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A Unifying Framework for Analysing Common Cyclical Features in Cointegrated Time Series*

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Abstract

This paper provides a unifying framework in which the coexistence of different form of common cyclical features can be tested and imposed to a cointegrated VAR model. This goal is reached by introducing a new notion of common cyclical features, namely the weak form of polynomial serial correlation common features, which encompasses most of the previous ones. Statistical inference is obtained by means of reduced-rank regression, and alternative forms of common cyclical features are detected by means of tests for over-identifying restrictions on the parameters of the new model. Some iterative estimation procedures are then proposed for simultaneously modelling different forms of common features. Concepts and methods are illustrated by an empirical investigation of the US business cycle indicators.

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1 Introduction

A large body of recent advances in modelling multiple time series is devoted to analyze co-movements among economic variables. A very popular notion of long-run comovements is cointegration, according to which a vector of I(1) time series is cointegrated when its elements share some common stochastic trends (Engle and Granger, 1987). However, detrended economic variables often display quite similar cyclical patterns (Lucas, 1977). This well-known "stylized fact" suggests that economic time series tend to share common transitory components as well. Engle and Kozicki (1993) proposed the notion of serial correlation common features as a measure of short-run comovements among I(1) variables. Indeed, common cycles exist in the multivariate Beveridge and Nelson (1981) decomposition of a multiple I(1) time series when its first differences exhibit common serial correlation (Vahid and Engle, 1993).[1]

From the statistical viewpoint, the presence of common cycles allows for rewriting the Vector Error Correction Model [VECM] as a Reduced Rank Regression [RRR] model. This implies that RRR techniques (see, *inter alia*, Johansen (1996), and Reinsel and Velu (1998)) can be used to obtain a more parsimonious model of the data. However, the notion of common cycles is somewhat limited since it is not able to detect the presence of non-contemporaneous cyclical comovements among I(1) time series (Ericsson, 1993). Consequently, some variants of the common cycles model have been suggested in order to overcome such limitation. In this paper, the focus is on the notions of common cyclical features that impose a partial reduced-rank structure on the VECM, namely the polynomial serial correlation common features by Cubadda and Hecq (2001), and weak form of common features by Hecq *et al.* (2000, 2006).

A serious limitation of the considered methods for common features analysis is that they cannot simultaneously model different forms of common features in the same VECM. Although the presence of alternative forms of common features can be tested, existing procedures do not allow for imposing the implied reduced rank structures on the estimated model. Hence, the most parsimonious model cannot be fitted to the data.

The goal of this paper is threefold. First, it is provided a new interpretations of the weak form of common features that has a meaningful implication for the short-run components of the series. Second, it is proposed a new notion of common cyclical features, namely the weak

[1] The idea of common cycles has later been extended to seasonally integrated series (Cubadda, 1999), I(2) systems (Paruolo, 2006a), and periodically integrated series (Haldrup *et al.*, 2007).

form of polynomial serial correlation common features, which encompasses all the considered ones. Third, it is shown how the coexistence of different forms of common cyclical features can be tested and imposed on the estimated VECM. Differently from the nested reduced rank autoregressive model by Ahn and Reinsel (1988), the new methods can be applied even when the coexisting forms of common features are not nested.

This paper is organized as follows. Section 1 reviews some forms of common cyclical features and introduces the notion of weak form of polynomial serial correlation common features. Section 2 deals with the issue of simultaneously modelling different forms of common features. In Section 3 the methodology is applied to some US business cycle indicators. Section 4 concludes.

2 Alternative notions of common cyclical features

Let us assume that an n-vector $\{y_t, t = 1, \ldots, T\}$ of cointegrated time series of order (1,1) is generated by the following VECM

$$\Gamma(L)\Delta y_t = \Phi_0 + \alpha\beta_*' y_{t-1}^* + \varepsilon_t, \tag{1}$$

for fixed values of $y_{-p+1}, ..., y_0$, where $\beta_*' = (\beta', \Phi_1')$, α and β are both $(n \times r)$ matrices of full rank r, Φ_1 is an r-vector, $\Gamma(L) = I_n - \sum_{i=1}^{p-1} \Gamma_i L^i$ is such that the matrix $\alpha_\perp' \Gamma(1)\beta_\perp$ has rank equal to $(n-r)$ and $\det(\Gamma(z)(1-z) - \alpha\beta' z) = 0$ implies that $z = 1$ or $|z| > 1$, $y_t^* = (y_t', t)'$, and ε_t are i.i.d. $N_n(0, \Sigma_\varepsilon)$ innovations if $t \geq 1$ and an n-vector of zeros otherwise.

Since Δy_t is a stationary stochastic process, it admits the following Wold representation

$$\Delta y_t = \mu + C(L)\varepsilon_t, \tag{2}$$

where $C(L) = I_n + \sum_{j=1}^{\infty} C_j$, the coefficient matrices C_j decrease exponentially fast, and $\mu = \Phi_0 + \alpha\beta' y_0 + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{1-i}$ (see e.g. Johansen, 1996).[2] From the expansion

$$C(L) = C(1) + \Delta C^*(L), \tag{3}$$

where $C_i^* = -\sum_{i+1}^{\infty} C_j$ for $i \geq 0$, we obtain the multivariate BN representation (Beveridge and

[2] The expression for μ is obtained by taking both equations (1) and (2) for $t = 1$, and equating them.

