that the metric (2.78) is given by the sum of two contributions, arising from the independent deformations of $\mathcal{J}_-$ and $\mathcal{J}_+$ (or, equivalently, of the associated polyforms $\Phi_\pm$):

$$-\frac{1}{2}{\rm Tr}(\delta\mathcal{G}\delta\mathcal{G})\;=\;-\frac{1}{2}{\rm Tr}\big[\mathcal{J}_+(\delta\mathcal{J}_-)\mathcal{J}_+(\delta\mathcal{J}_-)\big]-\frac{1}{2}{\rm Tr}\big[(\delta\mathcal{J}_+)\mathcal{J}_-(\delta\mathcal{J}_+)\mathcal{J}_-\big]\,. \tag{2.84}$$

Again we can rewrite these terms using the bispinor picture. For the first one we find

(omitting the slashes in order not to clutter the formulae)

$$\begin{aligned}
-\frac{1}{2}\mathrm{Tr}\big[\mathcal{J}_+(\delta\mathcal{J}_-)\mathcal{J}_+(\delta\mathcal{J}_-)\big] &= \frac{1}{8^6}\mathrm{tr}\,[\gamma(\mathrm{Re}\Phi^0_+)^T\gamma_{mp}\mathrm{Re}\Phi^0_+]\,\mathrm{tr}\,[\gamma(\mathrm{Re}\Phi^0_+)^T\mathrm{Re}\Phi^0_+\gamma_{nq}]\,\cdot \\
&\quad \cdot\mathrm{tr}\,[\gamma\mathrm{Re}(\delta\chi^0_-)^T\gamma^p\mathrm{Re}\Phi^0_-\gamma^n]\,\mathrm{tr}\,[\gamma\mathrm{Re}(\delta\chi^0_-)^T\gamma^m\mathrm{Re}\Phi^0_-\gamma^q] \\
&= 8(\delta\chi_-)_{mn}(\delta\bar\chi_-)_{pq}(g^{mp}-iJ_1^{mp})(g^{nq}-iJ_2^{nq}) \\
&= -8\frac{\langle\delta\chi_-,\delta\bar\chi_-\rangle}{\langle\Phi_-,\bar\Phi_-\rangle}\,. \qquad (2.85)
\end{aligned}$$

The computation for the term involving the variation of $\delta\mathcal{J}_+$ is completely analogous. This concludes the analysis started with eq. (2.78). We have obtained an expression for the metric on the space of g and b fluctuations at a point of M_6 in terms of pure spinor deformations. This reads

$$\frac{1}{8}g^{mp}g^{nq}(\delta g_{mn}\delta g_{pq}+\delta b_{mn}\delta b_{pq}) \;=\; -\frac{\langle\delta\chi_-,\delta\bar\chi_-\rangle}{\langle\Phi_-,\bar\Phi_-\rangle}-\frac{\langle\delta\chi_+,\delta\bar\chi_+\rangle}{\langle\Phi_+,\bar\Phi_+\rangle}\,. \qquad (2.86)$$

Since $i\langle\Phi_-,\bar\Phi_-\rangle = i\langle\Phi_+,\bar\Phi_+\rangle = 8e^{-2\phi}vol_6$ (recall (2.56)–(2.58)), we can also write the above relation in the following integrated form:

$$\frac{1}{8\int e^{-2\phi}vol_6}\int vol_6 e^{-2\phi}g^{mp}g^{nq}(\delta g_{mn}\delta g_{pq}+\delta b_{mn}\delta b_{pq}) \;=\; -\frac{\int\langle\delta\chi_-,\delta\bar\chi_-\rangle}{\int\langle\Phi_-,\bar\Phi_-\rangle}-\frac{\int\langle\delta\chi_+,\delta\bar\chi_+\rangle}{\int\langle\Phi_+,\bar\Phi_+\rangle}. \qquad (2.87)$$

The separation in two independent contributions is remarkable, and reminds the splitting of the moduli space of Calabi-Yau manifolds in the direct product $\mathscr{M}_{cs}\times\mathscr{M}_{ks}$ of complex structure and Kähler structure deformation spaces, reviewed in subsection 2.3.2. This is also consistent with $N=2$ supergravity in four dimensions: as we will see in the next chapter, the two terms in the r.h.s. of (2.86) are naturally associated with the kinetic terms of the scalars respectively in the vector multiplets and in the hypermultiplets, which by $N=2$ supersymmetry have to be independent. It is equally remarkable that these two contributions have an underlying special Kähler structure, as they are directly related to the special Kähler metrics (2.76). Again, this is analogous to the Calabi-Yau case, where both factors of the moduli space are special Kähler. This fact is also welcome from the 4d $N=2$ supergravity viewpoint, since at least the scalar manifold of the vector multiplets has to be special Kähler.[17]

An important precisation is however in order. The analysis done so far is very general – it just relies on the topological condition that M_6 admit an SU(3)×SU(3) structure on $TM_6\oplus T^*M_6$ – and possibly encompasses all $N=2$ preserving compactifications. However, it should be regarded just as a first step towards the definition of an actual four dimensional theory. Indeed, till now we have not truncated any of the modes of the type II NSNS fields along the internal manifold. Correspondingly, (2.76), (2.86) can be considered as deformation space metrics only at a point of M_6, and not globally on the manifold.[18]

[17]We will discuss the properties of the hyperscalar manifold later on, starting with subsection 3.2.

[18]Results regarding the global moduli space of manifolds admitting generalized structures are available for some special classes. In particular, the deformation space of generalized complex manifolds, char-

In order to obtain an $N = 2$ theory in 4d, a truncation ansatz selecting a finite set of modes has to be specified. In the next chapter we will investigate which are the general constraints that this truncation ansatz should satisfy for the dimensional reduction to go through, and in particular for the special Kähler geometry properties of the metric on the deformation space of pure spinors at a point of M_6 to be inherited by the 4d theory.

A further, related remark is the following. Eq. (2.86) doesn't exactly coincide with the sum of the special Kähler metrics (2.76) on the $\mathcal{J}_+$- and $\mathcal{J}_-$- deformation spaces, because of the terms containing δu and δv, which parameterize the simultaneous pure spinor variations $\delta_{\rm tr}\Phi_\pm$ transforming in the SU(3)×SU(3) triplets $(\mathbf{3},\mathbf{1}) \oplus (\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\bar{\mathbf{3}})$, recall eq. (2.75). It follows that if we wish to have a precise identification between (2.86) and the sum of the metrics (2.76), then these terms need to be projected out. This extends an analogous prescription given in ref. [45] for deformations of strict SU(3) structures.

The position we take here concerning the issue just described is the following: after having chosen a specific class of SU(3)×SU(3) structure manifolds on which to perform the dimensional reduction, one should specify a truncation ansatz. Once this is done, one should verify that *among the modes one keeps* none transform in the $O(6,6)$ vector representation. Of course, any truncation ansatz eventually needs a justification: typically, it is considered acceptable an ansatz which either captures the low energy physics in four dimensions, or provides a consistent truncation of the higher dimensional theory. The expansion in harmonic forms for compactifications on Calabi-Yau manifolds with no fluxes, to be reviewed in section 3.2, is an example of the first kind. As for the second option, we will discuss a concrete realization in chapter 5.

2.6 Discussion

Let us summarize the eventual outcome of our excursion on the generalized tangent bundle. We started considering the general spinor ansatz (2.1) for dimensional reductions leading to $N = 2$ supergravity in four dimensions, and we saw that this requires the internal manifold M_6 to admit a pair of SU(3) structures. This means that the representation in which a spinor or tensor field on M_6 transforms can be decomposed in irreps of either one of the two SU(3) groups identified by the spinors η^1_+ and η^2_+ in (2.1). We also reviewed how these data are encoded in an SU(3)×SU(3) structure on $TM_6 \oplus T^*M_6$, characterized by the compatible pure spinor pair Φ_+, Φ_-.

Conversely, the specification of an SU(3)×SU(3) structure on $TM_6 \oplus T^*M_6$ fixes all the NSNS degrees of freedom on the six dimensional compact space, i.e. it provides a metric g, a 2–form b and a dilaton ϕ on M_6. It also provides a pair of globally defined nowhere vanishing Spin(6) spinors (with positive chirality) η^1_+ and η^2_+, together with their (negative chirality) conjugates η^1_- and η^2_-.

This repackaging of the NSNS and spinorial degrees of freedom is useful for several reasons. In the first instance, the different situations arising from the various possible

acterized by $d\Phi = (\iota_v + \zeta\wedge)\Phi$, was analyzed in [23], while the moduli space of generalized Calabi-Yau manifolds, defined by $d\Phi = 0$, was studied in [22, 60]. These results have been applied in the context of supersymmetric string backgrounds in [35, 38], and to the study of the associated $N = 1$ 4d effective theories in [61].

intersections of the two SU(3) subgroups of Spin(6) encountered in section 2.1 are now unified in a single mathematical framework. Furthermore, the description of the internal NSNS sector in terms of $\Phi_\pm$ seems in general more promising than the one in terms of g, b and ϕ for the purpose of determining the degrees of freedom to be inherited by a 4d $N = 2$ effective action. Indeed, it is the description in terms of $\Phi_\pm$ that makes a special Kähler geometry structure emerge on the deformation space, matching the 4d $N = 2$ supergravity requirements. This is analogous to the Calabi-Yau case, in which the metric and B-field deformations are best described by considering variations of the complexified Kähler form $b + iJ$ and the holomorphic (3,0)–form Ω, whose moduli space exhibits a special Kähler structure. We will have more to say about these aspects in the next two chapters, where we'll also investigate how generalized geometry provides general formulae for the various 4d supergravity data, such as the Kähler potential, the scalar potential, and the Killing prepotentials.

We close with a comment concerning the RR degrees of freedom. Contrarily to the NS 2–form b, these are not naturally included in the generalized geometry formalism defined on $T \oplus T^*$, and this is the reason why we have not dealt with them in this chapter. However, we'd like to mention that progresses towards the 'geometrization' of the RR fields have recently been achieved by considering an extension of the generalized tangent bundle including higher exterior powers of T and T^* [62, 63, 64]. The natural group acting on this extended bundle is not just $\mathrm{O}(d, d)$, but the full U-duality group. In this thesis, we will not be concerned with these developements any further.

Chapter 3

The dimensional reduction

Equipped with the tools of generalized geometry, in this chapter we study dimensional reductions of type II supergravity leading to $N = 2$ gauged supergravity in four dimensions. The background is assumed to admit SU(3)×SU(3) structure, and we allow for general NSNS and RR fluxes. The truncation is implemented via the expansion of the higher dimensional fields in a finite basis of differential forms on the compact manifold, satisfying a system of properties which we analyze in detail. The complete four-dimensional bosonic action is derived, and emphasis is given on the way its data are determined by the generalized geometry formalism. While the NSNS contribution is obtained via a direct dimensional reduction of the action, the contribution of the RR sector is computed starting from the democratic formulation of type II theory, and demanding consistency with the reduced equations of motion.

3.1 Democratic formulation of type II supergravity

We start this chapter by presenting the higher dimensional theory we wish to compactify, namely type II supergravity. As mentioned in the introduction, this corresponds to the low energy limit of type II superstring theory. On the same footing as type II strings, it comes in two manifestations: type IIA and type IIB. These are the unique maximally supersymmetric supergravity theories in ten dimensions (they are $N = 2$ in 10d, corresponding to 32 supercharges). This thesis we will be mainly focused on type IIA, but most of the results can easily be transferred to type IIB as well.

Below we provide a brief review of the 'democratic' version of type II supergravity, established in ref. [48]. This formulation automatically incorporates Romans' massive deformation of type IIA [65], and is the most suitable for flux compactifications in the generalized geometry approach. Its compatibility with the standard formulation of type II supergravity is illustrated in appendix C.

Before starting the description of the bosonic part of the theory, let us just mention its fermionic spectrum. This is made of a doublet of Majorana-Weyl spin 3/2 fields, the gravitini, and a doublet of Majorana-Weyl spin 1/2 fields, the dilatini. In type IIA, the two gravitini have opposite chirality, while in type IIB they have the same chirality. In both theories, the dilatini have opposite chirality than the gravitini.

The bosonic sector of type II supergravity, on which we focus below, splits in an NSNS and in a RR sector. The field content of the NSNS sector consists of the 10d spacetime metric, of the NS 2–form $\hat{B}$ and of a real scalar, the dilaton ϕ. The corresponding action has the standard (string frame) form[1]

$$S_{\rm NS} = \frac{1}{2}\int_{M_{10}} e^{-2\phi}\Big(\hat{R}*1 + 4d\phi\wedge *d\phi - \frac{1}{2}\hat{H}\wedge *\hat{H}\Big)\ , \tag{3.1}$$

where $\hat{R}$ is the Ricci curvature of M_{10}. The 3–form $\hat{H}$ is subject to the Bianchi identity

$$d\hat{H} = 0\ , \tag{3.2}$$

which for topologically trivial configurations is globally solved by $\hat{H} = d\hat{B}$. For more general topologies the global solution is

$$\hat{H} = \hat{H}^{\rm fl} + d\hat{B}\ , \tag{3.3}$$

where $\hat{H}^{\rm fl}$ is a cohomologically non-trivial representative ('fl' stands for 'flux'). Notice that this splitting of $\hat{H}$ allows us to work with globally defined quantities: we could have insisted in writing $\hat{H} = d\hat{B}$, but in this case generically the form $\hat{B}$ wouldn't be globally defined.

Let us now consider the RR sector. In the democratic approach to type IIA (IIB) supergravity, it describes the dynamics of a field $\hat{\mathbf{F}}$ consisting of a formal sum of forms of all possible even (odd) degrees:

$$\hat{\mathbf{F}} = \hat{F}_0 + \hat{F}_2 + \ldots + \hat{F}_{10} \quad \text{in IIA}\ , \quad \text{while} \qquad \hat{\mathbf{F}} = \hat{F}_1 + \hat{F}_3 + \ldots + \hat{F}_9 \quad \text{in IIB.} \tag{3.4}$$

In order to avoid a doubling of the degrees of freedom with respect to the usual formulation in which only the forms of lower degree appear, a self-duality constraint is imposed on the RR field. In our conventions for the Hodge-$*$, given in section A.2 of the appendix, this constraint reads

$$\hat{\mathbf{F}} = \lambda(*\hat{\mathbf{F}})\ , \qquad \text{with} \qquad \lambda(\hat{F}_k) = (-)^{[\frac{k+1}{2}]}\hat{F}_k\ . \tag{3.5}$$

In the absence of localized sources, the dynamics of the field $\hat{\mathbf{F}}$ is described by the following equation of motion (EoM from now on):

$$(d + \hat{H}\wedge) * \hat{\mathbf{F}} = 0 \qquad \Leftrightarrow \qquad (d - \hat{H}\wedge)\hat{\mathbf{F}} = 0\ , \tag{3.6}$$

where the two expressions are equivalent due to (3.5). The second has the form of a Bianchi identity; for topologically trivial configurations it is globally solved by

$$\hat{\mathbf{F}} = (d - \hat{H}\wedge)\hat{\mathbf{C}} + e^{\hat{B}}\hat{F}_0\ , \tag{3.7}$$

where $\hat{\mathbf{C}}$ is a sum of RR potentials of odd (even) degree for type IIA (IIB), $\hat{F}_0$ is a constant (present only in type IIA, and to be identified with Romans' mass parameter [65]), and $e^{\hat{B}} \equiv 1 + \hat{B}\wedge + \frac{1}{2}\hat{B}\wedge\hat{B}\wedge + \ldots$.

[1] Here and in the following, the hat symbol denotes ten-dimensional fields (no hat is needed for the dilaton). Our conventions concerning the form fields and the Hodge star are spelled out in appendix A.

Once (3.7) is established, the first expression in (3.6) can be derived by varying the potentials $\hat{\mathbf{C}}$ in the following *pseudo*-action [48]:

$$S_{\rm RR} \;=\; -\frac{1}{8}\int_{M_{10}} \big[\,\hat{\mathbf{F}}\wedge *\hat{\mathbf{F}}\,\big]_{10}\;, \tag{3.8}$$

where the notation $[\;]_{10}$ means that we pick the form of maximal degree 10. The prefix 'pseudo-' means that (3.8) contains redundant RR degrees of freedom, and should be considered just as a device to obtain their EoM. The redundancy is then removed at the level of the EoM by the self-duality constraint (3.5), which does not descend from (3.8) and has be imposed by hand. A further peculiarity of this pseudo-action is that it does not contain any Chern-Simons term, which is instead present in the usual formulations of type II supergravities (see e.g. [4, vol. 2]).

A *bona fide* action, containing just the independent degrees of freedom, can be recovered by breaking the democracy among the RR differential forms: a half of the $\hat{F}_k$ has to be eliminated exploiting the self-duality relation. The choice of the forms to keep is not unique, and in some cases the presence of localized sources can suggest the most convenient option [48, 66]. In appendix C we discuss how the action of standard type IIA supergravity without localized sources can be recovered, also taking into account a deformation of the Chern-Simons term due to background fluxes.

Beside the RR field EoM seen above, from the complete democratic pseudo-action $S_{\rm NS}+S_{\rm RR}$ one also derives the EoM for the NSNS degrees of freedom, namely the B-field, the Einstein and the dilaton equations. After using the first of (3.6), these read

$$d(e^{-2\phi}*\hat{H}) - \frac{1}{2}[\hat{\mathbf{F}}\wedge *\hat{\mathbf{F}}]_8 \;=\; 0\;, \tag{3.9}$$

$$\hat{R}_{MN} + 2\hat{\nabla}_M\partial_N\phi - \frac{1}{2}\iota_M\hat{H}\lrcorner\iota_N\hat{H} - \frac{e^{2\phi}}{4}\sum_{k=0}^{10}\iota_M\hat{F}_{(k)}\lrcorner\iota_N\hat{F}_{(k)} \;=\; 0\;, \tag{3.10}$$

$$\hat{R} - \frac{1}{2}\hat{H}\lrcorner\hat{H} + 4\big(\hat{\nabla}^2\phi - \partial_M\phi\hat{\partial}^M\phi\big) = 0\;, \tag{3.11}$$

where M, N are 10d spacetime indices.

3.2 The archetype: type II on Calabi-Yau 3-folds

One of the main tasks of this thesis is to study general features of dimensional reductions of type II theories leading to $N=2$ supergravity in four dimensions. We approach this problem endowed with the tools of generalized geometry, introduced in the previous chapter. The compactification procedure we will develop is modeled on the well-known Calabi-Yau example, which represents the best studied case of dimensional reduction preserving $N=2$ supersymmetry in 4d. For this reason, before dealing with the general case, in the following we discuss Calabi-Yau compactifications first. This also gives us the opportunity to review some basic notions of $N=2$ supergravity.

Calabi-Yau manifolds, whose geometry and moduli space have been summarized in section 2.3, are ubiquitous in string theory. One of the main reasons is that these spaces support consistent flat vacua of string theory preserving a fraction of supersymmetry [6]. In particular, in type II theories a Minkowski$_4 \times$CY$_3$ bosonic background, with all the supergravity field strengths set to zero, preserves eight supercharges, i.e. $N = 2$ in four dimensions. The four-dimensional low energy effective theory describing the physics of massless fluctuations of the higher dimensional fields around this vacuum is an $N = 2$ supergravity [67, 68, 69, 70], and its properties are governed by the Calabi-Yau topology, more specifically by its cohomology [71]. The fact that the features of the effective action don't depend on the Calabi-Yau metric is particularly welcome, since no Ricci flat metrics on compact Calabi-Yau manifolds are explicitly known.

In the following we focus for definiteness on type IIA, and briefly comment on type IIB. Furthermore, as it is often the case in the context of compactifications, we consider just the bosonic sector of the supergravity theory.[2] The fermionic part of the action can in principle be obtained by acting with the appropriate supersymmetry transformations.

In this fluxless case it is more efficient to compactify type IIA supergravity in its standard formulation rather than in the democratic version introduced in the previous section. The latter will instead be advantageous when considering the general case of SU(3)×SU(3) structure backgrounds with arbitrary fluxes. The standard type IIA theory has the same NSNS spectrum described above, while the RR field content just amounts to the gauge potentials $\hat{C}_1$ and $\hat{C}_3$. The action can be recovered from its democratic counterpart as illustrated in appendix C, setting to zero the Romans' mass parameter F_0 (cf. eqs. (C.6), (C.7)). It reads

$$S = \frac{1}{2}\int_{M_{10}} \Big[e^{-2\phi}\big(\hat{R}*1 + 4d\phi\wedge *d\phi - \frac{1}{2}\hat{H}\wedge *\hat{H}\big) - \frac{1}{2}\big(\hat{F}_2\wedge *\hat{F}_2 + \hat{F}_4\wedge *\hat{F}_4 + \hat{B}\wedge d\hat{C}_3\wedge d\hat{C}_3\big)\Big] \tag{3.12}$$

with the field strengths being

$$\hat{H} = d\hat{B} \quad , \quad \hat{F}_2 = d\hat{C}_1 \quad , \quad \hat{F}_4 = d\hat{C}_3 - \hat{H}\wedge\hat{C}_1 \; . \tag{3.13}$$

The various terms in (3.12) are canonical kinetic terms, except the last one which is a Chern-Simons topological coupling, required by supersymmetry.

3.2.1 The Kaluza-Klein (on-shell) approach

Let us describe the Kaluza-Klein approach to Calabi-Yau compactifications, which defines a low energy effective theory for small field fluctuations around a given vacuum (recall the discussion of section 1.1). One starts from a Minkowski$_4 \times$CY$_3$ supergravity vacuum, with a chosen Calabi-Yau metric $\langle g_{mn}\rangle$ depending on the internal coordinates only, a constant dilaton $\langle\phi\rangle$, vanishing supergravity field strengths, and vanishing fermionic vevs. Then one linearizes the higher dimensional equations of motion by considering small field fluctuations around this vacuum, and identifies the degrees of freedom which are massless from the 4d viewpoint. As it is standard in Kaluza-Klein reductions, these turn out to correspond to zero modes of appropriate wave operators on the compact manifold. The field content of

[2]For the heterotic theory, the dimensional reduction of the fermionic supergravity sector on 6d manifolds with SU(3) structure, hence including Calabi-Yau 3–folds, has been performed in [72].

the low energy theory is then defined by retaining just these zero modes, the energy gap between the massless and the massive modes being controled by the inverse length scale of the compact manifold, assumed to be sufficiently small. More in detail, for the bosonic spectrum one finds what follows (see e.g. [3, vol. 2] for an extended discussion).

The fluctuations of the higher dimensional metric $\hat{g}_{MN}(x,y)$ separate in fluctuations with purely 4d indices $g_{\mu\nu}(x,y)$, with mixed indices $g_{\mu n}(x,y)$, and with purely internal indices $g_{mn}(x,y)$; we recall that coordinates along M_4 and M_6 are denoted by x^μ and y^m respectively. From the first kind of fluctuations, one retains the (single) four dimensional graviton, depending on x^μ only. No fluctuations with mixed indices μn are to be included in the Kaluza-Klein truncation ansatz, since massless 4d fields from these terms would be in correspondence with continuous isometries, which do not exist on Calabi-Yau manifolds: indeed, since the latter is compact and Ricci-flat, any Killing vector generating the isometry would be covariantly constant, and this is forbidden by strict SU(3) holonomy. Finally, one considers the fluctuations with purely internal indices mn, i.e. the deformations of the background Calabi-Yau metric $\langle g_{mn}(y)\rangle$. The equation for massless 4d scalar fields arising from these fluctuations turns out to coincide with the Lichnerowicz equation (2.12). It follows that the massless deformations of the background metric are the ones which preserve Ricci-flatness, and therefore the Calabi-Yau condition. By the discussion of subsection 2.3.2, these can be rewritten in terms of harmonic deformations of the background Calabi-Yau Kähler form J and holomorphic (3,0)–form Ω, respectively parameterized by the moduli v^a and z^i. These parameters are now promoted to scalar fields propagating along the 4d spacetime. The physical reason why the massless 4d fields arising from the internal metric fluctuations are in one-to-one correspondence with the Calabi-Yau moduli is clear: since all Ricci flat metrics on Minkowski$_4$ $\times$ CY$_3$ yield equivalent supergravity solutions, the moduli parameterize the vacuum degeneracy; once a certain vev is assigned, their fluctuations behave as Goldstone bosons, and are hence massless.

The remaining type IIA supergravity bosonic fields are the differential forms $\phi, \hat{B}, \hat{C}_1$ and $\hat{C}_3$ (here the dilaton ϕ is seen as a 0–form). The relevant wave operator, whose zero modes correspond to massless 4d fields, is the Laplacian $\Delta = -*d*d - d*d*$ on CY$_3$ (notice that the definition of this operator is metric-dependent, and it is understood that the involved metric is the background one $\langle g_{mn}(y)\rangle$). We conclude that fluctuations of the form fields are to be expanded in harmonic forms on the Calabi-Yau 3–fold. Denoting these as done for the Calabi-Yau cohomology representatives employed in subsection 2.3.2, we define

$$\begin{aligned} \phi = \phi(x) \qquad &, \qquad \hat{B} = B(x) + b^a(x)\omega_a\,, \\ \hat{C}_1 = A^0(x)\omega_0 = A^0(x) \qquad &, \qquad \hat{C}_3 - \hat{B}\wedge\hat{C}_1 \;=\; A^a(x)\wedge\omega_a + \xi^I(x)\alpha_I - \tilde{\xi}_I(x)\beta^I\,, \end{aligned} \tag{3.14}$$

where we recall that the range of the indices is: $a = 1, 2, \ldots h^{1,1}$ and $I = 0, 1, \ldots, h^{2,1}$. Notice that there are no fields with one internal index, since on Calabi-Yau manifolds $h^{1,0} = 0$. The $h^{1,1} + 1$ one-forms $A^A = (A^0, A^a)$ inherit part of the gauge symmetry associated with the type IIA RR potentials $\hat{C}_1, \hat{C}_3$, and are therefore (abelian) gauge bosons on the four dimensional spacetime.

supermultiplets	multiplicity	spin 2	spin 3/2	spin 1	spin 1/2	spin 0
gravitational	1	$g_{\mu\nu}$	$\psi_{\mathcal{A}\mu}$	A^0_μ	–	–
vector	$h^{1,1}$	–	–	A^a_μ	$\lambda^{a\mathcal{A}}$	$t^a = b^a + iv^a$
hyper	$h^{2,1}+1$	–	–	–	$\zeta_{\mathbb{I}}$	$\begin{cases} (\varphi, a, \xi^0, \tilde{\xi}_0) \\ (\, z^i,\, \xi^i,\, \tilde{\xi}_i\,) \end{cases}$

Table 3.1: $N = 2$ supermultiplets for type IIA compactifications on Calabi-Yau 3–folds. Beside the bosonic fields discussed in the text, here we also represent the fermionic content of each supermultiplet; these additional data will be needed in chapter 4. The index $\mathcal{A} = 1, 2$ labels the fundamental representation of the SU(2) R-symmetry group rotating the $N = 2$ supercharges. The two spin 3/2 fields $\psi_{\mathcal{A}\mu}$ in the gravitational multiplet are the gravitini. The two spin 1/2 fermions $\lambda^{a\mathcal{A}}$ in each vector multiplet are named the gaugini. Finally, the $\zeta_{\mathbb{I}}$ – living in the hypermultiplets – are called the hyperini. The index $\mathbb{I} = 1, \dots, 2h^{2,1}+2$ is an $\mathrm{Sp}(h^{2,1}+1)$ index, so that we have two hyperini per hypermultiplet.

The 4d fields defined by the above expansion assemble into $N = 2$ supermultiplets, as follows:

- ⋄ The gauge boson A^0 enters in the $N = 2$ gravitational multiplet, together with the 4d metric $g_{\mu\nu}$. For this reason, it is known as the graviphoton.
  The gravitational multiplet, containing the metric degrees of freedom, is the hallmark of any supergravity theory. Here, the presence of a spin 1 field beside the metric is a consequence of $N = 2$ supersymmetry.

- ⋄ The remaining gauge bosons A^a, accompanied by the 4d complex scalars $t^a = b^a + iv^a$ arising from the combination of the Calabi-Yau Kähler structure moduli v^a with the NS 2–form parameters b^a, define the bosonic content of $h^{1,1}$ vector multiplets.

- ⋄ The 10d dilaton $\phi(x)$ combines with the Calabi-Yau volume $Vol(x)$ to define the 4d scalar $\varphi := \phi - \frac{1}{2}\log(Vol)$, called the 4d dilaton. Furthermore, the 2–form $B(x)$ in 4d can be dualized to a scalar $a(x)$. The scalars φ and a, together with the scalars $\xi^0, \tilde{\xi}_0$ coming from the expansion of $\hat{C}_3$ on α_0 and β^0, define the bosonic part of the so called universal hypermultiplet. The remaining RR real scalars $\xi^i, \tilde{\xi}_i$ pair up with the Calabi-Yau complex structure moduli z^i and define $h^{2,1}$ hypermultiplets.
  We recall that a hypermultiplet is an $N = 2$ supermultiplet made of matter fields only; its bosonic content consists of four real scalars.

This $N = 2$ multiplet structure is summarized in table 3.1. While here above we discussed the derivation of the bosonic half of the supermultiplets, for completeness in the table we also include their fermionic counterpart, as resulting from the Kaluza-Klein reduction.

The strategy one adopts in order to derive the 4d effective action governing the dynamics of these massless excitations is to substitute the 'vev + massless fluctuations' Kaluza-Klein ansatz in the higher dimensional action, and to integrate over the compact space.

After this step, a standard $N = 2$ supergravity action, with canonically normalized Einstein-Hilbert term, is obtained by performing the following Weyl rescaling of the 4d metric

$$g_{\mu\nu}^{\text{new}} := e^{-2\varphi} g_{\mu\nu}^{\text{old}}\,.$$

The resulting bosonic action reads [67]:

$$\begin{aligned} S^{(4)} \;=\; \int_{M_4} \Big[& \frac{1}{2} R_4 * 1 \;+\; \frac{1}{2}\mathrm{Im}\mathcal{N}_{AB} F^A \wedge * F^B \;+\; \frac{1}{2}\mathrm{Re}\mathcal{N}_{AB} F^A \wedge F^B \\ & - \mathscr{G}^{ks}_{a\bar{b}} dt^a \wedge * d\bar{t}^{\bar{b}} \;-\; h_{uv} dq^u \wedge * dq^v \Big], \end{aligned} \tag{3.15}$$

Let us discuss how $N = 2$ supersymmetry determines the structure of the couplings in $S^{(4)}$. The first term is a standard 4d Einstein-Hilbert term, while the three following terms involve the vector multiplets. In $N = 2$ supergravity, the scalars in the vector multiplets are governed by a σ-model whose target manifold has to be special Kähler [73]. Furthermore, the gauge kinetic and topological terms are fixed respectively by the imaginary and real parts of the period matrix associated with the special Kähler geometry of the scalar manifold. In the present type IIA Calabi-Yau compactification, the space spanned by the scalars t^a in the vector multiplets is the complexified Kähler structure moduli space $\mathscr{M}_{ks}$, whose special Kähler geometry was discussed in subsection 2.3.2. Consistently, the kinetic matrix of the t^a is the special Kähler metric $\mathscr{G}^{ks}_{a\bar{b}}$ on $\mathscr{M}_{ks}$, and the gauge kinetic terms involving the field strengths $F^A = dA^A$ are determined by the period matrix $\mathcal{N}_{AB}$ for $\mathscr{M}_{ks}$; these quantities were given in (2.31) and (2.35) respectively. Notice that here the gauge group is $\mathrm{U}(1)^{h^{1,1}+1}$.

Finally, let us consider the last term in $S^{(4)}$. We collectively denote by q^u, with $u = 1, \ldots, 4(h^{2,1} + 1)$, the real scalars in the hypermultiplets. In $N = 2$ supergravity, the hyperscalar dynamics is governed by a σ-model whose target space is a *quaternionic manifold* [74], namely a $4n$ Riemannian manifold whose holonomy is contained in $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ (see appendix E for a review of this geometry). This means that the hyperscalar kinetic matrix h_{uv} has to be a quaternionic metric. The Calabi-Yau compactification we are considering yields

$$\begin{aligned} h_{uv} dq^u dq^v \;=\; & \mathscr{G}^{cs}_{i\bar{\jmath}} dz^i d\bar{z}^{\bar{\jmath}} + (d\varphi)^2 + \frac{e^{4\varphi}}{4}\big(da - \xi^I d\tilde{\xi}_I + \tilde{\xi}_I d\xi^I\big)^2 \\ & - \frac{e^{2\varphi}}{2}\Big[d\xi^I \mathrm{Im}\mathcal{M}_{IJ} d\xi^J + (d\tilde{\xi}_I - d\xi^K \mathrm{Re}\mathcal{M}_{KI}) \mathrm{Im}\mathcal{M}^{-1\,IJ} (d\tilde{\xi}_J - \mathrm{Re}\mathcal{M}_{JL} d\xi^L)\Big], \end{aligned} \tag{3.16}$$

where $\mathscr{G}^{cs}$ and $\mathcal{M}$ are respectively the special Kähler metric (2.20) and the period matrix (2.26) on the Calabi-Yau complex structure moduli space $\mathscr{M}_{cs}$ introduced in subsection 2.3.2. It can be shown that this metric is indeed quaternionic [68].

Notice that the quaternionic manifold we are encountering is of a very specific type, in that its metric is completely determined by the data of the special Kähler submanifold $\mathscr{M}_{cs}$ spanned by the z^i. These particular quaternionic manifolds are termed *dual* (or *special*) *quaternionic*. In general, a map sending a special Kähler manifold of complex dimension n to a quaternionic manifold of real dimension $4(n+1)$ can be algorithmically constructed by dimensional reduction of 4d, $N = 2$ supergravity with vector multiplets to three dimensions

[75, 68]. We will discuss further the quaternionic geometry associated with the metric h_{uv} in subsection 4.1.2 of the next chapter.

This concludes our survey of the action (3.15) stemming from the reduction of type IIA supergravity on Calabi-Yau 3–folds. Here we will not treat the details of Calabi-Yau compactifications of type IIB supergravity [69, 70]. Let us just mention that the resulting 4d theory is again an $N = 2$ supergravity, with the same features seen above for type IIA. More specifically, the 4d $N = 2$ effective action for type IIB compactified on a given Calabi-Yau manifold is obtained from the $N = 2$ action arising from compactification of type IIA on the same Calabi-Yau manifold simply by interchanging the roles of the Kähler structure and the complex structure moduli spaces $\mathscr{M}_{ks}$ and $\mathscr{M}_{cs}$. In particular, the 4d theory derived from type IIB has $h^{2,1}$ vector multiplets and $h^{1,1} + 1$ hypermultiplets.

3.2.2 Going off-shell, and gauging by fluxes

Let us dwell upon the procedure adopted to derive the action (3.15). The truncation ansatz was identified by linearizing the higher dimensional equations of motion around a chosen Minkowski$_4 \times$ CY$_3$ vacuum, and restricting to the field fluctuations which are massless in 4d. Then the 'vev + massless fluctuations' ansatz is substituted in the higher dimensional action, and the integration over the compact manifold is carried out. The 4d effective action obtained in this way refers to the selected vacuum, and is a priori valid only at quadratic order in the fluctuations. As a consequence, the various kinetic matrices in (3.15) should be evaluated in the vacuum.[3]

However, we know that there exists a continuous family of equivalent 10d backgrounds, parameterized by the vevs of the Calabi-Yau moduli. As a consequence, it would be more satisfactory to dispose of a 4d effective action having access to all these vacua. In order to achieve this, one should construct a truncation ansatz valid beyond linear order in the field fluctuations around a given vacuum. Alternatively, one can try to exhibit an 'off-shell' reduction ansatz, which does not refer to a specific vacuum, and which leads to a 4d theory correctly reproducing the low energy dynamics of the field fluctuations once a given vacuum has been selected. In general, the identification of such ansätze is highly non-trivial, but in the Calabi-Yau case this can be achieved: the recipe is to still expand the higher dimensional fields in harmonic forms as done above, releasing however the condition that harmonicity be defined with respect to the background metric $\langle g_{mn}(y)\rangle$. In other words, one promotes the expansion forms to be moduli dependent, in such a way that they be harmonic all over the Calabi-Yau moduli space. Notice that in this way the expansion forms end up depending on the 4d spacetime coordinates, since the Calabi-Yau moduli v^a and z^i are dynamical fields in 4d. Despite this non-trivial modification, it turns out that the reduction goes through anyway, basically thanks to the fact that in the Calabi-Yau case the various quantities entering in the 4d effective action are determined by the cohomology classes of the expansion forms (recall subsection 2.3.2). A recent discussion of how this occurs can be found in [49], where this off-shell reduction was termed 'base-point independent'. The resulting 4d effective action still reads as (3.15), but now is not limited

[3] This does not necessarily hold for the R_4 term, since the truncation ansatz $\hat{g}_{\mu\nu}(x, y) = g_{\mu\nu}(x)$ can straightforwardly be extended at all orders in the fluctuations.

to a chosen vacuum. It rather incorporates an infinite family of Poincaré-invariant vacua, parameterized by arbitrary constant values of the scalar fields. Notice that this feature is strictly connected with the fact that (3.15) does not contain any scalar potential. Clearly, all these solutions lift to a solution of the higher dimensional theory. On the other hand, to our knowledge it has not been demonstrated that *any* possible solution of the 4d equations of motion stemming from (3.15) lifts to 10d.

Fluxes and gaugings

The theory (3.15) derived from type II compactifications on Calabi-Yau manifolds is the most simple example of 4d, $N = 2$ supergravity coupled to $h^{1,1}$ vector multiplets and $h^{2,1} + 1$ hypermultiplets.[4] Indeed, here the gauge group is abelian, all matter fields are uncharged, and there is no scalar potential. It follows that, in the guise presented here, this compactification is not suitable for phenomenology. In chapter 1 we have briefly mentioned how these issues can be overcome by embedding this supergravity model in a string theory scenario, where further ingredients like localized sources play a crucial role. While we will not enter into the details of these constructions, let us come back to the moduli problem. In section 1.1 we discussed how the presence of massless scalar fields with no preferred vev is one of the main problems of compactifications, and in particular of Calabi-Yau compactifications. We also saw that in order to stabilize the moduli one should introduce a scalar potential, and that a possible way to do this is provided by the introduction of background fluxes on the compact manifold. Let us now discuss how this fits with the 4d $N = 2$ theory presented in this section.

For the four-dimensional $N = 2$ effective action arising from Calabi-Yau compactifications, a scalar potential can be introduced at tree level without breaking explicitly $N = 2$ supersymmetry by performing a *gauging* of the theory. This is a general technique of extended supergravity, by means of which a subgroup of the global symmetry group of the action becomes local (gauged), yielding new interactions among the fields already present in the spectrum of the theory. For the cases of interest in this thesis, the global symmetries to be gauged will be the isometries of the hyperscalar quaternionic manifold, discussed in chapter 4 below. By this gauging procedure, a subset of the previously neutral matter fields acquire charges under the interaction mediated by (some of) the gauge fields present in the theory. The introduction of a scalar potential follows by supersymmetry. For a review of 4d gauged $N = 2$ supergravity, we refer e.g. to [76].

When the supergravity action is derived from compactification of a higher dimensional theory, the deformation produced by a gauging can in some cases be understood as the introduction of background fluxes of the higher dimensional p–form field strengths, threading cycles of the compact manifold. The flux numbers, given by the integral of the field strengths over the threaded cycles, provide the charges needed for the gauging. On the other hand, not all gaugings can be understood as an insertion of fluxes: some gaugings rather correspond to the replacement of the internal manifold with a completely different

[4]Notice that the minimal content of a type IIA Calabi-Yau compactification is a single vector multiplet and a single hypermultiplet. Indeed, $h^{1,1}$ cannot vanish since the Kähler form J is cohomologically non-trivial. On the other hand, $h^{2,1}$ can vanish; in this case, the hypersector is made of just the universal hypermultiplet.

manifold. A classical example is the gauging of maximal $N = 8$ supergravity: the ungauged theory can be derived by dimensional reduction of eleven-dimensional supergravity on the torus T^7 [9], while its SO(8) gauging [77] stems from the reduction on the sphere S^7 [78]. Further gaugings of $N = 8$ supergravity can be understood as reductions on twisted tori [79, 80]. See e.g. [81] for an introduction to the relation among fluxes, torsion and gaugings of maximal and half-maximal supergravities. Finally, it is important to remark that not all gaugings currently have been given a higher dimensional origin.

Let us come back to our Calabi-Yau context, and illustrate with a simple example [82] the connection between gaugings and fluxes. Consider a flux of the NS 3–form H, defined by the following expansion in the Calabi-Yau harmonic 3–forms α_I:

$$H = m^I \alpha_I\,,$$

where the m^I are constant parameters, corresponding to flux numbers since[5]

$$\int_{A^I} H = \int H \wedge \beta^I = m^I,$$

where A^I denotes a basis for the A–cycles of the third homology, and property (2.22) has been used. Then look at the component of the RR field strength $\hat{F}_4$ yielding the kinetic term for the scalars ξ^I in 4d. Recalling (3.13) and (3.14), before turning on the NS flux this reads $d\xi^I \wedge \alpha_I$. In the presence of the NS flux, this is modified to

$$D\xi^I \wedge \alpha_I\,, \qquad \text{with } \; D\xi^I = d\xi^I + m^I A^0\,.$$

Hence we see that the scalars ξ^I are now charged under the interaction mediated by the graviphoton A^0, the charges corresponding to the flux parameters m^I. From the 4d viewpoint this is a gauging, since the previously global shift symmetry $\xi^I \to \xi^I + \lambda^I$ is now promoted to a local symmetry, related with the gauge symmetry of the graviphoton $A^0 \to A^0 + d\lambda(x)$ by $\lambda^I = m^I\lambda$. We remark that no new 4d fields have been introduced, so that the supermultiplet structure of the theory, summarized in table 3.1, has not been modified. Consistently with the $N = 2$ supersymmetry of the 4d action, the introduction of the NS flux also induces a nontrivial scalar potential in the 4d theory, as it can be verified by inspection of the H^2 term in the type IIA action (3.12). We will discuss the form of this potential in a much more general setting in the course of this chapter. In fact, already within Calabi-Yau compactifications, this example can be generalized by considering the presence of general fluxes for both the NSNS and RR field strengths [14, 15, 82, 83, 84, 85, 86]. Starting with the next section we will study how, allowing the internal manifold to depart from the Calabi-Yau geometry, and applying the tools of SU(3)×SU(3) structures, one can obtain even more interesting gaugings. Then, in the next chapter we will enter more deeply into the gauged supergravity formalism and we will study several properties of the obtained 4d theories.

The departure from the Calabi-Yau geometry is also motivated by the fact that a 'Calabi-Yau with fluxes' background does not solve the higher dimensional supergravity

[5] As mentioned in section 1.1, quantum conditions require the fluxes to come in a discrete family. However, at the supergravity level these can consistently be regarded as continuous parameters.

equations of motion, due to the backreaction on the geometry induced by the flux contribution to the energy-momentum tensor (in particular, this removes the Ricci-flatness condition). This is also reflected in the scalar potential generated at the 4d level. Indeed, by ignoring the backreaction of the fluxes one can still expand the higher dimensional fields in the same Calabi-Yau harmonic forms used in the fluxless case, as we have done in the example above. However, one can see that the flux-generated 4d scalar potential displays a runaway behavior corresponding to a decompactification limit, so that the theory does not have a stable ground state at finite radius.

While in the following we will explore the consequences of abandoning the Calabi-Yau geometry, let us mention that the situation outlined here above can be improved by passing from supergravity to string theory, and including in the background further ingredients, like localized sources. In particular, one is interested in considering orientifold planes, which have negative tension and can compensate the positive energy density associated with the fluxes. In this enlarged framework, very interesting consistent backgrounds with underlying Calabi-Yau geometry can be obtained [13, 87].

3.3 Defining the truncation ansatz

3.3.1 The philosophy

Having reviewed Calabi-Yau compactifications in the previous section, we now wish to enlarge our horizon, and study the general properties of dimensional reductions of type II supergravity preserving $N = 2$ supersymmetry in 4d. In doing this, we allow for arbitrary NSNS and RR fluxes.

In the previous chapter we discussed how the general spinor ansatz (2.1) for $N = 2$ reductions requires that the compact manifold M_6 admit an SU(3)×SU(3) structure on $TM_6 \oplus T^*M_6$. After having introduced the necessary preliminary mathematical notions, in section 2.4 we studied the relevant properties of SU(3)×SU(3) structures within the generalized geometry formalism, and in section 2.5 we focused on their deformations. As anticipated at the end of that section, the next step towards the definition of a 4d theory with a finite number of degrees of freedom is the truncation of the modes of the higher dimensional fields along the compact manifold. This is done via the identification of a reduction ansatz for the higher dimensional fields. Subsequently, one substitutes this ansatz in the higher dimensional action, and integrates over the compact manifold.

A physically well-motivated reduction ansatz can in principle be derived by a Kaluza-Klein approach, identifying the low energy physics for small field fluctuations around a given background. However, switching on fluxes induces several additional terms to the field equations of motion, and the analysis directed to the determination of the operators whose zero modes define the light 4d fields becomes very involved (see e.g. [12] for the case – still relatively simple – of Freund-Rubin compactifications). Furthermore, it is likely that the characterization of these zero modes be crucially background-dependent.

Given the difficulty to directly identify the light modes to keep, a reasonable way to proceed which has been carried out in the literature is to adopt an off-shell approach, and to assume the existence of a generic basis of internal differential forms on which to expand the higher dimensional fields. The basis forms are required to satisfy just the minimal amount of constraints necessary to define a sensible $N = 2$ supergravity theory in four dimensions.

This program was first carried out for manifolds admitting strict SU(3) structure: with restriction to $N = 2$ reductions of type II, see [47, 88, 89, 90, 45, 91, 92, 49]. For the particular SU(3) structure class of nearly Kähler manifolds, a satisfactory reduction ansatz has been implemented in [93]. This case can to some extent be treated analogously to Calabi-Yau compactifications, since the manifold is still characterized by a real 2–form J and a complex 3–form Ω. However, crucially now these need not be closed (recall subsection 2.2.2), hence the basis forms in which they are expanded don't need to be all closed either. Rather, the differential relations which are established among them define a set of 'geometric charges' parameterizing the SU(3) torsion classes, and encoding the departure from the Calabi-Yau geometry.

Subsequently, in [45, 46] this off-shell approach to type II dimensional reductions was extended to SU(3)×SU(3) structures. In this latter case, since SU(3)×SU(3) structures are characterized by the polyforms $\Phi_\pm$, it is natural to allow the basis forms to be of mixed degree as well.

In this context, the safest way to proceed in order that the result of the compactification displays the features of an $N = 2$ supergravity in 4d is to stay as close as possible to the well-known path of Calabi-Yau dimensional reductions. For example, one of the features of Calabi-Yau compactifications one wants to reproduce is the fact that the σ-model governing the kinetic terms of the scalars associated with the internal metric and B-field deformations has a target space consisting of the product of two special Kähler manifolds, to be identified as the scalar manifold for the vector multiplets and the special Kähler base of the dual quaternionic manifold spanned by the hypermultiplets. As we discussed in detail in section 2.5, SU(3)×SU(3) structures have this property: the spaces of both $\mathcal{J}_-$ and of $\mathcal{J}_+$ deformations at a point of M_6 admit a local special Kähler structure, and their metric describes the internal metric and B-field fluctuations. However, as we already stressed, this result just holds at a point of the internal manifold M_6. The constraints imposed on the basis of expansion forms are aimed to guarantee that this local special Kähler structure be inherited by the 4d theory for the finite set of modes identified by the truncation.

In the following of this section, we introduce the basis forms defining the truncation, and we study the conditions required for the reduction to go through similarly to the Calabi-Yau case. In the generalized geometry context, these condition were discussed in [45, 46]. For the SU(3) structure case, a thorough analysis with a complete list of the constraints on the basis forms was given in [49]. What we provide below is a more explicit, revisited version of the analysis of [45, 46], based on [P1, P2]. In doing this, we also stress the relevance of decomposing the pure spinor deformations in representations of SU(3)×SU(3). Then, in subsection 3.3.4 we study the consequences of demanding closure of the system of basis forms under the action of a b-twisted Hodge-star operator, while in subsection 3.3.5 we discuss the differential relations to be satisfied by the expansion forms. At that point, we will be ready to implement the truncation ansatz to dimensionally reduce type II supergravity. We will deal in turn with the NSNS sector in section 3.4, and with the RR sector in section 3.5.

3.3.2 The basis forms

Let us assume the compact manifold M_6 admits SU(3)×SU(3) structure on $TM_6 \oplus T^*M_6$, characterized by a pair of compatible Spin(6,6) pure spinors Φ_+, Φ_-. We also demand the existence of a finite basis of differential forms, split in two subsets Σ_- and Σ_+, composed of odd and even real forms respectively. A main point is that these forms need not be of pure degree, i.e. are in general polyforms. In order to preserve the symplectic structure defined by the Mukai pairing, the basis forms should arrange in symplectic vectors. In the notation of [46],

$$\Sigma_+^{\mathbb{A}} = \begin{pmatrix} \tilde{\omega}^A \\ \omega_A \end{pmatrix} \qquad , \qquad \Sigma_-^{\mathbb{I}} = \begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix} , \tag{3.17}$$

where the range of the indices is: $A, B = 0, 1, \ldots, b^+$ and $I, J = 0, 1, \ldots, b^-$. We also introduce the symplectic indices $\mathbb{A}, \mathbb{B} = 1, 2, \ldots, 2(b^+ + 1)$ and $\mathbb{I}, \mathbb{J} = 1, 2, \ldots, 2(b^- + 1)$. The actual value of $b^\pm$ is model-dependent, and – contrarily to the Calabi-Yau case – there are no known general prescriptions to determine it. The pairings of the basis forms are then required to satisfy the integrated relations

$$\begin{pmatrix} \int\langle\alpha_I, \alpha_J\rangle & \int\langle\alpha_I, \beta^J\rangle \\ \int\langle\beta^I, \alpha_J\rangle & \int\langle\beta^I, \beta^J\rangle \end{pmatrix} = \begin{pmatrix} 0 & \delta_I{}^J \\ -\delta^I{}_J & 0 \end{pmatrix} := (\mathbb{S}_-)_{\mathbb{IJ}} , \tag{3.18}$$

$$\begin{pmatrix} \int\langle\omega_A, \omega_B\rangle & \int\langle\omega_A, \tilde{\omega}^B\rangle \\ \int\langle\tilde{\omega}^A, \omega_B\rangle & \int\langle\tilde{\omega}^A, \tilde{\omega}^B\rangle \end{pmatrix} = \begin{pmatrix} 0 & \delta_A{}^B \\ -\delta^A{}_B & 0 \end{pmatrix} := (\mathbb{S}_+)_{\mathbb{AB}} , \tag{3.19}$$

where $\mathbb{S}_\pm$ are the symplectic metrics of $\mathrm{Sp}(2b^\pm + 2, \mathbb{R})$. More concisely, we can write

$$\int_{M_6} \langle\Sigma_+^{\mathbb{A}}, \Sigma_+^{\mathbb{B}}\rangle = (\mathbb{S}_+)^{-1\,\mathbb{AB}} \qquad , \qquad \int_{M_6} \langle\Sigma_-^{\mathbb{I}}, \Sigma_-^{\mathbb{J}}\rangle = (\mathbb{S}_-)^{-1\,\mathbb{IJ}} . \tag{3.20}$$

Notice that the integrated Mukai pairing $\int_{M_6}\langle\,\cdot\,,\,\cdot\,\rangle$ naturally provides 4d scalars.

These basis forms will be used to define a truncation ansatz for the higher dimensional supergravity fields. In particular, we are going to specify the truncation of the internal NSNS sector by expanding the pure spinors Φ_-, Φ_+, seen as polyforms, respectively in Σ_- and Σ_+. We will denote by $\mathscr{M}_-$ and $\mathscr{M}_+$ the truncated projective spaces of odd and even pure spinors (i.e., the spaces of pure spinor lines). Their dimensions are b^- and b^+ respectively.

In subsection 2.5.3 we found that physically relevant pure spinor deformations, i.e. the ones which modify the internal metric and B-field, are those parameterized by $\delta\chi_- \in \Gamma(U_{\mathbf{3},\bar{\mathbf{3}}})$ and $\delta\chi_+ \in \Gamma(U_{\bar{\mathbf{3}},\mathbf{3}})$. In order to ensure that $\mathscr{M}_-$ and $\mathscr{M}_+$ describe a subset of just these deformations, we should require that none of the pure spinor deformations we keep transforms in the $(\mathbf{3},\mathbf{1}) \oplus (\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\bar{\mathbf{3}})$ representation of SU(3)×SU(3). A condition on the expansion forms $\Sigma_\pm$ guaranteeing this is [46]

$$\langle\omega_A, \Gamma^\Lambda\alpha_I\rangle = \langle\omega_A, \Gamma^\Lambda\beta^I\rangle = \langle\tilde{\omega}^A, \Gamma^\Lambda\alpha_I\rangle = \langle\tilde{\omega}^A, \Gamma^\Lambda\beta^I\rangle = 0 , \tag{3.21}$$

where, as in the previous chapter, $\Gamma^\Lambda = (\Gamma^m, \Gamma_m) = (dx^m\wedge\,,\, \iota_{\partial_m})$. This condition also ensures that the pure spinor compatibility condition (2.55) is respected already at the level of the basis forms, preventing in this way a relation between the moduli of Φ_+ and Φ_-.

Actually, the requirement of dropping the SU(3)×SU(3) triplets doesn't concern just the pure spinor deformations, but the full set of type II supergravity polyforms, naturally decomposing under SU(3)×SU(3) according to (2.68). Indeed, in refs. [45, 46] it was argued that, if not truncated out, the fields transforming in the O(6, 6) vector representation would assemble to define spin 3/2 multiplets in 4d, yielding non-standard couplings of $N = 2$ supergravity. Again, condition (3.21) gets us out of trouble.

We conclude that the SU(3)×SU(3) representations relevant to the definition of the $N = 2$ effective action reside in the horizontal and vertical axis of the diamond (2.68). This is somehow analogous to the Calabi-Yau case, where however the diamond is a true Hodge diamond, in that it consists of (p, q)-cohomologies.[6]

3.3.3 Special Kähler geometry on the truncated space of pure spinors

Special Kähler geometry for $\mathscr{M}_-$

Using the basis forms and the Mukai pairing we define the periods of Φ_- as

$$Z^I := \int \langle \Phi_-, \beta^I \rangle \qquad , \qquad \mathcal{G}_I := \int \langle \Phi_-, \alpha_I \rangle \,. \tag{3.22}$$

Then Φ_- can be expanded on the basis forms as

$$\Phi_- = Z^I \alpha_I - \mathcal{G}_I \beta^I \,. \tag{3.23}$$

When the compact manifold is a Calabi-Yau 3–fold, this expansion corresponds to the expansion of the Calabi-Yau holomorphic (3,0)–form Ω on a basis for the 3–form real cohomology, see eqs. (2.23), (2.24); hence in this case $\mathscr{M}_-$ coincides with the complex structure moduli space $\mathscr{M}_{cs}$ discussed in subsection 2.3.2. In the strict SU(3) structure case, one still has $\Phi_- \sim \Omega$, but this is generically not closed (recall (2.10)). One can still expand it in a basis of real 3–forms, but these need not be cohomology representatives.

