A Kinetic Model Of Interface Motion

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Abstract

After a brief introduction on scale limits and processes of phase segregation, we present a model of interface motion between two different fluids, described by a pair of kinetic equations. They provide the late stages dynamics expected by the Cahn-Hilliard equation.

Phase segregation is a first order phase transition which happens in mixtures when the temperature is lowered below the coexistence curve. Domains rich in different components begin to arise with several mechanisms. Domains are separated by layers, which become sharp in the last stages of the process and then they are called interfaces.

The main tool used is Hilbert expansion, the small parameter being the ratio between kinetic and macroscopic unit of length. The problem can be faced to in several ways. We show how to compute the limiting dynamics and the first order corrections to it.

It turns out that the late stages of phase segregation are characterized by a geometrical rearrangement of the interfaces which tends to minimize the surface, while keeping fixed the volume of the domains. The normal velocity in each point of the interface is given by gradients of harmonic functions, whose value on the boundary depends on the curvature and on the surface tension. The limiting motion is thus given in terms of a free boundary problem. On the other hand the corrections to that motion are solution of a yet non linear problem, but with boundaries fixed.

Our model has only a kind of conserved quantity: the total mass of each component. So it can be interpreted to describe the behaviour of highly viscous fluids, where energy and momentum yielding fluidodynamic effects are dissipated on much smaller time scales than those we are interested in. An example of such mixtures is given by polymer blends.

Chapter 1

Introduction

1.1 Scales

The same physical system can be described on several scales. We live, for example, on a scale in which a typical length is that of a baguette (1 meter) and a typical time is that required by a digital tv receiver to switch between two channels (1 second). Such a scale is often referred to as macroscopic. Thus we can write equations for quantities that have significant variations only over a fraction of meter in a few seconds. Think to temperature profile in a pot of water over a flame or to the pressure field around the wings of an airplane. In general macroscopic does not mean simple. Indeed a lot of components contribute to build up a macroscopic phenomenon. We deal with the result of complex interactions between objects that, for us, are microscopic and live on a different scale. Matter is composed by molecules (we do not go further because we want to disregard quantum effects) moving according to Newton equations. A typical length is now the effective range of interaction (several angstrom). As to a typical time, it is not obvious how to define it, because of the wide range of oscillatory phenomena that happen at the microscopic scale. However it should be reasonable a time of order 10^{-10}s.

Which is a coherent scheme that links the macroscopic equation to the microscopic ones? Of course the validity of a theory ultimately relies upon the experiment, but a phenomenological construction should not be, as a first requirement, contradictory with more fundamental rules. Moreover, deriving macroscopic equation from microscopic ones allows to estimate the errors that are committed using the former instead of the latter and to establish which is the regime of applicability of a macroscopic theory.

At present time the connection between microscopic and macroscopic is thought to be build over two major hypotheses: a law of large number hypothesis and a local equilibrium hypothesis. The former recalls the statistical property of some stochastic processes for which if the number of tries is very high, the behaviour of almost the totality of realizations

is however near to the mean one. That allows to substitute the actual values of any quantity of interest with its mean value. The latter hypothesis is harder because it is not yet clear how to demonstrate it also in cases that might appear simple. Systems in thermodynamic equilibrium are described by the Gibbs measure with parameters the temperature, the activity and the velocity of the center of mass in a grancanonical setting. That measure is a solution of the Liouville equation. The hypothesis of local equilibrium states that also a sample which is not in global equilibrium, yet can be divided in small, but macroscopic, cells in which it has an equilibrium behaviour characterized by parameters whose values can change from a cell to another; moreover this situation is preserved by the dynamics in the sense that the cells stay in equilibrium and only the parameters change their values according to some rule, that at the end is the macroscopic equation describing the whole system. In such a way one can replace the true mean values with that computed with respect to that local equilibrium measure. However the latter is not a solution to the Liouville equation.

1.1.1 The Basic Model

Consider a system of N point particles of mass $m = 1$ interacting via a smooth and bounded potential ϕ which is different from zero only inside a compact set of $\mathbb{R}^d$. Denote with q_i and v_i the position and the velocity of the i-th particle. Let the particles evolve according to Newton law. That is the simplest model of a gas that we can build in a mechanical fashion.

$$\frac{dq_i}{d\tau}(\tau) = v_i(\tau)$$

$$\frac{dv_i}{d\tau}(\tau) = -\sum_{j \neq i} \nabla\phi(q_i - q_j)$$

If N is large (in common situations the right order of magnitude is $N \approx 10^{23}$), Newton equations are a huge system carrying a lot of information. A macroscopic description is not required to be so much detailed. The value of few observables are sufficient to describe the whole system. The scale separation and the great number of particles involved make all quantities oscillate very rapidly and have a vanishing mean, but those which are conserved. In fact those ones evolve slowly and can survive over a macroscopic scale. In our simple model, conserved quantities are the total mass, momentum and energy.

In order to take advantage of the scale separation, let us introduce macroscopic variables related to the microscopic ones by a small parameter ε whose value is given by the ratio between the microscopic unit and the macroscopic one.

$$x_i = \varepsilon q_i, \quad t = \varepsilon\tau$$

The equations now are written

$$\frac{dx_i}{dt}(t) = v_i(t)$$

$$\frac{dv_i}{dt} = -\varepsilon^{-1} \sum_{j \neq i} \nabla\phi \left(\frac{x_i - x_j}{\varepsilon} \right)$$

Suppose that the sample is contained in a torus Λ and that its microscopic volume is $|\Lambda|_{micr}$. Then, if we assume that the particle density stays constant, N has to scale as ε^{-d} because

$$n = \frac{N}{|\Lambda|_{micr}} = \frac{\varepsilon^d N}{|\Lambda|}$$

In this way the limit $\varepsilon \to 0$, i.e. the scale separation limit, yields the thermodynamic limit, in which the volume and the number of particles diverge.