Nelson, 1981) of the series y_t

$$y_t = \delta_t + \tau_t + \kappa_t,$$

where $\delta_t = y_0 + \mu t$, $\tau_t = C(1)\sum_{i=0}^{t-1}\varepsilon_{t-i}$, and $\kappa_t = C^*(L)\varepsilon_t$. Based on the popular view that the stochastic trend of an I(1) time series is a random walk, the processes τ_t and κ_t are respectively defined as the stochastic trends and cycles of variables y_t.[3]

It is well known that the presence of cointegration is equivalent to the existence of $(n-r)$ common stochastic trends since $\beta'\tau_t = 0$ (Engle and Granger, 1987). Hence, a reduced rank restriction on the coefficient matrix of the terms y_{t-1} in model (1) is associated with a reduced number of components that are responsible for the long-run behavior of series y_t.

The analysis of common cyclical features is instead concerned with the short-run components of series y_t. In particular, the focus is on additional reduced-rank restrictions on the parameters of model (1) that have interesting implications on the cycles κ_t. Let us briefly review the various forms of common cyclical features which gained some attention in the literature, starting from the seminal notion of common cyclical features proposed by Engle and Kozicki (1993):

Definition 1 ***Serial Correlation Common Feature*** *[SCCF]: series Δy_t have s $(s < n)$ SCCF's iff there exists an $n \times s$ matrix δ_S with full column rank such that the VECM in* (1) *can be rewritten as the following RRR model*

$$\Delta y_t = \Phi_0 + \delta_{S\perp}\psi_S' w_{t-1} + \varepsilon_t, \tag{4}$$

where for any full column rank matrix M we denote by $M_\perp$ a full column rank matrix such that $M'M_\perp = 0$, ψ_S is an $(np-n+r)\times(n-s)$ matrix with full column rank, and $w_{t-1} = (y_{t-1}^{*\prime}\beta_*, \Delta y_{t-1,}', \ldots, \Delta y_{t-p+1}')'$.

The distinctive property of model (4) is that the predictable variations of series Δy_t are entirely generated by the $(n-s)$ common factors $\psi_S' w_{t-1}$.[4] Indeed, by premultiplying both

[3] Proietti (1997) discusses in details the relations among the multivariate BN representation and other popular permanent-transitory decompositions.

[4] Another well-known notion of common autocorrelation is discussed in the so-called common factor analysis, see, *inter alia,* Sargan (1983) and Mizon (1995). However, it is easy to check there is no relation of implication between these two notions. A proof is available upon request.

sides of equation (4) by δ_S' it follows that

$$\delta_S'\Delta y_t = \delta_S'\Phi_0 + \delta_S'\varepsilon_t.$$

Hence, the SCCF requires that there exists a linear combination of series Δy_t that is an innovation with respect to Ω_{t-1}, where Ω_t is the σ-field generated by $\{y_{t-i}; i \geq 0\}$. Moreover, the presence of s SCCF's is equivalent to the existence of $(n-s)$ common cycles since, as shown by Vahid and Engle (1993), $\delta_S'\kappa_t = 0$.

A drawback of the above definition is that it is not able to detect the existence of common serial correlation among non-contemporaneous elements of series Δy_t (see, e.g., Ericsson, 1993). In order to overcome this limitation, Cubadda and Hecq (2001) introduced the following variant of the SCCF

Definition 2 ***Polynomial Serial Correlation Common Feature*** *[PSCCF]: series Δy_t have s PSCCF's iff there exists an $n \times s$ matrix δ_P with full column rank such that $\delta_P'\Gamma_1 \neq 0$, and the VECM in (1) can be rewritten as the following partial RRR model*

$$\Delta y_t = \Phi_0 + \Gamma_1 \Delta y_{t-1} + \delta_{P\perp}\psi_P'(\Delta y_{t-2}', \ldots, \Delta y_{t-p+1}', y_{t-1}^{*\prime}\beta_*)' + \varepsilon_t, \tag{5}$$

where ψ_P is an $(np - 2n + r) \times (n - s)$ matrix with full column rank.

In order to interpret the notion of PSCCF, let us premultiply both sides of equation (5) by δ_P'. We then obtain

$$\delta(L)'\Delta y_t = \delta_P'\Phi_0 + \delta_P'\varepsilon_t,$$

where $\delta(L) = \delta_P - \Gamma_1'\delta_P L$. Hence, the PSCCF requires that there exists a first-order polynomial matrix $\delta(L)$ such that $\delta(L)'\Delta y_t$ is unpredictable from the past.[5]

The existence of the PSCCF has an interesting implication for the BN cycles of series y_t. Indeed, Cubadda and Hecq (2001) proved that $\delta(L)'\kappa_t = -\delta_P'\Gamma_1 C(1)\varepsilon_t$. Hence, the same PSCCF relationships cancel the dependence from the past of both the first differences and cycles of series y_t.

