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Queueing models for air traffic

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I dedicate this thesis
to my mother Drande
and to my father Simon.

Abstract

In this thesis we study some queueing models that are worthwhile to understand the air-traffic congestion. From the point of view of classical queueing theory the air traffic system is difficult to study, mainly because it is hard even to define the basic quantities of the theory. The system becomes complex, since there are a many factor, that influence the air-traffic like weather conditions, technical problems, air turbulences caused by the different types of aircrafts. Thus is necessary to investigate the impact of the arrivals of aircraft on air traffic. A common hypothesis in literature is to assume that the arrivals of aircrafts are very well modeled by a Poisson process. This assumption is suitable for mathematical modelling, due to the memoryless property of Poisson process that simplifies the study of congestion in such systems.
Our first goal is to study the property of a model of the arrival process to a system and to compare its features to the Poisson process. We will show in this work why the Poissonian hypothesis for air-traffic is doomed to failure even if the Poisson process is very similar to our process if it is observed on a time scale sufficiently short. We found interesting connections of this model with the statistical mechanics of Fermi particles.
Once one understands the properties of arrival process to the system, to study its evolution we use the theory of Markov chain.
Our second goal is the study of the stochastic properties of other queueing systems, relevant in the applications, where the arrivals are described according general independent stochastic process and the service is delivered according to various disciplines. This corresponds to the study of the stationary measure of a Markov chain. In order to find the stationary distribution of such Markov chain we use the generating function technique. Part of this thesis is a discussion of the criteria usually presented in literature to evaluate the goodness of various approximation schemes. It will turn out, actually, that the generating function is not always possible to compute explicitly, and some numerical procedures are necessary in order to compute the relevant quantities of the system.

Keywords: Queueing system, air-traffic congestion, non Poissonian arrivals, tail approximation, two class queue in parallel, priority and Bernoulli scheduling.

Riassunto

In questa tesi studiamo dei modelli di coda che sono utili per capire la congestione del traffico aereo. Dal punto di vista della teoria delle code classica e' difficile studiare il sistema del traffico aereo, soprattutto perche' e' complesso definire le quantita' di base della teoria. Il sistema diventa complesso, poiche' ci sono molti fattori che influiscano sul traffico aereo, ad esempio le condizioni meteo, problemi tecnici, le turbolenze dell'aria causate da diversi tipi di aeromobili. Quindi per capire il traffico aereo diventa necessario studiare la distribuzione degli arrivi degli aeroplani. In letteratura l'ipotesi comune e' di assumere che gli arrivi degli aeroplani sono descritti molto bene dal processo di Poisson. Questa assunzione e' adatto per i modelli matematici, per la proprieta' di assenza di memoria del processo di Poisson che semplifica lo studio della congestione in tale sistema.
Il primo obiettivo di questa tesi e' di studiare le proprieta' di un processo degli arrivi degli aeromobili al sistema e di confrontare tale processo con il processo di Poisson. In questo lavoro mostriamo come l'ipotesi Poissoniana per il traffico aereo e' destinato a fallire anche se il processo Poissoniano e' molto simile al nostro modello degli arrivi se viene osservato su una scala di tempo opportunamente corta.
Troviamo poi, nella trattazione del nostro processo, una connessione interessante del nostro modello con la meccanica statistica di Fermioni.
Una volta comprese le proprieta' del processo degli arrivi al sistema, per studiare la sua evoluzione usiamo la teoria Markoviana.
Il secondo obiettivo di questa tesi e' lo studio delle proprieta' stocastiche di altri sistemi di coda, sempre rilevanti nelle applicazioni, in cui gli arrivi sono generali ma indipendenti, e discipline di servizio particolari rendono non banale lo studio della distribuzione stazionaria della catena. Questo corrisponde a studiare la misura stazionaria di certe catene di Markov. Per trovare la distribuzione stazionaria usiamo il metodo della funzione generatrice. Una parte di questa tesi e' la discussione dei criteri, di solito presentati in letteratura, per valutare l'ottimalita' dei vari schemi di approssimazione.

Parole chiave: Sistema di coda, traffico aereo congestionato, arrivi non Poissoniani, approssimazione coda, due classi di coda in parallelo, priorita' e scheduling Bernoulliano.

Contents

Chapter 1

Introduction and Motivation

To wait, or not to wait: that is the queue!

In this thesis we study some queueing models that are worthwhile to understand the air-traffic congestion. From the point of view of classical queueing theory the air traffic system is difficult to study, mainly because it is hard even to define the basic quantities of the theory. The system becomes complex, since there are a many factor, that influence the air-traffic like weather conditions, technical problems, air turbulences caused by the different types of aircrafts. For instance it is clear that there is some congestion for landing aircrafts, since they have to follow some holding paths, but it is not easy to quantify the actual time spent in queue or even its instant length. On the other hand, even assuming that the parameters of the system are known, it is not clear what kind of point processes are suitable to describe arrivals and service times. Thus is obviously necessary, to investigate the impact of the arrivals of aircrafts on air traffic. A common hypothesis in literature is to assume that the arrivals of aircrafts are very well modeled by a Poisson process. This assumption is very suitable for mathematical modelling, due to the memoryless property of Poisson process that simplifies the study of congestion in such system.
Our first goal is to study the property of arrival process to a system. We will show in this work why the Poissonian hypothesis for air-traffic is doomed to failure.
Once one understands the properties of arrival process to a system, to study the evolution of system we use the theory of Markov chain.
Our second goal is the study of the stochastic properties of other queueing system, where the arrivals are described according general independent stochastic process and the service is delivered according to various disciplines. This corresponds to the study of stationary measure of Markov chains. In order to find the stationary distribution of Markov chain we use the generating function technique. Part of this thesis is a discussion of the criteria usually presented in literature to evaluate the goodness of various approximation schemes. It will turn out, actually, that the generating function is not always possible to compute explicitly, and some numerical procedures are necessary in order to compute the relevant quantities of the system.

1.1 Stochastic point process as arrival process

Poissonian hypothesis for air traffic arrivals, to our knowledge, goes back to the 70's when Dunlay and Horonjeff gave in [17] a number of theoretical and statistical arguments to justify this assumption, and, since then, several other statistical studies have supported the same results. Even recently, see [16], a very careful study of the interarrival times of aircrafts to major US airports shows a small difference between the Poisson and the observed distribution, i.e. the actual arrivals are slightly less random than Poissonian ones, but the difference is quite small in all observed airports. On this ground, in various papers, see for instance [18], [21] and [22] and reference therein, Poisson arrivals have been assumed in the analysis of judicious management of service times. It should be stressed that in all these papers the statistical validation of the Poissonian hypothesis has been based on computations on time scales smaller than the intrinsic randomness of the system. The fact that arrivals are prescheduled clearly make the Poissonian hypothesis questionable. If we forecast a reduction of the intrinsic variability of arrival times, which could be achieved by various technical improvements (e.g. a rescheduling closer to the actual arrival times, or an en-route control of the paths of the aircrafts), we should expect the Poissonian assumption to fail, because it depends only on a single parameter ρ. About the stochastic models of aircraft arrivals we consider a point process defined as follows

$$t_i = \frac{i}{\lambda} + \xi_i \tag{1.1}$$

where $i \in \mathbb{Z}$, the ξ_i are i.i.d. random variables with variance σ^2 eventually much larger than $\frac{1}{\lambda}$ and $\frac{1}{\lambda}$ is the expected interarrival time between two aircrafts. From now on, we will call this process *pre-scheduled random arrivals* (PSRA) process. Note that this arrival process, excepted the presence of cancellations and pop-ups[1], is exactly the actual arrival process introduced in [4] using for ξ_i a uniform distribution. This process that we will study in chapter 2 is an arrivals model with two features. First, it shows a pattern of arrivals very close to a Poisson process when we look at time scales smaller than the standard deviation of aircraft delays, second, it provides the distribution of arrivals on time scales larger or comparable to the standard deviation of aircraft delays.

We study more rigorously the features of arrival process presented in [4], which we suitably generalize, and to understand its analytical properties, we show that this process, with a suitable rescaling of the distribution of ξ_i's, converges to the Poisson process in total variation for large σ, so

$$\sum_{n=0}^{\infty} |q_n^{(\sigma)} - q_n| \to 0 \quad \text{as} \quad \sigma \to \infty \tag{1.2}$$

where the sequence $q_n^{(\sigma)}$ and q_n are the coefficient of generating function of PSRA process and Poisson process respectively. Moreover, we show, both analytically and numerically, that the congestion related to this process is very different from the congestion of a Poisson process, on any time scale. This is due to the negative autocorrelation of the process, as we prove explicitly.

[1]Pop-ups are flights that arrive at the airport but were not expected on the time the GDP was implemented. A Ground Delay Program (GDP) is a traffic flow initiative that is instituted by the Federal Aviation Administrative (FAA) in the US.

1.2 Model with variable number of servers

The multi-server queueing system, with Poisson arrivals and deterministic service time, denoted usually by $M/D/c$ has a long history. The system was initially ideated by Erlang (see e.g. [11]) and later studied in [9, 10, 19], using theory of complex analysis. They describe the generating function of the system in terms of the root of the denominator inside the unit disc. Nevertheless, the problem to find the c complex root inside the unit disc is still studied. In this context, we will consider the systems with general arrivals, a variable number of servers, that is a system in which c is a random variable. This is because we want to study simple models of a discrete time services in which the number of servers at each time is the independent realization of a given random variable with distribution $\alpha_l = P(c_i = l)$ i.e. α_l is the probability that the users[2] find l available severs. A very natural example: in an airport, due to the safety rules for air traffic, some runways may be momentarily unavailable when some other runways are used by aircrafts in certain conditions. Hence the number of runways available in each time slot[3] may vary very rapidly.

We write the exact generating function in terms of c singularities $z_1, ...z_c$ of denominator in the unit disc, which represent the exact solution of problem, that gives us the knowledge of the queueing system. Nevertheless, the solution has two main disadvantages. First, it is not always easy to find complex zeroes of denominator, especially when c is large. Second, a small error (always present in numerical computations) on the values of $z_1, ...z_c$ generates a sequence of P_n that rapidly diverges from the true (probabilistic) expression. We show that if $z_1, ...z_c$ do not cancel exactly the zeroes of the denominator in the generating function, the latter diverges in some z_i and their coefficients P_n diverge exponentially, loosing their probabilistic interpretation. We therefore define a suitable approximation schemes for infinite Markov chain in order to avoid the explicit computation of such singularities.

The basic idea of this approximation scheme is as follows: we describe the system starting from the *cumulative probabilities* $\sigma_n := \sum_{k=0}^{n} P_n$ and in order to have a probabilistic interpretation, the sequence of σ_n has to be increasing, $\sigma_i \le \sigma_{i+1}$ and $\lim_{n\to\infty} \sigma_n = 1$. Let $\{\hat{\sigma}_n\}_{n\ge 0}$ and $\{\sigma\}_{n\ge 0}$ be the solution of the infinite system and truncated system respectively and also let us define by $\Delta_n := \sigma_n - \hat{\sigma}_n$ the errors of approximation; thus due to the linearity of system representing the σ_n's, we can write it in terms of Δ_n's. We will study the property of errors Δ_n's of approximation. We will present a rephrasing of one of the most used approximation schemes, the so-called *last column augmentation*, and we will discuss some theoretical and numerical results about the errors involved in this approximation.

We will show that an optimal approximation scheme, the censored Markov chain, gives often a bigger error than augmentation procedure in the part of the distribution that is relevant in the computation of the average values. Also with our approximation scheme we give an estimate on the errors in the sense of the L_∞ norm, then

$$|P_n - \hat{P}_n| = |\sigma_n - \sigma_{n-1} - (\hat{\sigma}_n - \hat{\sigma}_{n-1})| = |\Delta_n - \Delta_{n-1}| \le \Delta_{\bar{n}+1} \tag{2.1}$$

where $\bar{n}$ is the order of truncation of Markov chain, $\hat{P}_n$ and P_n are the stationary probability measure of infinite Markov chain and augmented Markov chain respectively.

[2]In this thesis we use the term "user" in a general sense.

[3]In this thesis we assume that the time is divided into fixed length intervals or slots. The users arrive in the system according to a general arrival process during the consideration and the consecutive slot, but they can receive service only at pre-defined discrete times, so at the beginning of slot

1.3 Two queues in parallel

Let us now consider a simplified description of the following problem: the air traffic is composed of different types of aircrafts, implying different queueing costs as well or different service times. Hence a non realistic description has to take into account the existence of different classes of users. Let us now consider the system, when two classes of users arrive to a single server, according to a general independent stochastic process. We can think this system composed of two separate queues in parallel, where the each one of users belonging to each queue are served according to FIFO discipline. We discus two cases: 1) users of first class have a higher priority than users of second class, or in other words the users of second class can be served only if the users of first class are absent. 2) users of first class receive service with probability p ($0 \leq p \leq 1$) and the users of second class receive service with probability $1-p$ (Bernoulli scheduling) if the two classes are both non empty. To understand these typology of system we describe it as a Markov chain with two dimensional probability generating function. In this way we obtain a generating function which contain some unknown *functions*. In order to find the unknown functions we use the theory of boundary value problem. So, the problem to finding the stationary joint probability distribution can be reduced to solve the boundary value problem.

Fayolle and Iasnogorodski in [12] studied two queueing in parallel, with Poisson arrivals and exponential service times and single server; using the Riemann Hilbert boundary value problem they found the joint generating function. Later Cohen and Boxma in [15] described in general the boundary value problem in queueing theory.

In the first case i.e. a simple priority rule, that will we study in chapter 4, the generating function contains a single unknown function. We study the condition for the existence of solution of boundary value problem and we solve it. In this way we control completely the generating function.

The study of second case is more difficult, because the generating function contain two unknown functions. The idea to bypass this difficulty is to use the perturbative method. Clearly the generating function depends on p and for $p = 1$ and $p = 0$ the generating function has the same structure as the generating functions of system discussed in the first case, hence we have the exact solution of problem for $p = 1$ and $p = 0$ as we will prove in chapter 4. Now expanding the generating function in powers of $1 - p$ and p on appropriate domain we obtain two sequence of functional coefficients that are symmetric in x and y. Using these coefficients and exact solution of problem for $p = 1$ and $p = 0$ we will give the approximate solution of generating function. In order to prove the validity of approximation method we present some numerical results for lower and heavy traffic intensity and varying of parameter p.

1.4 The map of thesis

The thesis is organized as follows:
In chapter 2, we propose the PSRA process as arrival process, that is alternative of Poisson process to describe the arrivals of aircrafts. We study the property of this stochastic point process through the use of generating function and compare its feature to Poisson process.
In chapter 3, we investigate the model with fixed and variable number of servers. We give a suitable approximation scheme for infinite Markov chain in order to obtain the average of the queue length.
In chapter 4, we study discrete time single server queueing systems with two classes of users,

where the users of first class have a higher priority than users of second class. We obtain the generating function of the joint stationary probability distribution through the solution of the boundary problem. Moreover we study the discrete time single server queueing with Bernoulli scheduling and two classes of users. We give the approximation solution of generating function through perturbative method.

Chapter 2

Queueing system with pre-scheduled random arrivals

In this chapter we consider a point process obtained summing a random variable ξ_i to each point i of the set of integer $\mathbb{Z}$. The ξ_i's are i.i.d. random variables with variance σ^2 eventually much larger than 1. We compare the process obtained with this construction with the Poisson process. Moreover, we show that, this process, with a suitable rescaling of the distribution of ξ_i's, converges to the Poisson process in total variation for large σ. We then study a simple queueing system with our process as arrival process, and we provide some analytical and numerical results.

2.1 Introduction

The main aim of this chapter is to define a stochastic point process and to compare its features to the Poisson process. It is well known that the memoryless property of the Poisson process simplifies many technical steps in the analysis of queueing systems, but there are arrival processes where such an assumption is not completely satisfied. In particular, we have in mind air traffic models.

Stochastic models of aircraft arrivals based on statistical analysis and on simulations have a long history. As a first attempt, Barnett et al. [1] studied the arrivals to Boston Logan Airport. A version of the alternative model of arrivals we propose in this chapter was introduced and studied numerically in [4]. The model is refined in [3], where seasonal and daily effects are taken into account to describe random delays of departure times and, with these corrections, the model is quite accurate in its predictions. The key feature of the model is a *soft* a-prior scheduling of arrivals: indeed, both in US and in Europe, aircrafts are supposed to take off and to land by a schedule dictated by the capacity constraint of the runways, and by the assumption that each aircraft would land in a very narrow time slot. However, on the day of operations, an aircraft will be declared "on time" if it lands in a time interval larger than ten times the original slot. In this sense the scheduling should be considered "soft".

The process we study below is an arrivals model with two features. First, it shows a pattern of arrivals very close to a Poisson process when we look at time scales smaller than the standard deviation of aircraft delays, second, it provides the distribution of arrivals on time scales larger or comparable to the standard deviation of aircraft delays.

Thus, the aim of this chapter is an attempt to study more rigorously the features of arrival process presented in [4], which we suitably generalize, and to understand its analytical properties.

Moreover, we compare, both analytically and numerically, the queueing system in which the arrivals are described according to a Poisson process and a PSRA process respectively and for the second system we give the expression for the stationary probability distribution, under the hypothesis that the number of arrivals in subsequent slot are independent random variable. We can consider that stationary distribution like lower bound for the stationary distribution for the $M/D/1$ queueing system.

The analytical description of the system clarifies many interesting features of this kind of traffic: for heavy traffic the system has a long memory of the initial conditions; its description is obtained by the superposition of two processes, living on different time scales. This give the possibility to investigate also systems with slowly variable traffic intensities.

The chapter is organized as follows: in section 2.2 we describe our arrival process, and we list some results on the comparison to the Poisson process. In section 2.3 we present a simplified computation, which shows that congestion levels according to our process are quite different from the Poisson process. However, in section 2.4 we show numerically that our approximation is bad for very congested systems, and the actual level of congestion is even more different than the Poissonian one. In section 2.5 we describe completely our queueing system at the price to enlarge suitably the state space of the Markov chain describing it. It turns out that for our process we have a finite value of the expected queue length even in the critical case $\varrho = 1$, while the Poisson queue diverges. Starting from the results on the critical case, we propose an approximation scheme that works very well for highly congested (ϱ near to 1) systems. In this description a nice connection with the statistical mechanics of Fermi gas emerges quite naturally. Section 2.6 is devoted to conclusions and open problems.

2.2 Description of the model: the arrival process

The queueing model we study is defined by a single server with deterministic service time and an arrival process, which we will call *pre-scheduled random arrivals* (PSRA) process, defined as follows. Let $\frac{1}{\lambda}$ be the expected interarrival time between two users, we define $t_i \in \mathbb{R}$ the actual arrival time of the i-th user by

$$t_i = \frac{i}{\lambda} + \xi_i \qquad i \in \mathbb{Z} \tag{2.1}$$

where ξ_i's are i.i.d. random variables.

If the ξ_i's are uniform, the model is the actual arrival times process introduced in [4] without cancellations and pop-ups, which could be easily integrated into the process. From now on, we will assume that ξ_i's have continuous probability density $f_\xi^{(\sigma)}(t)$ with variance σ^2, and we will set without loss of generality $E(\xi_i) = 0$, since $E(\xi_i) \neq 0$ affects only the initial configuration of the system. The main aim of this section is to compare the features of the PSRA process to the Poisson process when σ is large. It is well known, e.g. [2, p.447], that the Poisson arrival process is defined by the fact that probabilities $P_{j,j+1}(\Delta t) = P(n(t + \Delta t) = j + 1 | n(t) = j)$ of a "jump" from the state j to the state $j + 1$ in the time interval $(t, t + \Delta t]$ have the form

$$P_{j,j+1}(\Delta t) = P^+(\Delta t) = \lambda \Delta t + o(\Delta t) \tag{2.2}$$

where λ is a constant independent of t and j; λ has the meaning of velocity of arrivals, i.e. denoting with t_a the interarrival time, $E(t_a) = \frac{1}{\lambda}$. For pre-scheduled random arrivals the probability $P(i,t,\Delta t)$ that the i-th user arrives in the time interval $(t, t+\Delta t]$ is given by

$$P(i,t,\Delta t) = P\left(t < \frac{i}{\lambda} + \xi_i < t + \Delta t\right) = \tag{2.3}$$

$$= P\left(t - \frac{i}{\lambda} < \xi_i < t + \Delta t - \frac{i}{\lambda}\right) = \int_{t-\frac{i}{\lambda}}^{t+\Delta t-\frac{i}{\lambda}} f_\xi^{(\sigma)}(x)dx \tag{2.4}$$

and, for small Δt, it may be written as

$$P(i,t,\Delta t) = f_\xi^{(\sigma)}\left(t - \frac{i}{\lambda}\right)\Delta t + o(\Delta t) \tag{2.5}$$

By (2.5), the probability $P^+(t,\Delta t)$ of a single PSRA arrival in the interval $(t, t+\Delta t]$ is

$$P^+(t,\Delta t) = \sum_{i\in\mathbb{Z}} P(i,t,\Delta t)\prod_{j\neq i}(1 - P(j,t,\Delta t)) =$$

$$= \sum_{i\in\mathbb{Z}}\left[f_\xi^{(\sigma)}\left(t-\frac{i}{\lambda}\right)\Delta t + o(\Delta t)\right]\exp\left(\sum_{j\neq i}\log\left[1 - f_\xi^{(\sigma)}\left(t-\frac{j}{\lambda}\right)\Delta t + o(\Delta t)\right]\right) \tag{2.6}$$

and

$$P^+(t,\Delta t) = \sum_{i\in\mathbb{Z}}\left[f_\xi^{(\sigma)}\left(t-\frac{i}{\lambda}\right)\Delta t + o(\Delta t)\right]\exp\left(-\sum_{j\neq i}\left[f_\xi^{(\sigma)}\left(t-\frac{j}{\lambda}\right)\Delta t + o(\Delta t)\right]\right) \tag{2.7}$$

Hence up to the first order in Δt the rate of arrival $\lambda(t)$ of the pre-scheduled random arrivals is defined by

$$\lambda(t) = \sum_{i\in\mathbb{Z}} f_\xi^{(\sigma)}\left(t - \frac{i}{\lambda}\right) \tag{2.8}$$

This rate $\lambda(t)$ is periodic in t with period $\frac{1}{\lambda}$, but it has an explicit dependence on t. However we are interested in the dependence of $\lambda(t)$ on σ, in particular when σ is large with respect to $\frac{1}{\lambda}$. To prove limit properties for our process, we have to specify the way we want to send σ to infinity. We will require the following *scaling* property for the density $f_\xi^{(\sigma)}(t)$.

Assumptions 2.2.1. *The probability density of ξ has the form*

$$f_\xi^{(\sigma)}(t,\sigma^2) = \frac{1}{\sigma}f_\xi(t/\sigma) \tag{2.9}$$

i.e. it is the rescaling of a well defined continuous density $f_\xi(t)$ with finite variance. We will also write $\max_{t\in\mathbb{R}} f_\xi(t) = M$.

This assumption is introduced in order to exclude pathological ways to send σ to infinity, as, for instance, to have a bimodal distribution with fixed maxima, see figure 2.1.

For example Gaussian, Exponential, Gamma, and Uniform random variables satisfy this property. It follows that, in the limit σ very large the expression

$$R(\sigma, 1/\lambda) := \sum_{i\in\mathbb{Z}} \frac{1}{\lambda} f_\xi^{(\sigma)}\left(t - \frac{i}{\lambda}\right) \tag{2.10}$$

Figure 2.1: A bimodal distribution with fixed shapes shifting to infinity for $\sigma \to \infty$.

is the Riemann integral of the function $f_{\xi}^{(\sigma)}(t)$.
For example, let ξ be Gaussian $N(0, \sigma^2)$,

$$R(\sigma, 1/\lambda) = \sum_{i \in \mathbb{Z}} \frac{1}{\lambda} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\lambda t - i)^2}{2\sigma^2\lambda^2}} = \sum_{i \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t\lambda - i}{\lambda\sigma}\right)^2} \frac{1}{\lambda\sigma} = \sum_{i \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} \Delta x \longrightarrow 1$$

where $x_i = \frac{\lambda t - i}{\lambda\sigma}$ and $\Delta x = \frac{1}{\lambda\sigma}$ and the limit is for $\sigma \to \infty$.
For any random variable rescaled in the above sense it is clear that the result

$$\lim_{\sigma^2 \to \infty} R(\sigma, 1/\lambda) = 1 \tag{2.11}$$

holds, and therefore, in the same limit,

$$\lim_{\sigma^2 \to \infty} \lambda(t) = \lim_{\sigma \to \infty} \lambda R(\sigma, 1/\lambda) = \lambda \tag{2.12}$$

It is interesting, for Gaussian ξ, to check numerically how fast the limit is reached. Table 2.1 shows it. For simplicity, we set $\lambda = 1$.

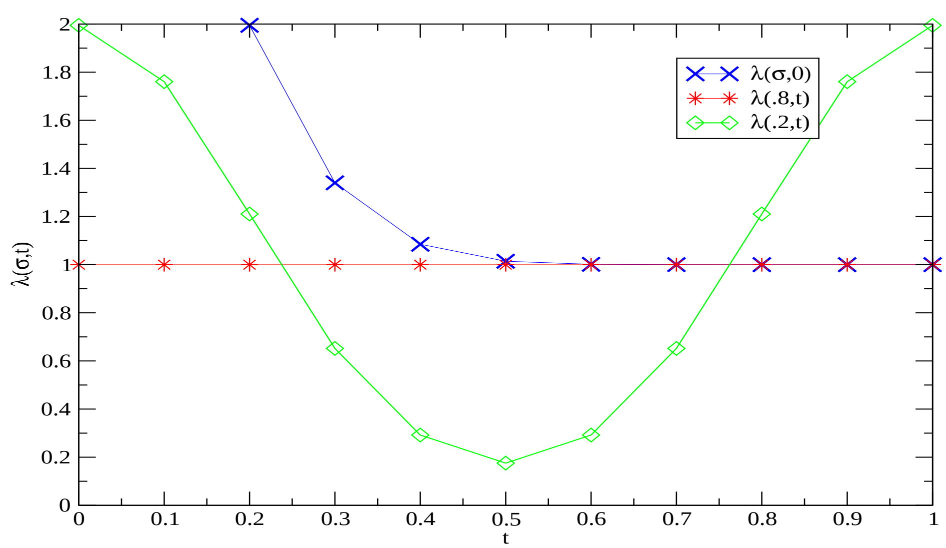

Figure 2.2: Behavior of the function $\lambda(\sigma, t)$

The graph in figure 2.2 shows that, in terms of rate of arrivals, the pre-scheduled random arrivals approach the Poisson process when σ is suitably large. In particular for Gaussian variables with standard deviation σ of order $1/\lambda$ we have that $\lambda(t)$ tends to be constant. Note

σ	$\lambda(0)$	$\lambda(0.1)$	$\lambda(0.2)$	$\lambda(0.3)$	$\lambda(0.4)$	$\lambda(0.5)$
.2	1.994726	1.760407	1.210523	0.651951	0.292114	0.175283
.3	1.340089	1.274318	1.103259	0.894087	0.726696	0.663191
.4	1.085005	1.068767	1.026261	0.973729	0.931237	0.915008
.5	1.014384	1.011637	1.004445	0.995555	0.988363	0.985616
.6	1.00164	1.001327	1.000507	0.999493	0.998673	0.99836
.7	1.000126	1.000102	1.000039	0.999961	0.999898	0.999874
.8	1.000007	1.000005	1.000002	0.999998	0.999995	0.999993
.9	1.	1.	1.	1.	1.	1.
1.	1.	1.	1.	1.	1.	1.

σ	$\lambda(0.6)$	$\lambda(0.7)$	$\lambda(0.8)$	$\lambda(0.9)$	$\lambda(1)$
.2	0.292114	0.651951	1.210523	1.760407	1.994726
.3	0.726696	0.894087	1.103259	1.274318	1.340089
.4	0.931237	0.973729	1.026261	1.068767	1.085005
.5	0.988363	0.995555	1.004445	1.011637	1.014384
.6	0.998673	0.999493	1.000507	1.001327	1.00164
.7	0.999898	0.999961	1.000039	1.000102	1.000126
.8	0.999995	0.999998	1.000002	1.000005	1.000007
.9	1.	1.	1.	1.	1.
1.	1.	1.	1.	1.	1.