From (3.22) we see that performing a constant rescaling $\Phi_- \to \lambda\Phi_-$ implies $Z^I \to \lambda Z^I$ and $\mathcal{G}_I \to \lambda \mathcal{G}_I$. We would like to conclude that Φ_- is a homogeneous function of degree 1 in the Z^I variables, and then see these as projective coordinates for $\mathscr{M}_-$. For this to be true, we need that the Z^I, $I = 0, 1, \ldots, b^-$, define b^- independent functions on $\mathscr{M}_-$ (then the $\mathcal{G}_I$ are holomorphically determined by the Z^J), and that the basis forms, which in general can be moduli-dependent, be homogeneous of degree 0 in the Z^I. Once this is satisfied, away from the $Z^0 = 0$ locus we can also introduce special coordinates $z^i = Z^i/Z^0$, $i = 1, 2, \ldots, b^-$, for $\mathscr{M}_-$.

Given (3.23), and recalling (3.18), the Kähler potential K_- written in (2.70), now integrated, takes the standard form of (local) special Kähler geometry:

$$K_- \equiv -\log i \int \langle \Phi_-, \bar{\Phi}_- \rangle = -\log i(\bar{Z}^I \mathcal{G}_I - Z^I \bar{\mathcal{G}}_I) \,. \tag{3.24}$$

[6] The diamond of section 2.3.1 displays the dimensions $h^{p,q}$ of the (p, q)-cohomologies $H^{p,q}(M_6)$.

In the generalized geometry literature, the functional $\int\langle\Phi,\bar{\Phi}\rangle$, introduced in [53, 22], is known as the *Hitchin functional*.

We still have to verify if $\mathcal{G}_I$ can be obtained by derivation from a prepotential, homogeneous of degree 2 in the Z^I. In subsection 2.3.2 we saw that in the Calabi-Yau case an essential tool to show that the space of complex structure deformations is special Kähler is represented by the Kodaira formula (2.19). Here we rewrite it in a slightly different way. We start appending some tildes: calling now $\tilde{\Omega}(z^i)$ the Calabi-Yau holomorphic $(3,0)$-form, we rewrite (2.19) as

$$\frac{\partial\tilde{\Omega}}{\partial z^i} = \tilde{\kappa}_i\tilde{\Omega} + \tilde{\chi}_i \qquad\qquad i = 1,\ldots,h^{2,1}\ , \tag{3.25}$$

where again the $\tilde{\kappa}_i$ can depend on z but not on the coordinates of M_6, and $\{\tilde{\chi}_i\}$ is a basis for the $(2,1)$-harmonic forms. Introducing then $\Omega(Z^I) = Z^0\tilde{\Omega}(z^i)$, we can recast (3.25) in terms of projective coordinates $Z^I = (Z^0,\, Z^i = Z^0 z^i)$ as

$$\frac{\partial\Omega}{\partial Z^I} = \kappa_I\Omega + \chi_I \qquad\qquad I = 0,1,\ldots,h^{2,1}\ , \tag{3.26}$$

where $\kappa_I = (\kappa_0,\kappa_i) = \frac{1}{Z^0}(1 - z^i\tilde{\kappa}_i\, ,\, \tilde{\kappa}_i)$ and $\chi_I = (\chi_0,\chi_i) = (-z^i\tilde{\chi}_i\, ,\, \tilde{\chi}_i)$. Notice that $\chi_i = \tilde{\chi}_i$ is homogeneous of degree 0.

We now reconsider deformations of pure spinors, which in subsection 2.5 we wrote in the form (2.73), and we rephrase them in a form analogous to (3.25) and (3.26). Parameterizing the truncated projective space of pure spinors $\mathscr{M}_-$ by the moduli z^i, or alternatively by the projective coordinates Z^I, we can write:

$$\frac{\partial\tilde{\Phi}_-}{\partial z^i} \sim \tilde{\kappa}_i^-\tilde{\Phi}_- + \tilde{\chi}_i^-\ , \qquad \text{or} \qquad \frac{\partial\Phi_-}{\partial Z^I} \sim \kappa_I^-\Phi_- + \chi_I^-\ , \tag{3.27}$$

where the tildes have the same meaning as above, and the relations between the κ_I, χ_I and the $\tilde{\kappa}_i$, $\tilde{\chi}_i$ are also the same. Referring to the deformations introduced in eq. (2.73), we identify $\delta\kappa_- = \tilde{\kappa}_i^-\delta z^i$, $\delta\chi_- = \tilde{\chi}_i^-\delta z^i$, and therefore at each point of M_6 we have $\chi_I^- \in U_{\bar{\mathbf{3}},\bar{\mathbf{3}}}$. Adopting the notation of [46], here and in the following by the symbol $\sim$ we mean 'equality up to terms that vanish in the integrated symplectic pairing'. In the above expression the $\sim$ is required because in principle the pure spinor variations contain a term transforming in the triplets of SU(3)×SU(3), and we are preventing its presence in the truncated spectrum by assumption (3.21).

Since (3.27) does not contain a term proportional to $\bar{\Phi}_-$, we have

$$\int\langle\Phi_-,\partial_I\Phi_-\rangle = 0 \qquad\qquad (\partial_I \equiv \tfrac{\partial}{\partial Z^I})\ , \tag{3.28}$$

which indeed is a necessary condition for special Kähler geometry, see eq. (D.2). From the expansion (3.23) we have

$$\partial_I\Phi_- = \alpha_I - \partial_I\mathcal{G}_K\beta^K + Z^K\partial_I\alpha_K - \mathcal{G}_K\partial_I\beta^K\ , \tag{3.29}$$

where the last two terms have been taken into account because in general the expansion forms are moduli dependent. This is true also when considering an off-shell reduction on a

Calabi-Yau 3-fold (see e.g. [49] for a recent discussion on this point), but in this case $\partial_I \alpha_J$ and $\partial_I \beta^J$ are exact and don't contribute to the integral. In the more general case this is not automatic, and we are led to require

$$\int \langle \alpha_J, \partial_I \alpha_K \rangle = \int \langle \alpha_J, \partial_I \beta^K \rangle = \int \langle \beta^J, \partial_I \beta^K \rangle = 0 \; . \tag{3.30}$$

This also guarantees constancy of the symplectic structure (3.18). Analogously to the Calabi-Yau case, (3.28) then gives

$$2\mathcal{G}_I = \partial_I (Z^K \mathcal{G}_K) \; , \tag{3.31}$$

which implies that $\mathcal{G} := \frac{1}{2} Z^K \mathcal{G}_K$ is a homogeneous function of degree 2 in the Z variables (the prepotential), and $\mathcal{G}_I = \partial_I \mathcal{G}$. Then $\mathcal{G}_I$ is homogeneous of degree 1: $\mathcal{G}_I = Z^K \partial_K \mathcal{G}_I$. We will denote $\mathcal{G}_{IJ} := \partial_I \mathcal{G}_J = \partial_I \partial_J \mathcal{G}$.

We can now derive an useful relation between the coefficient κ_I^- appearing in (3.27) and the special geometry data. Assuming that κ_I^- does not depend on the coordinates of M_6 (this condition is automatically verified in the Calabi-Yau case), we obtain

$$\kappa_I^- = \frac{\int \langle \partial_I \Phi_-, \bar{\Phi}_- \rangle}{\int \langle \Phi_-, \bar{\Phi}_- \rangle} = \frac{\mathrm{Im} \mathcal{G}_{IJ} \bar{Z}^J}{Z^K \mathrm{Im} \mathcal{G}_{KL} \bar{Z}^L} \; . \tag{3.32}$$

where for the first equality we used the orthogonality of the different representations in (2.68). Notice that $\kappa_I^- = -\partial_I K_-$ and therefore from (3.27)

$$\chi_I^- \sim D_I \Phi_- \sim D_I Z^J \alpha_J - D_I \mathcal{G}_J \beta^J ,$$

where $D_I = \partial_I + \partial_I K_-$. Again, these are direct generalizations of expressions valid in the Calabi-Yau case (see e.g. [56]).

Provided the whole set of conditions summarized in this subsection is satisfied, we can conclude that $\mathscr{M}_-$ has a local special Kähler structure. From (2.76), (2.87) it follows that the metric $\mathscr{G}_{i\bar{\jmath}}^-$ on $\mathscr{M}_-$ is given by

$$\mathscr{G}_{i\bar{\jmath}}^- \; = \; \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^{\bar{\jmath}}} K_- \; = \; -\frac{\int \langle \chi_i^- \, , \, \bar{\chi}_{\bar{\jmath}}^- \rangle}{\int \langle \Phi_-, \bar{\Phi}_- \rangle} \; . \tag{3.33}$$

In the Calabi-Yau case, (3.33) reduces to (2.18), with the χ_i corresponding to harmonic (2,1)-forms, recall the discussion of subsection 2.3.2.

Analogously to what we did for the Calabi-Yau complex structure moduli space $\mathscr{M}_{cs}$, we can introduce the period matrix relating the upper and lower components of the symplectic vector $(Z^I, \mathcal{G}_I)$. We continue to call $\mathcal{M}$ this matrix, and its formal definition is the same as in (2.26), where here the Kähler covariant derivative D_k is $D_k = \partial_{z^k} + \partial_{z^k} K_-$. As in the Calabi-Yau case, this period matrix is going to play an important role in the compactification. In the forthcoming subsection we will discuss how it can be derived from the generalized geometry on the internal manifold. Before doing this we briefly describe the truncated space $\mathscr{M}_+$ of even pure spinors.

Special Kähler geometry for $\mathscr{M}_+$

The requirements to be imposed in order to ensure the local special Kähler structure of $\mathscr{M}_+$ are completely parallel to the ones described above for $\mathscr{M}_-$. Here we summarize the important relations, mainly to fix our notation. The periods of Φ_+ are defined as

$$X^A := \int \langle \Phi_+, \tilde{\omega}^A \rangle \qquad , \qquad \mathcal{F}_A := \int \langle \Phi_+, \omega_A \rangle \ . \tag{3.34}$$

Hence Φ_+ is expanded on the truncated basis of forms as

$$\Phi_+ = X^A \omega_A - \mathcal{F}_A \tilde{\omega}^A \ . \tag{3.35}$$

The $\mathcal{F}_A$ are holomorphic functions of the X^A, identified with $\partial_{X^A}\mathcal{F}$, where $\mathcal{F}$ is the prepotential (holomorphic and homogeneous of degree two in the X^A). We denote the special coordinates for $\mathscr{M}_+$ as $t^a = X^a/X^0$. The Kähler potential K_+ is expressed as

$$K_+ = -\log i \int \langle \Phi_+, \bar{\Phi}_+ \rangle = -\log i (\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A) \ . \tag{3.36}$$

The metric $\mathscr{G}^+_{a\bar{b}}$ on $\mathscr{M}_+$ can be obtained from K_+ by

$$\mathscr{G}^+_{a\bar{b}} = \frac{\partial}{\partial t^a}\frac{\partial}{\partial \bar{t}^{\bar{b}}} K_+ \ = \ -\frac{\int \langle \chi^+_a \ , \ \bar{\chi}^+_{\bar{b}} \rangle}{\int \langle \Phi_+, \bar{\Phi}_+ \rangle} \ . \tag{3.37}$$

The period matrix for the special geometry on $\mathscr{M}_+$ will be denoted by $\mathcal{N}$. This is the same symbol used for the period matrix on the Calabi-Yau Kähler structure moduli space $\mathscr{M}_{ks}$, and its formal definition is the same as in (2.35), with the Kähler covariant derivative here being given by $D_a = \partial_{t^a} + \partial_{t^a} K_+$.

Refining the description of the Calabi-Yau moduli space

We remark that the formalism developed here above for the even forms can be fruitfully applied to the Calabi-Yau case. We have already seen around eq. (2.71) that the pure spinor $\Phi_+ = e^{-\phi} e^{-b-iJ}$ yields the correct Kähler potential for the Calabi-Yau Kähler structure moduli space. It is also straightforward to extend the basis $\{\omega_a\}$ for $H^2(M)$ employed in subsection 2.3.2 for the expansion of J and b to a basis Σ_+ for the full even cohomology $H^0 \oplus H^2 \oplus H^4 \oplus H^6$. Given the ω_a, one can always choose 4–forms $\tilde{\omega}^a$ spanning H^4 and satisfying (3.19). As for ω_0 and $\tilde{\omega}^0$, we take

$$\omega_0 = 1 \qquad , \qquad \tilde{\omega}^0 = \frac{vol_6}{Vol} \, , \tag{3.38}$$

spanning H^0 and H^6 respectively. We can then expand Φ_+ as [45]

$$\Phi_+ = e^{-\phi} e^{-b-iJ} \sim e^{-\phi} \Big(1 - t^a \omega_a + \frac{1}{2} \mathcal{K}_{abc} t^a t^b \tilde{\omega}^c - \frac{1}{6} \mathcal{K}_{abc} t^a t^b t^c \tilde{\omega}^0 \Big) \tag{3.39}$$

where as in subsection 2.3.2 $t^a = b^a + iv^a$, and we used $\omega_a \wedge \omega_b \sim \mathcal{K}_{abc} \tilde{\omega}^c$ as well as $\omega_a \wedge \omega_b \wedge \omega_c \sim \mathcal{K}_{abc} \tilde{\omega}^0$, both satisfied in cohomology (recall that $\mathcal{K}_{abc}$ are the triple intersection numbers, defined in (2.33)). Now, (3.39) is just expansion (3.35), with $X^A =$

$(X^0, X^a) = (e^{-\phi}, -e^{-\phi}t^a)$ and $\mathcal{F}_A = \partial_{X^A}\mathcal{F}$, where $\mathcal{F}$ is (minus) the cubic prepotential given in (2.32). Hence, besides the t^a, also X^0 and the $\mathcal{F}_A$ are now read directly from the differential form encoding the complexified Kähler structure on the Calabi-Yau manifold, which here is the polyform $\Phi_+ = e^{-\phi}e^{-b-iJ}$; this is in contrast with subsection 2.3.2, where the prepotential $\mathcal{F}$ (and by consequence the $\mathcal{F}_A$) was introduced somehow *ad hoc*. In other words, here the symplectic $\mathrm{Sp}(2h^{1,1}+2,\mathbb{R})$ structure of the complexified Kähler structure moduli space emerge directly from Φ_+, analogously to what was manifest for the holomorphic (3,0)–form Ω already in expansion (2.24). Hence we see that the construction just decribed, which was in part anticipated in [56, sect. 5] and arises in full naturalness in the context of generalized geometry, makes even more explicit the symmetry bewteen the complex structure and the Kähler structure moduli spaces of Calabi-Yau manifolds. The symplectic sections $(Z^I, \mathcal{G}_I)^T$ and $(X^A, \mathcal{F}_A)^T$ on the two special Kähler manifolds $\mathscr{M}_- = \mathscr{M}_{cs}$ and $\mathscr{M}_+ = \mathscr{M}_{ks}$ are just the periods of two geometric objects, the pure spinors Φ_- and Φ_+, which are treated on the same footing. Also the Kähler potentials have the perfectly symmetric expression $K_\pm = -\log i\int\langle\Phi_\pm, \bar{\Phi}_\pm\rangle$, with $K_- = K_{cs}$, and $K_+ = K_{ks}$.

If the compact manifold has strict SU(3) structure without being Calabi-Yau, we can still use for the expansion of e^{-b-iJ} a system of forms $\omega_A, \tilde{\omega}^A$ analogous to the one introduced above, but generically these will not all be closed. In subsection 3.3.5 below we will discuss in more detail the general differential conditions to be imposed on the basis forms.

A further constraint on the expansion forms

In order to derive eq. (3.32), together with its analogue on $\mathscr{M}_+$, we had to require that κ_I^- and κ_A^+ do not depend on the internal coordinates. A related condition that seems necessary for the dimensional reduction to proceed analogously to the Calabi-Yau case is that the ratios

$$\frac{\langle\Sigma_+^{\mathbb{A}}, \Phi_+\rangle}{\langle\Phi_+, \bar{\Phi}_+\rangle} \quad \text{and} \quad \frac{\langle\Sigma_-^{\mathbb{I}}, \Phi_-\rangle}{\langle\Phi_-, \bar{\Phi}_-\rangle} \quad \text{be constant on } M_6, \tag{3.40}$$

where as in the previous chapter by the ratio of two top forms we mean the ratio of their unique elements. Provided that $\langle Z^J\partial_{Z^I}\alpha_J - \mathcal{G}_J\partial_{Z^I}\beta^J, \bar{\Phi}\rangle = 0$ (and similarly for the even basis), this indeed implies that

$$\kappa_A^+ = \frac{\langle\partial_{X^A}\Phi_+, \bar{\Phi}_+\rangle}{\langle\Phi_+, \bar{\Phi}_+\rangle} \quad \text{and} \quad \kappa_I^- = \frac{\langle\partial_{Z^I}\Phi_-, \bar{\Phi}_-\rangle}{\langle\Phi_-, \bar{\Phi}_-\rangle} \quad \text{are constant on } M_6, \tag{3.41}$$

We notice that conditions (3.40) and (3.41) are satisfied when M_6 is a Calabi-Yau 3-fold. To verify (3.40) one uses a basis $\Sigma_\pm$ for the Calabi-Yau cohomology made of harmonic representatives. Recall also that the pure spinors take form $\Phi_+ = e^{-\phi}e^{-b-iJ}$ and $\Phi_- = -ie^{-\phi}\Omega$ (see (2.65)), where J is the Kähler form of the Calabi-Yau, Ω is the holomorphic $(3,0)$ form, and the dilaton ϕ is constant along M_6. For instance, for the harmonic (1,1)–forms ω_a the first expression in (3.40) reads

$$3\frac{\omega_a\wedge J\wedge J}{J\wedge J\wedge J} = \omega_a\lrcorner J\ , \tag{3.42}$$

where eqs. (A.23) and (A.9) were used. Now, harmonicity of ω_a implies $\partial_m(\omega_a\lrcorner J) = 0$ [71]. In the general SU(3)×SU(3) structure case (3.40) and (3.41) are non-trivial assumptions, and we are going to employ them at several points of the dimensional reduction.

3.3.4 The twisted Hodge star $*_b$

In the following we discuss the geometric origin of the period matrices $\mathcal{N}_{AB}$ and $\mathcal{M}_{IJ}$ associated with the special Kähler structure of $\mathscr{M}_+$ and $\mathscr{M}_-$ respectively. This important piece of information about the 4d $N = 2$ supergravity arising from SU(3)×SU(3) compactifications can be extracted from the study of the 6d b-twisted Hodge star operator [28, 24, 94]:

$$*_b := e^{-b} * \lambda\, e^b \,, \tag{3.43}$$

which is the covariant generalization of the usual Hodge $*$ when considering Spin(6,6) spinors containing the B-field, as $\Phi_\pm = e^{-b}\Phi^0_\pm$.

In particular, we are interested in identifying the action of $*_b$ on the basis of forms $\Sigma_\pm$ in terms of the special geometry data. In doing so, we will generalize some well-known results [95, 96] for the action of the usual Hodge $*$ on the harmonic 3-forms of a Calabi-Yau 3-fold.

We start with a couple of remarks. It is easy to check that $(*_b)^2 = -id$; therefore its eigenvalues are $\pm i$ and an almost complex structure is defined on $\wedge^\bullet T^*$. Using the bispinor picture, one can readily verify that the $U_{\mathbf{r},\mathbf{s}}$ defined in (2.68) are $\pm i$ eigenbundles for $*_b$. This can be seen as follows: in the differential form picture, consider the B-transformed of (2.69), and act on it with $*_b$; then pass to the bispinor picture, using (B.6) to evaluate $*\lambda$ under the Clifford map. One obtains the eigenvalues[7]

$$\begin{array}{ccccccc}
 & & & -i & & & \\
 & & -i & & i & & \\
 & -i & & i & & -i & \\
-i & & i & & -i & & i \\
 & i & & -i & & i & \\
 & & -i & & i & & \\
 & & & i & & &
\end{array} \tag{3.44}$$

In particular, we have $*_b\Phi_\pm = -i\Phi_\pm$, and therefore

$$*_b\mathrm{Re}(\Phi_\pm) = \mathrm{Im}(\Phi_\pm). \tag{3.45}$$

So we can conclude that once the metric has been fixed, $*_b$ behaves on pure spinors as the Hitchin 'hat operator' [22], since acting on the real part of the pure spinor it gives its imaginary part.[8]

Let us now determine the action of the $*_b$ operator on the elements of the basis $\Sigma_\pm$. As already mentioned, to achieve this we will generalize the analysis of refs. [95, 96] for the action of the usual Hodge $*$ on the Calabi-Yau harmonic 3-forms (see also [49] for the SU(3) structure case). In the Calabi-Yau case, the analysis builds on the simple observation that

---

[7] For $b = 0$, this can be found in [35]. The overall sign difference with respect to that paper is due to an opposite sign choice we have in the definition of the 6d chirality matrix γ, see (A.12)

[8] The same holds for the Hitchin hat operator : $\widehat{\mathrm{Re}\Phi} = \mathrm{Im}\Phi$. Anyway, the operator defined by Hitchin is more fundamental in that it does not need the metric. Furthermore, the hat operator acts differently than $*_b$ on the other entries of the generalized diamond.

the Hodge $*$ acts[9] as $-i$ on $(3,0)$-forms and as $+i$ on the $(2,1)$-harmonic forms which parameterize the complex structure deformations.

Our generalization employs the decomposition of $\wedge^\bullet T^*$ in terms of SU(3)×SU(3) representations instead of the (p,q)-decomposition of complex forms of pure degree.

As a starting point we need the assumption that the action of $*_b$ on the elements of $\Sigma^\pm$ can still be expanded on $\Sigma_\pm$. Focusing on Σ_-, we write

$$*_b\alpha_I \sim \mathcal{A}_I{}^J\alpha_J + \mathcal{B}_{IJ}\beta^J \qquad , \qquad *_b\beta^I \sim \mathcal{C}^{IJ}\alpha_J \ + \mathcal{D}^I{}_J\beta^J \ . \tag{3.46}$$

We also require that the matrices $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ do not depend on the coordinates of M_6. For a Calabi-Yau, expansion (3.46) is not an assumption but a matter of fact since Σ_- consists of harmonic 3-forms.

Using (3.18) and the fact that for any $A, C \in \wedge^\bullet T^*$, $\langle A, *_b C\rangle = -\langle *_b A, C\rangle$ (this descends from eqs. (B.2)-(B.4)), we see immediately that (I,J indices are understood)

$$\mathcal{B}^T = \mathcal{B} = \int\langle\alpha, *_b\alpha\rangle \quad , \quad \mathcal{C}^T = \mathcal{C} = -\int\langle\beta, *_b\beta\rangle \quad , \quad -\mathcal{A}^T = \mathcal{D} = \int\langle\alpha, *_b\beta\rangle \ . \tag{3.47}$$

Applying $*_b$ to (3.46), using $(*_b)^2 = -id$ and (3.18), one can see that the matrix

$$\mathbb{M} := \begin{pmatrix} \int\langle\alpha, *_b\beta\rangle & -\int\langle\beta, *_b\beta\rangle \\ \int\langle\alpha, *_b\alpha\rangle & -\int\langle\beta, *_b\alpha\rangle \end{pmatrix} = \begin{pmatrix} \mathcal{D} & \mathcal{C} \\ \mathcal{B} & \mathcal{A} \end{pmatrix} \tag{3.48}$$

is symplectic (i.e. $\mathbb{M}^T\mathbb{S}_-\mathbb{M} = \mathbb{S}_-$, with $\mathbb{S}_-$ given in (3.18)) and satisfies $\mathbb{M}^2 = -\mathbb{1}$.

Now, the key observation is that, as one sees from (3.44), $*_b$ acts as $-i$ on $\Phi_- \in U_{\mathbf{1},\mathbf{1}}$ and as $+i$ on $\chi_I^- \in U_{\bar{\mathbf{3}},\bar{\mathbf{3}}}$, so that, referring to eq. (3.27), we have

$$*_b(\partial_I\Phi_-) \ \sim \ i(\partial_I\Phi_- - 2\kappa_I^-\Phi_-) \tag{3.49}$$

On the other hand, recalling (3.29) and (3.30),

$$\partial_I\Phi_- \sim \alpha_I - \mathcal{G}_{IJ}\beta^J \ . \tag{3.50}$$

Substituting (3.50) into (3.49) and using (3.46) we get

$$(\mathcal{A}_I{}^J - \mathcal{G}_{IK}\mathcal{C}^{KJ})\alpha_J + (\mathcal{B}_{IJ} + \mathcal{A}_J{}^K\mathcal{G}_{KI})\beta^J \ \sim \ i(\delta_I{}^J - 2\kappa_I^- Z^J)\alpha_J - i(\mathcal{G}_{IJ} - 2\kappa_I^-\mathcal{G}_J)\beta^J \ . \tag{3.51}$$

Taking the integrated Mukai pairing of this expression with the basis forms α,β, separating into real and imaginary parts, and using the expression (3.32) for κ_I^-, we arrive at

$$\begin{aligned}
\mathcal{C}^{IJ} &= -(\mathrm{Im}\mathcal{G})^{-1\,IJ} + \frac{Z^I\bar{Z}^J + \bar{Z}^I Z^J}{Z^K\mathrm{Im}\mathcal{G}_{KL}\bar{Z}^L} \\
&= (\mathrm{Im}\mathcal{M})^{-1\,IJ} \\
\mathcal{A}_I{}^J &= -[\mathrm{Re}\mathcal{G}(\mathrm{Im}\mathcal{G})^{-1}]_I{}^J + \frac{\mathcal{G}_I\bar{Z}^J + \bar{\mathcal{G}}_I Z^J}{Z^K\mathrm{Im}\mathcal{G}_{KL}\bar{Z}^L} \\
&= [\mathrm{Re}\mathcal{M}(\mathrm{Im}\mathcal{M})^{-1}]_I{}^J \\
\mathcal{B}_{IJ} &= [\mathrm{Im}\mathcal{G} + \mathrm{Re}\mathcal{G}(\mathrm{Im}\mathcal{G})^{-1}\mathrm{Re}\mathcal{G}]_{IJ} - \frac{\mathcal{G}_I\bar{\mathcal{G}}_J + \bar{\mathcal{G}}_I\mathcal{G}_J}{Z^K\mathrm{Im}\mathcal{G}_{KL}\bar{Z}^L} \\
&= -[\mathrm{Im}\mathcal{M} + \mathrm{Re}\mathcal{M}(\mathrm{Im}\mathcal{M})^{-1}\mathrm{Re}\mathcal{M}]_{IJ} \ ,
\end{aligned} \tag{3.52}$$

[9]See (A.6) for our definition of the Hodge $*$.

where to write the second equalities we use (D.8). So the matrices $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are expressed in terms of the special Kähler geometry period matrix $\mathcal{M}$, and the result can be summarized in

$$\begin{aligned} \mathbb{M} &\equiv \begin{pmatrix} \int\langle\alpha,*_b\beta\rangle & -\int\langle\beta,*_b\beta\rangle \\ \int\langle\alpha,*_b\alpha\rangle & -\int\langle\beta,*_b\alpha\rangle \end{pmatrix} \\ &= \begin{pmatrix} -(\mathrm{Im}\mathcal{M})^{-1}\mathrm{Re}\mathcal{M} & (\mathrm{Im}\mathcal{M})^{-1} \\ -\mathrm{Im}\mathcal{M}-\mathrm{Re}\mathcal{M}(\mathrm{Im}\mathcal{M})^{-1}\mathrm{Re}\mathcal{M} & \mathrm{Re}\mathcal{M}(\mathrm{Im}\mathcal{M})^{-1} \end{pmatrix}. \end{aligned} \tag{3.53}$$

The associated matrix

$$\widetilde{\mathbb{M}} := -\mathbb{S}_-\mathbb{M} = \begin{pmatrix} 1 & -\mathrm{Re}\mathcal{M} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{Im}\mathcal{M} & 0 \\ 0 & (\mathrm{Im}\mathcal{M})^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\mathrm{Re}\mathcal{M} & 1 \end{pmatrix} \tag{3.54}$$

is symmetric and negative definite. The latter property is implied by the negative definiteness of $\mathrm{Im}\mathcal{M}$, guaranteed by

$$-(\mathrm{Im}\mathcal{M})^{-1\,IJ} = \int\langle\beta^I,*_b\beta^J\rangle = \int\langle\, e^b\beta^I, *\lambda(e^b\beta^J)\,\rangle = \sum_k\int(e^b\beta^I)_k\lrcorner(e^b\beta^J)_k\, vol_6\,,$$

where k runs over the different form degrees of the polyform $e^b\beta^I$. The matrix $\widetilde{\mathbb{M}}$ is an important piece of information in the definition of $N=2$ supergravity by compactification of type II theories to four dimensions. In particular, for type IIA compactifications it appears in the kinetic terms for the scalars $\xi^I,\tilde{\xi}_I$ coming from the expansion of the RR potentials. Namely, as we have seen for Calabi-Yau compactifications, it is one of the special geometry data that determine the quaternionic metric for the $N=2$ hypermultiplets σ-model. While this is familiar for dimensional reductions on a Calabi-Yau, we have shown that the same structure can be extended to more general settings, for instance to cases in which the basis forms are not of pure degree. In subsection (3.4.3) below we will see that $\widetilde{\mathbb{M}}$ appears in the scalar potential as well.

It is readily checked that when considering Calabi-Yau (or more generally SU(3) structure [49]) compactifications, (3.53) reduces to the well known expression for the action of the Hodge $*$ on the Calabi-Yau harmonic 3-forms. Indeed in this case, because of the constraint $b\wedge\alpha_I = b\wedge\beta^I = 0$, the action of $*_b$ on Σ_- simplifies to the action of the usual Hodge $*$, so that $\int\langle\alpha,*_b\beta\rangle = \int\alpha\wedge*\beta$ (similarly for the other pairings). Therefore in this case (3.53) specializes to the result of [95, 96].

One can now proceed in a completely parallel fashion to derive the action of $*_b$ on the even basis forms Σ_+. In this case, $\Phi_+\in U_{\mathbf{1},\bar{\mathbf{1}}}$ and its deformations are in $U_{\bar{\mathbf{3}},\mathbf{3}}$ (deformations in $U_{\mathbf{3},\bar{\mathbf{1}}}\oplus U_{\mathbf{1},\bar{\mathbf{3}}}$ are assumed to vanish in the Mukai pairing due to condition (3.21)). Again, these are two eigenbundles of $*_b$ associated with opposite eigenvalues.

Repeating the steps done for the odd forms, and adopting analogous assumptions, we find[10]

$$
\begin{aligned}
\mathbb{N} \;&:=\; \begin{pmatrix} \int\langle\omega, *_b\tilde{\omega}\rangle & -\int\langle\tilde{\omega}, *_b\tilde{\omega}\rangle \\ \int\langle\omega, *_b\omega\rangle & -\int\langle\tilde{\omega}, *_b\omega\rangle \end{pmatrix} \\
&=\; \begin{pmatrix} -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & (\mathrm{Im}\mathcal{N})^{-1} \\ -\mathrm{Im}\mathcal{N} - \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}, \qquad (3.55)
\end{aligned}
$$

where $\mathbb{N} \in \mathrm{Sp}(2b^+ + 2, \mathbb{R})$ and satisfies $\mathbb{N}^2 = -\mathbb{1}$. The analog of (3.54) is

$$
\widetilde{\mathbb{N}} \;:=\; -\mathbb{S}_+\mathbb{N} \;=\; \begin{pmatrix} 1 & -\mathrm{Re}\mathcal{N} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\mathrm{Re}\mathcal{N} & 1 \end{pmatrix}, \qquad (3.56)
$$

which again is symmetric and negative definite (the argument for the negative-definiteness of $\mathrm{Im}\mathcal{N}$ is perfectly analogous to the one for $\mathrm{Im}\mathcal{M}$). In type IIA compactifications, $\mathrm{Im}\mathcal{N}$ and $\mathrm{Re}\mathcal{N}$ define the $N = 2$ gauge vector kinetic and topological terms in 4d. The matrix $\widetilde{\mathbb{N}}$ also enters in the scalar potential, see subsection 3.4.3 as well as eq. (3.141) below.

Note that in the particular case of Calabi-Yau 3-folds one can check (3.55) explicitly by separate evaluation of the two matrices appearing there. In order to evaluate the first line of (3.55), one can choose the basis for the even cohomology introduced around eq. (3.38), then expand $-b - iJ = \frac{X^a}{X^0}\omega_a$ and use $\frac{1}{4Vol}\int\langle\omega_a, *\omega_b\rangle = \mathscr{G}_{ab}$ (recall (2.29)). On the other hand, the period matrix $\mathcal{N}_{AB}$ appearing in the second line can be obtained starting from (minus) the cubic prepotential (2.32), and applying the special geometry formula (D.5) (translated in the notation for $\mathscr{M}_+$).

3.3.5 Differential conditions on the basis forms

While in the previous subsection we studied the action of the $*_b$ operator on the basis forms, in this subsection we deal instead with the action of the exterior derivative. Above it was essential to require that the system of expansion forms be closed under the action of $*_b$, up to terms vanishing in the integrated Mukai pairing (recall eq. (3.46)). The same condition will now be needed for the exterior derivative. This subsection mainly follows refs. [45, 46].

Let us first consider a strict SU(3) structure on the compact manifold M_6, characterized by the complex 3–form Ω and the almost symplectic structure J. As already remarked, the expansion forms need not be closed. Seeing this as a deformation of the Calabi-Yau case, one can choose Σ_- in (3.17) as composed of 3-forms only. As for Σ_+, one takes $\omega_0, \tilde{\omega}^0$ as in (3.38). Given a set of 2–forms $\{\omega_a\}$, a set of 4–forms $\{\tilde{\omega}^a\}$ satisfying (3.19) can be built via [49]

$$
\tilde{\omega}^a \;:=\; -g^{ab} * \omega_b\,, \qquad (3.57)
$$

[10]For later use we notice that, recalling (3.17), the matrices $\mathbb{M}$ and $\mathbb{N}$ can be concisely defined via

$$
\mathbb{N}^{\mathbb{A}}{}_{\mathbb{B}} := \int\langle\,\Sigma_+^{\mathbb{A}}, *_b\Sigma_{+\mathbb{B}}\,\rangle \qquad , \qquad \mathbb{M}^{\mathbb{I}}{}_{\mathbb{J}} := \int\langle\,\Sigma_-^{\mathbb{I}}, *_b\Sigma_{-\mathbb{J}}\,\rangle\,,
$$

where $\Sigma_{+\mathbb{B}} = \mathbb{S}_{+\mathbb{B}\mathbb{C}}\Sigma_+^{\mathbb{C}}$ and $\Sigma_{-\mathbb{J}} = \mathbb{S}_{-\mathbb{J}\mathbb{K}}\Sigma_-^{\mathbb{K}}$.

g^{ab} being the inverse of the matrix $g_{ab} = \int \omega_a \wedge * \omega_b$, assumed non-degenerate. The analogue of (3.21) is now $\omega_a \wedge \alpha_I \sim 0 \sim \omega_a \wedge \beta^I$, which implies the usual SU(3) structure constraint $J \wedge \Omega = 0$. As anticipated at the end of subsection 2.4.4, we separate the internal NS 3-form H (satisfying the Bianchi identity $dH = 0$) into an exact and a flux piece:

$$H = H^{\text{fl}} + db \, . \tag{3.58}$$

While b enters in the definition of the pure spinors $\Phi_\pm$ (in particular, in the strict SU(3) structure context it pairs up with J to define the complex form $b + iJ$), the NS flux H^{fl} enters in the twisted differential

$$d_{H^{\text{fl}}} := d - H^{\text{fl}} \wedge \ . \tag{3.59}$$

Expanding it on the basis 3–forms as

$$H^{\text{fl}} = m_0^I \alpha_I - e_{I0} \beta^I \, , \tag{3.60}$$

where m_0^I, e_{I0} are constant parameters, and demanding closure of the system of forms $\Sigma_\pm$ under the action of $d_{H^{\text{fl}}}$, one is led to assume [47, 45, 90, 89]

$$\begin{array}{rclcrcl} d_{H^{\text{fl}}} \alpha_I & \sim & e_{IA} \tilde{\omega}^A & \quad , \quad & d_{H^{\text{fl}}} \beta^I & \sim & m_A^I \tilde{\omega}^A \\ d_{H^{\text{fl}}} \omega_A & \sim & m_A^I \alpha_I - e_{IA} \beta^I & \quad , \quad & d_{H^{\text{fl}}} \tilde{\omega}^A & \sim & 0 \, , \end{array} \tag{3.61}$$

where we recall that $\sim$ means equality up to terms vanishing inside the integrated symplectic pairing. The m_a^I and e_{Ia} are required to be constant parameters, and are sometimes called 'geometric fluxes'. They can be put in relation with the torsion classes characterizing the SU(3) structure under consideration, recall subsection 2.2.2. Note that $(d_{H^{\text{fl}}})^2 = 0$ implies

$$e_{IA} m_B^I - m_A^I e_{IB} = 0 \, . \tag{3.62}$$

On backgrounds more general than the strict SU(3) structure case just described, the basis forms are not necessarily of pure degree. Without needing to specify the details of the model, we are led to adopt the following general differential conditions for the basis $\Sigma_\pm$ [46]:

$$\begin{array}{rclcrcl} d_{H^{\text{fl}}} \alpha_I & \sim & p_I^A \omega_A + e_{IA} \tilde{\omega}^A & \quad , \quad & d_{H^{\text{fl}}} \beta^I & \sim & q^{IA} \omega_A + m_A^I \tilde{\omega}^A \\ d_{H^{\text{fl}}} \omega_A & \sim & m_A^I \alpha_I - e_{IA} \beta^I & \quad , \quad & d_{H^{\text{fl}}} \tilde{\omega}^A & \sim & -q^{IA} \alpha_I + p_I^A \beta^I \, . \end{array} \tag{3.63}$$

By introducing the $(2b^- + 2) \times (2b^+ + 2)$ rectangular matrix of constant parameters

$$\mathbb{Q} := \begin{pmatrix} m^I{}_A & q^{IA} \\ e_{IA} & p_I{}^A \end{pmatrix} , \tag{3.64}$$

one can summarize (3.63) in the following relations [46]:

$$d_{H^{\text{fl}}} \Sigma_- \sim \mathbb{Q} \Sigma_+ \quad , \quad d_{H^{\text{fl}}} \Sigma_+ \sim \widetilde{\mathbb{Q}} \Sigma_- \, . \tag{3.65}$$

The matrix $\widetilde{\mathbb{Q}}$ is related to $\mathbb{Q}$ in a simple way: indeed, since $\int \langle d_{H^{\text{fl}}} \Sigma_-, \Sigma_+ \rangle = \int \langle \Sigma_-, d_{H^{\text{fl}}} \Sigma_+ \rangle$, one has

$$\widetilde{\mathbb{Q}} \ = \ (\mathbb{S}_+)^{-1} \mathbb{Q}^T \mathbb{S}_- \, . \tag{3.66}$$

The quadratic relations among the charges arising from $(d_{H^{\rm fl}})^2 = 0$ here read

$$\mathbb{Q}\widetilde{\mathbb{Q}} \;=\; 0 \;=\; \widetilde{\mathbb{Q}}\mathbb{Q}\,. \tag{3.67}$$

As it will be apparent in section 3.5, these constraints are fundamental to guarantee the consistency of the 4d $N = 2$ supergravity action arising from the dimensional reduction [97, 98, 50].

The matrix $\mathbb{Q}$ encodes both the $H^{\rm fl}$ and the geometric fluxes. However, in [46] it was argued that the action of the differential operator $d_{H^{\rm fl}}$ cannot realize all the possible charges in $\mathbb{Q}$. This can be achieved only on a non-geometric background [99]. On such backgrounds, the $d_{H^{\rm fl}}$ operator can formally be extended to an operator $\mathcal{D}$, first introduced in [100], encoding both geometric and non-geometric fluxes:

$$d_{H^{\rm fl}} \;\rightarrow\; \mathcal{D} := d_{H^{\rm fl}} - Q\,\cdot - R\llcorner\;, \tag{3.68}$$

where in the notation of [100] the Q and R operators act on a differential k-form C as

$$(Q\cdot C)_{m_1\ldots m_{k-1}} = Q^{ab}{}_{[m_1}C_{|ab|m_2\ldots m_{k-1}]} \qquad , \qquad (R\llcorner C)_{m_1\ldots m_{k-3}} = R^{abc}C_{abcm_1\ldots m_{k-3}}\;, \tag{3.69}$$

and so they lower its degree by 1 and 3 respectively. Therefore, $\mathcal{D}$ can both increase and reduce the degree of a form, but still sends odd/even forms into even/odd forms. The nilpotency condition $(d_{H^{\rm fl}})^2 = 0$ is now extended to $\mathcal{D}^2 = 0$, so that the constraints (3.67) still hold. When considering the specific case of the SU(3) structure basis described above, one can identify q^{Ia} and p_I^a appearing in (3.63) as arising from the action of $Q\cdot$, while q^{I0} and p_I^0 are generated by $R\llcorner$.

Since in this thesis we are not concerned with nongeometric backgrounds, in the following we will use just the operator $d_{H^{\rm fl}}$. Even if we have to recall that we cannot realize all the possible charges in it, we find all the same advantageous to employ the general symplectically covariant form of $\mathbb{Q}$.

To summarize, in this subsection we required that the system of basis forms $\Sigma_\pm$ be closed under the action of the twisted exterior derivative $d_{H^{\rm fl}}$. The non-closure of the basis forms introduces a set of parameters – the geometric fluxes – which supplement the NS 3–form fluxes, and vanish in the Calabi-Yau case. From a 4d viewpoint, to be developed starting with next section, these parameters will play the role of electric and magnetic charges assigned to some of the previously neutral scalar fields in the theory. As a consequence, the latter will take the form of a gauged $N = 2$ supergravity.

3.4 Reduction of the NSNS sector

We can now implement the dimensional reduction of type II supergravity, starting from the NSNS sector. We assume a background topology of the type $M_{10} = M_4 \times M_6$, where M_4 is the 4d 'external' spacetime and M_6 is a 6d 'internal' compact manifold admitting SU(3)×SU(3) structure on $T \oplus T^*$. Coordinates along M_4 and M_6 are denoted by x^μ and y^m respectively. Furthermore, we assume that the conditions on the expansion forms spelled above in this chapter are satisfied.

We start introducing an ansatz for the NSNS fields. For the metric we take

$$d\hat{s}^2 = \hat{g}_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(x,y)dy^m dy^n \ , \tag{3.70}$$

where the hatted 4d metric $\hat{g}_{\mu\nu}$ will be soon Weyl-rescaled. The NS 3–form $\hat{H}$ splits as in (3.3). The cohomologically non-trivial part has just internal indices: $\hat{H}^{\text{fl}} \equiv H^{\text{fl}}$, while for the potential $\hat{B}$ we take

$$\hat{B} = B + b \ , \quad \text{with } B = \tfrac{1}{2}B_{\mu\nu}(x)dx^\mu \wedge dx^\nu \quad \text{and} \quad b = \tfrac{1}{2}b_{mn}(x,y)dy^m \wedge dy^n \ . \tag{3.71}$$

Finally, we allow a possible dependence of the 10d dilaton on both external and internal coordinates:

$$\phi = \phi(x,y) \ . \tag{3.72}$$

The mode truncation of g_{mn}, b_{mn} and ϕ will be specified later on, via the expansion of the pure spinors $\Phi_\pm$ on the basis forms described in the previous section. The absence of the terms with mixed indices $g_{\mu n}$ and $B_{\mu n}$ in the reduction ansatz is a feature typical of Calabi-Yau compactifications, in which case it is due to the absence of continuous isometries and of harmonic 1–forms (recall section 3.2). In the general SU(3) and SU(3)×SU(3) structure context a motivation for not to include $g_{\mu n}$ and $B_{\mu n}$ in the truncation ansatz was given in [45, 46] by observing that these fields transform in the 'triplets' of SU(3)×SU(3) (recall the discussion in subsection 3.3.2 above). Notice that this ansatz implies that the NSNS sector will provide no 4d gauge vectors; they will all be supplied by the RR sector.

One can now plug ansatz (3.70)–(3.72) in (3.1) and derive the NSNS sector decomposition. The treatment of the quadratic terms in the dilaton ϕ and NS 3–form $\hat{H}$ appearing in (3.1) being straightforward, we just have to focus on the Einstein-Hilbert term in the action. Under (3.70), the higher dimensional Ricci scalar becomes

$$\hat{R}_{10} = \hat{R}_4 + R_6 - \frac{1}{4}g^{mp}g^{nq}\left(\partial_\mu g_{mp}\hat{\partial}^\mu g_{nq} - 3\partial_\mu g_{mn}\hat{\partial}^\mu g_{pq}\right) - g^{mn}\hat{\nabla}_4^2 g_{mn} \ , \tag{3.73}$$

where $\hat{R}_4$ and R_6 are the Ricci scalars associated with the metrics on $(M_4, \hat{g}_{\mu\nu})$ and (M_6, g_{mn}) respectively, while $\hat{\nabla}_4^2$ is the Laplacian on $(M_4, \hat{g}_{\mu\nu})$. One now proceeds in two steps. First substitute (3.73) in $\frac{1}{2}\int_{M_{10}} vol_{10}e^{-2\phi}\hat{R}_{10}$ and perform the integration by parts (vol_d is the volume form on M_d):

$$-\frac{1}{2}\int_{M_4} vol_4 \int_{M_6} vol_6 e^{-2\phi}g^{mn}\hat{\nabla}_4^2 g_{mn} \quad = \quad \frac{1}{2}\int_{M_4} vol_4 \int_{M_6} \partial_\mu(vol_6 e^{-2\phi}g^{mn})\hat{\partial}^\mu g_{mn} \ .$$

Secondly, pass to the 4d Einstein frame by introducing the 4d Weyl rescaled metric (no rescaling is instead performed on the 6d metric):

$$g_{\mu\nu} := e^{-2\varphi}\hat{g}_{\mu\nu} \ , \tag{3.74}$$

where the 4d dilaton φ is defined as

$$e^{-2\varphi} := \int_{M_6} vol_6 e^{-2\phi} \ . \tag{3.75}$$

For later use, notice from (2.57), (2.58), (3.24), (3.36) and (3.75) that φ is related to the Kähler potentials $K_\pm$ by

$$e^{-K_+} \;=\; e^{-K_-} \;=\; 8e^{-2\varphi}\,. \tag{3.76}$$

Under rescaling (3.74), $\hat{R}_4 = e^{-2\varphi}(R_4 - 6\nabla_4^2\varphi - 6\partial_\mu\varphi\partial^\mu\varphi)$, where on the r.h.s. the indices are raised with the Einstein frame metric $g_{\mu\nu}$.

Putting everything together, the decomposition of (3.1) results then in:

$$\begin{aligned} S_{\rm NS} \;=\;& \frac{1}{2}\int_{M_4} vol_4\big(R_4 - 2\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{12}e^{-4\varphi}H_{\mu\nu\rho}H^{\mu\nu\rho}\big)\\ &-\frac{1}{8}\int_{M_4} vol_4 e^{2\varphi}\int_{M_6} vol_6 e^{-2\phi}g^{mp}g^{nq}\big(\partial_\mu g_{mn}\partial^\mu g_{pq} + \partial_\mu b_{mn}\partial^\mu b_{pq}\big)\\ &-\frac{1}{2}\int_{M_4} vol_4 e^{2\varphi}\int_{M_6} vol_6 e^{-2\phi}\nabla_4^2\log\big(e^{-2\phi}\sqrt{g_6}\big)\\ &-\int_{M_4} vol_4 V_{\rm NS}\;, \end{aligned} \tag{3.77}$$

where $g_6 \equiv \det(g_{mn})$, while $V_{\rm NS}$ is identified with the part of the reduced NSNS sector not containing any 4d spacetime derivative:

$$V_{\rm NS} \equiv -\frac{e^{4\varphi}}{2}\int_{M_6} vol_6 e^{-2\phi}\big(R_6 + 4\partial_m\phi\partial^m\phi - \frac{1}{12}H_{mnp}H^{mnp}\big)\;, \tag{3.78}$$

and therefore represents the contribution of the NSNS sector to the 4d scalar potential.[11]

The first line of (3.77) already contains 4d fields only, and is compatible with 4d $N = 2$ supergravity. In section 3.2 we saw that in standard fluxless Calabi-Yau compactifications the four dimensional NS 2–form B is dualized to an axion which, together with the 4d dilaton φ and two further scalars from the RR sector, defines the bosonic part of the universal hypermultiplet. However, as first observed in [82], in the presence of RR magnetic fluxes the NS 2–form acquires mass terms and therefore cannot be dualized to a scalar anymore. Anyway, as shown in [101, 97, 98], (massive) antisymmetric 2–tensors can be included consistently in an $N = 2$ supergravity action. We will have more to say about this in section 3.5.

The subsequent lines in (3.77) still need to be reformulated in terms of a truncated set of modes of the fields g_{mn}, b_{mn} and ϕ. For this purpose, in the forthcoming subsections first we translate these expressions in the language of generalized geometry, relating them with the SU(3)×SU(3) structure data. Then we implement the expansion in terms of the truncated set of modes introduced in section 3.3.

Before discussing the relation with SU(3)×SU(3) structures, let's briefly make the link with the fluxless Calabi-Yau dimensional reduction of section 3.2. As we discussed in subsection 2.3.2, by expressing the Calabi-Yau metric and b-field deformations in terms of harmonic forms, the second line of (3.77) can be reformulated as a σ–model whose metric splits in the sum of the special Kähler metrics on the spaces of complex– and Kähler–structure deformations. In section 3.2 we reviewed how these correspond to the kinetic

[11] A further contribution to the scalar potential is generated from the RR sector and will be derived in the next section. The total potential of the effective theory will hence be $V = V_{\rm NS} + V_{\rm RR}$.

matrices for the scalars in the vector multiplets and for a subset of the scalars in the hypermultiplets.

The last two lines of (3.77) vanish in Calabi-Yau dimensional reductions. The line involving $\nabla_4^2 \log\left(e^{-2\phi}\sqrt{g_6}\right)$ vanishes thanks to the internal coordinate independence of this last term: passing it out the integral over M_6 and recalling (3.75), one is left with the integral over M_4 of a total derivative. The constancy of $\nabla_4^2 \log\sqrt{g_6}$ along the Calabi-Yau can be seen as follows. Recall that $\sqrt{g_6}$ depends on the 4d coordinates through the moduli $v^a(x)$ parameterizing the Kähler form $J = v^a\omega_a$ ($\{\omega_a(v)\}$ is a basis of harmonic (1,1)–forms): the relevant relation is $vol_6 = \frac{1}{6}J\wedge J\wedge J$. Therefore one has[12]

$$\partial_\mu \log\sqrt{g_6} = \frac{\partial}{\partial v^a}(\log\sqrt{g_6})\partial_\mu v^a = 3\frac{\omega_a\wedge J\wedge J}{J\wedge J\wedge J}\partial_\mu v^a = (\omega_a \lrcorner J)\partial_\mu v^a\ . \tag{3.79}$$

The statement then follows recalling that below eq. (3.42) we deduced $\partial_m(\omega_a\lrcorner J) = 0$.

$V_{\rm NS}$ is zero due to the Ricci-flatness of Calabi-Yau manifolds, as well as to the closure of b and ϕ with respect to the exterior derivative on M_6. The absence of a scalar potential in the 4d effective action (there is no contribution from the RR sector either) is consistent with the fact that the dimensional reduction is performed on a class of equivalent *solutions* of the 10d theory (with vanishing 4d cosmological constant), so that the geometrical moduli correspond to massless 4d scalars with no preferred vev. This is in contrast with what expected for general SU(3)×SU(3) structure off-shell reductions: as we will show in subsection 3.4.3, in this case a non-trivial scalar potential is generated.

3.4.1 Scalar kinetic terms

The second line of (3.77) defines kinetic terms for the parameters of the internal metric and b-field propagating along the 4d spacetime. This has already been translated in the generalized geometry formalism in section 2.5, where we showed eq. (2.87). Subsequently, in subsection 3.3.3 we expressed the two addends appearing in the r.h.s. of (2.87) in terms of a finite set of deformations. These are parameterized by the $\mathscr{M}_-$ and $\mathscr{M}_+$ coordinates z^i and t^a, which are scalar fields from the 4d viewpoint. By the discussion of subsection 3.3.3, the two addends in the r.h.s. of (2.87) take the form (3.33), (3.37), where the Kähler potentials $K_\pm$ associated with the special Kähler metrics $\mathscr{G}^-_{i\bar{\jmath}}$ and $\mathscr{G}^+_{a\bar{b}}$ were given in (3.24) and (3.36). We conclude that the second line of (3.77) yields the kinetic terms for the scalars z^i, t^a, and can be rewritten as

$$\frac{e^{2\varphi}}{8}\int e^{-2\phi}vol_6\, g^{mp}g^{nq}(\partial_\mu g_{mn}\partial^\mu g_{pq} + \partial_\mu b_{mn}\partial^\mu b_{pq}) \;=\; \mathscr{G}^-_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu \bar{z}^{\bar{\jmath}} \;+\; \mathscr{G}^+_{a\bar{b}}\partial_\mu t^a\partial^\mu \bar{t}^{\bar{b}}\ , \tag{3.80}$$

where the definition of the 4d dilaton (3.75) has also been used.

3.4.2 Variations of $\sqrt{g_6}$ and the dilaton

In the following we discuss the condition under which the variation of $\log(e^{-2\phi}\sqrt{g_6})$, as induced by SU(3)×SU(3) structure deformations, is independent of the internal coordi-

[12]Notice that even if the harmonic forms ω_a depend on the moduli, as illustrated in [49, 93] on a Calabi-Yau one has $v^b\frac{\partial}{\partial v^a}\omega_b = 0$, and therefore $\frac{\partial}{\partial v^a}J = \omega_a$.

nates. As observed above eq. (3.79), this guarantees vanishing of the third line in (3.77), in analogy with the Calabi-Yau case.

Recalling the relation (2.58) between the dilaton ϕ and the pure spinor norm (defined in (2.57)), we immediately see that under a general pure spinor deformation (2.73) we have

$$\delta\log(e^{-2\phi}\sqrt{g_6}) = \frac{\delta\langle\Phi_\pm,\bar\Phi_\pm\rangle}{\langle\Phi_\pm,\bar\Phi_\pm\rangle} = 2\mathrm{Re}(\delta\kappa)\ ,$$

where we call $\mathrm{Re}(\delta\kappa)$ the equal real parts of $\delta\kappa_+$ and $\delta\kappa_-$ (recall (2.56)). Hence the third line of (3.77) vanishes if the function $\mathrm{Re}(\delta\kappa)$ associated with pure spinor rescalings is constant on M_6. For the truncated set of modes, this is guaranteed by our assumption (3.41).

In the remainder of this subsection we deal with the following related issue: we know from subsection 2.5.3 that deformations of the metric on M_6 are orthogonal to variations of the pairing $\langle\Phi,\bar\Phi\rangle$; on the other hand, they obviously modify the volume form $vol_6 = \sqrt{g_6}d^6y$ employed in the definition (2.57) of the pure spinor norm. It follows that deformations of the metric on M_6 affect $||\Phi||$ and so, by eq. (2.58), the dilaton. However, it is more natural to keep the deformations of ϕ and $\sqrt{g_6}$ as independent. This can be achieved by prescribing, simultaneously with a metric deformation, a real rescaling of $\Phi_\pm$, with $\delta\kappa = \frac{1}{2}\delta\log\sqrt{g_6}$. It is understood that any other independent pure spinor rescaling (having $\mathrm{Re}(\delta\kappa)\neq 0$) modifies ϕ without affecting the metric g_{mn}.