[5] Notice that the notion of PSCCF can be easily generalized to the case where the polynomial matrix $\delta(L)$ is of order m, where $m \leq (p-1)$. See Cubadda and Hecq (2001) for details.

Notice that equations (4) and (5) imply that both the matrices δ_S and δ_P must lie in the left-null space of the error-correction term loading matrix α. Hence, the number of the SCCF's or PSCCF's, s, cannot exceed the number of common trends $(n-r)$. In order to release this restriction, Hecq *et al.* (2000, 2006) proposed the following notion of weak form of SCCF

Definition 3 ***Weak Form of serial correlation common feature*** [*WF*]: *series* Δy_t *have* s *WF's iff there exists an* $n \times s$ *matrix* δ_W *with full column rank such that* $\delta_W' \alpha \neq 0$*, and the VECM in* (1) *can be rewritten as the following partial RRR model*

$$\Delta y_t = \Phi_0 + \alpha \beta_*' y_{t-1}^* + \delta_{W\perp} \psi_W' (\Delta y_{t-1}', \ldots, \Delta y_{t-p+1}')' + \varepsilon_t, \tag{6}$$

where ψ_W *is an* $(np-n) \times (n-s)$ *matrix with full column rank.*

The usual interpretation of the WF is that there exists a linear combination of series $(\Delta y_t - \alpha \beta_*' y_{t-1}^*)$ that is an innovation with respect to Ω_{t-1}. It is however possible to provide a new reading that permits one to uncover an interesting implication of the WF for the BN cycles κ_t. Indeed, premultiplying both sides of equation (6) by δ_W' yields

$$\delta_W(L)' y_t = \delta_W' (\Phi_0 + \alpha \Phi_1' t) + \delta_W' \varepsilon_t, \tag{7}$$

where $\delta_W(L) = \delta_W - (\beta\alpha' + I_n)\delta_W L$. By substituting (3) into (2) and premultiplying both sides of the resulting equation by $\delta_W(L)'$ one obtains

$$\delta_W(L)' \Delta y_t = \delta_W(1)' \mu + \delta_W(L)' [C(1) + \Delta C^*(L)] \varepsilon_t,$$

Finally, by taking first differences of both sides of (7) and comparing the resulting equation with the one above, it follows that

$$\delta_W(L)' \kappa_t \equiv \delta_W(L)' C^*(L) \varepsilon_t = \delta_W' (I_n - C(1)) \varepsilon_t.$$

The above results, which highlight that the WF is an analogous property to the PSCCF that applies to the levels rather than to the differences of series y_t, are summarized in the following proposition:

Proposition 4 *Series* Δy_t *have* s *WF's iff there exists a first-order polynomial matrix* $\delta_W(L)$

such that $(\delta'_W(L)y_t - \alpha\delta'_W\Phi'_1 t)$ is an innovation process with respect to Ω_{t-1}. Moreover, $\delta_W(L)'\kappa_t$ is also an innovation.[6]

Interestingly enough, it is possible to merge the notions of PSCCF and WF as follows.

Definition 5 ***Weak Form of Polynomial serial correlation common feature*** *[WFP]: series Δy_t have s WFP's iff there exists an $n \times s$ matrix δ_F with full column rank such that $\delta'_F\alpha \neq 0$, $\delta'_F\Gamma_1 \neq 0$, and the VECM in (1) can be rewritten as the following partial RRR model*

$$\Delta y_t = \Phi_0 + \alpha\beta'_* y^*_{t-1} + \Gamma_1\Delta y_{t-1} + \delta_{F\perp}\psi'_F(\Delta y'_{t-2}, \ldots, \Delta y'_{t-p+1})' + \varepsilon_t, \tag{8}$$

where ψ_F is an $(np-2n) \times (n-s)$ matrix with full column rank.

By premultiplying both sides of equation (8) by δ'_F we see that the WFP requires the existence of a second-order polynomial matrix $\delta_F(L) = \delta_F - (\beta\alpha' + I_n + \Gamma'_1)\delta_F L + \Gamma'_1\delta_F L^2$ such that

$$\delta_F(L)'y_t = \delta'_F(\Phi_0 + \alpha\Phi'_1 t) + \delta'_F\varepsilon_t, \tag{9}$$

In order to establish the implications of the WFP for the cycles κ_t, let us substitute (3) into (2) and premultiply both sides of the resulting equation by $\delta_F(L)'$. We obtain that

$$\delta_F(L)'\Delta y_t = \delta_F(1)'\mu + \delta_F(L)'[C(1) + \Delta C^*(L)]\varepsilon_t.$$

Finally, by taking first differences of both sides of (9) and in view of the above equation one obtains

$$\delta_F(L)'\kappa_t \equiv \delta_F(L)'C^*(L)\varepsilon_t = \delta'_F[I_n - C(1)]\varepsilon_t - \delta'_F\Gamma_1 C(1)\varepsilon_{t-1}.$$

Hence, the second-order polynomial matrix $\delta_F(L)$ transforms the BN cycles κ_t into a VMA(1) process.