Table 2.1:

that for applications mentioned in the introduction, we do expect the standard deviation to be much larger than $1/\lambda$. Note also that the explicit structure of the density of ξ does not play any particular role, and similar results may be obtained with different distributions. However it is clear that a small dependence on t is always present in the expression of $\lambda(t)$, and hence it is difficult to obtain a quantitative comparison between the pre-scheduled random arrivals and the Poisson process on this basis, therefore we look at the distribution of the random variable $n(t,t+T)$, number of arrivals in the *finite* interval $(t,t+T]$. So the random variable $n(t,t+T)$ count the number of arrivals in the interval $(t,t+T]$. Let us call $p_i(t,t+T)$ the probability that the i-th user arrives in the interval $(t,t+T]$. Clearly

$$p_i(t,t+T) = \int_t^{t+T} f_\xi^{(\sigma)}\left(x - \frac{i}{\lambda}\right) dx \tag{2.13}$$

Given the probabilities $p_i(t,t+T)$ we can write the generating function of the random variable $n(t,t+T)$, and, defining $q_n^{(\sigma)} = P(n(t,t+T)=n)$ we get

$$q_n^{(\sigma)} = \sum_{I=\{i_1,\dots,i_n\}} \prod_{i\in I} p_i(t,t+T) \prod_{j\notin I}(1-p_j(t,t+T)) \tag{2.14}$$

where the sum runs over all the possible distinct subsets I of indices of cardinality n. By mean of this expression one obtains the generating function

$$q^{(\sigma)}(z) = \sum_{n\geq 0} q_n^{(\sigma)} z^n = \prod_{i\in\mathbb{Z}}(1+(z-1)p_i(t,t+T)) \tag{2.15}$$

To take into account also the possibility of random independent *deletion* as in [4], let us outline here that a similar generating function can be introduced also when each arrival has an independent probability $1-\gamma$ to be deleted, and the complementary probability γ to be an actual arrival. In other words, we construct the PSRA process for $i \in \mathbb{Z}$ and then for each i we cancel the corresponding i-th arrival with independent probability $1-\gamma$. It is obvious that in this case the generating function is

$$q_\gamma^{(\sigma)}(z) = \sum_{n\geq 0} q_{\gamma,n}^{(\sigma)} z^n = \prod_{i\in\mathbb{Z}} (1 + (z-1)\gamma p_i(t, t+T)) \tag{2.16}$$

The expression (2.15) are exact, and gives us all the information on the distribution of $n(t, t+T)$, and it depends explicitly on t and T. However we can study $q^{(\sigma)}(z)$ and $q_\gamma^{(\sigma)}(z)$ for large σ, in the sense of the rescaling defined above, and show that they converges to a Poisson distribution with parameter λT and $\gamma\lambda T$. The main idea is to exploit the fact that, for large σ, $p_i(t, t+T)$ goes to zero as $\frac{1}{\sigma}$.

We now prove the following results.

Lemma 2.2.2.

$$\max_i p_i(t, t+T) \leq \frac{const(T)}{\sigma} \tag{2.17}$$

Proof.

$$p_i(t, t+T) = \int_t^{t+T} f_\xi^{(\sigma)}\left(x - \frac{i}{\lambda}\right) dx = \int_{t-\frac{i}{\lambda}}^{t-\frac{i}{\lambda}+T} f_\xi^{(\sigma)}(s) ds = \frac{1}{\sigma}\int_{t-\frac{i}{\lambda}}^{t-\frac{i}{\lambda}+T} f_\xi\left(\frac{s}{\sigma}\right) ds \tag{2.18}$$

by the Intermediate Value Theorem

$$p_i(t, t+T) = \frac{1}{\sigma} f_\xi\left(\frac{s_i^*}{\sigma}\right) T \leq \frac{MT}{\sigma} \tag{2.19}$$

where

$$\frac{s_i^*}{\sigma} \in \left(t - \frac{i}{\lambda}, t - \frac{i}{\lambda} + T\right)$$

□

Now we will use lemma 2.2.2 to bound the generating function

$$q^{(\sigma)}(z) = \exp\left[\sum_{i\in\mathbb{Z}} \ln(1 + (z-1)p_i(t, t+T))\right] = \tag{2.20}$$

$$= \exp\left[(z-1)\sum_{i\in\mathbb{Z}} p_i(t, t+T)\left(1 + (z-1)p_i(t, t+T)\int_0^1 \frac{s\, ds}{(1 + (z-1)(1-s)p_i(t, t+T))^2}\right)\right] \tag{2.21}$$

Lemma 2.2.3. *With $p_i(t, t+T)$ defined as above, the sum in* (2.21) *converges to* λT

$$\lim_{\sigma\to\infty} \sum_{i\in\mathbb{Z}} p_i(t, t+T)\left(1 + (z-1)p_i(t, t+T)\int_0^1 \frac{s\, ds}{(1 + (z-1)(1-s)p_i(t, t+T))^2}\right) = \lambda T \tag{2.22}$$

Proof. First we prove that

$$\lim_{\sigma\to\infty}\sum_{i\in\mathbb{Z}} p_i(t,t+T)=\lambda T. \tag{2.23}$$

Let us define $T := \frac{K+\Delta T}{\lambda}$, where $K \in \mathbb{Z}^+$ and $0 \le \Delta T < 1$. Then we can write

$$\sum_{i\in\mathbb{Z}} p_i(t,t+T)=\sum_{i\in\mathbb{Z}}\int_{t-\frac{i}{\lambda}}^{t-\frac{i}{\lambda}+T} f_\xi^{(\sigma)}(s)ds=\sum_{i\in\mathbb{Z}}\int_{t-\frac{i}{\lambda}}^{t+\frac{K-i}{\lambda}+\frac{\Delta T}{\lambda}} f_\xi^{(\sigma)}(s)ds=$$

$$=\sum_{i\in\mathbb{Z}}\int_{t-\frac{i}{\lambda}}^{t+\frac{K-i}{\lambda}} f_\xi^{(\sigma)}(s)ds+\sum_{i\in\mathbb{Z}}\int_{t+\frac{K-i}{\lambda}}^{t+\frac{K-i}{\lambda}+\frac{\Delta T}{\lambda}} f_\xi^{(\sigma)}(s)ds \tag{2.24}$$

The first term on the right hand side of (2.24) is K. Let $i = mK + l$, where $l \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$,

$$\sum_{i\in\mathbb{Z}}\int_{t-\frac{i}{\lambda}}^{t+\frac{K-i}{\lambda}} f_\xi^{(\sigma)}(s)ds=\sum_{l=0}^{K-1}\sum_{m\in\mathbb{Z}}\int_{t-\frac{mK+l}{\lambda}}^{t-\frac{(m-1)K+l}{\lambda}} f_\xi^{(\sigma)}(s)ds=\sum_{l=0}^{K-1}\int_{\mathbb{R}} f_\xi^{(\sigma)}(s)ds=K \tag{2.25}$$

The second term on the right hand side of (2.24) converges to ΔT for $\sigma\to\infty$:

$$\sum_{i\in\mathbb{Z}}\int_{t+\frac{K-i}{\lambda}}^{t+\frac{K-i}{\lambda}+\frac{\Delta T}{\lambda}} f_\xi^{(\sigma)}(s)ds=\sum_{i\in\mathbb{Z}}\int_{t+\frac{i}{\lambda}}^{t+\frac{i}{\lambda}+\frac{\Delta T}{\lambda}} f_\xi^{(\sigma)}(s)ds=\sum_{i\in\mathbb{Z}}\frac{1}{\sigma}\int_{t+\frac{i}{\lambda}}^{t+\frac{i}{\lambda}+\frac{\Delta T}{\lambda}} f_\xi\left(\frac{s}{\sigma}\right)ds \tag{2.26}$$

and, by the Intermediate Value Theorem we get

$$\sum_{i\in\mathbb{Z}}\frac{1}{\sigma}\int_{t+\frac{K-i}{\lambda}}^{t+\frac{K-i}{\lambda}+\frac{\Delta T}{\lambda}} f_\xi\left(\frac{s}{\sigma}\right)ds=\sum_{i\in\mathbb{Z}}\frac{1}{\sigma}f_\xi\left(\frac{s_i^*}{\sigma}\right)\frac{\Delta T}{\lambda} \tag{2.27}$$

where

$$\frac{s_i^*}{\sigma}\in\left(t+\frac{K-i}{\lambda},t+\frac{K-i}{\lambda}+\frac{\Delta T}{\lambda}\right)$$

and finally,

$$\Delta T\sum_{i\in\mathbb{Z}} f_\xi\left(\frac{s_i^*}{\sigma}\right)\frac{1}{\lambda\sigma}\longrightarrow\Delta T \tag{2.28}$$

as $\sigma\to\infty$, where the sum on the last equality is the Riemann sum of $f_\xi(t)$. This ends the proof of (2.23). In order to complete the lemma we need to show that, uniformly in i,

$$\lim_{\sigma\to\infty}(z-1)p_i(t,t+T)\int_0^1 ds\frac{s}{(1+(z-1)(1-s)p_i(t,t+T))^2}=0$$

but this follows from lemma 2.2.2 and from the fact that

$$(z-1)\int_0^1 ds\frac{s}{(1+(z-1)(1-s)p_i(t,t+T))^2}\le C$$

for any $p_i(t,t+T)<1/2$ and $|z|\le 1$.

□

Lemma 2.2.4. *Let $q(z) = \exp(\lambda T(z-1))$ be the probability generating function of the Poisson random variable ζ with intensity λT, and $q_\gamma(z) = \exp(\gamma\lambda T(z-1))$ be the probability generating function of the Poisson random variable ζ with intensity $\gamma\lambda T$, then*

$$\lim_{\sigma\to\infty} q^{(\sigma)}(z) = q(z); \qquad \lim_{\sigma\to\infty} q_\gamma^{(\sigma)}(z) = q_\gamma(z) \tag{2.29}$$

Proof. Follows immediately from lemma 2.2.3. □

Theorem 2.2.5. *If $q^{(\sigma)}(z) \longrightarrow q(z)$, then $\sum_{n=0}^{\infty} |q_n^{(\sigma)} - q_n| \longrightarrow 0$ as $\sigma \to \infty$. The same result holds for the arrivals with random deletions.*

Proof. The proof follows from the continuity theorem for probability generating function see Feller [2, p.280]. □

Hence the PSRA process converges in distribution to the Poisson process in total variation norm.

In order to show that the process has negative autocorrelation, we will compute the expected value, the variance σ_n of the number n of arrivals in a time slot $(t, t+T]$, and the covariance $Cov(n_1, n_2)$, where n_1 and n_2 are the numbers of arrivals in $(t, t+T]$ and $(t+T, t+2T]$, respectively.
Let $\chi_i(t_i \in (t, t+T])$ be the characteristic function of the event "user i arrives in the interval $(t, t+T]$", so that $\mathbb{E}(\chi_i) = p_i(t, t+T)$, then the expected number of arrivals in a time slot $(t, t+T]$ is

$$\mathbb{E}(n) = \mathbb{E}\left(\sum_i \chi_i\right) = \sum_i \mathbb{E}(\chi_i) = \sum_i p_i(t, t+T)$$

and also

$$\begin{aligned}
\mathbb{E}(n^2) = \mathbb{E}\left(\sum_i \chi_i \sum_j \chi_j\right) &= \mathbb{E}\left(\sum_i \chi_i + \sum_{i\neq j} \chi_i\chi_j\right) = \\
&= \sum_i p_i(t, t+T) + \sum_{i\neq j} p_i(t, t+T)p_j(t+T, t+2T) \\
&= \sum_i p_i(t, t+T) + \left(\sum_i p_i(t, t+T)\right)^2 - \sum_i (p_i(t, t+T))^2
\end{aligned}$$

Then the variance is:

$$\sigma_n^2 = \mathbb{E}(n^2) - (\mathbb{E}(n))^2 = \sum_i p_i(t, t+T) - \sum_i (p_i(t, t+T))^2 = \sum_i p_i(t, t+T)(1 - p_i(t, t+T))$$

and we see again that $\sigma_n^2 \to \lambda T$ in the limit $\sigma \to \infty$. Finally, let us define $\chi_i^{(1)} := \chi_i(t_i \in (t, t+T])$ and $\chi_i^{(2)} := \chi_i(t_i \in (t+T, t+2T])$

$$\begin{aligned}
\mathbb{E}(n_1 n_2) = \mathbb{E}\left(\sum_i \chi_i^{(1)} \sum_j \chi_j^{(2)}\right) &= \mathbb{E}(\sum_{i\neq j} \chi_i^{(1)}\chi_j^{(2)}) = \sum_{i\neq j} \mathbb{E}(\chi_i^{(1)})\mathbb{E}(\chi_j^{(2)}) = \\
&= \sum_{i\neq j} p_i(t, t+T)p_j(t+T, t+2T) \\
&= \sum_{i,j} p_i(t, t+T)p_j(t+T, t+2T) - \sum_i p_i(t, t+T)p_i(t+T, t+2T)
\end{aligned}$$

so that

$$Cov(n_1, n_2) = \mathbb{E}(n_1 n_2) - \mathbb{E}(n_1)\mathbb{E}(n_2) = -\sum_i p_i(t, t+T)p_i(t+T, t+2T)$$

A negative covariance means that n_1 and n_2 are inversely correlated, as we should expect in our arrival model: a congested time slot should be followed or preceded by a slot with lower than expected arrivals. Moreover, this is a clear indication that the hypothesis of independence for n_1 and n_2, numbers of arrivals in different time slots, is not correct, unless we are in the limit $\sigma \to \infty$.

2.3 Queueing systems with PSRA process: independence approximation

In this section we want to try to use the classical results of queueing theory for a system in which the arrivals are described in terms of our PSRA, there is a single server and the service time is deterministic. For the air traffic applications the deterministic service (landing) times are obviously an approximation, but neglecting the mix of aircrafts the actual landing times have a low variability.

In order to study the queueing system given by our PSRA process we set a service time T and define the instant traffic intensity $\varrho(\sigma, t) = E(n(t, t+T))$. In fig. 2.3 and table 2.2 we report numerical results for the convergence of $\varrho(\sigma, t)$ to λT, granted by lemma 2.2.3. For simplicity we consider ξ Gaussian, and $\lambda = 1$. In this case $\varrho(\sigma, t)$ converges as soon as σ gets close to 1.

σ	T	$\varrho(\sigma,0)$	$\varrho(\sigma,0.1)$	$\varrho(\sigma,0.2)$	$\varrho(\sigma,0.3))$	$\varrho(\sigma,0.4))$
.2	.9	0.808534	0.808534	0.850089	0.907951	0.954826
.3	.9	0.868214	0.868214	0.88048	0.900153	0.919615
.4	.9	0.892048	0.892048	0.895086	0.900001	0.904914
.5	.9	0.898654	0.898654	0.899168	0.9	0.900832
.6	.9	0.899847	0.899847	0.899905	0.9	0.900095
.7	.9	0.899988	0.899988	0.899993	0.9	0.900007
.8	.9	0.899999	0.899999	0.9	0.9	0.9
.9	.9	0.9	0.9	0.9	0.9	0.9
1.	.9	0.9	0.9	0.9	0.9	0.9
σ	T	$\varrho(\sigma,0.5)$	$\varrho(\sigma,0.6)$	$\varrho(\sigma,0.7)$	$\varrho(\sigma,0.8)$	$\varrho(\sigma,0.9)$
.2	.9	0.9786	0.9786	0.954826	0.907951	0.850089
.3	.9	0.931537	0.931537	0.919615	0.900153	0.88048
.4	.9	0.907951	0.907951	0.904914	0.900001	0.895086
.5	.9	0.901346	0.901346	0.900832	0.9	0.899168
.6	.9	0.900153	0.900153	0.900095	0.9	0.899905
.7	.9	0.900012	0.900012	0.900007	0.9	0.899993
.8	.9	0.900001	0.900001	0.9	0.9	0.9
.9	.9	0.9	0.9	0.9	0.9	0.9
1.	.9	0.9	0.9	0.9	0.9	0.9

Table 2.2:

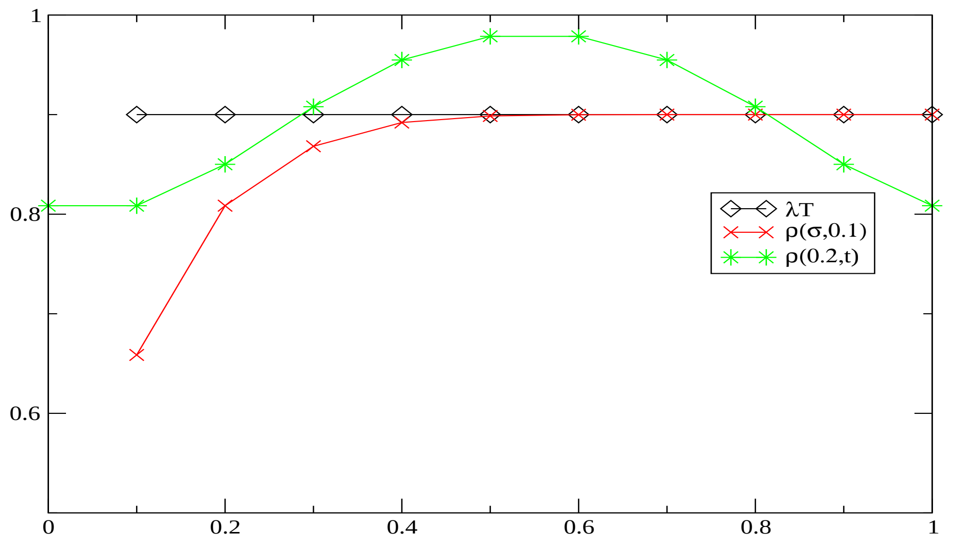

Figure 2.3: Behavior of the function $\varrho(\sigma, t)$. On the x axis we have time t for $\varrho(0.2, t)$ and standard deviation σ for $\varrho(\sigma, 0.1)$.

We want to compare the average queue size in $M/D/1$ queueing system (Poisson arrivals) with the $G/D/1$ queueing system in which the arrivals are described in terms of PSRA. It is well known, see e.g.[7], that the stationary probabilities for the discrete time $G/D/1$ queueing system are given by

$$\begin{aligned} &P_0 = (P_0 + P_1)Q_0 \\ &\vdots \\ &P_n = P_0 Q_n + \sum_{k=1}^{n+1} P_k Q_{n-k+1} \\ &\vdots \end{aligned} \tag{3.1}$$

where Q_n is the probability to have n arrivals in a service time slot.

The corresponding generating function is given by

$$P(z) = \frac{P_0(1-z)}{1 - \frac{z}{Q(z)}} \tag{3.2}$$

where

$$P_0 = 1 - \varrho \tag{3.3}$$

In the case of Poisson arrivals with traffic intensity ϱ, $Q(z) = q(z) = \exp(\varrho(z-1))$. Denoting by N the average queue size, after straightforward computations we get

$$P'(z)\big|_{z=1} = N = \frac{\varrho(2-\varrho)}{2(1-\varrho)} \tag{3.4}$$

Consider now the PSRA process. In this case we can try to compute (3.2) by means of the generating function (2.15). This is obviously an approximation, since the generating function (3.2) is obtained under the hypothesis that the number of arrivals in subsequent slots are independent variables. Indeed this is not the case for PSRA arrivals, as it has been shown in Section 2. However, assuming that such independence we neglect possible effects of autocorrelation, then we have that $Q(z) = q^{(\sigma)}(z)$. Now we employ the boundary condition $P(z)|_{z=1} = 1$ and l'Hôpital rule, also using the fact that $q^{(\sigma)}(z)\big|_{z=1} = 1$, we find that

$$P_0 = 1 - \sum_{i\in\mathbb{Z}} p_i(t, t+T) \tag{3.5}$$

Now denoting by $N(\sigma, t)$ the average queue size and applying l'Hôpital rule we get

$$N(\sigma, t) = P'(z)\big|_{z=1} = \left.\frac{zq^{(\sigma)}(z)q^{(\sigma)}_{zz}(z) - 2\big(zq^{(\sigma)}_z(z) - q^{(\sigma)}_z(z)\big)}{\big(q^{(\sigma)}_z(z) - zq^{(\sigma)}_z(z)\big)q^{(\sigma)}_z(z)}\right|_{z=1} \tag{3.6}$$

where

$$q^{(\sigma)}_z(z) = \prod_{i\in\mathbb{Z}}(1 + (z-1)p_i(t, t+T)) \sum_{i\in\mathbb{Z}} \frac{p_i(t, t+T)}{1 + (z-1)p_i(t, t+T)}$$
$$q^{(\sigma)}_{zz}(z) = \prod_{i\in\mathbb{Z}}(1 + (z-1)p_i(t, t+T)) \sum_{\substack{k\neq l\\ k,l\in\mathbb{Z}}} \frac{p_k(t, t+T)p_l(t, t+T)}{(1 + (z-1)p_k(t, t+T))(1 + (z-1)p_l(t, t+T))}$$

After few calculations we find

$$N(\sigma, t) = \frac{2\sum_{i\in\mathbb{Z}} p_i(t, t+T) - (\sum_{i\in\mathbb{Z}} p_i(t, t+T))^2 - \sum_{i\in\mathbb{Z}} p_i^2(t, t+T)}{2(1 - \sum_{i\in\mathbb{Z}} p_i(t, t+T))} \tag{3.7}$$

In order to give a complete description of our $GI/D/1$ queueing system in which the arrivals are described in terms of PSRA, we can find the stationary probability distributions P_n. We now consider (3.2). Using (3.5) and (2.15), we can rewrite (3.2) as

$$P(z) = \frac{(1 - \sum_{i\in\mathbb{Z}} p_i(t, t+T))(1-z)}{1 - \frac{z}{\prod_{i\in\mathbb{Z}}(1+(z-1)p_i(t,t+T))}} \tag{3.8}$$

Note that

$$\left|\frac{z}{\prod_{i\in\mathbb{Z}}(1 + (z-1)p_i(t, t+T))}\right| < 1 \tag{3.9}$$

for all $z \in \mathbb{D}$, hence we can expand the the inverse denominator of (3.8) as a geometric series.

Therefore,

$$
\begin{aligned}
P(z) &= P_0(1-z)\sum_{k=0}^{\infty}\frac{z^k}{\left(\prod_{i\in\mathbb{Z}}(1+(z-1)p_i(t,t+T))\right)^k}\\
&= P_0(1-z)\sum_{k=0}^{\infty}z^k\prod_{i\in\mathbb{Z}}\frac{1}{(1-(1-z)p_i(t,t+T)))^k}\\
&= P_0(1-z)\sum_{k=0}^{\infty}z^k\prod_{i\in\mathbb{Z}}\sum_{l=0}^{\infty}\binom{l+k-1}{l}((1-z)p_i(t,t+T))^l\\
&= P_0(1-z)\sum_{k=0}^{\infty}z^k\sum_{l=0}^{\infty}\binom{l+k-1}{l}(1-z)^l\prod_{i\in\mathbb{Z}}(p_i(t,t+T))^l\\
&= P_0(1-z)\sum_{k=0}^{\infty}z^k\sum_{l=0}^{\infty}\binom{l+k-1}{l}\sum_{j=0}^{l}\binom{l}{j}z^j(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}\\
&= P_0(1-z)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{l+k-1}{l}\sum_{j=0}^{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}z^{k+j}\\
&= P_0(1-z)\sum_{l=0}^{\infty}\sum_{j=0}^{l}\sum_{n=j}^{\infty}\binom{l+n-j-1}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}z^n\\
&= P_0(1-z)\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n\wedge l}\binom{l+n-j-1}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}z^n\\
&= P_0\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n\wedge l}\binom{l+n-j-1}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}z^n-\\
&- P_0\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n\wedge l}\binom{l+n-j-1}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}z^{n+1}\\
&= P_0\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n\wedge l}\binom{l+n-j-1}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}z^n-\\
&- P_0\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{(n-1)\wedge l}\binom{l+n-j-2}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}z^n
\end{aligned}
$$

Hence we expand $P(z)$ as a power series in z and consequently we obtain an exact result for P_n,

$$
\begin{aligned}
P_n = (1-\sum_{i\in\mathbb{Z}}p_i(t,t+T))\sum_{l=0}^{\infty}\sum_{j=0}^{n\wedge l}\binom{l+n-j-1}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}-\\
-(1-\sum_{i\in\mathbb{Z}}p_i(t,t+T))\sum_{l=0}^{\infty}\sum_{j=0}^{(n-1)\wedge l}\binom{l+n-j-2}{l}\binom{l}{j}(-1)^j\exp\left\{l\sum_{i\in\mathbb{Z}}\log p_i(t,t+T)\right\}
\end{aligned}
$$

Note that, it is well known the steady state probability distribution P_n for $M/D/1$ (see e.g. [6]) queueing system is given by

$$P_n = (1-\varrho)\left\{\sum_{k=0}^{n}(-1)^{n-k}e^{k\varrho}\frac{(k\varrho)^{n-k}}{(n-k)!} - \sum_{k=0}^{n-1}(-1)^{n-k-1}e^{k\varrho}\frac{(k\varrho)^{n-k-1}}{(n-k-1)!}\right\} \tag{3.10}$$

For σ large $N(\sigma,t)$ becomes independent of t, and it converges to N by (2.23). Table 2.3 shows that for Gaussian ξ and $\lambda = 1$ the convergence is quite fast.

σ	T	$N(\sigma,0)$	$N(\sigma,0.1)$	$N(\sigma,0.2)$	$N(\sigma,0.3)$	$N(\sigma,0.4)$	$N(\sigma,0.5)$
.1	.9	0.89105	0.89105	1.00493	1.04024	1.02267	1.00905
.2	.9	1.61425	1.61425	1.58187	1.51872	1.42902	1.32201
.3	.9	2.26812	2.26812	2.21399	2.10656	1.95949	1.83453
.4	.9	2.75253	2.75253	2.68673	2.57205	2.44587	2.36133
.5	.9	3.03548	3.03548	2.9955	2.92993	2.86327	2.82151
.6	.9	3.24502	3.24502	3.23019	3.20614	3.18205	3.16714
.7	.9	3.43207	3.43207	3.42809	3.42165	3.41521	3.41123
.8	.9	3.59488	3.59488	3.59405	3.5927	3.59134	3.59051
.9	.9	3.73131	3.73131	3.73117	3.73094	3.73071	3.73056
1.	.9	3.84462	3.84462	3.8446	3.84457	3.84454	3.84452

σ	T	$N(\sigma,0.6)$	$N(\sigma,0.7)$	$N(\sigma,0.8)$	$N(\sigma,0.9)$	$N(\sigma,1)$
.1	.9	1.00905	1.02267	1.04024	1.00493	0.89105
.2	.9	1.32201	1.42902	1.51872	1.58187	1.61425
.3	.9	1.83453	1.95949	2.10656	2.21399	2.26812
.4	.9	2.36133	2.44587	2.57205	2.68673	2.75253
.5	.9	2.82151	2.86327	2.92993	2.9955	3.03548
.6	.9	3.16714	3.18205	3.20614	3.23019	3.24502
.7	.9	3.41123	3.41521	3.42165	3.42809	3.43207
.8	.9	3.59051	3.59134	3.5927	3.59405	3.59488
.9	.9	3.73056	3.73071	3.73094	3.73117	3.73131
1.	.9	3.84452	3.84454	3.84457	3.8446	3.84462

Table 2.3:

2.4 Numerical results

In the previous section we have computed the average queue size for the $G/D/1$ queueing system in which the arrivals are described in terms of PSRA under the hypothesis that the number of arrivals in subsequent slots are independent variables. and we have remembered the formula for average queue size $M/D/1$ queueing system. In this section we compare the PSRA average queue size $N(\sigma,t)$ obtained by numerical simulations to (3.7) and (3.4). We recall that (3.7) is obtained assuming the independence of number of arrivals in different time slots; (3.4) is the length of the queue for Poissonian arrivals. The numerical results can be found in table 2.4. In this table we can compare the values of average queue obtained by formula (3.4), (3.7) and simulation, for low traffic intensity ($\varrho = 0.5$) and for heavy traffic intensity ($\varrho = 0.9$)

σ	T	$N(3.3)$	$N(3.4)$	$N(sim)$	σ	T	$N(3.3)$	$N(3.4)$	$N(sim)$
0.1	0.5	0.75	0.5	0.4963	0.1	0.9	4.95	1.00905	0.9153
0.5	0.5	0.75	0.614554	0.5096	0.5	0.9	4.95	2.8215	1.2258
1	0.5	0.75	0.680202	0.5618	1.	0.9	4.95	3.84452	1.5004
2	0.5	0.75	0.71483	0.6173	2.	0.9	4.95	4.38353	1.834
3	0.5	0.75	0.726519	0.6481	3.	0.9	4.95	4.57059	2.0145
4	0.5	0.75	0.732381	0.6621	4.	0.9	4.95	4.66498	2.2124
5	0.5	0.75	0.735901	0.6787	5.	0.9	4.95	4.72181	2.3278
6	0.5	0.75	0.738249	0.6821	6.	0.9	4.95	4.75976	2.4414
7	0.5	0.75	0.739927	0.6873	7.	0.9	4.95	4.7869	2.555
8	0.5	0.75	0.741186	0.6948	8.	0.9	4.95	4.80726	2.6249
9	0.5	0.75	0.742165	0.6974	9.	0.9	4.95	4.8231	2.7232
10	0.5	0.75	0.742949	0.7078	10.	0.9	4.95	4.83583	2.8007

Table 2.4:

In figure 2.4 $N(\sigma, t)$ is plotted as a function of σ, for different values of $\varrho = 0.5, 0.7, 0.9$, and $t = 0.5$. The dotted straight lines represent N obtained by (3.4) for different values of ϱ. As we can see from the graph, values of $N(\sigma, t = 0.5)$ for fixed ϱ given by (3.7) are larger than the corresponding ones obtained by simulation. This is due to the fact that we neglected the (negative) autocorrelations. Moreover, while for small ϱ (say $\varrho \leq 0.6$) approximation (3.7) gives relatively good results, the overestimate becomes very important when ϱ increases.