At this stage, empirical densities are distributions; thus we use a weak formalism to compute their derivatives. For any smooth function f consider

$$I_f^\alpha = \int_\Lambda dx \xi^\alpha(,t) f(x), \quad \alpha = 0, \ldots, d+1$$

where

$$\xi^0 = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(t))$$

$$\xi^\alpha(x,t) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(t)) v_i^\alpha(t), \quad \alpha = 1, \ldots, d$$

$$\xi^{d+1}(x,t) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(t)) \left[\frac{1}{2} v_i^2(t) + \frac{1}{2} \sum_{j \neq i} \phi(\varepsilon^{-1} |x_i(t) - x_j(t)|) \right]$$

Using Newton equation and the fact that $N\varepsilon^{-d}$ is bounded and the potential has compact support, it can be shown that

$$\frac{d}{dt} I_f^\alpha(t) = \int_\Lambda dx \sum_{\beta=1}^d \zeta_{\alpha,\beta}(x,t) \partial_{x_\beta} f(x) + O(\varepsilon) \tag{1.1}$$

where the empirical currents are

$$\zeta_{0,\beta}(x,t) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(t)) v_i^\beta(t)$$

$$\zeta_{\alpha,\beta}(x,t) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(t)) \left[v_i^\alpha(t) v_i^\beta(t) + \frac{1}{2} \sum_{j=1}^N \Psi_{\alpha,\beta}(\varepsilon^{-1}(x_i(t) - x_j(t))) \right]$$

$$\zeta_{d+1,\beta}(x,t) = \frac{1}{N}\sum_{i=1}^{d}\delta(x-x_i(t))\left[v_i^\beta(t)\left[\frac{1}{2}v_i^2(t)+\frac{1}{2}\sum_{j\neq i}\phi(\varepsilon^{-1}|x_i(t)-x_j(t)|)\right]\right.$$
$$\left.+\frac{1}{2}\sum_{j=1}^{N}\sum_{\gamma=1}^{d}\Psi_{\gamma,\beta}(\varepsilon^{-1}(x_i(t)-x_j(t)))\frac{1}{2}[v_i^\gamma(t)+v_j^\gamma(t)]\right]$$

with

$$\Psi_{\alpha,\beta}(z) = -z_\beta\partial_{x_\alpha}\phi(z)$$

Equation (1.1) looks very similar to the weak form of an hydrodynamical equation, but something is missing. The equation is not closed in any variable, because we still need to compute the empirical currents from the solution flow of Newton equations.

It is at that point that the hypotheses of a law of large numbers and of a local equilibrium become important. They have to be used together to obtain a system of equations for the hydrodynamical fields of density, velocity and energy. Instead of focusing on a particular realization of the newtonian motion, we may study how an initial distribution over the phase space evolves according to the underlying dynamics. Indeed we can never know exactly the initial configuration of a sample, but rather we expect that it is in a certain state with a given probability. The equation solved by the distribution evolving according to Newton law is called Liouville equation and has got the following aspect

$$(\partial_\tau + \mathcal{L}_N)f^N = 0$$

where

$$\mathcal{L}_N = \mathcal{L}_N^0 + \mathcal{L}_N^I$$

and

$$\mathcal{L}_N^0 = \sum_{i=1}^{N} v_i\cdot\nabla_{q_i}, \quad \mathcal{L}_N^I = \sum_{i=1}^{N}\sum_{j\neq i}F_{i,j}\cdot\nabla_{v_i}$$

with

$$F_{i,j} = -\nabla\phi(|q_i-q_j|)$$

In most cases it provides only a theoretical tool, its solution being as difficult as that of Newton equations. But working with probability distribution led to great achievements in understanding the mechanical nature of thermodynamics. For example, even if no one knows how to prove it, the thermodynamical equilibrium is precisely described by the Maxwell-Gibbs distribution, which actually is a solution of the Liouville equation. If a system is in contact with a reservoir at temperature T and it can exchange energy and particles with it, the phase space is

$$\Gamma = \bigcup_{N\geq 0}\Gamma_N$$

where $\Gamma_N = (\Lambda \times \mathbb{R}^d)^N$. Equilibrium states are parameterized by the temperature T, the velocity of the center of mass u and the activity z. If we put ourselves in a region of the phase diagram free from phase transitions, we can establish a one to one correspondence (at fixed temperature) between the activity and the particle mean density ρ. The local equilibrium hypothesis prescribe the following distribution over Γ_N

$$P^N_{z(\cdot,t),T(\cdot,t),u(\cdot,t)}(x_1, v_1, \ldots, x_N, v_N) =$$
$$= \frac{1}{Z} \prod_{i=1}^{N} \exp\left[\mu(x_i,t) - \frac{1}{2T(x_i,t)}\left[(v_i - u(x_i,t))^2 + \sum_{j\neq i} \phi(\varepsilon^{-1}|x_i - x_j|)\right]\right]$$

where Z is the partition function and represents a normalization factor. $\mu = \ln z$ is the chemical potential and together with T and u is a function of the point thought to vary very slowly on a microscopic scale, but in a regular fashion on the macroscopic one. If we replace the empirical densities and currents with their expectation values with respect to the local equilibrium distribution, we obtain from (1.1) equations for those fields. Then, using properties of isotropy of the potential and assuming some smoothness we can write

$$\partial_t \rho + \sum_{\alpha=1}^{d} \partial_{x_\alpha}[\rho u_\alpha] = 0$$
$$\partial_t[\rho u_\beta] + \sum_{\alpha=1}^{d} \partial_{x_\alpha}[\rho u_\alpha u\beta + P\delta_{\alpha,\beta}] = 0$$
$$\partial_t[\rho(\frac{1}{2}u^2 + e)] + \sum_{\alpha=1}^{d} \partial_{x_\alpha}[u_\alpha[\rho(\frac{1}{2}u^2 + e) + P]] = 0$$

where the energy density e and the pressure P are known functions of ρ and T. The equations just derived are called Euler equations and are usually used to describe not dissipating fluids.

We can consider a different point of view. Substantially to solve Liouville equation is equivalent to solve Newton system, thus if we show that using a local equilibrium distribution we get a set of phenomenological equations known as valid in a certain regime, we can conclude that our guess is correct and our microscopic picture is not too far from reality.