For the sake of completeness, in the following we derive the relation between the variation of $\sqrt{g_6}$ and $\delta\Phi_\pm$ in (2.73). Recalling (2.46), and assuming here $b=0$ for simplicity, we have that $g_{mn} = \mathcal{G}_{mn} = -(\mathcal{J}_+\mathcal{J}_-)_{mn}$. Using (2.42) we obtain[13,14]

$$2\,\delta\log\sqrt{g_6} \equiv g^{mn}\delta g_{mn} = \frac{1}{2}\big[(\delta\mathcal{J}_+)_{mn}(J_1+J_2)^{mn} + (\delta\mathcal{J}_-)_{mn}(J_1-J_2)^{mn}\big]\ , \tag{3.81}$$

so we see that in general both $\delta\mathcal{J}_+$ and $\delta\mathcal{J}_-$ will contribute. Now we express $\delta\mathcal{J}_\pm$ employing (2.82): as discussed in subsection 2.5.3, $\delta_{\rm tr}\mathcal{J}_\pm$ drop when computing variations of the generalized metric $\mathcal{G}$, so we are left with the deformations induced by $\delta\chi_\pm$. Performing as usual the computation in the bispinor picture, and recalling (2.74), we arrive at the result

$$\delta\log\sqrt{g_6} = 4g^{mn}\mathrm{Re}(\delta\chi_- - \delta\chi_+)_{mn}\ .$$

All this can be illustrated considering strict SU(3) structures, for which $J_1 = J_2 \equiv J$, $I_1 = I_2 \equiv I$, and $\mathcal{J}_\pm$ take the form (2.43). It is immediate to see that the term containing $\delta\mathcal{J}_-$ doesn't contribute to (3.81), so that

$$\delta\log\sqrt{g_6} = \frac{1}{2}(\delta\mathcal{J}_+)_{mn}J^{mn} = (\delta J)\lrcorner J\ .$$

In particular, only the rescalings $\delta J = \delta\lambda J$ (where $\delta\lambda$ is a function) contribute to $(\delta J)\lrcorner J$. Now we notice that this J–rescaling also implies a rescaling of Φ^0_+, which in the SU(3)

[13] If $b=0$ in $\mathcal{G}$, then in general the variation $\delta\mathcal{G}$ will contain a small δb. However, at first order this doesn't enter in $\delta\mathcal{G}_{mn}$, which is then identified with δg_{mn}.

[14] The supplementary term $g^{mn}[(\delta\mathcal{J}_+)_m{}^p\mathcal{J}_{-pn} + \mathcal{J}_{+mp}(\delta\mathcal{J}_-)^p{}_n]$ that should enter in (3.81) vanishes because $g^{nm}(\delta\mathcal{J}_+ - \delta_{\rm tr}\mathcal{J}_+)_m{}^p$ and $(\delta\mathcal{J}_- - \delta_{\rm tr}\mathcal{J}_-)^p{}_n g^{nm}$ turn out to be symmetric tensors while $\mathcal{J}_{-pn}$ and $\mathcal{J}_{+mp}$ are antisymmetric.

structure case reads $\Phi^0_+ = e^{-\phi}e^{-iJ}$ (recall (2.65)). Indeed, at first order we have

$$\delta e^{-iJ} = \frac{3}{2}\delta\lambda e^{-iJ} + \frac{1}{4}\delta\lambda(-6 + 2iJ - J^2 + iJ^3)\ , \tag{3.82}$$

where the second term in the r.h.s. is in the $(\bar{\mathbf{3}}, \mathbf{3})$ of SU(3)× SU(3). It is now immediate to check that, thanks to the presence of the rescaling term in (3.82), it is consistent to keep the pure spinor norm $||\Phi_\pm||$, viz. the dilaton, unmodified. Notice also that the condition $\mathrm{Re}(\delta\kappa) = \mathrm{const}$ in this case also requires $\delta\lambda$ to be constant along M_6. Choosing the basis of expansion forms described in ref. [49], we have $3\delta\lambda = (\delta J)\lrcorner J = \omega_a \lrcorner J \delta v^a$, and we recover the requirement $d(\omega_a \lrcorner J) = 0$ discussed in that paper. As seen below (3.42), this same requirement is satisfied on a Calabi-Yau 3–fold.

3.4.3 The scalar potential

In the following, first we derive a formula expressing the Ricci curvature R_6 of the compact manifold (supplemented by terms involving H_{mnp} and $\partial_m\phi$) as a function of the pure spinors $\Phi_\pm$. Then we apply this result to reformulate the NSNS contribution (3.78) to the 4d scalar potential. This allows us to make contact with an expression for the potential obtained via purely 4d gauged supergravity methods in [50].

At the end of this subsection we will prove that under the assumption

$$\langle d_H\Phi^0_+, \overrightarrow{\gamma}^m\bar{\Phi}^0_+\rangle + \langle d_H\Phi^0_-, \overrightarrow{\gamma}^m\bar{\Phi}^0_-\rangle = 0 \quad , \quad \langle d_H\Phi^0_+, \bar{\Phi}^0_+\overleftarrow{\gamma}^m\rangle + \langle d_H\bar{\Phi}^0_-, \Phi^0_-\overleftarrow{\gamma}^m\rangle = 0\ , \tag{3.83}$$

constraining a subset[15] of the SU(3)×SU(3) triplets in $d_H\Phi^0_\pm$, the following formula is valid:

$$R_6 - \frac{1}{12}H_{mnp}H^{mnp} + 4\partial_m\phi\partial^m\phi - 2e^{2\phi}\nabla_6^2 e^{-2\phi} = \tag{3.84}$$

$$= -4\frac{\langle d_H\Phi^0_+, *\lambda(d_H\bar{\Phi}^0_+)\rangle}{i\langle\Phi_\pm,\bar{\Phi}_\pm\rangle} - 4\frac{\langle d_H\Phi^0_-, *\lambda(d_H\bar{\Phi}^0_-)\rangle}{i\langle\Phi_\pm,\bar{\Phi}_\pm\rangle} + 16\Big|\frac{\langle d_H\Phi^0_+, \Phi^0_-\rangle}{i\langle\Phi_\pm,\bar{\Phi}_\pm\rangle}\Big|^2 + 16\Big|\frac{\langle d_H\Phi^0_+, \bar{\Phi}^0_-\rangle}{i\langle\Phi_\pm,\bar{\Phi}_\pm\rangle}\Big|^2,$$

where ∇_6^2 is the Laplacian on M_6 and $d_H = d - H\wedge$, with $H = H^{\mathrm{fl}} + d_{(6)}b$ purely internal. This completes and generalizes an expression given in the context of SU(3) structures in footnote 2 of ref. [49], referring to results in [102].

Before plugging (3.84) in the definition (3.78) of V_{NS}, let us comment on some of its features.

As a first thing, we remark that (3.84) is symmetric under the exchange $\Phi^0_+ \leftrightarrow \Phi^0_-$, in agreement with the formulation of mirror symmetry in the context of generalized structures [103, 26, 27]. Indeed we have $\langle d_H\Phi^0_+, \Phi^0_-\rangle = \langle\Phi^0_+, d_H\Phi^0_-\rangle$, thanks to the fact that Φ^0_+, Φ^0_- satisfy (2.55).

Furthermore, notice that while the last two terms in the r.h.s. of (3.84) are positive definite, the first two are instead negative definite: in fact for any complex polyform $C = \sum_k C_k$, one has $\langle C, *\lambda(\bar{C})\rangle = vol_6 \sum_k C_k \lrcorner \bar{C}_k$. The last two terms of (3.84) vanish

[15] Here we don't strictly need the condition projecting out *all* the SU(3)×SU(3) triplets in $d_H\Phi^0_\pm$, which would read: $\langle d_H\Phi^0_+, \Gamma^\Lambda\bar{\Phi}^0_+\rangle = 0 = \langle d_H\Phi^0_-, \Gamma^\Lambda\bar{\Phi}^0_-\rangle$, with $\Gamma^\Lambda = dy^m\wedge$ or ι_{∂_m} (the analogous relations containing $\Phi^0_\pm$ at the place of $\bar{\Phi}^0_\pm$ are automatically satisfied).

when at least one of the two pure spinors satisfies the condition $d_H\Phi^0 = (\iota_v + \zeta\wedge)\Phi^0$, where v is a vector and ζ a 1–form; this corresponds to a twisted integrability condition for the generalized almost complex structure associated with Φ^0 [23]. Finally, the r.h.s. of (3.84) vanishes identically when the pure spinors satisfy the 'generalized Calabi-Yau metric' condition $d_H\Phi^0_\pm = 0$ introduced in [23].[16] Then for these geometries we have an expression for the curvature R_6 in terms of $\partial_m\phi$ and H_{mnp} (playing the role of torsion).

The r.h.s. of (3.84) can also be expressed in terms of the SU(3)×SU(3) torsion classes introduced in [27, 33]. We refer to the parameterization provided by eqs. (6.14), (6.15) of ref. [33] (even if written for SU(3) structure pure spinors, that parameterization also applies to the general SU(3)×SU(3) structure case). Using (B.6), (B.8) we get

$$\begin{aligned}\text{r.h.s. of (3.84)} \quad &= \\ &= 8\big(\,|W^{30}|^2 + |W^{03}|^2\,\big) - 16\big(\,|W^{21}|^2 + |W^{12}|^2 + |W^{11}|^2 + |W^{22}|^2 + |W^{10}|^2 + |W^{01}|^2\,\big)\,,\end{aligned}$$

where expressions like for instance $|W^{12}|^2$ and $|W^{10}|^2$ mean $W^{12}_{\bar\imath_1 j_2}\overline{W}^{12\,i_1 j_2}$ and $W^{10}_{j_2}\overline{W}^{10\,j_2}$ respectively. Here the indices $\bar\imath_1, i_1$ are (anti)holomorphic with respect to the almost complex structure I_1, and analogously for $\bar\jmath_2, j_2$ w.r.t. I_2. Our constraint (3.83), which in terms of torsion classes reads $W^{01}_{\bar\imath_1} + W^{31}_{\bar\imath_1} = 0$ and $W^{10}_{j_2} - \overline{W}^{20}_{j_2} = 0$, has been used to eliminate W^{31} and W^{20}.

Now we multiply eq. (3.84) with $e^{-2\phi}vol_6$ and integrate over M_6, obtaining in this way a geometric expression for the NSNS contribution (3.78) to the 4d scalar potential. Using eqs. (2.57), (2.58), as well as (2.64) and the definition (3.43) of the $*_b$ operator, we arrive at

$$\begin{aligned} V_{\rm NS} \quad = \quad & \frac{e^{4\varphi}}{4}\int \Big[\,\langle\, d_{H^{\rm fl}}\Phi_+, *_b(d_{H^{\rm fl}}\bar\Phi_+)\,\rangle \;+\; \langle\, d_{H^{\rm fl}}\Phi_-, *_b(d_{H^{\rm fl}}\bar\Phi_-)\,\rangle\,\Big] \\ & - e^{4\varphi}\int \frac{\big|\langle d_{H^{\rm fl}}\Phi_+, \Phi_-\rangle\big|^2 \;+\; \big|\langle d_{H^{\rm fl}}\Phi_+, \bar\Phi_-\rangle\big|^2}{i\langle\Phi_\pm, \bar\Phi_\pm\rangle}\quad . \end{aligned} \tag{3.85}$$

Notice that this is written purely in terms of generalized geometry quantities.

Our next step is to reformulate expression (3.85) for $V_{\rm NS}$ in terms of the 4d degrees of freedom by substituting the expansions (3.23), (3.35) for $\Phi_\pm$, and exploiting the assumed properties of the basis polyforms. In order to keep the resulting expressions as compact as possible, it is convenient to repackage the $X^A, \mathcal{F}_A$ and $Z^I, \mathcal{G}_I$ appearing in (3.23), (3.35) in the following symplectic vectors

$$X^{\mathbb{A}} = \begin{pmatrix} X^A \\ \mathcal{F}_A \end{pmatrix} \qquad , \qquad Z^{\mathbb{I}} = \begin{pmatrix} Z^I \\ \mathcal{G}_I \end{pmatrix}, \tag{3.86}$$

so that the pure spinors can be written as $\Phi_+ = X^{\mathbb{A}}\mathbb{S}_{+\mathbb{A}\mathbb{B}}\Sigma_+^{\mathbb{B}}$ and $\Phi_- = Z^{\mathbb{I}}\mathbb{S}_{-\mathbb{I}\mathbb{J}}\Sigma_-^{\mathbb{J}}$. Then, recalling (3.65), the results of subsection 3.3.4 (condensed in footnote 10) as well as relation (3.76), we obtain for instance

$$e^{2\varphi}\int\langle d_{H^{\rm fl}}\Phi_+, *_b(d_{H^{\rm fl}}\bar\Phi_+)\rangle = -8e^{K_+}X^{\mathbb{A}}(\mathbb{Q}^T\widetilde{\mathbb{M}}\mathbb{Q})_{\mathbb{A}\mathbb{B}}\bar X^{\mathbb{B}}\,,$$

[16] This condition does not coincide with the notion of generalized Calabi-Yau manifold defined in [22], see e.g. [60, section 4] for a comparison.

To evaluate the second line of (3.85), we need requirement (3.40), implying

$$\frac{\langle \Sigma_-^{\mathbb{I}}, \Phi_- \rangle}{\langle \Phi_-, \bar{\Phi}_- \rangle} = \frac{\int \langle \Sigma_-^{\mathbb{I}}, \Phi_- \rangle}{\int \langle \Phi_-, \bar{\Phi}_- \rangle} = -ie^{K_-} Z^{\mathbb{I}} \, .$$

The resulting expression for $V_{\rm NS}$ is symplectically invariant, and reads

$$\begin{aligned} V_{\rm NS} \;\; = \;\; & -2e^{2\varphi}\Big[e^{K_+} X^{\mathbb{A}} (\mathbb{Q}^T \widetilde{\mathbb{M}} \mathbb{Q})_{\mathbb{A}\mathbb{B}} \bar{X}^{\mathbb{B}} + e^{K_-} Z^{\mathbb{I}} (\widetilde{\mathbb{Q}}^T \widetilde{\mathbb{N}} \widetilde{\mathbb{Q}})_{\mathbb{I}\mathbb{J}} \bar{Z}^{\mathbb{J}}\Big] \\ & - 8e^{2\varphi} e^{K_+ + K_-} \bar{Z}^{\mathbb{I}} (\mathbb{S}_- \mathbb{Q})_{\mathbb{I}\mathbb{A}} (X^{\mathbb{A}} \bar{X}^{\mathbb{B}} + \bar{X}^{\mathbb{A}} X^{\mathbb{B}}) (\mathbb{Q}^T \mathbb{S}_-)_{\mathbb{B}\mathbb{J}} Z^{\mathbb{J}} \, , \end{aligned} \qquad (3.87)$$

where we recall that $\widetilde{\mathbb{Q}}$ is given by (3.66). Notice that, since $\widetilde{\mathbb{N}}$ and $\widetilde{\mathbb{M}}$ are negative definite, the first line is positive definite, while the second line is obviously negative. Expression (3.87) for $V_{\rm NS}$ coincides precisely with a formula obtained in [50] by means of 4d gauged supergravity techniques, starting from the 4d effective action associated with Calabi–Yau compactifications (see next chapter for more details). Here we derived it directly from the dimensional reduction of the higher dimensional theory on manifolds admitting SU(3)×SU(3) structure on $TM_6 \oplus T^*M_6$.

Finally, we remark that the value of expression (3.84) *in a vacuum* is also related to the external spacetime Ricci curvature R_4. Indeed the string frame 10d dilaton equation (3.11) evaluated on a 4d×6d background preserving maximal 4d symmetry takes the form

$$-R_4 = R_6 - \frac{1}{12} H_{mnp} H^{mnp} + 4\partial_m \phi \partial^m \phi - 2e^{2\phi} \nabla_6^2 e^{-2\phi} \, ,$$

with no contributions from the RR sector. Acting on this equation with $\int_{M_6} e^{-2\phi} vol_6$ and rescaling the 4d metric as in (3.74), we obtain $R_4 = 2V_{\rm NS}$. On the other hand, from the trace of the 4d Einstein equation evaluated on a maximally symmetric vacuum, in general one has $R_4 = 4V$. Since the total potential of the reduced theory is $V = V_{\rm NS} + V_{\rm RR}$, then we can conclude that in a vacuum $2V_{\rm RR} = -V_{\rm NS}$, which is the same as $V = -V_{\rm RR}$. Since $V_{\rm RR}$ turns out to be positive definite, we conclude that the 4d vacua obtained within this framework are all AdS. In subsection 5.6.1 we will apply this result to a concrete example of dimensional reduction.

Proof of relation (3.84)

In the remainder of this section we give an account of the main computational steps proving eq. (3.84). These are based on the bispinor description of subsection 2.4.4. However here we relax condition (2.62), i.e. we work with unnormalized spinors, and we reabsorb the dilaton appearing in (2.60) in the product of the spinor norms. More specifically, we parameterize $||\eta_\pm^1|| = |a|$, $||\eta_\pm^2|| = |b|$ (the latter should not be confused with the internal NS 2–form, also called b), and we take $|ab| = e^{-\phi}$. Hence in the following computations the same $\Phi_\pm^0$ appearing in eq. (2.60) read $\not{\Phi}_\pm^0 = 8\eta_+^1 \otimes \eta_\pm^{2\dagger}$.

We start without imposing any constraint on the SU(3)×SU(3) triplets of $d_H \Phi_\pm^0$. The r.h.s. of (3.84) is evaluated recalling (2.57), (2.58), and using (B.8) for the Mukai pairing as well as

$$\begin{aligned}\frac{1}{4}\cancel{d}_H\Phi^0_\pm &= (\cancel{D} - \tfrac{1}{4}\cancel{H})\eta^1_+\eta^{2\dagger}_\pm \pm (D_m - \tfrac{1}{4}H_m)\eta^1_+\eta^{2\dagger}_\pm\gamma^m \\ &\pm \eta^1_+\big[(\cancel{D}+\tfrac{1}{4}\cancel{H})\eta^2_\pm\big]^\dagger + \gamma^m\eta^1_+\big[(D_m+\tfrac{1}{4}H_m)\eta^2_\pm\big]^\dagger\ ,\end{aligned} \tag{3.88}$$

where $\cancel{D} = \gamma^n D_n$, $\cancel{H} = \frac{1}{6}H_{mnp}\gamma^{mnp}$ and $H_m = \frac{1}{2}H_{mnp}\gamma^{np}$. Eq. (3.88) is directly derived (also recalling (2.60)) from the expressions for $\cancel{d}\Phi^0_\pm$ and $\cancel{H\wedge}\Phi^0_\pm$ given e.g. in appendix A of ref. [33]. For instance, the second-last term in (3.84) yields

$$\begin{aligned}16\Big|\frac{\langle d_H\Phi^0_+,\Phi^0_-\rangle}{i\langle\Phi_\pm,\bar\Phi_\pm\rangle}\Big|^2 &= 4|a|^{-4}\big[(\cancel{D}-\tfrac{1}{4}\cancel{H})\eta^1_+\big]^\dagger\eta^1_-\eta^{1\dagger}_-(\cancel{D}-\tfrac{1}{4}\cancel{H})\eta^1_+ \\ &= 4|a|^{-2}\big|(\cancel{D}-\tfrac{1}{4}\cancel{H})\eta^1_+\big|^2 - 2|a|^{-4}\big|\eta^{1\dagger}_+\gamma^m(\cancel{D}-\tfrac{1}{4}\cancel{H})\eta^1_+\big|^2\ ,\end{aligned}$$

where in the second equality we used identity (A.18) to reexpress $|a|^{-2}\eta^1_-\eta^{1\dagger}_-$. The computation of the terms in the r.h.s. of (3.84) containing $*\lambda$ is slightly more involved, but employs the same technique. For the image of $*\lambda$ under the Clifford map we use (B.6).

Resumming all the terms and taking a few cancellations into account we obtain

$$\begin{aligned}&-4\frac{\langle d_H\Phi^0_+,*\lambda(d_H\bar\Phi^0_+)\rangle}{i\langle\Phi_\pm,\bar\Phi_\pm\rangle} - 4\frac{\langle d_H\Phi^0_-,*\lambda(d_H\bar\Phi^0_-)\rangle}{i\langle\Phi_\pm,\bar\Phi_\pm\rangle} + 16\Big|\frac{\langle d_H\Phi^0_+,\Phi^0_-\rangle}{i\langle\Phi_\pm,\bar\Phi_\pm\rangle}\Big|^2 + 16\Big|\frac{\langle d_H\Phi^0_+,\bar\Phi^0_-\rangle}{i\langle\Phi_\pm,\bar\Phi_\pm\rangle}\Big|^2 = \\ &= |a|^{-2}\big[2D_m\eta^{1\dagger}_+\gamma^{mn}D_n\eta^1_+ + \frac{1}{8}\eta^{1\dagger}_+(H_mH^m - \cancel{H}\cancel{H})\eta^1_+ - \frac{1}{12}D_m(\eta^{1\dagger}_+\gamma^{mnpq}\eta^1_+)H_{npq}\big] \\ &- 4|a|^{-2}\mathrm{Re}\big[\eta^{1\dagger}_+\gamma^m(\cancel{D}-\tfrac{1}{4}\cancel{H})\eta^1_+\big]\partial_m\log|b| - 2|a|^{-4}\big|\eta^{1\dagger}_+\gamma^m(\cancel{D}-\tfrac{1}{4}\cancel{H})\eta^1_+\big|^2 \\ &+ \eta^1_+\to\eta^2_+\ ,\ |a|\leftrightarrow|b|\ ,\ H\to -H\end{aligned} \tag{3.89}$$

where the last line denotes the repetition of the two preceding lines performing the prescribed transformations.

Next we consider our requirement (3.83) on the SU(3)×SU(3) triplets of $d_H\Phi^0_\pm$: this can be translated as[17]

$$|a|^{-2}\eta^{1\dagger}_+\gamma^m(\cancel{D}-\tfrac{1}{4}\cancel{H})\eta^1_+ + 2P^m_{1\,n}\partial^n\log|b| = 0\ , \tag{3.90}$$

together with the analogous relation obtained implementing $1\to 2$, $|a|\leftrightarrow|b|$, $H\to -H$. Here P_1 is the holomorphic projector associated with the almost complex structure I_1.

Now, constraint (3.90) implies that the two terms in (3.89) containing $\cancel{D}-\frac{1}{4}\cancel{H}$ cancel each other. Then, using the following relations

$$\begin{aligned}[D_m,D_n]\eta_+ = \frac{1}{4}R_{mnpq}\gamma^{pq}\eta_+ \quad &\Rightarrow \quad D_m\eta^\dagger_+\gamma^{mn}D_n\eta_+ = D_m(\eta^\dagger_+\gamma^{mn}D_n\eta_+) + \frac{1}{4}||\eta_+||^2R_6 \\ H_mH^m - \cancel{H}\cancel{H} \quad &= \quad -\frac{1}{3}H_{mnp}H^{mnp} \\ dH = 0 \quad &\Leftrightarrow \quad D_{[m}H_{npq]} = 0\ ,\end{aligned}$$

[17] One can check that in the notation of ref.[33, app.A.4], this constraint corresponds to $T^1_{\bar\imath_1}+\partial_{\bar\imath_1}\log|b| = 0$ together with $T^2_{\bar\imath_2}+\partial_{\bar\imath_2}\log|a| = 0$.

we rewrite

$$\begin{aligned}\text{r.h.s. of (3.89)} &= R_6 - \frac{1}{12}H_{mnp}H^{mnp} + 2|a|^{-2}D_m\big(\eta_+^{1\dagger}\gamma^{mn}D_n\eta_+^1 - \frac{1}{24}H_{npq}\eta_+^{1\dagger}\gamma^{mnpq}\eta_+^1\big)\\ &\quad + \eta_+^1 \to \eta_+^2 \ , \ |a| \to |b| \ , \ H \to -H \ ,\end{aligned} \tag{3.91}$$

where only the term involving $|a|^{-2}$ needs to be repeated with the prescribed substitutions.

Now we observe that the real part of constraint (3.90) can be written as

$$|a|^{-2}\big[\mathrm{Re}(\eta_+^{1\dagger}\gamma^{mn}D_n\eta_+^1) - \frac{1}{24}H_{npq}\eta_+^{1\dagger}\gamma^{mnpq}\eta_+^1\big] + \partial^m \log|ab| = 0 \ .$$

Noticing that $D_m[\mathrm{Im}(\eta_+^\dagger\gamma^{mn}D_n\eta_+)]$ vanishes identically, and recalling $|ab| = e^{-\phi}$, we can use this equation, together with the analogous one obtained performing $1 \to 2 \ , \ |a| \leftrightarrow |b| \ , \ H \to -H$, to see that

$$\text{last two terms in (3.91)} \ = \ -4\partial_m\phi\partial^m\phi + 4\nabla_6^2\phi \ \equiv \ 4\partial_m\phi\partial^m\phi - 2e^{2\phi}\nabla_6^2 e^{-2\phi} \ .$$

This proves eq. (3.84).

3.5 Reduction of the RR sector

In this section we reduce the RR sector. Here we will focus on type IIA supergravity; however the procedure we develop can equally well be applied to type IIB.

We wish to reduce the RR democratic pseudo-action (3.8), also implementing the self-duality constraint (3.5) in an appropriate way (a direct substitution of (3.5) in (3.8) results indeed in a vanishing action). In principle we could follow a procedure similar to the one adopted in [83], and subsequently in [82, 88], to reduce the type IIB action taking into account the self-duality of the RR 5–form F_5. In [83], firstly the electric and magnetic 4d gauge field strengths descending from the expansion of F_5 on the Calabi-Yau harmonic 3–forms are regarded as independent and kept in the 4d action. Then, the addition of a suitable Lagrange multiplier term makes the equations of motion for the magnetic field strengths precisely correspond to the self-duality constraint. Integrating out the magnetic field strengths provides thus an action with electric fields only and the self-duality constraints correctly implemented. In our context, the generalization of this procedure would require a first step in which one keeps in the 4d action forms of every degree[18] (from 0 to 4) descending from the expansion of the RR field on the internal basis (3.17), and then a second step in which a subset of these forms is integrated out. However, in our case this direct approach to the reduction of the action turns out to be quite involved due to the large amount of fields and constraints, and we find it more efficient to proceed along the following alternative path.

First we reduce the self-duality constraint (3.5) for the democratic RR field, as well as its EoM/Bianchi identities (3.6). From the reduced Bianchi identities we isolate and solve a set of 4d Bianchi identities, defining in this way the fundamental dynamical fields

[18] It would be interesting to relate this with the tensor hierarchy proposed in [104, 105].

of the 4d effective theory. Using the relations obtained from the reduction of the 10d self-duality condition, the remaining 4d equations are interpreted as EoM associated with the identified dynamical degrees of freedom. The last step consists in the reconstruction of the four dimensional action leading precisely to such EoM.

We will work with the so-called G–basis for the RR field, defined by [48]:

$$\hat{\mathbf{F}} \equiv e^{\hat{B}}\hat{\mathbf{G}} \; . \tag{3.92}$$

In this basis, the self-duality constraints (3.5) and the Bianchi identities in (3.6) read respectively

$$e^{\hat{B}}\hat{\mathbf{G}} = \lambda * (e^{\hat{B}}\hat{\mathbf{G}}) \; , \tag{3.93}$$

$$(d - H^{\text{fl}}\wedge)\hat{\mathbf{G}} = 0 \; , \tag{3.94}$$

where as in the previous section we used the decomposition $\hat{H} = H^{\text{fl}} + d\hat{B}$, with $\hat{B} = B + b$. We recall that, due to the self-duality, the RR EoM are equivalent to the Bianchi identities.

3.5.1 Reduction of the RR self-duality constraint

We start expanding the RR field $\hat{\mathbf{G}}$ on the internal basis polyforms (3.17). Recalling (3.4) and (3.92), this expansion naturally leads to forms of any degree in the 4d spacetime M_4:

$$2^{-1/2}\hat{\mathbf{G}} = (G_0^A + G_2^A + G_4^A)\omega_A - (\tilde{G}_{0A} + \tilde{G}_{2A} + \tilde{G}_{4A})\tilde{\omega}^A + (G_1^I + G_3^I)\alpha_I - (\tilde{G}_{1I} + \tilde{G}_{3I})\beta^I , \tag{3.95}$$

where G_p denotes a p–form on M_4 depending on the x^μ coordinates only. The $2^{-1/2}$ factor is introduced just for later convenience (concerning the relative normalization of the reduced RR and NSNS sectors). We also introduce the following auxiliary expansion

$$2^{-1/2} e^{\hat{B}}\hat{\mathbf{G}} = e^b\big(K^A\omega_A - \tilde{K}_A\tilde{\omega}^A + L^I\alpha_I - \tilde{L}_I\beta^I\big) \; , \tag{3.96}$$

so that (the indices are understood and B is along M_4): $L = G_1 + (G_3 + B\wedge G_1)$, $K = G_0 + (G_2 + BG_0) + (G_4 + B\wedge G_2 + \frac{1}{2}B\wedge BG_0)$, and analogously for $\tilde{K}$ and $\tilde{L}$.

We now reduce the self-duality constraint (3.93). Substituting (3.96), this can be rewritten as

$$K^A\omega_A - \tilde{K}_A\tilde{\omega}^A + L^I\alpha_I - \tilde{L}_I\beta^I = -*\lambda(K^A)*_b\omega_A + *\lambda(\tilde{K}_A)*_b\tilde{\omega}^A - *\lambda(L^I)*_b\alpha_I + *\lambda(\tilde{L}_I)*_b\beta^I \tag{3.97}$$

where (A.11) has been used, as well as the definition (3.43) of the 6d operator $*_b$. Taking the Mukai pairings with the basis forms, integrating over M_6 and using the results for the action of $*_b$ derived in subsection 3.3.4, from (3.97) we get the 4d relations

$$\tilde{K}_A = -\text{Im}\mathcal{N}_{AB} * \lambda(K^B) + \text{Re}\mathcal{N}_{AB}K^B \tag{3.98}$$

$$\tilde{L}_I = -\text{Im}\mathcal{M}_{IJ} * \lambda(L^J) + \text{Re}\mathcal{M}_{IJ}L^J \; . \tag{3.99}$$

In order to keep the notation of the forthcoming expressions as compact as possible, we use the symplectic notation already employed above in this chapter, and we define the symplectic vectors

$$G_k^{\mathbb{A}} = \begin{pmatrix} G_k^A \\ \tilde{G}_{kA} \end{pmatrix} \quad \text{for } k = 0,2,4 \qquad \text{and} \qquad G_k^{\mathbb{I}} = \begin{pmatrix} G_k^I \\ \tilde{G}_{kI} \end{pmatrix} \quad \text{for } k = 1,3 \; .$$

Then, separating the different form degrees and rescaling the 4d metric as done in (3.74) when discussing the reduction of the NSNS sector, eq. (3.98) yields the following relations among the 4d fields

$$\tilde{G}_{2A} + B\tilde{G}_{0A} \;=\; \mathrm{Im}\mathcal{N}_{AB} * (G_2^B + BG_0^B) \;+\; \mathrm{Re}\mathcal{N}_{AB}(G_2^B + BG_0^B)\,, \tag{3.100}$$

$$G_4^{\mathbb{A}} + B\wedge G_2^{\mathbb{A}} + \frac{1}{2}B\wedge B\,G_0^{\mathbb{A}} \;=\; e^{4\varphi}\mathbb{N}^{\mathbb{A}}{}_{\mathbb{B}}\,G_0^{\mathbb{B}} * 1\,, \tag{3.101}$$

while from (3.99) we obtain

$$G_3^{\mathbb{I}} + B\wedge G_1^{\mathbb{I}} = -e^{2\varphi}\mathbb{M}^{\mathbb{I}}{}_{\mathbb{J}} * G_1^{\mathbb{J}}\,. \tag{3.102}$$

Eqs. (3.100)–(3.102) represent the 4d remains of the 10d RR self-duality condition (3.93).

3.5.2 Reduction of the equations of motion / Bianchi identities

We now pass to reduce eq. (3.94). This will provide a set of Bianchi identities for the 4d fields as well as the 4d EoM, once the relations (3.100)–(3.102) imposed by the reduced 10d self-duality will be used to eliminate the redundant 4d fields. Starting from the expansion (3.95) for $\hat{\mathbf{G}}$, we use the ansatz (3.65) to evaluate $d_{H^{\mathfrak{fl}}}$ on the internal basis of forms,[19] and then separate the different components by acting with $\int_{M_6}\langle\Sigma_\pm,\,\cdot\,\rangle$. The following set of four-dimensional equations is obtained (recall that $\widetilde{\mathbb{Q}}$ is related to $\mathbb{Q}$ as in (3.66)):

$$\mathbb{Q}^{\mathbb{I}}{}_{\mathbb{A}}G_0^{\mathbb{A}} \;=\; 0 \tag{3.103}$$

$$dG_0^{\mathbb{A}} - \widetilde{\mathbb{Q}}^{\mathbb{A}}{}_{\mathbb{I}}G_1^{\mathbb{I}} \;=\; 0 \tag{3.104}$$

$$dG_1^{\mathbb{I}} + \mathbb{Q}^{\mathbb{I}}{}_{\mathbb{A}}G_2^{\mathbb{A}} \;=\; 0 \tag{3.105}$$

$$dG_2^{\mathbb{A}} - \widetilde{\mathbb{Q}}^{\mathbb{A}}{}_{\mathbb{I}}G_3^{\mathbb{I}} \;=\; 0 \tag{3.106}$$

$$dG_3^{\mathbb{I}} + \mathbb{Q}^{\mathbb{I}}{}_{\mathbb{A}}G_4^{\mathbb{A}} \;=\; 0\,. \tag{3.107}$$

We immediately rewrite eq. (3.107): using (3.101) and (3.102) to eliminate $G_4^{\mathbb{A}}$ and $G_3^{\mathbb{I}}$, also employing (3.103), (3.105) to simplify the expression, we obtain

$$-d\big(e^{2\varphi}\mathbb{M}^{\mathbb{I}}{}_{\mathbb{J}} * G_1^{\mathbb{J}}\big) - dB\wedge G_1^{\mathbb{I}} + e^{4\varphi}(\mathbb{Q}\mathbb{N})^{\mathbb{I}}{}_{\mathbb{A}}G_0^{\mathbb{A}} * 1 = 0\,. \tag{3.108}$$

We also need to reduce the ten dimensional EoM (3.9) for the NS 2–form $\hat{B}$, which receives contributions from both the NSNS and the RR sectors. This is an 8–form equation, and here we consider just its piece with 2 legs along M_4 and six legs along M_6.[20] Furthermore, recalling (3.93), we rewrite the RR piece of (5.32) as

$$[\hat{\mathbf{F}}\wedge *\hat{\mathbf{F}}]_8 \;=\; [\hat{\mathbf{F}}\wedge\lambda(\hat{\mathbf{F}})]_8 \;=\; [\hat{\mathbf{G}}\wedge\lambda(\hat{\mathbf{G}})]_8\,. \tag{3.109}$$

[19] Due to a possible moduli dependence, the basis forms might not be closed even with respect to the 4d exterior derivative. However, recall that in subsection 3.3 we assumed that their derivatives with respect to the moduli vanish in the integrated Mukai pairing.

[20] See subsection 5.4.3 for a complete analysis of this equation in a concrete example of compactification.

Using the expansions in subsection 3.5.1, and taking the integral over M_6, we arrive at the 4d equation

$$\frac{1}{2}d(e^{-4\varphi} * dB) \, + \, G_0^A\tilde{G}_{2A} - \tilde{G}_{0A}G_2^A + \tilde{G}_{1I} \wedge G_1^I = 0\,, \tag{3.110}$$

where the 4d metric has been Weyl rescaled as in (3.74). This corresponds to the EoM for the 2–form B in the reduced 4d theory.

3.5.3 $p_I^A = 0 = q^{IA}$ case. SU(3) structure

We pursue the analysis by considering first the simpler case in which $p_I^A = 0 = q^{IA}$, namely, recalling (3.64), $\mathbb{Q}^{\mathbb{I}A} = 0$. As we will discuss below, this is particularly relevant for dimensional reductions on SU(3) structure manifolds.

We start by identifying and solving a set of Bianchi identities in the system of equations (3.103)–(3.107). From the components of (3.104) with upper A–indices we see that

$$G_0^A = const := m_{\rm RR}^A \tag{3.111}$$

(these parameters are associated with RR fluxes). Then (3.103) are just constraints among constants:

$$m_A^I m_{\rm RR}^A \,=\, 0 \,=\, e_{IA}m_{\rm RR}^A\,. \tag{3.112}$$

The upper components of (3.106) are solved by $G_2^A = dA^A$, defining the 1–forms A^A, corresponding to (electric) gauge potentials of the 4d theory. Then (3.105) are solved by $G_1^I = d\xi^I - m_A^I A^A$ and $\tilde{G}_{1I} = d\tilde{\xi}_I - e_{IA}A^A$, where ξ^I and $\tilde{\xi}_I$ are scalar fields. Finally, using also the quadratic constraint (3.62), from the lower components of (3.104) we find that

$$\tilde{G}_{0A} = e_{{\rm RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A^I\,, \tag{3.113}$$

where $e_{{\rm RR}A}$ are constant RR flux parameters.

At this point the only equations we still have to deal with are eq. (3.107) and the lower components of (3.106). Employing the relations descending from the RR self-duality constraint, these will now be interpreted as EoM for the fields $\tilde{\xi}_I, \xi^I$ and A^A. Eq. (3.107) has already been treated along these lines, yielding eq. (3.108), which we take as the EoM for the scalars $\xi^I, \tilde{\xi}_I$. Concerning the EoM for A^A, we use (3.100) and (3.102) to eliminate $\tilde{G}_{2A}, G_3^{\mathbb{I}}$ in the lower components of (3.106), and we get:

$$d[{\rm Im}\mathcal{N}_{AB} * F^B + {\rm Re}\mathcal{N}_{AB}F^B] - \tilde{G}_{0A}dB - e^{2\varphi}(\mathbb{Q}^T\widetilde{\mathbb{M}})_{A\mathbb{I}} * G_1^{\mathbb{I}} = 0\;,$$

where we introduced the modified field strengths F^A containing the 2–form B:

$$F^A \,:=\, G_2^A + G_0^A B \,=\, dA^A + m_{\rm RR}^A B\;. \tag{3.114}$$

One can now check that precisely the equations of motion just obtained, together with the EoM for B given in (5.35), can be derived from the 4d action[21]

$$\begin{aligned} S^{(4)}_{\rm RR} \;=\; \int_{M_4} \Big[\; & \frac{1}{2}{\rm Im}\mathcal{N}_{AB}F^A\wedge *F^B + \frac{1}{2}{\rm Re}\mathcal{N}_{AB}F^A\wedge F^B + \frac{e^{2\varphi}}{2}\widetilde{\mathbb{M}}_{\mathbb{IJ}}D\xi^{\mathbb{I}}\wedge *D\xi^{\mathbb{J}} \\ & + \frac{1}{2}dB\wedge\big[\xi^{\mathbb{I}}\mathbb{S}_{-\mathbb{IJ}}D\xi^{\mathbb{J}} + (2e_{{\rm RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A^I)A^A\big] - \frac{1}{2}m^A_{\rm RR}e_{{\rm RR}A}B\wedge B \\ & - V_{\rm RR} * 1 \;\Big] , \end{aligned} \tag{3.115}$$

where $\xi^{\mathbb{I}} = \binom{\xi^I}{\tilde{\xi}_I}$, and we have introduced the covariant derivatives

$$D\xi^I \equiv G_1^I = d\xi^I - m_A^I A^A \qquad , \qquad D\tilde{\xi}_I \equiv \tilde{G}_{1I} = d\tilde{\xi}_I - e_{IA}A^A \; . \tag{3.116}$$

Furthermore we defined

$$V_{\rm RR} = -\frac{e^{4\varphi}}{2}G_0^{\mathbb{A}}\widetilde{\mathbb{N}}_{\mathbb{AB}}G_0^{\mathbb{B}} \; , \tag{3.117}$$

where $G_0^A, \tilde{G}_{0A}$ are given in (3.111), (3.113). This is the contribution of the RR sector to the scalar potential of the reduced theory.[22] Notice that it is non-negative.

Since it yields the correct reduced EoM, we interpret the action (3.115) as the suitable one for the reduced type IIA RR sector. To check that $S^{(4)}_{\rm RR}$ reproduces the EoM written above, one needs the consistency constraints (3.67), as well as conditions (3.112).

As mentioned above, the present setting with $p_I^A = 0 = q^{IA}$ is relevant for SU(3) structure compactifications, once the specific basis of forms (of pure degree) described in the first part of subsection 3.3.5 is adopted. In this context, the parameters e_{IA}, m_A^I arise from relations (3.60)–(3.62). In fact, the action (3.115), which has the features of an $N = 2$ gauged supergravity, is in agreement with all the studies done for $N = 2$ type IIA compactifications on SU(3) structures [47, 89, 90, 45, 49, 92, 93].

It can be useful to see how several particular cases already described in the literature can be recovered. Let's take $m^A_{\rm RR} = 0$ first. In this case the 2–form B can be dualized to a scalar a. The terms in the action (3.115) containing dB, together with the kinetic term $-\frac{1}{4}\int e^{-4\varphi}dB\wedge *dB$ coming from the NSNS sector (see eq. (3.77)), are then replaced by

$$S_{\rm dual} = \int_{M_4} -\frac{e^{4\varphi}}{4}\big(Da - \xi^{\mathbb{I}}\mathbb{S}_{-\mathbb{IJ}}D\xi^{\mathbb{J}}\big)\wedge *\big(Da - \xi^{\mathbb{I}}\mathbb{S}_{-\mathbb{IJ}}D\xi^{\mathbb{J}}\big) \; , \tag{3.118}$$

where

$$Da = da - (2e_{{\rm RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A^I)A^A \; . \tag{3.119}$$

The term (3.118) contributes to define a quaternionic σ–model for the hyperscalars analogous to the one featured by the standard $N = 2$ effective action derived from Calabi–Yau

[21] The term $\frac{1}{2}d(e^{-4\varphi} * dB)$ in (5.35) is indeed derived from the piece of the 4d action associated with the reduction of the NSNS sector, see eq. (3.77). This also fixes the overall normalization of $S^{(4)}_{\rm RR}$.

[22] We remark that (3.117) contains a term $-\frac{e^{4\varphi}}{2}\binom{m_{\rm RR}}{e_{\rm RR}}^T\widetilde{\mathbb{N}}\binom{m_{\rm RR}}{e_{\rm RR}}$ which does not depend on the RR scalars ξ^I, $\tilde{\xi}_I$ and indeed does not contribute to their EoM. We have added it as the natural completion of the expression for $V_{\rm RR}$ directly reconstructed from these EoM. The correctness of (3.117) can also be verified studying the reduced Einstein equations. We will perform this study for a particular case in chapter 5.

dimensional reductions, reviewed in section 3.2. More specifically, the (RR sector of the) $N = 2$ supergravity obtained from Calabi-Yau compactifications with no fluxes is recovered by setting all the charges $e_{IA}, m^I_A, e_{\mathrm{RR}A}$ (as well as m_{RR}) to zero. This is consistent with the fact that all the basis forms (3.17) are then closed. Allowing for non-vanishing e_{I0}, m^I_0 yields the Calabi-Yau effective action with NS fluxes described in [82].[23]

Furthermore, taking just $e_{0A} \neq 0$, we find agreement with the results of [47] for type IIA reductions on half-flat manifolds (the parameter e_{00} being associated with an NS flux).

Finally, let's consider nonvanishing m^A_{RR}. These parameters generate some couplings for the NS 2–form B, including a mass term: then B cannot be dualized to an axion [82, 45]. If $m^I_A = 0 = e_{IA}$, eq. (3.115) precisely reproduces the RR part of the action derived in [82] for Calabi-Yau compactifications of type IIA with RR fluxes.

3.5.4 General case

Let us now consider the general charge matrix $\mathbb{Q}$ defined in (3.64). An $N = 2$ lagrangian including the same set of charges contained in $\mathbb{Q}$ was derived in [50] using purely 4d supergravity techniques, and building on results in [89, 97, 98]. Having the $N = 2$ effective theory arising from Calabi-Yau compactifications as a starting point (recall the outcome of section 3.2), the authors of [50] first deformed it by implementing a standard electric gauging of the quaternionic isometries (see subsection 4.1.2 for more details), and subsequently performed a dualization of a subset of the RR axions to antisymmetric 2–tensors in order to include the magnetic charges. In section 3.4 we found consistency between this procedure and the dimensional reduction of the NSNS sector, obtaining in particular eq. (3.87) for the NSNS scalar potential. Here we approach the same question for the RR sector. As in the previous subsection, we construct a 4d action via the analysis of the reduced RR EoM/Bianchi identities. A set of 2–form potentials, beside the vector and scalar fields, will emerge directly from the solution of the selected 4d Bianchi identities. The outcome of our analysis is summarized in Table 3.2.

Even if for a general $\mathbb{Q}$ all the equations (3.103)–(3.107) are symplectically covariant, we will anyway break this symmetry in order to establish a set of EoM associated with a 4d action written in terms of electric vectors only. For this task we introduce appropriate projectors with which we will act on eqs. (3.103)–(3.107). In the following computations, several technical steps are close to the ones employed in [50] for the dualization of the RR axions to antisymmetric 2–tensors.

We start splitting the charge matrix $\mathbb{Q}^{\mathbb{I}}{}_{\mathbb{A}}$ in the following $(2b^-+2)\times(b^++1)$ submatrices:

$$U^{\mathbb{I}}{}_A := \mathbb{Q}^{\mathbb{I}}{}_A = \begin{pmatrix} m^I{}_A \\ e_{IA} \end{pmatrix} \qquad , \qquad V^{\mathbb{I}A} := \mathbb{Q}^{\mathbb{I}A} = \begin{pmatrix} q^{IA} \\ p_I{}^A \end{pmatrix} . \tag{3.120}$$

With respect to the gauge vectors with upper indices A^A that we are going to define below, the elements of U are electric charges, while V contains magnetic charges.[24]

[23] With respect to [82], we have a sign difference in the definition of the RR scalars $\tilde{\xi}$.

[24] The latter matrix should not be confused with the scalar potential of the $N = 2$ theory, also called V.

Equation	yields:	Equation	yields:
(3.103) (3.104)	constraints among charges expression for $G_0^{\mathbb{A}}$	(3.106)	Bianchi for $G_2^A \to$ def. A^A EoM for A^A
(3.105)	Bianchi for $\widehat{G}_1^{\mathbb{I}} \to$ def. $\widehat{\xi}^{\mathbb{I}}$ EoM for $\check{C}_{2A}$	(3.107) (rewr. as (3.108))	Bianchi for $\check{G}_{3A} \to$ def. $\check{C}_{2A}$ EoM for $\widehat{\xi}^{\mathbb{I}}$

Table 3.2: Analysis of the reduced RR equations for a general charge matrix $\mathbb{Q}$.

As in [50], we adopt the working assumptions $b^+ \leq b^-$, and that the matrix U has maximal rank $b^+ + 1$. Then we introduce the matrix $\widetilde{U}^A{}_{\mathbb{I}}$, defined by

$$\widetilde{U}^A{}_{\mathbb{I}} U^{\mathbb{I}}{}_B = \delta^A_B \qquad , \qquad U^{\mathbb{I}}{}_A \widetilde{U}^A{}_{\mathbb{J}} = (\mathbb{P}_{\neq 0})^{\mathbb{I}}{}_{\mathbb{J}} \, , \tag{3.121}$$

$\mathbb{P}_{\neq 0}$ being the projector on the subspace corresponding to the non-vanishing minor of $U^{\mathbb{I}}{}_A$. We also define the orthogonal projector $(\mathbb{P}_0)^{\mathbb{I}}{}_{\mathbb{J}} \equiv \delta^{\mathbb{I}}_{\mathbb{J}} - (\mathbb{P}_{\neq 0})^{\mathbb{I}}{}_{\mathbb{J}}$.

An identity we will need is

$$V \;=\; V U^T \widetilde{U}^T \;=\; U V^T \widetilde{U}^T \, , \tag{3.122}$$

which is obtained recalling the first of (3.121) and then the first of (3.67). Notice that $(\widetilde{U}V)^{AB}$ is then symmetric.

Bianchi identities and fundamental 4d fields

With respect to the analysis of subsection 3.5.3, the presence of the p_I^A and q^{IA} charges makes less trivial the identification and the solution of a set of Bianchi identities for the fundamental 4d fields. For this purpose we make use of the matrices defined here above. As we will see, a set of 2–form degrees of freedom will be required.

We start introducing a set of scalar fields. Define [50]:

$$\check{G}_1^A := \widetilde{U}^A{}_{\mathbb{I}} G_1^{\mathbb{I}} \qquad , \qquad \widehat{G}_1^{\mathbb{I}} := \mathbb{P}_0{}^{\mathbb{I}}{}_{\mathbb{J}} G_1^{\mathbb{J}} \, , \tag{3.123}$$

so that

$$G_1^{\mathbb{I}} = U^{\mathbb{I}}{}_A \check{G}_1^A + \widehat{G}_1^{\mathbb{I}} \, . \tag{3.124}$$

We want to keep the $\widehat{G}_1^{\mathbb{I}}$, while we will deal with $\check{G}_1^A$ in the next paragraph. We act with $\mathbb{P}_0$ on eq. (3.105) and we observe that $\mathbb{P}_0\mathbb{Q} = 0$, due to the definition of $\mathbb{P}_0$ below eq. (3.121) and to identity (3.122). Then we get

$$d\widehat{G}_1^{\mathbb{I}} = 0 \qquad \Rightarrow \qquad \widehat{G}_1^{\mathbb{I}} = d\widehat{\xi}^{\mathbb{I}} \, , \tag{3.125}$$

with $\widehat{\xi}^{\mathbb{I}}$ being a set of real scalars satisfying $(\mathbb{P}_{\neq 0})^{\mathbb{I}}{}_{\mathbb{J}} \widehat{\xi}^{\mathbb{J}} = 0$ and corresponding therefore to $\mathrm{rank}(\mathbb{P}_0) = 2(b^- + 1) - (b^+ + 1)$ degrees of freedom.

Recalling (3.124) and (3.67), eq. (3.104) can then be written as

$$dG_0^{\mathbb{A}} - \widetilde{\mathbb{Q}}^{\mathbb{A}}{}_{\mathbb{I}} d\widehat{\xi}^{\mathbb{I}} = 0 \qquad \Rightarrow \qquad G_0^{\mathbb{A}} \;=\; c^{\mathbb{A}} + \widetilde{\mathbb{Q}}^{\mathbb{A}}{}_{\mathbb{I}} \widehat{\xi}^{\mathbb{I}} \, , \tag{3.126}$$

with $c^{\mathbb{A}} = \binom{m^A_{\rm RR}}{e_{{\rm RR}A}}$ a vector of constant charges, associated with general RR background fluxes. Again employing (3.67), eq. (3.103) translates in the following consistency condition among the different parameters [50]:

$$\mathbb{Q}^{\mathbb{I}}{}_{\mathbb{A}}c^{\mathbb{A}} = 0 \,. \tag{3.127}$$

Next we define the $b^+ + 1$ combinations

$$\breve{G}_{3A} := -(U^T\mathbb{S}_-)_{A\mathbb{I}}G_3^{\mathbb{I}} \,. \tag{3.128}$$

Multiplying eq. (3.107) by $U^T\mathbb{S}_-$ from the left, and recalling (3.67), we get

$$d\breve{G}_{3A} = 0 \,, \tag{3.129}$$

which we choose to solve as

$$\breve{G}_{3A} = d(\breve{C}_{2A} + \zeta_A B) \,, \tag{3.130}$$

where the 2–forms $\breve{C}_{2A}$ are new fields, B is the NS 2–form and ζ_A is a combination of the scalars $\widehat{\xi}^{\,\mathbb{I}}$ to be specified below. The 2–forms $\breve{C}_{2A}$ will be dynamical fields of our eventual 4d action.

Let's finally turn to gauge vectors. Here we choose to define fundamental vector potentials with upper indices only, so we keep all the G_2^A and dualize all the $\tilde{G}_{2A}$, breaking in this way the symplectic structure of the 2–forms $G_2^{\mathbb{A}}$. The components of (3.106) with upper indices can be read as Bianchi identities for G_2^A, while the dualization of the lower components will provide the EoM for the associated vector potentials. First we look at the Bianchi identities, which read

$$dG_2^A + (V^T\mathbb{S}_-)^A{}_{\mathbb{I}}G_3^{\mathbb{I}} = 0 \,. \tag{3.131}$$

Using (3.122) and (3.128), we rewrite this as $dG_2^A - (\widetilde{U}V)^{AB}\breve{G}_{3B} = 0$. Taking (3.130) into account, this last equation is solved introducing a set of vector potentials A^A:

$$G_2^A \;=\; dA^A + (\widetilde{U}V)^{AB}(\breve{C}_{2B} + \zeta_B B) \,. \tag{3.132}$$

We now fix the ζ_A introduced in (3.130). We choose

$$\zeta_A \equiv (U^T\mathbb{S}_-)_{A\mathbb{I}}\widehat{\xi}^{\,\mathbb{I}} \,,$$

in such a way that the $b^+ + 1$ two–forms

$$F^A \;:=\; G_2^A + G_0^A B \;=\; dA^A + (\widetilde{U}V)^{AB}\breve{C}_{2B} + m^A_{\rm RR}B \tag{3.133}$$

contain vectors and 2–form potentials only (to obtain this expression recall (3.122) and (3.126)). Thus the F^A are a set of field strengths for the vector potentials A^A, modified by the presence of the 2–forms B and $\breve{C}_2^A$. This modification of the field strengths (3.114) is required by the non-vanishing magnetic $V^{\mathbb{I}A}$ charges, and guarantees compatibility with the formalism of $N = 2$ supergravity with tensor multiplets developed in [97, 98, 106, 50].

Notice that one could also express the $\breve{C}_{2A}$ by introducing a redundant set of $b^- + 1$ two-forms $C_2^{\mathbb{I}} = \binom{C_2^I}{\tilde{C}_{2I}}$ and writing, in analogy with (3.128),

$$\breve{C}_{2A} = -(U^T \mathbb{S}_-)_{A\mathbb{I}} C_2^{\mathbb{I}} = C_2^I e_{IA} - \tilde{C}_{2I} m_A^I .$$

Then, recalling (3.122), eq. (3.133) would become

$$F_2^A = dA^A + C_2^I p_I^A - \tilde{C}_{2I} q^{IA} + m_{\mathrm{RR}}^A B \,.$$

However, the propagating degrees of freedom would be just the combinations of C_2^I and $\tilde{C}_{2I}$ equivalent to $\breve{C}_{2A}$ [46, 50]. Analogously, as in subsection 3.5.3 we could introduce a symplectic vector of $b^- + 1$ scalars $\xi^{\mathbb{I}} = \binom{\xi^I}{\tilde{\xi}_I}$ such that $\widehat{\xi}^{\,\mathbb{I}} = \mathbb{P}_0{}^{\mathbb{I}}{}_{\mathbb{J}} \xi^{\mathbb{J}}$. Then the result of (3.126) would read:

$$G_0^A = m_{\mathrm{RR}}^A + \xi^I p_I^A - \tilde{\xi}_I q^{IA} \qquad \text{and} \qquad \tilde{G}_{0A} = e_{\mathrm{RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A^I \,. \tag{3.134}$$

However, in these expressions the $\xi^{\mathbb{I}}$ actually appear only in the combinations corresponding to the $\widehat{\xi}^{\,\mathbb{I}}$.

To summarize, the outcome of this paragraph is a set of fundamental degrees of freedom $\widehat{\xi}^{\,\mathbb{I}}$, $\breve{C}_{2A}$ and A^A, related to $\widehat{G}_1^{\mathbb{I}}$, $\breve{G}_{3A}$ and G_2^A as in (3.125), (3.130) and (3.132). Furthermore in (3.133) we defined the proper modified field strengths for A^A, and in (3.126) we expressed $G_0^{\mathbb{A}}$ as a combination of scalars and charges. The charges have to satisfy conditions (3.127).

Equations of motion

We now establish the EoM associated with the identified fundamental 4d fields. For this purpose, we study the projections of eqs. (3.103)–(3.107), which are independent of the ones considered in the above study of the Bianchi identities.