2.5 Queueing systems with PSRA process: autocorrelated arrivals

As it is clear from the results of the previous section, neglecting the autocorrelation the estimate of the average queue length is grossly overestimated in the interesting cases. If we want to describe the system only by the length of the queue, the presence of autocorrelation implies the loss of Markov property. In this section we show that if we enlarge suitably the state space we may keep the Markov property, and describe completely the autocorrelation. With this description some interesting features of the system are clarified, but at the moment we are able to compute explicitly the quantities of interest with some approximations. Such approximations, however, turn out to give almost negligible errors.
To simplify the analytical treatment of the system, we will consider from now on densities $f_\xi^{(\sigma)}(t)$ of the random i.i.d. variables ξ_i that are compact support, i.e. such that $f_\xi^{(\sigma)}(t) = 0$ for $t > L$ for some $L < \infty$. We are setting $\lambda = 1$, and we take $L \in \mathbb{N}$. This implies that at a certain discrete time j the i'th user is certainly arrived to the system for all $i \leq j - L$, while for all $i \geq j + L$ it is certainly not yet arrived. Hence to completely describe the state of the system we have to specify, beside the number n of users waiting in queue right before the service at time j is delivered, also a *finite* set I_j of i's, $I_j \subset \{j-L+1, ..., j+L-1\}$, that are the users that are already arrived at the service at time j. Note that the users in the set I_j are not necessarily already served at time j, or, in other words, the set I_j is the set of the users with indices in $\{j - L + 1, ..., j + L - 1\}$ that are in the queue at time j, or that are already served at time j. Note also that $0 \leq |I_j| \leq 2L - 1$. Finally, we want to outline that due to the independence of the ξ's I_{j+i} is independent of I_j for all $i \geq 2L$.

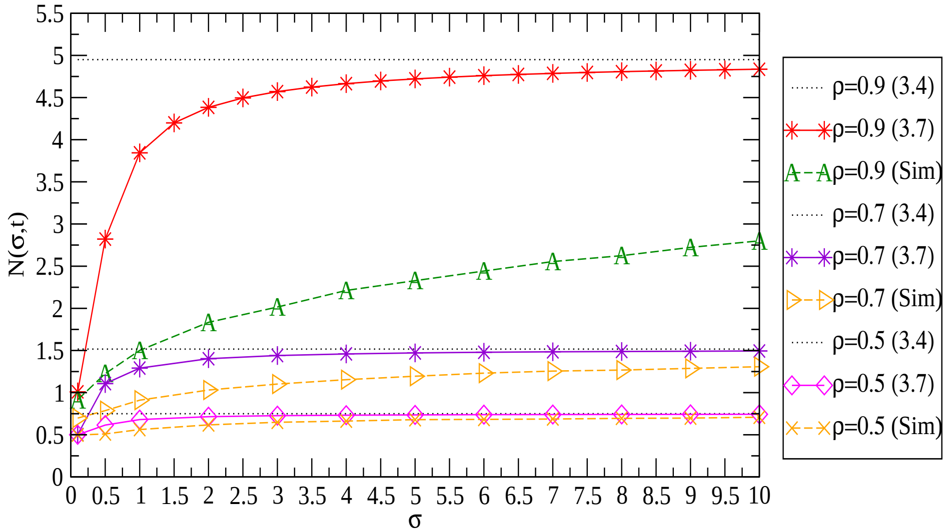

Figure 2.4: Behavior of the function $N(\sigma, 0.5)$, for different values of ϱ. Dotted lines refer to Poisson arrivals, continuous lines refer to approximation (3.7), dashed lines refer to simulations.

We will treat first the case $\varrho = 1$, or in other words, the case $\lambda = T = 1$ in (2.15). This special case is important for several reasons. First, we will prove that for PSRA arrivals the system has a *finite* average queue length, showing that, even if the PSRA process tends in distribution to the Poisson process, for finite variance of the ξ's the two systems are deeply different. Second, we will show that in the $\varrho = 1$ case there is a conserved quantity in the system, when the stationary distribution is reached. Third, it is possible, using an interest interpretation of the system in terms of Fermi statistics, to compute the (very long) times needed to the system to reach the stationary distribution. Fourth, and maybe more important, on the basis of the computation of this relaxation times it is possible to approximate efficiently the distribution of the length of the queue even for $\varrho < 1$.
Hence, we fix $\varrho = 1$ and we start from the obvious relation

$$n(j+1) = n(j) - (1 - \delta_{n(j)0}) + m(j) \tag{5.1}$$

where $n(j)$ is the length of the queue immediately before the service at time j, $m(j)$ is the number of users arrived in the time slot $[j, j+1)$, and the term $(1 - \delta_{n(j)0})$ indicates the fact that if there is some user in the queue at time j, i.e. $n(j) > 0$, the first of the queue is served, while if $n(j) = 0$ then $n(j+1) = m(j)$.
Now we observe that with our notations we can write

$$m(j) = |I_{j+1}| - |I_j| + 1 \tag{5.2}$$

This relation can be shown as follows: the total number $na(j)$ of users arrived to the service from a certain fixed time, say from time 1, to time j, is obviously $na(j) = j - L + |I_j|$, because all the users k up to user $j - L$'th are already arrived, due to the compactness of the support of $f_\xi^{(\sigma)}(t)$, while for $k > j - L$ the number of arrived users is $|I_j|$ by definition. Hence $m(j) = na(j+1) - na(j) = j + 1 - L + |I_{j+1}| - j + L - |I_j| = |I_{j+1}| - |I_j| + 1$. Putting (5.2) into(5.1) we obtain

$$n(j+1) = n(j) + |I_{j+1}| - |I_j| + \delta_{n(j)0} \tag{5.3}$$

This relation shows that the quantity $\alpha(j) = n(j) - |I_j|$ is constant during a busy period, and it increases by 1 at the end of each busy period. This implies that the stationary distribution is reached once $\alpha > 0$. If the initial value of α is strictly positive, the value $n(j) = 0$ is never realized, and then α remains constant and

$$N = E(n) = \alpha + E(|I|) \tag{5.4}$$

If the initial value of α is 0 or it is negative, a sequence of busy periods is realized, giving in the end the value $\alpha = 1$, and the expected queue length $N = E(n) = 1 + E(|I|)$. Once the stationary value of $\alpha > 0$ is reached, the probability distribution of n is given by

$$P_k = P(n = k) = P(|I| = k - \alpha) \tag{5.5}$$

giving the obvious result that $k \geq \alpha$. The explicit expression of the P_k depends therefore from the distribution of the $|I|$'s, and hence from the details of $f_\xi^{(\sigma)}(t)$. This solves completely the stationary problem in the $\varrho = 1$ case. For application to the air traffic, however, it could be also interesting to study some non stationary features of the system: in particular we want to compute the probability to pass from some negative value of α to the following value $\alpha + 1$. These quantities are interesting in this $\varrho = 1$ case because if the probability to reach the state $n = 0$ for a given $\alpha \leq 0$ is much smaller that the inverse of the number of operation in a single day of traffic, it is very likely that the system remains on states $n > 0$. These probability to jump from a definite value of α to the following one are important also in the description of the $\varrho < 1$ case, as it will be explained below.
Hence suppose that at time j the system is in the state $n(j) = 0$, with a given value of $\alpha < 0$. Call $t(\alpha)$ the quantity such that $n(j + i) > 0$ for all $0 < i < t(\alpha)$, and $n(j + t(\alpha)) = 0$. $t(\alpha)$ is therefore the length of the busy period with starting value α. We are interested to the quantities $T(\alpha) = E(t(\alpha))$. By the definition of α we have that $|I_j| = -\alpha + 1$ and that the instant $j + t(\alpha)$ is the first instant after j in which $|I_{j+t(\alpha)}| = -\alpha$, having $|I_{j+i}| > -\alpha$ for all $0 < i < t(\alpha)$. To compute $T(\alpha)$ we should evaluate the probability $P(|I_{j+i}| = -\alpha \big| |I_j| = -\alpha + 1)$. This probability are however hard to compute due to the conditioning. Here we introduce our approximation: we will measure $T(\alpha)$ in terms of

$$T(\alpha) \approx \frac{1}{P(|I| = -\alpha)} \tag{5.6}$$

i.e. we neglect the conditioning. This approximation is reasonable for α such that $P(|I| = -\alpha) \ll \frac{1}{2L}$: in these cases we have to expect that the probability to have $P(|I_{j+i}| = -\alpha \big| |I_j| = -\alpha + 1)$ for $i < 2L$ is very small, and since I_{j+i} is independent of I_j for the greater values of i, that gives the bigger contribution to $T(\alpha)$, we have that the conditioning is almost ineffective. On the other side, for α such that $P(|I| = -\alpha) \geq \frac{1}{2L}$ we have to expect a gross underestimate of $P(|I_{j+i}| = -\alpha \big| |I_j| = -\alpha + 1)$, and therefore a gross overestimate of $T(\alpha)$. We will return on this point later.
We want now to compute explicitly $P(|I| = -\alpha)$. We will write general formulas, valid for any density $f_\xi^{(\sigma)}(t)$, and we will also consider a concrete probability distribution for the delays ξ, namely the case of $f_\xi^{(\sigma)}(t)$ uniform in $[-L, L]$, in which many computations may be carried out explicitly.
By straightforward computations one can see that

$$P(|I| = 0) = \prod_{i=-L+1}^{L-1} (1 - F_\xi(i)) = \frac{(2L)!}{(2L)^{2L}} \approx e^{-2L}\sqrt{4\pi L} \tag{5.7}$$

where the last approximation is valid for uniform ξ's, using Stirling formula, and

$$P(|I|=k)=P(|I|=0)\sum_{-L+1\leq i_1<i_2<...<i_k\leq L-1}\frac{F_\xi(i_1)}{1-F_\xi(i_1)}...\frac{F_\xi(i_k)}{1-F_\xi(i_k)}=$$

$$=P(|I|=0)\sum_{-L+1\leq i_1<i_2<...<i_k\leq L-1}\frac{L-i_1}{L+i_1}...\frac{L-i_k}{L+i_k} \tag{5.8}$$

where $F_\xi(t)$ is the probability distribution of the ξ's, and the last equality is again valid for uniform distribution.

It is worthy to observe that (5.8) may be interpreted as the canonical partition function of a Fermi system with $2L$ energy level and k particles, where the i-th level has energy $\log(F_\xi(i))-\log(1-F_\xi(i))$. With this respect many computational techniques may be used in order to compute the probabilities $P(|I|=k)$. Note that, in the approximation (5.6), once we are able to compute the quantities $P(|I|=k)$ we know also the expected values $T(\alpha)$.

Let us list here a couple of possible way to evaluate $P(|I|=k)$ using the fact that, since it is possible to interpret it as a well known object in statistical mechanics, one can use computational results that are classical in that framework. The number of energy level, as mentioned above, is $2L$. In real traffic context one should expect that this value is of the order 20 or 30. One of the available approximation of the quantity $P(|I|=k)$, i.e. the so called equivalence with the *grand canonical ensemble*, uses a method that is roughly speaking the Lagrange multipliers method, giving very good approximations for $2L$ large (see e.g. [20, chapter 5, section 53]. Since in our case $2L$ is not large enough to ensure the goodness of the approximation, it is much better to use an exact expression for $P(|I|=k)$, due to Ginibre. For completeness, and for the fact that it is quoted in a very implicit sense in [43], we give the proof of this formula.

Calling $w_i=\frac{F_\xi(i)}{1-F_\xi(i)}$, one can prove the following equality

$$P(|I|=k)=\sum_{l=0}^{k}\sum_{\substack{1\leq j_1\leq...\leq j_l\\ \sum_m j_m=k}}C(j_1,...,j_l)\prod_{m=1}^{l}\sum_i(w_i)^{j_m} \tag{5.9}$$

with

$$C(j_1,...,j_l)=P(|I|=0)\frac{(-1)^{k-l}}{j_1.....j_l m_1!...m_k!} \tag{5.10}$$

where m_i is the number of j's equal to i. To prove (5.9) we observe that

$$P(|I|=k)=P(|I|=0)\frac{1}{k!}\frac{d^k}{dt^k}\prod_i(1+tw_i)\Big|_{t=0}$$

The quantity $\prod_i(1+tw_i)$ can be expanded in series as follows

$$\frac{1}{k!}\frac{d^k}{dt^k}\prod_i(1+tw_i)\Big|_{t=0}=\frac{1}{k!}\frac{d^k}{dt^k}e^{\sum_i\log(1+tw_i)}\Big|_{t=0}=\frac{1}{k!}\frac{d^k}{dt^k}e^{\sum_i\sum_{j=1}^{k}(-1)^{j-1}\frac{(tw_i)^j}{j}}\Big|_{t=0}=$$

$$=\frac{1}{k!}\frac{d^k}{dt^k}e^{\sum_{j=1}^{k}(-1)^{j-1}\frac{t^j}{j}\sum_i(w_i)^j}\Big|_{t=0}=\frac{1}{k!}\frac{d^k}{dt^k}\sum_{l=1}^{k}\frac{(\sum_{j=1}^{k}(-1)^{j-1}\frac{t^j}{j}\sum_i(w_i)^j)^l}{l!}\Big|_{t=0}=$$

$$= \sum_{l=0}^{k} \frac{(-1)^{k-l}}{l!} \sum_{\substack{j_1,\dots,j_l \\ \sum_m j_m = k}} \prod_{m=1}^{l} \sum_i \frac{(w_i)^{j_m}}{j_m} = \sum_{l=0}^{k} \frac{(-1)^{k-l}}{l!} \sum_{\substack{1 \le j_1 \le \dots \le j_l \\ \sum_m j_m = k}} \prod_{m=1}^{l} \sum_i \frac{(w_i)^{j_m}}{j_m} \frac{l!}{m_1!\dots m_k!}$$

which is (5.9).
We conclude then the discussion of the $\varrho = 1$ case observing that in a concrete framework of air traffic, if we want to avoid to have lost slot but we want to keep the queue as short as possible we have to choose initial condition in such a way that α is the smaller possible value such that $T(\alpha) > D$, where D is the number of operations in a day. This value of α gives the corresponding value of the length of the queue using (5.4).
A simple observation allows us to give an estimate of the average length of the queue also when $\varrho < 1$. Let us suppose that we impose the condition $\varrho < 1$ keeping the time between two expected arrivals equal to the service time, but assuming that the arrivals are described by PSRA process with random deletion (see (2.16)), with probability of deletion equal to $1 - \varrho$. It is easy to realize that this corresponds to say that the value of α has a probability $1 - \varrho$ to decrease by one. Hence we have this picture of our queueing system: the queue is described by a superposition of a slow varying process, the process that describes the value of α, and a fast varying process, the one describing the n for fixed α. If we are able to compute the distribution probabilities of the values of α, we can evaluate the expected length of the queue (and even its distribution) by (5.4), weighted with the probabilities of the various values of α.
In the unconditioned approximation (5.6), the computation of the stationary probabilities π_α of α is a standard task of the theory of the birth-and-death processes: the evolution of α is a discrete time birth-and-death process, with transition probabilities

$$P_{\alpha,\alpha'} = \begin{cases} 1 - \varrho \equiv \mu_\alpha & \text{if } \alpha' = \alpha - 1 \\ P(|I| = -\alpha) \equiv \lambda_\alpha & \text{if } \alpha' = \alpha + 1 \\ 1 - \lambda_\alpha - \mu_\alpha & \text{if } \alpha' = \alpha \\ 0 & \text{otherwise} \end{cases}$$

and boundary conditions $\mu_{-L+1} = \lambda_0 = 0$. We get the following linear system

$$\begin{aligned} \pi_{-L+1} &= \pi_{-L+1}(1 - \lambda_{-L+1}) + \pi_{-L+2}\mu_{-L+2} \\ \pi_i &= \pi_{i-1}\lambda_{i-1} + \pi_{i+1}\mu_{i+1} + \pi_i(1 - \lambda_i - \mu_i) \qquad -L+1 < i < 0 \\ \pi_0 &= \pi_{-1}\lambda_{-1} + \pi_0(1 - \mu_0) \end{aligned}$$

whose solution is

$$\pi_i = \pi_{-L+1} \prod_{k=-L+2}^{i} \frac{\lambda_{k-1}}{\mu_k}$$

The stationary distribution π is defined by the normalization condition $\sum_i \pi_i = 1$, then

$$\pi_{-L+1} = \frac{1}{1 + \sum_{n=-L+2}^{0} \prod_{k=-L+2}^{i} \frac{\lambda_{k-1}}{\mu_k}} \tag{5.11}$$

This approximation is good for $1 - \varrho$ sufficiently small, because the probability to increase $\alpha = -L + 1$ is much bigger than the probability to decrease it, and at the same time the unconditioned transition probabilities to increase α when $\alpha > -L + 1$ are a good approximation of the actual transition probabilities.

In the following figure we show the value of the expected length of the queue obtained by the formula

$$N = \sum_{\alpha} \pi_\alpha \left(\alpha + E_\alpha(|I|)\right) \tag{5.12}$$

Note that $E_\alpha(|I|)$ is α-dependent, because in its computation we neglect the terms with $|I| < -\alpha$, since they do not contribute to the evolution of the process with that value of α. As it can be seen from the figure, the estimate of the average length of the queue is extremely near to the simulations, also for highly congested systems. In the figure we have shown for completeness also the (wrong, for high ϱ) values of the length of the queue computed by means of formula (3.7), which neglects the autocorrelations.

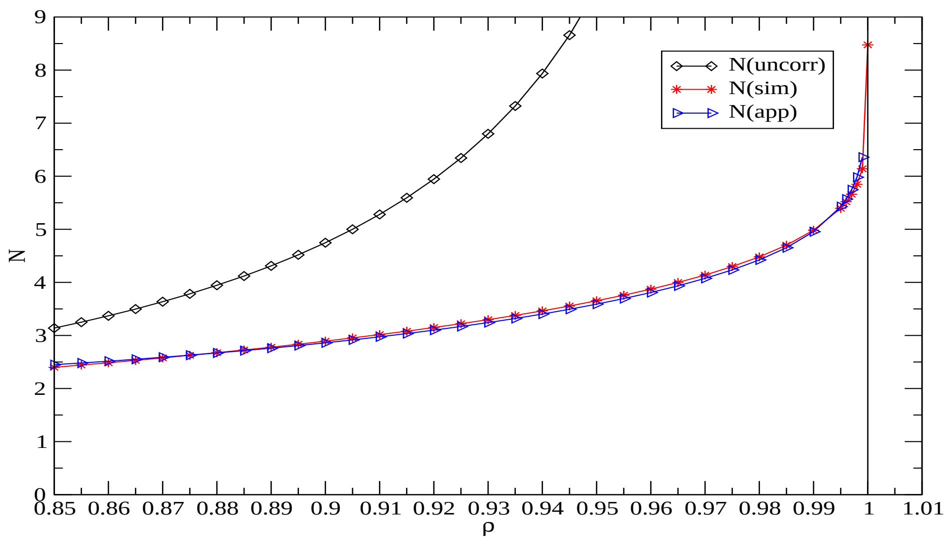

Figure 2.5: The length of the queue for highly congested systems, computed by means of numerical simulations (red line) and our analytical approximation (blue line). It can be seen that the uncorrelated approximation (black line) obtained by formula (3.7) gives for these values of ϱ a gross overestimate. The simulations are run for a time sufficiently long to have fluctuations on the result negligible in the scale of the figure.

2.6 Conclusions and open problems

The main aim of this chapter is to study a stochastic process close to the Poisson process, but more suitable to describe the arrivals to a queueing systems when such arrivals are scheduled in advance, and some randomness is added to the schedule. We looked into this problem as an attempt to describe the congestion in air traffic systems, but the same construction can be used in different contexts.

We found some analytical results, in particular we showed that our process can be indistinguishable from a Poisson process if one wants to study the distribution either of the number of arrivals or of the interarrival times in a time slot shorter than the standard deviation of the randomness imposed to the scheduled arrivals.

However we have shown that from the point of view of the resulting congestion, due to the autocorrelation of this stochastic process, the queueing properties of this model are quite different from the analogous problem with Poisson arrivals. Interesting connection with the statistical mechanics emerged in the analytical solution of the problem. We proposed some approximation in our computations, but the results we obtained are in very good agreement with numerical simulations. An important question is the discussion of the accuracy of this description with respect to actual air traffic data. We have with this respect some preliminary results showing that the description of the distribution of the length of the queue using the PSRA as arrival process is much more accurate than the description assuming Poisson process, that is well known to be unfit.

Chapter 3

Discrete time Queueing System With Variable Number of Servers

We consider in this chapter a multi-server queueing system in a discrete time framework such that the number of servers in each time slot is the independent realization of a random variable. For this model we give a complete theoretical description and we discuss some numerical approximations.

3.1 Introduction

In classical queueing theory the number of servers c is usually a fixed parameter of the theory. When $c = 1$, the single server queue case, many results may be obtained in a very general framework, by studying the generating function of the process (see e.g. [14]). For a multi servers queue, i.e. $c > 1$, on the other hand, there are few analytical results, due to the fact that for general service times the system is not Markovian. With the exception of the completely memoryless case, it is possible to obtain exact results only for deterministic service times, since in this case the natural language to describe the system is the discrete time Markov chain approach. Even in this particular case, however, the description of the system is complicated because for c servers the generating function has exactly c singularities in the closed complex circle of radius 1, and therefore its study becomes much more complicated. In general the direct approach, involving the explicit study of the singularities, is quite heavy, and ingenious numerical techniques (see e.g. [7]) have been created in order to bypass this difficulty. These techniques are also useful to obtain approximated formulas in the case of general service times.

In this context, the study of systems with a variable number of servers, that is a system in which c is a random variable, was not even tried so far. Nevertheless, it is not difficult to imagine concrete frameworks where the number of servers could vary with time. In addition to the example give in the introduction of this thesis, we point out an another example of possible application of our model in telecommunication systems, that is a set of servers which process simultaneously a real time and a best effort traffic, in such a way that the number of server available for the best effort traffic is a random variable, given by the difference between the total number of servers and the number of servers occupied by the real time traffic. In this chapter we study a relatively simple model of a discrete time service in which

the number of servers at each time is the independent realization of a given random variable. The choice of a discrete time service is motivated by the fact that in this framework the system is Markovian, and the equations for the stationary measure have a very transparent meaning. For the sake of simplicity, the actual value of c is chosen independently in each time slot. From a physical point of view the possibility of studying a service in which, for example, the number of servers is described by a Markov chain would be of great interest, basically because it would provide a tool to circumvent the difficulties of the description of systems with different class of users and non trivial rules of priority (see [15]). This point, however, is slightly more complicated and it will be the subject of further studies.
In the study of our model of queueing systems with a variable number of servers the computations of the probability distribution of the states of the system have to face a complicated structure, in terms of singularities of the generating function. From a numerical point of view it is then useful to define suitable approximation schemes in order to avoid the explicit computation of such singularities.
This is a classical problem discussed widely in literature (see e.g.[23, 24, 25, 27, 28]). The first idea of approximating an infinite Markov chain is due to Seneta [23, 24]. He introduced the augmentation concept to compute the finite approximation of the stationary distribution of an infinite Markov chain. The efficiency of the augmentation method is discussed in[25]. Wolf [27] used another approach to study the approximation of an infinite Markov chain and he investigated the conditions which assure that at least one subsequence of finite stationary distributions converges pointwise to the true infinite stationary distribution. Later Zhao and Liu [28] introduced the censoring method, that gives the best approximation in sense of the L_1 norm, but in general it is not easy to compute directly.
We will present a rephrasing of one of the most used approximation schemes, the so-called *last column augmentation*, and we will discuss some theoretical and numerical results about the errors involved in this approximation. We will compare also our results with the other approximation schemes usually discussed in literature.

The chapter is organized as follows. In section 3.2 we give a description of our model and introduce some basic notations. In section 3.3 we give the stability condition in order to obtain the equations of the stationary state of the system. In section 3.4 we give some details about the root of denominator of generating function and we write it in terms of root of denominator. In section 3.5 we introduce our approximation and and recall some results on augmentation concept. In sections 3.6 and 3.7 prove the basic theoretical results of this chapter for the system $GI/D/c$ and $GI/D/c_i$ respectively. In section 3.9 we give some analytical result for average queue size and variance. In section 3.10 there are some numerical results. The section 3.11 is devoted to the conclusions.

3.2 Description of the model

In this section we will describe a multi-server discrete time queueing system with variable number of identical servers. Users are processed accordingly to the *FIFO* discipline and each service starts at discrete time $t_i = iD$. For the sake of simplicity, we will make three assumptions: first, without loss of generality, that $D = 1$, i.e unit service time slots; second that the number of servers c_i at time i is the independent realization of a random variable with distribution $\alpha_l = P(c_i = l)$, and, finally, that there exists a finite maximum number of servers c, i.e. $\alpha_c > 0$ and $\alpha_i = 0$ for all $i > c$. We call q_n the probability of n arrivals in a single time slot, and X_i the number of users waiting in queue at time i, and ready to be served in the time interval $(i, i+1)$. $\{X_i\}_{i=0}^{\infty}$ is a Discrete Time Markov Chain with countable state

space $\{0, 1, 2, \ldots\}$. This queueing system will be denoted from now on as $GI/D/c_i$.
Let P_n^i be the probability of n users in the system at time i, immediately before the beginning of the time slot $(i, i+1)$.

$$P_n^{(i)} = P[X(i) = n] \tag{2.1}$$

Our interest is the distribution of $\{X_i\}$. The description above maps into the following linear system for the probabilities $P_n^{(i)}$

$$\begin{aligned}
P_0^{(i+1)} &= \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k^{(i)} q_0 \\
P_1^{(i+1)} &= \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k^{(i)} q_1 + \sum_{l=0}^{c} \alpha_l P_{l+1}^{(i)} q_0 \\
&\vdots \\
P_n^{(i+1)} &= \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k^{(i)} q_n + \sum_{l=0}^{c} \alpha_l \sum_{k=l+1}^{n+l} P_k^{(i)} q_{n-k+l} \qquad n \geq 1 \\
&\vdots
\end{aligned} \tag{2.2}$$

Remark 3.2.1. *We are assuming that probabilities α_l are time independent. One might think on interesting cases of time-dependent probabilities $\alpha_l(i)$, e.g., arrivals with retrials or even finite waiting room. This will be the subject of further studies.*

Example 1: As an example of application of the above system, let us describe a model of the landing procedure of aircraft at an airport, equipped with c runways[1]. Accordingly the procedure and management in air traffic control each of runways can be used in either direction, hence for takeoff or landing of aircraft. If in each time slot the runways is empty[2] with probability p, then the probability that the aircraft find l runways available has binomial distribution with parameter c and p,

$$\alpha_l = \binom{c}{l} p^l (1-p)^{c-l} \tag{2.3}$$

Example 2: In order to give an another example of application of the system introduced in this section, let us describe a model of the realtime best-effort traffic on same cable in UMTS systems. We assume that there are two classes of users. The first class has the priority over the second. Let Z_T be the number of arrivals of first class users in each time slot. We assume Z_T be a truncated version of a Poisson RV Z, with parameter λ.

$$p_k := P(Z_T = k) = \begin{cases} \frac{\frac{\lambda^k}{k!}}{\sum_{n=0}^{c} \frac{\lambda^n}{n!}} & k \leq c \\ 0 & k > c \end{cases} \tag{2.4}$$

with

$$E(Z_T) = \lambda \left(1 - \frac{\frac{\lambda^c}{c!}}{\sum_{n=0}^{c} \frac{\lambda^n}{n!}} \right) := \lambda_T < \lambda \tag{2.5}$$

[1] We assume that in each time slot the operations on each runways do not take into account the operations on other runways

[2] In sense of there are not takeoffs aircrafts.

The truncation has been chosen so as to describe, naively, either a system where the probability of a number of first class users exceeding, at each time t_i, the number of servers c is very small or (in the case of large λ) a system in which the exceeding real time traffic cannot wait in queue and it is lost. When we look at the second class users, the relevant information is the number of servers available after first class users have been served. We define the random variable $Y = c - Z_T$ and the probability $\alpha_l := P(Y = c - Z_T = l) = P(Z_T = c - l)$, i.e. α_l is the probability that *second class* users find l available servers.

This model, hence, can be thought of as a service with a variable number of servers, assuming the following:

- the arrival process has a well defined distribution as far as the number of users in each time slot is concerned;
- service time is deterministic and the service may start only at pre-defined discrete times;
- the waiting room for the second class users has an infinite capacity, so that users are not lost.