1.1.2 The Boltzmann Equation

In this section we briefly describe an heuristic derivation of Boltzmann equation (BE) and provide a sketchy idea of its rigorous proof [EP]. The focus is on the meaning of Lanford's theorem as a law of large numbers. Indeed since the beginning we assume that $f(x, v)$, the distribution function on the single particle phase space, coincides with the fraction of

particles in the cell of size $dxdv$ around (x, v). That is true only for a diverging number N of particle and for vanishing density with the mean free path that stays constant.

Boltzmann equation gives a description of dilute gases which is midway between the microscopic and macroscopic ones. For that reason, as we will see in next section, it can be used as an intermediate step in linking microscopic dynamics to hydrodynamics. The scale of BE is called kinematic and a typical length is represented by the mean free path, i.e. the average distance covered by a particle between two consecutive collisions.

The heuristic derivation of BE is very intuitive. The probability density f in a point (x, v) can change in a time dt because of three different reasons: particles with velocity v leave the cell centered in (x, v) or enter it; a collision changes the velocity of a particle that was moving in x with velocity v; a collision produces a particle with velocity v. The first contribution is known as transport term. Its integral version, given a small space region Ω, is given by

$$-\int_{\partial\Omega} dl\, f(x, v) v \cdot \nu(x)$$

where $\nu(x)$ is the outward normal to the surface $\partial\Omega$ in the point x. By the divergence theorem we can write BE as

$$(\partial t + v \cdot \nabla_x) f = G - L$$

where G and L are the loss and gain terms due to collisions. The left hand side is a substantial derivative. In other words it is the time derivative that a lagrangian observer would compute. If no collision was present, nothing would happen and the density probability would be transported unchanged along the characteristics of the flow, i.e. along simple lines $x + vt$. Only collisions modify substantially the probability density f. For simplicity we consider hard elastic collisions between particles. An impact happens when a particle is in x and a second one is in $x + rn$, where r is the diameter of the hard potential and n is the unit vector parallel to the line joining the centers of the two particles. For a fixed n, the particle in x can interact in a time dt with any other particle with velocity v_2 in the cylinder of volume $|(v_2 - v) \cdot n| r^2 dn dv_2 dt$. Then the loss term is obtained integrating over all possible velocities and direction that provides a collision and finally multiplying by the total number of particle with which a fixed particle can interact, i.e. $N - 1$:

$$L = (N-1) r^2 \int dn \int_{S_-} dv_2 f_2(x, v, x + nr, v_2) |(v_2 - v) \cdot n|$$

Here $f_2(x, v, x_2, v_2)$ is the joint probability that particle one is in x with velocity v and particle two is in (x_2, v_2). S_- is the set of velocities such that $(v_2 - v) \cdot n < 0$. For the gain term one can proceed in the same way, but it would not be useful. The problem is represented by the closure of the equation. In fact we need a reasonable ansatz to express f_2 as a function of f only. Boltzmann proposed to set $f_2(x, v, x_2, v_2) = f(x, v) f(x_2, v_2)$ as if the particles were independent. It can be accepted before a collision, but after that

it is certainly incorrect. Fortunately Boltzmann's ansatz refers to two fixed particle and the probability that a collision between them can happen is vanishing if we take the limit $N \to \infty$ with $Nr^2 \approx const$ because it is of order r^2 (the surface of a particle). The probability that a particle collides with any other particle is, on the other hand, of order Nr^2 and thus it is finite. However, writing the gain term as the loss one, we are sure that a collision happens, thus we can not use Boltzmann's ansatz. But we can overcome this problem using the continuity property

$$f_2(x, v, x + nr, v_2) = f_2(x, v', x + nr, v_2')$$

where v' and v_2' are the incoming velocities. Now it is correct to use the independence hypothesis for the two particles with velocities v' and v_2' because they never met before, almost surely. At the end

$$G - L = (N-1)r^2 \int dn \int_{S_+} dv_2 (v - v_2) \cdot n [f(x, v') f(x - nr, v_2') - f(x, v) f(x + nr, v_2)]$$

where S_+ is defined as S_- but with the reversed inequality. In the limit $N \to \infty$, $r \to 0$ with $Nr^2 = \lambda^{-1}$ one obtains the Boltzmann equation:

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f)$$

where the collision operator Q is defined by

$$Q(f, f) = \lambda^{-1} \int dn \int_{S_+} dv_2 (v - v_2) \cdot n [f(x, v') f(x, v_2') - f(x, v) f(x, v_2)]$$

In literature Boltzmann's ansatz is known as Stosszahlansatz, that is propagation of initial caos. Indeed BE generates a solution flow such that if initially the particles of the gas are moving not correlated to each other, they persist in this state although collisions happen. From a mathematical point of view, that means that if the initial measure over the phase space is product, its evolution is product too. This proposition is part of a theorem, proved by Lanford, which deduces BE from Newton equations.

First step is establishing the correct limit. This is done by rewriting Liouville's equation in a different form and scaling variables as

$$x = \varepsilon q, \quad t = \varepsilon \tau$$

In order to operate with meaningful quantities, we introduce marginal distributions $\tilde{f}_j^N$ of the full probability density f^N over the N-particle phase space:

$$f_j^N(q_1, v_1, \ldots, q_j, v_j, t) = \int dq_{j+1} dv_{j+1} \ldots dq_N dv_N f^N(q_1, v_1, \ldots, q_N, v_N, t)$$

All functions are symmetric under the exchange of two particles. The pair interaction potential has range 1 in microscopic unit; thus in macroscopic ones, it is vanishing as ε. The total volume of the N particles is order $N\varepsilon^3$ and has to go to zero if we want to approach the situation described by BE, as seen above. However, let us introduce scaled marginal distribution

$$g_j^N(x_1, v_1, \ldots, x_j, v_j) = \varepsilon^{-3j} \tilde{f}_j^N(\varepsilon^{-1}x_1, v_1, \ldots, \varepsilon^{-1}x_j, v_j)$$

which satisfies a BBGKY hierarchy induced by Liouville's equation:

$$\left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}\right) g_j^N + \sum_{i=1}^{j} \sum_{k \neq i} \varepsilon^{-1} F\left(\frac{x_i - x_k}{\varepsilon}\right) \cdot \nabla_{v_i} g_j^N =$$
$$\frac{N-j}{\varepsilon} \int dx_{j+1} \int dv_{j+1} F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right) \cdot \nabla_{v_{j+1}} g_{j+1}^N$$