Let CanCor$\{\Delta y_t, x_t \mid z_t\}$ denote the partial canonical correlations between series Δy_t and x_t having removed the linear dependence on z_t. Maximum Likelihood [ML] inference on the various forms of common features is obtained by solving CanCor$\{\Delta y_t, x_t \mid z_t\}$ for proper choices

[6] *As correctly pointed out by a referee, the original definition of WF is not invariant to reparametrizations of the VECM such as the one where the EC terms appear as $\beta' y_{t-p}$ in place of $\beta' y_{t-1}$. However, it is easy to see that the definition of WF in terms of the polynomial matrix $\delta_W(L)$ does not suffer from this non-uniqueness problem. A proof is available upon request.*

of the variables x_t and z_t. In particular, let $\widehat{\lambda}_i$ denotes the i-th smallest squared partial canonical correlation for $i = 1, ...n$. Under the null that s common features of a given form exist, the test statistic

$$LR_1 = -T \sum_{i=1}^{s} \ln(1 - \widehat{\lambda}_i), \qquad s = 1, \dots, n, \tag{10}$$

is asymptotically distributed as a $\chi^2(d_1)$ as detailed in Table 1, see, *inter alia*, Anderson (2002), and Paruolo (2003).

Table 1

Canonical correlations and tests for common features

Model	x_t	z_t	d_1
(4)	$(\Delta y'_{t-1}, ..., \Delta y'_{t-p+1}, y^{*\prime}_{t-1}\beta_*)'$	1	$s \times (n(p-2) + r + s)$
(6)	$(\Delta y'_{t-1}, ..., \Delta y'_{t-p+1})'$	$(1, y^{*\prime}_{t-1}\beta_*)'$	$s \times (n(p-2) + s)$
(5)	$(\Delta y'_{t-2}, ..., \Delta y'_{t-p+1}, y^{*\prime}_{t-1}\beta_*)'$	$(1, \Delta y'_{t-1})'$	$s \times (n(p-3) + r + s)$
(8)	$(\Delta y'_{t-2}, ..., \Delta y'_{t-p+1})'$	$(1, \Delta y'_{t-1}, y^{*\prime}_{t-1}\beta_*)'$	$s \times (n(p-3) + s)$

Moreover, let $\widehat{\varphi}_i^{\Delta y}$ and $\widehat{\varphi}_i^{x}$ respectively denote the partial canonical coefficients of Δy_t and x_t associated with $\widehat{\lambda}_i$. Optimal estimates of both the common features vectors and (partial) RRR coefficients are then obtained as described in Table 2.

Table 2

Estimators of the common features vectors and RRR coefficients

Model	$(\widehat{\varphi}_1^{\Delta y}, \dots, \widehat{\varphi}_s^{\Delta y})$	$(\widehat{\varphi}_{s+1}^{x}, \dots, \widehat{\varphi}_n^{x})$
(4)	$\widehat{\delta}_S$	$\widehat{\psi}_S$
(6)	$\widehat{\delta}_W$	$\widehat{\psi}_W$
(5)	$\widehat{\delta}_P$	$\widehat{\psi}_P$
(8)	$\widehat{\delta}_F$	$\widehat{\psi}_F$

Finally, the remaining parameters of the RRR models (4), (5), (6) and (8) are then estimated by OLS after fixing the various matrices ψ's to their estimated values.

3 Simultaneously modelling different forms of common features

A serious limitation of the existing methods for common features analysis is that they cannot handle the possible coexistence of different types of reduced rank restrictions in the same VECM. Consider, for instance, the following model

$$\Delta y_t = \Phi_0 + \delta_{A\perp}\psi_A'\Delta y_{t-1} + \delta_{B\perp}\psi_B'\beta_*'y_{t-1}^* + \delta_{F\perp}\psi_F'(\Delta y_{t-2}', \ldots, \Delta y_{t-p+1}')' + \varepsilon_t, \tag{11}$$

where $\delta_F = (\delta_A, \delta_B)$, δ_A is an $n \times s_1$ matrix, δ_B is an $n \times s_2$ matrix, the rank of matrix δ_F equals $(s_1 + s_2)$, and ψ_A and ψ_B are, respectively, $r \times (n - s_1)$ and $n \times (n - s_2)$ matrices with full column ranks. Hence, model (11) exhibit both s_1 WF's and s_2 PSCCF's.

Assume now that series Δy_t are instead generated by the model below

$$\Delta y_t = \Phi_0 + \delta_{C\perp}\psi_C'(\Delta y_{t-1}', y_{t-1}^{*\prime}\beta_*)' + \delta_{F\perp}\psi_F'(\Delta y_{t-2}', \ldots, \Delta y_{t-p+1}')' + \varepsilon_t, \tag{12}$$

where $\delta_C = \delta_F\omega$, ω is a full-rank $s \times s_1$ matrix, and ψ_C is an $(n + r) \times (n - s_1)$ matrix with full column rank. It is clear that s_1 out of the s WFP's of model (12) are indeed SCCF's.

