Chapter 2

Dynamical Micromagnetics

2.1 The Generalized Gilbert Equation

In a saturated, undeformable ferromagnet occupying the region Ω, the evolution of the magnetization vector is ruled by the Gilbert equation:

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m} \times (\mathbf{h} + \mathbf{d})\,, \tag{2.1}$$

where $\gamma > 0$ is the *gyromagnetic ratio*, $\mathbf{h}$ the effective magnetic field (cf. eq. (1.3)) and $\mathbf{d}$ the *dissipation field*. This equation is a formal generalization of the classical model equation due to Landau & Lishitz and Gilbert, a generalization that can be given a precise physical status (cf. §2.3 and the references quoted therein); provided $\dot{\mathbf{m}} \neq \mathbf{0}$, it can be written as the following system of two scalar equations:

$$\begin{aligned} -\gamma^{-1}\dot{\mathbf{m}} \cdot \dot{\mathbf{m}} &= (\mathbf{h} + \mathbf{d}) \cdot \mathbf{m} \times \dot{\mathbf{m}}\,, \\ 0 &= (\mathbf{h} + \mathbf{d}) \cdot \dot{\mathbf{m}}\,. \end{aligned} \tag{2.2}$$

The *effective magnetic field* $\mathbf{h}$ is, we recall, the variational derivative of the *free-energy functional* $\Psi\{\mathbf{m}\}$ (cf. §1.2 and §1.2.2), while the dissipation field $\mathbf{d}$ is the variational derivative of the *dissipation potential* $X\{\dot{\mathbf{m}}\}$:

$$\mathbf{h} = -\delta_m \Psi\,, \quad \Psi\{\mathbf{m}\} = \int_\Omega \psi(\mathbf{m}, \nabla\mathbf{m}, \nabla\nabla\mathbf{m})\,, \tag{2.3}$$

$$\mathbf{d} = -\delta_{\dot{m}} X\,, \quad X\{\dot{\mathbf{m}}\} = \int_\Omega \chi(\dot{\mathbf{m}}, \nabla\dot{\mathbf{m}})\,. \tag{2.4}$$

Thermodynamic compatibility is guaranteed if the *dissipation density* is non-negative:

$$\mathrm{d} := \partial_{\dot{m}}\chi \cdot \dot{\mathbf{m}} + \partial_{\nabla\dot{m}}\chi \cdot \nabla\dot{\mathbf{m}} \geq 0\,. \tag{2.5}$$

Provided that the appropriate homogeneous Neumann conditions prevail at the boundary of Ω, it follows from $(2.2)_2$, (2.3), and (2.4), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \int_\Omega \mathrm{d} = 0\,, \tag{2.6}$$

a relation that embodies the *Liapounov structure* intrinsic to the Gilbert equation, a structure that all of its generalizations must retain.

2.2 Standard Form of the Gilbert Equation

We now consider the classical choices of ψ and χ to which both Landau & Lifshitz and Gilbert confined themselves. More general choices that have been recently suggested will be illustrated at a later stage.

Free energy
As to the free energy Ψ, we keep the constitutive prescritions outlined in §1.2.2 (see eqs. (1.4) and (1.7)), which give, for the effective field:

$$\mathbf{h} = \alpha\Delta\mathbf{m} + \beta(\mathbf{m}\cdot\mathbf{e})\mathbf{e} + \mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}} = 0\,, \tag{2.7}$$

where, for the reader's sake, we recall that $\mathbf{e}$ is a constant unit vector (the easy axis), α and β are two positive constants, $\mathbf{h}^{\mathrm{e}}$ is a prescribed external field, and $\mathbf{h}^{\mathrm{s}}$ is the unique solution of the Maxwell equations

$$\begin{aligned} \operatorname{curl}\mathbf{h}^{\mathrm{s}} &= 0\,,\\ \operatorname{div}\mathbf{h}^{\mathrm{s}} &= -\operatorname{div}(\chi_\Omega\mathbf{m})\,, \end{aligned} \tag{2.8}$$