The second term on the left hand side has a L^1 vanishing norm; thus the relevant term is that on the right hand side. There the integral is order ε^3, so we get a finite object if N scales as ε^{-2}. That is exactly the low density limit quoted above. Summarizing

$$N \to \infty, \quad \varepsilon \to 0, \quad N\varepsilon^2 = \lambda^{-1}$$

Second step is to write the BBGKY hierarchy again in a different way. Now we change slightly notation, rewriting Liouville's equation in macroscopic variables and setting the range of the potential equal to ε, with N diverging in such a way that $N\varepsilon^2 = \lambda^{-1}$. Moreover let us define

$$f_j^N(x_1, v_1, \ldots, x_j, v_j, t) =$$
$$\int dx_{j+1} dv_{j+1} \ldots dx_N dv_N \prod_{i=1}^{j} \prod_{k=j+1}^{N} \chi(\{|x_i - x_k| > \varepsilon\}) f^N(x_1, v_1, \ldots, x_N, v_N, t)$$

The functions f_j^N are asymptotically equivalent to the marginal distribution $\tilde{f}_j^N$. They satisfy the modified BBGKY hierarchy

$$\partial_t f_j^N + \mathcal{L}_j^\varepsilon f_j^N = \varepsilon^2 (N-j) C_{j,j+1}^\varepsilon f_{j+1}^N$$

with

$$C_{j,j+1}^\varepsilon f_{j+1}^N(x_1, v_1, \ldots, x_j, v_j) =$$
$$\frac{1}{\varepsilon^2} \sum_{i=1}^{j} \int_{\sigma_i(X_j)} d\sigma(x_{j+1}) \int dv_{j+1} n_{i,j+1} \cdot (v_i - v_{j+1}) f_{j+1}^N(x_1, v_1, \ldots, x_{j+1}, v_{j+1})$$

where $\sigma_i(X_j) = \{x \in \bigcap_{k=1}^{j}\{y||y-x_k| > \varepsilon\}||x-x_i| = \varepsilon\}$ and $n_{i,j} = (x_i - x_j)/|x_i - x_j|$. A solution can be found in form of a perturbative expansion:

$$f_j^N = \sum_{m\geq 0} \varepsilon^{2m}(N-j)(N-j-1)\dots(N-j-m+1)\int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m$$
$$U^\varepsilon(t-t_1)C^\varepsilon_{j,j+1}\dots U^\varepsilon(t_{m-1}-t_m)C^\varepsilon_{j+m-1,j+m}U^\varepsilon(t_m)f^N_{o,m+j}$$

where $f^N_{0,j}$ is the initial datum and $U^\varepsilon(t)g_j(x_1,v_1,\dots,x_j,v_j) := g_j(\Phi_j^{-t}(x_1,v_1,\dots,x_j,v_j)$; Φ is the j-particle hamiltonian flow. It can be proved that the sequence $\{f_j^N\}$ converges as N goes to infinity to the solution of the so called Boltzmann's hierarchy

$$\partial_t f_j + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i} f_j = C_{j,j+1} f_{j+1}$$

where

$$C_{j,j+1}f_{j+1}(x_1,v_1,\dots,x_j,v_j) = \lambda^{-1}\sum_{k=1}^{j}\int_{S_+} dv_{j+1}\int d\omega B(|v_k - v_{j+1}|,\omega)$$
$$[f_{j+1}(x_1,v_1',\dots,x_k,v_k',\dots,x_{j+1},v_{j+1}') - f_{j+1}(x_1,v_1,\dots,x_k,v_k,\dots,x_{j+1},v_{j+1})]$$

Indeed if we write f_j as a perturbative series

$$f_j = \sum_{m\geq 0}\lambda^{-m}\int_0^t dt_1\int_0^{t_1} dt_2\dots\int_0^{t_{m-1}} dt_m$$
$$U(t-t_1)C_{j,j+1}\dots U(t_{m-1}-t_m)C_{j+m-1,j+m}U(t_m)f^N_{0,m+j}$$

where U is the formal limit of U^ε, then it can be shown, for suitable short times, the convergence term by term of the series defining f_j^N and the existence of a dominating convergent series.

A remarkable property of the Boltzmann's hierarchy is the propagation of chaos: if the initial datum is a product measure

$$f_{0,j}(x_1,v_1,\dots,x_j,v_j) = \prod_{k=1}^{j} f_0(x_k,v_k)$$

then the solution is still a product

$$f_j(x_1,v_1,\dots,x_j,v_j) = \prod_{k=1}^{j} f(x_k,v_k,t)$$

where f is the solution of the BE with initial datum f_0.

Boltzmann equation, which describes an irreversible dynamics, thus arises from microscopic reversible equations as a limiting process characterized by a vanishing density of particles.

If we want to measure the average behaviour of some system observables, we can define an empirical distribution

$$\mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j,v_j}$$

and use it to compute the mean values. Lanford's theorem is really a law of large numbers because it allows to prove that those mean values, which depend on the specific motion of each particle, can be approximated by the mean values computed with respect to the solution of BE. Indeed as a consequence of the convergence of marginal distributions one have that for any smooth function ϕ and integer m

$$\mathbb{E}\left(\left|\int f(x,v,t)\phi(x,v) - \int \mu_N(dx,dv,t)\phi(x,v)\right|^m\right) \to 0$$

when N goes to infinity. That means two things: the behaviour of each particle is substantially the same of each other. It is not really surprising because they are indistinguishable and in the low density limit they become practically independent. Second, in the limit $N \to \infty$, the behaviour of each realization becomes identical to the mean one.