Even if the presence of these different forms of common features can be tested by means of the statistic (10), it is not possible to impose the implied reduced rank structure on the estimated model. In this section we try to overcome such a limitation. Based on Cubadda (2007), one can use the following RRR model

$$u_t = \widetilde{\Phi}_0 + \delta_\perp\Psi'w_{t-1} + \widetilde{\varepsilon}_t, \tag{13}$$

where $u_t = (\Delta y_t', y_{t-1}^{*\prime}\beta_*, \Delta y_{t-1,}')'$, $\widetilde{\Phi}_0 = (\Phi_0', 0_{1\times(r+n)})'$, $\widetilde{\varepsilon}_t = (\varepsilon_t', 0_{1\times(r+n)})'$, δ is an $(2n+r) \times s$ matrix with $s < n$, and Ψ is a $(r + pn - n) \times (2n + r - s)$ matrix such that

$$\delta_\perp\Psi' = \begin{pmatrix} (\alpha, \Gamma_1) & (\Gamma_2, \ldots, \Gamma_{p-1}) \\ I_{r+n} & 0_{(r+n)\times(pn-2n)} \end{pmatrix}.$$

Since model (13) is an isomorphic representation of model (8), statistical inference based on the solution of

$$\text{CanCor}\{u_t, w_{t-1} \mid 1\} \tag{14}$$

is identical to that for the existence of s WFP's.[7] However, since the other forms of common features are nested in model (13), inference on all of them can be conducted by means of a restricted solution of the canonical correlation problem (14).

In a similar fashion as Johansen (1996), let us consider linear restrictions of the form $\delta = H\theta$, where H is a known $(2n+r) \times g$ matrix with full column rank, and θ is a $g \times s$ matrix to be estimated. Let $\widehat{\nu}_i$ denotes the i-th smallest squared canonical correlation, and $\widehat{\varphi}_i^{H'u}$ denote the associated canonical coefficients of $H'u_t$ drawn from the following canonical correlation program

$$\text{CanCor}\left\{H'u_t, w_{t-1} \mid 1\right\}. \tag{15}$$

Then the LR test statistic for the null hypothesis $\delta = H\theta$ is given by

$$LR_2 = T\sum_{i=1}^{s} \ln\left(\frac{1-\widehat{\omega}_i}{1-\widehat{\nu}_i}\right), \qquad s = 1, \ldots, n, \tag{16}$$

where $\widehat{\omega}_i$ denotes the i-th smallest squared canonical correlation drawn from the solution of (14), and the estimates of the parameters θ are given by $[\widehat{\varphi}_1^{H'u}, \ldots, \widehat{\varphi}_s^{H'u}]$. Under the null hypothesis the test statistic (16) has a $\chi^2(d_2)$ limit distribution, where $d_2 = s(2n+r-g)$.

Let us suppose that s WFP's exist and one wishes to tests if a more restricted form of common features exists in the data. For this purpose, it is required to solve the restricted canonical correlation program (15) for proper choices of the matrix H and use the test statistic (16) as detailed in Table 3.

TABLE 3

Tests for overidentifying restrictions in the WFP model

Model	H's matrices	d_2
(4)	$H_1 = (I_n, 0_{n\times(n+r)})'$	$s \times (n+r)$
(6)	$H_2 = (I_{n+r}, 0_{(n+r)\times n})'$	$s \times n$
(5)	$H_3 = \begin{pmatrix} I_n & 0_{n\times r} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times r} & I_n \end{pmatrix}'$	$s \times r$

When different forms of common features are simultaneously present in the data, a more

[7] Given that the terms $(y_{t-1}^{*\prime}\beta_*, \Delta y_{t-1,}')'$ are present in both u_t and w_{t-1}, the $(n+r)$ largest canonical correlation coming from (13) are exactly equal to one. However, the n smallest of such canonical correlations are the same as those required for statistical inference on model (8) in Table 1.

elaborated statistical approach is called for. For the sake of simplicity, the focus is only on the case where two different types of common features coexist but the proposed methods can be easily generalized. It is convenient to treat separately the case where PSCCF's and WF's are both present as in model (11) from the case of nested common features structures as occurs in model (12).

3.1 Coexistence of PSCCF's and WF's

Let us start by reparametrizing model (11) in terms of model (13). In view of Table 3, this is obtained by writing $\delta = (\delta_2, \delta_3)$ where $\delta_2 = H_2\theta_2$ and $\delta_3 = H_3\theta_3$. Hence, premultiplying both sides of model (13) by, respectively, H_2' and H_3' yields

$$\begin{aligned} H_2'u_t &= H_2'\widetilde{\Phi}_0 + H_2'\delta_\perp\Psi'w_{t-1} + H_2'\widetilde{\varepsilon}_t, \\ H_3'u_t &= H_3'\widetilde{\Phi}_0 + H_3'\delta_\perp\Psi'w_{t-1} + H_3'\widetilde{\varepsilon}_t. \end{aligned}$$