such that

$$\int_{\mathcal{E}} |\mathbf{h}^{\mathrm{s}}|^2 < \infty\,. \tag{2.9}$$

Dissipation
As to the density of the dissipation potential, the standard choice is

$$\chi = \frac{1}{2}\,\mu\,|\dot{\mathbf{m}}|^2\,, \quad \mu > 0\,, \tag{2.10}$$

the so-called *relativistic* dissipation. According to (2.4) and (2.10), the relativistic dissipation field has the form

$$\mathbf{d} = -\,\mu\,\dot{\mathbf{m}}\,, \tag{2.11}$$

and hence, by the definition in (2.5),

$$\mathbf{d} \cdot \dot{\mathbf{m}} + \mathrm{d} = 0\,. \tag{2.12}$$

Substituting the standard prescriptions (2.7) and (2.11) in (2.1), we arrive at the *Gilbert equation with standard free energy and dissipation*:

$$\gamma^{-1}\dot{\mathbf{m}} + \mu\mathbf{m} \times \dot{\mathbf{m}} = \mathbf{m} \times (\alpha\Delta\mathbf{m} + \beta(\mathbf{m} \cdot \mathbf{e})\mathbf{e} + \mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}})\,. \tag{2.13}$$

2.3 The Gilbert Equation as a Balance Law

Within the framework of a broader theory, which account for both magnetic and mechanical interactions [10, 18, 33, 16], the Gilbert equation can be interpreted as the specialization of a balance law. As anticipated in Chapter 1, ferromagnetic bodies are modeled as the composition of a *lattice continuum* and a *spin continuum*. Their kinematics is described by the deformation with respect to a reference configuration and by the magnetization. Their dynamics is described by systems of generalized forces which expend power in a typical process, both on the lattice and on the spin continuum.[1] Following a procedure which is customary in continuum mechanics, balance equations for the dynamical descriptors can be obtained by postulating translational and rotational invariance of the power ([16]). Evolution equations obtain when the balance equations are completed with constitutive prescriptions, that is, a set of relations between the kinematical and the dynamical descriptors. Those constitutive prescriptions are supposed to model a specific class of materials.

In order to determine the evolution of the magnetization vector in an undeformable ferromagnetic body it suffices to consider the *balance of torques* for the spin continuum ([16], [33]):

$$\mathbf{m} \times (\operatorname{div} \mathbf{C} + \mathbf{k} + \mathbf{b}) = \mathbf{0}\,. \tag{2.14}$$

Here $\mathbf{C}$ is the *couple stress*, $\mathbf{m} \times \mathbf{k}$ is the *interaction couple* and $\mathbf{m} \times \mathbf{b}$ is the *distance couple*. By means of the decomposition

$$\mathbf{b} = \mathbf{b}^{in} + \mathbf{b}^{ni}\,, \tag{2.15}$$

we split the distance couple in two contributions: an *inertial part* $\mathbf{m} \times \mathbf{b}^{in}$, which is related to the motion through a constitutive specification, and a *non-inertial* part $\mathbf{m} \times \mathbf{b}^{ni}$, which can be thought as a control field. Constitutive

[1] As pointed out in [16], the definition of the dynamical descriptors requires care, due to the presence of magnetostatic interactions; see also [17].

prescriptions are also needed for the interaction couple, the couple stress and the free-energy density, respectively:

$$\begin{aligned} \mathbf{k} &= \hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \nabla\dot{\mathbf{m}}) \,; \\ \mathbf{C} &= \hat{\mathbf{C}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \nabla\dot{\mathbf{m}}) \,; \\ \psi &= \hat{\psi}(\mathbf{m}, \nabla\mathbf{m}) \,. \end{aligned} \tag{2.16}$$