1.1.3 From Boltzmann Equation To Euler Equation

In practical situations, the mean free path, although the density is very low, is still far shorter than any macroscopic length as, for example, the size of the container of the gas. That scale separation can be exploited to derive hydrodynamics from BE through a rigorous proof of the local equilibrium hypothesis. The validity of such a result is only limited by the existence of smooth solution to Euler equations (EE), being matter of present research how to extend it to the case with shocks and singularities.

Boltzmann's collision operator $Q(f,f)$ admits a more general formulation:

$$Q(f,g)(v) = \frac{1}{2}\int d\omega \int_{S_+} dv_* B(|v-v_*|,\omega)[f(v')g(v'_*) + f(v'_*)g(v') - f(v)g(v_*) - f(v_*)g(v)]$$

where B is the differential cross section of the scattering between two particles and the apex denotes outgoing velocities. Now, by symmetry arguments, one can write

$$\int dv h(v)Q(f,g)(v) = \frac{1}{8}\int dv dv_* d\omega B(|v-v_*|,\omega)[h(v)+h(v_*)-h(v')-h(v'_*)]$$
$$[f(v')g(v'_*) + f(v'_*)g(v') - f(v)g(v_*) - f(v_*)g(v)]$$

which shows that $\int dvh(v)Q(f,g)(v) = 0$ if and only if h is a collision invariant, that is

$$h(v) + h(v_*) - h(v') - h(v'_*) = 0$$

We suppose that the collisions between particles are elastic, so collision invariants are mass, momentum and energy and any linear combination of them:

$$h(v) = \sum_{\alpha=0}^{4} c_\alpha \chi_\alpha(v)$$

where

$$\chi_0(v) = 1, \quad \chi_\alpha(v) = v_\alpha,\ \alpha = 1,\dots,3, \quad \chi_4(v) = \frac{1}{2}|v|^2$$

If we multiply BE by χ_α and integrate over v, we obtain the system

$$\partial_t \rho + \nabla \cdot [\rho u] = 0 \tag{1.2}$$

$$\partial_t[\rho u_i] + \sum_{j=1}^{3} \partial_{r_j}[\rho u_i u_j + \Pi_{i,j}] = 0,\ i = 1,\dots,3 \tag{1.3}$$

$$\partial_t \left[\rho\left(\frac{u^2}{2} + e\right)\right] + \nabla \cdot \left[\rho u\left(\frac{u^2}{2} + \frac{5}{3}e\right) + q\right] = 0 \tag{1.4}$$

where we introduced

$$\rho(r,t) = \int dv f(r,v,t)$$

$$\rho u(r,t) = \int dv v f(r,v,t)$$

$$\rho e(r,t) = \int dv \frac{1}{2}(v-u)^2 f(r,v,t)$$

the density, stream velocity and energy and

$$\Pi_{i,j} = \int dv f(r,v,t)(v_i - u_i)(v_j - u_j)$$

the stress tensor,

$$q = \int dv f(r,v,t)(v-u)^2(v-u)$$

the heat current vector. System (1.2) is not closed in ρ, u and e because we need to know f in order to compute the stress tensor and the heat current vector.

In the kinetic context, because of the low density regime we are dealing with, the local equilibrium distribution simplifies to the so called local maxwellians, the potential part

being irrelevant. Local maxwellians are maxwellians with respect to the velocity, but the parameters depend on space and time:

$$M(\rho, u, T, v) = \frac{\rho}{[2\pi T]^{\frac{3}{2}}} e^{-\frac{(v-u)^2}{2T}}$$

If we set $f = M$, the heat current vector vanishes, the stress tensor yields the equation of state of perfect gases $\Pi_{i,j} = \rho T \delta_{i,j}$ and $e = 3T/2$, while the system (1.2) becomes Euler's system of equations. But there is a major problem: local maxwellians are not solution of BE. Just in case all parameters are constant, M can solve BE.

However that difficulty is not surprising because EE belongs to macroscopic world and cannot be linked to BE, which lives on a kinetic scale, but in a suitable limit. In other words, if we call ε the ratio between kinetic and macroscopic units of length, then we expect that local maxwellians approximate the real solution of BE when ε goes to zero. Then let us define

$$f^\varepsilon(x, v, t) = f(\varepsilon^{-1}x, v, \varepsilon^{-1}t)$$

which solves the scaled BE

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon)$$

It is clear that, when $\varepsilon \to 0$, to avoid singularities f^ε has to be such that

$$\lim_{\varepsilon \to 0} Q(f^\varepsilon, f^\varepsilon) = 0$$

that is f^ε has to approach a local maxwellian, because it can be proved that the only solutions to equation (1.5) below are local maxwellians . All this stuff can be made rigorous by mean of the mathematical tool known as Hilbert expansion. The idea is to find a solution to the scaled BE as a power series

$$f^\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k f_k$$

In order to compute the f_k coefficients we plug the series in the equation and equate terms of the same order in ε. The order ε^{-1} gives

$$Q(f_0, f_0) = 0 \tag{1.5}$$

that is f_0 is a local maxwellian. Now, where do Euler equations arise from? They are simply solvability conditions for the equation of the following order ε^0

$$2Q(f_0, f_1) = \partial_t f_0 + v \cdot \nabla_x f_0 \tag{1.6}$$

Recalling that f_0 is a maxwellian M, we recognize in the left hand side the Boltzmann's linear operator

$$Lf = 2Q(M, f)$$

This operator has very good properties over the Hilbert's space defined by the inner product

$$(f, g) = \int dv M^{-1}(v) f(v) g(v)$$

The kernel of L is spanned by the collision invariants multiplied by M:

$$\mathrm{Ker}L = \{f | f = \sum_{\alpha=0}^{4} c_\alpha M \chi_\alpha,\ c_\alpha \in \mathbb{R},\ \alpha = 0, \ldots, 4\}$$

Thus the equation

$$Lf = g$$

may have a solution only if g stays in the orthogonal space to Ker$L$. In this case, because of the quoted good properties of L, we can state that there is a function f_g in $(\mathrm{Ker}L)^\perp$ such that $Lf_g = g$ and it is unique. Then we can add to f_g any function $\hat{f}$ in Ker$L$. $\hat{f}$ is known as hydrodynamical part and its arbitrariness allows to fulfill the compatibility conditions of the following orders equations.