3.3 Steady state probability distribution

It is not difficult to think of pathological arrival processes q_n and distributions of servers α_l such that the Markov chain $\{X(t_i)\}$ is periodic or reducible. However under mild conditions the aperiodicity and the irreducibility are guaranteed. In the literature, in section 6 of [27] the condition

$$q_0, ..., q_c > 0 \tag{3.1}$$

has been proposed in the case of the $GI/D/c$ service. This condition is sufficient, but it is not necessary. In our case, in which c represents the maximum number of available servers, the condition (3.1) is sufficient too, but milder conditions can be imposed. For example it is enough in order to guarantee the aperiodicity that for some l one has $\alpha_l > 0$ and $q_l > 0$ simultaneously. Once the aperiodicity is guaranteed, a necessary and sufficient condition for the irreducibility is the following. Call $J \subset \mathbb{N}$ the set of integers j such that for some l and n such that $n - l = j$ we have $\alpha_l > 0$ and $q_n > 0$ simultaneously. J is then the set of the possible jumps of the state of the system in a unit time. The chain is irreducible if and only if the greatest common factor of J is 1.
In what follows we will assume that the choice of q_n and α_l is "reasonable" in the sense that $\{X(t_i)\}$ is aperiodic and irreducible, and we need to impose a further technical condition, i.e.

$$\alpha_l > 0, \quad 0 < \sum_{i=0}^{l-1} q_i < 1 \qquad \text{for some } l \leq c \tag{3.2}$$

Note that (3.1) implies condition (3.2), too. If one of the relations above is satisfied the system is ergodic under the *stability condition*

$$\sum_{n \geq 0} n q_n < \bar{c} \tag{3.3}$$

where

$$\bar{c} = \sum_{l=0}^{c} l \alpha_l \tag{3.4}$$

The physical meaning of this condition is that the average value of available servers has to exceed the average numbers of arriving users in each time slot. We will refer to the case

$$\sum_{n\geq 0} nq_n = \bar{c} \tag{3.5}$$

as the **critical regime**. Define the stationary probability P_n by

$$P_n = \lim_{i\to\infty} P[X_i = n] \tag{3.6}$$

Under the stability condition (3.3), the limit exists for any n, and the stationary probability satisfies the following infinite linear system

$$\begin{aligned}
P_0 &= \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_0 \\
P_1 &= \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_1 + \sum_{l=0}^{c} \alpha_l P_{l+1} q_0 \\
&\vdots \\
P_n &= \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_n + \sum_{l=0}^{c} \alpha_l \sum_{k=l+1}^{n+l} P_k q_{n-k+l} \qquad n \geq 1 \\
&\vdots
\end{aligned} \tag{3.7}$$

Define the generating functions

$$P(z) = \sum_{n=0}^{\infty} P_n z^n \qquad q(z) = \sum_{n=0}^{\infty} q_n z^n \qquad |z| \leq 1 \tag{3.8}$$

Now we multiply both side of (3.7) by z^n and summing from $n = 0$ to ∞ we have

$$\begin{aligned}
P(z) &= \sum_{n=0}^{\infty}\sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_n z^n + \sum_{n=0}^{\infty}\sum_{l=0}^{c} \alpha_l \sum_{k=l+1}^{n+l} P_k q_{n-k+l} z^n \\
&= q(z) \sum_{l=0}^{c}\sum_{k=0}^{l} \alpha_l P_k + \sum_{n=1}^{\infty}\sum_{l=0}^{c}\sum_{i=1}^{n} \alpha_l P_{i+l} q_{n-i} z^{n-i} z^i \\
&= q(z) \sum_{l=0}^{c}\sum_{k=0}^{l} \alpha_l P_k + \sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\sum_{l=0}^{c} \alpha_l P_{i+l} q_{n-i} z^{n-i} z^i \\
&= q(z) \sum_{l=0}^{c}\sum_{k=0}^{l} \alpha_l P_k + q(z) \sum_{i=1}^{\infty}\sum_{l=0}^{c} \frac{\alpha_l}{z^l} P_{i+l} z^{i+l} \\
&= q(z) \sum_{l=0}^{c}\sum_{k=0}^{l} \alpha_l P_k + q(z) \sum_{k=l+1}^{\infty}\sum_{l=0}^{c} \frac{\alpha_l}{z^l} P_k z^k
\end{aligned} \tag{3.9}$$

A straightforward computation shows that

$$P(z) = \frac{n(z)}{d(z)} \qquad |z| \leq 1 \tag{3.10}$$

where

$$
\begin{aligned}
n(z) &= q(z)z^c \sum_{l=0}^{c}\sum_{k=0}^{l-1} \alpha_l P_k (z^{k-l}-1)\\
d(z) &= q(z)\sum_{l=0}^{c} \alpha_l z^{c-l} - z^c
\end{aligned}
\tag{3.11}
$$

The explicit expression of $n(z)$ depends on the first c probabilities $P_0, ..., P_{c-1}$: we only need to know $P_0, ..., P_{c-1}$ in order to describe explicitly the whole distribution P_n. In next section we give the solution of problem using the same idea given by Crommelin [9] and by Bailey [19].

3.4 Some details about the root of denominator

The correct values of $P_0, ..., P_{c-1}$ may be computed on the basis of the zeroes of $d(z)$ inside the complex disk of radius 1, see e.g. [7]. Indeed the function $P(z)$ has to be holomorphic in such a disk in order to have a probabilistic interpretation, and hence the parameters $P_0, ..., P_{c-1}$ have to be such that the zeroes of $d(z)$ are exactly canceled. Since (as it is always done because $q(z)$ has a probabilistic interpretation) that $q(z)$ is holomorphic in the unit disc $\overline{\mathbb{D}}$, by Rouché's theorem (Ahlfors [5, p.152]), $q(z)\sum_{l=0}^{c}\alpha_l z^{c-l} - z^c$ has $c-1$ complex zeros in $|z| < 1$, and a single zero in $z = 1$ see [8, Lemma 2.2, 3.1]. Let call $z_0, z_1, \ldots, z_{c-1}$ the zeros of such denominator and we assume that $z_0 = 1$. In such zeroes the numerator has to vanish. So in this way we obtain the following $c-1$ equations.

$$
\sum_{l=0}^{c}\sum_{k=0}^{l-1} \alpha_l P_k (z_j^c - z_j^{c+k-l}) = 0 \qquad \text{for } j = 1, 2, \ldots c-1 \tag{4.1}
$$

Now for $z_0 = 1$ we employ the boundary condition $P(z)|_{z=1} = 1$ and L'Hôpital's rule, getting the following condition

$$
\sum_{l=0}^{c}\sum_{k=0}^{l-1} (l-k)\alpha_l P_k = \sum_{l=0}^{c} l\alpha_l - q'(z)\big|_{z=1} \tag{4.2}
$$

We remark that the condition (4.2) make sense if and only if

$$
\frac{q'(z)\big|_{z=1}}{\sum_{l=0}^{c} l\alpha_l} < 1 \tag{4.3}
$$

which ensure us the existence of stationary probability distribution, otherwise the queue length tends to infinity. As we see the condition (4.3) is the stability condition given in (3.3)

When we make together the equations (4.1) and (4.2), these constitute a system of c equations with c unknown $P_0, ..., P_{c-1}$. The matrix of this system is given by

$$
\mathbf{A} = \begin{bmatrix}
\sum_{k=1}^{c} k\alpha_k & \sum_{k=2}^{c}(k-1)\alpha_k & \ldots & \alpha_c \\
\sum_{k=1}^{c} \alpha_k (z_1^c - z_1^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_1^c - z_1^{c-k+1}) & \ldots & \alpha_c (z_1^c - z_1^{c-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=1}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k+1}) & \ldots & \alpha_c (z_{c-1}^c - z_{c-1}^{c-1})
\end{bmatrix} \tag{4.4}
$$

and the column vector of constant terms with all entries zero except for the first entry that is $\sum_{l=0}^{c} l\alpha_l - q'(z)\Big|_{z=1}$. It possible to solve the system of c equations in c unknowns by using the Cramer rule. To do this we need to show that the determinant of matrix $\mathbf{A}$, $\det(\mathbf{A})$ do not to vanish. We assume that the zeros of denominator are simple.

$$\det(\mathbf{A}) = \begin{vmatrix} \sum_{k=1}^{c} k\alpha_k & \sum_{k=2}^{c}(k-1)\alpha_k & \dots & \alpha_c \\ \sum_{k=1}^{c} \alpha_k(z_1^c - z_1^{c-k}) & \sum_{k=2}^{c} \alpha_k(z_1^c - z_1^{c-k+1}) & \dots & \alpha_c(z_1^c - z_1^{c-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{c} \alpha_k(z_{c-1}^c - z_{c-1}^{c-k}) & \sum_{k=2}^{c} \alpha_k(z_{c-1}^c - z_{c-1}^{c-k+1}) & \dots & \alpha_c(z_{c-1}^c - z_{c-1}^{c-1}) \end{vmatrix} \tag{4.5}$$

Now we can write the matrix $\mathbf{A}$ in following way,

$$\mathbf{A} = \mathbf{BC} \tag{4.6}$$

where

$$\mathbf{B} = \begin{bmatrix} c & c-1 & \dots & 1 \\ z_1^c - 1 & z_1^c - z_1 & \dots & z_1^c - z_1^{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{c-1}^c - 1 & z_{c-1}^c - z_{c-1} & \dots & z_{c-1}^c - z_{c-1}^{c-1} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \alpha_c & 0 & \dots & 0 \\ \alpha_{c-1} & \alpha_c & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \dots & \alpha_c \end{bmatrix}$$

Since $\det(\mathbf{A}) = \det(\mathbf{B})\det(\mathbf{C})$, we can calculate easily the $\det(\mathbf{B})$ and the $\det(\mathbf{C})$, we have

$$\det(\mathbf{B}) = \begin{vmatrix} c & c-1 & \dots & 1 \\ z_1^c - 1 & z_1^c - z_1 & \dots & z_1^c - z_1^{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{c-1}^c - 1 & z_{c-1}^c - z_{c-1} & \dots & z_{c-1}^c - z_{c-1}^{c-1} \end{vmatrix} =$$

$$= \begin{vmatrix} c & z_1^c - 1 & \dots & z_{c-1}^c - 1 \\ c-1 & z_1^c - z_1 & \dots & z_{c-1}^c - z_{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_1^c - z_1^{c-1} & \dots & z_{c-1}^c - z_{c-1}^{c-1} \end{vmatrix} = \begin{vmatrix} 1 & z_1 - 1 & \dots & z_{c-1} - 1 \\ 1 & z_1^2 - z_1 & \dots & z_{c-1}^2 - z_{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_1^c - z_1^{c-1} & \dots & z_{c-1}^c - z_{c-1}^{c-1} \end{vmatrix} =$$

$$= \prod_{j=1}^{c-1}(z_j - 1) \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & z_1 & \dots & z_{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_1^{c-1} & \dots & z_{c-1}^{c-1} \end{vmatrix} = \prod_{j=1}^{c-1}(z_j - 1) \prod_{0 \le l < j \le c-1} (z_j - z_l)$$

where we have used the fact that: $\det(\mathbf{A}) = \det(\mathbf{A^T})$ on the first equality, then we added a row j the row $j+1$ multiplied by (-1), for $j = 1, \dots, c-1$ on the second equality; finally we used the expression for determinant of a Vandermonde matrix applies on the last equality. Finally we find

$$\det(\mathbf{A}) = \alpha_c^c \prod_{j=1}^{c-1}(z_j - 1) \prod_{0 \le l < j \le c-1} (z_j - z_l) \qquad j = 1, \dots, c-1 \tag{4.7}$$

Then we have shown that the $\det(\mathbf{A}) \neq 0$, so the system of c equations has a unique solution $P_0, \ldots, P_{c-1}$. Now we rewrite the generating function (3.11) as follows

$$P(z) = \frac{n(z)}{d(z)} \qquad |z| \leq 1 \tag{4.8}$$

where

$$\begin{aligned} n(z) &= \sum_{l=0}^{c} \sum_{k=0}^{l-1} \alpha_l P_k (z^c - z^{c+k-l}) \\ d(z) &= \frac{z^c}{q(z)} - \sum_{l=0}^{c} \alpha_l z^{c-l} \end{aligned} \tag{4.9}$$

By the Fundamental Theorem of Algebra there are c roots of the numerator $n(z)$, then we can write the $n(z)$ on factorial form

$$n(z) = \sum_{l=0}^{c} \sum_{k=0}^{l-1} \alpha_l P_k (z^c - z^{c+k-l}) = \sum_{n=0}^{c-1} P_n \sum_{k=n+1}^{c} \alpha_k \prod_{l=1}^{c} (z - z_l) = (z-1)\beta \prod_{l=1}^{c} (z - z_l) \tag{4.10}$$

with

$$\beta = \sum_{n=0}^{c-1} P_n \sum_{k=n+1}^{c} \alpha_k \tag{4.11}$$

Now consider the system of $c+1$ equations (4.11), (4.1) and (4.2) with c unknowns. This system has unique solution if it satisfy the following condition

$$\begin{vmatrix} \sum_{k=1}^{c} \alpha_k & \sum_{k=2}^{c} \alpha_k & \ldots & \alpha_c & \beta \\ \sum_{k=1}^{c} k\alpha_k & \sum_{k=2}^{c} (k-1)\alpha_k & \ldots & \alpha_c & \gamma \\ \sum_{k=1}^{c} \alpha_k (z_1^c - z_1^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_1^c - z_1^{c-k+1}) & \ldots & \alpha_c (z_1^c - z_1^{c-1}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{k=1}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k+1}) & \ldots & \alpha_c (z_{c-1}^c - z_{c-1}^{c-1}) & 0 \end{vmatrix} = 0 \tag{4.12}$$

where $\gamma = \sum_{l=0}^{c} \alpha_l - q'(z)|_{z=1}$.
After a few calculations we obtain

$$\beta \begin{vmatrix} \sum_{k=1}^{c} k\alpha_k & \sum_{k=2}^{c} (k-1)\alpha_k & \ldots & \alpha_c \\ \sum_{k=1}^{c} \alpha_k (z_1^c - z_1^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_1^c - z_1^{c-k+1}) & \ldots & \alpha_c (z_1^c - z_1^{c-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k+1}) & \ldots & \alpha_c (z_{c-1}^c - z_{c-1}^{c-1}) \end{vmatrix} =$$

$$= \gamma \begin{vmatrix} \sum_{k=1}^{c} \alpha_k & \sum_{k=2}^{c} \alpha_k & \ldots & \alpha_c \\ \sum_{k=1}^{c} \alpha_k (z_1^c - z_1^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_1^c - z_1^{c-k+1}) & \ldots & \alpha_c (z_1^c - z_1^{c-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k}) & \sum_{k=2}^{c} \alpha_k (z_{c-1}^c - z_{c-1}^{c-k+1}) & \ldots & \alpha_c (z_{c-1}^c - z_{c-1}^{c-1}) \end{vmatrix}$$

$$
\begin{aligned}
&\beta\alpha_c^c\prod_{j=1}^{c-1}(z_j-1)\prod_{0\le l<j\le c-1}(z_j-z_l)=\\
&=\gamma\begin{vmatrix}1 & 1 & \dots & 1\\ z_1^c-1 & z_1^c-z_1 & \dots & z_1^c-z_1^{c-1}\\ \vdots & \vdots & \ddots & \vdots\\ z_{c-1}^c-1 & z_{c-1}^c-z_{c-1} & \dots & z_{c-1}^c-z_{c-1}^{c-1}\end{vmatrix}\begin{vmatrix}\alpha_c & 0 & \dots & 0\\ \alpha_{c-1} & \alpha_c & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_1 & \alpha_2 & \dots & \alpha_c\end{vmatrix}=\\
&=\gamma\alpha_c^c\begin{vmatrix}1 & z_1^c-1 & \dots & z_{c-1}^c-1\\ 1 & z_1^c-z_1 & \dots & z_{c-1}^c-z_{c-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & z_1^c-z_1^{c-1} & \dots & z_{c-1}^c-z_{c-1}^{c-1}\end{vmatrix}=\gamma\alpha_c^c\begin{vmatrix}0 & z_1-1 & \dots & z_{c-1}-1\\ 0 & z_1^2-z_1 & \dots & z_{c-1}^2-z_{c-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & z_1^c-z_1^{c-1} & \dots & z_{c-1}^c-z_{c-1}^{c-1}\end{vmatrix}\\
&=(-1)^{c+1}\gamma\alpha_c^c\prod_{j=1}^{c-1}(z_j-1)\prod_{0\le l<j\le c-2}(z_j-z_l)
\end{aligned}
$$

Therefore

$$\beta=\left(\sum_{l=0}^{c}l\alpha_l-q'(z)|_{z=1}\right)\prod_{l=1}^{c-1}\frac{1}{1-z_l} \tag{4.13}$$

In conclusion, from (3.11), (4.10) and (4.13), we obtain the exact generating function of the system

$$P(z)=\frac{q(z)(z-1)\left(\sum_{l=0}^{c}l\alpha_l-q'(z)|_{z=1}\right)}{z^c-q(z)\sum_{l=0}^{c}\alpha_l z^{c-l}}\prod_{l=1}^{c-1}\frac{z-z_l}{1-z_l} \tag{4.14}$$

From (4.14) one can write the expression for all P_n, we give only the expression for P_0 evaluating $P(z)$ at $z=0$

$$P_0=(-1)^{c-1}\frac{\sum_{l=0}^{c}l\alpha_l-q'(z)|_{z=1}}{\alpha_c}\prod_{l=1}^{c-1}\frac{z_l}{1-z_l} \tag{4.15}$$

Note that the $GI/D/c$ service is a particular case of our system, namely

$$\alpha_l=\begin{cases}0 & l<c\\ 1 & l=c\end{cases} \tag{4.16}$$

In this case the (4.14) becomes

$$P(z)=\frac{q(z)(z-1)\left(c-q'(z)|_{z=1}\right)}{z^c-q(z)}\prod_{l=1}^{c-1}\frac{z-z_l}{1-z_l} \tag{4.17}$$

as it is well known by Crommelin [9]. Than in this way we give the generalization of the results given in [9] and in [19].

3.5 The idea of approximation

Although the construction described in the above section is in principle the complete solution of the problem, it has two main disadvantages.

- First of all it is not always easy to find complex zeroes $z_1, ...z_c$ of $d(z)$, especially when c is large.
- Second, a small error (always present in numerical computations) on the values of $z_1, ...z_c$ generates a sequence of P_n that rapidly diverges from the true (probabilistic) expression. We will show that if $z_1, ...z_c$ do not cancel exactly the zeroes of the denominator in the generating function, the latter diverges in some z_i and their coefficients P_n diverge exponentially, loosing their probabilistic interpretation.

Suitable approximation schemes have to be found in order to overcome these issues. For memoryless arrivals and a fixed number of servers, the problem is solved in [7]p.119 by the so called *geometric tail approach*. The idea is to reduce the the system (3.7) to a finite linear algebraic system fixing a suitably chosen $\bar{n}$ and assuming that for $n > \bar{n}$ the following relation is approximately true

$$P_n = P_{\bar{n}} \tau^{n-\bar{n}} \quad \tau < 1 \tag{5.1}$$

It is easy to see that τ is the inverse of the absolute value of the zero of the denominator *outside* the circle of radius 1. The solution of the finite algebraic linear system obtained in this way converges numerically very fast to a set of fixed values, which are then assumed to be a good approximated solutions of the first $\bar{n}$ values of P_n, and in particular of $P_0, ..., P_{c-1}$.

There are other possible approaches to the solution of the problem above. An approximated solution of the probability distribution P_n can be obtained from the infinite set of equations of the process by restricting it to a finite set of states $E_f = \{0, 1, ..., \bar{n}\}$. When the distribution of the (true) infinite system is such that the probability of being in the complement of E_f is suitably small, then the probability of the truncated finite system should be quite near to the true probabilities of the infinite system. The transition probability matrix of the finite system, however, may not be just the finite truncation of the infinite matrix, since in this way the resulting finite matrix would not be Markovian. Indeed the terms of the truncated matrix have to be suitably increased in order to restore the Markov property, i.e. the sum of each row has to be one. This process is usually called in literature *augmentation* (see [25, 28]). Since the first idea of approximating an infinite Markov chain is due to Seneta and the much of literature on this topic is collected in his book we recall some definition and theorem given on his book [26]. Let X_i be a positive recurrent discrete time Markov chain on the countable state space $\{0, 1, 2, \dots\}$, with transition matrix $\mathbf{P}$ and stationary distribution $\boldsymbol{\pi}$. Let $\mathbf{P}^{(\bar{n})}$ denote the truncation of size $\bar{n}$. Consider $\boldsymbol{\pi}^{(\bar{n})}$ obtained from a $\bar{n}$x$\bar{n}$ stochastic matrix $\hat{\mathbf{P}}^{(\bar{n})}$ where $\hat{\mathbf{P}}^{(\bar{n})} \geq \mathbf{P}^{(\bar{n})}$ elementwise.

Definition 3.5.1. *A stochastic matrix $\mathbf{P} = \{p_{ij}\}$ is said to be a Markov matrix if the elements of at least one column are away from 0, i.e. there exists a j_0 and $\epsilon > 0$ such that $p_{ij_0} > \epsilon$, all i.*

Such matrix has a single essential class, which is positive recurrent and aperiodic and contains j_0.

Definition 3.5.2. *A stochastic matrix $\mathbf{P} = \{p_{ij}\}$ is said to be upper Hessenberg if $p_{ij} > 0$ for $i > j + 1$.*

Theorem 3.5.3. *Let $\mathbf{P} = \{p_{ij}\}$ be a Markov matrix and for each $\bar{n} \in \mathbb{N}$, let $\hat{\mathbf{P}}^{(\bar{n})}$ be an $\bar{n}x\bar{n}$ stochastic matrix satisfying $\hat{\mathbf{P}}^{(\bar{n})} \geq \mathbf{P}^{(\bar{n})}$. Then for all $\bar{n}$ sufficiently large $\hat{\mathbf{P}}^{(\bar{n})}$ has a unique stationary distribution $\boldsymbol{\pi}^{(\bar{n})}$ and $\boldsymbol{\pi}^{(\bar{n})} \to \boldsymbol{\pi}$ as $\bar{n} \to \infty$.*

Proof. See [25]. □

There has been, also recently, a certain emphasis on the problem of finding the augmentation procedure that gives the smaller errors with respect to the true infinite distribution. If the notion of distance between the finite distribution $P_n^{(f)}$ and the infinite distribution P_n is the so called *total variation distance*

$$d_{TV}(P^{(f)}, P) = \frac{1}{2} \sum_{n \geq 0} |P_n^{(f)} - P_n| \tag{5.2}$$

A consequence of theorem 1 in [28] for the errors is

$$d_{TV}(P^{(f)}, P) = \sum_{\substack{n=0 \\ P_n^{(f)} > P_n}}^{\bar{n}} |P_n^{(f)} - P_n| \tag{5.3}$$

It is clear that a lower bound of this distance is given by the total weight of the tail of the infinite distribution:

$$d_{TV}(P^{(f)}, P) \geq \frac{1}{2} \sum_{n > \bar{n}} P_n \tag{5.4}$$

An augmentation is considered *optimal* if the equality is realized in (5.4). Two different augmentation procedures are particularly discussed in literature: the *last column augmentation* and the augmentation given by the *censored Markov chain*. The following lemma is stated in [31, Lemma 6.6].

Lemma 3.5.4. *Let* $\mathbf{P}$ *be the transition probability matrix of an arbitrary Markov chain partitioned according to subsets* E_f *and* E_f^c

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} E_f & E_f^c \end{array} \\ \begin{array}{c} {\scriptstyle E_f} \\ {\scriptstyle E_f^c} \end{array} & \left[\begin{array}{cc} \mathbf{T} & \mathbf{U} \\ \mathbf{D} & \mathbf{Q} \end{array} \right] \end{array} \tag{5.5}$$

Then, the censored process is Markov chain and its transition probability matrix is given by

$$\mathbf{P_f}^E = \mathbf{T} + \mathbf{U}\hat{\mathbf{Q}}\mathbf{D} \tag{5.6}$$

with $\hat{\mathbf{Q}} = \sum_{k=0}^{\infty} \mathbf{Q}^k$

The *last column augmentation* is easy to implement numerically, as we shall see explicitly below, but it not always optimal. The *censored Markov chain*, having the first $\bar{n}$ probabilities proportional to the true P_n is always optimal, but it is hard to compute explicitely, unless it is exactly the augmentation of the last column, e.g the case of an upper Hessenberg matrix.

In this chapter we will present an approach that is actually a different way to write the last column augmentation. The basic idea of this approach is simple: we describe the system starting from the *cumulative probabilities* σ_n defined by

$$\sigma_n = \sum_{k=0}^{n} P_k.$$

In order to have a probabilistic interpretation, the sequence of σ_n has to be increasing, $\sigma_i \leq \sigma_{i+1}$, and $\lim_{n\to\infty} \sigma_n = 1$. We can rewrite the equations (3.7) in terms of the σ's to get

$$
\begin{aligned}
\sigma_0 &= \sum_{l=0}^{c} \alpha_l \sigma_l q_0 \\
\sigma_1 &= \sum_{l=0}^{c} \alpha_l (\sigma_l q_1 + \sigma_{l+1} q_0) \\
&\vdots \\
\sigma_n &= \sum_{l=0}^{c} \alpha_l (\sigma_l q_n + \sigma_{l+1} q_{n-1} + \cdots + \sigma_{n+l} q_0) \qquad n \geq 1 \\
&\vdots
\end{aligned}
\tag{5.7}
$$

that we can rewrite in a compact form as

$$
\sigma_n = \sum_{l=0}^{c} \alpha_l \sum_{k=l}^{n+l} \sigma_k q_{n+l-k} \tag{5.8}
$$

Now multiplying both sides of (5.8) by z^n and summing from 0 to ∞ we have

$$
\begin{aligned}
\sigma(z) &= \sum_{n=0}^{\infty} \sum_{l=0}^{c} \alpha_l \sum_{k=l}^{n+l} \sigma_k q_{n-k+l} z^n \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{c} \sum_{i=0}^{n} \alpha_l \sigma_{i+l} q_{n-i} z^{n-i} z^i \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \sum_{l=0}^{c} \alpha_l \sigma_{i+l} q_{n-i} z^{n-i} z^i \\
&= q(z) \sum_{i=1}^{\infty} \sum_{l=0}^{c} \frac{\alpha_l}{z^l} \sigma_{i+l} z^{i+l} \\
&= q(z) \sum_{k=l}^{\infty} \sum_{l=0}^{c} \frac{\alpha_l}{z^l} \sigma_k z^k
\end{aligned}
\tag{5.9}
$$

So the explicit form of the generating function $\sigma(z) = \sum_{n=0}^{\infty} \sigma_n z^n$ is given by

$$
\sigma(z) = \frac{m(z)}{d(z)} \qquad |z| \leq 1 \tag{5.10}
$$

where

$$
m(z) = q(z) z^c \sum_{l=0}^{c} \sum_{k=0}^{l-1} \alpha_l \sigma_k z^{k-l} \tag{5.11}
$$

and $d(z)$ is defined as in (3.11). Since we know that the asymptotic value of the $\sigma's$ is $\lim_{n\to\infty} \sigma_n = 1$, we can construct an approximated solution of the system (5.7) imposing that $\sigma_i = 1$ for all $i > \bar{n}$. In this way the system (5.7) becomes a finite dimensional linear algebraic system, which can be solved numerically.

3.6 Theorical results: case $GI/D/c$

The main results of this section are:

1) A generic choice of the initial probabilities $\sigma_0, ..., \sigma_{c-1}$ induces on *all* terms of the sequence $\sigma_n,\ n \geq c$, an error that increases exponentially in n. Hence the idea of computing approximately the first c σ's and then use them to solve the system (5.7) is completely useless if we are interested in a numerical solution. The proof of this theorem is straightforward but complicated. Hence we decided to prove that even in a simplified case, the $GI/D/c$ queueig system with constant c, the same problem arises.

2) Fixing, as we outlined in the above section, the value $\bar{n}$ such that we assume the differences $1 - \sigma_n$ negligible for all $n > \bar{n}$ is, on the other side, an approach that allows us to control explicitly the error we do on all the values $\sigma_n,\ n < \bar{n}$. We prove this result first for the $GI/D/c$ queueig system, where the expressions are less heavy, and then we show how the similar proof can be used to control the flow of our $GI/D/c_i$, as we see in section 3.7.