The EoM for the vector potentials A^A are obtained from the lower components of (3.106). Using the duality relation (3.100) to eliminate $\tilde{G}_{2A}$, recalling expressions (3.128), (3.130) as well as the definition of F^A in (3.133), and noticing that $\breve{G}_{0A} = e_{\mathrm{RR}A} + \zeta_A$, one arrives at

$$d\big(\mathrm{Im}\mathcal{N}_{AB} * F^B + \mathrm{Re}\mathcal{N}_{AB} F^B + \breve{C}_{2A} - e_{\mathrm{RR}A} B\big) = 0 \,. \tag{3.135}$$

Next we find an expression for the $\breve{G}_1^A$ defined in (3.123). Multiplying relation (3.102) by $U^T \mathbb{S}_-$ from the left, substituting (3.124) in it and recalling (3.67), (3.128) as well as the expressions for $\breve{G}_{3A}$, $\widehat{G}_1^{\mathbb{I}}$ and ζ_A obtained in the study of the Bianchi identities, we get

$$\breve{G}_1^A = -\Delta^{-1\,AB} \big[* d\breve{C}_{2B} + \zeta_B * dB + e^{2\varphi} (U^T \widetilde{\mathbb{M}})_{B\mathbb{I}} \, d\widehat{\xi}^{\,\mathbb{I}} \big] \,, \tag{3.136}$$

where we introduced the symmetric matrix [50]

$$\Delta_{AB} := e^{2\varphi} (U^T)_A{}^{\mathbb{I}} \, \widetilde{\mathbb{M}}_{\mathbb{I}\mathbb{J}} \, U^{\mathbb{J}}{}_B \,.$$

In order to get the EoM associated with $\breve{C}_{2A}$, we start acting with $\widetilde{U}$ from the left on eq. (3.105), and exploiting (3.100) to eliminate $\tilde{G}_{2A}$. After some steps involving the expressions arising from the Bianchi identites above, we obtain

$$d\breve{G}_1^A + dA^A + (\widetilde{U}V)^{AB} \big[\mathrm{Im}\mathcal{N}_{BC} * F^C + \mathrm{Re}\mathcal{N}_{BC} F^C + \breve{C}_{2B} - e_{\mathrm{RR}B} B\big] \;=\; 0 \,, \tag{3.137}$$

where $\breve{G}_1^A$ should be read as (3.136).

The EoM for the scalars $\widehat{\xi}^{\mathbb{I}}$ are obtained substituting (3.124) in (3.108), and lowering the symplectic index with $\mathbb{S}_-$:

$$-\,d\big[\,e^{2\varphi}\widetilde{\mathbb{M}}_{\mathbb{IJ}}*(d\widehat{\xi}^{\mathbb{J}}+U^{\mathbb{J}}{}_A\breve{G}_1^A)\,\big]\;+\;dB\wedge[\,(\mathbb{S}_-U)_{\mathbb{I}A}\breve{G}_1^A+(\mathbb{S}_-d\widehat{\xi}\,)_{\mathbb{I}}\,]$$
$$-\,e^{4\varphi}(\mathbb{S}_-\mathbb{Q}\mathbb{N})_{\mathbb{I}\mathbb{A}}G_0^{\mathbb{A}}*1\;=\;0\,,\quad(3.138)$$

where again expression (3.136) for $\breve{G}_1^A$ should be substituted. Once this is done,[25] the piece of (3.138) associated with a kinetic term for the $\widehat{\xi}^{\mathbb{I}}$ reads $-d\big(\widetilde{\Delta}_{\mathbb{IJ}}*d\widehat{\xi}^{\mathbb{J}}\big)$, with [50]:

$$\widetilde{\Delta}_{\mathbb{IJ}}=e^{2\varphi}\big(\widetilde{\mathbb{M}}-e^{2\varphi}\widetilde{\mathbb{M}}U\Delta^{-1}U^T\widetilde{\mathbb{M}}\big)_{\mathbb{IJ}}\,.$$

Finally, we rewrite the EoM for the four dimensional B–field given in (5.35) by substituting the expressions for the fundamental 4d fields. After some steps we arrive at

$$\frac{1}{2}d(e^{-4\varphi}*dB)\;+\;m^A_{\rm RR}\big(\mathrm{Im}\mathcal{N}_{AB}*F^B+\mathrm{Re}\mathcal{N}_{AB}F^B\big)\;-\;e_{{\rm RR}A}F^A$$
$$-\,\frac{1}{2}d\widehat{\xi}^{\mathbb{I}}\mathbb{S}_{-\mathbb{IJ}}d\widehat{\xi}^{\mathbb{J}}\;+\;d(\zeta_A\breve{G}_1^A)\;\;=\;\;0\,.\quad(3.139)$$

<u>*4d action for the reduced RR sector*</u>

We can now reconstruct the action yielding the EoM (3.135), (3.137), (3.138) and (3.139), respectively associated with the fields A^A, $\breve{C}_{2A}$, $\widehat{\xi}^{\mathbb{I}}$ and B (for this last remind footnote 21). We find:

$$\begin{aligned}S^{(4)}_{\rm RR}=\int_{M_4}\Big\{\;&\frac{1}{2}\mathrm{Im}\mathcal{N}_{AB}F^A\wedge*F^B+\frac{1}{2}\mathrm{Re}\mathcal{N}_{AB}F^A\wedge F^B+\frac{1}{2}\widetilde{\Delta}_{\mathbb{IJ}}d\widehat{\xi}^{\mathbb{I}}\wedge*d\widehat{\xi}^{\mathbb{J}}\\&+\frac{1}{2}\Delta^{-1AB}(d\breve{C}_{2A}+\zeta_AdB)\wedge*(d\breve{C}_{2B}+\zeta_BdB)\\&+(d\breve{C}_{2A}+\zeta_AdB)\wedge(e^{2\varphi}\Delta^{-1}U^T\widetilde{\mathbb{M}})^A{}_{\mathbb{I}}d\widehat{\xi}^{\mathbb{I}}\;+\;\frac{1}{2}dB\wedge\widehat{\xi}^{\mathbb{I}}\,\mathbb{S}_{-\mathbb{IJ}}d\widehat{\xi}^{\mathbb{J}}\\&+(\breve{C}_{2A}-e_{{\rm RR}A}B)\wedge\big[dA^A+\frac{1}{2}(\widetilde{U}V)^{AB}\breve{C}_{2B}+\frac{1}{2}m^A_{\rm RR}B\big]\;-\;V_{\rm RR}*1\Big\}.\end{aligned}\quad(3.140)$$

In order to derive the EoM, constraint (3.127) (written in the form $Um_{\rm RR}+Ve_{\rm RR}=0$) should be recalled. The RR contribution to the 4d scalar potential is defined as in (3.117):

$$V_{\rm RR}=-\frac{e^{4\varphi}}{2}G_0^{\mathbb{A}}\widetilde{\mathbb{N}}_{\mathbb{AB}}G_0^{\mathbb{B}}\,,\quad(3.141)$$

[25]Taking into account the explicit expression for $\breve{G}_1^A$, one can see that the b^++1 linear combinations of the equations (3.138) obtained via multiplication by $(U^T)_A{}^{\mathbb{I}}$ vanish identically, as it should: we have already exploited these combinations to write (3.129).

but in the present general case expression (3.126) for $G_0^{\mathbb{A}}$ should be used (alternatively, expression (3.134) can be adopted). Recalling the results of subsection 3.3.4, eq. (3.141) can be derived from the geometric formula

$$V_{\rm RR} \;=\; \frac{e^{4\varphi}}{4}\int_{M_6}\langle G, *_b G\rangle\;, \tag{3.142}$$

where

$$G := \sqrt{2}(G_0^A\omega_A - \tilde{G}_{0A}\tilde{\omega}^A) \tag{3.143}$$

is the formal sum of RR field strengths with all the indices along M_6, corresponding to the purely internal part of the RR field $\hat{\mathbf{G}}$, expanded as in (3.95). Notice that $V_{\rm RR}$ is non-negative.

The dimensionally reduced action (3.140) coincides with the one found in [50] using purely four dimensional $N = 2$ supergravity techniques. It contains topological as well as mass terms for the 2–forms B and $\breve{C}_{2A}$, with mass matrix:

$$M^2 = -\begin{pmatrix} m_{\rm RR}^T{\rm Im}\mathcal{N}m_{\rm RR} & m_{\rm RR}^T{\rm Im}\mathcal{N}\widetilde{U}V \\ (\widetilde{U}V)^T{\rm Im}\mathcal{N}m_{\rm RR} & (\widetilde{U}V)^T{\rm Im}\mathcal{N}\widetilde{U}V \end{pmatrix}\,.$$

3.6 Summary and discussion

In this chapter we established a general framework for dimensional reductions of type II supergravity leading to $N = 2$ supergravity in four dimensions. Specifically, we derived the complete 4d bosonic action associated with $N = 2$ flux compactifications of type IIA supergravity on SU(3)×SU(3) structures. This is given by the union of the results of section 3.4, where the reduction of the NSNS sector is established, and section 3.5, where we deal with the RR sector.

The dimensional reduction has been implemented by expanding the higher dimensional fields on a finite basis of differential (poly)forms. As we discussed, for the dimensional reduction to go through and yield an $N = 2$ supergravity in 4d, the basis forms need to satisfy a series of constraints. Though in general these constraints appear to be very restrictive, in chapter 5 we will exhibit a simple class of examples in which the expansion forms are well characterized, and the dimensional reduction goes through, even *consistently*.

The generalized geometry formalism described in the previous chapter played here a crucial role, especially in the study of the dimensional reduction of the NSNS sector, common to type IIA and type IIB supergravity. Building on the fact that the full internal NSNS sector is encoded in the two Spin(6,6) pure spinors $\Phi_\pm$ characterizing the SU(3)×SU(3) structure on $TM_6\oplus T^*M_6$, we established the relation between the kinetic terms for the 4d scalar fields describing the fluctuations of the internal metric and B-field, and the special Kähler metrics derived from the logarithm of the Hitchin functionals for even/odd pure spinors. An essential tool used in this analysis was the decomposition of the space of poly-forms in SU(3)×SU(3) representations, already employed in section 2.5. The same tool proved very useful for studying the action of the $*_b$ operator on the expansion forms. This allowed to derive an expression in terms of internal geometry for the period matrices $\mathcal{N}$

and $\mathcal{M}$ of the special Kähler spaces associated with even and odd pure spinor deformations. We recall that in type IIA compactifications the matrix $\mathcal{N}$ corresponds to the gauge kinetic matrix of the $N=2$ theory, the matrix $\mathcal{M}$ contributes to define the quaternionic metric for the hyperscalar kinetic terms, and finally both $\mathcal{N}$ and $\mathcal{M}$ appear in the scalar potential.

The application of the generalized geometry formalism also allowed to derive a geometric formula for the full 4d scalar potential $V = V_{\rm NS} + V_{\rm RR}$. By (3.85) and (3.142), this reads

$$\begin{aligned} V \; &= \; \frac{e^{4\varphi}}{4}\int \Big[\,\langle\, d_{H^{\rm fl}}\Phi_+, *_b(d_{H^{\rm fl}}\bar{\Phi}_+)\,\rangle \; + \; \langle\, d_{H^{\rm fl}}\Phi_-, *_b(d_{H^{\rm fl}}\bar{\Phi}_-)\,\rangle\,\Big] \\ &\quad - e^{4\varphi}\int \frac{\big|\langle d_{H^{\rm fl}}\Phi_+, \Phi_-\rangle\big|^2 \; + \; \big|\langle d_{H^{\rm fl}}\Phi_+, \bar{\Phi}_-\rangle\big|^2}{i\langle\Phi_\pm, \bar{\Phi}_\pm\rangle} \; + \; \frac{e^{4\varphi}}{4}\int\langle G, *_b G\rangle\,. \end{aligned}$$

Expanding the pure spinors as well as the internal RR field strengths in the basis polyforms, and integrating over the compact manifold, we recovered the symplectically invariant scalar potential derived in ref. [50]. The NSNS contribution to the potential is mirror symmetric under the exchange $\Phi_+ \leftrightarrow \Phi_-$, while we expect the type IIB RR contribution still read as the type IIA expression (3.142), modulo the substitution of the even internal RR field strengths G with the corresponding odd ones. We remark that a thorough understanding of this scalar potential may open new avenues for the study of non-supersymmetric string vacua on SU(3)×SU(3) structure manifolds. In [41] this program has been started, taking also into account a possible non-trivial warp factor, as well as the contributions to V arising from the addition of localized sources.

A complication arising in the derivation of the lower-domensional action is associated with the fact that the general system of fluxes we considered yields both electric and magnetic couplings at the 4d level. This issue is overcome by including a set of 2–forms in the 4d theory. As we will discuss further in the next chapter, the 4d action we obtain is consistent with the $N=2$ supergravity formalism with tensor multiplets, and coincides with the one that ref. [50] derived by starting from the Calabi-Yau 4d effective action, then gauging the quaternionic isometries, and finally dualizing a set of axions in order to introduce the magnetic charges. However, in our approach to the reduction of the RR sector we didn't need to perform any *a posteriori* dualization of scalars: reducing the RR EoM/Bianchi identities of democratic type IIA supergravity, we identified and solved a set of 4d Bianchi identities, already encoding the appropriate degrees of freedom. In particular, we selected physically propagating degrees of freedom only, and we avoided to introduce the magnetic counterparts of the electric vector potentials. Then we reconstructed the 4d action leading to the correct equations of motion, which were also deduced from the higher dimensional RR EoM/Bianchi identities.

While this chapter was mainly concerned with the dimensional reduction procedure, and in particular with the relations between the internal geometry and the features of the lower dimensional $N=2$ theory, in the next chapter we will rather concentrate on the formalism of 4d, $N=2$ supergravity, applied to the theory derived here.

Chapter 4

The 4d $N = 2$ supergravity picture, and the $N = 1$ vacuum conditions

In this chapter we clarify the consistency between the theory derived above via dimensional reduction and the formalism of gauged $N = 2$ supergravity including tensor multiplets. In particular, we focus on the quantities determining the gauging, and we spell out the fermionic shifts in the supersymmetry variations. Then we derive the $N = 1$ vacuum conditions within the 4d $N = 2$ theory, and we establish a precise matching with the pure spinor equations characterizing the $N = 1$ backgrounds at the ten dimensional level. We also perform a similar analysis by considering $N = 1$ truncations of the $N = 2$ action, and imposing the F-flatness and D-flatness conditions.

4.1 $N = 2$ structure of the 4d theory

While in the previous chapter we established the dimensional reduction of type II theory to 4d $N = 2$ supergravity with focus on the bosonic sector, in this chapter we turn to the fermionic side of these theories. In particular, we consider the supersymmetry transformation rules of the fermions in the 4d $N = 2$ supergravity. We derive the scalar parts of these transformations, known as the fermionic shifts, which encode important informations about the structure of the gauged $N = 2$ supergravity. For instance, as a consequence of supersymmetry, the algebraic sum of their squares fixes the scalar potential V. In the next section, starting from the fermionic shifts discussed here, we will establish the $N = 1$ vacuum conditions arising within the 4d $N = 2$ theory, and we will confront them with the $N = 1$ background conditions derived at the 10d level using the generalized geometry formalism [26, 27].

Along the way in this section, we clarify the $N = 2$ structure of the 4d theory, with focus on the quantities defining the gauging, namely the Killing vectors of the quaternionic isometries being gauged, and the associated Killing prepotentials. We also illustrate the necessity to include antisymmetric rank-2 tensors in the supergravity description, due to the fact that the general system of fluxes introduced in the previous chapter implies the simultaneous presence of electric and magnetic charges under the gauge fields appearing in the 4d action.

4.1.1 $N=2$ Killing prepotentials from the dimensional reduction

The $N=2$ fermionic shift we are most interested in is the one in the gravitino susy transformation, whose relevant part reads [76]

$$\delta\psi_{\mathcal{A}\mu} = \ldots + \nabla_\mu \varepsilon_{\mathcal{A}} - S_{\mathcal{A}\mathcal{B}}\gamma_\mu \varepsilon^{\mathcal{B}}, \tag{4.1}$$

where $\psi_{\mathcal{A}\mu}$ $(\mathcal{A},\mathcal{B}=1,2)$ are the $N=2$ gravitini,[1] and $\varepsilon_{\mathcal{A}}$, $\varepsilon^{\mathcal{B}}$ are the 4d $N=2$ susy parameters. According to a standard notation in 4d $N=2$ supergravity, lower indices on Spin(3,1) spinors denote positive chirality, while upper indices refer to negative chirality. The spinors $\varepsilon^1,\varepsilon^2$ are related to $\varepsilon_1,\varepsilon_2$ by charge conjugation, which in our conventions just amounts to complex conjugation (see appendix A.3.2). Moreover, in (4.1) ∇_μ is the usual 4d spacetime covariant derivative for $\mathrm{Spin}(3,1)$ spinors. Both ∇_μ and the $\mathrm{Cliff}(3,1)$ gamma matrix γ_μ are defined with respect to the 4d Einstein frame metric, introduced by the Weyl rescaling (3.74). Finally, $S_{\mathcal{A}\mathcal{B}}$ is the gravitino fermionic shift, also known as the gravitino mass matrix since it appears as a gravitino mass term in the $N=2$ lagrangian. It contains the triplet of $N=2$ Killing prepotentials $\mathcal{P}^x$, $x=1,2,3$, which, as we are going to see in next subsection, are the building blocks of gaugings. Indeed, its general form is

$$S_{\mathcal{A}\mathcal{B}} \;=\; \frac{i}{2}e^{\frac{K_V}{2}}(\sigma_x)_{\mathcal{A}}^{\;\;\mathcal{C}}\epsilon_{\mathcal{B}\mathcal{C}}\mathcal{P}^x \;=\; \frac{i}{2}e^{\frac{K_V}{2}}\begin{pmatrix} \mathcal{P}^1 - i\mathcal{P}^2 & -\mathcal{P}^3 \\ -\mathcal{P}^3 & -(\mathcal{P}^1 + i\mathcal{P}^2)\end{pmatrix}, \tag{4.2}$$

where $\epsilon_{\mathcal{A}\mathcal{B}} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ is the $\mathrm{SU}(2)\cong\mathrm{Sp}(1)$ metric, $(\sigma_x)_{\mathcal{A}}^{\;\;\mathcal{B}}$, $x=1,2,3$, are the standard Pauli matrices and K_V is the Kähler potential for the scalar manifold of the $N=2$ vector multiplets; for the type IIA compactifications on which we are focusing, $K_V \equiv K_+$.

The $N=2$ gravitino mass matrix $S_{\mathcal{A}\mathcal{B}}$ can be considered as the fundamental fermionic shift, since its derivatives determine the shifts appearing in the supersymmetry transformations of the remaining $N=2$ supergravity fermions (see subsection 4.1.3 below).

The matrix $S_{\mathcal{A}\mathcal{B}}$ arising from compactifications of type II theory on $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure backgrounds was determined in ref. [46] via dimensional reduction of the type II gravitino susy variations, generalizing a previous study [45] done for SU(3) structures, and applying the generalized geometry formalism. For type IIA, the result of that analysis was[2]

$$S_{\mathcal{A}\mathcal{B}} = ie^{\frac{K_+}{2}}\begin{pmatrix} e^{\frac{K_-}{2}+\varphi}\int\langle\Phi_+\,,\, d_{H^{\mathrm{fl}}}\Phi_-\rangle & \frac{e^{2\varphi}}{\sqrt{8}}\int\langle\Phi_+\,,\,G\rangle \\ \frac{e^{2\varphi}}{\sqrt{8}}\int\langle\Phi_+\,,\,G\rangle & -e^{\frac{K_-}{2}+\varphi}\int\langle\Phi_+\,,\, d_{H^{\mathrm{fl}}}\bar{\Phi}_-\rangle\end{pmatrix}. \tag{4.3}$$

Let us discuss the various ingredients in this expression. The flux part H^{fl} of the NS field strength is contained in the $d_{H^{\mathrm{fl}}}$ operator defined in (3.59), while the remaining piece of the internal NS 2–form b is included in the pure spinors $\Phi_\pm$ as in (2.64). The latter are built via (2.60) from the two globally defined Spin(6) spinors η^1,η^2 appearing in the ansatz (2.1)

[1] Our R-symmetry SU(2) indices are $\mathcal{A},\mathcal{B}=1,2$, while we reserved the letters $A,B,\ldots$ (running over $0,1,\ldots,b^+$) for the projective coordinates of $\mathscr{M}_+$.

[2] A few remarks are in order for the comparison with ref. [46]. Our matrix $S_{\mathcal{A}\mathcal{B}}$ corresponds to the matrix called $S^{(4)}_{AB}(\mathrm{IIA})$ there. The differences in the numerical factors are due to different choices of normalization for the pure spinors. Finally, here we have already taken the integral over M_6.

for the susy parameters entering in the type II gravitino variation. Furthermore, G is the formal sum of purely internal RR field strengths, and the 4d dilaton φ was defined in (3.75); relation (3.76) holds.

Comparing (4.3) with (4.2), one deduces a geometric expression for the Killing prepotentials [46]:

$$\begin{aligned} \mathcal{P}^1 - i\mathcal{P}^2 &= 2e^{\frac{K_-}{2}+\varphi}\int\langle\Phi_+\,,\,d_{H^{\text{fl}}}\Phi_-\rangle \qquad , \qquad \mathcal{P}^1 + i\mathcal{P}^2 = 2e^{\frac{K_-}{2}+\varphi}\int\langle\Phi_+\,,\,d_{H^{\text{fl}}}\bar{\Phi}_-\rangle \\ \mathcal{P}^3 &= -\frac{e^{2\varphi}}{\sqrt{2}}\int\langle\Phi_+\,,\,G\rangle\,. \end{aligned} \tag{4.4}$$

Finally, expanding $\Phi_\pm$ and G in terms of the basis forms as in (3.23), (3.35) and (3.143), and recalling subsection 3.3.5, one obtains the $\mathcal{P}^x$ in terms of the quantities entering in the 4d effective action

$$\begin{aligned} \mathcal{P}^1 - i\mathcal{P}^2 &= 2e^{\frac{K_-}{2}+\varphi}\,Z^{\mathbb{I}}(\mathbb{S}_-\mathbb{Q})_{\mathbb{I}\mathbb{A}}X^{\mathbb{A}} \\ &= 2e^{\frac{K_-}{2}+\varphi}\big[(Z^I e_{IA} - \mathcal{G}_I m^I_A)X^A + (Z^I p^A_I - \mathcal{G}_I q^{IA})\mathcal{F}_A\big]\,, \\ \mathcal{P}^1 + i\mathcal{P}^2 &= 2e^{\frac{K_-}{2}+\varphi}\,\bar{Z}^{\mathbb{I}}(\mathbb{S}_-\mathbb{Q})_{\mathbb{I}\mathbb{A}}X^{\mathbb{A}} \\ &= 2e^{\frac{K_-}{2}+\varphi}\big[(\bar{Z}^I e_{IA} - \bar{\mathcal{G}}_I m_A{}^I)X^A + (\bar{Z}^I p^A_I - \bar{\mathcal{G}}_I q^{IA})\mathcal{F}_A\big]\,, \\ \mathcal{P}^3 &= -e^{2\varphi}\,G^{\mathbb{A}}\mathbb{S}_{+\mathbb{A}\mathbb{B}}X^{\mathbb{B}} \\ &= e^{2\varphi}\big[(e_{\text{RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m^I_A)X^A - (m^A_{\text{RR}} + \xi^I p^A_I - \tilde{\xi}_I q^{IA})\mathcal{F}_A\big], \end{aligned} \tag{4.5}$$

where the symplectic vectors $X^{\mathbb{A}}$, $Z^{\mathbb{I}}$ were introduced in (3.86). In order to be consistent with [45, 46, P1], for $G^{\mathbb{A}} \equiv (G^A, \tilde{G}_A)^T$ we are using expression (3.134), but expression (3.126) would be equally fine (recall the comment below (3.134)). Furthermore, since here there is no risk of confusion with other p-forms $G^A_p, \tilde{G}_{p\,A}$ on the 4d spacetime, in this chapter we drop the label 0 appearing in (3.134).

4.1.2 Gauging the quaternionic isometries

In this subsection we discuss how the Killing prepotentials given above fit into the general formalism of 4d $N = 2$ gauged supergravity. This will allow us to derive the form of all the fermionic shifts in the $N = 2$ theory, given in eqs. (4.18)-(4.21) below. The latter will be the building blocks to establish the supersymmetric vacuum conditions, to be studied in section 4.2.

In the previous chapter we investigated the dimensional reduction of type IIA supergravity on SU(3)×SU(3) structure backgrounds with fluxes, eventually recovering the 4d, $N = 2$ supergravity action constructed in ref. [50] by means of purely 4d supergravity techniques. In order to clarify the $N = 2$ structure of this action, let us come back on the approach followed in [50], already very concisely summarized at the beginning of subsection 3.5.4. Here we will highlight the way the Killing prepotentials emerge in that picture.

The general framework is the one of gauged $N=2$ supergravity with (massive) tensor multiplets, which was first suggested in [82], and then constructed in [101, 97, 98]. As above, we will choose a setting corresponding to a type IIA compactification (the discussion for IIB would proceed in a perfectly mirror symmetric way). The strategy adopted in [50] was to start from an ungauged $N=2$ supergravity of the kind obtained in type II compactifications on Calabi-Yau 3-folds (recall section 3.2), and then deform it by gauging the abelian isometries of the quaternionic metric h_{uv} corresponding to the hyperscalar kinetic matrix (cf. (3.16)). A second step, allowing to introduce further interactions, was the dualization of a subset of the hyperscalars to antisymmetric rank-2 tensors.

We recall that the quaternionic manifold which is relevant for the theory under consideration is a *special*, or *dual* one: the metric h_{uv} is determined by the data of a special Kähler submanifold, which for a type IIA Calabi-Yau compactification is the complex structure moduli space $\mathscr{M}_{cs}$. In the present more general case, it is $\mathscr{M}_{-}$, describing the metric deformations of the pure spinor Φ_- (the Φ_+-moduli t^a, $a=1,\dots,b^+$, enter instead in the $N=2$ vector multiplets). Let us first recall the principal features of this dual quaternionic manifold. Its coordinates are the scalars $q^u=(\varphi,a,\xi^I,\tilde{\xi}_I,z^i)$, $u=1,\dots,4(b^-+1)$, representing the bosonic components of the $N=2$ hypermultiplets. The quadruple $(\varphi,a,\xi^0,\tilde{\xi}_0)$ corresponds to the universal hypermultiplet, where a is the axion coming from the dualization of the NS 2-form $B_{\mu\nu}$ extending along the 4d spacetime. As in the previous chapter, the complex scalars z^i are the coordinates of the special Kähler manifold $\mathscr{M}_-$. Let us introduce the 1-forms [68, 50]:

$$\begin{aligned} u &= ie^{\frac{K_-}{2}+\varphi} Z^I (d\tilde{\xi}_I - \mathcal{M}_{IJ} d\xi^J) \\ v &= \frac{e^{2\varphi}}{2}\left[de^{-2\varphi} - i(da + \tilde{\xi}_I d\xi^I - \xi^I d\tilde{\xi}_I)\right] \\ E &= -\frac{i}{2} e^{\varphi-\frac{K_-}{2}} P_I (\mathrm{Im}\mathcal{G})^{-1\,IJ}(d\tilde{\xi}_J - \mathcal{M}_{JL} d\xi^L) \\ e &= P_I dZ^I\ , \end{aligned} \tag{4.6}$$

with

$$P_I = (P_0^{\underline{j}}, P_i^{\underline{j}}) = (-e_i^{\underline{j}} Z^i, e_i^{\underline{j}})\,, \tag{4.7}$$

where $e_i^{\underline{j}}$, $(i,j=1,\dots,b^-)$ are the vielbeine of the special Kähler manifold $\mathscr{M}_-$ (the underlined indices are flat), and $\mathcal{M}$ is the period matrix on it. The choice of special coordinates $Z^I=(1,z^i)$ is assumed.[3] The quaternionic metric h_{uv} is then given by

$$\begin{aligned} h_{uv} dq^u \otimes dq^v &= \bar{u}\otimes u + \bar{v}\otimes v + \bar{E}\otimes E + \bar{e}\otimes e \\ &= \mathscr{G}^-_{i\bar{\jmath}}\, dz^i d\bar{z}^{\bar{\jmath}} + (d\varphi)^2 + \frac{e^{4\varphi}}{4}\big(da + d\xi^{\mathbb{I}}\mathbb{S}_{-\mathbb{I}\mathbb{J}}\xi^{\mathbb{J}}\big)^2 - \frac{e^{2\varphi}}{2} d\xi^{\mathbb{I}}\widetilde{\mathbb{M}}_{\mathbb{I}\mathbb{J}}\, d\xi^{\mathbb{J}}\ , \end{aligned} \tag{4.8}$$

where $\mathscr{G}^-_{i\bar{\jmath}}$ is the metric on $\mathscr{M}_-$, $\widetilde{\mathbb{M}}$ corresponds to the symmetric, negative definite matrix built in terms of $\mathcal{M}$ in (3.54), and $\xi^{\mathbb{I}}=(\xi^I,\tilde{\xi}_I)^T$ is the symplectic vector of RR scalars. Of course, when the compact manifold M_6 is Calabi-Yau, this is nothing else than the metric given in (3.16).

[3] The matrix which in our conventions corresponds to $-2\mathrm{Im}\mathcal{G}_{IJ}$ was called N in [68, 50].

As we review in some more detail in appendix E, due to the fact that the holonomy of the quaternionic manifold is in $\mathrm{Sp}(1) \times \mathrm{Sp}(b^- + 1)$, by introducing the $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ indices $\mathcal{A}, \mathcal{B} = 1, 2$ and the $\mathrm{Sp}(b^- + 1)$ indices $\mathbb{I}, \mathbb{J} = 1, \ldots, 2b^- + 2$, one can define the natural vielbeine $\mathcal{U}_u^{\mathcal{A}\mathbb{I}}$, relating the metric h_{uv} to the flat $\mathrm{Sp}(1)$ and $\mathrm{Sp}(b^- + 1)$ invariant metrics $\epsilon_{\mathcal{AB}}$ and $\mathbb{S}^-_{\mathbb{IJ}}$:

$$h_{uv} = \mathcal{U}_u^{\mathcal{A}\mathbb{I}} \, \mathcal{U}_v^{\mathcal{B}\mathbb{J}} \, \mathbb{S}^-_{\mathbb{IJ}} \, \epsilon_{\mathcal{AB}} \, .$$

For the metric (4.8), we can choose the vielbein 1-forms:

$$\mathcal{U}^{\mathcal{A}\mathbb{I}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{u} & \bar{e} & -v & -E \\ \bar{v} & \bar{E} & u & e \end{pmatrix} . \tag{4.9}$$

As we are going to see below, this will appear in the hyperino mass matrix.

The last ingredient we need is the triplet of connection 1-forms ω^x ($x = 1, 2, 3$) for the SU(2)-bundle over the quaternionic manifold (again cf. appendix E). In the present case, these are given by [68, 14, 15]:

$$\begin{aligned} \omega^1 &= i(\bar{u} - u) \qquad , \qquad \omega^2 = u + \bar{u} \\ \omega^3 &= \frac{i}{2}(v - \bar{v}) + \frac{i}{2} \frac{Z^I \mathrm{Im}\mathcal{G}_{IJ} d\bar{Z}^J - \bar{Z}^I \mathrm{Im}\mathcal{G}_{IJ} dZ^J}{\bar{Z}^K \mathrm{Im}\mathcal{G}_{KL} Z^L} \, . \end{aligned} \tag{4.10}$$

Starting from the ungauged $N = 2$ theory, having the form (3.15) and containing the quaternionic σ-model outlined above, a first deformation was obtained in [89, 50] by gauging the global symmetries corresponding to the abelian isometries of the metric (4.8) generated by the Killing vectors[4]

$$k_A = (2e_{\mathrm{RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A^I) \frac{\partial}{\partial a} + m_A^I \frac{\partial}{\partial \xi^I} + e_{IA} \frac{\partial}{\partial \tilde{\xi}_I} \; , \qquad A = 0, 1, \ldots, b^+ \; , \tag{4.11}$$

where the $e_{\mathrm{RR}A}, e_{IA}, m_A^I$ are half of the parameters associated with the general set of fluxes introduced in subsection 3.3.5. The gauging couples the hyperscalars q^u to the gauge fields A^A via a minimal substitution, i.e. the standard differentials dq^u appearing in the ungauged action (3.15) are now replaced by exterior covariant derivatives:

$$dq^u \to Dq^u = dq^u - k_A^u A^A \, .$$

The resulting $N = 2$ gauged supergravity action is in agreement with the results of sections 3.4 and 3.5. In particular, the relevant piece of the action stemming from the type IIA RR sector is the one worked out in subsection 3.5.3, with the further requirement $m^A_{\mathrm{RR}} = 0$. The hyperscalar covariant derivatives introduced by the gaugings correspond exactly to expressions (3.116), (3.119). We conclude that the scalars $a, \xi^I, \tilde{\xi}_I$ are electrically charged under the gauge fields A^A, the electric charges being provided by $e_{\mathrm{RR}A}, e_{IA}, m_A^I$.

Let us also describe the consistency with the gravitino mass matrix $S_{\mathcal{AB}}$ derived in the previous subsection. As we review in appendix E, in gauged supergravity to each of the quaternionic isometries generated by Killing vectors k_A is associated a set of three

[4]The abelianity $[k_A, k_B] = 0$ follows from the quadratic constraints written below eq. (3.17).

momentum maps $\mathcal{P}^x_A$, which actually are the proper Killing prepotentials, related to the $\mathcal{P}^x$ appearing in the gravitino mass matrix by $\mathcal{P}^x = \mathcal{P}^x_A X^A + \ldots$ (see below for the missing symplectic completion). In our case, these are given by the formula (cf. appendix E.2)

$$\mathcal{P}^x_A = \omega^x_u k^u_A \, . \tag{4.12}$$

Plugging (4.10) and (4.11) in (4.12), one obtains

$$\begin{aligned}
\mathcal{P}^1_A &= 2e^{\frac{K_-}{2}+\varphi}(\mathrm{Re}Z^I e_{IA} - \mathrm{Re}\mathcal{G}_I m^I_A) \qquad , \qquad \mathcal{P}^2_A = -2e^{\frac{K_-}{2}+\varphi}(\mathrm{Im}Z^I e_{IA} - \mathrm{Im}\mathcal{G}_I m^I_A) \\
\mathcal{P}^3_A &= e^{2\varphi}(e_{\mathrm{RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m^I_A) \, ,
\end{aligned} \tag{4.13}$$

and we immediately recognize in the sums $\mathcal{P}^x_A X^A$ the part of the $\mathcal{P}^x$ in (4.5) containing the charges $e_{\mathrm{RR}A}, e_{IA}, m^I_A$.

Let us now consider the second half of flux parameters $m^A_{\mathrm{RR}}, p^A_I, q^{IA}$. In order to take these into account, the authors of ref. [50] performed a dualization of a subset of the scalars $\{\xi^I, \tilde{\xi}_I\}$, together with the axion a, to antisymmetric 2-tensors. Then the charges $m^A_{\mathrm{RR}}, p^A_I, q^{IA}$ could be introduced as mass terms for these tensors, in a way which is consistent with $N=2$ supersymmetry [97, 98]. The action yielded by this procedure is the same that in the previous chapter we derived by dimensional reduction; see in particular the RR piece (3.140), containing a set of undualized RR scalars $\xi^{\mathbb{I}}$ and a set of 2–forms B and $\breve{C}^A_2$. Alternatively, using the 'redundant' formalism described in [106], one could generate the same interactions by performing a gauging involving a set of magnetic gauge potentials and the quaternionic Killing vectors

$$\tilde{k}^A = (2m^A_{\mathrm{RR}} + \xi^I p^A_I - \tilde{\xi}_I q^{IA})\frac{\partial}{\partial a} - q^{IA}\frac{\partial}{\partial \xi^I} - p^A_I \frac{\partial}{\partial \tilde{\xi}_I} \, , \tag{4.14}$$

and then integrating out the magnetic vector potentials, leaving in this way a theory with electric vectors and antisymmetric tensors (together with the other fields already present in the original action). In this sense, the flux parameters $m^A_{\mathrm{RR}}, p^A_I, q^{IA}$ can be interpreted as magnetic charges from the 4d $N=2$ viewpoint, assigned to the same hyperscalars $a, \xi^I, \tilde{\xi}_I$ that are already electrically charged. We can then define the symplectic completion $\tilde{\mathcal{P}}^{xA}$ of the $\mathcal{P}^x_A$ introduced above [15, 106]:

$$\tilde{\mathcal{P}}^{xA} = \omega^x_u \tilde{k}^{uA} \, , \tag{4.15}$$

yielding

$$\begin{aligned}
\tilde{\mathcal{P}}^{1A} &= -2e^{\frac{K_-}{2}+\varphi}(\mathrm{Re}Z^I p^A_I - \mathrm{Re}\mathcal{G}_I q^{IA}) \qquad , \qquad \tilde{\mathcal{P}}^{2A} = 2e^{\frac{K_-}{2}+\varphi}(\mathrm{Im}Z^I p^A_I - \mathrm{Im}\mathcal{G}_I q^{IA}) \\
\tilde{\mathcal{P}}^{3A} &= e^{2\varphi}(m^A_{\mathrm{RR}} + \xi^I p^A_I - \tilde{\xi}_I q^{IA}) \, .
\end{aligned} \tag{4.16}$$

It is worth to remark that the combinations of the $\xi^I, \tilde{\xi}_I$ entering in $\mathcal{P}^3_A$ and $\tilde{\mathcal{P}}^{3A}$ correspond to the $\widehat{\xi}^{\mathbb{I}}$ of subsection 3.5.4, and do not contain the scalars which have been dualized to antisymmetric tensors.

It is now easy to see that the symplectic invariant expressions

$$\mathcal{P}^x = \mathcal{P}^x_A X^A - \tilde{\mathcal{P}}^{xA}\mathcal{F}_A \tag{4.17}$$

precisely reproduce the full Killing prepotentials (4.5) provided by the compactification.

4.1.3 $N = 2$ fermionic shifts with electric and magnetic charges

The $N = 2$ structure of the theory implies that the same charges introduced above appear in the fermionic supersymmetry variations of the $N = 2$ theory [97, 98]. We have already seen that this is the case for the gravitini, whose fermionic shift $S_{\mathcal{AB}}$ was investigated in subsection 4.1.1. As anticipated in table 3.1, besides the gravitini $\psi_{\mathcal{A}\mu}\,,\ \mathcal{A} = 1, 2$, the (positive chirality) fermions contained in the $N = 2$ theory under consideration are the hyperini $\zeta_{\mathbb{I}}$, labeled by the symplectic index $\mathbb{I} = 1, \ldots, 2b^- + 2$, and the gaugini $\lambda^{a\mathcal{A}}$, $a = 1, \ldots, b^+$; these are the fermionic components of the hypermultiplets and of the vector multiplets respectively. To be precise, the $\zeta_{\mathbb{I}}$ are the hyperini of the theory prior to the dualization of the axions: after the dualization, the $\zeta_{\mathbb{I}}$ belong to a scalar-tensor multiplet containing the undualized scalars as well as the antisymmetric 2-tensors [98]; however, for simplicity we will continue to call them hyperini. The $N = 2$ fermionic transformation laws read

$$\begin{aligned} \delta\psi_{\mathcal{A}\mu} &= \ldots + \nabla_\mu \varepsilon_{\mathcal{A}} - S_{\mathcal{AB}}\gamma_\mu \varepsilon^{\mathcal{B}} \\ \delta\zeta_{\mathbb{I}} &= \ldots + N_{\mathbb{I}}^{\mathcal{A}} \varepsilon_{\mathcal{A}} \\ \delta\lambda^{a\mathcal{A}} &= \ldots + W^{a\mathcal{AB}}\varepsilon_{\mathcal{B}}\,. \end{aligned} \tag{4.18}$$

The "..." refer to terms which vanish on a bosonic, maximally symmetric spacetime and which therefore will not be relevant for the supersymmetric vacuum conditions we are going to analyze in the forthcoming section. We recall that the 4d metric we use here is rescaled with respect to the original string frame metric $\hat{g}_{\mu\nu}$ as in (3.74). Hence, $\gamma_\mu = e^{-\varphi}\hat{\gamma}_\mu$. Finally, $S_{\mathcal{AB}}, N_{\mathbb{I}}^{\mathcal{A}}$ and $W^{a\mathcal{AB}}$ are the fermionic shifts, also termed mass matrices because they appear as fermionic mass terms in the supergravity lagrangian. They contain the charges provided by the fluxes, and their expression is [98, 15]:

$$S_{\mathcal{AB}} = \frac{i}{2} e^{\frac{K_+}{2}} (\sigma_x)_{\mathcal{A}}{}^{\mathcal{C}} \epsilon_{\mathcal{BC}} (\mathcal{P}^x_A X^A - \tilde{\mathcal{P}}^{xA}\mathcal{F}_A) \tag{4.19}$$

$$N_{\mathbb{I}}^{\mathcal{A}} = 2e^{\frac{K_+}{2}}\, \mathcal{U}^{\mathcal{A}}_{\mathbb{I}u} (k^u_A \bar{X}^A - \tilde{k}^{uA}\bar{\mathcal{F}}_A) \tag{4.20}$$

$$W^{a\mathcal{AB}} = ie^{\frac{K_+}{2}} \mathscr{G}_+^{a\bar{b}} (\sigma_x)_{\mathcal{C}}{}^{\mathcal{B}} \epsilon^{\mathcal{CA}} (\mathcal{P}^x_C D_{\bar{b}} \bar{X}^C - \tilde{\mathcal{P}}^{xC} D_{\bar{b}}\bar{\mathcal{F}}_C)\,. \tag{4.21}$$

Notice that the vielbeine $\mathcal{U}^{\mathcal{A}}_{\mathbb{I}u}$ of the quaternionic manifold prior to the dualization of the axions appear in the hyperino mass matrix $N_{\mathbb{I}}^{\mathcal{A}}$.

Of course all the mass matrices vanish in the absence of fluxes. In this case we would have a continuum of $N = 2$ supersymmetric vacuum configurations (with vanishing cosmological constant), all the scalar fields corresponding to massless moduli. In the presence of fluxes, the mass matrices (4.19)-(4.21) are non-trivial, and determine a potential for the 4d supergravity action, in this way lifting a certain number of previously flat scalar directions. Indeed, a general Ward identity of extended supergravities relates the scalar potential V to the squares of the fermionic shifts. In the present $N = 2$ setting, it reads [76, 97, 98]

$$V\,\delta^{\mathcal{A}}_{\mathcal{B}} = \mathscr{G}^+_{a\bar{b}} W^{a\mathcal{CA}}\overline{W}^{\bar{b}}_{\mathcal{CB}} + 2N_{\mathbb{I}}^{\mathcal{A}} N^{\mathbb{I}}_{\mathcal{B}} - 12\overline{S}^{\mathcal{CA}} S_{\mathcal{CB}}\,. \tag{4.22}$$

Notice that the gaugino and hyperino mass matrices yield a positive contribution to V, while the gravitino contribution is negative. From the derivation of [50] it follows that, upon substitution of the quantities above, this formula yields the sum of expressions (3.87), (3.141) which in the previous chapter we derived by dimensional reduction.

4.2 The $N = 1$ vacuum conditions

This section, based on publication [P1], confronts the 4d and 10d approaches to $N = 1$ backgrounds.[5] In subsection 4.2.1 we consider the equations characterizing the $N = 1$ vacua at the 10d level, translated in the generalized geometry formalism in [26, 27], and we rewrite them in a way which is suitable for the comparison with the conditions arising in the 4d approach. The latter are analyzed in subsection 4.2.2, having as a starting point the $N = 2$ theory derived in the previous chapter, and further discussed here above.

Starting from an $N = 2$ theory, an $N = 1$ vacuum can be obtained by spontaneous partial supersymmetry breaking. This is a concrete possibility when considering compactifications with fluxes, since the associated 4d supergravities possess a non-trivial scalar potential due to the flux-generated gaugings. However, spontaneous partial susy breaking is non-generic: the old no-go theorem of [107] forbidding such phenomenon on Minkowski vacua can be circumvented only by the choice of a degenerate symplectic section in the vector multiplet sector, such that a prepotential doesn't exist [108]. On the other hand, the no-go theorem does not constrain AdS vacua, which represent therefore an available possibility, as we will see in the next chapter.

Such an obstruction for $N = 1$ solutions with vanishing vacuum energy is somehow reflected at the 10d level: it is well known that tadpole cancellation in a background with fluxes consisting of the product of Minkowski$_4$ with a compact M_6 manifold requires the presence of negative tension sources, such as orientifold planes. With an appropriate choice of the orientifold, the resulting 4d effective theory takes an $N = 1$ form, and corresponds to a truncation of the previously $N = 2$ action. At this point, the $N = 1$ vacuum condition amounts just to the requirement of unbroken supersymmetry.[6]

The two possibilities we have mentioned (spontaneous partial susy breaking and the $N = 2 \to N = 1$ truncation) are not unrelated, since the physics around an $N = 1$ vacuum for energies well below the partial susy breaking scale has to be described by an $N = 1$ theory (see [109] for a discussion), and in some cases such low energy theory can correspond to a truncation of the $N = 2$ action.

Here however we don't need to specify which is the mechanism leading to the $N = 1$ vacua, and it will be sufficient to observe that a supersymmetric (bosonic) vacuum is characterized by the vanishing of the fermionic variations under the preserved supersymmetries. In particular, starting from an $N = 2$ theory one has an (at least) $N = 1$ vacuum if such a condition is satisfied by the variations given by any chosen linear combination of the $N = 2$ spinorial parameters $\varepsilon_{\mathcal{A}}$, $\mathcal{A} = 1, 2$. This characterization applies to $N = 2 \to N = 1$ truncations as well, provided the linear combination of the two susy generators under which the vacuum is required to be invariant coincides with the supersymmetry being preserved by the truncation at the level of the action.

[5] See also [37] for an analysis having a similar task, but adopting a different approach to the 4d theory.

[6] Here we are considering dimensional reductions of 10d supergravity on compact manifolds. Further possibilities are opened by allowing for a decompactification limit freezing a part of the moduli [86].

We can therefore proceed introducing a two-component vector $n_{\mathcal{A}} = \binom{\bar{a}}{b}$, where a and b are complex constants,[7] satisfying $|a|^2 + |b|^2 = 1$. Then, we select the preserved positive-chirality $N = 1$ susy parameter ε by

$$\varepsilon = \bar{n}^{\mathcal{A}} \varepsilon_{\mathcal{A}} \qquad \Longleftrightarrow \qquad \varepsilon_{\mathcal{A}} = n_{\mathcal{A}} \varepsilon \;, \tag{4.23}$$

where $\bar{n}^{\mathcal{A}} = \binom{\bar{a}}{b}^{\dagger}$, and the two expressions (4.23) are equivalent since we put to zero the independent linear combination $b\varepsilon_1 - \bar{a}\varepsilon_2$. The conjugated spinors $\varepsilon^{\mathcal{A}}$ can be written as $\varepsilon^{\mathcal{A}} = \varepsilon^c n^{*\mathcal{A}}$, where $\varepsilon^c \equiv \varepsilon^*$ has negative 4d chirality and $n^{*\mathcal{A}} = \binom{a}{\bar{b}}$.

Let us reconsider the spinor ansatz (2.1) adopted for the off-shell reduction. For type IIA, it reads

$$\begin{aligned} \epsilon^1 &= \varepsilon_1 \otimes \eta^1_- + \varepsilon^1 \otimes \eta^1_+ \\ \epsilon^2 &= \varepsilon_2 \otimes \eta^2_+ + \varepsilon^2 \otimes \eta^2_- \;. \end{aligned} \tag{4.24}$$

We recall that the Spin$(3,1)$ spinors $(\varepsilon_1, \varepsilon_2)$ have positive chirality, while $(\varepsilon^1, \varepsilon^2)$ have negative chirality; hence the Spin$(9,1)$ spinor ϵ^1 has negative chirality, while ϵ^2 has positive chirality. Now, (4.23) fixes the form of the 10d spinorial parameters in the $N = 1$ vacuum to

$$\begin{aligned} \epsilon^1 &= \varepsilon \otimes \bar{a}\eta^1_- + \varepsilon^c \otimes a\eta^1_+ \\ \epsilon^2 &= \varepsilon \otimes b\eta^2_+ + \varepsilon^c \otimes \bar{b}\eta^2_- \;. \end{aligned} \tag{4.25}$$

Since we wish to describe vacua, we require the 4d spacetime be maximally symmetric.[8] This will actually be either Minkowski or AdS, the dS case being ruled out. Hence we can choose ε to satisfy the Killing spinor equation

$$\hat{\nabla}_\mu \varepsilon = \frac{1}{2} \bar{\mu} \hat{\gamma}_\mu \varepsilon^c \;, \tag{4.26}$$

where the hat denotes that here we are using the string frame metric $\hat{g}_{\mu\nu}$. Using the relation $[\nabla_\mu, \nabla_\nu]\varepsilon = \frac{1}{4} R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\varepsilon$, we see that the complex parameter μ is related to the 4d spacetime cosmological constant $\Lambda \equiv \hat{R}_4/4$ by

$$\Lambda = -3|\mu|^2,$$

so that we have Minkowski$_4$ if $\mu = 0$, and AdS$_4$ otherwise.

[7] The choice of writing $n_1 = \bar{a}$ instead of a is dictated by later convenience, see the forthcoming eq. (4.25). Of course, the a here has nothing to do with the axion considered in subsection 4.1.2. Furthermore, the parameter b should not be confused with the internal NS 2–form, also called b.

[8] Requiring maximal symmetry of a d-dimensional Lorentzian spacetime amounts to demand Poincaré invariance in the flat case, SO$(2, d-1)$ invariance in the negative curvature case and SO$(1, d)$ invariance in the positive curvature case.

4.2.1 $N=1$ equations from the ten dimensional analysis

Before establishing the 4d $N=1$ vacuum conditions arising from the 4d action, let us illustrate which is the outcome of the 10d analysis for $N=1$ backgrounds.

A supersymmetric background configuration[9] of the 10d supergravity with four preserved supercharges is obtained by imposing the vanishing of the 10d fermionic transformations under the supersymmetry parameterized by the spinor ansatz (4.25). Having this as a starting point, it has been argued in [27] (and further illustrated in [33]) that $N=1$ backgrounds of type II theories have an internal manifold whose tangent plus cotangent bundle admits an SU(3)×SU(3) structure. The supersymmetry equations can then be rephrased in the framework of generalized geometry as differential conditions for the pair of Spin(6,6) pure spinors associated with the SU(3)×SU(3) structure. With reference to the decomposition (4.25), such pure spinors can be written as the following bispinors:

$$e^{-\phi}a\eta^1_+\otimes(b\eta^2_+)^\dagger=\frac{a\bar{b}}{8}\Phi^0_+\qquad,\qquad e^{-\phi}a\eta^1_+\otimes(\bar{b}\eta^2_-)^\dagger=\frac{ab}{8}\Phi^0_-\ ,\tag{4.27}$$

where for $\Phi^0_\pm$ we are using definition (2.60). The complex parameters a and b could in general depend on the internal coordinates, and indeed this would be the case for supersymmetric solutions on warped backgrounds. However, here we are interested in a comparison with what results from the effective action approach. For this reason we restrict ourselves to a vanishing warp factor,[10] and we assume both a and b to be constant. In contrast with [P1], here the 10d dilaton ϕ is included into $\Phi^0_\pm$, and we don't explicitly demand it to be constant along the compact manifold.

Finally, we have to pay attention to the 10d spinor conventions. Indeed, in [27, 33], the type IIA 'pure spinor equations' were derived assigning positive chirality to ϵ^1 and negative chirality to ϵ^2, while in (4.25) we have done the opposite choice (following the conventions of [46]). Furthermore, the conventions we are using here are not precisely the same as the ones in [27, 33] (the relevant differences are in the choice of the 6d chirality matrix γ, of the Hodge-$*$ and of the involution λ, cf. our appendix A). Taking all this into account, we find that our type IIA supersymmetry equations for the ansatz (4.25) are obtained from the ones given in [33] upon implementing the following transformation:

$$a\bar{b}\Phi^0_+\ \to\ \bar{a}b\bar{\Phi}^0_+\quad,\quad ab\Phi^0_-\ \to\ -\overline{ab\Phi^0_-}\quad,\quad H\ \to\ -H\quad,\quad \lambda(F)\ \leftrightarrow\ F\ ,\tag{4.28}$$

where the RR field $F=F_0+F_2+F_4+F_6$ is the purely internal part of the democratic RR field $\hat{\mathbf{F}}$ introduced in section 3.1, while the involution λ is defined in (2.52).

We remark that the type IIA pure spinor equations obtained in this way correspond precisely to the ones given in [33] for type IIB, provided we exchange the chirality of the Spin(6,6) pure spinors and of the RR field strengths and we conjugate the complex parameter μ given in (4.26) (this last transformation is harmless, because it does not modify

[9] At this stage, we cannot speak of a full solution of the 10d (classical) action, since the supersymmetry conditions alone do not imply all the equations of motion and Bianchi identities for the bosonic fields. See e.g. [36] for a recent discussion, with some new results in this sense.

[10] For recent progresses in the study of dimensional reductions on warped backgrounds see for instance [110, 111, 112, 37, 113, 61].

the physical quantity associated with μ, which is the 4d spacetime cosmological constant $\Lambda = -3|\mu|^2$). So type IIA with the ansatz (4.25) and type IIB with a positive chirality choice for both $\epsilon^{1,2}$ lead to the same pure spinor equations.

Starting from the pure spinor equations of [33], performing the transformations (4.28) and taking into account the assumptions on the warp factor, we arrive at:

$$(d - H\wedge)(a\bar{b}\Phi^0_+) = -2\bar{\mu}\mathrm{Re}(ab\Phi^0_-) \tag{4.29}$$

$$(d - H\wedge)(ab\Phi^0_-) = -3i\,\mathrm{Im}(\mu a\bar{b}\Phi^0_+) + \frac{1}{2}\big(c_-F - ic_+ * \lambda(F)\big)\,, \tag{4.30}$$

where $c_\pm = |a|^2 \pm |b|^2$. Consistently with our definition of a and b, (see above eq. (4.23)), we will fix $c_+ = 1$. Of course, any other choice for c_+ can be recovered by the redefinition $n^{\mathrm{old}}_{\mathcal{A}} = n^{\mathrm{new}}_{\mathcal{A}}/\sqrt{c_+}$.

We now rewrite (4.29) and (4.30) in a form more suitable for the forthcoming comparison with the effective theory approach. Separating the background flux and the exact pieces of the NS 3-form as $H = H^{\mathrm{fl}} + db$, acting with e^{-b} on the equations, and recalling defs. (3.59) for $d_{H^{\mathrm{fl}}}$ and (3.43) for the twisted Hodge operator $*_b$, we get

$$a\bar{b}\, d_{H^{\mathrm{fl}}}\Phi_+ = -2\bar{\mu}\mathrm{Re}(ab\Phi_-) \tag{4.31}$$

$$ab\, d_{H^{\mathrm{fl}}}\Phi_- = -3i\mathrm{Im}(\mu a\bar{b}\Phi_+) + \frac{1}{2}\big(c_-G - i *_b G\big)\,, \tag{4.32}$$

where the sum of purely internal RR field strengths G is related to F by $G = e^{-b}F$ (recall (3.92)), and $\Phi_\pm := e^{-b}\Phi^0_\pm$ are the same pure spinors appearing in the previous chapter.[11]

As already announced, our purpose is to compare these equations with the $N = 1$ vacuum conditions arising from the 4d effective action. In order to do this, we need only the pure spinor modes preserved by the truncation defining the 4d theory. Hence, we can implement here the expansions (3.23), (3.35) of $\Phi_\pm$ in terms of the basis forms $\Sigma_\pm$, introduced in (3.17). Using the properties of the basis forms, it is also possible to perform the integral over the compact manifold M_6.