If we go back to (1.6), we see that the solvability conditions

$$(M\chi_\alpha, g),\ \alpha = 0, \ldots, 4$$

yield exactly Euler equations. The picture is the following: at the lowest order in ε the solution to BE is given by a maxwellian, whose parameters satisfy EE. It means that the local equilibrium hypothesis become true in the scale separation limit $\varepsilon \to 0$ and the smooth changes between the parameters of small macroscopic cells are ruled by EE.

But something is still missing, because we do not know whether Hilbert's series converges or not and because it is not clear how to assign initial conditions to completely fix the hydrodynamical parts of the following orders. The latter problem can be fixed by a suitable initial layer expansion, that is by supplementing the solution with corrections which are effective only for kinetic times, far shorter than $\varepsilon^{-1}t$. For any macroscopic time, these corrections disappear exponentially fast in ε, but when $t \approx \varepsilon$ they provide the correct link with the initial conditions.

As to the convergence of Hilbert's series, it can be replaced by a truncated series with a reminder:

$$f^\varepsilon = \sum_{n=0}^{N} \varepsilon^n f_n + \varepsilon^m R^\varepsilon_{N,m}$$

What is unusual is that in general $m \neq N + 1$. The problem can be definitively solved by showing that a suitable $R^\varepsilon_{N,m}$ does exist and it is bounded.

All that program has been realized. Thus in the particular regime of low density, through the proof of the hypotheses of a law of large number and of a local equilibrium, a rigorous link between microscopic equation and macroscopic ones has been established.

1.1.4 Problems Related To The Derivation Of Navier-Stokes Equation

Boltzmann's H-theorem shows that there is a Lyapunov functional for the system, that is a quantity which is monotone in time, increasing or decreasing along the solution flow. Then BE has a nature intrinsically irreversible, in contrast with the reversibility of Newton equation. The relaxation to equilibrium may be thought to come with some dissipating effect; but we saw that with an hyperbolic scaling ($x = \varepsilon^{-1}x',\ t = \varepsilon^{-1}t'$) we get EE that is not dissipating anything. That should not be surprising, however, because we know the deep relationship between dissipation and diffusion. The latter is characterized by a growth of the size of domain where the gas is diffusing which goes as $t^{\frac{1}{2}}$. Then we expect to see dissipating effects arising from BE only after a much longer time. For this reason one can try a so called parabolic scaling: $x = \varepsilon^{-1}x',\ t = \varepsilon^{-2}t'$. Unfortunately this scaling does not yield any regular behaviour for the system.

Of course we started that story from the end. Actually also in the frame of the hyperbolic scaling one can try to compute corrections to EE which provide a dissipating trend. This is the most natural thing to do, having worked with Hilbert expansion, because it allows in principle to derive all the corrections to the local equilibrium.

Denote the substantial derivative with $D_t := \partial_t + v \cdot \nabla_x$; then the function f_1 is given by

$$f_1 = (f_1)_{D_t M} + \hat{f}_1$$

where the parameters of M solve EE and $\hat{f}_1$ is a linear combination of collision invariants multiplied by M:

$$\hat{f}_1 = M \sum_{\alpha=0}^{4} c_\alpha^{(1)} \chi_\alpha$$

The functions $c_\alpha^{(1)}$ are determined by the solvability conditions of the next order equation. They give the order ε corrections ρ_1, u_1 and T_1 to the hydrodynamical fields ρ, u and T. So we can introduce

$$\rho_\varepsilon = \rho + \varepsilon\rho_1,\quad u_\varepsilon = u + \varepsilon u_1,\quad T_\varepsilon = T + \varepsilon T_1$$

and evaluate their time derivatives in order to find a system of closed equations which they satisfy. But it is not possible because in order to close the equations one obtains greater order corrections are needed.

A solution to that obstacle is provided by a suitable resummation of Hilbert's series, called Chapmann-Enskog expansion. Its starting point is the following expression for f^ε

$$f^\varepsilon = M + \varepsilon G^\varepsilon$$

where the parameters of M have to be yet determined and G^ε is required to have null projection on the kernel of L_M, that is the linear Boltzmann's operator associated to M. The scaled BE is now

$$D_t M + \varepsilon D_t G^\varepsilon = L_M G^\varepsilon + \varepsilon Q(G^\varepsilon, G^\varepsilon) \tag{1.7}$$

If we denote by P_M and $P_M^\perp$ the projectors on $\mathrm{Ker}L_M$ and $(\mathrm{Ker}L_M)^\perp$ respectively, then we can subdivide the above equation in the two following relations

$$P_M D_t M + \varepsilon P_M D_t G^\varepsilon = 0 \tag{1.8}$$

$$P_M^\perp D_t M + \varepsilon P_M^\perp D_t G^\varepsilon = L_M G^\varepsilon + \varepsilon Q(G^\varepsilon, g^\varepsilon) \tag{1.9}$$

because the right hand side of (1.7) lives entirely in $(\mathrm{Ker}L_M)^\perp$. Then, thanks to the assumption $P_M G^\varepsilon = 0$, we can rewrite (1.9) as

$$G^\varepsilon = L_M^{-1} P_M^\perp D_t M + \varepsilon L_M^{-1}(P_M^\perp D_t G^\varepsilon - Q(G^\varepsilon, g^\varepsilon))$$