Hence we start choosing ($GI/D/c$ service) the 4.16, hence in this case the stationary probability satisfies the following linear system

$$
\begin{aligned}
P_0 &= \sum_{k=0}^{c} P_k q_0 \\
P_1 &= \sum_{k=0}^{c} P_k q_1 + P_{c+1} q_0 \\
P_2 &= \sum_{k=0}^{c} P_k q_2 + P_{c+1} q_1 + P_{c+2} q_0 \\
&\vdots \\
P_n &= \sum_{k=0}^{c} P_k q_n + \sum_{k=c+1}^{n+c} P_k q_{n-k+c} \qquad n \geq 1
\end{aligned}
\tag{6.1}
$$

Let again σ_n be defined by, $\sigma_n = \sum_{k=0}^{n} P_k$. We obtain directly from (6.1) the following infinite linear system for $\{\sigma_n\}_{n\geq 0}$,

$$
\begin{aligned}
&\sigma_0 = \sigma_c q_0 \\
&\sigma_1 = \sigma_c q_1 + \sigma_{c+1} q_0 \\
&\sigma_2 = \sigma_c q_2 + \sigma_{c+1} q_1 + \sigma_{c+2} q_0 \\
&\vdots \\
&\sigma_n = \sigma_c q_n + \sum_{k=c+1}^{n+c} \sigma_k q_{n-k+c} = \sum_{k=c}^{n+c} \sigma_k q_{n-k+c} \qquad n \geq 1 \\
&\vdots
\end{aligned}
\tag{6.2}
$$

Define the generating functions

$$
\sigma(z) = \sum_{n=0}^{\infty} \sigma_n z^n \qquad q(z) = \sum_{n=0}^{\infty} q_n z^n
\tag{6.3}
$$

we can derive easily the following functional equation from (6.2) multiplying both side of (6.2) by z^n and summing from 0 to ∞ or using the condition (4.16) in (5.11), to obtain

$$\sigma(z) = \frac{\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}}{1 - z^c q^{-1}(z)} \tag{6.4}$$

From (6.4), writing the denominator as a geometric series in $|z^c/q(z)| < 1$, we obtain

$$\sigma(z) = (\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}) \sum_{k=0}^{\infty} \left[\frac{z^c}{q(z)}\right]^k \tag{6.5}$$

which can be rewritten as

$$\sigma(z) = \sum_{k=0}^{c-1} \sigma_k z^k \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{c-1} a_n \sigma_k z^{n+k} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\min(c-1,n)} a_{n-k} \sigma_k\right] z^n \tag{6.6}$$

Thus, we get for σ_n:

$$\sigma_n = \sum_{k=0}^{\min(c-1,n)} a_{n-k} \sigma_k \tag{6.7}$$

from this expression we can find the explicit form of the probabilities σ_n as functions of $\sigma_0, \ldots, \sigma_{c-1}$. To be explicit, let us consider first the case of Poisson arrivals, $q(z) = e^{\varrho(z-1)}$.

$$\begin{aligned}
\sigma(z) &= (\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}) \sum_{k=0}^{\infty} \left[e^{\varrho(1-z)} z^c\right]^k \\
&= (\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}) \sum_{k=0}^{\infty} e^{k\varrho} e^{-k\varrho z} z^{kc} \\
&= (\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}) \sum_{k=0}^{\infty} e^{k\varrho} z^{kc} \sum_{m=0}^{\infty} \frac{(-k\varrho z)^m}{m!} \\
&= (\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m e^{k\varrho} \frac{(k\varrho)^m}{m!} z^{m+kc} \\
&= (\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}) \sum_{k=0}^{\infty} \sum_{n=kc}^{\infty} (-1)^{n-kc} e^{k\varrho} \frac{(k\varrho)^(n-kc)}{(n-kc)!} z^n
\end{aligned} \tag{6.8}$$

We have

$$\sigma(z) = (\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1}) \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{c} \rfloor} (-1)^{n-kc} e^{k\varrho} \frac{(k\varrho)^{n-kc}}{(n-kc)!} z^n \tag{6.9}$$

Hence in this case

$$a_n = \sum_{k=0}^{\lfloor \frac{n}{c} \rfloor} (-1)^{n-kc} e^{k\varrho} \frac{(k\varrho)^{n-kc}}{(n-kc)!} \tag{6.10}$$

From (6.7) it is clear that the explicit expression of all the σ's is known once we know the first c sigmas, i.e. the set $\{\sigma_0, \ldots, \sigma_{c-1}\}$. In what follows we shall see that an infinite set $\{\sigma\} = \{\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots\}$ is *generated* by a choice $\{\sigma_0, \ldots, \sigma_{c-1}\}$. In other words the equations (6.2) may be interpreted as the equations of a *flow* $\{\sigma\}$ which is uniquely determined by its initial condition $\{\sigma_0, \ldots, \sigma_{c-1}\}$. Note that the flow $\{\sigma\}$ is linear in any of the element of the set $\{\sigma_0, \ldots, \sigma_{c-1}\}$, and hence the flow generated by $\{r\sigma_0, \ldots, r\sigma_{c-1}\}$ with $r \in \mathbb{R}^+$ is simply $\{r\sigma\}$. This means that the initial conditions are determined modulo an overall constant factor r. Such factor will be fixed by the normalization condition $\lim_{n\to\infty} \sigma_n = 1$. This behavior does not depend on the explicit form of $q(z)$ as we see below: the set $\{\sigma\}$ is always generated modulo constant factor by $\{\sigma_0, \ldots, \sigma_{c-1}\}$.

It is now clear that the flow generated by an initial condition $\{\sigma_0, \ldots, \sigma_{c-1}\}$ such that the generating function $\sigma(z)$ has singularities in the circle $|z| < 1$ has to be such that a subsequence of σ_n diverges exponentially. Such a choice of $\{\sigma_0, \ldots, \sigma_{c-1}\}$ has hence to be excluded if we want to keep a probabilistic meaning for the flow $\{\sigma\}$.

Let us go back to the general form of $\sigma(z)$

$$\sigma(z) = \frac{q(z)(\sigma_0 + \sigma_1 z + \cdots + \sigma_{c-1} z^{c-1})}{q(z) - z^c} = \frac{N(\sigma_0, \ldots, \sigma_{c-1}, z)}{d(z)} \tag{6.11}$$

In what follows we assume that $q(z)$ is holomorfic in the unit disc $\overline{\mathbb{D}}$, hence by Rouché's theorem (Ahlfors [5, p.152]), $q(z) - z^c$ has $c-1$ zeros in $|z| < 1$, and a single zero in $z = 1$ see [8, Lemma 2.2, 3.1]. Therefore $d(z) = q(z) - z^c$ will be of the following form

$$d(z) = h(z)(z-1)(z-\hat{z}_1)\cdots(z-\hat{z}_{c-1})$$

Where $h(z)$ is an analytic function and $h(z) \neq 0$ on $|z| \leq 1$.

Since $\sigma(z)$ must be analytic in $\mathbb{D}$, there must be values $\{\hat{\sigma}_0, \ldots, \hat{\sigma}_{c-1}\}$ for the coefficients so that

$$\hat{\sigma}_0 + \hat{\sigma}_1 z + \cdots + \hat{\sigma}_{c-1} z^{c-1} = \hat{\sigma}_{c-1}(z-\hat{z}_1)\cdots(z-\hat{z}_{c-1})$$

in order to cancel the divergences corresponding to the roots of the denominator $d(z)$. Since there are $c-1$ zeros in $\mathbb{D}$ the choice $\{\hat{\sigma}_0 \ldots, \hat{\sigma}_{c-1}\}$ is unique modulo the overall factor. For sake of simplicity we define

$$\hat{\sigma}(z) := \frac{N(\hat{\sigma}_0, \ldots, \hat{\sigma}_{c-1}, z)}{d(z)}$$

as the actual generating function, analytic in $\mathbb{D}$, and from now on we will consider $\sigma(z)$ as a function of the parameters $\{\sigma_0, \cdots, \sigma_{c-1}\}$. Such a dependence can be translated into a variational factor $g(z)$:

$$\sigma(z) = q(z)\left[\frac{(z-\hat{z}_1)\cdots(z-\hat{z}_{c-1})}{q(z)-z^c}\right]\left[\frac{\sigma_0 + \sigma_1 z^1 + \cdots + \sigma_{c-1} z^{c-1}}{(z-\hat{z}_1)\cdots(z-\hat{z}_{c-1})}\right] := \hat{\sigma}(z) g(z) \tag{6.12}$$

where

$$g(z) = \frac{1}{\sigma_{c-1}}\left[\frac{\sigma_0 + \sigma_1 z^1 + \cdots + \sigma_{c-1} z^{c-1}}{(z-\hat{z}_1)\cdots(z-\hat{z}_{c-1})}\right]$$

By the Fundamental Thm of Algebra, there are $c-1$ roots of the numerator of $g(z)$, which we write as $\hat{z}_i + \Delta_i$ for $i = 1, \ldots, c-1$, then

$$g(z) = \frac{(z-\hat{z}_1-\Delta_1)\cdots(z-\hat{z}_{c-1}-\Delta_{c-1})}{(z-\hat{z}_1)\cdots(z-\hat{z}_{c-1})} \tag{6.13}$$

and $\sigma(z)$ is now written in terms of a function of $\Delta_1, \ldots, \Delta_{c-1}$.

Theorem 3.6.1. *Let $\{\Delta_1, \dots, \Delta_{c-1}\}$ be given as above in (6.13), and let the parameters Δ_i satisfy the following conditions:*

when the smaller root $\hat{z}_1$ is real

$$-\Delta_1 = \hat{z}_1 - \hat{z}_j \qquad 0 < j < c, j \neq i \tag{6.14}$$

when the smaller roots $\hat{z}_1$ and $\hat{z}_2$ are complex conjugates

$$|\Delta_1 \prod_{j \neq 1} (\Delta_j + \hat{z}_j - \hat{z}_1)| \neq |\Delta_2 \prod_{j \neq 2} (\Delta_j + \hat{z}_j - \hat{z}_2|) \tag{6.15}$$

Then there exist constants $A > 0$ and $K > 1$ such that asymptotically in n

$$|\sigma_n| \geq AK^n$$

The proof of this theorem is straightforward but quit delicate, we give the proof of this theorem in section 3.8 The meaning of the result, on the other side, is quite clear: not only a subsequence of σ_n diverges exponentially, but every term of the sequence. Hence the rounding error due to the numerical evaluation of the zeroes of the denominator of (6.4) brings to errors on the σ's that tend to become exponentially large, and it is difficult to evaluate a priori the numerical relevance of this errors up to a defined n.

For these reason we propose in what follows a different approximation scheme. We us introduce the following notations

- $\{\hat{\sigma}_n\}_{n \geq 0}$ is the solution of the infinite system (6.2), satisfying $\sigma_i \leq \sigma_{i+1}$, and $\sigma_i \to 1$ for $i \to \infty$.
- $\{\sigma\}_{n \geq 0}$ is the solution of the truncated system, where $\sigma_{\bar{n}+k} = 1$ for any $k \geq 1$.
- $$\Delta_n = \sigma_n - \hat{\sigma}_n \tag{6.16}$$
  and $\Delta_{\bar{n}+k}$ is a non negative decreasing sequence in $k \geq 1$.

By linearity, $\{\Delta_n\}_{n \leq \bar{n}}$ satisfy the same linear system of equations (6.2) truncated at $\bar{n}$.

$$\Delta_i = \sum_{j=c}^{i+c} \Delta_j q_{i+c-j} \qquad 0 \leq i \leq \bar{n} \tag{6.17}$$

Theorem 3.6.2. *$\Delta_n = \sigma_n - \hat{\sigma}_n$ is non negative for any $n \leq \bar{n}$*

Proof. We know that $\Delta_{\bar{n}+1} \geq \cdots \geq \Delta_{\bar{n}+c} \geq 0$ and $\lim_{\bar{n} \to \infty} \Delta_{\bar{n}+1} = 0$.
Let $\Delta \equiv \min_{0 \leq i \leq \bar{n}+c} \{\Delta_i\}$. Let $j = \max\{0, 1, \cdots, \bar{n} + c | \Delta_j = \Delta\}$

From the definition it follows that

$$\Delta_j < \Delta_i \qquad \text{for } i \text{ such that } j < i \leq \bar{n} + c \tag{6.18}$$
$$\Delta_j \leq \Delta_i \qquad \text{for } i \text{ such that } i < j \tag{6.19}$$

We will prove that $\Delta_n \geq 0$ by contradiction: Let $\sum_{i=0}^{c-1} q_i = \beta$. By the *stability condition* (3.3), we can assume $\beta > 0$. Consider the jth equation in 6.17, if $j < c$

$$\Delta_j = \sum_{k=c}^{j+c} \Delta_k q_{j+c-k} > \Delta_j \sum_{k=c}^{j+c} q_{j+c-k} \tag{6.20}$$

which is false if $\Delta_j < 0$. The case $j > c$ is treated similarly. Since for all $i > j$ $\Delta_j < \Delta_i$,

$$\Delta_j = \sum_{k=c}^{j} \Delta_k q_{j+c-k} + \sum_{k=j+1}^{j+c} \Delta_k q_{j+c-k} > \Delta_j \sum_{k=c}^{j+c} q_{j+c-k} \tag{6.21}$$

we have strict inequality since $\beta > 0$; again the statement is false if $\Delta_j < 0$, given that the sum in the last term is at most one. $\square$

Similarly we will prove, in the next theorem, that $\Delta_i \leq \Delta_{\bar{n}+1}$ $\forall 0 \leq i \leq \bar{n}$

Theorem 3.6.3. *For Δ_n, as defined in* (6.16)

$$\Delta_i \leq \Delta_{\bar{n}+1} \qquad \forall 0 \leq i \leq \bar{n} \tag{6.22}$$

Proof. As already noted in Thm 3.6.2, $\Delta_{\bar{n}+1} \geq \cdots \geq \Delta_{\bar{n}+c} \geq 0$, and we have by stability condition $\sum_{i=0}^{c-1} q_i = \beta > 0$.

Let

$$\Delta = \max_{0 \leq i \leq \bar{n}+1} \{\Delta_i\}$$

and let

$$j = \max\{1, 2, \ldots, \bar{n}+1 | \Delta_j = \Delta\}$$

From the definition it follows that

$$\Delta_j > \Delta_i \qquad \text{for } i \text{ such that } j < i \leq \bar{n}+1 \tag{6.23}$$

$$\Delta_j \geq \Delta_i \qquad \text{for } i \text{ such that } i < j \tag{6.24}$$

Consider the jth equation in 6.17, if $j < c$ then from (6.23) we have

$$\Delta_j = \sum_{i=c}^{j+c} \Delta_i q_{j+c-i} < \Delta_j \sum_{i=c}^{j+c} q_{j+c-i} \leq \Delta_j \beta \tag{6.25}$$

The case $j > c$ is treated similarly. Since, for all $i > j$, $\Delta_j > \Delta_i$,

$$\Delta_j = \sum_{i=c}^{j} \Delta_i q_{j+c-i} + \sum_{i=j+1}^{j+c} \Delta_i q_{j+c-i} < \Delta_j \sum_{i=c}^{j+c} q_{j+c-i} \leq \Delta_j \beta \tag{6.26}$$

Thus in both case we have strict inequality, since $\beta > 0$. On the other side $\beta \leq 1$, and hence (6.18) cannot be true, and we have proven that there are no $j < \bar{n}+1$ satisfying (6.23). Hence

$$\Delta_i \leq \Delta_{\bar{n}+1}$$

$\square$

Using the results of the two theorem above, we have shown that the errors that we have on the first c σ's $\sigma_0, \ldots, \sigma_{c-1}$ are bounded by $\Delta_{\bar{n}+1} = 1 - \hat{\sigma}_{\bar{n}+1} = \sum_{k=\bar{n}+1}^{\infty} P_k$. For a generic choice of the arrival process, assuming that the stability condition is satisfied, P_n vanishes exponentially because there exists a zero of the denominator of the generating function on the real axis for $|z| > 1$, and hence the error on $\sigma_0, \ldots, \sigma_{c-1}$ vanish exponentially in $\bar{n}$ as well.

3.7 Theorical results: case $GI/D/c_i$

In this section we will give some results for $GI/D/c_i$ queueing system that are a generalization of results obtained in section 3.6.
As in subsection 3.6 we use the same notations

- $\{\hat{\sigma}_n\}_{n\geq 0}$ is the solution of the infinite system (5.8), satisfying $\hat{\sigma}_i \leq \hat{\sigma}_{i+1}$, and $\hat{\sigma}_i \to 1$ for $i \to \infty$. Such solution exists and it is unique since the system is ergodic.
- $\{\sigma\}_{n\geq 0}$ is the solution of the truncated system, where $\sigma_{\bar{n}+k} = 1$ for any $k \geq 1$.
- $$\Delta_n = \sigma_n - \hat{\sigma}_n \tag{7.1}$$
  and $\Delta_{\bar{n}+k}$ is a non negative decreasing sequence in $k \geq 1$.

By linearity, $\{\Delta_n\}_{n\leq\bar{n}}$ satisfy the same linear system of equations (5.8) truncated at $\bar{n}$.

$$\Delta_i = \sum_{l=0}^{c} \alpha_l \sum_{k=l}^{i+l} \Delta_k q_{i+l-k} \qquad 0 \leq i \leq \bar{n} \tag{7.2}$$

where the terms $\Delta_{\bar{n}+1} \geq \cdots \geq \Delta_{\bar{n}+c} \geq 0$ have to be considered fixed.

Theorem 3.7.1. *If the system is ergodic $\Delta_n = \sigma_n - \hat{\sigma}_n$ is non negative for any $n \leq \bar{n}$*

Proof. Fix l such that $0 < \alpha_l < 1$ for $l \in \{1, 2 \ldots c\}$ and $\sum_{i=0}^{l} q_i = \beta > 0$. Such an l exists because if for all l such that $\alpha_l \neq 0$ we have $\sum_{i=0}^{l} q_i = 0$ then the stability condition (3.3) can not be satisfied.
By definition $\Delta_{\bar{n}+1} \geq \cdots \geq \Delta_{\bar{n}+c} \geq 0$ and $\lim_{\bar{n}\to\infty} \Delta_{\bar{n}+1} = 0$.
Let

$$\Delta \equiv \min_{0\leq i\leq \bar{n}+c} \{\Delta_i\}$$

and let

$$j = \max\{0, 1, \cdots, \bar{n}+c | \Delta_j = \Delta\}$$

From the definition it follows that

$$\begin{array}{ll} \Delta_j < \Delta_i & \text{for } i \text{ such that } j < i \leq \bar{n}+c \\ \Delta_j \leq \Delta_i & \text{for } i \text{ such that } i < j \end{array} \tag{7.3}$$

We will prove that $\Delta_j \geq 0$ by contradiction. Considering the *j-th* equation in (7.2), we can write it in the following way:

$$\begin{aligned} \Delta_j &= \sum_{l=0}^{c} \alpha_l \sum_{k=l}^{j+l} \Delta_k q_{j+l-k} \\ &= \alpha_l \sum_{k=l}^{j+l} \Delta_k q_{j+l-k} + \sum_{\substack{i=0 \\ i\neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} \Delta_k q_{j+i-k} \qquad (7.4) \\ &\geq \Delta_j \gamma(j,l) \qquad (7.5) \end{aligned}$$

where $\gamma(j,l)$ is a positive number and we have used that Δ_j is minimum. We want to show that $\gamma(j,l) < 1$. In this case the inequality $\Delta_j \geq \Delta_j \gamma(j,l)$ can not be satisfied if $\Delta_j < 0$.

Consider two cases, $j < l$ and $j > l$. If $j < l$ we have again two cases: if $q_i = 0$ for $i = 0, 1, ..., j$ then

$$\gamma(j,l) = \alpha_l \sum_{k=l}^{j+l} q_{j+l-k} + \sum_{\substack{i=0 \\ i \neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} = \sum_{\substack{i=0 \\ i \neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} \leq 1 - \alpha_l < 1$$

if $\sum_{k=l}^{j+l} q_{j+l-k} = \sum_{i=0}^{j} q_i > 0$

$$\Delta_j > \Delta_j \left[\alpha_l \sum_{k=l}^{j+l} q_{j+l-k} + \sum_{\substack{i=0 \\ i \neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} \right]$$

because $\Delta_j < \Delta_k$ for $k = l, ..., j + l$, and then $\gamma(j, l) < 1$. The case $j > l$ is treated similarly

$$\Delta_j = \alpha_l \left(\sum_{k=l}^{j} \Delta_k q_{j+l-k} + \sum_{k=j+1}^{j+l} \Delta_k q_{j+l-k} \right) + \sum_{\substack{i=0 \\ i \neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k}$$

and since $\Delta_j < \Delta_k$ for the Δ_k appearing in the second sum in parentheses, we have again

$$\Delta_j > \Delta_j \left[\alpha_l \sum_{k=l}^{j+l} q_{j+l-k} + \sum_{\substack{i=0 \\ i \neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} \right]$$

and then $\gamma(j, l) < 1$. □

Similarly we will prove, in the next theorem, that $\Delta_i \leq \Delta_{\bar{n}+1}$ $\forall 0 \leq i \leq \bar{n}$

Theorem 3.7.2. *If the system is ergodic and the condition* (3.2) *is verified then*

$$\Delta_i \leq \Delta_{\bar{n}+1} \qquad \forall\ 0 \leq i \leq \bar{n} \tag{7.6}$$

Proof. Fix an l satisfying condition (3.2). As already noted $\Delta_{\bar{n}+1} \geq \cdots \geq \Delta_{\bar{n}+c} \geq 0$. Let

$$\Delta = \max_{0 \leq i \leq \bar{n}+1} \{\Delta_i\}$$

and let

$$j = \max\{1, 2, \ldots, \bar{n} + 1 | \Delta_j = \Delta\}$$

Assume that $j < \bar{n} + 1$. From the definition it follows that

$$\begin{array}{ll} \Delta_j > \Delta_i & \text{for } i \text{ such that } j < i \leq \bar{n} + 1 \\ \Delta_j \geq \Delta_i & \text{for } i \text{ such that } i < j \end{array} \tag{7.7}$$

Consider the *j-th* equation in 7.2, we can write it in following way

$$\begin{aligned} \Delta_j &= \sum_{l=0}^{c} \alpha_l \sum_{k=l}^{j+l} \Delta_k q_{j+l-k} \\ &= \alpha_l \sum_{k=l}^{j+l} \Delta_k q_{j+l-k} + \sum_{\substack{i=0 \\ i \neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} \Delta_k q_{j+i-k} && (7.8) \\ &\leq \Delta_j \gamma(j, l) && (7.9) \end{aligned}$$

where in the last line we have used that Δ_j is max and we defined

$$\gamma(j,l) = \alpha_l \sum_{k=l}^{j+l} q_{j+l-k} + \sum_{\substack{i=0\\i\neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} \leq 1 \tag{7.10}$$

We want to show that inequality $j < \bar{n}+1$ is a contradiction, and therefore the maximum of the Δ's must be $\Delta_{\bar{n}+1}$. Consider two cases, $j < l$ and $j > l$. If $j < l$ we have again two cases: if $q_i = 0$ for $i = 0, 1, ..., j$ then

$$\gamma(j,l) = \alpha_l \sum_{k=l}^{j+l} q_{j+l-k} + \sum_{\substack{i=0\\i\neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} = \sum_{\substack{i=0\\i\neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} \leq 1 - \alpha_l < 1$$

and then the inequality (7.9) $\Delta_j \leq \Delta_j \gamma(j,l)$ can not be satisfied. If $\sum_{k=l}^{j+l} q_{j+l-k} = \sum_{i=0}^{j} q_i > 0$ then the inequality (7.9) becomes

$$\Delta_j < \Delta_j \left[\alpha_l \sum_{k=l}^{j+l} q_{j+l-k} + \sum_{\substack{i=0\\i\neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} \right] = \Delta_j \gamma(j,l)$$

because $\Delta_j > \Delta_k$ for $k = l, ..., j+l$ and again we have a contradiction. The case $j > l$ is treated similarly:

$$\Delta_j \quad = \quad \alpha_l \left(\sum_{k=l}^{j} \Delta_k q_{j+l-k} + \sum_{k=j+1}^{j+l} \Delta_k q_{j+l-k} \right) + \sum_{\substack{i=0\\i\neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} \Delta_k q_{j+i-k}$$

and since $\forall k > j$ $\Delta_j > \Delta_k$ and the condition (3.2) is satisfied, the second sum in parentheses gives again the strict inequality

$$\Delta_j < \Delta_j \left[\alpha_l \sum_{k=l}^{j+l} q_{j+l-k} + \sum_{\substack{i=0\\i\neq l}}^{c} \alpha_i \sum_{k=i}^{j+i} q_{j+i-k} \right] = \Delta_j \gamma(j,l)$$

Hence the inequality $j < \bar{n}+1$ cannot hold. Thus

$$\Delta_i \leq \Delta_{\bar{n}+1}$$

□

Using the results of the two theorem above, we have shown that the errors we have on the first c σ's $\sigma_0, ..., \sigma_{c-1}$ are bounded by $\Delta_{\bar{n}+1} = 1 - \hat{\sigma}_{\bar{n}+1} = \sum_{k=\bar{n}+1}^{\infty} P_k$. For a choice of the arrival process such that there exists a zero of the denominator of the generating function on the real axis for $|z| > 1$, we have that P_n vanishes exponentially, and hence the error on $\sigma_0, ..., \sigma_{c-1}$ vanish exponentially in $\bar{n}$ as well.

The results above give in our case an estimate on the errors of our approximation scheme in the sense of the L_∞ norm. We can write

$$|P_n - \hat{P}_n| = |\sigma_n - \sigma_{n-1} - (\hat{\sigma}_n - \hat{\sigma}_{n-1})| = |\Delta_n - \Delta_{n-1}| \leq \Delta_{\bar{n}+1} \tag{7.11}$$

where in the last equality we used the fact that both Δ_n and Δ_{n-1} are positive (theorem 3.7.1) and the fact that they are both bounded by $\Delta_{\bar{n}+1}$ (theorem 3.7.2).

Our approach using the cumulative probabilities σ's gave us some results about the probability distribution P_n. It is easy to show, however, that our approximation scheme is completely equivalent to a procedure well known in the literature: the last column augmentation. To see this, note that the solution of the system (5.7) truncated at $\bar{n}$ is equivalent to the solution of the following system for the P_n

$$\begin{aligned}
&P_0 = \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_0 \\
&P_1 = \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_1 + \sum_{l=0}^{c} \alpha_l P_{l+1} q_0 \\
&\vdots \\
&P_n = \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_n + \sum_{l=0}^{c} \alpha_l \sum_{k=l+1}^{n+l} P_k q_{n-k+l} \qquad 1 < n \leq \bar{n} - c + 1 \\
&\vdots \\
&P_{\bar{n}} = \sum_{l=0}^{c} \alpha_l \sum_{k=0}^{l} P_k q_{\bar{n}} + \sum_{l=0}^{c} \alpha_l \sum_{k=l+1}^{\bar{n}+1} P_k q_{\bar{n}-k+l}
\end{aligned} \tag{7.12}$$

where the P_n's with $n > \bar{n} + 1$ in (3.7) vanish due to the fact that the σ_n are all equal to 1 for $n > \bar{n}$. The system (7.12) is an homogeneous system of $\bar{n} + 1$ equations in $\bar{n} + 2$ unknowns, and, since it has rank $\bar{n} + 1$, it has a unique solution if we impose further the normalization condition $P_0 + P_1 + ... + P_{\bar{n}+1} = 1$. Such solution is exactly the one that would be obtained augmenting the last column (the one giving the equation for $P_{\bar{n}+1}$) and finding the corresponding stationary measure for the finite Markov chain so obtained. Actually, this means to solve an homogeneous system of $\bar{n} + 2$ equations in $\bar{n} + 2$ unknowns. The first $\bar{n} + 1$ equations of such system are exactly the ones in (7.12). The last one is the equation for $P_{\bar{n}+1}$ in which the coefficients are chosen in such a way that the matrix has vanishing determinant. Hence the system, having again rank $\bar{n} + 1$, may be solved neglecting one equation and imposing the normalization condition $P_0 + P_1 + ... + P_{\bar{n}+1} = 1$. If the equation neglected is the last one, the two systems become identical.

Hence our results can be interpreted as bounds on the L_∞ norm of the last column augmentation procedure. Such procedure is well known in literature, and it is known that, despite its simplicity, it is not always optimal in the sense of the total variation distance. However, as we will show in the last part of this section, it is not at all obvious that the total variation distance is the more natural notion of distance when, for instance, we want to compute the average value of the queue's length or its variance. We will show in the section 3.10 that an optimal approximation procedure, the censored Markov chain (which is very hard to compute explicitly in our model, and which we compute only indirectly using its properties), gives often a bigger error in the part of the distribution that is relevant in the computation of the average values.