By taking, respectively, the expectation of $H_2'u_t$ conditional to $\delta_3'u_t$ and that of $H_3'u_t$ conditional to $\delta_2'u_t$ one obtains

$$\begin{aligned} H_2'u_t &= H_2'\widetilde{\Phi}_0 + H_2'\delta_\perp\Psi'w_{t-1} + \mathrm{E}(H_2'\widetilde{\varepsilon}_t|\delta_3'u_t) + \xi_{2,t} \equiv \mu_2 + H_2'\delta_\perp\Psi'w_{t-1} + \gamma_2\delta_3'u_t + \xi_{2,t}, \\ H_3'u_t &= H_3'\widetilde{\Phi}_0 + H_3'\delta_\perp\Psi'w_{t-1} + \mathrm{E}(H_3'\widetilde{\varepsilon}_t|\delta_2'u_t) + \xi_{3,t} \equiv \mu_3 + H_3'\delta_\perp\Psi'w_{t-1} + \gamma_3\delta_2'u_t + \xi_{3,t}, \end{aligned}$$

where $\xi_{2,t}$ and $\xi_{3,t}$ are i.i.d. Gaussian innovations with respect to Ω_{t-1}.

In view of the above partial RRR models, ML inference on the parameters θ_3 for fixed δ_2 is obtained by solving

$$\mathrm{CanCor}\left\{H_3'u_t, w_{t-1}|(1, u_t'\delta_2)'\right\}, \tag{17}$$

and, *vice versa*, ML inference on θ_2 having fixed δ_3 is obtained by the solution of

$$\mathrm{CanCor}\left\{H_2'u_t, w_{t-1}|(1, u_t'\delta_3)'\right\}. \tag{18}$$

Hence, the likelihood function of δ can be maximized by a linear switching algorithm similar to that proposed for cointegration analysis by Johansen and Juselius (1992), Johansen (1996), and Paruolo (2006b, 2006c).[8] This algorithm, which increases the likelihood function in each

[8] Notice that the necessary and sufficient condition for identification of the parameters (θ_2', θ_3') (see Johansen,

step, proceeds as follows:

1. Estimate δ unrestricted and obtain an initial estimate of δ_2 as the orthogonal projection of δ onto H_2.This is obtained as $\widehat{\delta}_2 = H_2(H_2'H_2)^{-1}H_2'\widehat{\delta}$.[9]

2. For fixed $\delta_2 = \widehat{\delta}_2$, obtain $\widehat{\delta}_3 = H_3\widehat{\theta}_3$, where $\widehat{\theta}_3$ are the canonical coefficients of $H_3'u_t$ associated with the s_2 smallest eigenvalues drawn from the solution of (17)

3. For fixed $\delta_3 = \widehat{\delta}_3$, obtain $\widehat{\delta}_2 = H_2\widehat{\theta}_2$, where $\widehat{\theta}_3$ are the canonical coefficients of $H_2'u_t$ associated with the s_1 smallest eigenvalues drawn from the solution of (18)

4. Repeat 2 and 3 until numerical convergence occurs.

The LR test statistic for the null hypothesis $\delta = (H_2\theta_2, H_3\theta_3)$ versus the alternative that δ is unrestricted is then given by

$$LR_3 = T\log\left(\det\left(\widehat{\Sigma}_\varepsilon\right)\det\left(\widetilde{\Sigma}_\varepsilon\right)^{-1}\right), \qquad (19)$$

where $\widehat{\Sigma}_\varepsilon$ and $\widetilde{\Sigma}_\varepsilon$ are the residual covariance matrices of models (8) and (11) respectively. The test statistic (19) follows asymptotically a $\chi^2(d_3)$ distribution, where $d_3 = (s_1 n + s_2 r - s)$.[10]

Regarding the estimators of the parameters of model (11), $\widehat{\psi}_A$ is given by the canonical coefficients of Δy_{t-1} associated with the $(n - s_1)$ largest canonical correlations drawn from (17), $\widehat{\psi}_B$ is given by the canonical coefficients of $\beta_*' y_{t-1}^*$ associated with the $(n - s_2)$ largest canonical correlations drawn from (18), and $\widehat{\psi}_F$ is obtained by regressing $(\widehat{\delta}_{F\perp}'\widehat{\delta}_{F\perp})^{-1}\widehat{\delta}_{F\perp}' r_{0t}$ on r_{1t}, where $\widehat{\delta}_F$ is given by the first n rows of the matrix $(\widehat{\delta}_2, \widehat{\delta}_3)$, and r_{0t} and r_{1t} are, respectively, the residuals of a regression of Δy_t and $(\Delta y_{t-2}', \ldots, \Delta y_{t-p+1}')'$ on $(\Delta y_{t-1}'\widehat{\psi}_A, y_{t-1}^{*\prime}\beta_*\widehat{\psi}_B)'$. Finally, the remaining parameters of model (11) are estimated by OLS after fixing the parameter matrices ψ_A, ψ_B and ψ_F to their estimated values.

3.2 Nested forms of common features

In order to simplify notation, let us suppose that the statistical problem consists of testing whether s_1 out of the s WFP's are indeed common features of a restricted form. However, it

1995) is here satisfied since $\text{rank}(H_{2\perp}'H_3) = \text{rank}(H_{3\perp}'H_2) = n \geq s$.