In the right hand side, the second term is surely of order greater or equal to ε; thus we can choose as a first approximation to G^ε the function G_1 defined by

$$G_1 = L_M^{-1} P_M^\perp D_t M$$

G_1 still depends on ε and this is the major difference with Hilbert expansion. We denoted, committing a slight abuse of notation, the inverse operator of the restriction to $(\mathrm{Ker}L_M)^\perp$ of L_M with L_M^{-1}. G_1 can be written explicitly as a functions of the hydrodynamical fields which parameterize M. Thus, replacing G^ε with G_1 in (1.8), we get

$$P_M D_t M + \varepsilon P_M D_t G_1 = 0$$

which is equivalent to the system

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0 \\ \rho \partial_t u + \rho (u \cdot \nabla_x) u + \nabla_x P &= \nabla_x \cdot (\nu_\varepsilon \nabla_x u) + \nabla_x (\sigma_\varepsilon \nabla_x \cdot u) \\ \frac{3}{2}\rho[\partial_t T + \rho(u \cdot \nabla_x)T] + P \nabla_x \cdot u &= \nabla_x \cdot (\kappa_\varepsilon \nabla_x T) + \nu_\varepsilon (\nabla_x u)^2 + \sigma_\varepsilon (\nabla_x u)^2 \end{aligned} \tag{1.10}$$

where P is function of ρ and T through the equation of state of ideal gases and

$$\nu_\varepsilon = \varepsilon \nu, \quad \sigma_\varepsilon = \varepsilon \sigma, \quad \kappa_\varepsilon = \varepsilon \kappa$$

with ν, σ and κ, respectively the shear viscosity, the volume viscosity and the heat conduction coefficient, given by formulas of Green-Kubo type.

The system of equations obtained is known as Navier-Stokes-Fourier system. But the coefficients ν_ε, σ_ε and κ_ε are of order ε, while in the true Navier-Stokes-Fourier equations they are finite. Only in presence of very large spatial gradients or waiting for a sufficiently long time, needed in order to make the effect of the second order differential operators appreciable, the dissipative nature of those equations arises.

Obviously, we could consider different scheme of resummation from Chapmann-Enskog's, leading to macroscopic equations other than those above. But Navier-Stokes (NS) equation is confirmed by a huge number of experiments and this is the reason why one wants to derive it and not something else. The great difficulty in deriving NS from a scaling argument is its lack of scale invariance. All equations describing a dynamics resulting from a limit of scale separation are invariant under such scaling. Let us see what happens if we try to scale the variables in NS as

$$x = \varepsilon^{-1} x', \quad t = \varepsilon^{-a} t'$$

The equations, which in their natural form are as in (1.10) but with finite transport coefficients, become

$$\begin{gathered}
\partial_{t'} \hat\rho + \varepsilon^{1-a} \nabla_{x'} \cdot (\hat\rho \hat u) = 0 \\
\hat\rho \partial_{t'} \hat u + \varepsilon^{1-a} [\hat\rho (\hat u \cdot \nabla_{x'}) \hat u + \nabla_{x'} \hat P] = \varepsilon^{2-a} [\nabla_{x'} \cdot (\hat\nu \nabla_{x'} \hat u) + \nabla_{x'} (\hat\sigma \nabla_{x'} \cdot \hat u)] \\
\frac{3}{2} \hat\rho [\partial_{t'} \hat T + \varepsilon^{1-a} [\hat\rho (\hat u \cdot \nabla_{x'}) \hat T] + \hat P \nabla_{x'} \cdot \hat u] = \varepsilon^{2-a} [\nabla_{x'} \cdot (\hat\kappa \nabla_{x'} \hat T) + \hat\nu (\nabla_{x'} \hat u)^2 + \hat\sigma (\nabla_{x'} \hat u)^2]
\end{gathered}$$

The hat denotes the scaled functions: $\hat f(x', t') = f(\varepsilon x', \varepsilon^a t')$. It is apparent that no choice of a can make the equations invariant. But if we scale the velocity field in a different manner, that is

$$u(\varepsilon^{-1} x', \varepsilon^{-a} t') = \varepsilon^{a-1} \hat u(x', t') \tag{1.11}$$

and we take $a = 2$, there are some chances, in a suitable regime, that the equations result invariant. In fact with that scaling and removing the hat over the scaled functions, the equations become

$$\begin{gathered}
\partial_{t'} \rho + \nabla_{x'} \cdot (\rho u) = 0 \\
\rho \partial_{t'} u + \rho (u \cdot \nabla_{x'}) u + \varepsilon^{-2} \nabla_{x'} P = \nabla_{x'} \cdot (\nu \nabla_{x'} u) + \nabla_{x'} (\sigma \nabla_{x'} \cdot u) \\
\frac{3}{2} \rho [\partial_{t'} T + \rho (u \cdot \nabla_{x'}) T] + P \nabla_{x'} \cdot u = \nabla_{x'} \cdot (\kappa \nabla_{x'} T) + \varepsilon^2 [\nu (\nabla_{x'} u)^2 + \sigma (\nabla_{x'} u)^2]
\end{gathered}$$

The additional scaling (1.11) implies that the velocities have to be small compared to a macroscopic reference velocity as, for example, the sound speed. The ratio between the

average velocity of the system and the sound speed is known as Mach number. So condition (1.11) determines a regime of low Mach number.

Moreover we see that the term containing the pressure P in the second equation above might exhibit a divergent behaviour. Then it is required that the pressure is constant till the second order, that is, if

$$P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots$$

then it is necessary that

$$\nabla_{x'} P_0 = 0, \quad \nabla_{x'} P_1 = 0$$

If the gas is ideal then there is a relationship between P and ρ and T: $P = \rho T$. Above requirements can be fulfilled supposing that the density and the temperature are constant at the lowest order

$$\rho = 1 + \varepsilon \rho_1 + \dots, \quad T = 1 + \varepsilon \theta + \dots$$