3.8 Proof of theorem 3.6.1

Proof. We will consider first the case in which we have simple roots. Later we will show the general case of multiple roots.
Simple roots: Assume that $\hat{z}_i \neq \hat{z}_j$ for $i \neq j$. By (6.12)

$$g(z) = \frac{(z-\hat{z}_1-\Delta_1)\cdots(z-\hat{z}_{c-1}-\Delta_{c-1})}{(z-\hat{z}_1)...(z-\hat{z}_{c-1})} = \sum_{I\subseteq\{1,\ldots,c-1\}} \prod_{i\in I} \frac{\Delta_i}{(\hat{z}_i - z)} =$$
$$= \sum_{I\subseteq\{1,\ldots,c-1\}} \sum_{i\in I} \frac{\Delta_i}{\hat{z}_i - z} \prod_{\substack{j\in I \\ j\neq i}} \frac{\Delta_j}{\hat{z}_j - \hat{z}_i} := 1 + \sum_{I\neq\emptyset} \sum_{i\in I} \frac{\Delta_I}{\hat{z}_i - z} S(i,I) \tag{8.1}$$

where: 1) the first sum in each term is over all ordered subsets of indices, i.e. $I = \{i_1,\ldots,i_k\}$ with $i_j < i_{j+1}$, with the understanding that, for $I = \emptyset$, the empty product is 1;
2) the Heaviside *Cover-up* method applies on the third equality;
3) last term is to define Δ_I and $S(i,I)$:

$$\Delta_I = \prod_{i\in I} \Delta_i \qquad \text{and} \qquad S(i,I) = \prod_{\substack{j\in I \\ j\neq i}} \frac{1}{\hat{z}_j - \hat{z}_i} \tag{8.2}$$

Plugging (8.1) in (6.12) we get

$$\sigma(z) = \hat{\sigma}(z) + \sum_{I\neq\emptyset} \sum_{i\in I} \frac{\Delta_I \hat{\sigma}(z) S(i,I)}{\hat{z}_i - z} \tag{8.3}$$

and for any I and $i \in I$ we analyze the z-dependent term $\hat{\sigma}(z)\frac{1}{\hat{z}_i - z}$

$$\hat{\sigma}(z)\frac{1}{\hat{z}_i - z} = \frac{1}{\hat{z}_i} \sum_{m=0}^{\infty} \frac{\hat{\sigma}_m z^m}{(1-\frac{z}{\hat{z}_i})} = \frac{1}{\hat{z}_i} \sum_{m=0}^{\infty} \hat{\sigma}_m z^m \sum_{n=0}^{\infty} \left[\frac{z}{\hat{z}_i}\right]^n =$$
$$= \frac{1}{\hat{z}_i} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{\sigma}_m \frac{z^{m+n}}{\hat{z}_i^n} = \frac{1}{\hat{z}_i} \sum_{k=0}^{\infty} \left[\frac{z}{\hat{z}_i}\right]^k \sum_{m=0}^{k} \hat{\sigma}_m \hat{z}_i^m := \frac{1}{\hat{z}_i} \sum_{k=0}^{\infty} \left[\frac{z}{\hat{z}_i}\right]^k T_k\hat{\sigma}(\hat{z}_i) \tag{8.4}$$

where last step is a definition for $T_k\hat{\sigma}(\hat{z}_i)$, which is the power expansion truncated at k-th order of $\hat{\sigma}(z)$, computed in $\hat{z}_i$. Note that for k_0 large enough and for all $k > k_0$, $T_k\hat{\sigma}(\hat{z}_i) \neq 0$ Hence we got an expression for σ_k:

$$\sum \sigma_k z^k = \sum_k \left[\hat{\sigma}_k + \sum_I \sum_{i\in I} \frac{\Delta_I S(i,I) T_k\hat{\sigma}(\hat{z}_i)}{\hat{z}_i^{k+1}}\right] z^k \tag{8.5}$$

and therefore

$$\sigma_k = \hat{\sigma}_k + \sum_I \sum_{i\in I} \frac{\Delta_I S(i,I) T_k\hat{\sigma}(\hat{z}_i)}{\hat{z}_i^{k+1}} \tag{8.6}$$

Let us define the complex coefficients $a(I,i,k) := \Delta_I S(i,I) T_k\hat{\sigma}(\hat{z}_i)$. Clearly

$$\sup_I \sup_i \sup_k |a(I,i,k)| \leq Const$$

To show that the coefficient σ_k in (8.6) is exponentially divergent let us now focus the second term in right hand side of (8.6). We can sort the roots $\hat{z}_1, \hat{z}_2, \dots \hat{z}_{c-1}$ so that $|\hat{z}_1| < |\hat{z}_2| \leq \cdots \leq |\hat{z}_{c-1}|$.
Assume first that $\hat{z}_1$ is a real root, then the integral in right hand side of (8.6) by after elementary algebraic computation yields

$$
\begin{aligned}
\sum_I \sum_{i\in I} \frac{a(I,i,k)}{\hat{z}_i^{k+1}} &= \frac{1}{\hat{z}_1^{k+1}} \sum_I \sum_{i\in I} \frac{a(I,i,k)}{\left(\frac{\hat{z}_i}{\hat{z}_1}\right)^{k+1}} \\
&= \frac{1}{\hat{z}_1^{k+1}} \left[\sum_{I\ni 1} \left[a(I,1,k) + \sum_{j\in I} \frac{a(I,j,k)}{\left(\frac{\hat{z}_j}{\hat{z}_1}\right)^{k+1}} \right] + \sum_{I\not\ni 1} \sum_{i\in I} \frac{a(I,i,k)}{\left(\frac{\hat{z}_i}{\hat{z}_1}\right)^{k+1}} \right] \\
&= \frac{1}{\hat{z}_1^{k+1}} \left[\sum_{I\ni 1} a(I,1,k) + o\left(\frac{\hat{z}_1}{\hat{z}_j}\right)^{k+1} \right]
\end{aligned}
$$

We need to show that $|\sum_{I\ni 1} a(I,1,k)| \neq 0$. To show this we can write $\sum_{I\ni 1} a(I,1,k)$ in following way

$$
\sum_{I\ni 1} a(I,1,k) = \sum_{I\ni 1} \Delta_I S(1,I) T_k \hat{\sigma}(\hat{z}_1) = T_k \hat{\sigma}(\hat{z}_1) \sum_{I\ni 1} \Delta_I S(1,I) \tag{8.7}
$$

$$
= T_k \hat{\sigma}(\hat{z}_1) \sum_{I\ni 1} \Delta_1 \prod_{\substack{j\in I \\ j\neq 1}} \frac{\Delta_j}{\hat{z}_j - \hat{z}_1} = T_k \hat{\sigma}(\hat{z}_1) \Delta_1 \prod_{j=2}^{c-1} \left[\frac{\Delta_j}{\hat{z}_j - \hat{z}_1} + 1 \right] = T_k \hat{\sigma}(\hat{z}_1) \Delta_1 \prod_{j=2}^{c-1} \rho_j e^{i\theta_j} \tag{8.8}
$$

Now we take the norm of the last term

$$
|T_k \hat{\sigma}(\hat{z}_1) \Delta_1| \prod_{j=2}^{c-1} \rho_j \tag{8.9}
$$

Clearly $\rho_j \neq 0$ unless $\Delta_1 = \hat{z}_1 - \hat{z}_j$, thus $|\sum_{I\ni 1} a(I,1,k)| \neq 0$.
Let us now consider the case in which $\hat{z}_1$ and $\hat{z}_2$ are complex conjugate and they are the smallest zero in I; clearly $|\hat{z}_1| = |\hat{z}_2|$. We can partition the set I of subset $\{1, 2, \dots, c-1\}$ by $I_1 = \{I | I \in 1, I \not\ni 2\}$, $I_2 = \{I | I \not\ni 1, I \in 2\}$, $I_{12} = \{I | I \in 1, I \in 2\}$ and $I_0 = \{I | I \not\ni 1, I \not\ni 2\}$. Then in this case the second term in right of (8.6) becomes

$$
\begin{aligned}
\sum_{I}\sum_{i\in I}\frac{a(I,i,k)}{\hat{z}_i^{k+1}} &= \sum_{I_1}\sum_{i\in I_1}\frac{a(I_1,i,k)}{\hat{z}_i^{k+1}} + \sum_{I_2}\sum_{i\in I_2}\frac{a(I_2,i,k)}{\hat{z}_i^{k+1}} + \\
&+ \sum_{I_{12}}\sum_{i\in I_{12}}\frac{a(I_{12},i,k)}{\hat{z}_i^{k+1}} + \sum_{I_0}\sum_{i\in I_0}\frac{a(I_0,i,k)}{\hat{z}_i^{k+1}} \\
&= \frac{1}{\hat{z}_1^{k+1}}\sum_{I_1}\sum_{i\in I_1}\frac{a(I_1,i,k)}{\left(\frac{\hat{z}_i}{\hat{z}_1}\right)^{k+1}} + \frac{1}{\hat{z}_2^{k+1}}\sum_{I_2}\sum_{i\in I_2}\frac{a(I_2,i,k)}{\left(\frac{\hat{z}_i}{\hat{z}_2}\right)^{k+1}} + \sum_{I_0}\sum_{i\in I_0}\frac{a(I_0,i,k)}{\hat{z}_i^{k+1}} + \\
&+ \sum_{I_{12}}\left[\frac{a(I_{12},1,k)}{\hat{z}_1^{k+1}} + \frac{a(I_{12},2,k)}{\hat{z}_2^{k+1}} + \sum_{1,2\neq i\in I_{12}}\frac{a(I_{12},i,k)}{\hat{z}_i^{k+1}}\right] \\
&= \frac{1}{\hat{z}_1^{k+1}}\sum_{I_1}\left[a(I_1,1,k) + o\left(\frac{\hat{z}_1}{\hat{z}_i}\right)^{k+1}\right] + \frac{1}{\hat{z}_2^{k+1}}\sum_{I_2}\left[a(I_2,2,k) + o\left(\frac{\hat{z}_2}{\hat{z}_i}\right)^{k+1}\right] + \\
&+ \sum_{I_{12}}\left[\frac{a(I_{12},1,k)}{\hat{z}_1^{k+1}} + \frac{a(I_{12},2,k)}{\hat{z}_2^{k+1}}\right] + \sum_{I_{12}}\sum_{\substack{i\in I_{12}\\ i\neq 1,2}}\frac{a(I_{12},i,k)}{\hat{z}_i^{k+1}} + \sum_{I_0}\sum_{i\in I_{12}}\frac{a(I_0,i,k)}{\hat{z}_i^{k+1}}
\end{aligned}
$$

Neglecting now the terms in $o\left(\frac{\hat{z}_1}{\hat{z}_i}\right)^{k+1}$ and $o\left(\frac{\hat{z}_2}{\hat{z}_i}\right)^{k+1}$

$$
\begin{aligned}
\sum_{I}\sum_{i\in I}\frac{a(I,i,k)}{\hat{z}_i^{k+1}} &= \frac{1}{\hat{z}_1^{k+1}}\left[\sum_{I_1}a(I_1,1,k) + \sum_{I_{12}}a(I_{12},1,k)\right] + \\
&+ \frac{1}{\hat{z}_2^{k+1}}\left[\sum_{I_2}a(I_2,2,k) + \sum_{I_{12}}a(I_{12},2,k)\right] \\
&= \frac{1}{\hat{z}_1^{k+1}}\sum_{I\ni 1}a(I,1,k) + \frac{1}{\hat{z}_2^{k+1}}\sum_{I\ni 2}a(I,2,k) \\
&= \frac{1}{\hat{z}_1^{k+1}}\sum_{I\ni 1}\Delta_1\prod_{\substack{j\in I\\ j\neq 1}}\frac{\Delta_j}{\hat{z}_j-\hat{z}_1}T_k\hat{\sigma}(\hat{z}_1) + \frac{1}{\hat{z}_2^{k+1}}\sum_{I\ni 2}\Delta_2\prod_{\substack{j\in I\\ j\neq 2}}\frac{\Delta_j}{\hat{z}_j-\hat{z}_2}T_k\hat{\sigma}(\hat{z}_2) \\
&= \frac{T_k\hat{\sigma}(\hat{z}_1)}{\hat{z}_1^{k+1}}\Delta_1\prod_{j\neq 1}\left[\frac{\Delta_j}{\hat{z}_j-\hat{z}_1}+1\right] + \frac{T_k\hat{\sigma}(\hat{z}_2)}{\hat{z}_2^{k+1}}\Delta_2\prod_{j\neq 2}\left[\frac{\Delta_j}{\hat{z}_j-\hat{z}_2}+1\right] \\
&= \frac{T_k\hat{\sigma}(\hat{z}_1)}{\hat{z}_1^{k+1}}\Delta_1\prod_{j\neq 1}\frac{1}{\hat{z}_j-\hat{z}_1}\prod_{j\neq 1}(\Delta_j+\hat{z}_j-\hat{z}_1) + \\
&+ \frac{T_k\hat{\sigma}(\hat{z}_2)}{\hat{z}_2^{k+1}}\Delta_2\prod_{j\neq 2}\frac{1}{\hat{z}_j-\hat{z}_2}\prod_{j\neq 2}(\Delta_j+\hat{z}_j-\hat{z}_2)
\end{aligned}
$$

Since the roots are real or pairs of complex conjugates we have that

$$
\left|\prod_{j\neq 1}\frac{1}{\hat{z}_j-\hat{z}_1}\right| = \left|\prod_{j\neq 2}\frac{1}{\hat{z}_j-\hat{z}_2}\right| \tag{8.10}
$$

Then we deduce that for

$$|\Delta_1 \prod_{j\neq 1}(\Delta_j + \hat{z}_j - \hat{z}_1)| \neq |\Delta_2 \prod_{j\neq 2}(\Delta_j + \hat{z}_j - \hat{z}_2|) \tag{8.11}$$

the term $\sum_I \sum_{i\in I} \frac{a(I,i,k)}{\hat{z}_i^{k+1}}$ grows exponentially for k large enough.

Multiple roots: Let us now consider the case of multiple roots. The function $g(z)$ becomes

$$\begin{aligned}
g(z) &= \frac{(z-\hat{z}_1-\Delta_1)^{k_1}\cdots(z-\hat{z}_m-\Delta_m)^{k_m}}{(z-\hat{z}_1)^{k_1}\cdots(z-\hat{z}_m)^{k_m}} && (8.12)\\
&= \sum_{I\subseteq\{1,\dots,m\}} \sum_{l_1,\dots,l_{|I|}=1}^{k_1,\dots,k_{|I|}} \prod_{i\in I}\binom{k_i}{l_i}\left(\frac{\Delta_i}{\hat{z}_i - z}\right)^{l_i} = \\
&= \sum_I \sum_{l_I} \prod_{i\in I} \frac{b(k_i,l_i,\Delta_i)}{(\hat{z}_i-z)^{l_i}} = && (8.13)\\
&= \sum_I \sum_{l_I} \sum_{i\in I} \sum_{j=1}^{l_i} \frac{A_j^{(i)}}{(\hat{z}_i-z)^j} = && (8.14)\\
&= \sum_I \sum_{i\in I} \sum_{j=1}^{k_i} \frac{C_j^{(i)}}{(\hat{z}_i-z)^j} && (8.15)
\end{aligned}$$

where: 1) the first sum in each term is over all ordered subsets of indices, i.e. $I = \{i_1,\dots,i_k\}$ with $i_j < i_{j+1}$, with the understanding that, for $I=\emptyset$, the empty product is 1 and $\sum_{i=1}^m k_i = c-1$;
2) the Heaviside *Cover-up* method applies on the fourth equality;
3) we defined $b(k_i,l_i,\Delta_i)$, $A_j^{(i)}$ and $C_j^{(i)}$

$$b(k_i,l_i,\Delta_i) := \binom{k_i}{l_i}{\Delta_i}^{l_i} \qquad A_j^{(i)} := \prod_{i\in I} b(k_i,l_i,\Delta_i)B_j^{(i)} \qquad C_j^{(i)} := \sum_{k_i\geq l_i\geq j} A_j^{(i)}$$

where

$$B_j^{(i)} = \prod_{\substack{k\in I\\ k\neq i}}\left(\frac{1}{\hat{z}_k-\hat{z}_i}\right)^{l_k} \sum_{\substack{p_1\cdots p_{|I|}\geq 0\\ \sum_k p_1\dots p_{|I|}=l_i-j}} \frac{1}{p_i!}\prod_{\substack{k\in I\\ k\neq i}}\left[\binom{l_k+p_k-1}{l_k-1}\frac{1}{(\hat{z}_k-\hat{z}_i)^{p_k}}\right] \tag{8.16}$$

From (8.1) and (8.12) we have

$$\sigma(z) = \hat{\sigma}(z) + \sum_{I\neq\emptyset}\sum_{i\in I}\sum_{j=1}^{k_i} \frac{C_j^{(i)}\hat{\sigma}(z)}{(\hat{z}_i-z)^j} \tag{8.17}$$

Now we study the term $\hat{\sigma}(z)\left(\frac{1}{\hat{z}_i-z}\right)^j$ for any I and $i\in I$. For the sake of simplicity in this proof we consider the case $k_i\leq 2$, but the case $k_i \geq 2$ can be easily treated along the same way.

$$\hat{\sigma}(z)\left(\frac{1}{\hat{z}_i - z}\right)^2 = \hat{\sigma}(z)\frac{d}{dz}\left(\frac{1}{\hat{z}_i - z}\right) = \hat{\sigma}(z)\frac{1}{\hat{z}_i}\frac{d}{dz}\sum_{k=0}^{\infty}\left(\frac{z}{\hat{z}_i}\right)^k = \frac{1}{\hat{z}_i^2}\sum_{m=0}^{\infty}\hat{\sigma}_m z^m \sum_{k=0}^{\infty}(k+1)\left(\frac{z}{\hat{z}_i}\right)^k$$
$$= \frac{1}{\hat{z}_i^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n-m+1)\frac{\hat{\sigma}_m}{\hat{z}_i^{n-m}}z^n = \sum_{n=0}^{\infty}\left(\frac{1}{\hat{z}_i}\right)^{n+2}\left[(n+1)T_n\hat{\sigma}(\hat{z}_i) - \hat{z}_i\frac{d}{dz}T_n\hat{\sigma}(z)\Big|_{z=\hat{z}_i}\right] \tag{8.18}$$

Hence equation (8.17) in this case becomes

$$\sigma(z) = \sum_{k=0}^{\infty}\left[\hat{\sigma}_k + \sum_{I\neq\emptyset}\sum_{i\in I}\frac{1}{\hat{z}_i^{k+1}}\left\{C_1^{(i)}T_k\hat{\sigma}(\hat{z}_i) + C_2^{(i)}\frac{1}{\hat{z}_i}\left((k+1)T_k\hat{\sigma}(\hat{z}_i) - \hat{z}_i\frac{d}{dz}T_k\hat{\sigma}(\hat{z})\Big|_{z=\hat{z}_i}\right)\right\}\right]z^k$$

thus

$$\sigma_k = \hat{\sigma}_k + \sum_{I\neq\emptyset}\sum_{i\in I}\frac{1}{\hat{z}_i^{k+1}}\left\{C_1^{(i)}T_k\hat{\sigma}(\hat{z}_i) + C_2^{(i)}\frac{1}{\hat{z}_i}\left((k+1)T_k\hat{\sigma}(\hat{z}_i) - \hat{z}_i\frac{d}{dz}T_k\hat{\sigma}(\hat{z})\Big|_{z=\hat{z}_i}\right)\right\} \tag{8.19}$$

From now on, the control of the lower bound of σ_k proceeds as in the case of the simple roots. The final condition is the same since the leading term in (8.19) is $C_2^{(i)}\frac{1}{\hat{z}_i}(k+1)T_k\hat{\sigma}(\hat{z}_i)$. This concludes the proof.

□

3.9 Average queue size and variance

Consider the generating function (3.11) obtained in section 2. In order to simplify the calculation we rewrite (3.11) in the following way

$$P(z) = \frac{N(z)}{D(z)} \qquad |z| \leq 1 \tag{9.1}$$

where

$$\begin{aligned} N(z) &= \sum_{l=0}^{c}\sum_{k=0}^{l-1}\alpha_l P_k(z^{k-l} - 1) \\ D(z) &= -\frac{1}{q(z)} + \sum_{l=0}^{c}\frac{\alpha_l}{z^l} \end{aligned} \tag{9.2}$$

Let L be the average queue size (expected number of users in the system at steady state) given by definition

$$L = \lim_{z\to 1}P'(z) = P'(z)\big|_{z=1} = \lim_{z\to 1}\frac{N'(z)D(z) - N(z)D'(z)}{D(z)^2} \tag{9.3}$$

To calculate $P'(z)\big|_{z=1}$ we apply De Hôpital's rule so as to get

$$\lim_{z\to 1}P'(z) = \lim_{z\to 1}\frac{1}{2}\Big[\frac{N''(z)}{D'(z)} - \frac{N(z)}{D(z)}\frac{D''(z)}{D'(z)}\Big] \tag{9.4}$$

Hence from 9.4 and by the boundary condition $P(z)\big|_{z=1} = 1$ we get

$$P'(z)\big|_{z=1} = \frac{1}{2}\Big[\frac{N''(z)}{D'(z)} - \frac{D''(z)}{D'(z)}\Big]_{z=1} \tag{9.5}$$

where

$$\begin{aligned}
N'(z) &= \sum_{l=0}^{c}\sum_{k=0}^{l-1}(k-l)P_k\alpha_l z^{k-l-1} \\
N''(z) &= \sum_{l=0}^{c}\sum_{k=0}^{l-1}(k-l)(k-l-1)P_k\alpha_l z^{k-l-2} \\
D'(z) &= \frac{q'(z)}{(q(z))^2} - \sum_{l=1}^{c} l\alpha_l z^{-(l+1)} \\
D''(z) &= \frac{q''(z)}{(q(z))^2} - \frac{2(q'(z))^2}{(q(z))^3} + \sum_{l=1}^{c} l(l+1)\alpha_l z^{-(l+2)}
\end{aligned} \tag{9.6}$$

After few calculations we find

$$L = \frac{\sum_{l=0}^{c}\sum_{k=0}^{l-1}(k-l)(k-l-1)P_k\alpha_l - q''(1) + 2(q'(1))^2 - \sum_{l=1}^{c} l(l+1)\alpha_l}{2(q'(1) - \sum_{l=1}^{c} l\alpha_l)} \tag{9.7}$$

in the Poisson case we have

$$L_P = \frac{\sum_{l=0}^{c}\sum_{k=0}^{l-1}(k-l)(k-l-1)P_k\alpha_l + \rho_2^2 - \sum_{l=1}^{c} l(l+1)\alpha_l}{2(\rho_2 - \sum_{l=1}^{c} l\alpha_l)} \tag{9.8}$$

3.9.1 Variance

By definition the variance is given by

$$Var = P''(1) + P'(1) - P'(1)^2 \tag{9.9}$$

To compute it, we need $P''(z)\big|_{z=1}$:

$$P''(z) = \frac{N''(z)D(z)^2 - N(z)D(z)D''(z) - 2N'(z)D'(z)D(z) + 2N(z)D'(z)^2}{D(z)^3} \tag{9.10}$$

To find $P''(z)\big|_{z=1}$ we use De L'Hôpital rule

$$\begin{aligned}
&P''(z)\big|_{z=1} = \\
&= \frac{N'''(z)D(z)^2 - 3D''(z)\Big[N'(z)D(z) - N(z)D'(z)\Big] - N(z)D(z)D'''(z)}{3D(z)^2 D'(z)}\Bigg|_{z=1} \\
&= \frac{N'''(z) - 3D''(z)\Big[\frac{N'(z)D(z)-N(z)D'(z)}{D(z)^2}\Big] - \frac{N(z)}{D(z)}D'''(z)}{3D'(z)}\Bigg|_{z=1}
\end{aligned}$$

where

$$
\begin{aligned}
N'''(z) &= \sum_{l=0}^{c}\sum_{k=0}^{l-1}(k-l)(k-l-1)(k-l-2)P_k\alpha_l z^{k-l-3} \\
D'''(z) &= \frac{q'''(z)}{(q(z))^2} - \frac{6q'(z)q''(z)}{(q(z))^3} + \frac{6(q'(z))^3}{(q(z))^4} - \sum_{l=1}^{c} l(l+1)(l+2)\alpha_l z^{-(l+2)}
\end{aligned}
\tag{9.11}
$$

hence

$$
P''(z)\big|_{z=1} = \frac{1}{3D'(1)}\Big[N'''(1) - 3D''(1)P'(1) - P(1)D'''(1)\Big] \tag{9.12}
$$

By 9.12 and 9.3 we get

$$
\begin{aligned}
V \;=\; & \frac{\sum_{l=0}^{c}\sum_{k=0}^{l-1}(k-l)(k-l-1)(k-l-2)P_k\alpha_l}{3(q'(1)-\sum_{l=1}^{c} l\alpha_l)} + L - L^2 \\
& \frac{-3L(q''(1) - 2(q'(1))^2 - \sum_{l=1}^{c} l(l+1)\alpha_l) - q'''(1)}{3(q'(1)-\sum_{l=1}^{c} l\alpha_l)} + \\
& + \frac{6q'(1)q''(1) - 6(q'(1))^3 + \sum_{l=1}^{c} l(l+1)(l+2)\alpha_l}{3(q'(1)-\sum_{l=1}^{c} l\alpha_l)}
\end{aligned}
$$

in the Poisson case we have

$$
\begin{aligned}
V_P \;=\; & \frac{\sum_{l=0}^{c}\sum_{k=0}^{l-1}(k-l)(k-l-1)(k-l-2)P_k\alpha_l}{3(\rho_2-\sum_{l=1}^{c} l\alpha_l)} \\
& \frac{-3L(\rho_2^2 - \sum_{l=1}^{c} l(l+1)\alpha_l) - \rho_2^3 + \sum_{l=1}^{c} l(l+1)(l+2)\alpha_l}{3(\rho_2-\sum_{l=1}^{c} l\alpha_l)} + L_P - L_P^2
\end{aligned}
$$

Remark 3.9.1. *Note that only the first c values of the distribution of the P_n appear in the expected queue length and in its variance .*

3.10 Numerical results

In this section we will present some numerical results obtained with the approximation define in section 3.5, moreover we calculate the average queue size and its variance as a function of the first c probabilities $P_0, \ldots, P_{c-1}$. The approximated (truncated) system (5.7), as described on section 3.5, can be written as

$$
(\mathbf{QA} - \mathbf{I})\,\sigma = \mathbf{b} \tag{10.1}
$$

where Q, A, and b are given below and I is the identity matrix.

$$
\mathbf{Q} = \begin{pmatrix} q_0 & 0 & 0 & \ldots & 0 \\ q_1 & q_0 & 0 & \ldots & 0 \\ q_2 & q_1 & q_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_n & q_{n-1} & q_{n-2} & \ldots & q_0 \end{pmatrix} \qquad
\mathbf{A} = \begin{pmatrix} \alpha_0 & \ldots & \alpha_c & 0 & \ldots \\ 0 & \alpha_0 & \ldots & \alpha_c & \ldots \\ 0 & 0 & \alpha_0 & \ldots & \alpha_c \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \alpha_0 \end{pmatrix} \tag{10.2}
$$

$$
\mathbf{b}^T = \Big(0, \cdots, 0, -q_0\alpha_c, -q_0\alpha_{c-1} - (q_0+q_1)\alpha_c, \cdots, -q_0\alpha_1 - \ldots - (q_0 + \ldots + q_{c-1})\alpha_c\Big)
$$

Such a system can be solved numerically, so that we can get the values of the first c cumulative probabilities σ_n and, by definition of σ_n, the values of the first c probabilities $P_0, \ldots, P_{c-1}$ (at least to six decimals). We'll show in two tables the values of σ_n listed as a function of the dimension n of the linear system. In both tables the values of the parameters $\{c, \rho_1, \rho_2\}$ are such that

$$\frac{c - \rho_1 - \rho_2}{c} << 1$$

and therefore the system is very close to its critical condition. It is clear that the value $\bar{n}$ such that the exact probabilities are found up to the requested precision is very sensitive to the value of $\rho = \rho_1 + \rho_2$, but $\bar{n}$ remains reasonable small even for very small values of $\frac{c-\rho_1-\rho_2}{c}$

c	ρ_1	ρ_2	$\bar{n}$	σ_0	σ_1	σ_2	σ_3	σ_4
5	1.5	3.2	15	0.011935	0.054177	0.132078	0.233745	0.34185
5	1.5	3.2	25	0.01057	0.04798	0.116971	0.20701	0.30275
5	1.5	3.2	35	0.010263	0.046583	0.113565	0.200982	0.293935
5	1.5	3.2	45	0.010185	0.04623	0.112705	0.199459	0.291708
5	1.5	3.2	55	0.010165	0.046139	0.112481	0.199064	0.291129
5	1.5	3.2	65	0.010159	0.046115	0.112423	0.19896	0.290978
5	1.5	3.2	75	0.010158	0.046108	0.112408	0.198933	0.290938
5	1.5	3.2	85	0.010158	0.046107	0.112404	0.198926	0.290928
5	1.5	3.2	95	0.010157	0.046106	0.112403	0.198924	0.290925
5	1.5	3.2	105	0.010157	0.046106	0.112402	0.198924	0.290924
5	1.5	3.2	115	0.010157	0.046106	0.112402	0.198924	0.290924
c	ρ_1	ρ_2	$\bar{n}$	P_0	P_1	P_2	P_3	P_4
5	1.5	3.2	115	0.010157	0.035949	0.066296	0.086522	0.092000

Table 3.1:

c	ρ_1	ρ_2	$\bar{n}$	σ_0	σ_1	σ_2	σ_3	σ_4
5	3.	1.7	15	0.086316	0.25791	0.444882	0.600331	0.715614
5	3.	1.7	25	0.085818	0.256424	0.442318	0.596871	0.71149
5	3.	1.7	35	0.085801	0.256371	0.442227	0.596749	0.711344
5	3.	1.7	45	0.0858	0.25637	0.442224	0.596744	0.711339
5	3.	1.7	55	0.0858	0.25637	0.442224	0.596744	0.711339
5	3.	1.7	65	0.0858	0.25637	0.442224	0.596744	0.711339
5	3.	1.7	75	0.0858	0.25637	0.442224	0.596744	0.711339
5	3.	1.7	85	0.0858	0.25637	0.442224	0.596744	0.711339
5	3.	1.7	95	0.0858	0.25637	0.442224	0.596744	0.711339
5	3.	1.7	105	0.0858	0.25637	0.442224	0.596744	0.711339
5	3.	1.7	115	0.0858	0.25637	0.442224	0.596744	0.711339
c	ρ_1	ρ_2	$\bar{n}$	P_0	P_1	P_2	P_3	P_4
5	3	1.7	115	0.0858	0.17057	0.185854	0.15452	0.114595

Table 3.2:

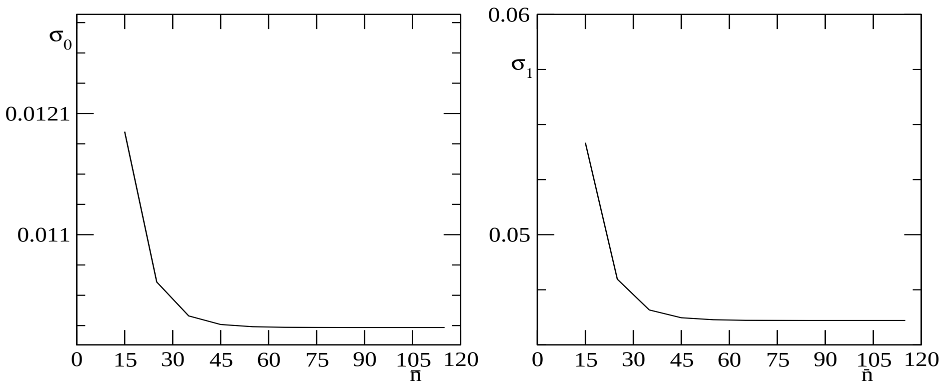

Figure 3.1: On the left σ_0 vs $\bar{n}$, on the right σ_1 vs $\bar{n}$

The following numerical results show that the errors on the first c components of the stationary probability distribution of the finite Markov Chain by augmenting the last column are smaller than the errors on the the first c components of the stationary probability distribution of the censored Markov Chain. This fact shows that the censoring is an augmentation method optimal in the sense of the L_1 norm, but may lead to approximations far from optimal when the average values of the relevant quantities of the system are computed.