[9] Alternative choices of the starting values are discussed in details in Paruolo (2006c).

[10] Notice that Paruolo (2006b) corrected a common error in previous formulations of such a LR test statistic.

will be clear that the proposed solution applies to any case of nested common features. Hence, let us write $\delta = (\delta_r, \delta_u)$, where $\delta_r = H_j\theta_j$ for $j = 1, 2, 3$, θ_j is a $g_j \times s_1$ matrix with full column rank, and δ_u is an $n \times s_2$ matrix with full column rank.

A similar reasoning as in the previous subsection yields to the following equations

$$\begin{aligned} H_j u_t &= H_2'\widetilde{\Phi}_0 + H_j'\delta_\perp\Psi' w_{t-1} + \mathrm{E}(H_j'\widetilde{\varepsilon}_t|\delta_u' u_t) + \xi_{r,t} \equiv \mu_r + H_j'\delta_\perp\Psi' w_{t-1} + \gamma_r\delta_u' u_t + \xi_{r,t}, \\ H_{j\perp}' u_t &= H_{j\perp}'\widetilde{\Phi}_0 + H_{j\perp}'\delta_\perp\Psi' w_{t-1} + \mathrm{E}(H_{j\perp}'\widetilde{\varepsilon}_t|\delta_r' u_t) + \xi_{u,t} \equiv \mu_u + H_{j\perp}'\delta_\perp\Psi' w_{t-1} + \gamma_u\delta_r' u_t + \xi_{u,t}, \end{aligned}$$

where $\xi_{r,t}$ and $\xi_{u,t}$ are i.i.d. Gaussian innovations with respect to Ω_{t-1}. Hence, the statistical problem is solved by the following switching algorithm[11]:

I. Estimate δ unrestricted and obtain an initial estimate of δ_r as $\widehat{\delta}_r = H_j(H_j'H_j)^{-1}H_j'\widehat{\delta}$.

II. For fixed $\delta_r = \widehat{\delta}_r$, obtain $\widehat{\delta}_u = H_{j\perp}\widehat{\theta}_u$, where $\widehat{\theta}_u$ are the the canonical coefficients of $H_{j\perp}' u_t$ associated with the s_2 smallest eigenvalues drawn from the solution of

$$\mathrm{CanCor}\left\{H_{j\perp}' u_t, w_{t-1}|(1, u_t'\delta_r)'\right\}. \tag{20}$$

Notice that $\widehat{\delta}_u$ is restricted to $H_{j\perp}'$ in order to avoid a singularity problem in the canonical correlation problem (20).

III. For fixed $\delta_u = \widehat{\delta}_u$, obtain $\widehat{\delta}_r = H_j\widehat{\theta}_j$, where $\widehat{\theta}_j$ as the the canonical coefficients of $H_j' u_t$ associated with the s_1 smallest eigenvalues drawn from the solution of

$$\mathrm{CanCor}\left\{H_j' u_t, w_{t-1} \mid (1, u_t'\delta_u)'\right\}. \tag{21}$$

IV. Repeat II and III until numerical convergence occurs.

The LR test statistic for the null hypothesis $\delta_r = H_j\theta_j$ versus the alternative that δ_r is unrestricted is again given by (19), where $\widetilde{\Sigma}_\varepsilon$ is in this case the residual covariance matrix of the model associated with matrix H_j in Table 3, and $d_3 = s_1(d_2/s - 1)$, see again Table 3.

Regarding the estimators of the RRR parameters, let us focus on model (12), i.e., $j = 1$. Then $\widehat{\psi}_C$ is given by the canonical coefficients of $(\Delta y_{t-1}', y_{t-1}^{*\prime}\beta_*)'$ associated with the

[11] Again, the necessary and sufficient condition for identification of the parameters θ_j (see Johansen, 1995) is satisfied since $\mathrm{rank}(H_j'H_j) \geq n \geq s_1$.

$(n - s_1)$ largest canonical correlations drawn from (20), and $\widehat{\psi}_F$ is obtained by regressing $(\widehat{\delta}'_{F\perp}\widehat{\delta}_{F\perp})^{-1}\widehat{\delta}'_{F\perp}r_{0t}$ on r_{1t}, where $\widehat{\delta}_F$ is given by the first n rows of the matrix $(\widehat{\delta}_r, \widehat{\delta}_u)$, and r_{0t} and r_{1t} are, respectively, the residuals of a regression of Δy_t and $(\Delta y'_{t-2}, \ldots, \Delta y'_{t-p+1})'$ on $\widehat{\psi}'_C(\Delta y'_{t-1}, y^{*\prime}_{t-1}\beta_*)'$. The other coefficient matrices of models (12) are then estimated by OLS after fixing the parameter matrices ψ_C and ψ_F to their estimated values. With a similar reasoning one obtains the estimators of the RRR parameters when $j = 2, 3$.