We obtain the version 'in components' of the pure spinor equations by taking the integrated Mukai pairing of the first and the second pure spinor equations – eqs. (4.31) and (4.32) – with the basis $\Sigma_\pm$. Adopting the symplectic notation already used above (see

[11] Although the pure spinor equations were derived in [27] assuming the background to be fully geometric, it is formally possible to substitute the differential operator $d_{H^{\mathrm{fl}}}$ with the more general operator $\mathcal{D}$ defined in (3.68), containing non-geometric fluxes. This is suggested by what is done in the effective action approach, along the lines of [46] (see also [34]). We therefore obtain the following generalized version of the pure spinor equations:

$$a\bar{b}\,\mathcal{D}\Phi_+ = -2\bar{\mu}\mathrm{Re}(ab\Phi_-)$$

$$ab\,\mathcal{D}\Phi_- = -3i\mathrm{Im}(\mu a\bar{b}\Phi_+) + \frac{1}{2}(c_-G - i *_b G)\,.$$

For example, in the SU(3) structure case, in which $\Phi_- = -i\Omega$, we have $(\mathcal{D}\Omega)_0 = R\llcorner\Omega$, $(\mathcal{D}\Omega)_2 = Q\cdot\Omega$, $(\mathcal{D}\Omega)_4 = d\Omega$ and $(\mathcal{D}\Omega)_6 = -H^{\mathrm{fl}}\wedge\Omega$. While here we will continue to work with the standard operator $d_{H^{\mathrm{fl}}}$, we actually showed in [P1] that there are no obstructions in performing the extension $d_{H^{\mathrm{fl}}} \to \mathcal{D}$.

In the context of generalized geometry, the role of the $\mathcal{D}$ operator has been investigated in [40].

in particular (3.18), (3.19) for $\mathbb{S}_\pm$, (3.55) for $\mathbb{N}$, (3.64) for $\mathbb{Q}$, as well as (3.86) for $X^{\mathbb{A}}$, $Z^{\mathbb{I}}$), by a straightforward computation one can see that

$$\int \langle\, 1^{\text{st}} \text{ pure sp.eq.}\,,\, \Sigma_-\,\rangle \qquad \Longrightarrow \qquad \boxed{a\bar{b}\,\mathbb{Q}^{\mathbb{I}}{}_{\mathbb{A}}X^{\mathbb{A}} = -2\bar{\mu}\mathrm{Re}(abZ^{\mathbb{I}})}\,, \tag{4.33}$$

$$\int \langle\, 2^{\text{nd}} \text{ pure sp.eq.}\,,\, \Sigma_+\,\rangle \;\Rightarrow\; \boxed{ab\,\widetilde{\mathbb{Q}}^{\mathbb{A}}{}_{\mathbb{I}}Z^{\mathbb{I}} = -3i\mathrm{Im}(\mu a\bar{b}X^{\mathbb{A}}) + \tfrac{c_-}{\sqrt{2}}G^{\mathbb{A}} + \tfrac{i}{\sqrt{2}}(\mathbb{N}G)^{\mathbb{A}}}\,. \tag{4.34}$$

where, as above, $G^{\mathbb{A}} = (G^A, \tilde{G}_A)^T$, coming from the expansion (3.143) of the sum G of the purely internal RR field strengths.

In this last derivation, it has been essential to dispose of eq. (3.55), expressing the action of $*_b$ in terms of the special Kähler geometry data: it allowed to compute

$$\int \langle *_b G, \Sigma_+^{\mathbb{A}}\rangle \;=\; -\sqrt{2}\,\mathbb{N}^{\mathbb{A}}{}_{\mathbb{B}}G^{\mathbb{B}}\,. \tag{4.35}$$

4.2.2 $N=1$ conditions from the effective action, and matching

We now study the $N=1$ vacuum conditions arising from the effective action approach, showing that they precisely satisfy the integrated version of the pure spinor equations established here above.

At the end of subsection 4.1.2 we wrote the form of the fermionic susy variations for the 4d $N=2$ effective theory corresponding to the type IIA compactification we considered. As it should be clear from the discussion at the beginning of this section, the 4d $N=1$ vacuum conditions amount to the vanishing of these fermionic variations under the *single* preserved supersymmetry, parameterized as in (4.23). From (4.18) we read:

$$\langle\delta_\varepsilon\psi_{\mathcal{A}\mu}\rangle = 0 \qquad \Longleftrightarrow \qquad 2e^{-\varphi}S_{\mathcal{AB}}n^{*\mathcal{B}} = n_{\mathcal{A}}\bar{\mu} \tag{4.36}$$

$$\langle\delta_\varepsilon\zeta_{\mathbb{I}}\rangle = 0 \qquad \Longleftrightarrow \qquad N_{\mathbb{I}}^{\mathcal{A}}n_{\mathcal{A}} = 0 \tag{4.37}$$

$$\langle\delta_\varepsilon\lambda^{a\mathcal{A}}\rangle = 0 \qquad \Longleftrightarrow \qquad W^{a\mathcal{AB}}n_{\mathcal{B}} = 0\,. \tag{4.38}$$

To get condition (4.36) we used (4.26) and $\gamma_\mu = e^{-\varphi}\hat{\gamma}_\mu$. Eq. (4.36) relates the Killing prepotentials to the spacetime curvature parameter μ. Recalling (4.19) and (4.17), its explicit form is

$$ie^{\frac{K_+}{2}-\varphi}\begin{pmatrix} a(\mathcal{P}^1 - i\mathcal{P}^2) - \bar{b}\mathcal{P}^3 \\ -a\mathcal{P}^3 - \bar{b}(\mathcal{P}^1 + i\mathcal{P}^2)\end{pmatrix} = \begin{pmatrix}\bar{a}\bar{\mu} \\ b\bar{\mu}\end{pmatrix}\,. \tag{4.39}$$

Let us now analyze the implications following from the vanishing of the hyperini variation, eq. (4.37). Recalling (4.20), this reads

$$n_{\mathcal{A}}\mathcal{U}^{\mathcal{A}}_{\mathbb{I}u}(k^u_A\bar{X}^A - \tilde{k}^{uA}\bar{\mathcal{F}}_A) = 0\,.$$

Using (4.9) and (4.6), substituting the expressions (4.11), (4.14) for k_A and $\tilde{k}^A$, and recognizing the form (4.5) of the $\mathcal{P}^x$, we obtain the following set of conditions:

$$\begin{aligned} a(\mathcal{P}^1 - i\mathcal{P}^2) - 2\bar{b}\mathcal{P}^3 &= 0 \qquad (4.40)\\ 2a\mathcal{P}^3 + \bar{b}(\mathcal{P}^1 + i\mathcal{P}^2) &= 0 \qquad (4.41)\\ \bar{b}P_I(\mathrm{Im}\mathcal{G})^{-1\,IJ}\big[(e_{JA} - \mathcal{M}_{JK}m^K_A)X^A + (p^A_J - \mathcal{M}_{JK}q^{KA})\mathcal{F}_A\big] &= 0 \qquad (4.42)\\ a\bar{P}_I(\mathrm{Im}\mathcal{G})^{-1\,IJ}\big[(e_{JA} - \overline{\mathcal{M}}_{JK}m^K_A)X^A + (p^A_J - \overline{\mathcal{M}}_{JK}q^{KA})\mathcal{F}_A\big] &= 0. \qquad (4.43) \end{aligned}$$

The first two equations come from the vielbeine $\mathcal{U}^{\mathcal{A}}_{\mathbb{I}}$ corresponding to the 1-forms u and v given in (4.6), while the last two are the conditions involving E (e doesn't contribute). The P_I are the Kählerian vielbeine, defined in (4.7).

Comparing (4.40) and (4.41) with (4.39) we get

$$i\frac{a}{2}(\mathcal{P}^1 - i\mathcal{P}^2) = i\bar{b}\mathcal{P}^3 = \bar{a}\bar{\mu}e^{\varphi - \frac{K_+}{2}} \qquad , \qquad -\frac{i\bar{b}}{2}(\mathcal{P}^1 + i\mathcal{P}^2) = ia\mathcal{P}^3 = b\bar{\mu}e^{\varphi - \frac{K_+}{2}} \ , \quad (4.44)$$

which implies $(|a|^2 - |b|^2)\bar{\mu} = 0$; then if the vacuum is AdS, necessarily[12] $|a| = |b|$. Furthermore, notice that on a Minkowski vacuum ($\mu = 0$) and for a and b being nonzero we have $\mathcal{P}^x = 0$; therefore the gravitino mass matrix $S_{\mathcal{AB}}$ vanishes (see (4.2)) and we cannot have spontaneous partial susy breaking in the $N = 2$ theory. In order to obtain $N = 1$ Minkowski vacua, an $N = 2 \to N = 1$ truncation of the action is required.

From now on we will assume $a \neq 0\,,\ b \neq 0$. The cases in which $a = 0$ or $b = 0$ could be studied separately; however, they are not relevant for the comparison with the pure spinor equations of the previous subsection, which were indeed established for nonvanishing a and b.

Multiply eqs. (4.42) by $\frac{1}{2}e^{-K}(\mathrm{Im}\mathcal{G})^{-1\,LM}\bar{P}_M$ and (4.43) by $\frac{1}{2}e^{-K}(\mathrm{Im}\mathcal{G})^{-1\,LM}P_M$, then use the relations [68]:

$$\begin{aligned} \frac{1}{2}e^{-K_-}[(\mathrm{Im}\mathcal{G})^{-1}P^\dagger P(\mathrm{Im}\mathcal{G})^{-1}]^{LJ} &= (\mathrm{Im}\mathcal{G})^{-1\,LJ} + 2e^{K_-}Z^L\bar{Z}^J\\ &= -(\mathrm{Im}\mathcal{M})^{-1\,LJ} - 2e^{K_-}\bar{Z}^L Z^J \end{aligned}$$

(see (D.8) for the second equality). Recognizing expressions (4.5) for $\mathcal{P}^1 \pm i\mathcal{P}^2$, we arrive at

$$\begin{aligned} \bar{b}(\mathrm{Im}\mathcal{M})^{-1\,IJ}\big[(e_{JA} - \mathcal{M}_{JK}m^K_A)X^A + (p^A_J - \mathcal{M}_{JK}q^{KA})\mathcal{F}_A\big] + \bar{Z}^I e^{\frac{K_-}{2} - \varphi}\bar{b}(\mathcal{P}^1 - i\mathcal{P}^2) &= 0\\ a(\mathrm{Im}\mathcal{M})^{-1\,IJ}\big[(e_{JA} - \overline{\mathcal{M}}_{JK}m^K_A)X^A + (p^A_J - \overline{\mathcal{M}}_{JK}q^{KA})\mathcal{F}_A\big] + Z^I e^{\frac{K_-}{2} - \varphi}a(\mathcal{P}^1 + i\mathcal{P}^2) &= 0\,. \end{aligned} \qquad (4.45)$$

Multiplying from the left (4.45) by $\binom{a}{-\bar{b}}^T$ and using (4.44), we conclude

$$a\bar{b}(m^I_A X^A + q^{IA}\mathcal{F}_A) = -2\bar{\mu}\mathrm{Re}(abZ^I)\ , \qquad (4.46)$$

[12] $|a| = |b|$ is also necessary for a Minkowski background; however in this case the condition doesn't arise from the susy equations, but rather from the orientifold projection one is led to consider in order to cancel the tadpoles [33]. Another way to arrive at the same conclusion is described in [30].

where we have also used the fact that, because of the normalizations we adopted for the pure spinors $\Phi_\pm$, we have $e^{K_+}=e^{K_-}$ (recall 3.76).

A second independent linear combination of the two equations (4.45) can be obtained multiplying them by $\binom{a}{\bar b}^T$. Plugging (4.46) in, using again (4.44) and recalling that $\mathcal{M}_{IJ}Z^J=\mathcal{G}_I$, we arrive at

$$a\bar b(e_{IA}X^A+p_I^A\mathcal{F}_A)=-2\bar\mu\mathrm{Re}(ab\mathcal{G}_I)\ . \tag{4.47}$$

Employing (3.64) and the symplectic vectors $X^{\mathbb{A}}$, $Z^{\mathbb{I}}$ introduced in (3.86), our conditions (4.46) and (4.47) can be summarized in the single equation

$$\boxed{a\bar b\mathbb{Q}X=-2\bar\mu\mathrm{Re}(abZ)}\ . \tag{4.48}$$

As it is clear from a comparison with eq. (4.33), the present condition precisely corresponds to the integrated first pure spinor equation.

The last condition to be analyzed is the variation of the gaugini, eq. (4.38). Using (4.21) this reads

$$ie^{\frac{K_+}{2}}\mathscr{G}_+^{a\bar b}D_{\bar b}\bar X^C(\mathcal{P}_C^x-\mathcal{N}_{CE}\tilde{\mathcal{P}}^{xE})\sigma_x^{\mathcal{AB}}n_{\mathcal{B}}=0\ ,$$

where $\sigma_x^{\mathcal{AB}}=(\sigma_x)_{\mathcal{C}}{}^{\mathcal{B}}\epsilon^{\mathcal{CA}}$, and we have used $D_{\bar a}\bar{\mathcal{F}}_B=\mathcal{N}_{BC}D_{\bar a}\bar X^C$ (recall D.3) in order to factorize $D_{\bar b}\bar X^C$. Multiply this expression by $e^{\frac{K_+}{2}}D_aX^D$ in order to trade a lower case index with an upper case one; then, using the special Kähler geometry relation

$$e^{K_+}D_aX^D\mathscr{G}_+^{a\bar b}D_{\bar b}\bar X^C=-\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1\,DC}-e^{K_+}\bar X^DX^C \tag{4.49}$$

(corresponding to the $\mathscr{M}_+$ version of (D.7)), and recalling that $(\mathcal{P}_B^x-\mathcal{N}_{BC}\tilde{\mathcal{P}}^{xC})X^B=\mathcal{P}^x$ (see (D.3) and (4.17)), we get

$$\sigma_x^{\mathcal{AB}}n_{\mathcal{B}}\big[(\mathrm{Im}\mathcal{N})^{-1\,AB}(\mathcal{P}_B^x-\mathcal{N}_{BC}\tilde{\mathcal{P}}^{xC})+2e^{K_+}\bar X^A\mathcal{P}^x\big]=0\ . \tag{4.50}$$

This is a vector of two equations ($\mathcal{A}=1,2$). Multiply it from the left by $\binom{-\bar b}{a}^T=\bar n^{\mathcal{C}}\epsilon_{\mathcal{CA}}$. Using (4.39) one sees that $\bar n^{\mathcal{C}}\epsilon_{\mathcal{CA}}\sigma_x^{\mathcal{AB}}n_{\mathcal{B}}\mathcal{P}^x=0$, therefore we are left with

$$\big\{2\mathrm{Re}[ab(\delta_x^1-i\delta_x^2)]+c_-\delta_x^3\big\}(\mathrm{Im}\mathcal{N})^{-1\,AB}(\mathcal{P}_B^x-\mathcal{N}_{BC}\tilde{\mathcal{P}}^{xC})=0\ , \tag{4.51}$$

where we have introduced the parameter $c_-:=|a|^2-|b|^2$. Separating into imaginary and real parts we arrive respectively at

$$\mathrm{Re}[ab(\tilde{\mathcal{P}}^{1A}-i\tilde{\mathcal{P}}^{2A})]+\frac{c_-}{2}\tilde{\mathcal{P}}^{3A}=0\qquad,\qquad\mathrm{Re}[ab(\mathcal{P}_A^1-i\mathcal{P}_A^2)]+\frac{c_-}{2}\mathcal{P}_A^3=0\ .$$

Substituting the expressions (4.13), (4.16) for $\mathcal{P}_A^x$ and $\tilde{\mathcal{P}}^{xA}$, and using (3.76) as well as expression (3.134) for G^A and $\tilde G_A$, we obtain the couple of equations

$$\mathrm{Re}(abZ^I)p_I^A-\mathrm{Re}(ab\mathcal{G}_I)q^{IA}=\frac{c_-}{2}G^A\quad,\quad\mathrm{Re}(abZ^I)e_{IA}-\mathrm{Re}(ab\mathcal{G}_I)m_A^I=\frac{c_-}{2}\tilde G_A\ ,$$

which can be assembled in a single equation for the symplectic vectors $Z^{\mathbb{I}}$ and $G^{\mathbb{A}}$:

$$\widetilde{\mathbb{Q}}^{\mathbb{A}}{}_{\mathbb{I}}\, \mathrm{Re}(abZ^{\mathbb{I}}) = \frac{c_-}{\sqrt{2}} G^{\mathbb{A}} \,. \tag{4.52}$$

Multiplying the two equations (4.50) by $\binom{\bar{b}}{a}^T$ and using once again constraint (4.44), we get a second independent (recall that $a \neq 0\,,\ b \neq 0$) combination:

$$\left\{2\,\mathrm{Im}[ab(\delta^1_x - i\delta^2_x)] - i\delta^3_x\right\}(\mathrm{Im}\mathcal{N})^{-1\,AB}(\mathcal{P}^x_B - \mathcal{N}_{BC}\tilde{\mathcal{P}}^{xC}) = 12 e^{\frac{K_+}{2}+\varphi}\bar{a}b\bar{\mu}\bar{X}^A \,. \tag{4.53}$$

Following analogous steps to the ones which led us from (4.51) to (4.52), we arrive at

$$\widetilde{\mathbb{Q}}^{\mathbb{A}}{}_{\mathbb{I}}\, \mathrm{Im}(abZ^{\mathbb{I}}) = -3\mathrm{Im}(\mu a\bar{b}X^{\mathbb{A}}) + \frac{1}{\sqrt{2}}\mathbb{N}^{\mathbb{A}}{}_{\mathbb{B}} G^{\mathbb{B}} \,, \tag{4.54}$$

where the symplectic matrix $\mathbb{N}$ is given in (3.55).

Conditions (4.52), (4.54) can be seen as the real and the imaginary parts of the single complex equation involving the symplectic vectors $X^{\mathbb{A}}$, $Z^{\mathbb{I}}$ and $G^{\mathbb{A}}$:

$$\boxed{ab\,\widetilde{\mathbb{Q}}Z = -3i\mathrm{Im}(\mu a\bar{b}X) + \tfrac{c_-}{\sqrt{2}}G + \tfrac{i}{\sqrt{2}}\mathbb{N}G} \,. \tag{4.55}$$

In this way we obtain a condition which exactly corresponds to the integrated second pure spinor equation, as it can be seen by comparison with eq. (4.34).

Let us summarize the outcome of this section. At the 10d background level, we expanded the pure spinor equations on the basis of forms $\Sigma_\pm$ and we took the integral over the internal manifold, obtaining eqs. (4.33) and (4.34). At the level of the four-dimensional theory, we started from the vev of the fermionic variations under an arbitrary linear combination of the two $N = 2$ supersymmetries, eqs. (4.36)-(4.38), and we exploited the properties of special Kähler geometry to rewrite the conditions in a more compact way. From the hyperini variation we obtained eq. (4.48), corresponding to the integrated first pure spinor equation, while the gaugini transformation (together with constraint (4.44)) yields eq. (4.55), which coincides with the integrated second pure spinor equation. The gravitini variation has been used to simplify the expressions, in particular to obtain constraint (4.44), which relates $\mathcal{P}^1 \pm i\mathcal{P}^2$ or $\mathcal{P}^3$ to the spacetime curvature parameter μ.

4.3 Aspects of $N = 2 \to N = 1$ theories

In section 4.2 we studied the conditions to have an $N = 1$ vacuum starting from the $N = 2$ effective supergravity defined by the compactification of type IIA on an SU(3)×SU(3) background. Physically, such solutions can be realized either by spontaneous partial supersymmetry breaking in the $N = 2$ theory, or as supersymmetry-preserving solutions of an $N = 1$ theory obtained as a consistent truncation of the $N = 2$ action. In string theory, such $N = 2 \to N = 1$ truncations can be realized including appropriate orientifold planes in the 10d background. Truncations can also be relevant for spontaneous partial susy breaking, in the sense that the $N = 1$ theory describing the low energy physics around an

$N=1$ vacuum which breaks $N=2$ spontaneously can in some special cases correspond to a truncation of the $N=2$ action. An example of this was provided in ref. [92], where an $N=1$ effective action containing the $N=1$ AdS$_4$ vacuum of [114] was obtained by truncating an $N=2$ theory (however, no fluctuations analysis aiming to determine the light degrees of freedom was done there).

In this section we study some aspects of the $N=1$ theory obtained as a generic truncation of the $N=2$ supergravity action derived in the previous chapter. In particular, we focus on the way the $N=1$ superpotential and D-terms are determined as linear combinations of the three $N=2$ Killing prepotentials $\mathcal{P}^x$ introduced above. Then we write the F- and D- flatness conditions for $N=1$ vacua, establishing their relation with the $N=1$ conditions of the previous section. In this way we will we able to nicely reinterpret the matching with the pure spinor equations coming from the 10d analysis.

A thorough analysis of the conditions allowing to define a consistent $N=2 \to N=1$ truncation has been performed in [115], and extended in [116] to the case in which tensor multiplets are also present.

4.3.1 $N=1$ superpotential

In subsection 4.1.1 we briefly reviewed how refs. [45, 46] derived the Killing prepotentials of the $N=2$ theory which is defined starting from a 10d background preserving eight supercharges. This strategy was further pursued in the same papers by restricting the background to preserve four supercharges rather than eight. In this way, as we will recall next, an expression for the $N=1$ superpotential $\mathcal{W}$ was obtained (see (4.58)).

The preserved $N=1$ spinor parameter can be chosen as in (4.23), and the correspondent linear combination of the $N=2$ gravitini defines the positive-chirality $N=1$ gravitino: $\psi_\mu = \bar{n}^{\mathcal{A}}\psi_{\mathcal{A}\mu}$. Then, recalling the general form of the $N=2$ gravitini variation, given in eq. (4.1), one has

$$\delta_\varepsilon \psi_\mu = \bar{n}^{\mathcal{A}}\delta_\varepsilon\psi_{\mathcal{A}\mu} = \nabla_\mu\varepsilon - \bar{n}^{\mathcal{A}} S_{\mathcal{A}\mathcal{B}} n^{*\mathcal{B}}\gamma_\mu\varepsilon^c \, . \tag{4.56}$$

On the other hand, the general form of the gravitino transformation in $N=1$ supergravity is

$$\delta_\varepsilon\psi_\mu = \nabla_\mu\varepsilon - e^{\frac{K}{2}}\mathcal{W}\gamma_\mu\varepsilon^c \, , \tag{4.57}$$

where the combination $e^{\frac{K}{2}}\mathcal{W}$ involving the $N=1$ Kähler potential K and the superpotential $\mathcal{W}$ corresponds to a gravitino mass term in the $N=1$ supergravity action [117].

Comparing (4.56) and (4.57), one arrives at the identification [45, 46]:

$$e^{\frac{K}{2}}\mathcal{W} \;=\; \bar{n}^{\mathcal{A}} S_{\mathcal{A}\mathcal{B}} n^{*\mathcal{B}} \;=\; \frac{i}{2} e^{\frac{K_+}{2}}\left[a^2(\mathcal{P}^1 - i\mathcal{P}^2) - \bar{b}^2(\mathcal{P}^1 + i\mathcal{P}^2) - 2a\bar{b}\mathcal{P}^3\right] , \tag{4.58}$$

where in the second equality eq. (4.2) has been used.

At this point let us make a comment. The combination of the $N=2$ gravitini which is orthogonal to the one defining ψ_μ is $\tilde{\psi}_\mu := b\psi_{1\mu} - \bar{a}\psi_{2\mu}$. From the point of view of the $N=1$ theory, $\tilde{\psi}_\mu$ would be a component of a (possibly massive) spin 3/2 multiplet. Such multiplets are usually not included in the standard supergravity action, and should therefore be truncated out of the spectrum. However, the truncation is consistent only if the variation of $\tilde{\psi}_\mu$ under the preserved supersymmetry vanishes identically: $\delta_\varepsilon\tilde{\psi}_\mu \equiv 0$.

Using the general form of the $N=2$ gravitini variation and mass matrix, eqs. (4.1) and (4.2), this can be written as

$$e^{\frac{K_+}{2}}\Big[ab(\mathcal{P}^1-i\mathcal{P}^2)+\bar{a}\bar{b}(\mathcal{P}^1+i\mathcal{P}^2)+c_-\mathcal{P}^3\Big] \;=\; 0\;, \tag{4.59}$$

where as before $c_-=|a|^2-|b|^2$. Exploiting this constraint, we rewrite the combination $e^{\frac{K}{2}}\mathcal{W}$ in a slightly different form. Assuming $a\neq 0, b\neq 0$, multiplying (4.59) by $\frac{ic_-}{4\bar{a}b}$ and subtracting it from (4.58), we get the more symmetric looking expression

$$e^{\frac{K}{2}}\mathcal{W} \;=\; \frac{i}{4\bar{a}b}e^{\frac{K_+}{2}}\Big[ab(\mathcal{P}^1-i\mathcal{P}^2)-\bar{a}\bar{b}(\mathcal{P}^1+i\mathcal{P}^2)-\mathcal{P}^3\Big]\;. \tag{4.60}$$

Notice that if $c_-=0\Leftrightarrow |a|^2=|b|^2=1/2$, then eq. (4.58) already has this form. Substituting the geometric expressions (4.4) for the three $\mathcal{P}^x$, we conclude that

$$e^{\frac{K}{2}}\mathcal{W} \;=\; \frac{i}{4\bar{a}b}e^{\frac{K_+}{2}+2\varphi}\Big[\sqrt{2}i\int\langle\Phi_+\,,\,d_{H^{\rm fl}}{\rm Im}(ab\Phi_-)\rangle+\frac{1}{\sqrt{2}}\int\langle\Phi_+,G\rangle\Big]\;.$$

We identify the $N=1$ Kähler potential K as [118, 94, 92, 66]:

$$K \;=\; K_+ + 4\varphi\;. \tag{4.61}$$

This yields the compact expression for the superpotential

$$\mathcal{W}=\frac{i}{4\bar{a}b}\int\langle\Phi_+\,,\,\frac{1}{\sqrt{2}}G^{\rm fl}+d_{H^{\rm fl}}\Pi_-\rangle\;, \tag{4.62}$$

where we have defined

$$\Pi_- := \frac{1}{\sqrt{2}}A+i{\rm Im}(C\Phi_-)\;, \tag{4.63}$$

$A\equiv A_1+A_3+A_5$ being the sum of the internal RR potentials, such that $G=G^{\rm fl}+d_{H^{\rm fl}}A$. We also introduced

$$C:=\sqrt{2}ab\,. \tag{4.64}$$

In [118, 94], C was termed compensator, and also contained the 10d dilaton $e^{-\phi}$. Here instead we are including the dilaton in Φ_-, and C is just a constant. However, for ease of comparison with the literature we will continue to employ it.

In ref. [94] the form (4.62) of the $N=1$ superpotential was derived in the context of type IIA compactifications in the presence of an $O6$ orientifold. Here we have a slightly different perspective, in that we are just requiring an $N=2\to N=1$ truncation, not necessarily induced by an orientifold. This is in principle more general: for instance, the orientifold requires $|a|=|b|$, while here we are not imposing $c_-=0$. It is not clear to us whether this really allows for more general constructions. An argument against this is that a 10d analysis indicates that c_- should vanish for all compact $N=1$ solutions [33, 30]. Restricting to $c_-=0$, anyway, does not necessarily mean considering an orientifold, and eq. (4.62) should also give the correct superpotential of those $N=1$ low energy effective theories valid around $N=1$ AdS$_4$ vacua breaking $N=2$ spontaneously (at least for the cases in which these $N=1$ theories correspond to $N=2$ truncations). This seems to be confirmed by the fact that in the geometric SU(3) structure case, the superpotential (4.62) reduces to the one appearing in the example of ref. [92] mentioned at the beginning of this section. Clearly, it would be interesting to find a concrete new example.

We obtain the form of the superpotential in terms of the flux charges and the 4d fields if we substitute into (4.60) the explicit expressions (4.5) of the $N = 2$ Killing prepotentials:

$$\begin{aligned}\mathcal{W} &= \frac{i}{4\bar{a}b}\Big\{\big[i\mathrm{Im}(CZ^I)e_{IA} - i\mathrm{Im}(C\mathcal{G}_I)m_A^I\big]X^A + \big[i\mathrm{Im}(CZ^I)p_I^A - i\mathrm{Im}(C\mathcal{G}_I)q^{IA}\big]\mathcal{F}_A \\ &- (e_{\mathrm{RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A{}^I)X^A + (m_{\mathrm{RR}}^A + \xi^I p_I^A - \tilde{\xi}_I q^{IA})\mathcal{F}_A\Big\}\,. \end{aligned} \tag{4.65}$$

Eq. (4.65) is still written in terms of the $N = 2$ degrees of freedom, while we should restate it in $N = 1$ variables. Recall that, as discussed in subsection 4.1.2, in the $N = 2$ theory a subset of the scalars $\xi^I, \tilde{\xi}_I$, together with the axion a, has been dualized to antisymmetric 2-tensors in order to allow the introduction of the magnetic charges $m_{\mathrm{RR}}^A, p_I^A, q^{IA}$. However, according to the remark below eq. (4.16), the Killing prepotential $\mathcal{P}^3$ just contains the combinations of the $\xi^I, \tilde{\xi}_I$ which have not been dualized to antisymmetric tensors (these correspond to the scalars $\widehat{\xi}^{\mathbb{I}}$ of section 3.5). Hence the same will be true for the expression (4.65) of the superpotential. These scalars need to be recombined with the other $N = 2$ degrees of freedom z^i, φ contained in (4.65) in order to define appropriate holomorphic $N = 1$ variables for the superpotential. Inspection shows that $\mathcal{W}$ depends holomorphically on the following combinations[13]

$$U^I := \xi^I + i\mathrm{Im}(CZ^I) \qquad , \qquad \tilde{U}_I := \tilde{\xi}_I + i\mathrm{Im}(C\mathcal{G}_I). \tag{4.66}$$

Instead no redefinition is needed for the scalars t^a coming from the $N = 2$ vector multiplets, since they appear in (4.65) only through the holomorphic functions $X^A(t)$ and $\mathcal{F}_A(t)$. From (4.63), with the expansion $A = \xi^I\alpha_I - \tilde{\xi}_I\beta^I$, we can see that U^I and $\tilde{U}_I$ are precisely the coefficients of the expansion of Π_- on the basis of odd forms:

$$\Pi_- = U^I\alpha_I - \tilde{U}_I\beta^I\ . \tag{4.67}$$

Therefore Π_- defines the correct $N = 1$ coordinates, and is the $N = 1$ analog of Φ_- [94].

The form of the field redefinition (4.66) was already identified in [118, 94, 92]. Here we have verified that it is appropriate for any $N = 2 \to N = 1$ truncation, even in the presence of the general set of fluxes defined in subsection 3.3.5.

Substituting (4.66) into (4.65), we have [34]

$$\mathcal{W} = \frac{i}{4\bar{a}b}\Big[U^I e_{IA}X^A - \tilde{U}_I m_A^I X^A + U^I p_I^A\mathcal{F}_A - \tilde{U}_I q^{IA}\mathcal{F}_A - X^A e_{\mathrm{RR}A} + \mathcal{F}_A m_{\mathrm{RR}}^A\Big]\,, \tag{4.68}$$

which now depends on holomorphic variables only. Notice that this form of the superpotential directly descends from (4.62) if the expansion (4.67) is used.

[13]Of course, these variables have nothing to do with the matrices $U^{\mathbb{I}}{}_A, \widetilde{U}^A{}_{\mathbb{I}}$ employed in subsection 3.5.4.

4.3.2 D-terms from $N = 2 \to N = 1$ truncations

Having as a starting point the 4d $N = 2$ supergravity defined by the SU(3)×SU(3) compactification of type IIA, we now derive the general form of the D-terms arising from an $N = 2 \to N = 1$ truncation. As the superpotential, the D-terms are determined by a linear combination of the three $N = 2$ Killing prepotentials. If the superpotential was obtained by looking at the gravitini variations, we will identify the D-terms by studying the gaugini transformations.

Before going into this, we need some more notions about $N = 2 \to N = 1$ truncations. Unlike rigid supersymmetry, one cannot rewrite an $N = 2$ supergravity in an $N = 1$ form unless some restrictions are imposed. We have already discussed the necessity of truncating the spin 3/2 multiplet. Consistency with supersymmetry then imposes a series of constraints involving the other fields appearing in the action [115, 116].

For the sake of writing an expression for the D-terms, we won't need to consider the whole set of constraints, rather we can restrict to the ones involving the $N = 2$ vector multiplets. In particular, it is not necessary to deal with the more involved part of the story, namely the fact that (leaving aside the further complication due to the possible dualization to antisymmetric 2-tensors) the $N = 2$ quaternionic manifold parameterized by the scalar components of the hypermultiplets has to reduce to a submanifold respecting the Kähler-Hodge structure required by $N = 1$ supersymmetry. Some aspects of this will be needed in subsection 4.3.3, where we will study the F-flatness conditions in the case of an orientifold-induced truncation.

An $N = 2$ vector multiplet is composed of one vector, one complex scalar and two Weyl fermions (the gaugini), and splits in an $N = 1$ vector multiplet and an $N = 1$ chiral multiplet. The consistent truncation acts in such a way that out of n_V $N = 2$ vector multiplets (for us $n_V = b^+ \equiv \dim\mathscr{M}_+$), the resulting $N = 1$ theory inherits just $n_{Ch} \leq n_V$ chiral multiplets and $\widehat{n}_V = n_V - n_{Ch}$ vector multiplets. In more detail, splitting the indices as $A = (\check{A}, \widehat{A})$, with $A = 0, \ldots, n_V$, $\check{A} = 0, \ldots, n_{Ch}$ and $\widehat{A} = 1, \ldots, \widehat{n}_V = n_V - n_{Ch}$, we have the following conditions [115]:

$$A_\mu^{\check{A}} = 0 \quad , \quad X^{\widehat{A}} = 0 \tag{4.69}$$

Notice that A_μ^0 is always truncated. If we use special coordinates $t^a = X^a/X^0$ for $\mathscr{M}_+$, then the submanifold inherited by the $N = 1$ theory is parametrized by the $t^{\check{a}}$. Further conditions are

$$\begin{aligned} \mathcal{F}_{\widehat{A}} = 0 \qquad &; \qquad \mathcal{N}_{\check{A}\widehat{B}} = 0 \\ \mathscr{G}^+_{\check{a}\hat{\bar{b}}} = 0 \qquad &; \qquad D_{\check{a}} X^{\widehat{B}} = D_{\widehat{a}} X^{\check{B}} = 0\,. \end{aligned} \tag{4.70}$$

We now deduce the expression of the $N = 1$ D-terms by studying the gaugino variations, adapting an analogous derivation performed in [115]. In [115] this was done for the choice of the susy parameters $\varepsilon = \varepsilon_1$ and $\varepsilon_2 = 0$, while here we allow for an arbitrary linear combination $\varepsilon_{\mathcal{A}} = n_{\mathcal{A}}\varepsilon$, and moreover we set everything in the context of flux compactifications.

When splitting each $N = 2$ vector multiplet in two $N = 1$ supermultiplets, a linear combination of the two gaugini $\lambda^{a\mathcal{A}}$, $\mathcal{A} = 1, 2$ pairs up with the vector A_μ^a and becomes the gaugino of the $N=1$ vector multiplet, while the orthogonal combination enters in the chiral multiplet together with the scalar t^a. In order to recognize which combination of the gaugini belonging to a given $N=2$ vector multiplet corresponds to the $N=1$ chiral fermion and which other should be identified with the $N=1$ gaugino, it is sufficient to study the $N=2$ gaugini variation under the one preserved supersymmetry. Indeed, the chiral fermion has to transform into the scalar, while the $N=1$ gaugino goes into the vector field strength. The general form (ignoring three fermions terms) of the (positive-chirality) gaugini variation for the $N=2$ theory we are considering is[14] [76]

$$\delta\lambda^{a\mathcal{A}} \;=\; \partial_\mu t^a \gamma^\mu \varepsilon^{\mathcal{A}} - G^{(-)a}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{\mathcal{A}\mathcal{B}}\varepsilon_{\mathcal{B}} + W^{a\mathcal{A}\mathcal{B}}\varepsilon_{\mathcal{B}} \,. \tag{4.71}$$

While the gaugino mass matrix $W^{a\mathcal{A}\mathcal{B}}$ is defined in (4.21), we won't need the precise definition of $G^{(-)a}_{\mu\nu}$, corresponding to the anti self-dual part of the "dressed field strength" for the vectors inside the $N=2$ vector multiplets.

With our definition (4.23) of the $N=1$ susy parameter ε, we see that the relevant linear combination for the $N=1$ gaugino is

$$\tilde{\lambda}^a \equiv \bar{n}^{\mathcal{A}}\epsilon_{\mathcal{A}\mathcal{B}}\lambda^{a\mathcal{B}} \,, \tag{4.72}$$

Indeed this projects (4.71) on the term containing the field strength, excluding the term containing the scalar t^a:

$$\delta_\varepsilon\tilde{\lambda}^a \;=\; G^{(-)a}_{\mu\nu}\gamma^{\mu\nu}\varepsilon + \bar{n}^{\mathcal{A}}\epsilon_{\mathcal{A}\mathcal{B}}W^{a\mathcal{B}\mathcal{C}}n_{\mathcal{C}}\varepsilon \,. \tag{4.73}$$

The projection on the term containing $\partial_\mu t^a$ is instead obtained by considering

$$\rho^a \equiv n_{\mathcal{A}}\lambda^{a\mathcal{A}} \;,$$

so that

$$\delta_\varepsilon\rho^a \;=\; \partial_\mu t^a\gamma^\mu\varepsilon^c + n_{\mathcal{A}}W^{a\mathcal{A}\mathcal{B}}n_{\mathcal{B}}\varepsilon \,.$$

Two steps are still needed in order to get the identification of the $N=1$ gaugini. First, we should recall that conditions (4.69) imply that (with the special coordinates choice $t^a = X^a/X^0$) from a given $N=2$ vector multiplet we retain either the $N=1$ vector multiplet or the chiral multiplet. In particular, requiring $A^{\check{A}}_\mu = 0$ requires $\tilde{\lambda}^{\check{a}} = 0$ too. So we are left with the $\tilde{\lambda}^{\widehat{a}}$ only. Second, by looking at the variations of the surviving vectors $\delta_\varepsilon A^{\widehat{A}}_\mu$, and comparing with the generic susy transformation of an $N=1$ vector, one realizes that the correct identification for the $N=1$ gaugini $\lambda^{\widehat{A}}$ is [115]:

$$\lambda^{\widehat{A}} = -2e^{\frac{K_+}{2}}D_{\widehat{b}}X^{\widehat{A}}\,\tilde{\lambda}^{\widehat{b}} \,. \tag{4.74}$$

Similar arguments lead us to put $\rho^{\widehat{a}} = 0$ and to identify the n_{Ch} $N=1$ chiral fermions with the $\rho^{\check{a}}$.

[14] The derivative of the t^a is not covariantized since in the $N=2$ effective action obtained from flux compactifications as described in this paper one does not have gaugings of the special Kähler isometries.

Having now the expression (4.74) for the $N=1$ gaugini arising from the $N=2 \to N=1$ truncation, we can compare their supersymmetry variation with the general form of the gaugini variation in 4d $N=1$ supergravity, which reads (up to three fermions terms):

$$\delta\lambda^{\widehat{A}} = F^{(-)\widehat{A}}_{\mu\nu}\gamma^{\mu\nu}\varepsilon + iD^{\widehat{A}}\varepsilon \ , \tag{4.75}$$

where $F^{(-)\widehat{A}}_{\mu\nu}$ is the (anti self-dual) $N=1$ field strengths and $D^{\widehat{A}}$ are the D-terms, whose generic form is

$$D^{\widehat{A}} = -2(\mathrm{Im} f_{\widehat{A}\widehat{B}})^{-1}\mathscr{P}_{\widehat{B}} \ , \tag{4.76}$$

where $\mathscr{P}_{\widehat{B}}$ is the Killing prepotential of the $N=1$ theory depending on the scalars in the chiral multiplets and $f_{\widehat{A}\widehat{B}}$ is the vector kinetic matrix, which is holomorphic in the $N=1$ scalars.

Comparison of (4.75) with $\delta_\varepsilon\lambda^{\widehat{A}} = -2e^{\frac{K_+}{2}}D_{\widehat{b}}X^{\widehat{A}}\delta_\varepsilon\tilde{\lambda}^{\widehat{b}}$, $\delta_\varepsilon\tilde{\lambda}^a$ being given in (4.73), with the further information that $-2e^{\frac{K_+}{2}}D_{\widehat{b}}X^{\widehat{A}}G^{(-)\widehat{b}}_{\mu\nu}$ reduces to $F^{(-)\widehat{A}}_{\mu\nu}$ [115], provides the identification

$$\begin{aligned} D^{\widehat{A}} &= 2ie^{\frac{K_+}{2}}D_{\widehat{c}}X^{\widehat{A}}\bar{n}^{\mathcal{C}}\epsilon_{\mathcal{C}\mathcal{A}}W^{\widehat{c}\mathcal{A}\mathcal{B}}n_{\mathcal{B}} \\ &= -2e^{K_+}D_{\widehat{c}}X^{\widehat{A}}\mathscr{G}_+^{\widehat{c}\widehat{\bar{d}}}D_{\widehat{\bar{d}}}\bar{X}^{\widehat{B}}\left(\bar{n}^{\mathcal{C}}(\sigma_x)_{\mathcal{C}}{}^{\mathcal{B}}n_{\mathcal{B}}\right)\left(\mathcal{P}^x_{\widehat{B}} - \mathcal{N}_{\widehat{B}\widehat{C}}\tilde{\mathcal{P}}^{x\widehat{C}}\right) . \end{aligned} \tag{4.77}$$

We have also used $D_{\bar{a}}\bar{\mathcal{F}}_B = \mathcal{N}_{BC}D_{\bar{a}}\bar{X}^C$ (recall D.3) in order to factorize $D_{\bar{b}}\bar{X}^C$ in the expression (4.21) for $W^{a\mathcal{A}\mathcal{B}}$. Recalling the special geometry formula (4.49) and the fact that $X^{\widehat{A}}=0$, we obtain

$$D^{\widehat{A}} = (\mathrm{Im}\mathcal{N})^{-1\,\widehat{A}\widehat{B}}\Big\{2\mathrm{Re}\big[ab(\mathcal{P}^1_{\widehat{B}} - i\mathcal{P}^2_{\widehat{B}})\big] - \mathcal{N}_{\widehat{B}\widehat{C}}2\mathrm{Re}\big[ab(\tilde{\mathcal{P}}^{1\widehat{C}} - i\tilde{\mathcal{P}}^{2\widehat{C}})\big] + c_-(\mathcal{P}^3_{\widehat{B}} - \mathcal{N}_{\widehat{B}\widehat{C}}\tilde{\mathcal{P}}^{3\widehat{C}})\Big\} . \tag{4.78}$$

In [115] it is shown that $\overline{\mathcal{N}}_{\widehat{A}\widehat{B}}$ is holomorphic on the reduced manifold,[15] and by comparison with (4.76) it can then be identified with the holomorphic kinetic matrix $f_{\widehat{A}\widehat{B}}$ of the $N=1$ theory.

Substituting the expressions (4.5) for the Killing prepotentials and using the definition (4.64) of C, we finally obtain our expression for the D-terms:

$$\begin{aligned} D^{\widehat{A}} &= \sqrt{2}e^{2\varphi}(\mathrm{Im}\mathcal{N})^{-1\,\widehat{A}\widehat{B}}\Big\{\mathrm{Re}(CZ^I)e_{I\widehat{B}} - \mathrm{Re}(C\mathcal{G}_I)m^I_{\widehat{B}} - \mathcal{N}_{\widehat{B}\widehat{C}}\big[\mathrm{Re}(CZ^I)p^{\widehat{C}}_I - \mathrm{Re}(C\mathcal{G}_I)q^{I\widehat{C}}\big] \\ &- \frac{c_-}{2}(\tilde{G}_{\widehat{B}} - \mathcal{N}_{\widehat{B}\widehat{C}}G^{\widehat{C}})\Big\} . \end{aligned} \tag{4.79}$$

Since the $N=2 \to N=1$ truncation reduces also the hypersector, it is understood that the index I runs now over the surviving fields only.

Notice that, due to the fact that the graviphoton A^0_μ is always projected out by the $N=2 \to N=1$ truncation, the charges e_{I0}, m^I_0 do not appear in the expression for the D-terms. In the specific context of SU(3) structure compactifications, these charges are associated with the NS 3-form flux, recall (3.60). We conclude that H^{fl} contributes to the superpotential only.

[15] When the $N=2$ prepotential exists, this can be seen from the $\mathscr{M}_+$ analogous of (D.5): one checks that $\overline{\mathcal{N}}_{\widehat{A}\widehat{B}} = \mathcal{F}_{\widehat{A}\widehat{B}}$, which is holomorphic in the t^a.

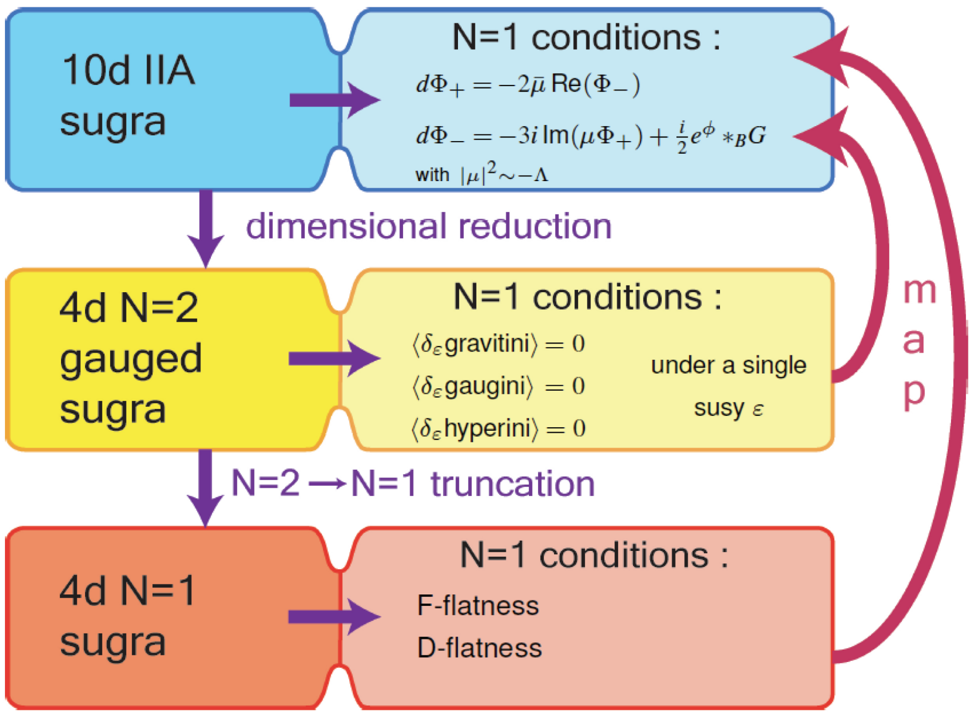

Figure 4.1: Summary of $N = 1$ conditions.

Furthermore, we can check that the D-terms vanish when considering a (geometric) Calabi-Yau orientifold with general RR fluxes [118]; this is because in the Calabi-Yau case all the basis forms are closed, i.e. $e_{Ia} = m_a^I = 0$ (recall the ansatz (3.61)), while the RR fluxes contained in $G^A, \tilde{G}_A$ (see below eq. (3.134)) don't contribute because the orientifold condition imposes $|a| = |b| \Leftrightarrow c_- = 0$.

More generally, we observe that the $N = 1$ theory does not have D-terms if $c_- = 0$ and $d_{H^{\rm fl}}\mathrm{Re}(ab\Phi_-) = 0$. This is a 'generalized half-flatness' condition for the manifold M_6 [94, 33]. In the SU(3) structure case this yields $d\mathrm{Re}(iab\Omega) = 0$, which, together with the constraint $d(J \wedge J) = 0$ (being always satisfied when adopting the ansatz (3.61) for the expansion forms), characterizes a half-flat manifold [21].

Finally, let us compare the D-flatness condition with the results of section 4.2. The combination of the Killing prepotentials defining the D-terms corresponds exactly to the one appearing in the vacuum condition (4.51); indeed, the D-terms are defined precisely by the same combination of the $N = 2$ gaugino variations which has been taken to write (4.51). The only difference is that here a part of the degrees of freedom has been eliminated by the $N = 2 \to N = 1$ truncation. According to the computation we did below eq. (4.51), we conclude that the D-flatness equation for $N = 1$ supersymmetric solutions corresponds to the real part of the second pure spinor equation, eq. (4.34), once this last is expanded in terms of the $N = 1$ degrees of freedom.

4.3.3 Supersymmetric vacua from $O6$-induced truncations

The $N = 1$ vacuum conditions we have analyzed in subsection 4.2.2 are also valid for the case of $N = 2 \to N = 1$ truncations. Of course, the truncation reduces the number of degrees of freedom and, since it has to be consistent with the preserved supersymmetry, part of the constraints presented in subsection 4.2.2 will be automatically satisfied. For instance, as we have discussed above eq. (4.59), truncating the $N = 2$ gravitini combination corresponding to $\tilde{\psi}_\mu$ goes together with $\delta_\varepsilon \tilde{\psi}_\mu = 0$, and this has to be imposed already at the level of the action.

Here we want to reinterpret the conditions of subsection 4.2.2 in the language of $N = 1$ supergravity. We will also re-establish the correspondence with the pure spinor equations, this time expanded in terms of the $N = 1$ degrees of freedom.

We have already seen in the previous subsection how the vanishing of the ε-generated susy variation of the $N = 2$ gaugini combination (4.72) corresponds in the truncated theory to the D-flatness condition, which therefore yields the real part of the second pure spinor equation.

The $\langle \delta_\varepsilon \psi_\mu \rangle = 0$ condition concerning the $N = 1$ gravitino is also readily treated using (4.57) and (4.26), yielding a relation between the spacetime curvature parameter μ and the vev of $e^{\frac{K}{2}} \mathcal{W}$ (the gravitino mass term):

$$\bar{\mu} = 2 \langle e^{\frac{K}{2} - \varphi} \mathcal{W} \rangle \ . \tag{4.80}$$

In order to write the F-flatness conditions associated with the chiral multiplets, one needs a more detailed knowledge of the $N = 2 \to N = 1$ truncation, and in particular of the way the $N = 2$ hypermultiplet sector is reduced to $N = 1$ chiral multiplets (or better, since antisymmetric 2-tensors are in principle present, how the $N = 2$ scalar-tensor multiplet reduces to $N = 1$ chiral and linear multiplets). For this reason we restrict ourselves to the explicit example of truncation provided by the inclusion of an $O6$ orientifold in the IIA background.

Orientifolds are string theory objects which correspond to hypersurfaces of the ten-dimensional background, and which determine a projection of the 10d supergravity fields, breaking at least half of the supersymmetry of the background. An $O6$ orientifold is a 6+1 dimensional object, that in our case fills the 4d spacetime, and wraps some 3-dimensional subspace of the compact manifold. The latter has to satisfy certain calibration conditions in order that the truncation induced by the orientifold breaks precisely half of the supersymmetry.

The 4d $N = 1$ action associated with the introduction of $O6$ planes was derived in [118] for Calabi-Yau compactifications (see also [119] for the related toroidal orbifolds), while the generalization to SU(3) and SU(3)×SU(3) structures has been discussed in [94]. In the following we summarize just the features that will be needed in order to compute the supersymmetric vacuum conditions.

The BPS condition associated with the $O6$ orientifold gives $a = \bar{b} e^{i\theta}$, where θ is an arbitrary phase. This implies $c_- = 0$ and $2ab = e^{i\theta}$.

Beside constraints (4.69), (4.70) concerning the $N = 2$ vector multiplet sector, even/odd parity of the internal forms under the orientifold projection imposes the following constraints on the $N=2$ hypermultiplet sector[16] (prior to the dualization of the axions):

$$\xi^{\widehat{I}} = \tilde{\xi}_{\check{I}} = \mathrm{Im}(CZ^{\widehat{I}}) = \mathrm{Im}(C\mathcal{G}_{\check{I}}) = \mathrm{Re}(CZ^{\check{I}}) = \mathrm{Re}(C\mathcal{G}_{\widehat{I}}) = 0,$$

where the index $I = 0, 1, \ldots, b^-$ has been split as $I = (\check{I}, \widehat{I}\,)$.

The $N=1$ scalar degrees of freedom are then encoded in

$$\Phi_+ = X^{\check{A}}\omega_{\check{A}} - \mathcal{F}_{\check{A}}\tilde{\omega}^{\check{A}} \qquad , \qquad \Pi_- = U^{\check{I}}\alpha_{\check{I}} - \tilde{U}_{\widehat{I}}\beta^{\widehat{I}} \; .$$

For the case in which no axions are dualized, the (real) dimension of the scalar manifold parameterized by $U^{\check{I}}, \tilde{U}_{\widehat{I}}$ is $2b^- + 2$, equal to half the dimension of the $N=2$ original quaternionic manifold.

The Kähler potential (4.61) of the $N=1$ theory reads

$$K = -\log i(\bar{X}^{\check{A}}\mathcal{F}_{\check{A}} - X^{\check{A}}\bar{\mathcal{F}}_{\check{A}}) + 4\varphi \; . \tag{4.81}$$

Its dependence on the $N=1$ chiral scalars $U^{\check{I}}$, $\tilde{U}_{\widehat{I}}$ is implicit in φ. Indeed, recalling (4.64) and using the fact that $i\int\langle\Phi_-, \bar{\Phi}_-\rangle = e^{-K_-}$, one has

$$\begin{aligned} e^{-2\varphi} = \frac{i}{4}\int\langle C\Phi_-, \overline{C\Phi_-}\,\rangle \;&=\; \frac{1}{2}\int\langle\, \mathrm{Re}(C\Phi_-)\,,\, \mathrm{Im}(C\Phi_-)\,\rangle \\ &=\; \frac{1}{2}\big[\mathrm{Im}(CZ^{\check{I}})\mathrm{Re}(C\mathcal{G}_{\check{I}}) \;-\; \mathrm{Re}(CZ^{\widehat{I}})\mathrm{Im}(C\mathcal{G}_{\widehat{I}})\big] \; . \end{aligned} \tag{4.82}$$

From the first line of (4.82), it follows [94] that $e^{-2\varphi}$ takes the form of a Hitchin functional. The real and imaginary parts of the pure spinor $C\Phi_-$ are related through the Hitchin map, which can also be expressed as $\mathrm{Re}(C\Phi_-) = *_b\mathrm{Im}(C\Phi_-)$. Hence $\mathrm{Re}(C\mathcal{G}_{\check{I}})$ and $\mathrm{Re}(CZ^{\widehat{I}})$ are functions of $\mathrm{Im}(CZ^{\check{I}})$ and $\mathrm{Im}(C\mathcal{G}_{\widehat{I}})$.

Recalling that $\Pi_- = \frac{1}{\sqrt{2}}A + i\mathrm{Im}(C\Phi_-) = U^{\check{I}}\alpha_{\check{I}} - \tilde{U}_{\widehat{I}}\beta^{\widehat{I}}$, we can see that $e^{-2\varphi}$ depends only on the imaginary parts of $U^{\check{I}}$ and $\tilde{U}_{\widehat{I}}$. Shifts of the RR scalars corresponding to the real parts of $U^{\check{I}}$ and $\tilde{U}_{\widehat{I}}$ are therefore isometries of the Kähler metric.