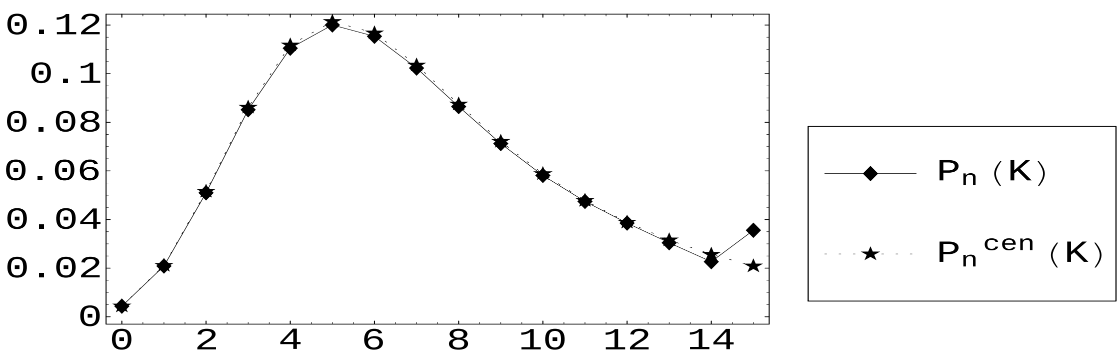

Figure 3.2:

3.11 Conclusions

The results of this chapter are a first encouraging attempt to describe a queueing system with a variable number of servers. We want to outline that the good numerical results we obtained are explained from an analytical point of view, and the accuracy of the approximation is in this case completely under control. Since the expected length of the queue and its variance are linear in the first c probabilities computed in our approximation scheme, it is trivial to give an a priori estimate of the error in the expectations computed in this way.

The work presented in this chapter could be generalized in order to describe systems interesting in the applications. In particular the explicit study of the case of bulk arrivals can be easily performed, and other interesting generalizations are possible. For example the

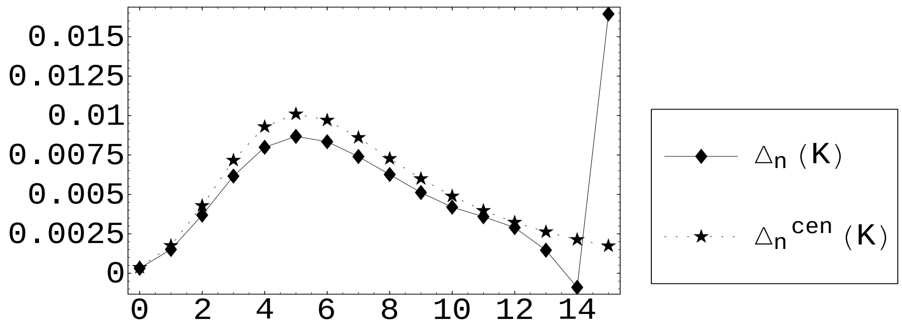

Figure 3.3:

number of servers may vary accordingly to a Markov chain. These generalizations will be the subject of further studies.

Chapter 4

Two class queue in parallel: Priority and Bernoulli scheduling

We study a discrete time single server system with generic independent distribution of the number of arrivals in a time slot and two classes of users. We consider two models: the first one has geometric distribution of the service time and the user of second class can be served only if the users of the first class are absent. The users can be served only when each time slot begins. We give a complete description of this model. The second model has deterministic sevice times and the server serves the user of first class with probability p ($0 \leq p \leq 1$) and serves the users of second class with probability $1-p$. This model is more complicated to study due to the complicated structure of the generating functional equation. We give the approximation solution of the generating functional equation using perturbative method by starting from the exact solution of first model. Moreover we give some numerical results.

4.1 Introduction

This chapter is a simplified description of the following problem: the air traffic is composed of different types of aircrafts, implying different queueing costs as well or different sevice times. Hence a non realistic description has to take into account the existence of different classe of users. We will introduce the variable service time simply assuming geometric distribution. In this chapter we study two discrete time models: the first is GI/Geom/1 queueing model, when the users belong to two different priority classes and the first class has to be served with priority. The second is the single server model Bernoulli scheduling with two different class of users. In both cases the resulting generating function is not simply the product of two functions, one for each class of users. Even it the first model presented in this chapter is clearly simple, this computation shows that in discrete time queues it is possible to solve the boundary value problem, always arising in multiclass queues, when the service discipline is a simple priority rule. In other words, this work points out that: 1) even for a simple priority rule the generating function does not factorize, preventing the possibility to treat the systems as the superposition of two independent probability distributions. 2) A judicious treatment of the zeroes of the denominator of the generating function allows nevertheless to control completely the problem. This is not completely trivial, because the zeroes lie on a manifold, but it can be solved.

These problems, in context of discrete time services, are widely studied in literature (see e.g.[32]) Some amount of results about geometric or general service time distribution appeared in literature. Some of these results deals with the GI/Geom/c system, with general arrivals, geometric service time and c servers (see [33, 34, 35]), but various generalization of this model has been proposed. For example in [36] a model with variable arrival rates has been studied, in [37] has been attached the problem of random vacations of the server, while [39] and [41] deal with finite buffers models. All these papers study systems with a single class of users. The problem to describe a system with more than one class of users has been attached in [38], where however the generatingfunction has to be of product form.

The second model is also simple, but more complicated to study because the generating functional equation contains two unknown function that are not easily to find. Usually these problems are solved by the means of the solution of a boundary value problem (see e.g. [15]). In this chapter we propose an alternative method to study these generating functionals that is a perturbative method, starting from the exact solution of generating functional of first model. In this way we obtain an approximation solution of generating functional which give a very good results for lower traffic intensity, and is accurate also in some cases with heavy traffic intensity.

A more complex and realistic model can be studied, taking into account the structure of the arrivals and the number of user classes, which in general is greater than two. However for simple priority the structure of the zeroes of the denominator of the generating function is invertible, and therefore the resulting queueing model will be more complicated but conceptually analogous to the one presented here. Clearly, the possibility to compute analytically the stationary distribution of the system has the great advantage that the dependence on the relevant parameters is much more clear and useful than in a simulative approach.

The chapter is organized as follows: in section 4.2 we describe the first model, we write the evolution equations of the joint probabilities of the process and we write the two dimensional generating function of the system. The generating function so obtained can be completely determined solving a boundary value problem. In section 4.3 we study the derivatives of the generating function around the point $(1, 1)$, in order to obtain by elementary considerations the expected value of the length of the queues of the two classes of users. In section 4.4 we give the complete form of the generating function, solving the boundary value problem, by means of the explicit computation of the zeroes of the kernel of the generating function. In section 4.5 we describe the second model. In section 4.6 we derive the functional equation under stationary condition. In section 4.7 we study the generating function using perturbative method. In section 4.8 we give the approximation formula for average queue length of both classes of users. In section 4.9 we present some numerical results in order to check the validity of perturbative method on the second model. The last section is devoted to the summary of our results and to the description of some possible development of these models.

4.2 The model I: Discrete time GI/Geom/1 queueing system with priority

In our system two classes of users arrive to a single server, according to a general stochastic process, that may be obviously different for the two classes. The server may start his service only at discrete time $t_l = lD$, and hence the service time can be always decomposed in a integer number of time slots. Without loss of generality we will assume in the rest of the paper $D = 1$, i.e unit service time slots. The service time for each class of users is distributed according to a geometrical probability distribution. This means that the probability $P_k^{(i)}$ that the service time of the users of the class i, $i = 1, 2$ has length k is given by the expression

$$P_k^{(i)} = (1 - \mu_i)^{k-1}\mu_i \tag{2.1}$$

Note that denoting with $t_s^{(i)}$ the service time of the class i we have

$$\mathbb{E}(t_s^{(i)}) = \sum_{k\geq 0} kP_k^{(i)} = \sum_{k\geq 0} k(1-\mu_i)^{k-1}\mu_i = -\mu_i \sum_{k\geq 0} \frac{d}{d\mu_i}(1-\mu_i)^k = -\mu_i \frac{d}{d\mu_i}\left(\frac{1}{\mu_i}\right) = \frac{1}{\mu_i} \tag{2.2}$$

and hence $\mu_i < 1$ is the service rate of the users of the class i. Note also that the distribution (2.1) means that each user has probability μ_i independent on its past to finish its service at the end of each time slot. Since the time is discretized by the server, the relevant description of the arrival process is the distribution of the number of arrivals in each time slot. We assume that the number of arrivals in each time slot are independent realization of a well defined random variable with integer values. We will denote by $q_k^{(i)}$ the (stationary) probability to have in a single time slot k arrivals of users of class i. In this paper the arrivals of the two classes are therefore independent, but this assumption may be weakened, see section 4.10 for more details. As we will see in section 4.3, in order to compute the expected value of the length of the queues we just need to know the expected value $\lambda_i = \sum_{k\geq 0} kq_k^{(i)}$ of the number of arrivals of users of class i in each time slot, and its variance σ_i^2. The users are served on the basis of a FIFO discipline, but a user of class 2 may start its service at the beginning of a time slot only if there are no users of class 1 in the system. The service of a user of class 2 is interrupted when a time slot begins if in the previous time slot a user of class 1 arrived, and it starts again when the queue of users of class 1 is empty We will denote with $P_{m,n}^l$ the probability to have m users of class 1 and n users of class 2 in the system immediately before the beginning of the time slot $[l, l+1]$, and with $L_{m,n}^l$ the probability to have m users of class 1 and n users of class 2 in the system immediately after the beginning of the time slot $[l, l+1]$. By the description above it is easy to see that the system obeys to the following evolution equations

$$P_{m,n}^{l+1} = \sum_{i=0}^{m}\sum_{j=0}^{n} L_{i,j}^l q_{m-i}^{(1)} q_{n-j}^{(2)} \tag{2.3}$$

where

$$\begin{aligned}
L_{m,n}^l &= P_{m,n}^l(1-\mu_1) + P_{m+1,n}^l\mu_1 \qquad \text{for } m > 0, n \geq 0\\
L_{0,n}^l &= P_{1,n}^l\mu_1 + P_{0,n}^l(1-\mu_2) + P_{0,n+1}^l\mu_2 \quad \text{for } n > 0\\
L_{0,0}^l &= P_{0,0}^l + P_{1,0}^l\mu_1 + P_{0,1}^l\mu_2
\end{aligned} \tag{2.4}$$

The stationary distribution has to solve the set of equations

$$P_{m,n}^{l+1} = P_{m,n}^l = P_{m,n}, \qquad L_{m,n}^{l+1} = L_{m,n}^l = L_{m,n} \tag{2.5}$$

So the system (2.4) becomes

$$\begin{aligned}
L_{m,n} &= P_{m,n}(1-\mu_1) + P_{m+1,n}\mu_1 \qquad \text{for } m > 0, n \geq 0\\
L_{0,n} &= P_{1,n}\mu_1 + P_{0,n}(1-\mu_2) + P_{0,n+1}\mu_2 \quad \text{for } n > 0\\
L_{0,0} &= P_{0,0} + P_{1,0}\mu_1 + P_{0,1}\mu_2
\end{aligned} \tag{2.6}$$

and therefore, defining the generating functions

$$P(x,y) = \sum_{m,n\geq 0} P_{m,n}x^m y^n, \quad L(x,y) = \sum_{m,n\geq 0} L_{m,n}x^m y^n \tag{2.7}$$

$$q(x,y) = q^{(1)}(x)q^{(2)}(y) = \sum_{m,n\geq 0} q_m^{(1)}q_n^{(2)}x^m y^n \tag{2.8}$$

we have by (2.3)

$$P(x,y) = L(x,y)q(x,y) \tag{2.9}$$

and by (2.4)
First we compute the $L(x,y)$. Now multiplying both side of (2.6) by x^m and y^n and summing over all m and n we have

$$\begin{aligned} L(x,y) &= \sum_{m\geq 1}\sum_{n\geq 0}(P_{m,n}(1-\mu_1)x^m y^n + P_{m+1,n}\mu_1 x^m y^n) + \\ &+ \sum_{n\geq 1}(P_{1,n}\mu_1 y^n + P_{0,n}(1-\mu_2)y^n + P_{0,n+1}\mu_2 y^n) + P_{0,0} + P_{1,0}\mu_1 + P_{0,1}\mu_2 \\ &= \sum_{m\geq 0}\sum_{n\geq 0}(P_{m,n}(1-\mu_1)x^m y^n + P_{m+1,n}\mu_1 x^m y^n) - \sum_{n=0}(P_{0,n}(1-\mu_1)y^n + P_{1,n}\mu_1 y^n) + \\ &+ \sum_{n\geq 0}(P_{1,n}\mu_1 y^n + P_{0,n}(1-\mu_2)y^n + P_{0,n+1}\mu_2 y^n) - P_{0,0}(1-\mu_2) + P_{0,0} \end{aligned} \tag{2.10}$$

After elementary algebra we get

$$L(x,y) = \left[1-\mu_1\left(1-\frac{1}{x}\right)\right]P(x,y) + \left[\mu_1\left(1-\frac{1}{x}\right) - \mu_2\left(1-\frac{1}{y}\right)\right]P(0,y) + \mu_2\left(1-\frac{1}{y}\right)P(0,0) \tag{2.11}$$

Now plugging this value of $L(x,y)$ into (2.9) to finally obtain

$$\left[\frac{xy}{q(x,y)} - \mu_1 y - (1-\mu_1)xy\right]P(x,y) = [\mu_1(xy-y) - \mu_2(xy-x)]P(0,y) + \mu_2(xy-x)P(0,0) \tag{2.12}$$

and hence

$$P(x,y) = \frac{[\mu_1(xy-y) - \mu_2(xy-x)]P(0,y) + \mu_2(xy-x)P(0,0)}{\frac{xy}{q(x,y)} - \mu_1 y - (1-\mu_1)xy} \tag{2.13}$$

As we see the functional equation (2.13) contain two unknown that $P(0,0)$ and $P(0,y)$. $P(0,0)$ it easy to calculate using the boundary condition $P(x,y)|_{(1,1)} = 1$ and remand the computation of $P(0,y)$ in section 4.4.

4.3 The average length of the queue

In this section we will compute the average number of users of the two classes in the system by a simple study of the behavior of the numerator of (2.13) around the point $(1,1)$, which is obviously a zero of the denominator. The next section will be devoted to the explicit computation of the function $P(0,y)$, i.e. to the solution of the (easy) boundary value problem represented by equation (2.12).

To fulfill this program we start with the normalization condition. Defining $N(x,y)$ and $D(x,y)$ by

$$N(x,y) = [\mu_1(xy-y) - \mu_2(xy-x)]P(0,y) + \mu_2(xy-x)P(0,0)$$
$$D(x,y) = \frac{xy}{q(x,y)} - \mu_1 y - (1-\mu_1)xy$$

and hence

$$P(x,y) = \frac{N(x,y)}{D(x,y)} \tag{3.1}$$

and denoting $\frac{\partial}{\partial x}f(x,y) = f_x(x,y)$, $\frac{\partial}{\partial y}f(x,y) = f_y(x,y)$, we have the obvious relations (l'Hopital rule)

$$\lim_{x,y\to 1,1} P(x,y) = \left.\frac{N_x(x,y)}{D_x(x,y)}\right|_{1,1} = \left.\frac{N_y(x,y)}{D_y(x,y)}\right|_{1,1} = 1 \tag{3.2}$$

To compute the derivatives in (3.2) we note that $q_x(1,1) = \lambda_1$, $q_y(1,1) = \lambda_2$, and denoting with $\varrho_i = \frac{\lambda_i}{\mu_i}$, $i = 1,2$, $\varrho = \varrho_1 + \varrho_2$, we get

$$1 = \left.\frac{N_x(x,y)}{D_x(x,y)}\right|_{1,1} = \frac{\mu_1 P(0,1)}{\mu_1 - \lambda_1} \Rightarrow P(0,1) = 1 - \varrho_1 \tag{3.3}$$

and equivalently

$$1 = \left.\frac{N_y(x,y)}{D_y(x,y)}\right|_{1,1} = \frac{-\mu_2 P(0,1) + \mu_2 P(0,0)}{-\lambda_2} \Rightarrow P(0,0) = 1 - \varrho \tag{3.4}$$

From a physical point of view the meaning of (3.3) and (3.4) is quite obvious, since they represent the traffic intensity in the system, respectively of users of class 1 and of total traffic. This implies that the existence condition of a stationary distribution is $\varrho < 1$. Now we compute the expected values of the length of the queues. We first note that

$$\begin{aligned} \sum_{m,n\geq 0} mP_{m,n} &\equiv N_1 = P_x(1,1) = \frac{N_{xx}(1,1) - D_{xx}(1,1)}{2D_x(1,1)} \\ \sum_{m,n\geq 0} nP_{m,n} &\equiv N_2 = P_y(1,1) = \frac{N_{yy}(1,1) - D_{yy}(1,1)}{2D_y(1,1)} \end{aligned} \tag{3.5}$$

where in the last equalities of (3.5) we made use of (3.2). Moreover we have that, always by l'Hopital rule and again by (3.2),

$$D_x(1,1)N_2 + D_y(1,1)N_1 = N_{xy}(1,1) - D_{xy}(1,1) \tag{3.6}$$

To compute N_1 and N_2 we need also the derivatives $q_{xx}(1,1) = \sigma_1^2 - \lambda_1(1-\lambda_1)$, $q_{yy}(1,1) = \sigma_2^2 - \lambda_2(1-\lambda_2)$. Then we get from (3.5)

$$\begin{aligned} N_1 &= \frac{\lambda_1(1-\lambda_1) + \sigma_1^2}{2(\mu_1 - \lambda_1)} \\ N_2 &= \frac{2\mu_2 P_y(0,1) - \lambda_2(1-\lambda_2) - \sigma_2^2}{2\lambda_2} \end{aligned} \tag{3.7}$$

and from (3.6)

$$(\mu_1-\lambda_1)N_2-\lambda_2 N_1 = (\mu_1-\mu_2)(1-\varrho_1)+\mu_1 P_y(0,1)+\mu_2(1-\varrho)-(1-\lambda_1)(1-\lambda_2)+1-\mu_1 \quad (3.8)$$

Hence we can write explicitly N_2 solving (3.7)-(3.8), which represent a linear system in the unknown $N_1,\ N_2$ and $P_y(0,1)$. We obtain

$$N_2 = \frac{1}{1-\varrho}\left[\frac{-(1-\lambda_1)(1-\lambda_2)+1-\mu_1}{\mu_1} + \frac{\lambda_2\lambda_1(1-\lambda_1)+\lambda_2\sigma_1^2}{2\mu_1(\mu_1-\lambda_1)} + \frac{\sigma_2^2}{2\mu_2}\right] + \quad (3.9)$$

$$+\left[\frac{\mu_2(1-\varrho)}{\mu_1} + \frac{\mu_1-\mu_2}{\mu_1}(1-\varrho_1) + \frac{\varrho_2}{2}(1-\lambda_2)\right] \quad (3.10)$$

$$(3.11)$$

By formula (3.7) we remark that the users of first class do not are influenced by presence of user of second class, so we can sei that the number of users of first class present in the system are the same of the single server queue $GI/Geom/1$. Note that in the simplified case of the service $M/D/1$ with priority, i.e. in the case of poissonian arrivals and deterministic services ($\mu_1 = \mu_2 = 1$), the formulas (3.7) and (3.9) becomes

$$N_1 = \frac{2\varrho_1-\varrho_1^2}{2(1-\varrho_1)}, \quad N_2 = \frac{\varrho_2}{2(1-\varrho_1)(1-\varrho)} + \varrho_2 \quad (3.12)$$

4.4 The boundary value problem

In order to obtain the complete stationary distribution of the system we need to compute the generic derivative of the generating function, according to the well-known relation

$$P_{mn} = \frac{d^m}{dx^m}\frac{d^n}{dy^n}P(x,y)\bigg|_{\substack{x=0\\y=0}} \quad (4.1)$$

To do this we have to impose that $P(x,y)$ is analytic for all x,y in the complex unit circle $D=\{x,y:|x|\le 1,|y|\le 1\}$. Since we shall easily see that the denominator $D(x,y)$ vanishes at least on a real curve $\mathcal{C}\subset D$, we have to impose that the numerator $N(x,y)$ vanishes on the same set . This will give us a condition on $P(0,y)$. We first show the existence of $\mathcal{C}$. For a given $0\le y<1$, $D(x,y)$ vanishes for the values of $0\le x<1$ satisfying

$$\frac{x}{q(x,y)} = \mu_1 + (1-\mu_1)x \quad (4.2)$$

i.e.

$$\frac{x}{\mu_1+(1-\mu_1)x} = q^{(1)}(x)\ q^{(2)}(y) \quad (4.3)$$

For $0\le y<1$ fixed, we can think to $q^{(2)}(y)<1$ as a fixed parameter. The function

$$f(x) = \frac{x}{\mu_1+(1-\mu_1)x} \quad (4.4)$$

has the following properties:

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f'(x) &= \frac{\mu_1}{(\mu_1+(1-\mu_1)x)^2} < \mu_1 \quad \forall\ 0<x<1 \end{aligned} \quad (4.5)$$

On the other side the function $q^{(1)}(x)\ q^{(2)}(y)$ satisfies

$$\begin{aligned}
q^{(1)}(0)\ q^{(2)}(y) &> 0 \\
q^{(1)}(1)\ q^{(2)}(y) &< 1 \quad \forall\ 0 \le y < 1 \\
\frac{d}{dx} q^{(1)}(x)\ q^{(2)}(y) &< \lambda_1 < \mu_1 \quad \forall\ 0\ \le x, y < 1
\end{aligned} \tag{4.6}$$

It is then obvious that (4.5) and (4.5) imply that, for each $0 \le y < 1$, (4.3) is satisfied for a unique $0 \le x < 1$. We shall denote such value of x with $x(y)$, i.e. $x(y)$ is the solution of

$$x(y) = \frac{\mu_1}{\mu_1 - 1 + q^{-1}(x(y), y)} \tag{4.7}$$

Note that in case of service $M/D/1$ with priority, we have

$$q(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_{mn} z^m z^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\varrho_1 x)^m}{m!} e^{-\varrho_1} \frac{(\varrho_2 y)^n}{n!} e^{-\varrho_2} z^m z^n = e^{\varrho_1(x-1)} e^{\varrho_2(y-1)} \tag{4.8}$$

so in this case the equation (4.2) becomes

$$x = e^{\varrho_1(x-1)} e^{\varrho_2(y-1)} \tag{4.9}$$

We remark that the root $x(y)$ in case of $M/D/1$ queue can be interpreted as the generating function of arrival process. In figure 4.1 we have plotted the solution of equation (4.9).

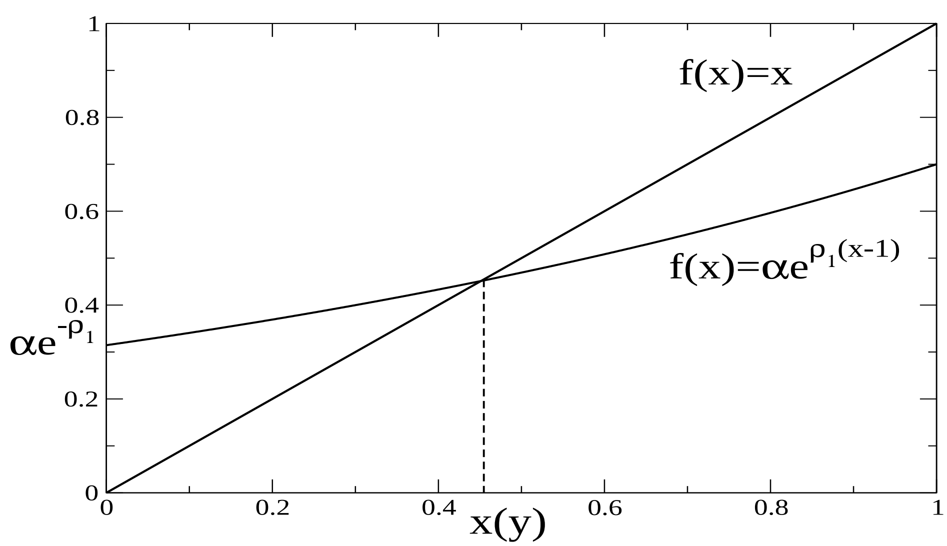

Figure 4.1: Graphical representation of the solution of (4.9).

It is now clear that in the points $(x(y), y)$ The numerator $N(x,y)$ has to vanish. This implies that

$$P(0,y) = \frac{\mu_2(1-y)(1-\varrho)}{\mu_1(y - y/x(y)) + \mu_2(1-y)} \tag{4.10}$$

This expression, together with (4.7), represents the solution of the boundary value problem given by the expression (2.13). In this case the solution of this problem had been possible because $P(x,y)$ is independent on $P(x,0)$, and therefore the singularity of the denominator was easy to study in terms of condition on the numerator. To obtain explicitly the whole stationary distribution P_{mn} using (4.1), we have now to perform explicitly the derivatives of $x(y)$. To do this we observe that by (4.7) we have

$$x'(y) = \frac{x(y)\frac{d}{dy}\ln q^{(2)}(y)}{\frac{\mu_1}{\mu_1+(1-\mu_1)x(y)} - x(y)\frac{d}{d\xi}\ln q^{(1)}(\xi)|_{\xi=x(y)}} \tag{4.11}$$

The relation (4.11) may be now used iteratively to compute all the derivatives of $x(y)$.