Consistency with the second principle of thermodynamics, requires that

$$-\mathbf{k} \cdot \dot{\mathbf{m}} + \mathbf{T} \cdot \nabla\dot{\mathbf{m}} - \dot{\psi} \geq 0 \tag{2.17}$$

for every process. It can be shown [16, 38] that the above *dissipation inequality* sets the following *constitutive restrictions* on the material response:

$$\begin{aligned} \mathbf{k}^{eq} &:= \hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) = -\partial_{\mathbf{m}}\psi \,; \\ \mathbf{C}^{eq} &:= \hat{\mathbf{C}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) = \partial_{\nabla\mathbf{m}}\psi \,. \end{aligned} \tag{2.18}$$

Restrictions (2.18) are only a part of the information that can be drawn from the dissipation inequality. The remaining information can be expressed in a compact form if we introduce the *viscous response*, defined as

$$\begin{aligned} \mathbf{k}^{vs}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \nabla\dot{\mathbf{m}}) &:= \hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \nabla\dot{\mathbf{m}}) - \mathbf{k}^{eq}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) \,, \\ \mathbf{C}^{vs}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \nabla\dot{\mathbf{m}}) &:= \hat{\mathbf{C}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \nabla\dot{\mathbf{m}}) - \mathbf{C}^{eq}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) \,. \end{aligned} \tag{2.19}$$

which gives the following decomposition of the dynamical descriptors:

$$\begin{aligned} \mathbf{k} &= \mathbf{k}^{eq} + \mathbf{k}^{vs} \,, \\ \mathbf{C} &= \mathbf{C}^{eq} + \mathbf{C}^{vs} \,. \end{aligned} \tag{2.20}$$

Then, granted (2.18), the dissipation inequality (2.17) is equivalent to:

$$-\mathbf{k}^{vs} \cdot \dot{\mathbf{m}} + \mathbf{C}^{vs} \cdot \dot{\nabla}\mathbf{m} \geq 0 \,. \tag{2.21}$$

Now, if we define

$$\begin{aligned} \mathbf{h} &:= \operatorname{div} \mathbf{C}^{eq} + \mathbf{k}^{eq} + \mathbf{b}^{ni} \,, \\ \mathbf{d} &:= \operatorname{div} \mathbf{C}^{vs} + \mathbf{k}^{vs} \,, \end{aligned} \tag{2.22}$$

we can write the balance equation (2.14) as

$$-\mathbf{m} \times \mathbf{b}^{in} = \mathbf{m} \times (\mathbf{h} + \mathbf{d}) \,, \tag{2.23}$$

and dynamical micromagnetics obtains from the constitutive prescription[2]

$$\mathbf{b}^{in} = -\gamma^{-1}\mathbf{m} \times \dot{\mathbf{m}} \,, \tag{2.24}$$

[2]A justification of such a choice in a mechanical setting is found in [37].

which leads to the generalized Gilbert equation (2.1) and renders it apparent the physical interpretation of the equilibrium and viscous part of the dynamical descriptors.

Remark. Taking into account the constitutive restrictions (2.18), the definition $(2.22)_1$ of the effective field $\mathbf{h}$ coincides with (2.3). This confirms that the latter is a necessity ruled by thermodynamics, and not an arbitrary assumption. On the contrary, the choice (2.4) of the dissipation vector $\mathbf{d}$ as the variational derivative of a potential is not mandatory. However, when making a choice for $\mathbf{d}$, one should keep in mind the restrictions which (2.21) sets on $\mathbf{d}$ through $(2.22)_2$.

With respect to variational approaches, the above treatment has the advantage that, once the basic principles have been stated, the evolution laws can be deduced without recourse to any *ad hoc* reasoning. In addition, since balance laws and constitutive prescriptions are kept distinct, it becomes easier to generalize the standard Gilbert equation in a manner consistent with the scope of the theory, that is, without violating the principles on which the theory is based: balance equations are stated once and for all, and we are left with a great deal of freedom in the choice of the constitutive prescriptions that model one or another material class.[3]

[3]In addition to obeying the general thermodynamic restrictions, those prescriptions must also obey the principle of frame-indifference. The reader is referred to [16, 12] for further details.