As an aside, we remark that the above also describes the example of the $N=2 \to N=1$ truncation exhibited in [92], even if no orientifold was introduced there. This example is based on compactifications on half-flat manifolds leading to $N=2$ theories without hypermultiplets, except the universal one (so $\dim\mathscr{M}_- = 0$); in the $N=1$ truncation only $U^0 = \xi^0 + i\mathrm{Im}(CZ^0)$ is kept.

F-flatness in the $U^{\check{I}}$ and $\tilde{U}_{\widehat{I}}$ directions

The F-flatness condition associated with the chiral multiplets coming from the $N=2$ hypersector could be studied demanding the vanishing of the chiral fermion susy transformations, and then exploiting the results of subsection 4.2.2. Equivalently, we choose to

[16] In this section the real and the imaginary parts of $C\Phi_-$ (and of its coefficients) are exchanged with respect to [118, 94]. This harmless difference can be traced back to the fact that in the SU(3) structure (or Calabi-Yau) case our $\Phi_- = Z^I\alpha_I - \mathcal{G}_I\beta^I$ reduces to $i\Omega$ instead of Ω.

evaluate the Kähler covariant derivatives of the superpotential with respect to the chiral scalars $U^{\check{I}}, \tilde{U}_{\widehat{I}}$, and impose

$$0 = D_{U^{\check{I}}}\mathcal{W} \equiv (\partial_{U^{\check{I}}} + \partial_{U^{\check{I}}}K)\mathcal{W} \qquad , \qquad 0 = D_{\tilde{U}_{\widehat{I}}}\mathcal{W} \equiv (\partial_{\tilde{U}_{\widehat{I}}} + \partial_{\tilde{U}_{\widehat{I}}}K)\mathcal{W}\ . \tag{4.83}$$

From (4.68) we immediately find the partial derivatives of the superpotential:

$$\partial_{U^{\check{I}}}\mathcal{W} = \frac{i}{4\bar{a}b}(e_{\check{I}\check{A}}X^{\check{A}} + p_{\check{I}}^{\check{A}}\mathcal{F}_{\check{A}}) \qquad , \qquad \partial_{\tilde{U}_{\widehat{I}}}\mathcal{W} = -\frac{i}{4\bar{a}b}(m^{\widehat{I}}_{\check{A}}X^{\check{A}} + q^{\widehat{I}\check{A}}\mathcal{F}_{\check{A}})\ . \tag{4.84}$$

The derivatives of the Kähler potential (4.81) are less trivial. Since K depends implicitly on $\mathrm{Im}U^{\check{I}} = \mathrm{Im}(CZ^{\check{I}})$ and $\mathrm{Im}\tilde{U}_{\widehat{I}} = \mathrm{Im}(C\mathcal{G}_{\widehat{I}})$ through $\varphi = \varphi(\mathrm{Im}(CZ^{\check{I}}), \mathrm{Im}(C\mathcal{G}_{\widehat{I}}))$, we have

$$\partial_{U^{\check{I}}}K = 4\partial_{U^{\check{I}}}\varphi \ = \ -2i\partial_{\mathrm{Im}(CZ^{\check{I}})}\varphi \qquad , \qquad \partial_{\tilde{U}_{\widehat{I}}}K = 4\partial_{\tilde{U}_{\widehat{I}}}\varphi \ = \ -2i\partial_{\mathrm{Im}(C\mathcal{G}_{\widehat{I}})}\varphi\ . \tag{4.85}$$

In order to evaluate this, we use the following property for the variation of a Hitchin functional

$$\delta e^{-2\varphi} \equiv \frac{i}{4}\delta\int\langle C\Phi_-, \overline{C\Phi_-}\,\rangle = \int\langle \mathrm{Re}(C\Phi_-), \delta\mathrm{Im}(C\Phi_-)\,\rangle\ , \tag{4.86}$$

which can be derived considering the decomposition under representations of SU(3)×SU(3) and recalling the fact that the Mukai pairing picks just the singlet. In terms of the moduli of $\mathrm{Im}(C\Phi_-) = \mathrm{Im}(CZ^{\check{I}})\alpha_{\check{I}} - \mathrm{Im}(C\mathcal{G}_{\widehat{I}})\beta^{\widehat{I}}$, (4.86) is rewritten as[17]

$$\begin{aligned} \frac{\partial\, e^{-2\varphi}}{\partial \mathrm{Im}(CZ^{\check{I}})} \ &= \ \int\langle \mathrm{Re}(C\Phi_-), \alpha_{\check{I}}\,\rangle \ = \ \mathrm{Re}(C\mathcal{G}_{\check{I}}) \\ \frac{\partial\, e^{-2\varphi}}{\partial \mathrm{Im}(C\mathcal{G}_{\widehat{I}})} \ &= \ -\int\langle \mathrm{Re}(C\Phi_-), \beta^{\widehat{I}}\,\rangle \ = \ -\mathrm{Re}(CZ^{\widehat{I}})\ . \end{aligned} \tag{4.87}$$

We conclude that

$$\partial_{U^{\check{I}}}K = ie^{2\varphi}\mathrm{Re}(C\mathcal{G}_{\check{I}}) \qquad , \qquad \partial_{\tilde{U}_{\widehat{I}}}K = -ie^{2\varphi}\mathrm{Re}(CZ^{\widehat{I}})\ .$$

Recalling the definition (4.64) of C, the fact that with our choice for the normalization of the pure spinors $e^{K_-} = e^{K_+}$, and eqs. (4.61), (4.80), we obtain

$$(\partial_{U^{\check{I}}}K)\mathcal{W} = 2i\bar{\mu}\mathrm{Re}(ab\mathcal{G}_{\check{I}}) \qquad , \qquad (\partial_{\tilde{U}_{\widehat{I}}}K)\mathcal{W} = -2i\bar{\mu}\mathrm{Re}(abZ^{\widehat{I}})\ .$$

It is now straightforward to see that the two sets of conditions (4.83) precisely give

$$\boxed{a\bar{b}\mathbb{Q}X = -2\bar{\mu}\mathrm{Re}(abZ)}\ . \tag{4.88}$$

Here $X^{\mathbb{A}} = \left(X^{\check{A}}, \mathcal{F}_{\check{A}}\right)^T$ and $Z^{\mathbb{I}} = \left(Z^{\widehat{I}}, \mathcal{G}_{\check{I}}\right)^T$, i.e. they are the remnants of the $N=2$ symplectic sections, containing only the truncated fields. The charge matrix is also reduced accordingly. In agreement with our discussion of subsection 4.2.1, eq. (4.88) corresponds to

[17] We also checked this explicitly by computing and inverting the jacobian for the change of variables $\left(e^{-\varphi}, \mathrm{Im}(a\bar{b}Z^{\check{\imath}}), \mathrm{Re}(a\bar{b}Z^{\hat{\imath}})\right) \longrightarrow \left(\mathrm{Im}(CZ^{\check{I}}), \mathrm{Im}(C\mathcal{G}_{\widehat{I}})\right)$, where the unphysical Z^0 has not been included in the old variables. The result confirms (4.87).

the first pure spinor equation, expanded in the $N = 1$ degrees of freedom and integrated over the internal manifold.

F-flatness in the $t^{\check{a}}$ directions

In order to write the F-flatness condition associated with the $N = 1$ chiral multiplets $(t^{\check{a}}, \rho^{\check{a}})$ descending from the $N = 2$ vector multiplet sector ($\rho^{\check{a}}$ are the chiral fermions), we will build on the results of subsection 4.2.2. Imposing $\langle \delta_\varepsilon \rho^{\check{a}} \rangle = 0$ is clearly the same thing as requiring $D_{\check{a}}\mathcal{W} \equiv (\partial_{t^{\check{a}}} + \partial_{t^{\check{a}}} K)\mathcal{W} = 0$. Indeed, the form of the variations of the chiral fermions dictated by $N = 1$ supergravity is

$$\delta_\varepsilon \rho^{\check{a}} = \partial_\mu t^{\check{a}} + 2e^{\frac{K}{2}} \mathscr{G}_{+}^{\check{a}\bar{\check{b}}} D_{\bar{\check{b}}} \overline{\mathcal{W}} \,. \tag{4.89}$$

The chiral fermions $\rho^{\check{a}}$ have been identified in subsection 4.3.2 with the $N = 2$ gaugini combination $n_{\mathcal{A}} \lambda^{\check{a}\mathcal{A}}$. Therefore we have the F-flatness condition $0 = \langle \delta_\varepsilon \rho^{\check{a}} \rangle = n_{\mathcal{A}} \langle \delta_\varepsilon \lambda^{\check{a}\mathcal{A}} \rangle$, where in $\delta_\varepsilon \lambda^{\check{a}\mathcal{A}}$ one should consider only the non-truncated degrees of freedom. Since $c_- = 0\,,\ n_{\mathcal{A}} \langle \delta_\varepsilon \lambda^{\check{a}\mathcal{A}} \rangle = 0$ is equivalent to $\bar{b} \langle \delta_\varepsilon \lambda^{\check{a}1} \rangle + a \langle \delta_\varepsilon \lambda^{\check{a}2} \rangle = 0$, and this corresponds to eq. (4.53).

At this point the computation becomes identical to the one in subsection 4.2.1, and we conclude that $\langle \delta_\varepsilon \rho^{\check{a}} \rangle = 0$ leads to

$$\boxed{\widetilde{\mathbb{Q}}\, \mathrm{Im}(abZ) = -3\mathrm{Im}(\mu a \bar{b} X) + \tfrac{1}{\sqrt{2}} \mathbb{N} G} \,. \tag{4.90}$$

Here again the symplectic vectors $X^{\mathbb{A}}$, $Z^{\mathbb{I}}$ and $G^{\mathbb{A}}$ contain just the components surviving the $N = 2 \to N = 1$ truncation. Eq. (4.90) corresponds to the imaginary part of the second pure spinor equation, expanded in the $N = 1$ degrees of freedom and integrated over the internal manifold.

Let us summarize the correspondence between the supersymmetric vacuum conditions arising in the $N = 1$ effective action and the pure spinor equations resulting from the 10d approach.

In order to perform the comparison, the pure spinor equations have to be expanded on the basis $\Sigma_\pm$, truncated to the $N = 1$ degrees of freedom only, and then integrated over the compact 6d manifold. The D-flatness constraint matches the real part of the second pure spinor equation, while the F-flatness condition for the chiral multiplets coming from the $N = 2$ vector multiplets corresponds to its imaginary part. F-flatness with respect to the chiral multiplets descending from the $N = 2$ hypersector provides instead the first pure spinor equation.

Even though we have performed the analysis of the present subsection for the orientifold case, it is pretty clear that it should be applicable more generally to any $N = 2 \to N = 1$ truncation.

4.4 Summary and discussion

In the first part of this chapter we shed more light on the $N = 2$ structure of the 4d theory which was obtained in the previous chapter by dimensional reduction of type II supergravity on SU(3)×SU(3) structure backgrounds with general fluxes. In doing this, we also discussed how the gravitino mass matrix $S_{\mathcal{AB}}$ derived from the higher dimensional supergravity in [45, 46] is recovered within the 4d, $N = 2$ theory. The $N = 2$ formalism then fixes the remaining fermionic mass matrices appearing in the supersymmetry transformations, and these determine the scalar potential via relation (4.22).

In the second part of the chapter, we confronted the 4d and 10d approaches to $N = 1$ vacua of type II theories. Starting from the $N = 2$ fermionic shifts, we established the $N = 1$ vacuum conditions, and we showed they satisfy an integrated version of the 10d $N = 1$ equations, written in the generalized geometry formulation of [27, 33]. Subsequently, we considered generic $N = 2 \to N = 1$ truncations of the 4d supergravity action, and we derived the $N = 1$ constraints within this $N = 1$ theory as D-flatness and F-flatness conditions. Again, we established the matching with the equations holding at the 10d level.

We remark that we have verified the correspondence in the presence of a very large set of fluxes, composed by the RR ones as well as by all the charges generated by the action of the twisted differential $d_{H^{\mathrm{fl}}}$. These include the NS and the geometric fluxes (see [P1] for the further inclusion of a complementary set of charges, which – as argued in [46] – turn out to be associated with nongeometric backgrounds).

Finally let us notice that, although we have explicitly performed the comparison in a type IIA setting, we expect the matching be the same for type IIB. Indeed, both the 10d pure spinor equations and the Killing prepotentials leading to the 4d $N = 1$ vacuum conditions display a very mirror symmetric aspect: to pass from IIA to IIB and back again, basically one just has to exchange the pure spinors $\Phi_+ \leftrightarrow \Phi_-$ and the RR fluxes $F^{\mathrm{even}} \leftrightarrow F^{\mathrm{odd}}$. As seen in the previous chapter, this symmetric structure is reflected in the 4d action, and in particular in the scalar potential V.

Chapter 5

Consistent reductions on cosets with SU(3) structure

As an application of the formalism developed in the previous chapters, we present here three concrete examples of dimensional reduction, based on SU(3) structures on coset spaces. The model is simple enough to allow full control on the compact geometry. Nevertheless, the associated 4d, $N = 2$ supergravity displays interesting features, like gaugings and tensor multiplets. We perform a detailed study, and establish the consistency of the truncation based on left-invariance. The scalar potential is non-trivial, and we explore his supersymmetric and non-supersymmetric extrema. Finally, exploiting the constraints imposed by $N = 2$ supersymmetry, we investigate the effects of string loops on the vacuum structure of the theory.

5.1 Introduction and overview

While the previous chapters covered the content of refs. [P1, P2], together with much background material, this chapter reproduces publication [P3], with just a few changes. In the context of this thesis, it can be seen as an application of the general picture developed above.

We consider concrete examples of SU(3) structures on coset spaces G/H. In contrast with the Calabi-Yau case (recall section 3.2), here the forms J and Ω defining the SU(3) structure are not closed, hence the truncation cannot be defined by an expansion in a basis consisting of harmonic forms only. For our expansion basis, we instead select the forms being invariant under the left action of the group G. This reduction ansatz is not motivated by a Kaluza-Klein analysis aimed at the identification of the light fluctuations around a single 10d vacuum, however it finds a solid justification in the fact that it yields a *consistent truncation*. We recall that a truncation is called consistent when all solutions to the lower dimensional equations of motion lift to solutions of the higher dimensional theory.

The resulting 4d, $N = 2$ supergravity is in agreement with the framework developed in the previous chapters, and displays several of the features described there. Specifically, it is a gauged $N = 2$ supergravity with one up to three vector multiplets (depending on which coset space is chosen), and a single tensor multiplet (replacing the universal hypermultiplet). In particular, we will focus on the off-shell scalar potential, and explore its extrema, which are also solutions of the 10d theory thanks to consistency.

Thanks to the full control on the internal geometry allowed by coset spaces, we can push our analysis beyond the supergravity approximation, and study the corrections that string theory induce to the higher dimensional supergravity action. Indeed, string theory is defined perturbatively as a double expansion in the string coupling constant $g_s = e^{\phi}$ and in the parameter α' controlling the string length. Supergravity is recovered by truncating the expansion at lowest order (tree level). Higher order corrections to the supergravity action are only partially known. E.g., various leading non-trivial contributions in α' to the 10d type II supergravity action have been determined [120, 121, 122, 123]. One may hope to establish the complete action to this order by 10d supersymmetric completion [124]. However, the 10d supersymmetry equations have simply proved too cumbersome to date. By contrast, the supersymmetric completion of the contribution of these terms to the 4d $N = 2$ supergravity action is readily available, yielding the full string tree level and one loop corrected action. In fact, in 4d we can, as we will discuss, even draw conclusions regarding the all string loop corrected action.

The chapter is structured as follows. In section 5.2 we introduce our coset spaces and, after having worked out the basis of left-invariant forms, we discuss their SU(3) structure. In section 5.3 we revisit the family of $N = 1$ backgrounds of massive type IIA supergravity found in [125, 126], and reparameterize the solution in terms of the fluxes. The discreteness of this family as a result of flux quantization is thus manifest, and we perform the required K-theory analysis. Furthermore, we show that multiple $N = 1$ solutions are permitted for a given choice of flux numbers. By the analysis of chapter 4.2 and [93], these solutions can be recovered from the 4d point of view. Then we turn to the dimensional reduction. In section 5.4, we demonstrate that the left-invariant coset reductions represent a consistent truncation by establishing that the 10d equations of motion reduce to the 4d equations following from the appropriate $N = 2$ action. The underlying property explaining the consistency of the dimensional reduction is the invariance under the group G of the modes we keep in the truncation ansatz; it follows that these fields define a closed system, which decouples from the remaining non-invariant modes. In section 5.5 we determine the off-shell $N = 2$ scalar potential of the four-dimensional theory. We do this first at tree level, and then including string loop corrections. Extrema of this potential are investigated in section 5.6. We find several non-supersymmetric Nearly Kähler companions to the solution of section 5.3 and study their stability. We also consider the question of the existence of de Sitter vacua, which has received some attention recently in the type IIA context [127, 128, 129, 130, 131, 132, 133]. We demonstrate that such vacua are absent at string tree level (we prove this result in greater generality than the coset context: it is valid for any gauged supergravity with merely the universal tree-level hypermultiplet, irrespective of the specifics of the vector multiplet sector). The inclusion of string loop corrections evades this no-go result, and we uncover a necessary condition on the contribution of the NSNS sector to the potential for de Sitter vacua to be possible.

Further material referring to this chapter is collected in appendices F and G, where in turn we fill in the details of the dimensional reduction leading to the 4d $N = 2$ theory, and study the string loop corrected 4d $N = 1$ conditions.

5.2 Introducing the internal geometries

We consider dimensional reductions of type IIA supergravity on left coset spaces $M_6 = G/H$ endowed with a left-invariant SU(3) structure. An exhaustive list of such cosets was provided in ref. [126] (see section 1 and in particular table 1 therein). In the following, we are going to focus on the cosets whose SU(3) structure cannot be further reduced to SU(2), namely[1]

$$\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)} \ , \quad \frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))} \ , \quad \frac{\mathrm{G}_2}{\mathrm{SU}(3)} \ , \tag{5.1}$$

where $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$ is non-maximally embedded in Sp(2). Topologically, these three cosets are identified respectively with the 'flag manifold' $\mathbb{F}(1,2;3)$, with $\mathbb{CP}^3$, and with S^6.

It is easy to see that a reduction performed on these manifolds by expanding the higher dimensional fields in a basis of left-invariant forms satisfies the constraints of [49] (and hence the contraints of section 3.3) and therefore yields a gauged $N = 2$ supergravity in 4d.

The remaining cosets listed in [126] have vanishing Euler characteristic and admit a left-invariant vector: this means that their SU(3) structure group is therefore further reduced to at least SU(2). For these cosets, the $N = 2$ reduction ansatz based on the presence of SU(3) structure can be more naturally enlarged to include the whole set of left-invariant forms, possibly yielding a further extended supergravity ($N \geq 4$) in 4d.

Recall that in subsection 2.2.2 we characterized an SU(3) structure by its torsion classes $W_i, i = 1, \dots, 5$. For the SU(3) structures on the cosets (5.1), the only non-vanishing torsion classes are W_1 and W_2, i.e. the SU(3) invariant 2- and 3-form J and Ω satisfy

$$\begin{aligned} dJ &= \frac{3}{2}\mathrm{Im}(\bar{W}_1\Omega) \ , \\ d\Omega &= W_1 J\wedge J + W_2\wedge J \ . \end{aligned} \tag{5.2}$$

In fact, $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$ allows just $W_1 \neq 0$. The manifolds whose SU(3) structure satisfies this condition are called Nearly Kähler. The cosets $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ and $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$ also admit a region in the SU(3) structure parameter space in which they are Nearly Kähler, but in general, their W_2 torsion class does not vanish. Since W_1 and W_2 can be chosen purely imaginary, these cosets fall into the class of 'half-flat' manifolds, characterized by $\mathrm{Re}W_1 = \mathrm{Re}W_2 = W_4 = W_5 = 0$ [21].

A description of the coset spaces (5.1) was given e.g. in [134]. In the context of SU(3) structure compactifications of (massive) type IIA supergravity, supersymmetric AdS_4 backgrounds on these manifolds have recently been found in [135, 125, 126] and further discussed in [136], while refs. [137, 131] study the properties of the associated effective 4d $N = 1$ supergravity in the presence of orientifold projections (see also [92] for a previous work considering the coset $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$). Type IIA reduction on Nearly Kähler manifolds has been

[1] The group $\mathrm{Sp}(n)$ is defined as $\mathrm{Sp}(n) := \mathrm{Sp}(2n,\mathbb{C})\cap\mathrm{U}(2n)$, and can also be denoted as $\mathrm{USp}(2n)$. It is compact, and has real dimension $n(2n+1)$. It does not coincide with the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ defined in footnote 4 of chapter 2, which has the same dimension, but is noncompact.

worked out in [93]; furthermore, in this paper the general analysis presented in section 4.2 was applied, and it was demonstrated that 4d solutions preserving $N=1$ supersymmetry lift to 10d. The cosets (5.1) appeared in the string literature in [138, 139] in the heterotic context, and have also been employed recently in [140] for heterotic dimensional reductions.

5.2.1 Coset spaces and expansion forms

In this subsection, after having introduced some preliminary notions about coset spaces, we provide the most general left-invariant positive-definite metric for each coset (5.1), as well as a basis for all the left-invariant differential forms, on which we are going to expand the supergravity fields.

We define the 6d coset spaces (5.1) as in ref. [126], and in particular we adopt the set of group structure constants listed therein. Following [126], we now give a concise account of the needed mathematical notions about coset spaces. A more extended discussion can be found e.g. in [134].

Consider a coset space G/H, and let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{h}$ the Lie algebra of H. Choose bases of generators[2]

$$\{\mathcal{H}_i\}\,,\quad i=1,\ldots,\dim H \qquad \text{and} \qquad \{\mathcal{K}_{\underline{m}}\}\,,\quad \underline{m}=1,\ldots,\dim G-\dim H$$

respectively for $\mathfrak{h}$ and for the complement $\mathfrak{k}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Then the structure constant of G are defined by

$$[\mathcal{H}_i,\mathcal{H}_j]=f^k{}_{ij}\mathcal{H}_k \quad , \quad [\mathcal{H}_i,\mathcal{K}_{\underline{m}}]=f^{\underline{n}}{}_{i\underline{m}}\mathcal{K}_{\underline{n}}+f^j{}_{i\underline{m}}\mathcal{H}_j \quad , \quad [\mathcal{K}_{\underline{m}},\mathcal{K}_{\underline{n}}]=f^{\underline{p}}{}_{\underline{mn}}\mathcal{K}_{\underline{p}}+f^i{}_{\underline{mn}}\mathcal{H}_i\,.$$

In the cases of interest for us (actually, in any compact case), the coset is reductive, i.e. $\mathfrak{k}$ can be chosen in such a way that $[\mathfrak{h},\mathfrak{k}]\subset\mathfrak{k}$, or in other words $f^i{}_{j\underline{n}}=0$.

A coframe on G/H is then introduced as follows. Let y^m, $m=1,\ldots,\dim G-\dim H$ be local coordinates on G/H, and let $L(y)$ be a coset representative. The $\mathfrak{g}$-valued 1–form $L^{-1}dL$ decomposes according to

$$L^{-1}dL=e^{\underline{m}}\mathcal{K}_{\underline{m}}+\theta^i\mathcal{H}_i\,.$$

The 1–forms $e^{\underline{m}}(y)$ define a local coframe on G/H. Furthermore, the Maurer-Cartan structure equation for $L^{-1}dL$ fixes the action of the exterior derivative on the coframe:

$$de^{\underline{m}} \;=\; -\frac{1}{2}f^{\underline{m}}{}_{\underline{np}}e^{\underline{n}}\wedge e^{\underline{p}} \;-\; f^{\underline{m}}{}_{i\underline{n}}\theta^i\wedge e^{\underline{n}}\,.$$

We now come to the notion of most interest for us, namely the one of *left invariance*: given a differential form on G/H

$$\omega_k=\frac{1}{k!}\omega_{\underline{m}_1\cdots,\underline{m}_k}e^{\underline{m}_1}\wedge\cdots\wedge e^{\underline{m}_k},$$

[2]As it will be immediately clear, the indices associated with $\mathfrak{k}$ are going to be frame indices on G/H. According to the conventions adopted in this thesis, they are hence underlined.

this is invariant under the left action of G if its components are constant and satisfy

$$f^{\underline{p}}{}_{i[\underline{m}_1}\,\omega_{\underline{m}_2\ldots\underline{m}_k]\underline{p}} = 0\ , \tag{5.3}$$

It is easy to see that the exterior derivative of a left-invariant form is still left-invariant.

For the coset metric $ds^2 = g_{\underline{mn}} e^{\underline{m}} \otimes e^{\underline{n}}$ the condition for left-invariance is analogous to (5.3), with a symmetrization of indices replacing the antisymmetrization.

Finally, we define the 'standard volume' on a 6d coset as

$$I := \int e^{123456}\ .$$

Let us now analyze the three cosets (5.1) in turn. As a first thing, we remark that none of them admits left-invariant 1– or 5–forms.

$\frac{\mathbf{SU(3)}}{\mathbf{U(1)\times U(1)}}$

Left-invariant metric:

$$g_{\underline{mn}} = \mathrm{diag}(v^1, v^1, v^2, v^2, v^3, v^3)\ , \quad v^1 > 0,\ v^2 > 0,\ v^3 > 0\ . \tag{5.4}$$

The left-invariant forms are spanned by

$$\begin{aligned}
&\omega_0 = 1 \quad , \quad \omega_1 = -e^{12} \quad , \quad \omega_2 = e^{34} \quad , \quad \omega_3 = -e^{56}\ ,\\
&\alpha = \frac{1}{2\sqrt{I}}(e^{135} + e^{146} - e^{236} + e^{245}) \quad , \quad \beta = \frac{1}{2\sqrt{I}}(-e^{136} + e^{145} - e^{235} - e^{246})\ ,\\
&\tilde{\omega}^0 = \frac{1}{I}e^{123456} \quad , \quad \tilde{\omega}^1 = \frac{1}{I}e^{3456} \quad , \quad \tilde{\omega}^2 = -\frac{1}{I}e^{1256} \quad , \quad \tilde{\omega}^3 = \frac{1}{I}e^{1234}\ .
\end{aligned} \tag{5.5}$$

$\frac{\mathrm{Sp(2)}}{\mathrm{S(U(2)\times U(1))}}$

Left-invariant metric:

$$g_{\underline{mn}} = \mathrm{diag}(v^1, v^1, v^1, v^1, v^2, v^2)\ , \qquad v^1 > 0,\ v^2 > 0\ . \tag{5.6}$$

Basis of left-invariant forms:

$$\begin{aligned}
&\omega_0 = 1 \quad , \quad \omega_1 = -e^{12} - e^{34} \quad , \quad \omega_2 = e^{56}\ ,\\
&\alpha = \frac{1}{2\sqrt{I}}(e^{135} + e^{146} + e^{236} - e^{245}) \quad , \quad \beta = \frac{1}{2\sqrt{I}}(e^{136} - e^{145} - e^{235} - e^{246})\ ,\\
&\tilde{\omega}^0 = \frac{1}{I}e^{123456} \quad , \quad \tilde{\omega}^1 = \frac{1}{2I}(e^{1256} + e^{3456}) \quad , \quad \tilde{\omega}^2 = -\frac{1}{I}e^{1234}\ .
\end{aligned} \tag{5.7}$$

$\frac{\mathbf{G_2}}{\mathbf{SU(3)}}$

Left-invariant metric:

$$g_{\underline{mn}} = \mathrm{diag}(v^1, v^1, v^1, v^1, v^1, v^1)\ , \qquad v^1 > 0\ . \tag{5.8}$$

Basis of left-invariant forms:

$$\omega_0 = 1 \quad , \quad \omega_1 = -e^{12} + e^{34} - e^{56} \, ,$$
$$\alpha = \frac{1}{2\sqrt{I}}(e^{135} + e^{146} - e^{236} + e^{245}) \quad , \quad \beta = \frac{1}{2\sqrt{I}}(-e^{136} + e^{145} - e^{235} - e^{246}) \, ,$$
$$\tilde{\omega}^0 = \frac{1}{I} e^{123456} \quad , \quad \tilde{\omega}^1 = \frac{1}{3I}(e^{3456} - e^{1256} + e^{1234}) \, . \tag{5.9}$$

Properties

The overall factors in the basis forms (5.5), (5.7), and (5.9) have been chosen in such a way that

$$\int \langle \omega_A, \tilde{\omega}^B \rangle = \delta_A^B \quad , \quad \int \alpha \wedge \beta = 1 \, , \tag{5.10}$$

where $A = (0, a)$, $B = (0, b)$ and a, b label the left-invariant 2– and 4–forms. The anti-symmetric pairing $\langle \, , \, \rangle$ is defined as in eq. (2.51).

The basis forms define a closed differential system,

$$\begin{aligned} d\omega_a &= q_a \alpha \, , \\ d\alpha = 0 \quad &, \quad d\beta = q_a \tilde{\omega}^a \, , \\ d\tilde{\omega}^A &= 0 \, , \end{aligned} \tag{5.11}$$

which is also closed under the action of the Hodge star operator,

$$*\alpha = \beta \quad , \quad *\tilde{\omega}^0 = \frac{1}{Vol} \quad , \quad *\tilde{\omega}^a = -\frac{1}{4Vol} \mathscr{G}^{ab} \omega_b \, .$$

Here, the q_a correspond to the geometric fluxes m_a^0 of subsection 3.3.5, cf. eq. (3.61) (while the e parameters of eq. (3.61) vanish here). Furthermore, Vol denotes the volume of the coset, and the matrix $\mathscr{G}^{ab}$ is the inverse of

$$\mathscr{G}_{ab} = \frac{1}{4Vol} \int \omega_a \wedge *\omega_b \, , \tag{5.12}$$

corresponding to the special Kähler metric on the space of the internal metric and B-field deformations [49]; see section F.1 of the appendix for more details.

In table 5.1, we give the values of the quantities introduced above for each coset. The standard volume I was computed following ref. [134].[3] Its evaluation requires knowledge of the Euler characteristic of our cosets. Since the harmonic forms on a compact coset reside among the left-invariant forms, we can read off the cohomology from the differential relations (5.11). We immediately conclude that all our cosets have trivial odd cohomology. Concerning the even cohomology, for $\frac{\mathrm{SU(3)}}{\mathrm{U(1)\times U(1)}}$, with

$$\omega_1' = \omega_1 - \omega_3 \, , \quad \omega_2' = \omega_2 - \omega_3 \, , \tag{5.13}$$

[3] We have a 2^6 supplementary factor in I with respect to [134]. This is due to the fact that for the normalization of the group structure constants we follow the choice of [126], and this differs from the one of [134] by a factor 1/2.

	$\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$	$\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	$\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$
range of a :	$1,2,3$	$1,2$	1
geometric flux q_a :	$q_1=q_2=q_3=-\sqrt{I}$	$q_1=2\sqrt{I}\ ,\ q_2=\sqrt{I}$	$q_1=2\sqrt{3I}$
$\mathscr{G}^{ab}\ =$	$\mathrm{diag}\Big(4(v^1)^2\,,\,4(v^2)^2\,,\,4(v^3)^2\Big)$	$\mathrm{diag}\Big(2(v^1)^2\,,\,4(v^2)^2\Big)$	$\frac{4}{3}(v^1)^2$
$Vol=$	$v^1v^2v^3I$	$(v^1)^2v^2I$	$(v^1)^3I$
$I=$	$2^5\pi^3$	$\frac{2^7\pi^3}{3}$	$\frac{144\pi^3}{5}$

Table 5.1: Values of the different quantities introduced in subsection 5.2.1.

we have

$$H^2=\mathrm{Span}\left([\omega_1'],[\omega_2']\right)\ ,\quad H^4=\mathrm{Span}\left([\tilde{\omega}^1],[\tilde{\omega}^2]\right)\ ,$$

hence the Euler characteristic is $\chi=6$.

For $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$, we have $b_2=1$ and $\chi=4$, while for $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$, $b_2=0$ and $\chi=2$.

5.2.2 The SU(3) structure

For each coset in (5.1), the pair of left-invariant forms parametrized by v^a,

$$J=v^a\omega_a\qquad,\qquad\Omega=2\sqrt{Vol}(\alpha+i\beta)\,,\tag{5.14}$$

satisfies the relations $J\wedge\Omega=0$ and $\frac{3i}{4}\Omega\wedge\bar{\Omega}=J\wedge J\wedge J$ and hence determines a left-invariant SU(3) structure (recall section 2.2). The metric specified by J and Ω is precisely the one given in eq. (5.4), (5.6), and (5.8) respectively for the three cosets. Using the properties of the basis forms listed in subsection 5.2.1 above, one can see that the differential relations (5.2) are satisfied, with torsion classes[4]

$$\begin{aligned} W_1 &= -\frac{iv^aq_a}{3\sqrt{Vol}}\ , \\ W_2 &= -\frac{2i}{3\sqrt{Vol}}q_a\big(v^av^b-\frac{3}{4}\mathscr{G}^{ab}\big)\omega_b. \end{aligned}\tag{5.15}$$

Substituting the quantities given in the table of subsection 5.2.1, we see that the Nearly Kähler condition $W_2=0$ is identically satisfied on $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$. For $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$ and $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$, this condition is satisfied on a line in the parameter space determined by $v^1=v^2$ and $v^1=v^2=v^3$ respectively. In this Nearly Kähler limit the cosets are Einstein manifolds (the only other loci at which the Einstein condition is satisfied are $2v^1=v^2$ for $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$ and $2v^1=2v^2=v^3$, or cyclic permutations of this, for $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ [134]).

The forms (5.14) are the most general left-invariant pair satisfying the SU(3) structure defining relations (the overall phase of Ω is unphysical; requiring the torsion classes to be

[4] The evaluation of W_2 is performed rewriting the second line of (5.2) as $W_2=2W_1J-*d\Omega$.

purely imaginary, as we have done, fixes it up to a sign). In particular, since the volume *Vol* is fixed by the v^a, we see that Ω identifies a rigid SL(3,$\mathbb{C}$) structure, and there are no almost complex structure moduli.

5.2.3 An alternative basis?

In the analysis of SU(3) structure reductions done in [49], conditions on the expansion forms were emphasized that arise when these are moduli dependent, as is the case with the basis of harmonic forms on which Calabi-Yau reductions are based (the *-ed conditions in section 2 of [49]). For the set of expansion forms that we have introduced above, these conditions are trivially satisfied, as the forms are moduli independent. In this sense, our expansion ansatz here is technically simpler than in the Calabi-Yau case. However, in a small flux approximation, the Laplacian $\Delta = -{*d*d} - d{*d*}$ becomes the mass operator for the modes of the 10d supergravity fields, and an expansion in eigenforms of it is physically motivated. Can we replace the forms introduced above by such a basis of eigenforms?

In the Nearly Kähler case, the expansion in eigenforms of the Laplacian is further motivated by the fact that both J and Ω are themselves eigenforms of Δ [93]. In the more general case $W_2 \neq 0$, this still holds for Ω,[5]

$$\Delta\Omega = \big(3|W_1|^2 + \frac{1}{4}W_2 \lrcorner \bar{W}_2\big)\Omega \,, \tag{5.16}$$

but not for J, which instead satisfies

$$\Delta J = 3|W_1|^2 J - \frac{3}{2}\mathrm{Re}(\bar{W}_1 W_2) \,.$$

Considering e.g. the coset $\frac{\mathrm{SU(3)}}{\mathrm{U(1)\times U(1)}}$, a change of basis sending the 2–forms introduced in (5.5) to a set of eigenforms of the Laplacian is

$$\omega_1' = \omega_1 - \omega_3 \quad , \quad \omega_2' = \omega_2 - \omega_3 \quad , \quad \omega_3' = \frac{\sum_a (v^a)^2 \omega_a}{\sum_b (v^b)^2} \,, \tag{5.17}$$

where $\Delta\omega_1' = \Delta\omega_2' = 0$, while $\Delta\omega_3' = \frac{(v^1)^2+(v^2)^2+(v^3)^2}{v^1 v^2 v^3}\omega_3'$. The harmonic 4–forms are spanned by

$$*\omega_1' \propto \frac{v^3}{v^1}\tilde{\omega}^1 - \frac{v^1}{v^3}\tilde{\omega}^3 \qquad , \qquad *\omega_2' \propto \frac{v^3}{v^2}\tilde{\omega}^2 - \frac{v^2}{v^3}\tilde{\omega}^3 \,, \tag{5.18}$$

while $*\omega_3' \propto -\sqrt{I}(\tilde{\omega}^1 + \tilde{\omega}^2 + \tilde{\omega}^3) = d\beta$ is exact.

The condition $v^a \partial_{v^b} \omega_a$ (*7 of [49]) gives rise to a complicated set of equations for possible v^a dependent normalization factors of the primed basis. However, it is easy to see upon inspection that the moduli independence of the triple intersection product (condition *8 of [49]) cannot be satisfied for any such choice. The question whether the choice of left-invariant expansion forms can be motivated from a Kaluza-Klein reduction point of view hence remains an interesting open question.

[5] One needs the relation $dW_2 = \frac{i}{4}(W_2 \lrcorner \bar{W}_2)\mathrm{Re}\Omega$, satisfied by the cosets (5.1).

5.3 Supersymmetric 10d solutions parametrized by fluxes

In this section, we will rewrite the family of $N = 1$ solutions of the 10d supergravity equations found in [126] in a manner which makes the discreteness of this family as a result of flux quantization manifest. By the analysis of chapter 4.2 and [93], these solutions can be recovered from the 4d point of view. After proving the full consistency of our reduction in section 5.4, we will proceed to complement these solutions with their non-supersymmetric relatives in section 5.6.

5.3.1 Flux quantization and K-theory

RR-fields are classified topologically by K-theory classes [141, 142]. This has two consequences for the choice of fluxes associated to the RR field-strengths. Firstly, the naive integer quantization of fluxes must be replaced by quantization in multiples of fractions determined also by the topology of the compactification manifold. Secondly, not every choice of flux number satisfying these quantization conditions will possess a K-theory lift and hence be permissible. We will now study these two points in turn.

In [141], fluxes were conjectured to take values in the image of the map

$$\sqrt{\hat{A}(X)}\,\mathrm{ch}(\cdot) : K(X) \to H^{\mathrm{even}}(X, \mathbb{Q}) \,.$$

$\mathrm{ch}(x)$ is the Chern character as extended to a K-theory element $x = E - F$ via $\mathrm{ch}(x) = \mathrm{ch}(E) - \mathrm{ch}(F)$. Hence,

$$\frac{[F(x)]}{2\pi} \;=\; \sqrt{\hat{A}}\,\mathrm{ch}(x)\,, \tag{5.19}$$

where $F = \sum_{i=0}^{5} F_{2i}$ denotes a formal sum of all RR field strengths, and $[\cdot]$ indicates rational cohomology class (rational rather than integral due to the fractional coefficients of Chern classes that appear in the expansion of the Chern character). When $H \neq 0$, the equations of motion and Bianchi identity of F are modified from the naive Maxwell form, enforcing harmonicity of F, to a version of these equations twisted by H. In particular, F now satisfies $(d - H)F = 0$. When H is exact, as will be the case in our study, H-twisted cohomology maps to ordinary cohomology via $F \to e^{-b}F$, where $H = db$. It hence proves convenient to introduce a basis of RR fields given by $G = e^{-b}F$. Equation (5.19) then holds for G rather than F, and the term ‘fluxes’ refers to the cohomology classes $[G]$.

To decide which fluxes we can choose as boundary conditions of our physical system (and then parametrize our solutions by this choice), we need to decide on electric vs. magnetic variables. Ignoring subtleties related to torsion, which does not enter in a supergravity analysis, we can choose the electric basis to lie in $\oplus_{i=1}^{3} H^{2i}(X, \mathbb{Q})$.

Let us now consider the question of flux quantization. To this end, we expand the right hand side of (5.19) in terms of Chern classes for x the class of a vector bundle F on X,

$$\mathrm{ch}_0(F) = \mathrm{rank}(F)\,, \qquad \mathrm{ch}_1(F) = c_1(F)\,, \qquad \mathrm{ch}_2(F) = \frac{1}{2}[c_1(F)^2 - 2c_2(F)]\,,$$
$$\mathrm{ch}_3(F) = \frac{1}{3!}[c_1(F)^3 - 3c_1(F)c_2(F) + 3c_3(F)]\,,$$
$$\hat{A} = 1 - \frac{p_1}{24} + \dots\,.$$

Hence,

$$\frac{[G_0]}{2\pi} = \mathrm{rank}(F)\,, \qquad \frac{[G_2]}{2\pi} = c_1(F)\,, \qquad \frac{[G_4]}{2\pi} = \frac{1}{2}[c_1(F)^2 - 2c_2(F)] - \frac{p_1(X)}{48}\mathrm{rank}(F)\,,$$
$$\frac{[G_6]}{2\pi} = \frac{1}{3!}[c_1(F)^3 - 3c_1(F)c_2(F) + 3c_3(F)] - \frac{p_1(X)}{48}c_1(F)\,.$$

As Chern classes take value in integral cohomology, it follows that, ignoring gravitational effects, in the presence of G_2 flux, $G_4/2\pi$ is generically half-integrally quantized, and $G_6/2\pi$ is quantized in multiples of $\frac{1}{6}$. Incorporating the $\hat{A}$-genus generically yields quantization in multiples of $\frac{1}{48}$ for both $G_4/2\pi$ and $G_6/2\pi$. In particular, for the cosets we are considering, the Pontrjagin classes are given by

$$p\left(\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}\right) = 1 \quad , \quad p\left(\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}\right) = (1+x^2)^4 \quad , \quad p\left(\frac{\mathrm{G}_2}{\mathrm{SU}(3)}\right) = 1\,. \tag{5.20}$$

The first result follows from a theorem of Borel and Hirzebruch, according to which the Pontrjagin class of a coset G/U, with U a maximal torus of G, is trivial. The latter two follow from the identification of the two cosets topologically with $\mathbb{CP}^3$ and S^6 respectively. The x that occurs is the generator of the integer cohomology of $\mathbb{CP}^3$. It follows that $G_6/2\pi$ is quantized in multiples of $\frac{1}{6}$ for the cosets $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ and $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$, and in multiples of $\frac{1}{12}$ for $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$. For $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$, we can go further. In [143], the following mod 2 relation among Chern classes is derived

$$c_3(E) \quad = \quad c_1(E)c_2(E) + Sq^2c_2(E) \qquad \mathrm{mod}\ 2\,.$$

Since $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$ has no 2- and 4-cohomology, it follows that $c_3(E)$ must be even for any vector bundle on this space ([143] provide an index theory argument for this conclusion). We conclude that on this coset, G_6 is integrally quantized. These results are summarized in table 5.2.

We turn to the second question raised above: given an element of $H^*(X,\mathbb{Q})$ satisfying the integrality conditions just discussed, when does it lie in the image of the map $\sqrt{\hat{A}}\,\mathrm{ch}(\cdot)$, thus qualifying as a viable choice of flux? We will not provide a general answer, but address the following two subquestions which will be relevant in the next subsection.

Is it possible to have only G_0 and G_6 non-vanishing? It is a theorem (see e.g. Thm. V.3.25 in [144]) that the map (5.19) provides an isomorphism when the domain is extended to rational K-theory, $K(X)\otimes\mathbb{Q}$. It follows that any class in $H^*(X,\mathbb{Q})$ lifts to a fractional K-theory class. Multiplying our choice of G_0 and G_6 with an appropriate integer hence always provides a viable choice of flux.

	$\frac{SU(3)}{U(1)\times U(1)}$	$\frac{Sp(2)}{S(U(2)\times U(1))}$	$\frac{G_2}{SU(3)}$
G_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
G_2	$\mathbb{Z}$	$\mathbb{Z}$	$-$
G_4	$\frac{1}{2}\mathbb{Z}$	$\frac{1}{12}\mathbb{Z}$	$-$
G_6	$\frac{1}{6}\mathbb{Z}$	$\frac{1}{12}\mathbb{Z}$	$\mathbb{Z}$

Table 5.2: Quantization condition on fluxes.

Given $G_2 = 0$, which G_4 are permissible? Let us consider the class

$$x = \frac{[G_4]}{2\pi} - \frac{p_1(X)}{48}\mathrm{rank}(F)\,.$$

Among the geometries we consider, x differs from $[G_4]/(2\pi)$ only for $\mathbb{CP}^3$. x is integrally quantized whenever G_2 vanishes. For the two cosets with non-trivial 2- and 4-cohomology, this is the only restriction on x, i.e. x can take values in all of $H^4(X,\mathbb{Z})$. As pointed out in [143], this situation arises whenever the cohomology of the manifold is generated in second degree. If we call the generators x_i, line bundles L_i exist with $c_1(L_i) = x_i$. The K-theory classes $x_{ij} = L_i \otimes L_j - L_i \oplus L_j$ can then be used as building blocks for lifting x to a K-theory class, by

$$\mathrm{ch}(x_{ij}) \;=\; x_i x_j + \frac{1}{2}(x_i^2 x_j + x_i x_j^2)\,.$$

If we choose G_0-flux as a multiple of 12, we can ignore the gravitational contribution which accounts for the difference between x and $[G_4]/(2\pi)$ in the case of $\mathbb{CP}^3$. Then, $[G_4]/(2\pi)$ takes values in $H^4(X,\mathbb{Z})$ also for this case.

5.3.2 The solution

The $N = 1$ supersymmetry conditions for an AdS$_4$ vacuum with internal SU(3) structure have been determined by [114] (see [27, 33] for generalization to the SU(3)×SU(3) structure context). A non-trivial warp factor is not allowed, and the dilaton ϕ has to be constant. Furthermore, in our conventions[6] the equations governing the H-field and the internal RR field strengths read

$$\begin{aligned} H &= (-1)^s \frac{2m}{5} e^{\phi}\,\mathrm{Re}\Omega\,, \\ F_0 &= m\,, \qquad F_2 = -\frac{f}{9}J + (-1)^s i e^{-\phi} W_2\,, \qquad F_4 = \frac{3m}{10} J\wedge J\,, \qquad F_6 = \frac{f}{6} J\wedge J\wedge J\,, \end{aligned} \tag{5.21}$$

[6]Here, the susy conditions are derived consistently with the conventions adopted all over this thesis. In particular, recall that our ansatz for the two type IIA susy parameters ϵ^1, ϵ^2 assigns negative chirality to ϵ^1 and positive chirality to ϵ^2. The resulting equations (5.21) and the SU(3) torsion classes differ from the ones in [126] by just a few minus signs. The factor of $(-1)^s = \pm 1$ arises in the following equations as unlike [114], we have fixed the phase of Ω once and for all in (5.14); see also [33]. Both signs are consistent with supersymmetry.

where the only non-vanishing purely imaginary torsion classes are $W_1 = (-1)^s \frac{4i}{9} e^{\phi} f$ and W_2. The only Bianchi identity which is not automatically satisfied is $dF_2 - HF_0 = 0$. This imposes

$$dW_2 = ie^{2\phi}\left(\frac{2}{27}f^2 - \frac{2}{5}m^2\right)\mathrm{Re}\Omega\,. \tag{5.22}$$

The AdS cosmological constant is determined by

$$\Lambda = -3e^{2\phi}\left(\frac{m^2}{25} + \frac{f^2}{9}\right)\,. \tag{5.23}$$

Following work of [125], [126] showed that these equations can be solved on the cosets we introduced in the previous section, by expanding all fields in forms invariant under the left group action. We will repeat this analysis, but parametrize the solutions by the fluxes $[G]$, as introduced in the previous subsection, rather than the parameter f and the dilaton. This is the favored approach as it allows us to take flux quantization into account naturally (from a 4d point of view, the distinction between fluxes and parameters such as f and the dilaton is most striking, as the former correspond to charges, the latter to vevs; in 10d, while fluxes can also be considered as vevs, they are distinguished by encoding topological information).

We will focus on $\frac{\mathrm{SU(3)}}{\mathrm{U(1)}\times\mathrm{U(1)}}$ for concreteness. This example is the most rich among the three cosets we are considering, as it has the largest set of left-invariant forms, and the largest cohomology.

The ansatz (5.14) already led to the expressions (5.15) for W_1 and W_2 in terms of the metric parameters v^a. It will prove convenient for this section to express the internal component b of the B-field using the closed 2-forms (5.13),

$$b \;=\; b'^1\omega'_1 + b'^2\omega'_2 + b'^3\omega_3\,.$$

Thus, b'^1 and b'^2 capture topological information about the B-field, while b'^3 enters in H. Likewise, our ansatz for G is

$$\begin{aligned}
G_0 &= m\,,\\
G_2 &= m'^1\omega'_1 + m'^2\omega'_2\,,\\
G_4 &= -e_1\tilde{\omega}^1 - e_2\tilde{\omega}^2 - \tilde{\xi}\,d\beta\,,\\
G_6 &= -e\tilde{\omega}^0\,.
\end{aligned}$$

The equations of motion for G are complicated, and are encoded in the equations (5.21). By contrast, the Bianchi identities are already guaranteed by the ansatz (hence the use of primed forms).

To solve (5.21) in terms of the flux parameters, we begin by solving (5.22) in term of ϕ, invoking the relation between W_1 and f,

$$e^{2\phi} \;=\; \frac{5}{16m^2v^1v^2v^3}\Big[6\sum_{a<b} v^a v^b - 5\sum (v^a)^2\Big]\,.$$

For the rest of this section, ϕ will denote this solution.

Utilizing the equation for H, this allows us to solve for b'^3 in terms of the metric parameters,

$$
\begin{aligned}
b'^3 &= (-1)^{s+1}\frac{4m}{5}\sqrt{v^1v^2v^3}e^{\phi} \\
&= (-1)^{s+1}\frac{m}{|m|}\sqrt{5\left(6\sum_{a<b}v^av^b-5\sum(v^a)^2\right)}\,.
\end{aligned}
$$

We next want to solve for f, starting with

$$
F_6 = G_6 + B\wedge G_4 + \frac{1}{2}B^2\wedge G_2 + \frac{1}{3!}B^3\wedge G_0 = \frac{f}{6}J\wedge J\wedge J\,. \tag{5.24}
$$

We eliminate the B^3 term via

$$
\begin{aligned}
F_4 = G_4 + B\wedge G_2 + \frac{1}{2}B\wedge B\wedge G_0 &= \frac{3m}{10}J\wedge J \\
\Leftrightarrow\quad mB^3 &= \frac{3m}{5}B\wedge J\wedge J - 2B\wedge G_4 - 2B^2\wedge G_2\,.
\end{aligned}
$$

Hence,

$$
\frac{f}{6}J\wedge J\wedge J = G_6 + \frac{2}{3}B\wedge G_4 + \frac{1}{6}B^2\wedge G_2 + \frac{m}{10}B\wedge J\wedge J\,,
$$

and substituting f into

$$
F_2 = G_2 + B\wedge G_0 = -\frac{f}{9}J + (-1)^s i e^{-\phi}W_2^-
$$

yields three equations which can be solved for b'^1, b'^2 and $\tilde{\xi}$,

$$
\begin{aligned}
b'^1 &= (-1)^s\frac{(5v^1-3(v^2+v^3))\sqrt{v^1v^2v^3}}{4v^2v^3m}e^{-\phi} - \frac{m'^1}{m}\,, \\
b'^2 &= (-1)^s\frac{(5v^2-3(v^1+v^3))\sqrt{v^1v^2v^3}}{4v^1v^3m}e^{-\phi} - \frac{m'^2}{m}
\end{aligned}
$$

We omit the expression for $\tilde{\xi}$, which is lengthy and not illuminating.

At this stage, we have expressed $\tilde{\xi}, b'^a, e^{\phi}$ in terms of v^a. Substituting these into the F_4 equation,

$$
G_4 + G_2\wedge B + \frac{1}{2}B\wedge B\wedge G_0 = \frac{3m}{10}J\wedge J\,, \tag{5.25}
$$

yields three independent equations for v^a, two of which take the simple form

$$
\begin{aligned}
\frac{(v^1-v^3)(v^1v^2+v^2v^3-3v^1v^3)}{v^1v^3} - e^{2\phi}\left(\frac{me_1}{I} + m'^2(2m'^1+m'^2)\right) &= 0\,, \\
\frac{(v^2-v^3)(v^1v^2+v^1v^3-3v^2v^3)}{v^2v^3} - e^{2\phi}\left(\frac{me_2}{I} + m'^1(m'^1+2m'^2)\right) &= 0\,.
\end{aligned} \tag{5.26}
$$

The main new feature we wish to demonstrate, as compared to the Nearly Kähler analysis of [93], is the presence of several supersymmetric vacua of a given theory, i.e. upon a fixed choice of fluxes. This phenomenon already occurs at $e_a = m'^a = 0$, which is a permissible choice of flux by the previous subsection. The third equation following from (5.25) here takes the form

$$15e^{\phi}\sqrt{v^1v^2v^3}\, e + (-1)^{s+1}8I\, v^2v^3(v^2v^3 - 3v^1v^2 - 3v^1v^3) = 0\,.$$

It is easy to see that this system of equations has, aside from the Nearly Kähler solution[7]

$$v^1 = v^2 = v^3 = \frac{\sqrt{15}}{2}\left(\frac{1}{20I}\left|\frac{e}{m}\right|\right)^{\frac{1}{3}}\,,$$

the solution

$$v^1 = v^2 = 2v^3 = \frac{\sqrt{15}}{4}\left(\frac{1}{2I}\left|\frac{e}{m}\right|\right)^{\frac{1}{3}}\,,$$

as well as two others which arise upon cyclic permutation of v^1, v^2, v^3.

The symmetry between the three metric parameters v^1, v^2, v^3 can be broken by considering backgrounds with G_4 flux. E.g., maintaining $G_2 = 0$, we obtain from (5.26)

$$\begin{aligned} e_1 \neq 0 \quad &\rightarrow \quad v^1 \neq v^3\,,\\ e_2 \neq 0 \quad &\rightarrow \quad v^2 \neq v^3\,,\\ e_1 \neq e_2 \quad &\rightarrow \quad v^1 \neq v^2\,. \end{aligned}$$

We have checked numerically that e.g. at $e_1 \neq 0, e_2 = 0$, solutions with $v^2 = v^3$ exist.

5.4 The dimensional reduction

5.4.1 The truncation scheme

As announced, we will adopt a reduction prescription in which the higher dimensional supergravity fields are expanded on a basis for the left-invariant tensors admitted by the coset. This expansion basis was introduced in subsection 5.2.1 for the three cosets (5.1).

We stress again that this G-invariant truncation does not coincide with a massless Kaluza-Klein ansatz, of the type adopted e.g. for the Calabi-Yau reduction (see subsection 3.2.1). We can illustrate the differences between the two schemes e.g. by considering the gauge vectors of the dimensionally reduced theory arising from the decomposition of the higher dimensional metric. The conventional massless Kaluza-Klein ansatz associates a gauge vector of the truncated theory to each Killing vector on the compact manifold, the gauge symmetry being inherited from the reparameterization invariance of the higher dimensional spacetime.[8] On the other hand, the G-invariant ansatz preserves just a subgroup of the full isometry group of the internal manifold G/H. The theory of compact left

[7]Note that physicality (positivity of v^a) determines the appropriate choice of s depending on the sign of e.

[8]In principle, non-vanishing background values of the non-metric supergravity fields may break the gauge symmetry to a subgroup of the isometry group, however this is guaranteed not to happen as far as these vevs are invariant under the isometries [12, pag. 16].

coset spaces endowed with a left-invariant metric (such are the cosets we consider) states that in general the isometry group of G/H is $G \times N(H)/H$, where $N(H)$ is the normalizer of H in G, defined as $N(H) := \{g \in G : gH = Hg\}$. The G factor in $G \times N(H)/H$ is associated with the left action of G on the coset, while the $N(H)/H$ factor derives from the right action of G. The Killing vectors generating the right isometries are left-invariant, while this is not the case for the ones generating the left isometries.[9] It follows that a left-invariant reduction ansatz keeps only the former, and the gauge group descending from the higher dimensional metric sector is just $N(H)/H$.