4 Empirical example: common features of the US business cycle indicators

In order to illustrate the practical value of the proposed methods, let us consider the monthly indicators that The Conference Board uses to build the composite coincident indicator of the business cycle in the US. In particular, the empirical analysis concerns the logarithms of Employees on non-agricultural payrolls, Personal income less transfer payments,[12] Industrial production, and Manufacturing and trade sales for the period 1974.1-2003.7. These series are graphed in Figure 1. Although the data are available from 1959.1, only the post first oil-shock period is used because a preliminary application of the test by Bai *et al.* (1998) revealed that a VAR model of these series is affected by a structural break occurring in the late 1973.

According to the longest significant lag rule, a VAR(6) with a linear trend is fitted to the data. This model seems to appropriately reproduce the dynamic features of the data since the null hypothesis of no residuals autocorrelation is clearly not rejected by both the single-equation and multivariate tests. Table 4 reports the results of the Johansen's LR tests for cointegration, which suggest the existence of one cointegrating vector.

TABLE 4

Trace tests for cointegration

$r = 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
72.33**	38.26	21.20	9.236

* (**) Significant at the 5% (10%) confidence level

Having fixed $r = 1$, the presence of the various forms of common cyclical features is scrutinized. The results, reported in Table 5, indicate $s = 1$ for the SCCF, WF and PSCCF, and

[12] This series was corrected for two additive outliers corresponding to 1992.12 and 1993.12.

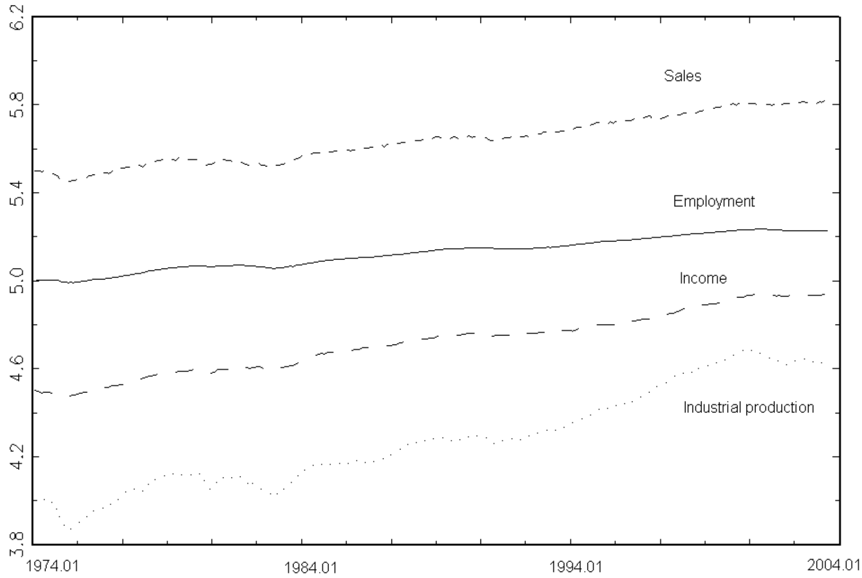

Figure 1: The US business cycle coincident indicators

$s = 2$ for the WFP. Overall, the evidence favors the existence of one unrestricted WFP, and one common feature of a restricted form.

TABLE 5

Common features tests

	$s \geq 1$	$s \geq 2$	$s \geq 3$	$s = 4$
SCCF	20.20**	88.70	187.8	495.7
WF	17.14**	60.24	155.7	421.1
PSCCF	16.25**	59.74	118.4	257.4
WFP	15.56**	36.16**	90.36	207.7

* (**) Significant at the 5% (10%) confidence level

Since the presence of two unrestricted WFP's and one cointegration vector implies that one PSCCF exists,[13] it is of interest to check whether the restricted form of common feature is either a WF or a SCCF. A test for the former hypothesis produces a test statistic equal to 7.64

[13]Indeed, there exists a direction lying in the space spanned by $\widehat{\delta}$ that has a null coefficient for the error-correction term $\widehat{\beta}_*' y_{t-1}^*$.

with a $p-$value equal to 0.27, whereas a test for the latter hypothesis produces a test statistic equal to 8.65 with a $p-$value equal to 0.28. Since the SCCF is nested within the WF, these results put forward the coexistence of one unrestricted WFP and one SCCF. The estimates of the associated common feature vectors are reported in Table 6.[14]

Table 6

Estimates of the common features relationships

SCCF	$(1, -1.571, 0.423, 0.038)'\Delta y_t$
WFP	$(1, 0.445, -0.668, -0.469)'\Delta y_t + (0.673, 0.191, 0.031, -0.100)'\Delta y_{t-1} - 0.004\widehat{\beta}'_* y^*_{t-1}$

Remarkably, the model that incorporates the above common features relationships has 52 parameters, whereas the model that only satisfies the SCCF restrictions calls for the estimation of 74 parameters.

5 Conclusions

This paper offers an approach for simultaneously modelling different forms of common cyclical features among I(1) time series. After showing that several existing forms of common features are nested within a new model, namely the weak form of polynomial serial correlation common features, some iterative procedures are proposed for testing and imposing diverse forms of common features to a cointegrated VAR model. The empirical application reveals that the new methods provide a model of the US business cycle indicators that is considerably more parsimonious than those obtained by previous notions of common cyclical features.

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