4.5 The model II: Bernoulli schedules in two class $GI/D/1$ queueing system

We consider a single server queue with two class of users. Users of class i $(i = 1, 2)$ arrive at system according arbitrary general independent stochastic processes. Let $q_{n,i}$ the probability to have n arrivals of ith class of users in single time slot. The class of users are served according the following discipline. If only a single queue is nonempty then the first user is served. Whenever both queues are nonempty the users of class 1 receive service with probability p $(0 \le p \le 1)$ and the users of class 2 receive service with probability $1-p$ independently in each time slot. The server serves the users according to the $FIFO$ discipline on each class of users and each service starts only at discrete time $t_j = jD$. Without loss of generality we will assume that $D = 1$, i.e unit service time slots. Let us denote by $X_i(j)$ the number of ith class users waiting in queue at time t_j, and ready to be served in the time interval $[j, j+1)$. The two dimensional stochastic process $\{X_i(j)\}_{j=0}^{\infty}$ is a Discrete Time Markov Chain with countable state space $\{0,1,2,\ldots\}x\{0,1,2,\ldots\}$. Further, we denote with $P^j_{m,n}$ the probability to have m users of first class and n users of second class in the system at time j, i.e. immediately before the beginning of the time slot $[j, j+1)$,

$$P^j_{m,n} = P[X_1(j) = m, X_2(j) = n] \tag{5.1}$$

In view of the description above, the following linear system describes the probability distribution $P^j_{m,n}$.

$$\begin{aligned}
P^j_{00} &= \left(P^j_{00} + P^j_{10} + P^j_{01}\right) q_{00} \\
&\vdots \\
P^j_{m,n} &= P^j_{00} q_{m,n} + \sum_{k=1}^{m+1} P^j_{k0} q_{m-k+1,n} + \sum_{l=1}^{n+1} P^j_{0l} q_{m,n-l+1} + \\
&+ p \sum_{k=1}^{m+1}\sum_{l=1}^{n} P^j_{kl} q_{m-k+1,n-l} + (1-p)\sum_{k=1}^{m}\sum_{l=1}^{n+1} P^j_{kl} q_{m-k,n-l+1} \\
&\vdots
\end{aligned} \tag{5.2}$$

4.6 Derivation of functional equation under stationary condition

The Markov chain $\{X_i(j)\}_{j=0}^{\infty}$ for $i = 1, 2$ is aperiodic and positive recurrent, therefore it is ergodic under the stability condition

$$\sum_{i=1}^{2} \sum_{n \geq 0} n q_{n,i} < 1 \tag{6.1}$$

Let us define the stationary probability $P_{m,n}$ by

$$P_{m,n} = \lim_{j \to \infty} P[X_1(j) = m, X_2(j) = n] \tag{6.2}$$

Under the stability condition (6.1), the limit exists for any m n, and the stationary probability satisfies the following linear system

$$\begin{aligned}
P_{00} &= \Big(P_{00} + P_{10} + P_{01}\Big) q_{00} \\
&\vdots \\
P_{m,n} &= P_{00} q_{m,n} + \sum_{k=1}^{m+1} P_{k0} q_{m-k+1,n} + \sum_{l=1}^{n+1} P_{0l} q_{m,n-l+1} + \\
&+ p \sum_{k=1}^{m+1} \sum_{l=1}^{n} P_{kl} q_{m-k+1,n-l} + (1-p) \sum_{k=1}^{m} \sum_{l=1}^{n+1} P_{kl} q_{m-k,n-l+1} \\
&\vdots
\end{aligned} \tag{6.3}$$

Now define the generating function

$$P(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{m,n} x^m y^n \qquad q(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_{m,n} x^m y^n \qquad |x| \leq 1 \qquad |y| \leq 1 \tag{6.4}$$

Multiplying both side of (6.3) by $x^i y^j$ and summing over all i and j we have

$$\begin{aligned}
P(x,y) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{00} q_{ij} x^i y^j + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=1}^{i+1} P_{k0} q_{i+1-k,j} x^i y^j + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=1}^{j+1} P_{0l} q_{i,j+1-l} x^i y^j + \\
&\quad + p \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=1}^{i+1} \sum_{l=1}^{j} P_{kl} q_{i+1-k,j-l} x^i y^j + (1-p) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=1}^{i} \sum_{l=1}^{j+1} P_{kl} q_{i-k,j+1-l} x^i y^j \\
&= P_{00} q(x,y) + \frac{1}{x} \sum_{k=1}^{\infty} P_{k0} x^k q(x,y) + \frac{1}{y} \sum_{l=1}^{\infty} P_{0l} y^l q(x,y) + \\
&+ \frac{p}{x} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P_{kl} x^k y^l q(x,y) + \frac{(1-p)}{y} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P_{kl} x^k y^l q(x,y)
\end{aligned}$$

After standard computation we obtain the following generating functional equation

$$P(x,y) = \frac{N(x,y)}{D(x,y)} \tag{6.5}$$

where

$$N(x,y) := (x-y)\left(pP(0,y)-(1-p)P(x,0)\right)+(xy-(1-p)y-px)P(0,0)$$
$$D(x,y) := \frac{xy}{q(x,y)} - py - (1-p)x$$

The functional equation (6.5) contains two unknown functions $P(x,0)$ and $P(0,y)$ which are crucial to determine completely $P(x,y)$. We give the following theorem that is stated in [13].

Theorem 4.6.1. *For the irreducible aperiodic Markov chain to be ergodic, it is necessary and sufficient that there exist $P(x,y)$, $P(x,0)$, $P(0,y)$ holomorphic in $|x|,|y|<1$ and a constant $P(0,0)$ satisfying the generating functional equation* (6.5) *together with the L_1 condition*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_{m,n} < \infty \tag{6.6}$$

In this case the function are unique.

It is easy to compute the constant $P(0,0)$ we employing the normalization condition rule $\lim_{\substack{x\to 1\\ y\to 1}} P(x,y)=1$ and we obtain

$$P(0,0) = 1 - \left.\frac{\partial q(x,y)}{\partial x}\right|_{\substack{x=1\\ y=1}} - \left.\frac{\partial q(x,y)}{\partial y}\right|_{\substack{x=1\\ y=1}} \tag{6.7}$$

In case of independent arrivals the equation (6.7) becomes

$$P_{00} = 1-\rho_1-\rho_2 \tag{6.8}$$

Clearly the constant $P(0,0)$ is independent of p and this fact is reasonable considering the first equation of the system (6.3).

4.7 Study of functional equation

In this section we will study the generating function $P(x,y)$, by using an approach different to the one given by Cohen and Boxma [15].

4.7.1 Case $p=0$ and $p=1$

If $p=1$ we can see our model like two class of users in parallel, where the users of first class has to be served with priority over users of second class and the users of second class received service whenever the users of first class are absent. This case is well known see e.g [42], when the service time has geometric distribution. In this case the (6.5) becomes

$$P(x,y) = \frac{(x-y)P(0,y)+(xy-x)P(0,0)}{xyq^{-1}(x,y)-y} \tag{7.1}$$

To find the unknown function $P(0,y)$ in (7.1) we need to study the zero of denominator of (7.1). Let $\mathcal{C}\subset D$ be a real curve given by equation

$$xyq^{-1}(x,y)-y=0 \tag{7.2}$$

Since $P(x,y)$ is regular in $|x|<1$, $|y|<1$, we have that for any couple (x,y) such that the denominator of (7.1) vanishes, the numerator of (7.1) must be vanish

$$(x-y)P(0,y)+(xy-x)P(0,0)=0 \tag{7.3}$$

This fact give us a condition to obtain $P(0,y)$. Thus the unknown function $P(0,y)$ in (7.1) is given by

$$P(0,y)=\frac{x(y)(1-y)P(0,0)}{x(y)-y} \tag{7.4}$$

where $x(y)$ is the zero of the denominator of functional equation (7.1),and then it is solution of

$$x(y)=q(x(y),y) \tag{7.5}$$

Now plugging this value of $P(0,y)$ into (7.1), we obtain

$$P(x,y)=\frac{(x(y)-x)(y-1)P(0,0)}{(xq^{-1}(x,y)-1)(x(y)-y)} \tag{7.6}$$

If $p=0$ the (6.5) becomes

$$P(x,y)=\frac{(y-x)P(x,0)+(xy-y)P(0,0)}{xyq^{-1}(x,y)-x} \tag{7.7}$$

and in the some way we have,

$$P(x,0)=\frac{y(x)(1-x)P(0,0)}{y(x)-x} \tag{7.8}$$

where $y(x)$ is solution of

$$y(x)=q(x,y(x)) \tag{7.9}$$

Now plugging (7.8) in (7.7), finally to get

$$P(x,y)=\frac{(y(x)-y)(x-1)P(0,0)}{(yq^{-1}(x,y)-1)(y(x)-x)} \tag{7.10}$$

4.7.2 Case $0<p<1$

In this subsection we want to study the functional equation (6.5) using perturbative method. Let $\epsilon := 1-p$ perturbation parameter. Expanding (6.5) in powers of ϵ and ignoring ϵ^n we can write,

$$P_\epsilon(x,y)=P_0(x,y)+\epsilon P_1(x,y)+\epsilon^2P_2(x,y)+\cdots+O(\epsilon^n) \tag{7.11}$$

and the expansion of $P(x,0)$ and $P(0,y)$ is

$$\begin{aligned}
P_\epsilon(x,0)&=P_0(x,0)+\epsilon P_1(x,0)+\epsilon^2P_2(x,0)+\cdots+O(\epsilon^n)\\
P_\epsilon(0,y)&=P_0(0,y)+\epsilon P_1(0,y)+\epsilon^2P_2(0,y)+\cdots+O(\epsilon^n)\\
P_\epsilon(0,0)&=P_0(0,0)
\end{aligned} \tag{7.12}$$

Consider the inverse of denominator $D(x,y)$. We can write it in the following way

$$\begin{aligned}
\frac{1}{D(x,y)}&=\frac{1}{(xyq^{-1}(x,y)-y)\left(1-\frac{\epsilon(x-y)}{xyq^{-1}(x,y)-y}\right)}= \\
&=\frac{1}{(xyq^{-1}(x,y)-y)}\left(1+\epsilon\frac{(x-y)}{xyq^{-1}(x,y)-y}+\epsilon^2\frac{(x-y)^2}{(xyq^{-1}(x,y)-y)^2}+\cdots+O(\epsilon^n)\right)
\end{aligned} \tag{7.13}$$

In the last equality we use the expansion of geometric series in some region granted by the statement of following lemma.

Lemma 4.7.1. *For $0 < \epsilon < 1/2$ and for $\delta_1 > 0$ sufficiently small one can find a $\delta_2 > 0$ such that*

$$\Big|\frac{\epsilon(x-y)}{xyq^{-1}(x,y)-y}\Big| < 1 \tag{7.14}$$

Proof. For $|x| = 1-\delta_1$ and $|y| = 1-\delta_2$, we study (7.14) neglecting the terms up to the second order, we have

$$\Big|\frac{\epsilon(\delta_2-\delta_1)}{(1-\delta_1)(1-\delta_2)(1+\varrho_1\delta_1+\varrho_2\delta_2)-1+\delta_2}\Big| = \Big|\frac{\epsilon(\delta_2-\delta_1)}{(\varrho_1-1)\delta_1+\varrho_2\delta_2}\Big| < 1 \tag{7.15}$$

It easy to see that the (7.14) is satisfied in the region given by solution of inequality (7.15). □

Substituting the (7.12) in (6.5) and considering the expansion in (7.13), after some algebraic calculation we find for instance

$$P_0(x,y) = \frac{(y-x)P_0(0,y)+(x-xy)P(0,0)}{y-xyq^{-1}(x,y)} \tag{7.16}$$

$$P_1(x,y) = \frac{(x-y)\Big(P_0(x,y)-P_0(0,y)-P_0(x,0)+P_1(0,y)+P(0,0)\Big)}{y-xyq^{-1}(x,y)} \tag{7.17}$$

and

$$P_2(x,y) = \frac{(x-y)\Big(P_1(x,y)-P_1(0,y)-P_1(x,0)+P_2(0,y)\Big)}{y-xyq^{-1}(x,y)} \tag{7.18}$$

where $P_0(x,y)$, $P_1(x,y)$ and $P_2(x,y)$ are defined in (7.11),(7.12). Itering this procedure we obtain the following theorem.

Theorem 4.7.2. *The functional coefficient of the expansion of* (6.5) *in powers of* ϵ

$$P_\epsilon(x,y) = \sum_{k=0}^{n} \epsilon^k P_k(x,y) + O(\epsilon^n) \tag{7.19}$$

is given by

$$\begin{aligned} P_k(x,y) = {} & \frac{x-y}{xyq^{-1}(x,y)-y}\Big[P_{k-1}(x,y)-P_{k-1}(0,y)-P_{k-1}(x,0)+P_k(0,y)+\delta_{k,1}P(0,0)\Big] \\ & + \frac{\delta_{k,0}(xy-x)P(0,0)}{xyq^{-1}(x,y)-y} \end{aligned} \tag{7.20}$$

with condition $P_{k-1}(x,y) = P_{k-1}(0,y) = P_{k-1}(x,0) = 0$ for $k = 0$.

Using the recurrent formula for $P_k(x,y)$ we can write the functional equation (6.5) in terms of the well known function $P_0(x,y)$ given by (7.1). In a similar way expanding (6.5) in powers of p and ignoring p^n we obtain the following theorem

Theorem 4.7.3. *The functional coefficient of the expansion of* (6.5) *in powers of* ϵ

$$P_p(x,y) = \sum_{k=0}^{n} p^k P_k(x,y) + O(p^n) \tag{7.21}$$

is given by

$$\begin{aligned} P_l(x,y) = {} & \frac{y-x}{xyq^{-1}(x,y)-x}\Big[P_{l-1}(x,y) - P_{l-1}(0,y) - P_{l-1}(x,0) + P_l(x,0) + \delta_{l,1}P(0,0)\Big] \\ & + \frac{\delta_{l,0}(xy-y)P(0,0)}{xyq^{-1}(x,y)-x} \end{aligned} \tag{7.22}$$

with condition $P_{l-1}(x,y) = P_{l-1}(0,y) = P_{l-1}(x,0) = 0$ *for* $l=0$.

4.8 The queue length

In this section we give the expression for the queue length of class 1 and class 2 users. In the begin we work with expansion (7.19), so by starting the exact generating function (7.6) obtained from (6.5) for $p=1$. Denote by $N_1^{(n)}$ the average values of the queue length of class 1 user, where n give the order of expansion. Differentiating (7.19) whith respect to x and evaluating in point $(1,1)$ and applying l'Hôpital rule gives

$$\begin{aligned} N_1^{(n)} &= \sum_{k=0}^{n} \epsilon^k \frac{\partial}{\partial x} P_k(x,y)\Big|_{(1,1)} \\ &= \sum_{k=0}^{n} \epsilon^k \frac{1}{1+\frac{\partial}{\partial x}q^{-1}(x,y)|_{(1,1)}}\Big[\frac{\partial}{\partial x}P_{k-1}(x,y) - \frac{\partial}{\partial x}P_{k-1}(x,0)\Big]_{(1,1)} \\ &= \sum_{k=0}^{n} \epsilon^k \Big(\frac{1}{1+\frac{\partial}{\partial x}q^{-1}(x,y)|_{(1,1)}}\Big)^k N_1^{(0)} - \sum_{k=0}^{n}\sum_{l=0}^{k-1} \epsilon^k \Big(\frac{1}{1+\frac{\partial}{\partial x}q^{-1}(x,y)|_{(1,1)}}\Big)^{k-l} \frac{\partial}{\partial x}P_l(x,0)|_{(1,1)} \end{aligned} \tag{8.1}$$

with

$$N_1^{(0)} = -\frac{2\frac{\partial}{\partial x}q^{-1}(x,y)|_{(1,1)} + \frac{\partial^2}{\partial x^2}q^{-1}(x,y)|_{(1,1)}}{2(1+\frac{\partial}{\partial x}q^{-1}(x,y)|_{(1,1)})} \tag{8.2}$$

Remains to determinate $P_l(x,0)$. To do this, now take the limit of (7.20) as y tends to zero for $k \geq 1$

$$\begin{aligned} P_k(x,0) &= \lim_{y\to 0} \frac{x-y}{xyq^{-1}(x,y)-y}\Big[P_{k-1}(x,y) - P_{k-1}(0,y) - P_{k-1}(x,0) + P_k(0,y) + \delta_{k,1}P(0,0)\Big] \\ &= \frac{x}{xq^{-1}(x,0)-1}\lim_{y\to 0}\frac{P_{k-1}(x,y) - P_{k-1}(0,y) - P_{k-1}(x,0) + P_k(0,y) + \delta_{k,1}P(0,0)}{y} \\ &= \frac{x}{xq^{-1}(x,0)-1}\frac{\partial}{\partial y}P_{k-1}(x,y)|_{y=0} \\ &= \Big(\frac{x}{xq^{-1}(x,0)-1}\Big)^k \frac{\partial^k}{\partial y^k}P_0(x,y)|_{y=0} \end{aligned} \tag{8.3}$$

Plugging the last result for $P_l(x,0)$ into (8.1) we obtain

$$\begin{aligned} N_1^{(n)} &= N_1^{(0)} \sum_{k=0}^{n} \epsilon^k \Big(\frac{1}{1+\frac{\partial}{\partial x} q^{-1}(x,y)|_{(1,1)}} \Big)^k - \\ &- \sum_{k=0}^{n} \sum_{l=0}^{k-1} \epsilon^k \Big(\frac{1}{1+\frac{\partial}{\partial x} q^{-1}(x,y)|_{(1,1)}} \Big)^{k-l} \Big(\frac{1}{q^{-1}(1,0)-1} \Big)^l \frac{\partial^{l+1}}{\partial x \partial^l y} P_0(x,y)|_{(1,0)} \end{aligned} \tag{8.4}$$

In order to obtain the queue length $N_2^{(n)}$ we use the following observation: we compute the total average queue of the system described in section 4.5. Considering the generating function (6.5), if we pose $y = x$ into the (6.5), we have

$$P(x,x) = \frac{(x-1)P(0,0)}{xq^{-1}(x,x)-1} \tag{8.5}$$

Differentiating both side of (8.5) with respect to x and taking the limit as x tends to one, we obtain

$$N_1 + N_2 = - \frac{2\frac{\partial}{\partial x} q^{-1}(x,x) + \frac{\partial^2}{\partial x^2} q^{-1}(x,x)}{2(1+\frac{\partial}{\partial x} q^{-1}(x,x))} \Big|_{(1,1)} = \frac{2(\varrho_1+\varrho_2) - (\varrho_1+\varrho_2)^2}{2(1-\varrho_1-\varrho_2)} \tag{8.6}$$

where the second equality is implied by the fact the arrivals are independent. Since we have just computed the average of first queue, we obtain by (8.6) also the value of $N_2^{(n)}$. In this way we have the approximate solution of problem near $p = 1$.

Now we proceed with expansion (7.21) in order to have the approximate solution of problem near $p = 0$. Differentiating (7.21) whith respect to y and evaluating in point $(1,1)$ and applying l'Hôpital rule gives

$$\begin{aligned} N_2^{(n)} &= \sum_{l=0}^{n} p^l \frac{\partial}{\partial y} P_l(x,y) \Big|_{(1,1)} \\ &= \sum_{l=0}^{n} p^l \frac{1}{1+\frac{\partial}{\partial y} q^{-1}(x,y)|_{(1,1)}} \Big[\frac{\partial}{\partial y} P_{l-1}(x,y) - \frac{\partial}{\partial y} P_{l-1}(0,y) \Big]_{(1,1)} \\ &= \sum_{l=0}^{n} p^k \Big(\frac{1}{1+\frac{\partial}{\partial y} q^{-1}(x,y)|_{(1,1)}} \Big)^k N_2^{(0)} - \sum_{l=0}^{n} \sum_{m=0}^{l-1} p^k \Big(\frac{1}{1+\frac{\partial}{\partial y} q^{-1}(x,y)|_{(1,1)}} \Big)^{l-m} \frac{\partial}{\partial y} P_m(0,y)|_{(1,1)} \end{aligned} \tag{8.7}$$

with

$$N_2^{(0)} = - \frac{2\frac{\partial}{\partial y} q^{-1}(x,y)|_{(1,1)} + \frac{\partial^2}{\partial y^2} q^{-1}(x,y)|_{(1,1)}}{2(1+\frac{\partial}{\partial y} q^{-1}(x,y)|_{(1,1)})} \tag{8.8}$$

Remains to determinate $P_m(0,x)$. To do this, now take the limit of (7.22) as x tends to zero for $l \geq 1$

$$\begin{aligned}
P_l(0,y) &= \lim_{x\to 0} \frac{y-x}{xyq^{-1}(x,y)-x}\Big[P_{l-1}(x,y) - P_{l-1}(0,y) - P_{l-1}(x,0) + P_l(x,0) + \delta_{l,1}P(0,0)\Big] \\
&= \frac{y}{yq^{-1}(0,y)-1} \lim_{x\to 0} \frac{P_{l-1}(x,y) - P_{l-1}(0,y) - P_{l-1}(x,0) + P_l(x,0) + \delta_{l,1}P(0,0)}{x} \\
&= \frac{y}{yq^{-1}(0,y)-1} \frac{\partial}{\partial x} P_{k-1}(x,y)|_{x=0} \\
&= \Big(\frac{y}{yq^{-1}(0,x)-1}\Big)^l \frac{\partial^l}{\partial y^l} P_0(x,y)|_{x=0}
\end{aligned} \tag{8.9}$$

Plugging the last result for $P_l(0,y)$ into (8.1) we obtain

$$\begin{aligned}
N_2^{(n)} = N_2^{(0)} \sum_{l=0}^{n} p^l \Big(\frac{1}{1+\frac{\partial}{\partial y}q^{-1}(x,y)|_{(1,1)}}\Big)^l - \\
- \sum_{l=0}^{n}\sum_{m=0}^{l-1} p^l \Big(\frac{1}{1+\frac{\partial}{\partial y}q^{-1}(x,y)|_{(1,1)}}\Big)^{l-m} \Big(\frac{1}{q^{-1}(0,1)-1}\Big)^m \frac{\partial^{m+1}}{\partial y \partial^m x} P_0(x,y)|_{(0,1)}
\end{aligned} \tag{8.10}$$

4.9 Numerical results

In this section we present some numerical results on the average queue size obtained with approximation (8.4) and by simulation. To do this we consider the case of Poisson arrivals for users of both classes with different traffic intensity ϱ_1 and ϱ_2 for the first and second class respectively. Clearly the parameters which influences the average of the queues are ϱ_1, ϱ_2 and p. We did the numerical results varying these parameters. In particular we compare these results in order to see the efficiency of approximation (8.4) taking into account the case of lower traffic and heavy traffic intensity and varying parameter $p \in [0,1]$. In Table 4.1 we have listed the results of both average queues for lower traffic intensity ($\varrho_1 + \varrho_2 = 0.6$) and in figure 4.2 we have plotted these results. As we see comparing results in this table and in figure 4.2 for the respective average queue the perturbative method give us a good result excepted for p near the point 0.7 ($p \in [0.66, 0.74]$). Note that the numerical results for the case $p = 0$ and $p = 1$ are obtained by the complete solution of problem given in section 4.7. In Table 4.2 we have listed the results of both average queue for heavy traffic intensity ($\varrho_1 + \varrho_2 = 0.9$) and in figure 4.3 we have plotted these results. In this case as we see comparing results in this table and in figure 4.2 for the respective average queue the perturbative method give us a good results for $p \in [0.3]$ and ($p \in [0.76, 1]$).

4.10 Conclusions

In this chapter we have studied two models of two class queue in parallel: priority and Bernoulli scheduling in which the generating function is not in the factorized form. We have obtained a complete solution for the discrete time $GI/Geom/1$ queueing system with priority. This has been possible because of the simple structure of the service discipline. For the single server with Bernoulli scheduling we have obtained an approximation solution of the generating function near the value $p = 0$ and $p = 1$. This approximation give a good results for lower intensity traffic and relatively good results for heavy traffic intensity.

ρ_1	ρ_2	p	$N_1(sim)$	$N_1(app)$	$N_2(sim)$	$N_2(app)$
0.2	0.4	0	0.514	0.516	0.531	0.533
0.2	0.4	0.1	0.479	0.484	0.570	0.569
0.2	0.4	0.2	0.439	0.444	0.607	0.605
0.2	0.4	0.3	0.401	0.408	0.645	0.641
0.2	0.4	0.4	0.366	0.372	0.684	0.677
0.2	0.4	0.5	0.333	0.336	0.713	0.713
0.2	0.4	0.6	0.305	0.300	0.742	0.749
0.2	0.4	0.66	0.289	0.278	0.758	0.771
0.2	0.4	0.7	0.28	0.264	0.767	0.785
0.2	0.4	1.	0.225	0.225	0.8232	0.825
0.2	0.4	0.9	0.241	0.239902	0.8055	0.810098
0.2	0.4	0.8	0.259	0.25499	0.7869	0.793491
0.2	0.4	0.74	0.272	0.264131	0.7774	0.785869
0.2	0.4	0.7	0.28	0.270263	0.7671	0.779737

Table 4.1:

ρ_1	ρ_2	p	$N_1(sim)$	$N_1(app)$	$N_2(sim)$	$N_2(app)$
0.6	0.3	0	4.573	4.58571	0.364	0.364286
0.6	0.3	0.04	4.5541	4.56656	0.3826	0.383437
0.6	0.3	0.1	4.515	4.53401	0.4158	0.415987
0.6	0.3	0.14	4.476	4.50944	0.4411	0.440564
0.6	0.3	0.2	4.4508	4.46769	0.4851	0.482311
0.6	0.3	0.24	4.4037	4.43627	0.5207	0.513725
0.6	0.3	0.3	4.3211	4.38322	0.5839	0.566785
0.6	0.3	0.34	4.2722	4.34355	0.6364	0.606445
0.6	0.3	0.4	4.1778	4.27707	0.7314	0.672932
0.6	0.3	1	1.0447	1.05	3.8695	3.9
0.6	0.3	0.94	1.196	1.19858	3.7455	3.75142
0.6	0.3	0.9	1.3191	1.3182	3.5725	3.6318
0.6	0.3	0.84	1.5542	1.53348	3.335	3.41652
0.6	0.3	0.8	1.7678	1.70383	3.1336	3.24617
0.6	0.3	0.76	2.0331	1.89797	2.8596	3.05203

Table 4.2:

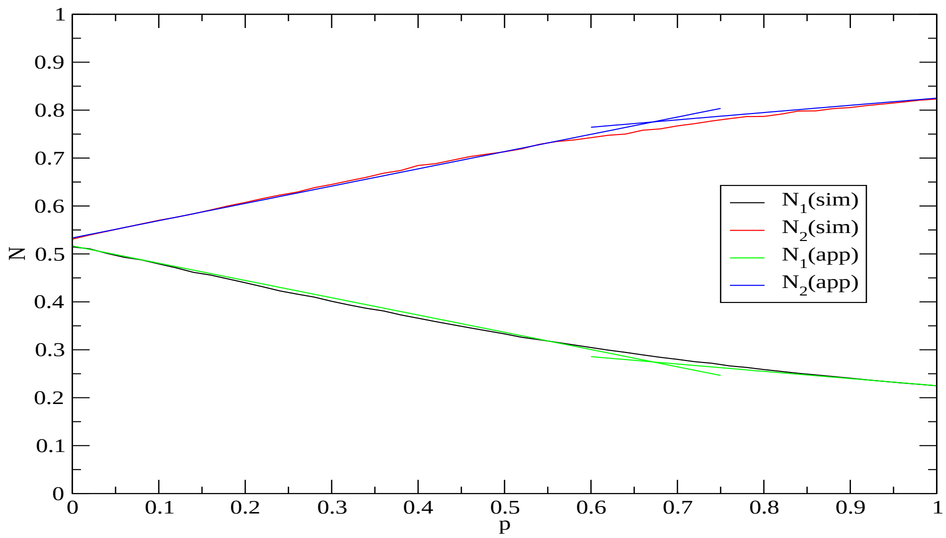

Figure 4.2: Average queue for $\varrho_1 = 0.2$ and $\varrho_1 = 0.4$

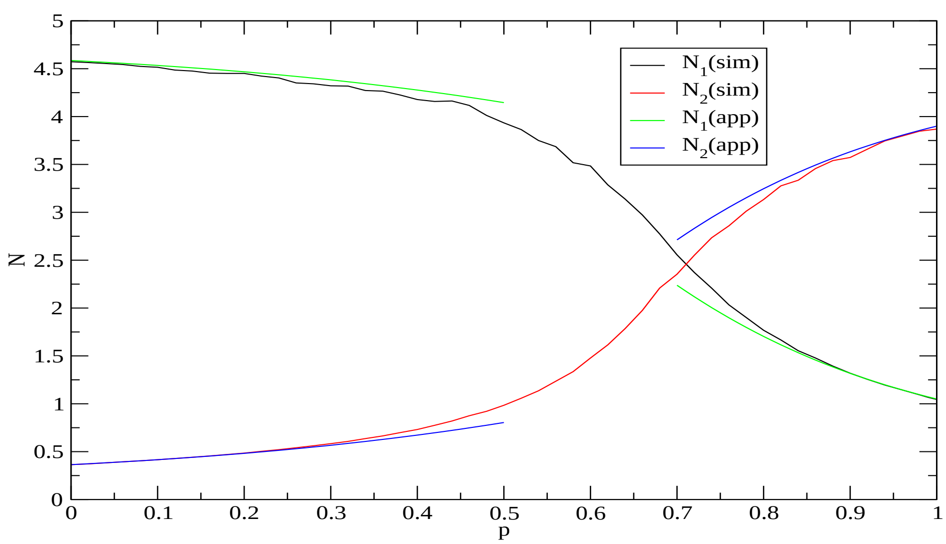

Figure 4.3: Average queue for $\varrho_1 = 0.6$ and $\varrho_1 = 0.3$

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