For the cosets we consider the G-invariant ansatz is particularly simple, because $N(H)/H$ turns out to be trivial. This can be seen either by observing that $\operatorname{rank} G = \operatorname{rank} H$ [134], or by noticing that our cosets do not admit left-invariant vectors at all. We conclude that no gauge vectors will descend from the dimensional reduction of the type II supergravity NSNS sector, and the whole (abelian) gauge group will be provided by the RR sector. This is analogous to what is realized in Calabi-Yau compactifications (in the latter case, this feature is due to the absence of continuous isometries, cf. subsection 3.2.1).

Though physically well motivated, dimensional reductions based on the full massless KK ansatz have a drawback: they are generically inconsistent [146, 12]. Rare exceptions are known, an example being the S^7 reduction of [78] (see [147] for a discussion of consistent KK sphere reductions). The G-invariant reduction scheme is instead believed to provide consistent truncations, due to the fact that the preserved invariant fields never generate the truncated non-invariant modes. A further argument for consistency is that the substitution of a G-invariant ansatz guarantees the dropping of the dependence on the internal coordinates y from the higher dimensional Lagrangian, see e.g. [148, 12] for more details. The consistency of the G-invariant scheme was explicitly shown in ref. [149] for a reduction of the pure gravity action. Recent related discussions can be found in [150] (for coset space reductions of Einstein-Yang-Mills theories), in [151, 152] (for Scherk-Schwarz reductions on group manifolds), and in [153, 154] (for consistent reductions on spaces supporting AdS solutions, and their relation with the dual superconformal field theory). However, an explicit check of consistency in the context of SU(3) structure compactifications with fluxes had not been performed to date. In subsection 5.4.3 we will work out the reduction of the higher dimensional equations of motion in detail, and prove the consistency of the truncation of the full type IIA bosonic sector for the cosets (5.1).

5.4.2 The 4d action

The general dimensional reduction established in chapter 3 applies to our coset spaces (5.1). In particular, for the RR sector the relevant analysis is the one of subsection 3.5.3.

Performing the necessary specializations, and separating the contributions of the NSNS and RR sectors as in chapter 3, the complete 4d bosonic action arising from the reduction of type IIA supergravity on our cosets reads then $S^{(4)} = S^{(4)}_{\rm NS} + S^{(4)}_{\rm RR}$, with

[9] A detailed discussion of the isometries of G/H can be found in section 2 of ref. [145].

$$S^{(4)}_{\rm NS} = \int_{M_4} \Big(\frac{1}{2} R_4 * 1 - \frac{1}{4} e^{-4\varphi} dB \wedge * dB - d\varphi \wedge * d\varphi - \mathscr{G}_{ab} dt^a \wedge * d\bar{t}^b - V_{\rm NS} * 1 \Big), \quad (5.27)$$

$$\begin{aligned} S^{(4)}_{\rm RR} &= \int_{M_4} \Big\{ \frac{1}{4} {\rm Im}\mathcal{N}_{AB} F^A \wedge * F^B + \frac{1}{4} {\rm Re}\mathcal{N}_{AB} F^A \wedge F^B - \frac{e^{2\varphi}}{4} (D\xi \wedge * D\xi + d\tilde{\xi} \wedge * d\tilde{\xi}) \\ &\quad + \frac{1}{4} dB \wedge \big[\xi d\tilde{\xi} - \tilde{\xi} D\xi + 2 e_A A^A + \tilde{\xi}\, q_a A^a \big] - \frac{1}{4} m^A e_A B \wedge B - V_{\rm RR} * 1 \Big\} . \end{aligned} \quad (5.28)$$

In appendix F we make the link with the analysis of chapter 3, illustrating how it specializes to the present cosets. Let us recall that the 4d degrees of freedom descending from the NSNS sector are the metric $g_{\mu\nu}$, the 2–form B, the complex scalars $t^a = b^a + i v^a$ and the 4d dilaton φ, defined in (F.2). The RR sector yields the scalars ξ and $\tilde{\xi}$ introduced in the first line of (F.11), as well as the gauge potentials A^A, whose modified field strengths F^A are defined in (F.12) (recall that the index A runs over $(0, a)$). Notice that the m^A and e_A used here correspond to the $m^A_{\rm RR}$ and $e_{\rm RRA}$ of chapters 3, 4.

The $N = 2$ action $S^{(4)}$ contains the gravitational multiplet $(g_{\mu\nu}, A^0)$, a number of vector multiplets (t^a, A^a) (see table 5.1 for the coset dependent range of a), and one tensor multiplet $(B, \varphi, \xi, \tilde{\xi})$. When $m^A = 0$ the antisymmetric tensor B becomes massless and can be dualized to a scalar, yielding the universal hypermultiplet. From $D\xi = d\xi - q_a A^a$ it follows that ξ is charged under the A^a, the charges being provided by the geometric fluxes q_a given in table 5.1. The graviphoton A^0 instead does not participate to this gauging (due to the fact that the compactification manifolds (5.1) do not allow for a flux of the NSNS 3–form [47]).

The special Kähler metric $\mathscr{G}_{ab}$ governing the kinetic terms for the scalars in the vector multiplets is given in table 5.1, and further discussed in section F.1 of the appendix, together with the period matrix $\mathcal{N}_{AB}$ describing the kinetic and topological terms for the gauge potentials.

The full 4d scalar potential reads $V = V_{\rm NS} + V_{\rm RR}$. Reduction of the internal NSNS sector on our coset spaces yields

$$\begin{aligned} V_{\rm NS} &\equiv -\frac{e^{2\varphi}}{2} \big(R_6 - \frac{1}{2} H \lrcorner H \big) \\ &= \frac{e^{2\varphi}}{4 Vol} q_a q_b \big(\mathscr{G}^{ab} - 3 v^a v^b + b^a b^b \big) , \end{aligned} \quad (5.29)$$

where the 6d Ricci scalar R_6 has been evaluated in terms of the torsion classes expressed in eq. (5.15) via the formula[10] [102]

$$R_6 = \frac{15}{2} |W_1|^2 - \frac{1}{2} W_2 \lrcorner \overline{W}_2 , \quad (5.30)$$

[10] An equivalent expression for R_6 was given in [134] using a general formula relating the Riemann tensor of G/H to the structure constants of G. The 4 factor mismatch we have with respect to that expression is due to the different normalization of the SU(3) structure constants already mentioned in footnote 3.

while for the internal NSNS 3–form we have $H = d_6 b = b^a q_a \alpha$. Equivalently, in order to evaluate $V_{\rm NS}$ we could have applied the general formula (3.85) derived in chapter 3.

The RR contribution to the scalar potential is obtained from the general expression given in eq. (3.117) by substitution of the expressions for $G_{(0)}$ and $\tilde{G}_{(0)}$ given in (F.11), and reads

$$V_{\rm RR} = -\frac{e^{4\varphi}}{4}\big[m^A {\rm Im}\mathcal{N}_{AB} m^B + (e_A + q_A\tilde{\xi} - m^C {\rm Re}\mathcal{N}_{CA})({\rm Im}\mathcal{N})^{-1\,AB}(e_B + q_B\tilde{\xi} - {\rm Re}\mathcal{N}_{BD} m^D)\big], \tag{5.31}$$

where $q_A = (0, q_a)$. Notice that while $\tilde{\xi}$ appears in the potential, the other RR scalar ξ is a flat direction (however, ξ is not a modulus, since it is charged under the A^a). We recall that, since the matrix ${\rm Im}\mathcal{N}$ is negative, $V_{\rm RR}$ is positive semi-definite.

5.4.3 Consistency of the truncation

We now prove the consistency of the dimensional reduction leading to the 4d action $S^{(4)}$ introduced in the previous subsection. To this end, we plug the G-invariant reduction ansatz into the bosonic equations of motion (EoM) of type IIA supergravity, and show that these yield the EoM following from the reduced action $S^{(4)}$.

The reduction of the equations for the RR degrees of freedom was already described in the general analysis of section 3.5; in appendix F.2 we provide the specialization of that analysis to the compactification on the coset spaces (5.1). In fact, the piece (5.28) of the 4d action has been established requiring its compatibility with the EoM for the 4d fields $A^A, \xi, \tilde{\xi}$ as obtained from the higher dimensional equations (3.93), (3.94). It follows that, as far the RR sector is concerned, the reduction is consistent by construction.

Hence, we just have to analyze the equations of motion for the NSNS degrees of freedom, namely the B-field, the Einstein and the dilaton equations. For the democratic formulation of type II supergravity, these have already been given at the end of section 3.1. However, for the reader's comfort we reproduce them here:

$$d(e^{-2\phi} * \hat{H}) - \frac{1}{2}[\hat{\mathbf{F}} \wedge *\hat{\mathbf{F}}]_8 \;=\; 0\,, \tag{5.32}$$

$$\hat{R}_{MN} + 2\hat{\nabla}_M\partial_N\phi - \frac{1}{2}\iota_M\hat{H}\lrcorner\iota_N\hat{H} - \frac{e^{2\phi}}{4}\sum_{k=0,2}^{10}\iota_M\hat{F}_{(k)}\lrcorner\iota_N\hat{F}_{(k)} \;=\; 0\,, \tag{5.33}$$

$$\hat{R} - \frac{1}{2}\hat{H}\lrcorner\hat{H} + 4\big(\hat{\nabla}^2\phi - \partial_M\phi\hat{\partial}^M\phi\big) = 0\,, \tag{5.34}$$

where we recall that the hat denotes 10d quantities, M, N are 10d spacetime indices, and in type IIA $\hat{\mathbf{F}} \equiv \sum_{k=0,2}^{10}\hat{F}_{(k)}$ is the sum of the even RR field-strengths.

$\hat{B}$-field EoM

The $\hat{B}$-field EoM (5.32) is an 8–form equation, and its expansion in the left-invariant forms on M_6 yields two independent equations: the first exhibiting two indices along 4d spacetime M_4 and 6 indices along M_6, and the second with 4 indices along M_4 and 4 indices along M_6. We get no equation with 5 indices along M_6 due to the absence of invariant 5-forms on the cosets (5.1). Rewriting the RR piece of (5.32) as in (3.109), expanding $\hat{B}$ as in (F.4) and $\hat{\mathbf{G}}$ as in (3.95), we see that eq. (5.32) reduces to

$$\Big[d(e^{-4\varphi} * dB) \;+\; G^A_{(0)}\tilde{G}_{(2)A} - \tilde{G}_{(0)A}G^A_{(2)} + \tilde{G}_{(1)} \wedge G_{(1)}\Big]\tilde{\omega}^0 = 0 \tag{5.35}$$

and

$$\begin{aligned} &-\; 4d_4(\mathscr{G}_{ab} *_4 d_4 b^b)\tilde{\omega}^a \;+\; e^{-2\phi+4\varphi}vol_4 \wedge d_6(*_6 d_6 b) \;+ \qquad (5.36)\\ &+\; \Big[G^0_{(0)}\tilde{G}_{(4)a} + G^0_{(4)}\tilde{G}_{(0)a} - \mathcal{K}_{abc}G^b_{(0)}G^c_{(4)} - G^0_{(2)} \wedge \tilde{G}_{(2)a} + \frac{1}{2}\mathcal{K}_{abc}G^b_{(2)} \wedge G^c_{(2)}\Big]\tilde{\omega}^a = 0 \;, \end{aligned}$$

where the 4d forms $G_{(p)}, \tilde{G}_{(p)}$ are expressed in (F.11), and we used $\omega_a \wedge \omega_b = -\mathcal{K}_{abc}\tilde{\omega}^c$, with the $\mathcal{K}_{abc}$ explicitly given in (F.7).

As in section 3.5, eq. (5.35) provides the EoM for the 2–form B in 4d, and has been employed in order to deduce the 4d action $S^{(4)}$ written in subsection 5.4.2 above (notice however that here we are not integrating over M_6). It follows that, on the same footing as the RR equations, consistency of this equation with the action $S^{(4)}$ is guaranteed by construction.

Eq. (5.36) (which was not analyzed in section 3.5) corresponds to the EoM for the 4d scalars b^a defined by the expansion of the internal B-field b on the basis 2–forms. Using $d_6 *_6 d_6 b = q_b b^b q_a \tilde{\omega}^a$, substituting the expressions (F.11) for $G_{(2)}, \tilde{G}_{(2)}, G_{(4)}, \tilde{G}_{(4)}$ and the definition (3.114) of F^A, eq. (5.36) reads

$$\begin{aligned} &4\nabla_\mu(\mathscr{G}_{ab}\partial^\mu b^b) \;-\; e^{2\varphi}\frac{q_b b^b q_a}{Vol} \;-\; \mathrm{Im}\mathcal{N}_{aB} * (F^0 \wedge *F^B) - \mathrm{Re}\mathcal{N}_{aB} * (F^0 \wedge F^B)\\ &+\; \frac{1}{2}\mathcal{K}_{abc} * (F^b \wedge F^c) + e^{4\varphi}\big[G^0_{(0)}(\mathrm{Im}\mathcal{N}G_{(0)} - \mathrm{Re}\mathcal{N}L)_a - \tilde{G}_{(0)a}L^0 + \mathcal{K}_{abc}G^b_{(0)}L^c\big] \;=\; 0, \end{aligned}$$

where we denote $L \equiv (\mathrm{Im}\mathcal{N})^{-1}(\tilde{G}_{(0)} - \mathrm{Re}\mathcal{N}G_{(0)})$. Recalling the form of $\mathrm{Im}\mathcal{N}$ and $\mathrm{Re}\mathcal{N}$ in (F.8) and (F.9), as well as V_{NS} in (5.29) and V_{RR} in (5.31), one checks that this equation can be reformulated as

$$2\nabla_\mu(\mathscr{G}_{ab}\partial^\mu b^b) - \frac{1}{4}\partial_{b^a}\mathrm{Im}\mathcal{N}_{BC} * (F^B \wedge *F^C) - \frac{1}{4}\partial_{b^a}\mathrm{Re}\mathcal{N}_{BC} * (F^B \wedge F^C) - \partial_{b^a}(V_{\mathrm{NS}} + V_{\mathrm{RR}}) = 0$$

which is precisely the EoM obtained varying $S^{(4)}$ in (5.27), (5.28) with respect to b^a.

10d Einstein equation

We first deal with the term $\hat{R}_{MN} + 2\hat{\nabla}_M\partial_N\phi$ in eq. (5.33). Starting from the G–invariant metric ansatz (F.1) and recalling that the 4d dilaton $\varphi(x)$ satisfies (F.3), we derive the following decomposition[11]

$$
\begin{aligned}
\hat{R}_{\mu\nu} + 2\hat{\nabla}_\mu \partial_\nu \phi &= R_{\mu\nu} - \frac{1}{4} g^{\underline{mp}} g^{\underline{nq}} \partial_\mu g_{\underline{mn}} \partial_\nu g_{\underline{pq}} - 2\partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \nabla_4^2 \varphi \ , \\
\hat{R}_{\mu n} &= 0 = \hat{\nabla}_\mu \partial_n \phi \ , \\
\hat{R}_{\underline{mn}} + 2\hat{\nabla}_{\underline{m}} \partial_{\underline{n}} \phi &= R_{\underline{mn}} + \frac{1}{2} e^{-2\varphi} \big(g^{\underline{pq}} \partial_\mu g_{\underline{mp}} \partial^\mu g_{\underline{nq}} - \nabla_4^2 g_{\underline{mn}} \big) \ .
\end{aligned}
\tag{5.37}
$$

Taking the trace, we get

$$
\hat{R} + 4\hat{\nabla}^2 \phi - 4\partial_M \phi \hat{\partial}^M \phi = e^{-2\varphi} \Big(R_4 + e^{2\varphi} R_6 - \frac{1}{4} g^{\underline{mp}} g^{\underline{nq}} \partial_\mu g_{\underline{mn}} \partial^\mu g_{\underline{pq}} - 2\nabla_4^2 \varphi - 2\partial_\mu \varphi \partial^\mu \varphi \Big) \ .
\tag{5.38}
$$

In the previous expressions, quantities labeled with 4 or 6 are associated to $(M_4, g_{\mu\nu})$ or (M_6, g_{mn}) respectively. The 4d indices on the r.h.s. are raised using the rescaled metric $g^{\mu\nu}$ of eq. (F.1). Notice that all the terms depend just on x^μ: indeed, thanks to G-invariance, the whole dependence on the internal coordinates drops out.

Let us now consider the $\mu\nu$ components of the 10d Einstein equation (5.33). Using (5.37), (5.38) we find (we reinstate in the Einstein equation the term proportional to $\hat{g}_{\mu\nu}$, which actually vanishes thanks to the dilaton EoM (5.34)),

$$
\begin{aligned}
\hat{R}_{\mu\nu} \ &+ \ 2\hat{\nabla}_\mu \partial_\nu \phi - \frac{1}{2} \iota_\mu \hat{H} \lrcorner \iota_\nu \hat{H} - \frac{1}{2} \hat{g}_{\mu\nu} \Big(\hat{R} + 4\hat{\nabla}^2 \phi - 4\partial_\rho \phi \hat{\partial}^\rho \phi - \frac{1}{2} \hat{H}^2 \Big) = \\
&= \ R_{\mu\nu} - \frac{1}{4} e^{-4\varphi} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - 2\partial_\mu \varphi \partial_\nu \varphi - 2\mathscr{G}_{ab} \partial_{(\mu} t^a \partial_{\nu)} \bar{t}^b \\
&- \ g_{\mu\nu} \Big(\frac{1}{2} R_4 - \frac{1}{24} e^{-4\varphi} H_{\mu\nu\rho} H^{\mu\nu\rho} - \partial_\mu \varphi \partial^\mu \varphi - \mathscr{G}_{ab} \partial_\mu t^a \partial^\mu \bar{t}^b - V_{\rm NS} \Big) \ .
\end{aligned}
\tag{5.39}
$$

For the RR piece, taking into account all the terms of the expansion described in section F.2 of the appendix, we arrive at

$$
\begin{aligned}
-\frac{e^{2\phi}}{4} \sum_{k=0}^{10} \iota_\mu \hat{F}_{(k)} \lrcorner \iota_\nu \hat{F}_{(k)} \ &= \ \frac{1}{2} {\rm Im} \mathcal{N}_{AB} \iota_\mu F^A \lrcorner \iota_\nu F^B - \frac{1}{2} e^{2\varphi} (D_\mu \xi D_\nu \xi + \partial_\mu \tilde{\xi} \partial_\nu \tilde{\xi}) \\
&- g_{\mu\nu} \Big\{ \frac{1}{4} \, {\rm Im} \mathcal{N}_{AB} F^A \lrcorner F^B - \frac{e^{2\varphi}}{4} [(D_\mu \xi)^2 + (\partial_\mu \tilde{\xi}\,)^2] - V_{\rm RR} \Big\} \ .
\end{aligned}
\tag{5.40}
$$

From (5.39), (5.40) we see that the equation arising from the $\mu\nu$ components of (5.33) precisely reproduces the 4d Einstein equation following from $S^{(4)}$.

Since there are no left-invariant 1–forms on the cosets (5.1), the 10d Einstein equation with μn indices is trivialized by our left-invariant truncation prescription, and does not yield any constraint at the 4d level. Indeed, one can check that all the μn terms in (5.33) vanish once the truncation ansatz is plugged in.

[11] The non-vanishing higher dimensional Christoffel symbols are:

$$
\hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \partial_\mu \varphi \delta^\rho_\nu + \partial_\nu \varphi \delta^\rho_\mu - g_{\mu\nu} \partial^\rho \varphi \ , \quad \hat{\Gamma}^\rho_{mn} = -\frac{1}{2} e^{-2\varphi} \partial^\rho g_{mn} \ , \quad \hat{\Gamma}^p_{\mu n} = \frac{1}{2} g^{pq} \partial_\mu g_{nq} \ , \quad \hat{\Gamma}^p_{mn} = \Gamma^p_{mn} \ .
$$

In the derivation of $\hat{R}_{\mu n} = 0$ we assume $\nabla_m e^{\underline{p}}{}_n = 0$.

Finally, we study the purely internal components of (5.33) in flat $\underline{mn}$ indices. Depending on which of the cosets (5.1) we consider, these yield just one, two or three 4d scalar equations, labeled by the index a. On our cosets, any left-invariant symmetric rank-2 tensor has the same diagonal structure as the invariant metric $g_{\underline{mn}}$ given in subsection 5.2.1. Furthermore, the left-invariant Ricci tensor on coset spaces satisfies $R_{\underline{mn}} = \frac{\partial}{\partial g^{\underline{mn}}} R_6$. Focusing for definiteness on $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$, we have (recall $\mathscr{G}^{ab}$ in table 5.1)

$$R_{\underline{2a-1}\,\underline{2a-1}} \equiv R_{\underline{2a}\,\underline{2a}} \;=\; -\frac{1}{8}\mathscr{G}^{ab}\partial_{v^b} R_6 \;, \qquad a = 1,2,3 \,.$$

Then, using the last line of (5.37), we get

$$\hat{R}_{\underline{2a}\,\underline{2a}} + 2\hat{\nabla}_{\underline{2a}}\partial_{\underline{2a}}\phi - \frac{1}{2}\iota_{\underline{2a}}\hat{H}\lrcorner\iota_{\underline{2a}}\hat{H} \;=\; \frac{e^{-2\varphi}\mathscr{G}^{ab}}{4}\big[-2\nabla_\mu(\mathscr{G}_{bc}\partial^\mu v^c) + \partial_{v^b}\mathscr{G}_{cd}\partial_\mu t^c\partial^\mu \bar{t}^d + \partial_{v^b}V_{\mathrm{NS}}\big]. \tag{5.41}$$

Concerning the RR term, a tedious computation gives

$$-\frac{e^{2\phi}}{4}\sum_{k=0}^{10}\iota_{\underline{2a}}\hat{F}_{(k)}\lrcorner\iota_{\underline{2a}}\hat{F}_{(k)} \;=\; \frac{e^{-2\varphi}\mathscr{G}^{ab}}{4}\big[\partial_{v^b}V_{\mathrm{RR}} - \frac{1}{4}\partial_{v^b}(\mathrm{Im}\mathcal{N}_{CD})F^C\lrcorner F^D\big] \,. \tag{5.42}$$

Analogous steps can be repeated for the cosets $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$ and $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$, leading to the same r.h.s. of the equations here above.

From (5.41), (5.42) we conclude that the components of the 10d Einstein equation (5.33) with two internal indices precisely match the EoM for the scalars v^a following from $S^{(4)}$:

$$-\;2\nabla_\mu(\mathscr{G}_{ab}\partial^\mu v^b) + \partial_{v^a}\mathscr{G}_{bc}\partial_\mu t^b\partial^\mu\bar{t}^c + \partial_{v^a}(V_{\mathrm{NS}} + V_{\mathrm{RR}}) - \frac{1}{4}\partial_{v^a}(\mathrm{Im}\mathcal{N}_{BC})F^B\lrcorner F^C \;=\; 0 \,.$$

Dilaton equation

Subtracting the trace over the $\mu\nu$ components of (5.33) from the 10d dilaton equation (5.34), we eventually obtain

$$2\nabla_4^2\varphi + \frac{1}{6}e^{-4\varphi}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{e^{2\varphi}}{2}\big[\,(D_\mu\xi)^2 + (\partial_\mu\tilde{\xi})^2\,\big] - 2V_{\mathrm{NS}} - 4V_{\mathrm{RR}} = 0 \,, \tag{5.43}$$

which is the EoM for the 4d dilaton φ following from $S^{(4)}$.

This concludes the consistency proof of the dimensional reduction.

5.5 The 4d potential via $N = 2$

In this section, we recast the scalar potential obtained in (5.29) and (5.31) in 4d $N = 2$ language. In this framework, given the prepotential $\mathcal{F}$ governing the special geometry data of the vector multiplet sector and the quaternionic metric h_{uv} of the hypermultiplet sector, the potential is uniquely determined by the gauged isometries of h_{uv}. This structure allows us to incorporate string loops into our considerations, which correct the hypermultiplet metric. As the 4 dimensional quaternionic metrics with the isometry structure imposed by our compactifications are highly constrained, we use the results of [155, 156, 157] to write down the general form of the all-loop string corrected potential in subsection 5.5.2. We analyze this potential further in subsection 5.6.3.

From the Ward identity (4.22) it follows that the general form of the potential in 4d $N=2$ gauged supergravity is [76, 15, 97, 98]

$$\begin{aligned} V &= 4e^K h_{uv}(X^A k^u_A - \tilde{k}^{uA}\mathcal{F}_A)(\bar{X}^B k^u_B - \tilde{k}^{uB}\bar{\mathcal{F}}_B) \\ &\quad - \left[\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1\,AB} + 4e^K X^A \bar{X}^B\right](\mathcal{P}^x_A - \tilde{\mathcal{P}}^{xC}\mathcal{N}_{CA})(\mathcal{P}^x_B - \tilde{\mathcal{P}}^{xD}\bar{\mathcal{N}}_{DB})\,. \end{aligned} \tag{5.44}$$

The coordinates X, the prepotential $\mathcal{F}$, and the gauge coupling matrix $\mathcal{N}$ encode special geometry data and are discussed further in appendix F. h_{uv} refers to the universal hypermultiplet metric, which is expressed in terms of the quaternionic vielbein components as

$$h \;=\; u\otimes\bar{u} + v\otimes\bar{v}\,.$$

We will denote the quaternionic coordinates collectively by q^u. k^u_A and $\tilde{k}^{uA}$ are the components of the Killing vectors describing the isometries of the hypermultiplet metric being gauged by the A^{th} gauge vector. The Sp(1) factor ω of the spin connection of the hypermultiplet metric enters in the potential via its relation to the Killing prepotentials. For the case that the 3 components of the curvature of ω each are invariant under an isometry $k^u\partial_{q^u}$ of the metric, the corresponding Killing prepotential is given by

$$\mathcal{P}^x \;=\; \omega^x_u k^u\,. \tag{5.45}$$

In this case, one can rewrite the potential in a more convenient form. Introducing

$$Q^u_A = k^u_A - \tilde{k}^{uB}\mathcal{N}_{BA}\,,$$

we obtain

$$V = Q^u_A \bar{Q}^v_B\Big[4e^K X^A\bar{X}^B\big(u\otimes\bar{u}+v\otimes\bar{v}\big)_{uv} - \big(4e^K X^A\bar{X}^B + \frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1\,AB}\big)\sum_x\big(\omega^x\otimes\omega^x\big)_{uv}\Big]\,. \tag{5.46}$$

5.5.1 Tree level

At tree level, the quaternionic vielbein is given by [68][12]

$$\begin{aligned} u &= \frac{1}{2}e^{\varphi}(d\tilde{\xi} - i d\xi)\,, \\ v &= d\varphi - i\frac{e^{2\varphi}}{2}\left(da + \tilde{\xi}d\xi\right)\,. \end{aligned}$$

[12] $\varphi, \xi, \tilde{\xi}$ were introduced above. The coordinate a is related to the dual a_B of the spacetime component of the B-field via $a_B = a + \frac{\xi\tilde{\xi}}{2}$.

The Sp(1) connection has the following form in terms of these quaternionic vielbein components[13]

$$\omega^1 = i(\bar{u} - u) \qquad , \qquad \omega^2 = -(u + \bar{u}) \qquad , \qquad \omega^3 = \frac{i}{2}(v - \bar{v})\,. \tag{5.47}$$

In the class of theories we are considering, the Killing vectors (4.11), (4.14), generating the isometries being gauged, reduce to

$$k_A = \sqrt{2}\left(e_A \frac{\partial}{\partial a} + q_A \frac{\partial}{\partial \xi}\right) \qquad , \qquad \tilde{k}^A = \sqrt{2} m^A \frac{\partial}{\partial a}\,. \tag{5.48}$$

Since Q^u does not contain a non-vanishing entry for $u = \varphi$, the real part of v does not enter upon contraction with Q^u, hence we can substitute

$$\sum_x \big(\omega^x \otimes \omega^x\big) \quad \sim \quad 4u \otimes \bar{u} + v \otimes \bar{v}$$

in the potential, obtaining

$$\begin{aligned} V \;\; = \;\; Q^u_A \bar{Q}^v_B \Big[- e^{2\varphi} \big(\frac{1}{2} (\mathrm{Im}\mathcal{N})^{-1\,AB} + 3 e^K X^A \bar{X}^B \big) \big(d\xi^2 + d\tilde{\xi}^2\big)_{uv} \\ - \frac{1}{8} e^{4\varphi} (\mathrm{Im}\mathcal{N})^{-1\,AB} \big(da + \tilde{\xi} d\xi\big)^2_{uv} \Big]\,. \end{aligned}$$

This coincides with (5.29) and (5.31) obtained above via reduction from 10 dimensions.

5.5.2 All string loop

For the case of the universal hypermultiplet with 3 isometries, the quaternionic metric is of the Calderbank-Pedersen form [155]. It comes in a 1-parameter family [156, 157], determined by

$$\begin{aligned} u \;\; &= \;\; \frac{\sqrt{\rho^2 + c}}{2(\rho^2 - c)} (d\tilde{\xi} - i d\xi)\,, \\ v \;\; &= \;\; -\frac{\rho}{2(\rho^2 - c)\sqrt{\rho^2 + c}} \left[2 \frac{\rho^2 + c}{\rho} d\rho + i (da + \tilde{\xi} d\xi) \right]\,. \end{aligned} \tag{5.49}$$

The metric at string tree level lies at $c = 0$, and the variable identification

$$\rho = e^{-\varphi}$$

takes us back to the expression for the metric introduced above.[14]

[13]The components ω^x of the Sp(1) curvature ω should not be confused with the expansion forms ω_a.

[14]The coordinates used in [157] are related to our choice via $\psi = \frac{a + \xi\tilde{\xi}}{2}\,, \eta = -\frac{\xi}{2}\,, \phi = \tilde{\xi}$.

In terms of the quaternionic vielbein components (5.49), the Sp(1) connection of the Calderbank-Pedersen metric is [155]

$$
\begin{aligned}
\omega^1 &= \frac{\rho}{\sqrt{\rho^2+c}}\, i(\bar{u}-u) &&= -\frac{\rho}{\rho^2-c}\, d\xi\,, \\
\omega^2 &= -\frac{\rho}{\sqrt{\rho^2+c}}(u+\bar{u}) &&= -\frac{\rho}{\rho^2-c}\, d\tilde{\xi}\,, \\
\omega^3 &= \frac{\sqrt{\rho^2+c}}{\rho}\frac{i}{2}(v-\bar{v}) &&= \frac{1}{2(\rho^2-c)}(da+\tilde{\xi}d\xi)\,.
\end{aligned} \tag{5.50}
$$

The $N = 2$ potential (5.46) for this choice of metric becomes

$$
\begin{aligned}
V = \frac{Q^u_A \bar{Q}^v_B}{(\rho^2-c)^2}\Big[&\big(-\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1\,AB} - 3e^K X^A \bar{X}^B\big)\rho^2(d\xi^2+d\tilde{\xi}^2)_{uv} \\
&-\frac{1}{8}(\mathrm{Im}\mathcal{N})^{-1\,AB}(da+\tilde{\xi}d\xi)^2_{uv} + c\, e^K X^A \bar{X}^B (d\xi^2+d\tilde{\xi}^2)_{uv} \\
&-\frac{c}{\rho^2+c} e^K X^A \bar{X}^B (da+\tilde{\xi}d\xi)^2_{uv}\Big]\,.
\end{aligned} \tag{5.51}
$$

In the case of Calabi-Yau compactifications, the metric is corrected away from $c = 0$ in passing from tree level to 1-loop [157]. Beyond 1-loop, all corrections can be captured by field redefinitions. This means that the quaternionic metric (i.e. the value of c) remains unchanged, the identification $\rho = e^{-\varphi}$ however is modified (note that the isometry structure of the metric determines the identification of the other 3 Calderbank-Pedersen coordinates with the 10d variables as indicated in footnote 14; this is why we have not introduced separate notation for them).

To study perturbative string corrections in the case of interest, let us review the argument of [157]. The 1-loop correction to the four dimensional Einstein-Hilbert term can be determined by reduction of the 1-loop R^4 correction in 10d. In the normalization of [157], this yields

$$
S_{\mathrm{Einstein-Hilbert}} = \int d^4x\sqrt{g}\Big(e^{-2\varphi} - \frac{4\zeta(2)\chi}{(2\pi)^3}\Big)R\,.
$$

Unfortunately, the full 1-loop corrected 10d action is not available as a means towards obtaining the 1-loop completion of the 4d action. Nonetheless, after parametrizing the ignorance regarding this action and comparing to the 4d effective action obtained by choosing the Calderbank-Pedersen metric on the universal hypermultiplet scalar manifold, [157] finds that only two possible values for c are possible,

$$
c = 0 \quad \text{or} \quad c = -\frac{4\zeta(2)\chi}{(2\pi)^3}\,, \tag{5.52}
$$

with χ the Euler characteristic of the Calabi-Yau. A perturbative string calculation then establishes that it is the latter value that is correct beyond tree level. Such a calculation in the case of the coset backgrounds with RR-flux that we are interested in is very challenging, and beyond our scopes. However, the first part of the analysis of [157] goes through also for these more general backgrounds. In particular, the 10d R^4 term is proportional to [157]

$$t_8 t_8 R^4 + \frac{1}{4} E_8 \, .$$

The first term is shorthand for $t_8 t_8 R^4 = t^{M_1 \cdots M_8} t^{N_1 \cdots N_8} R_{M_1 M_2 N_1 N_2} \cdots R_{M_7 M_8 N_7 N_8}$, which is expanded in terms of scalars built out of contractions of four Riemann tensors in eq. (A.12) of [157]. The second term can be written compactly in form notation as

$$E_8 \sim \Omega_{AB} \wedge \Omega_{CD} \wedge \Omega_{EF} \wedge \Omega_{GH} \wedge *(e^A \wedge \cdots \wedge e^H) \, ,$$

with $\Omega^A{}_B = \frac{1}{2} R^A{}_{BCD} e^C e^D$ the curvature 2-form and e^A, $A = 1, \ldots, 10$ a local coframe basis. From the expansion of the t_8 term in [157], we see that in each scalar invariant, contractions pair at least two Riemann tensors. Hence, this term does not contribute to the 4d Einstein-Hilbert term upon reduction. The contribution from E_8 to the Einstein-Hilbert term stems, exactly as in the Ricci flat case, from

$$\Omega_{ab} \wedge *_4(e^a \wedge e^b) \wedge \Omega_{mn} \wedge \Omega_{pq} \wedge \Omega_{rs} \wedge *_6(e^m \wedge \cdots \wedge e^s) \, ,$$

with a, b flat spacetime and $m, n, \ldots$ flat internal indices. We recognize the internal contribution as proportional to the 6 dimensional Euler density. The conclusion of our analysis is hence that in generalizing beyond Calabi-Yau manifolds, the same two possibilities for the Calderbank-Pedersen parameter c exist as in the Calabi-Yau case (and await a perturbative string calculation as arbiter).

5.6 Non-supersymmetric vacua

As an application of our consistent truncation result, we will search for non-supersymmetric vacua of the 4d effective action. By the analysis of section 5.4, these are guaranteed to lift to 10d solutions.

5.6.1 Tree level

The potential we obtained at tree level above has the form

$$V = A_1 e^{2\varphi} + A_2 e^{4\varphi} \, , \tag{5.53}$$

with

$$\begin{aligned} A_1 &= -Q^u_A \bar{Q}^v_B \big(\frac{1}{2} (\mathrm{Im}\mathcal{N})^{-1\,AB} + 3 e^K X^A \bar{X}^B \big) \big(d\xi^2 + d\tilde{\xi}^2 \big)_{uv} \, , \\ A_2 &= -Q^u_A \bar{Q}^v_B \frac{1}{8} (\mathrm{Im}\mathcal{N})^{-1\,AB} \big(da + \tilde{\xi} d\xi \big)^2_{uv} \, . \end{aligned} \tag{5.54}$$

Minimizing the potential with regard to the 4d dilaton yields [158]

$$V_\varphi = -\frac{A_1^2}{4 A_2} \, .$$

As A_2 is positive definite, the potential at tree level is negative semi-definite on-shell. In fact, this result generalizes immediately to any hypermultiplet metric of the general form (3.16) that arises upon Calabi-Yau and SU(3) structure compactifications, and the respective gaugings. The corresponding potential is obtained by appropriately modifying u and v in (5.54) along the lines of eq. (4.6). A_2 hence remains positive also in this more general case.

We have thus proved that $N = 2$ gauged supergravity as it arises in Calabi-Yau like compactifications at string tree level (i.e. with hypermultiplet metric as given in [68], and gaugings of abelian quaternionic isometries) does not permit de Sitter solutions. Due to the consistency of the truncation, this 4d result also follows from the 10d no-go theorem of Maldacena-Nuñez [159]. Note however that our 4d reasoning continues to hold for an *arbitrary* vector multiplet sector, i.e. including all possible worldsheet instanton corrections.

The two contributions to (5.53) arise upon compactification from the NSNS and the RR sector respectively, see (5.29) and (5.31). The positivity of A_2 is also manifest here.

5.6.2 Non-supersymmetric Nearly Kähler companions

The 10d analysis of subsection 5.3.2 reveals that, given a choice of the RR fluxes G_0 and G_6, with all the other fluxes vanishing, there exists a single Nearly Kähler supersymmetric vacuum on the cosets (5.1). This solution is also recovered adopting the 4d approach, as discussed in [92, 93].

It is possible to show that, under the same conditions, the 4d tree level scalar potential V also admits non-supersymmetric Nearly Kähler extrema. In the following formulae, we introduce the sum of the geometric fluxes $q \equiv \sum_a q_a$, we rename the RR fluxes as $e_0 \to e$, $m^0 \to m$, and we call the equal v^a and the equal b^a respectively v and b.

We obtain three Nearly Kähler extrema, lying at

$$v = \frac{\sqrt{15}}{2}\left(\frac{1}{20I}\left|\frac{e}{m}\right|\right)^{1/3} , \quad b = \frac{1}{2}\left(\frac{1}{20I}\frac{e}{m}\right)^{1/3} , \quad \tilde{\xi} = \frac{24Imb^2}{q} , \quad e^{2\varphi} = \frac{5q^2}{48I^2m^2v^4} , \tag{5.55}$$

$$v = \sqrt{3}\left(\frac{1}{20I}\left|\frac{e}{m}\right|\right)^{1/3} , \quad b = -\left(\frac{1}{20I}\frac{e}{m}\right)^{1/3} , \quad \tilde{\xi} = -\frac{12Imb^2}{q} , \quad e^{2\varphi} = \frac{q^2}{12I^2m^2v^4} , \tag{5.56}$$

and

$$v = \left(\frac{1}{\sqrt{5}I}\left|\frac{e}{m}\right|\right)^{1/3} , \quad b \;=\; 0 \;=\; \tilde{\xi} \;, \quad e^{2\varphi} = \frac{5q^2}{36I^2m^2v^4} . \tag{5.57}$$

By comparing to section 5.3.2, we learn that the only extremum preserving supersymmetry is (5.55).

Thanks to the consistency of the reduction, the non-supersymmetric extrema of V found here also solve the 10d equations of motion, and actually turn out to coincide with the solutions previously found in ref. [41] via a 10d approach (see subsection 11.4 therein).

Unlike the situation for the supersymmetric solution (5.55), for (5.56) and (5.57) stability is of course no longer guaranteed. As in any truncation scheme, a full stability analysis can only take place in the higher dimensional theory. What we can offer in our 4 dimensional theory is a stability analysis with regard to the modes we retain. To this end, we

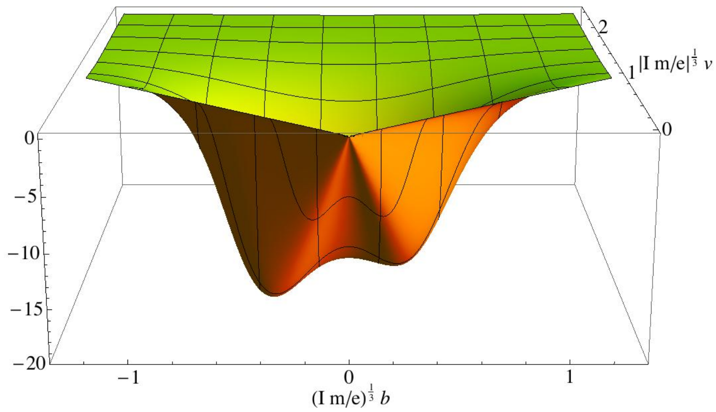

Figure 5.1: The potential for $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$: we plot the rescaled potential $e^{\frac{5}{3}}m^{\frac{1}{3}}I^{\frac{4}{3}}V$ as a function of $(Im/e)^{\frac{1}{3}}b$ and $|Im/e|^{\frac{1}{3}}v$, at the extremum of φ and $\tilde{\xi}$. The deepest minimum corresponds to solution (5.56). The cut of the plot at $V = 0$ is due to the constraint $e^{\varphi(b,v)} > 0$.

rescale the scalar fields[15] $(v^a, b^a, \varphi, \tilde{\xi})$ to obtain canonically normalized kinetic terms, and then diagonalize the mass matrix at the respective solutions.

The case $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$ is depicted in figure 5.1: the first two extrema (5.55) and (5.56) are minima, while the remaining extremum is a saddle point. For $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ and $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$, (5.56) is a minimum, whereas due to modes leading away from the Nearly Kähler locus $v^a = v$ for all a, (5.55) is merely a saddle point, as is (5.57). To analyze stability, we compare the magnitude of the negative masses at the saddle points with the Breitenlohner-Freedman bound

$$m^2_{\text{tachyonic}} \geq -\frac{3}{4}|V|\,.$$

All extrema (including the saddle point depicted in figure 5.1) prove stable.

Finally, we remark that α' and string loop corrections can be safely neglected for the solutions above by tuning the RR fluxes e and m in such a way that the internal volume $Vol \equiv v^3 I \sim e/m$ becomes sufficiently large and the string coupling constant $e^{\phi} \equiv e^{\varphi}\sqrt{Vol} \sim e^{-\frac{1}{6}}m^{-\frac{5}{6}}$ becomes small (recall the definition (F.2) of the 4d dilaton). We can study moderately large string coupling by invoking the corrected potential (5.51). A numerical analysis indicates that all three AdS extrema survive string loop corrections. For the supersymmetric extremum, we push beyond numerics in appendix G, and establish analytically that it persists, as expected, in the face of string loop corrections.

5.6.3 de Sitter vacua at all string loop order?

In face of the no-go result for de Sitter vacua obtained in subsection 5.6.1, we would like to analyze how loop corrections modify the outcome of this study. Of course, to guarantee the consistency of the truncation, the analysis in section 5.4 must be extended beyond the two derivative case. However, the arguments put forth in subsection 5.4.1 in favor of consistency apply to the additional terms as well. We will also assume in this section

[15]Note that the shift symmetry of a and ξ is gauged, the background value of these fields is hence a gauge choice.

that $c \neq 0$, as in the Calabi-Yau case. Note that by the results above, we can perform an (almost) complete analysis of the full loop corrected potential. The identification of the physical coordinate φ and the Calderbank-Pedersen coordinate ρ, which is modified order by order in the string coupling and is not available, merely enters in identifying the range of the CP coordinate, see below. Away from very strong coupling (in which brane instanton corrections would have to be considered regardless), this does not affect the search for de Sitter minima.

Focusing on the ρ dependence of the potential (5.51) and taking the obvious positivity constraints on the coefficients into account does not rule out de Sitter vacua. One can then proceed to derive various constraints on these coefficients. E.g., by noting that the potential (5.51) has the form

$$V(\rho) \;=\; P(\rho)Q(\rho)\,,$$

with $P(\rho) = \frac{1}{(\rho^2-c)^2}$, we obtain

$$\begin{aligned} V(\rho_0) \;&=\; -\frac{P^2}{P'}Q'|_{\rho_0} \\ &=\; \frac{Q^u_A\bar{Q}^v_B}{2(\rho_0^2-c)}\Big[\big(-\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1\,AB}-3e^K X^A\bar{X}^B\big)(4d\xi^2)_{uv} \\ &\qquad\qquad +\frac{c}{(\rho_0^2+c)^2}e^K X^A\bar{X}^B(da+\tilde{\xi}d\xi)^2_{uv}\Big]\,, \end{aligned}$$

where ρ_0 signifies the value of ρ at a minimum of the potential. Since c is negative for the cosets we are considering, a de Sitter vacuum requires the first term in the square bracket to be positive at the minimum of the potential. This term is proportional to the tree level NSNS contribution to V, given in eq. (5.29). Hence, our necessary condition translates into the following inequality involving the internal NSNS 3–form and Ricci scalar

$$H \lrcorner H - 2R_6 > 0\,.$$

Recalling eq. (5.30), this is obviously true whenever the non-vanishing SU(3) torsion classes satisfy $15|W_1|^2 < W_2 \lrcorner \overline{W}_2$. For the simple case of Nearly Kähler manifolds (i.e. when $W_2 = 0$) the inequality is however non-trivial, and reads $3b^2 - 5v^2 > 0$.

It would be interesting to analyze further this loop-corrected scalar potential, possibly including non-perturbative corrections due to brane instantons.

Chapter 6

Conclusions

In this thesis we have studied dimensional reductions of type II string theory yielding $N = 2$ supergravity in four dimensions. We have illustrated how the formalism of generalized geometry allows to approach the problem on general grounds, and provides the expressions for the various data of the $N = 2$ theory. Beside the previously known formulae for the Kähler potentials of special Kähler geometry and for the Killing prepotentials determining the $N = 2$ gaugings [45, 46], the expressions established in the context of this thesis were the metric on the space of deformations of the internal metric and B-field, the period matrices of special Kähler geometry, and the scalar potential [P1, P2]. In particular, the expression for the scalar potential can be helpful for the study of non-supersymmetric string backgrounds with SU(3)×SU(3) structure. See sections 2.6, 3.6, 4.4 and 5.1 for a more detailed discussion of the contents of the respective chapters.

Despite the progresses made over the last years, several aspects of flux compactifications deserve further study. For instance, a systematic characterization of the light degrees of freedom associated with a Kaluza-Klein truncation in the presence of fluxes has not been derived to date. A related open question is the identification of the moduli space of the geometries supporting flux backgrounds. A further point which is essential in the construction of realistic models is the role of the warp factor. Indeed, warping is generic in flux compactifications including localized sources, and its effect on the low energy effective physics in four dimensions is not completely understood yet.

From a more formal viewpoint, another question arising in the context of flux compactifications concerns the relations existing between lower dimensional supergravities and higher dimensional supergravity/string theories. It would be very interesting to understand which supergravities can be derived via dimensional reduction from a higher dimensional theory. Here, we have approached a more circumscribed question, namely which deformations of the ungauged $N = 2$ theory associated with Calabi-Yau compactifications can be realized via string compactifications with fluxes. As we have seen, the answer should be searched in the realm of SU(3)×SU(3) structures. We remark that the gaugings realized in this context are all associated with the hypermultiplet sector of $N = 2$ supergravity, hence the gauge group of the 4d theory remains abelian. It would be interesting to study if it is possible to obtain $N = 2$ gaugings which determine a non-abelian gauge group (see [160] for an M-theory derivation).

In order to gain a deeper understanding of these issues, it would be important to elaborate new explicit examples of string backgrounds with fluxes, possibly in a generalized geometry framework. For instance, in order to fully exploit the SU(3)×SU(3) structure formalism, one should find an example in which the basis defining the truncation consists of forms of mixed degree (not simply related to a basis of forms of pure degree by a symplectic rotation). In this thesis, we have studied in detail the case of dimensional reduction on certain coset spaces with SU(3) structure [P3]. These comprise the currently known examples of $N = 1$ AdS_4 vacua of type II supergravity in the absence of sources, and have the remarkable property of yielding a consistent truncation. The constraints imposed both by the coset structure and by $N = 2$ supersymmetry allow to take stringy corrections into account, going in this way beyond the supergravity approximation. It would be interesting to derive a thorough description of these corrections in the compactification context. Though not directly applicable to phenomenology, these AdS_4 supersymmetric solutions, together with their non-supersymmetric companions existing on the same coset spaces, are relevant [161] for the recent activity on the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence [162].

Appendix A

Notation and conventions

A.1 Indices

We summarize here the meaning of the several different indices used all over this thesis:

letters	range	labeling
$M, N, \ldots$	$0, \ldots, 9$	10d spacetime coords.
$\mu, \nu, \ldots$	$0, \ldots, 3$	4d spacetime coords.
$m, n, \ldots$	$1, \ldots, 6$	6d space coords.
$\Lambda, \Sigma, \ldots$	$1, \ldots, 12$	vector repr. of O(6,6), i.e. $T \oplus T^*$ coords.
$a, b, \ldots$	$1, \ldots, b^+$	coordinates for $\mathscr{M}_+$ ($\mathscr{M}_{ks}$ for Calabi-Yau 3–folds)
$A, B, \ldots$	$0, 1, \ldots, b^+$	projective coords. for $\mathscr{M}_+$ ($\mathscr{M}_{ks}$ for Calabi-Yau 3–folds)
$\mathbb{A}, \mathbb{B}, \ldots$	$1, \ldots, 2(b^- + 1)$	fundamental repr. of $\mathrm{Sp}(2b^+ + 2, \mathbb{R})$
$i, j, \ldots$	$1, \ldots, b^-$	coordinates for $\mathscr{M}_-$ ($\mathscr{M}_{cs}$ for Calabi-Yau 3–folds)
$I, J, \ldots$	$0, 1, \ldots, b^-$	projective coords. for $\mathscr{M}_-$ ($\mathscr{M}_{cs}$ for Calabi-Yau 3–folds)
$\mathbb{I}, \mathbb{J}, \ldots$	$1, \ldots, 2(b^- + 1)$	fundamental repr. of $\mathrm{Sp}(2b^- + 2, \mathbb{R})$
$u, v, \ldots$	$1, \ldots, 4(b^- + 1)$	quaternionic coordinates
$\mathcal{A}, \mathcal{B}, \ldots$	$1, 2$	fundamental repr. of SU(2)

In section 2.3, and only there, the greek indices $\mu, \nu \ldots$ label the complex coordinates on the Calabi-Yau manifold M_6.

Flat indices are denoted by underlining the letters associated with the corresponding curved indices. For instance, we denote the 10d vielbein by $e^{\underline{M}}{}_{N}$.

A.2 Differential forms and the Hodge dual

In the main text we deal with a 10d spacetime M_{10}, decomposed into a 4d spacetime M_4 and an internal, compact 6d space M_6. The manifold M_6 is Riemannian, while M_{10} and M_4 are Lorentzian, with a $(-+\ldots+)$ signature.

The manifold M_d is assumed orientable, i.e. it admits a globally defined and nowhere vanishing top form vol_d (the volume form). We fix the orientation requiring the coefficient of vol_d to be positive. Given a local coframe $\{e^{\underline{M}}\}$, we also take

$$e^0 \wedge e^1 \wedge \ldots \wedge e^{d-1} = +\,vol_d \,. \tag{A.1}$$

The Levi-Civita tensor is defined in flat indices by

$$\epsilon_{\underline{M_1\ldots M_d}} := \epsilon_{[\underline{M_1\ldots M_d}]} \ , \qquad \epsilon_{\underline{01\ldots d-1}} := +1 \,. \tag{A.2}$$

Hence in curved indices we have

$$\epsilon_{M_1\ldots M_d} \equiv \epsilon_{\underline{N_1\ldots N_d}} e^{\underline{N_1}}{}_{M_1} \cdots e^{\underline{N_d}}{}_{M_d} \qquad \Rightarrow \qquad \epsilon_{01\ldots d-1} = \det e^{\underline{N}}{}_M = \sqrt{|\det g_{MN}|} \,. \tag{A.3}$$

With this definition, ϵ transforms tensorially, and indices can be raised using the metric. Contractions of two Levi-Civita symbols yield

$$\epsilon^{M_1\ldots M_r P_1\ldots P_{d-r}} \epsilon_{N_1\ldots N_r P_1 \ldots P_{d-r}} = (-1)^t\, r!(d-r)!\, \delta^{M_1\ldots M_r}_{N_1\ldots N_r} \,, \tag{A.4}$$

where $\delta^{M_1\ldots M_r}_{N_1\ldots N_r} := \delta^{[M_1}_{N_1}\cdots\delta^{M_r]}_{N_r}$, while $t=0$ if M_d is Riemannian and $t=1$ if Lorentzian.

Differential p-forms come with a factor of $p!$, i.e.

$$A_p = \frac{1}{p!} A_{M_1\ldots M_p} dx^{M_1} \wedge \ldots \wedge dx^{M_p} \,. \tag{A.5}$$

We define the Hodge dual by

$$*A_p := \frac{1}{p!(d-p)!} A_{M_1\ldots M_p} \epsilon^{M_1\ldots M_p}{}_{M_{p+1}\ldots M_d} dx^{M_{p+1}} \wedge \ldots \wedge dx^{M_d} \,. \tag{A.6}$$

It satisfies (again, $t=0$ if M_d is Riemannian, and $t=1$ if Lorentzian):

$$**\,A_p = (-)^{p(d-p)+t} A_p \,. \tag{A.7}$$

If A_p and B_q are p– and q– forms respectively, with $p \leq q$, we define the contraction

$$A_p \lrcorner B_q := \frac{1}{p!(q-p)!} A^{M_1\ldots M_p} B_{M_1\ldots M_p M_{p+1}\ldots M_q} dx^{M_{p+1}} \wedge \cdots \wedge dx^{M_q} \,. \tag{A.8}$$

Then we have

$$A_p \wedge *B_p = A_p \lrcorner B_p * 1 \,, \tag{A.9}$$

so that the kinetic term of a p–form potential A_p can be written as $-\frac{1}{2}\int dA \wedge *dA$.

In the main text, ten dimensional quantities are denoted by a $\hat{}$ symbol.

If $\hat{F}_p = F_{p-k} \wedge \omega_k$ is a p–form living on M_{10}, while F_{p-k} lives on M_4 and ω_k lives on M_6, then the 10d Hodge dual splits into 4d and 6d Hodge duals as

$$*\hat{F}_n \;=\; (-1)^{k(n-k)} * F_{n-k} \wedge *\omega_k \; . \tag{A.10}$$

Recalling the definition of the involution λ in eq. (3.5) we also deduce

$$*\lambda(\hat{F}_n) \;=\; *\lambda(F_{n-k}) \wedge *\lambda(\omega_k) \; . \tag{A.11}$$

A.3 Clifford algebra and spinors

A.3.1 Gamma matrices

The 10d gamma matrices Γ^M generate the Cliff$(9,1)$ algebra, and satisfy

$$\{\Gamma^M, \Gamma^N\} = 2g^{MN} \; ,$$

where g_{MN} is the metric on M_{10}. The 4d and 6d gamma matrices γ^μ and γ^m generate respectively the Cliff$(3,1)$ and the Cliff(6) algebras, i.e.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad , \qquad \{\gamma^m, \gamma^n\} = 2g^{mn} \; .$$

We choose a Majorana representation for the Cliff$(3,1)$ and Cliff(6) gamma matrices. The γ^μ are all real; they are hermitian, except γ^0 which is antihermitian. The γ^m are all purely imaginary and hermitian.

Note that the 4d supergravity literature often adopts a $(+---)$ signature convention for the 4d metric. When switching from this mostly $-$ to our mostly $+$ signature, the γ^μ get multiplied by a factor of i. The original 4d supergravity formulae reported in this thesis have been retouched accordingly.

The 4d and 6d chirality matrices are respectively

$$\gamma_5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma} = -i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad , \qquad \gamma = \frac{i}{6!}\epsilon_{mnpqrs}\gamma^{mnpqrs} \; , \tag{A.12}$$

so that both γ_5 and γ are purely imaginary and hermitian, and square to $+\mathbb{1}$. An identity which will be useful in appendix B is

$$\gamma\gamma_{m_1\ldots m_k} = \frac{i(-)^{[\frac{k+1}{2}]}}{(6-k)!}\epsilon_{m_1\ldots m_k m_{k+1}\ldots m_6}\gamma^{m_{k+1}\ldots m_6} \; . \tag{A.13}$$

Concerning the Cliff$(9,1)$ gamma matrices, we build $\Gamma_M = (\Gamma_\mu, \Gamma_m)$ by tensoring the Cliff$(3,1)$ and Cliff(6) gamma matrices:

$$\Gamma_\mu = \gamma_\mu \otimes \mathbb{1} \qquad , \qquad \Gamma_m = \gamma_5 \otimes \gamma_m \; .$$

Notice that the Γ_M obtained in this way are all real. The 10d chirality matrix is given by

$$\Gamma_{11} = \gamma_5 \otimes \gamma \, , \tag{A.14}$$

and is real and hermitian.

A.3.2 Weyl spinors and Majorana spinors

We recall that on Spin(3,1) spinors one can impose either the Weyl *or* the Majorana condition. Similarly, on Spin(6) spinors one can impose either the Weyl *or* a reality condition. On the other hand, for Spin(9,1) spinors the Majorana and Weyl conditions are allowed simultaneously (see e.g. [4, vol. 2, appendix B]).

3+1 dimensions

For Spin(3,1) spinors, the Majorana (charge) conjugation can be defined using the charge conjugation matrix C, which satisfies

$$(\gamma^\mu)^T = -C^{-1}\gamma^\mu C \quad , \quad C^T = -C \quad , \quad C^{-1} = -C^* . \tag{A.15}$$

For any spinor ε, its charge conjugate ε^c is defined by

$$\varepsilon^c := C(\bar{\varepsilon})^T ,$$

where $\bar{\varepsilon} := i\varepsilon^\dagger\gamma^0$. Adopting the Majorana representation for the Cliff(3, 1) gamma matrices specified above, we take $C = -i\gamma^0$, in such a way that the charge conjugation acts just by complex conjugation:

$$\varepsilon^c \equiv C(\bar{\varepsilon})^T = \varepsilon^* .$$

With this choice, a Spin$(3, 1)$ Majorana spinor, i.e. a spinor satisfying $\varepsilon^c = \varepsilon$, is just a real spinor.

Notice that if ε_+ is a Weyl spinor with positive chirality: $\gamma_5\varepsilon_+ = \varepsilon_+$, then $\varepsilon_- := \varepsilon_+^c$ is again a Weyl spinor, with opposite chirality (and vice versa). Indeed, from (A.12), (A.15), we have that $\gamma_5^T = C^{-1}\gamma_5 C$. Then

$$\gamma_5\varepsilon_+^c \equiv \gamma_5 C(\bar{\varepsilon}_+)^T = C(\bar{\varepsilon}_+\gamma_5)^T = C(i\varepsilon_+^\dagger\gamma^0\gamma_5)^T = -C(i(\gamma_5\varepsilon_+)^\dagger\gamma^0)^T = -\varepsilon_+^c .$$

In our Majorana representation of the Cliff$(3, 1)$ gamma matrices, we have $\varepsilon_- \equiv \varepsilon_+^*$; its opposite chirality with respect to ε_+ follows immediately by the fact that γ_5 is purely imaginary.

6 dimensions

On Spin(6) spinors, one can impose a reality condition as follows. For any Spin(6) spinor η, define the conjugate $\eta^c := D\eta^*$, where D is the intertwiner relating the γ^m to their complex conjugate: $-\gamma^{m*} = D^{-1}\gamma^m D$. Then η satisfies the reality condition if $\eta^c = \eta$.

Now, if η_+ has positive Spin(6) chirality ($\gamma\eta_+ = \eta_+$), then $\eta_- := \eta_+^c$ has negative chirality. Indeed, recalling (A.12) we have: $\gamma\eta_+^c = \gamma D\eta_+^* = -D(\gamma\eta_+)^* = -\eta_+^c$.

In our Majorana representation of the gamma matrices, in which all the γ^m are imaginary, D can be taken as the identity, hence $\eta_- \equiv \eta_+^*$.

9+1 dimensions

Since our Cliff(9,1) gamma matrices are all real, a Majorana Spin$(9, 1)$ spinor is just a real spinor. If ζ is any Spin$(9, 1)$ Majorana spinor, then we can introduce Majorana-Weyl spinors of positive/negative chirality by $\zeta_\pm := (1 \pm \Gamma_{11})\zeta$. This preserves the Majorana condition because the Γ_{11} given in (A.14) is real.

A.4 SU(3) structure conventions

In this subsection we work with normalized spinors: $\eta_+^\dagger \eta_+ = 1$. We relate the different SU(3)–invariant objects on M_6 as follows

$$g_{mn} = J_{mp} I^p{}_n \, , \tag{A.16}$$

$$J_{mn} = \mp i \eta_\pm^\dagger \gamma_{mn} \eta_\pm \qquad , \qquad \Omega_{mnp} = -i \eta_-^\dagger \gamma_{mnp} \eta_+ \, . \tag{A.17}$$

where, as in subsection 2.2.2, $\eta_\pm$ are globally defined nowhere vanishing chiral spinors, I is the almost complex structure, J is the almost symplectic 2–form, and Ω is the decomposable (3,0)–form. J and Ω satisfy the compatibility condition $J \wedge \Omega = 0$, so that J is (1,1) with respect to I.

A useful decomposition of the chirality projectors on the basis of eigenstates $\{\eta_\pm, \gamma^m \eta_\mp\}$ is

$$\frac{1 \pm \gamma}{2} = \left(\eta_\pm \eta_\pm^\dagger + \frac{1}{2} \gamma^m \eta_\mp \eta_\mp^\dagger \gamma_m \right) \, . \tag{A.18}$$

This can be used to show that

$$\begin{aligned} \gamma_m \eta_+ &= -i J_{mn} \gamma^n \eta_+ && \text{(A.19)} \\ \gamma_{mn} \eta_+ &= i J_{mn} \eta_+ + \frac{i}{2} \Omega_{mnp} \gamma^p \eta_- && \text{(A.20)} \\ \gamma_{mnp} \eta_+ &= i \Omega_{mnp} \eta_- + 3 i J_{[mn} \gamma_{p]} \eta_+ \, . && \text{(A.21)} \end{aligned}$$

Using the holomorphic projector

$$P = \frac{1}{2} (\mathbb{1} - i I) \, ,$$

existing also if the almost complex structure is not integrable, we introduce the gamma matrices with holomorphic/antiholomorphic indices $i, \bar{\imath} = 1, 2, 3$:

$$\gamma^i := P^i{}_n \gamma^n \quad \text{and} \quad \gamma^{\bar{\imath}} := \bar{P}^{\bar{\imath}}{}_n \gamma^n \, . \tag{A.22}$$

From (A.19) and (A.16) we see that $\gamma^i \eta_+ = 0$. Instead $\gamma^{\bar{\imath}} \eta_+$ transforms in the $\bar{\mathbf{3}}$ of SU(3).

With the conventions listed above, one also has

$$*J = \frac{1}{2} J \wedge J \qquad , \qquad *1 \equiv vol_6 = \frac{1}{6} J \wedge J \wedge J = \frac{i}{8} \Omega \wedge \bar{\Omega} \, , \tag{A.23}$$

as well as, using the fierzing (2.61):

$$8 \eta_+ \otimes \eta_+^\dagger = e^{-iJ} \qquad , \qquad 8 \eta_+ \otimes \eta_-^\dagger = -i \, \Omega \, . \tag{A.24}$$

Appendix B

Mukai pairing and Clifford map

In this appendix we collect some relations involving the Mukai pairing and the Clifford map (defined in (2.51) and (2.59) respectively), which turn out to be useful in the generalized geometry computations performed in the main text. We also include a computation making explicit the relation between Spin(6,6) pure spinors and generalized almost complex structures.

We recall that the Mukai pairing is antisymmetric in six dimensions. Further simple properties are

$$\begin{aligned}
\lambda(e^b A) &= e^{-b}\lambda(A) \ , &\text{(B.1)}\\
\langle e^{-b}A, e^{-b}C\rangle &= \langle A, C\rangle \ , &\text{(B.2)}\\
\langle A, *C\rangle &= \langle C, *A\rangle \ , &\text{(B.3)}\\
\langle A_\pm, \lambda(C_\pm)\rangle &= \pm\langle C_\pm, \lambda(A_\pm)\rangle &\text{(B.4)}\\
\langle A, \Gamma^\Lambda C\rangle &= \langle C, \Gamma^\Lambda A\rangle \ , &\text{(B.5)}
\end{aligned}$$

where $A, C \in \wedge^\bullet T^* M_6$, $b \in \wedge^2 T^* M_6$, and the Cliff$(6,6)$ gamma matrices Γ^Λ correspond to $dy^m\wedge$ or ι_{∂_m}, as in the main text (see (2.50)).

The action of the operator $*\lambda$ on the forms can be evaluated using the Clifford map (2.59) and identity (A.13):

$$*\not{\lambda}(\not{A}) = -i\gamma\not{A} \ . \tag{B.6}$$

Since $\not{\lambda}(\not{A}) = \not{A}^T$, this also reads

$$*\not{A} = -i\gamma\not{A}^T \ . \tag{B.7}$$

Then the Mukai pairing translates under the Clifford map as [33]

$$\langle A_k, C_{6-k}\rangle \ = \ \frac{i}{8}\mathrm{tr}\left(\gamma\not{A}_k^T\,\not{C}_{6-k}\right)vol_6 \ , \tag{B.8}$$

where vol_6 is the volume form of M_6 and the trace is taken over the spinorial indices of the Cliff(6) gamma matrices. Indeed,

$$\begin{aligned}
\mathrm{tr}\left(\gamma \slashed{A}_k^T\, \slashed{C}_{6-k}\right) vol_6 &= \frac{i}{[(6-k)!]^2}(*A)_{m_1\dots m_{6-k}} C^{n_1\dots n_{6-k}} \mathrm{tr}\left[\gamma^{m_1\dots m_{6-k}}\gamma_{n_1\dots n_{6-k}}\right] vol_6 \\
&= 8i\,(-)^{\left[\frac{6-k}{2}\right]}\left(C_{6-k}\lrcorner(*A_k)\right) vol_6 \\
&= -(-)^{\left[\frac{k+1}{2}\right]} 8i\, A_k\wedge C_{6-k} \\
&= -8i\,\langle A_k, C_{6-k}\rangle\,,
\end{aligned}$$

where for the first equality we used (B.7), while the following two steps involve relations (A.7)–(A.9). For the last equality recall the definition (2.51) of the Mukai pairing.

If $C_\pm$ is an even/odd polyform, the operator corresponding to the action of the Cliff(6) gamma matrices on $\slashed{C}_\pm$ is given by [33]

$$\gamma^m \slashed{C}_\pm = \ \slashed{(dy^m\wedge + g^{mn}\iota_{\partial_n})C_\pm} \qquad , \qquad \slashed{C}_\pm\gamma^m = \pm\, \slashed{(dy^m\wedge - g^{mn}\iota_{\partial_n})C_\pm}\,. \tag{B.9}$$

With an abuse of notation, in the main text sometimes we write expressions like $\overrightarrow{\gamma}^m C_\pm$ and $C_\pm \overleftarrow{\gamma}^m$, to be read as the Clifford map counter-image of (B.9). From (B.9) we see that the Cliff(6,6) action on the polyforms, introduced in (2.48), translates under the Clifford map as

$$\slashed{dy^m\wedge C_\pm} = \frac{1}{2}[\gamma^m, \slashed{C}_\pm]_\pm \qquad , \qquad \slashed{\iota_{\partial_m} C_\pm} = \frac{1}{2}[\gamma_m, \slashed{C}_\pm]_\mp\,, \tag{B.10}$$

where $[\ ,\ ]_\pm$ stands for anticommutator/commutator. In the main text we also need the action of the antisymmetrized product of two Cliff(6,6) gamma matrices

$$\Gamma^{\Lambda\Sigma} = \left(dy^m\wedge dy^n\wedge\ ,\ \frac{1}{2}[dy^m\wedge, \iota_{\partial_n}]\ ,\ \frac{1}{2}[\iota_{\partial_m}, dy^n\wedge]\ ,\ \iota_{\partial_m}\iota_{\partial_n}\right). \tag{B.11}$$

Under the Clifford map this becomes:

$$\begin{aligned}
\slashed{dy^m\wedge dy^n\wedge C_\pm} &= \frac{1}{4}\left[\gamma^{mn}\slashed{C}_\pm \pm \gamma^m\slashed{C}_\pm\gamma^n \mp \gamma^n\slashed{C}_\pm\gamma^m - \slashed{C}_\pm\gamma^{nm}\right] \\
\slashed{\tfrac{1}{2}[dy^m\wedge, \iota_{\partial_n}]C_\pm} &= \frac{1}{4}\left[\gamma^m{}_n\slashed{C}_\pm \mp \gamma^m\slashed{C}_\pm\gamma_n \mp \gamma_n\slashed{C}_\pm\gamma^m + \slashed{C}_\pm\gamma_n{}^m\right] \\
\slashed{\tfrac{1}{2}[\iota_{\partial_m}, dy^n\wedge]C_\pm} &= \frac{1}{4}\left[\gamma_m{}^n\slashed{C}_\pm \pm \gamma_m\slashed{C}_\pm\gamma^n \pm \gamma^n\slashed{C}_\pm\gamma_m + \slashed{C}_\pm\gamma^n{}_m\right] \\
\slashed{\iota_{\partial_m}\iota_{\partial_n}C_\pm} &= \frac{1}{4}\left[\gamma_{mn}\slashed{C}_\pm \mp \gamma_m\slashed{C}_\pm\gamma_n \pm \gamma_n\slashed{C}_\pm\gamma_m - \slashed{C}_\pm\gamma_{nm}\right].
\end{aligned} \tag{B.12}$$

If C is one of the pure spinors $\Phi^0_\pm$ defining the SU(3)×SU(3) structure, then it's straightforward to see how each term appearing in (B.12) transforms under SU(3)×SU(3), and therefore to locate its position in the diamond (2.68). Indeed, recalling the explicit basis given in (2.69), we see that for instance $\gamma^m\Phi^0_+\gamma^n \in U_{\bar{\mathbf{3}},\mathbf{3}}$, while $\Phi^0_+\gamma^{mn}$ contains a term proportional to $\Phi^0_+ \in U_{\mathbf{1},\bar{\mathbf{1}}}$ and a term belonging to $U_{\mathbf{1},\bar{\mathbf{3}}}$.

As an example of the use of the technology introduced above, we now check the correspondence between the generalized almost complex structures defined from the pure spinors

(2.64) via the formula (2.54), and the matrices $\mathcal{J}_\pm$ given in eq. (2.42). We start from the case of vanishing b, and write (2.54) for $\Phi^0_\pm$. Recalling eq. (2.63) and the basis (2.50) for the Cliff(6,6) gamma matrices, we have

$$\mathcal{J}_{\pm}{}^{\Lambda}{}_{\Sigma} = -\frac{1}{2vol_6}\begin{pmatrix} \langle \mathrm{Re}\Phi^0_\pm\,,\, \frac{1}{2}[dy^m\wedge, \iota_{\partial_n}]\mathrm{Re}\Phi^0_\pm\rangle & \langle \mathrm{Re}\Phi^0_\pm\,,\, dy^m\wedge dy^n \wedge \mathrm{Re}\Phi^0_\pm\rangle \\ \langle \mathrm{Re}\Phi^0_\pm\,,\, \iota_{\partial_m}\iota_{\partial_n}\mathrm{Re}\Phi^0_\pm\rangle & \langle \mathrm{Re}\Phi^0_\pm\,,\, \frac{1}{2}[\iota_{\partial_m}, dy^n\wedge]\mathrm{Re}\Phi^0_\pm\rangle \end{pmatrix} . \qquad \text{(B.13)}$$

We evaluate this in the bispinor picture, using eqs. (B.8) and (B.12). For instance, for the 'south-west' block we have

$$\begin{aligned} -\frac{1}{2vol_6}\langle \mathrm{Re}\Phi^0_\pm, \iota_{\partial_m}\iota_{\partial_n}\mathrm{Re}\Phi^0_\pm\rangle &= -\frac{i}{8^2}\mathrm{tr}\left[\gamma(\mathrm{Re}\Phi^0_\pm)^T(\gamma_{mn}\,\mathrm{Re}\Phi^0_\pm + \mathrm{Re}\Phi^0_\pm\gamma_{mn})\right] \\ &= -\frac{i}{2}(\eta^{1\dagger}_+\gamma_{mn}\eta^1_+ + \eta^{2\dagger}_\pm\gamma_{mn}\eta^2_\pm) \\ &= \frac{1}{2}(J_1 \pm J_2)_{mn} \, . \end{aligned}$$

In the first equality we have written only the non-vanishing terms; to obtain the second line we substituted (2.60) and used the first of (A.17). The evaluation of the other blocks is analogous, and we obtain eq. (2.42) with $b = 0$. When considering pure spinors with nonvanishing b, in (B.13) $\Phi^0_\pm$ is replaced by $\Phi_\pm = e^{-b}\Phi^0_\pm$. We wish to make e^{-b} pass through the $dy^m\wedge$ and ι_{∂_m} and then use (B.2). While the $dy^m\wedge$ commute with e^{-b}, for the contractions we have $\iota_{\partial_m}e^{-b} = e^{-b}(\iota_{\partial_m} - b_{mn}dy^n\wedge)$. Taking this into account we recover the two matrices $\left(\begin{smallmatrix} 1 & 0 \\ -b & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ b & 1\end{smallmatrix}\right)$ of eq. (2.42).

Appendix C

Type IIA action with fluxes

In this appendix we make explicit the compatibility of the system of democratic EoM/Bianchi identities (with no localized sources) considered in section 3.1 with the standard formulation of the type IIA action.[1] In doing so, we reconsider an issue already discussed in the literature [85, 87] concerning the expression for the Chern-Simons piece of the action when NSNS and RR background fluxes are switched on. We derive a general form of this Chern-Simons term by requiring consistency with the equations of motion.

In order to make contact with the standard formulation of (massive) type IIA supergravity, we need to break the democracy among the RR fields stated in section 3.1. Eliminating via the self-duality relations (3.5) the forms[2] F_6, F_8, F_{10} from eqs. (3.6) and (3.9), we are left with the following set of independent equations in terms of H, F_0, F_2 and F_4 only:

$$dH = 0 \, , \tag{C.1}$$

$$dF_0 = 0 \quad , \quad dF_2 - HF_0 = 0 \quad , \quad dF_4 - H\wedge F_2 = 0 \, , \tag{C.2}$$

$$d(e^{-2\phi} * H) - F_0 \wedge *F_2 - F_2 \wedge *F_4 - \frac{1}{2} F_4 \wedge F_4 = 0 \, , \tag{C.3}$$

$$d * F_2 + H \wedge *F_4 = 0 \quad , \quad d * F_4 + H \wedge F_4 = 0 \, . \tag{C.4}$$

In a topologically trivial background (where no fluxes can be switched on), the Bianchi identities (C.1) and (C.2) are solved in terms of globally defined NS 2–form B as well as 1– and 3–form RR potentials C_1 and C_3 :

$$H = dB \quad , \quad F_0 = \text{const} \quad , \quad F_2 = dC_1 + BF_0 \quad , \quad F_4 = dC_3 - H\wedge C_1 + \frac{1}{2} B^2 F_0 \, . \tag{C.5}$$

Now we can immediately check that the remaining equations (C.3) and (C.4) correspond to the EoM for the potentials B, C_1 and C_3 descending from the standard massive type IIA (bosonic) action S_{IIA}, with mass parameter F_0. Denoting $S_{\text{IIA}} = S_{\text{kinetic}} + S_{\text{CS}}$, we have (see e.g. [48]):

$$S_{\text{kinetic}} = \frac{1}{2}\int \Big[e^{-2\phi}\big(R*1 + 4d\phi\wedge *d\phi - \frac{1}{2} H\wedge *H\big) - \frac{1}{2}\big(F_0\wedge *F_0 + F_2\wedge *F_2 + F_4 \wedge *F_4\big)\Big], \tag{C.6}$$

[1] The problem of writing a supergravity action in the presence of general D-branes is studied e.g. in [163, 164]. These papers also discuss a possible background independent formulation.

[2] In this appendix all the forms are ten dimensional. Since there is no risk of confusion, we omit the hat symbol over them.

$$S_{\rm CS} = -\frac{1}{4}\int \left[BdC_3dC_3 + \frac{1}{3}F_0B^3dC_3 + \frac{1}{20}F_0^2B^5\right] \tag{C.7}$$

(the $\wedge$ symbol is understood in $S_{\rm CS}$). Notice that the $F_0 = 0$ limit yields the standard massless type IIA action [4, vol. 2].

Things become more subtle if one looks for general global solutions of the Bianchi identities (C.1) and (C.2) on topologically non-trivial backgrounds, allowing for fluxes of the NS and RR field-strengths. In this case the expressions in (C.5) are modified as follows (F_0 is still a constant parameter):

$$\begin{aligned} H &= H^{\rm fl} + dB\ , \\ F_2 &= dC_1 + F_2^{\rm fl} + BF_0\ , \\ F_4 &= dC_3 - H\wedge C_1 + F_4^{\rm fl} + B\wedge F_2^{\rm fl} + \frac{1}{2}B^2F_0\ , \end{aligned} \tag{C.8}$$

where the forms labeled with 'fl' are defined as the non-exact parts of the solutions, satisfying the conditions

$$H^{\rm fl}F_0 = 0 \quad , \quad dH^{\rm fl} = 0 \quad , \quad dF_2^{\rm fl} = 0 \quad , \quad dF_4^{\rm fl} - H^{\rm fl}\wedge F_2^{\rm fl} = 0\ . \tag{C.9}$$

The first condition holds because if $F_0 \neq 0$, then the Bianchi identity $dF_2 - HF_0 = 0$ implies that H is exact and therefore $H^{\rm fl} = 0$. In the expression (C.10) below we will however keep both $H^{\rm fl}$ and F_0, also because the $F_0H^{\rm fl} = 0$ constraint can be invalidated by the possible introduction of localized sources such as O6 planes,[3] which modify the Bianchi identity for F_2 (see for instance [87, 66, 164, 165]).

Let's now consider how the new expressions (C.8) for the field-strengths enter in the type IIA action. While we can simply substitute such new expressions into the kinetic terms (C.6), the determination of the Chern-Simons action (C.7) is more delicate. In [85] a modified form of the Chern-Simons term was obtained by requiring consistency with the structure of the expected 4d $N = 2$ gauged supergravity after compactification on a Calabi-Yau three-fold, while in appendix A of [87] it was deduced by properly modifying the M-theory Chern-Simons term in order to accomodate for a 4-form flux, and then performing the reduction to ten dimensions.

Here we propose a general expression for $S_{\rm CS}$ by imposing that the equations of motion derived from the action still have the form (C.3), (C.4). We can see that this requirement is satisfied if we preserve the form (C.6) for $S_{\rm kinetic}$, and modify the Chern-Simons term as follows:

$$\begin{aligned} S_{\rm CS} \;=\; -\frac{1}{4}\int \Big[& C_3H^{\rm fl}(dC_3 + 2F_4^{\rm fl}) + B(dC_3 + F_4^{\rm fl})(dC_3 + F_4^{\rm fl}) + B^2F_2^{\rm fl}(dC_3 + F_4^{\rm fl}) \\ & + \frac{1}{3}B^3F_2^{\rm fl}F_2^{\rm fl} + \frac{1}{3}F_0B^3(dC_3 + F_4^{\rm fl}) + \frac{1}{4}F_0B^4F_2^{\rm fl} + \frac{1}{20}F_0^2B^5\Big]\ . \end{aligned} \tag{C.10}$$

This expression agrees with - and extends - the results of [85, 87], which just considered the vanishing $F_2^{\rm fl}$ case.

[3] In this case of course the action needs to be completed with the terms describing the couplings to the localized sources.

Finally, one can verify that the field-strengths H, F_2, F_4, as well as the action S_{IIA}, are invariant under the following globally defined gauge transformations involving the k–form (infinitesimal) parameters Λ_k:

$$\delta B = d\Lambda_1 \quad , \quad \delta C_1 = d\Lambda_0 - \Lambda_1 F_0 \quad , \quad \delta C_3 = d\Lambda_2 - H\Lambda_0 - \Lambda_1(F_2^{\mathrm{fl}} + BF_0)\,. \tag{C.11}$$

The EoM (C.3), (C.4) are of course gauge-invariant due to the invariance of the field-strengths.

Appendix D

Geometry of $N = 2$ supergravity I: Special Kähler manifolds

Special Kähler manifolds are the scalar manifolds of vector multiplets in 4d $N = 2$ supergravity [73]. In this appendix we provide a definition of this geometry, and we collect the properties needed in the main text. More thorough discussions can be found, for instance, in refs. [76, 166, 167]. Here we present the formulae in the notation referring to the special Kähler manifold $\mathscr{M}_-$ introduced in subsection 3.3.2. Modulo switching the notation, it is understood that the same relations hold for $\mathscr{M}_+$ too. We recall that for Calabi-Yau 3–folds we identify $\mathscr{M}_- = \mathscr{M}_{cs}$ and $\mathscr{M}_+ = \mathscr{M}_{ks}$.

A characterization of special Kähler manifolds, based on the notion of prepotential, has already been given in subsection 2.3.2, below eq. (2.25). In the following we provide an alternative definition [168], which is more intrinsic and does not rely on the prepotential.

A local special Kähler manifold $\mathscr{M}_-$ of complex dimension b^- is a Hodge-Kähler manifold (with line bundle $\mathcal{L}$)[1] with the further structure of a holomorphic flat $\mathrm{Sp}(2b^- + 2, \mathbb{R})$ vector bundle $\mathcal{S}$ over it. Furthermore, the $\mathcal{S} \otimes \mathcal{L}$ bundle has to admit a symplectic section V, which locally we denote as

$$V = \begin{pmatrix} Z^I(z) \\ \mathcal{G}_J(z) \end{pmatrix} , \qquad I, J = 0, \ldots, b^- ,$$

such that the Kähler potential reads

$$K = -\log(i\bar{V}^T \mathbb{S}\, V) \ = \ -\log i(\bar{Z}^I \mathcal{G}_I - \bar{\mathcal{G}}_J Z^J) \ , \tag{D.1}$$

and satisfying

$$V^T \mathbb{S}\, \partial_i V = 0 \, . \tag{D.2}$$

Here, $\mathbb{S} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the $\mathrm{Sp}(2b^- + 2, \mathbb{R})$ metric, defining the symplectic bilinear product on the fibers of $\mathcal{S}$, the $z^i, i = 1, \ldots, b^-$, are complex coordinates on $\mathscr{M}_-$, and $\partial_i = \partial/\partial z^i$. Notice that K is a symplectic invariant.

[1] A Hodge-Kähler manifold $\mathscr{M}$ is a Kähler manifold admitting a line bundle $\mathcal{L} \to \mathscr{M}$ whose first Chern class equals the cohomology class of the Kähler form. It follows that the Kähler potential K can be written as $K = -\log h$, where h is the hermitian metric on the fibers of $\mathcal{L}$ (see e.g. [76] for details). This is the geometry of the scalar manifolds in 4d $N = 1$ supergravity.

The advantage of avoiding to define special Kähler manifolds via the prepotential is related to the fact that this is not a symplectic invariant. In fact there are cases where, starting from a holomorphic section V for which there exists a given prepotential, by a symplectic rotation one can move to a new holomorphic section for which a prepotential does not exist [169]. It can therefore be more convenient to dispose of a definition which is independent of the explicit choice of the holomorphic section.

In the context of SU(3)×SU(3) structure compactifications (see subsection 3.3.2), the symplectic structure is provided by the Mukai pairing as in eq. (3.18), and the holomorphic section is encoded in $\Phi_- = Z^I\alpha_I - \mathcal{G}_I\beta^I$, so that $K_- = -\log i\int\langle\Phi_-,\bar{\Phi}_-\rangle$. As we discussed, this is true in particular for a Calabi-Yau 3-fold, where $\Phi_- = \Omega$ and $\Phi_+ = e^{-b-iJ}$.

The following relations define the *period matrix* $\mathcal{M}_{IJ}$, relating the upper and lower components of the holomorphic section:

$$\mathcal{G}_I = \mathcal{M}_{IJ}Z^J \qquad , \qquad D_i\mathcal{G}_J = \overline{\mathcal{M}}_{JK}D_iZ^K \, , \tag{D.3}$$

where the Kähler covariant derivative acting on the holomorphic section is $D_i = \partial_i + \partial_i K$.

Whenever a prepotential $\mathcal{G}$ can be introduced, we have $\mathcal{G}_I = \partial_I\mathcal{G}$ and

$$\mathcal{G}_I = \mathcal{G}_{IJ}Z^J \qquad , \qquad \text{where } \; \mathcal{G}_{IJ} := \partial_I\partial_J\mathcal{G} \; . \tag{D.4}$$

In this case the period matrix $\mathcal{M}_{IJ}$ can be expressed as

$$\mathcal{M}_{IJ} = \overline{\mathcal{G}}_{IJ} + 2i\frac{(\mathrm{Im}\mathcal{G}_{IK})Z^K(\mathrm{Im}\mathcal{G}_{JL})Z^L}{Z^M(\mathrm{Im}\mathcal{G}_{MN})Z^N} \; . \tag{D.5}$$

One can also prove the identities

$$Z^I\mathrm{Im}\mathcal{G}_{IJ}\bar{Z}^J = -\frac{1}{2}e^{-K} \qquad \text{(following directly from (D.1) and (D.4))} \tag{D.6}$$

$$D_kZ^I\mathscr{G}^{k\bar{l}}D_{\bar{l}}\bar{Z}^J = -\frac{1}{2}e^{-K}(\mathrm{Im}\mathcal{M})^{-1\,IJ} - \bar{Z}^IZ^J \; . \tag{D.7}$$

Finally, using (D.5) and (D.6), one can see that

$$\begin{aligned}
(\mathrm{Im}\mathcal{M})^{-1\,IJ} &= -(\mathrm{Im}\mathcal{G})^{-1\,IJ} - 2e^K(Z^I\bar{Z}^J + \bar{Z}^IZ^J) \\
[\mathrm{Re}\mathcal{M}(\mathrm{Im}\mathcal{M})^{-1}]_I{}^J &= -[\mathrm{Re}\mathcal{G}(\mathrm{Im}\mathcal{G})^{-1}]_I{}^J - 2e^K(\mathcal{G}_I\bar{Z}^J + \bar{\mathcal{G}}_IZ^J) \\
[\mathrm{Im}\mathcal{M} + \mathrm{Re}\mathcal{M}(\mathrm{Im}\mathcal{M})^{-1}\mathrm{Re}\mathcal{M}]_{IJ} &= -[\mathrm{Im}\mathcal{G} + \mathrm{Re}\mathcal{G}(\mathrm{Im}\mathcal{G})^{-1}\mathrm{Re}\mathcal{G}]_{IJ} - 2e^K(\mathcal{G}_I\bar{\mathcal{G}}_J + \bar{\mathcal{G}}_I\mathcal{G}_J) \; .
\end{aligned} \tag{D.8}$$

Appendix E

Geometry of $N = 2$ supergravity II: Quaternionic-Kähler manifolds

Quaternionic-Kähler manifolds are the scalar manifolds of the hypermultiplets in 4d $N = 2$ supergravity [74]. In the following we review their definition and some relevant properties, mainly following [76].

E.1 Definition

A *quaternionic-Kähler manifold* is a $4n$-dimensional Riemannian manifold whose Levi-Civita connection has holonomy group contained in $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$.

The degenerate case in which the holonomy is contained just in $\mathrm{Sp}(n)$ defines a *hyperkähler manifold*, which is the scalar manifold of hypermultiplets in $N = 2$ rigid supersymmetry. For the definition of $\mathrm{Sp}(n)$, recall footnote 1 in chapter 5.

An alternative definition of quaternionic-Kähler manifolds is the following. Let Q be a $4n$-dimensional manifold parameterized by coordinates $q^u,\ u = 1, \ldots, 4n$, and endowed with a Riemannian metric h:

$$ds^2 = h_{uv}(q)dq^u \otimes dq^v \,. \tag{E.1}$$

For Q to be quaternionic-Kähler, it has to locally admit three almost complex structures $I^x : TQ \to TQ$ $(x = 1, 2, 3)$ that satisfy the quaternionic ($\mathrm{Sp}(1) \cong \mathrm{SU}(2)$) algebra

$$I^x I^y \ = \ -\delta^{xy}\mathbb{1} + \epsilon^{xyz} I^z, \tag{E.2}$$

and with respect to which the metric is hermitian: $h(I^x X, I^x Y) = h(X, Y)\ \ \forall X, Y \in TQ$ (recall the notions introduced in section 2.2). To complete the definition, we introduce a triplet of locally defined 2–forms, known as the $\mathfrak{su}(2)$-valued *hyperkähler form*:

$$K^x = K^x_{uv} dq^u \wedge dq^v \quad , \quad K^x_{uv} = h_{uw}(I^x)^w{}_v \,, \tag{E.3}$$

as well as a non-trivial principal $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ bundle over Q, of which the hyperkähler form is a local section. The latter is required to be covariantly constant with respect to the connection ω^x on the bundle:

$$\nabla K^x \equiv dK^x + \varepsilon^{xyz}\omega^y \wedge K^z = 0 \,, \tag{E.4}$$

and proportional to its curvature Ω^x:

$$\Omega^x \equiv d\omega^x + \frac{1}{2}\varepsilon^{xyz}\omega^y \wedge \omega^z = \lambda K^x \,, \tag{E.5}$$

where λ is a real number. The $\lambda = 0$ limit corresponds to the hyperkähler case. Notice that when $\lambda \neq 0$ eq. (E.5) implies (E.4).

E.2 Properties

The SU(2) defining the bundle over Q introduced above is the same SU(2) $\cong$ Sp(1) factor in the holonomy group of Q. In $N = 2$ supergravity, this also corresponds to the R-symmetry group rotating the supersymmetries.

For $n > 1$, quaternionic-Kähler manifolds are Einstein spaces, with Ricci tensor

$$R_{uv} = 2\lambda(n+2)h_{uv}\,.$$

In particular, the quaternionic manifolds arising in 4d, $N = 2$ supergravity have negative curvature. Usually the normalization $\lambda = -1$ is chosen.

Quaternionic vielbein

Let us introduce the Sp(1) indices $\mathcal{A}, \mathcal{B} = 1, 2$ and the Sp(n) indices $\alpha, \beta = 1, \ldots, 2n$, as well as the flat Sp(1) and Sp(n) invariant metrics[1]

$$\epsilon_{\mathcal{AB}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbb{S}_{\alpha\beta} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

Recalling that Q has Sp(1) $\times$ Sp(n) holonomy, it is natural to define the vielbein

$$\mathcal{U}^{\mathcal{A}\alpha} = \mathcal{U}_u^{\mathcal{A}\alpha}(q)dq^u\,,$$

relating the metric h_{uv} with $\epsilon_{\mathcal{AB}}$ and $\mathbb{S}_{\alpha\beta}$:

$$h_{uv} = \mathcal{U}_u^{\mathcal{A}\alpha}\mathcal{U}_v^{\mathcal{B}\beta}\mathbb{S}_{\alpha\beta}\epsilon_{\mathcal{AB}}\,,$$

and satisfying the reality condition

$$\epsilon_{\mathcal{AB}}\mathbb{S}_{\alpha\beta}\mathcal{U}^{\mathcal{B}\beta} \equiv \mathcal{U}_{\mathcal{A}\alpha} = (\mathcal{U}^{\mathcal{A}\alpha})^*\,.$$

One can show that

$$\Omega^x = -i\lambda\mathbb{S}_{\alpha\beta}(\sigma_x)_{\mathcal{A}}{}^{\mathcal{C}}\epsilon_{\mathcal{CB}}\mathcal{U}^{\mathcal{A}\alpha} \wedge \mathcal{U}^{\mathcal{B}\beta}\,,$$

which also yields

$$\frac{i}{2}\Omega^x(\sigma_x)_{\mathcal{A}}{}^{\mathcal{B}} = -\lambda\mathcal{U}_{\mathcal{A}\alpha} \wedge \mathcal{U}^{\mathcal{B}\alpha}\,.$$

[1] In the main text (see chapter 4), the indices $\alpha, \beta = 1, \ldots, 2n$ are replaced by $\mathbb{I}, \mathbb{J} = 1, 2, \ldots, 2b^- + 2$.

Triholomorphic isometries

The isometries of the quaternionic manifold that respect the quaternionic nature of Q are called 'triholomorphic'. Suppose there are r such isometries, and denote by k_A, $A = 1, \dots r$ the Killing vectors generating them.[2] Then, the Lie derivative with respect to k_A of the hyperkähler form $K^x = \lambda^{-1}\Omega^x$ and of the SU(2) connection ω^x must vanish up to an SU(2) rotation in the SU(2) bundle introduced above:

$$\mathcal{L}_{k_A}\Omega^x = \epsilon^{xyz}\Omega^y W_A^z \quad , \quad \mathcal{L}_{k_A}\omega^x = \nabla W_A^x \,, \tag{E.6}$$

where W_A^x is an SU(2) compensator associated with k_A, and $\nabla = dq^u \nabla_u$ is the SU(2) covariant exterior derivative on Q.

To each triholomorphic isometry, one can associate a triplet of real functions $\mathcal{P}_A^x(q)$ via the momentum map equation

$$-\iota_{k_A} K^x = \nabla \mathcal{P}_A^x \equiv d\mathcal{P}_A^x + \varepsilon^{xyz}\omega^y \mathcal{P}_A^z. \tag{E.7}$$

The momentum maps $\mathcal{P}_A^x$, $x = 1, 2, 3$, are known as the *Killing prepotentials*, and can be seen as the fundamental objects determining the triholomorphic isometries, in a fashion analogous to the relation existing between the metric and the Kähler potential on a Kähler manifold.

From (E.7) and the first of (E.6), one finds [170]

$$\mathcal{P}_A^x = \iota_{k_A}\omega^x - W_A^x \,.$$

For the isometries we consider in the main text, it turns out that [15, 89]

$$\mathcal{L}_{k_A}\omega^x = 0 \quad \Rightarrow \quad \mathcal{L}_{k_A}\Omega^x = 0 \,,$$

so that the compensator W_A^x vanishes. Hence the formula determining the Killing prepotentials simplifies to

$$\mathcal{P}_A^x = \omega_u^x k_A^u \,.$$

This is the relation we use in the main text, cf. eq. (4.12).

[2] In the main text we have $A = 0, 1, \dots b^+$, where b^+ is the number of vector multiplets in the $N = 2$ theory.

Appendix F

Details of the dimensional reduction on coset spaces

In this appendix we collect some details of the dimensional reduction on the coset spaces considered in chapter 5. This can be seen as an application of the general procedure worked out in chapter 3.

The G-invariant reduction ansatz adopted in chapter 5 strongly constrains the dependence of all the higher dimensional fields on the G/H coordinates, relegating it into the coframe $e^{\underline{m}}$ introduced in subsection 5.2.1. In particular, the most general G-invariant 10d metric is (we recall that by the hat symbol we denote the 10d fields):

$$d\hat{s}^2 = e^{2\varphi(x)} g_{\mu\nu}(x) dx^\mu \otimes dx^\nu + g_{\underline{mn}}(x) e^{\underline{m}}(y) \otimes e^{\underline{n}}(y) \ , \tag{F.1}$$

where x^μ and y^m are respectively coordinates on the 4d spacetime and the internal manifold M_6, and $g_{\underline{mn}}$ satisfies the G-invariance condition discussed in subsection 5.2.1. Components of the 10d metric with mixed 4d-6d indices are not allowed since there are no left-invariant 1–forms on our coset manifolds (5.1); this feature is common to the massless Kaluza-Klein ansatz for Calabi-Yau compactifications described in subsection 3.2.1, though here it is justified differently. Since the invariant scalars on the coset are necessarily constant, a nontrivial warp factor is also not permitted. As illustrated in chapter 3, the Weyl rescaling factor $e^{2\varphi(x)}$ in front of the 4d metric is needed in order to obtain a canonical lower dimensional Einstein-Hilbert term $\int_{M_4} vol_4 R_4$ from the string frame higher dimensional action $\int_{M_{10}} vol_{10} e^{-2\phi} \hat{R}$. Recalling (3.75), the ansatz $\phi = \phi(x)$ for the 10d dilaton implies

$$\varphi(x) = \phi(x) - \frac{1}{2} \log \int_{M_6} d^6 y \sqrt{g_6} \ , \tag{F.2}$$

where $\sqrt{g_6} \equiv \sqrt{\det g_{mn}(x,y)} = \sqrt{\det g_{\underline{mn}}(x)}\, |\det e^{\underline{p}}{}_q(y)|$. Notice that, thanks to this factorization of the x and y dependence, $\partial_\mu \log \sqrt{g_6}$ does not depend on the internal coordinates (this was one of the requirements arising from the analysis of section 3.4), and

$$\partial_\mu \varphi = \partial_\mu \phi - \frac{1}{2} \partial_\mu \log \sqrt{g_6} \ . \tag{F.3}$$

The ansatz for the 10d supergravity field strengths must be chosen consistently with their Bianchi identities. For instance, from the Bianchi identity $d\hat{F}_2 = \hat{H}\hat{F}_0$, one sees that if

$\hat{F}_0 \neq 0$, then the NSNS 3–form $\hat{H}$ has to be exact: $\hat{H} = d\hat{B}$, with a globally defined 2–form potential $\hat{B}$. The most general $\hat{B}$ respecting left-invariance on M_6 is

$$\hat{B} = B + b\,, \tag{F.4}$$

where $B(x)$ is along 4d spacetime, while $b(x,y) = b^a(x)\omega_a(y)$ lives on M_6 (the left-invariant 2–forms ω_a were given in subsection 5.2.1). Again, our ansatz (3.71) here finds a concretization in the left-invariance requirement.

We deal with the expansion of the RR fields in subsection F.2.

F.1 Special Kähler geometry from the NSNS sector

Combining the 2–form J of subsection 5.2.2 and the internal NS field b we introduce $t = b + iJ$, whose expansion $t = t^a\omega_a$ on the basis 2–forms defines the complex 4d scalars $t^a = b^a + iv^a$. The associated kinetic term is determined by

$$\frac{1}{8}g^{mp}g^{nq}\big(\partial_\mu g_{mn}\partial^\mu g_{pq} + \partial_\mu b_{mn}\partial^\mu b_{pq}\big) \;=\; \frac{1}{4Vol}\int_{M_6}\partial_\mu t\wedge *\partial^\mu \bar{t} \;=\; \mathscr{G}_{ab}\partial_\mu t^a\partial^\mu \bar{t}^b\,, \tag{F.5}$$

where the l.h.s. originates from the reduction of the 10d Ricci scalar and $\hat{H}^2$ terms, while the σ-model metric $\mathscr{G}_{ab}$ was introduced in eq. (5.12). The first equality in (F.5) is derived recalling that the internal metric is fixed by the forms J and Ω defining the SU(3) structure: indeed, calling I the almost complex structure induced by Ω, we have $g_{mn} = J_{mp}I^p{}_n$. Notice that we get no contribution from the variation of I since the associated Ω, given in eq.(5.14), is rigid.

The metric $\mathscr{G}_{ab}$ is special Kähler: indeed, it can be obtained via $\mathscr{G}_{ab} = \frac{\partial^2 K}{\partial t^a \partial \bar{t}^b}$ from the Kähler potential

$$K = -\log\frac{4}{3}\int J\wedge J\wedge J \;=\; -\log 8Vol\,. \tag{F.6}$$

It in turn is determined by a prepotential $\mathcal{F}$ via the special Kähler geometry formula $K = -\log i(\,\overline{X}^A\mathcal{F}_A - X^A\overline{\mathcal{F}}_A\,)$, where $X^A \equiv (X^0, X^a) = (1, -t^a)$ and $\mathcal{F}_A = \frac{\partial\mathcal{F}(X)}{\partial X^A}$.

For each of the cosets we consider, the explicit expressions of $\mathscr{G}_{ab}$ and Vol were given in table 5.1. The (cubic) prepotential reads

$$\mathcal{F}(X) = \frac{1}{6}\mathcal{K}_{abc}\frac{X^aX^bX^c}{X^0}\,,$$

where the non-vanishing triple intersection numbers $\mathcal{K}_{abc} := \int\omega_a\wedge\omega_b\wedge\omega_c$ (recall the 2–forms ω_a in subsection 5.2.1) are

$$\begin{aligned}
\mathcal{K}_{123} &= I \qquad \text{for} \quad \tfrac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}\\
\mathcal{K}_{112} &= 2I \qquad \text{for} \quad \tfrac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}\\
\mathcal{K}_{111} &= 6I \qquad \text{for} \quad \tfrac{\mathrm{G}_2}{\mathrm{SU}(3)}\,.
\end{aligned} \tag{F.7}$$

The period matrix $\mathcal{N}_{AB}$ of special Kähler geometry is given by the formula (D.5), which here reads

$$\mathcal{N}_{AB} = \overline{\mathcal{F}}_{AB} + 2i\frac{\mathrm{Im}(\mathcal{F}_{AC})X^C\mathrm{Im}(\mathcal{F}_{BD})X^D}{X^E\mathrm{Im}(\mathcal{F}_{EF})X^F}\,, \quad \text{where } \mathcal{F}_{AB} \equiv \frac{\partial^2\mathcal{F}}{\partial X^A\partial X^B}\,.$$

Equivalently, we can compute it directly from the coset geometry via relation (3.55), derived using the tools of generalized geometry. We obtain the matrices

$$\mathrm{Im}\mathcal{N} \;=\; -Vol \begin{pmatrix} 1+4\mathcal{G}_{ab}b^ab^b & 4\mathcal{G}_{ab}b^b \\ 4\mathcal{G}_{ab}b^b & 4\mathcal{G}_{ab} \end{pmatrix} , \tag{F.8}$$

$$\mathrm{Re}\mathcal{N} \;=\; -\begin{pmatrix} \frac{1}{3}\mathcal{K}_{abc}b^ab^bb^c & \frac{1}{2}\mathcal{K}_{abc}b^bb^c \\ \frac{1}{2}\mathcal{K}_{abc}b^bb^c & \mathcal{K}_{abc}b^c \end{pmatrix} . \tag{F.9}$$

F.2 The RR sector

In order to reduce the RR sector we specialize the general procedure described in section 3.5 for M_6 corresponding to our coset spaces (5.1).

The expansion ansatz (3.95) for $\hat{\mathbf{G}}$ here reads

$$\hat{\mathbf{G}} = (G^A_{(0)}+G^A_{(2)}+G^A_{(4)})\omega_A-(\tilde{G}_{(0)A}+\tilde{G}_{(2)A}+\tilde{G}_{(4)A})\tilde{\omega}^A+(G_{(1)}+G_{(3)})\alpha-(\tilde{G}_{(1)}+\tilde{G}_{(3)})\beta \tag{F.10}$$

where the expansion forms are the left-invariant basis introduced in subsection 5.2.1, and $G_{(p)}(x)$ and $\tilde{G}_{(p)}(x)$ are p–forms in 4d spacetime. Notice that, in contrast to (3.95), here we are not taking a $\sqrt{2}$ factor multiplying the expansion of the RR field $\hat{\mathbf{G}}$. This difference should be recalled when comparing the expressions of chapter 5 which involve RR fields with the corresponding ones appearing in the previous chapters.

Going through the derivation of subsections 3.5.1–3.5.3, we identify the 4d variables

$$\begin{aligned} G^A_{(0)} &= m^A \quad , \quad \tilde{G}_{(0)A} = e_A + q_A\,\tilde{\xi} \\ G_{(1)} &= D\xi \equiv d\xi - q_aA^a \quad , \quad \tilde{G}_{(1)} = d\tilde{\xi} \\ G^A_{(2)} &= dA^A \quad , \quad \tilde{G}_{(2)A} + B\tilde{G}_{(0)A} = \mathrm{Im}\mathcal{N}_{AB} * F^B + \mathrm{Re}\mathcal{N}_{AB}F^B \\ G_{(3)} &= -B\wedge D\xi + e^{2\varphi} * d\tilde{\xi} \quad , \quad \tilde{G}_{(3)} = -B\wedge d\tilde{\xi} - e^{2\varphi} * D\xi \end{aligned} \tag{F.11}$$

$$G^A_{(4)} + B\wedge G^A_{(2)} + \frac{1}{2}B^2\,G^A_{(0)} = e^{4\varphi}\big[(\mathrm{Im}\mathcal{N})^{-1}(\tilde{G}_{(0)} - \mathrm{Re}\mathcal{N}G_{(0)})\big]^A * 1$$

$$\tilde{G}_{(4)A} + B\wedge\tilde{G}_{(2)A} + \frac{1}{2}B^2\tilde{G}_{(0)A} = e^{4\varphi}\big[-\mathrm{Im}\mathcal{N}G_{(0)} + \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}(\tilde{G}_{(0)} - \mathrm{Re}\mathcal{N}G_{(0)})\big]_A * 1$$

where the propagating fields are the two real scalars $\xi,\tilde{\xi}$ and the 1–forms A^A. The modified field strengths read as in (3.114):

$$F^A \;\equiv\; dA^A + m^AB\ . \tag{F.12}$$

Furthermore we introduce $q_A = (0,q_a)$, the q_a being the geometric fluxes defined in subsection 5.2.1, while m^A, e_A are constant flux parameters satisfying $q_am^a = 0$. Notice that the q_a correspond to the m^0_a introduced in eq. (3.61), and that the m^A and e_A used here correspond to the m^A_{RR} and $e_{\mathrm{RR}A}$ of chapters 3, 4. We observe that one of the e_a is redundant, since it can be eliminated via a constant shift of $\tilde{\xi}$. This reflects the fact that on our cosets the linear combination $q_a \tilde{\omega}^a$ is exact (see eq. (5.11)), and therefore doesn't support any flux.

Applying the procedure of subsection 3.5.3, we reconstruct the 4d action $S^{(4)}_{\mathrm{RR}}$ from the EoM for $\xi, \tilde{\xi}$ and A^A. The resulting expression is given in (5.28).

As a last remark, we stress that the whole procedure of section 3.5 applies here with no need to take any integral over M_6. In other words, once the left-invariant truncation ansatz has been plugged in, the dependence of eqs. (3.93), (3.94) on the internal coordinates automatically factorizes out.

Appendix G

String loop corrections to the $N = 1$ coset vacua

In this appendix, we study how string loop corrections affect the tree level supersymmetric AdS_4 solutions of massive type IIA supergravity compactified on the cosets of chapter 5.

As discussed in section 4.2, within an $N = 2$ theory the $N = 1$ vacuum conditions arise by requiring the vanishing of the fermionic (i.e. the gravitino-, hyperino- and gaugino-) variations under a single linear combination of the two $N = 2$ supersymmetry parameters. The conditions associated with general SU(3)×SU(3) structure compactifications, spelled out in section 4.2, were solved for the specific case of Nearly Kähler manifolds in ref. [93]. Here, we extend the latter analysis employing the string loop corrected quaternionic vielbein (5.49) and the associated Sp(1) connection. In particular, the Killing prepotentials associated with our electric and magnetic gaugings of the quaternionic isometries become, recalling relation (5.45), the Killing vectors (5.48), and the Calderbank-Pedersen Sp(1) connection (5.50),

$$\mathcal{P}^1_A = -\frac{\sqrt{2}\rho}{\rho^2 - c} q_A \quad , \quad \tilde{\mathcal{P}}^{1A} = \mathcal{P}^2_A = \tilde{\mathcal{P}}^{2A} = 0\,,$$
$$\mathcal{P}^3_A = \frac{\sqrt{2}}{2(\rho^2 - c)}(e_A + \tilde{\xi} q_A) \quad , \quad \tilde{\mathcal{P}}^{3A} = \frac{\sqrt{2}}{2(\rho^2 - c)} m^A\,. \tag{G.1}$$

The tree level Killing prepotentials are recovered by taking $c = 0$ (recall the possible values of c, given in (5.52)), together with the identification $\rho^2 = e^{-2\varphi}$. The first part of the analysis performed in subsection 6.1 of [93] goes through in the present case, the only substantial difference being that the relation between the quaternionic vielbein u, v and the Sp(1) connection ω^x is here slightly more involved than (5.47); this leads to a modification of the equations arising from the hyperino variation. After a few manipulations, we arrive at the following $N = 1$ AdS vacuum condition for our coset reductions (both $\pm$ signs are allowed by susy),

$$-\left[(\mathrm{Im}\mathcal{N})^{-1\,AB} + \frac{3\rho^2 + c}{\rho^2} e^K \bar{X}^A X^B\right] \mathcal{P}^1_B \pm i(\mathrm{Im}\mathcal{N})^{-1\,AB}\left(\mathcal{P}^3_B - \mathcal{N}_{BC}\tilde{\mathcal{P}}^{3C}\right) = 0\,, \tag{G.2}$$

the (string frame) AdS cosmological constant being given by

$$\Lambda = -\frac{3}{2} e^{K} |q_A X^A|^2 \; . \tag{G.3}$$

We now solve the susy condition in the Nearly Kähler limit. As in subsection 5.6.2, we define $q \equiv \sum_a q_a$, we rename the only non-vanishing fluxes as $e_0 \to e$, $m^0 \to m$, and we set $v^a = v$ and $b^a = b$ for all a. Separating (G.2) into real and imaginary parts, and recalling (F.6) for K, as well as (F.8), (F.9) for $\mathcal{N}$, we obtain the four real equations

$$b = \pm \frac{4\rho}{5\rho^2 - c} \frac{mIv^3}{q} \qquad , \qquad b^2 = \frac{\rho^2 + 3c}{15\rho^2 - 3c} v^2$$

$$-be + \big(b^2 + \frac{v^2}{3}\big) q\tilde{\xi} + mI(b^4 + v^2 b^2) = 0 \qquad , \qquad -e + bq\tilde{\xi} + mIb^3 \pm \frac{3\rho^2 + c}{4\rho} qv = 0 \; .$$

This system of equations is solved by

$$v = v_T \frac{5x - \tilde{c}}{(5x + 3\tilde{c}) x^{\frac{1}{2}}} \quad , \quad b = b_T \left[\frac{(x + 3\tilde{c})^3}{x(x - \tilde{c}/5)} \right]^{\frac{1}{4}} \quad , \quad \tilde{\xi} = \xi_T \left[\frac{x(x + 3\tilde{c})}{x - \tilde{c}/5} \right]^{\frac{1}{2}} \quad , \quad \rho^2 = \rho_T^2 x \; , \tag{G.4}$$

where by $v_T, b_T, \tilde{\xi}_T, \rho_T^2$ we denote the tree level values (5.55) (recall that at tree level ρ^2 is identified with $e^{-2\varphi}$). We have also defined $\tilde{c} = \rho_T^{-2} c$ (note that this depends on the values of the fluxes appearing in $\rho_T^2 \sim (me^2 I)^{\frac{2}{3}} q^{-2}$), while x is the unique positive solution to the equation

$$(5x + 3\tilde{c})^4 (x + 3\tilde{c}) x - 5(5x - \tilde{c})^3 = 0 \, , \tag{G.5}$$

and can easily be determined numerically. The cosmological constant (G.3) here reads

$$\Lambda = -\frac{q^2}{5Iv_T} \frac{(x + 3\tilde{c}/5) x^{\frac{3}{2}}}{(x - \tilde{c}/5)^2} \; .$$

The tree level result is recovered by taking $\tilde{c} = 0$, in which case (G.5) is solved by $x = 1$. We have also checked that (G.4), (G.5) extremize the all loop scalar potential (5.51).

We conclude that string loops preserve the main outcome of the tree level analysis: for any choice of the fluxes e, m, there exists a unique Nearly Kähler supersymmetric solution. This is however shifted from the tree level position as shown in (G.4). It would be interesting to study the lifting of this result to a 10d framework.

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