(with a suitable choice of units) and that the following equation holds

$$\nabla_{x'}(\rho_1 + \theta) = 0$$

That relation is known as Boussinesq condition. The asymptotic analysis of the system described by above equations yields a subset of NS called incompressible Navier-Stokes equation (INS). Thus we expect to obtain INS by starting from BE and scaling space and time as

$$q = \varepsilon^{-1} x, \quad \tau = \varepsilon^{-2} t$$

where (q, τ) are the microscopic variables and (x, t) the macroscopic ones. Of course a specific choice of the initial datum is needed to ensure the smallness of the velocity and the validity of the Bossinesq condition. Let us introduce

$$f^\varepsilon(x, v, t) = f(\varepsilon^{-1} x, v, \varepsilon^{-2} t)$$

Then BE is rewritten as

$$\left(\partial_t + \frac{1}{\varepsilon} v \cdot \nabla_x\right) f^\varepsilon = \frac{1}{\varepsilon^2} Q(f^\varepsilon, f^\varepsilon)$$

The initial datum is $f^\varepsilon(\cdot, \cdot, 0) = f_\varepsilon^{(0)}(\cdot, \cdot)$ and is such that

$$\int dv f_\varepsilon^{(0)} = 1 + \varepsilon \rho_1^{(0)} + O(\varepsilon^2), \quad \int dv \frac{v^2}{3} f_\varepsilon^{(0)} = 1 + \varepsilon \theta^{(0)} + O(\varepsilon^2)$$

with the Boussinesq condition

$$\nabla_x(\rho_1^{(0)} + \theta^{(0)}) = 0$$

and

$$\int dvv f_\varepsilon^{(0)} = \varepsilon u^{(0)} + O(\varepsilon^2)$$

which implements the low Mach number regime. A possible choice of $f_\varepsilon^{(0)}$ that fits all the above requirements is

$$f_\varepsilon^{(0)} = M_0^\varepsilon := \frac{1+\varepsilon\rho_1^{(0)}}{[2\pi(1+\varepsilon\theta^{(0)})]^{3/2}} e^{-\frac{(v-\varepsilon u^{(0)})^2}{2(1+\varepsilon\theta^{(0)})}}$$

with $\rho_1^{(0)} = -\theta^{(0)}$. Now one can proceeds as in the case of Euler equation, by expanding the solution as a power series in ε:

$$f^\varepsilon = M_t^\varepsilon + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \dots$$

where M_t^ε has the same functional form as M_0^ε but all its parameters depend on time. The ε order term is absent because if we wrote it, by the equation would follow that it lives in the null space of $L_{M_t^\varepsilon}$ and as a consequence it can be absorbed by M_t^ε itself.

As we have just seen deriving EE, all relevant relations come from solvability conditions for the equations that specify the functions f_k. Thus from the compatibility condition for the equation of f_2 we have that

$$\nabla_x \cdot u = 0$$
$$\nabla_x(\theta + \rho_1) = 0$$

They mean that the fluid is incompressible and the Boussinesq condition has to be satisfied at any time. Going on to the next equation, we get

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \triangle u$$
$$\frac{5}{2}(\partial_t \theta + u \cdot \nabla_x \theta) = \kappa \triangle \theta$$

The first equation together with the incompressibility condition gives the INS. The function p is now an unknown term, there is not an equation of state. The last equation is the heat equation with a transport term and can be solved after INS and independently of it. At the end one can choose $\rho_1 = -\theta$ and the system is completely satisfied. Again ν and κ are given by formulas of Green-Kubo type.

A theorem can be proved which formalize above sentences.

1.1.5 Vlasov Equation

Boltzmann equation can be derived successfully if the interaction potential is short ranged. Indeed the low density regime would be ineffective if particles were able to interact at long

distance too. But if the potential is long ranged and weak, then one can hope to describe its effect as a cumulative contribution from all interacting particles. In other words it become possible to replace the complex interaction of a particle with all the others with an average potential.

In presence of an external force field we can repeat the heuristic derivation of BE; besides the transport term, now a new component appears that measures the rate of change of the one particle probability density due to an increase or decrease of velocity produced by the force F_t. Thus we can write

$$\partial_t f_t = -v \cdot \nabla_x f_t - F \cdot \nabla_v f_t \tag{1.12}$$

Now if the particles interact via a potential U, the average force acting in the point x is

$$F_t(x) = -\int dx'dv' f_t(x', v')\nabla_{x'} U(x - x')$$

To be precise, the force so obtained is that produced by a single particle distributed according to f_t in the point x. In fact f_t is normalized to one. In order to get the total force we should multiply by the number of particles N. Moreover we expect a good approximation only for weak potentials. A way to implement these considerations is to replace U with U/N. Thus the force is really F_t.

We obtain Vlasov equation (VE) by using this definition of F_t in (1.12):

$$\partial_t f_t(x, v) + v \cdot \nabla_x f_t(x, v) = \left(\int dx'dv' f_t(x', v')\nabla_{x'} U(x - x')\right) \cdot \nabla_v f_t(x, v)$$

VE can be easily generalized to measures that are not absolutely continuous with respect to $dxdv$, by replacing $f_t dxdv$ with $\mu_t(dxdv)$, multiplying by a test function g and formally integrating by parts:

$$\partial_t \mu_t(g) = \mu_t(v \cdot \nabla_x g) - \mu_t \times \mu_t(\nabla_{x'} U(x - x') \cdot \nabla_v g) \tag{1.13}$$

where

$$\mu_t(g) = \int \mu_t(dxdv) g(x, v)$$

That form of VE allows an easy verification of the validity of the procedure followed above to derive it. Indeed if we choose as initial datum

$$\mu_0(dxdv) = \frac{1}{N}\sum_{j=1}^{N} \delta(q_j - x)\delta(p_j - v)dxdv$$

then the solution of (1.13) at time t is

$$\mu_t(dxdv) = \frac{1}{N}\sum_{j=1}^{N} \delta(q_j(t) - x)\delta(p_j(t) - v)dxdv$$

where the functions $q_j(t)$ and $p_j(t)$ verify a set of equations that we are going to deduce by inserting the above expression of μ_t in (1.13). We obtain

$$\sum_{j=1}^{N} \partial_t g(q_j(t), p_j(t)) = \sum_{j=1}^{N} p_j(t) \cdot \nabla_x g(q_j(t), p_j(t)) - \frac{1}{N} \sum_{j,k=1}^{N} \nabla_x U(q_j(t) - q_k(t)) \cdot \nabla_v g(q_j(t), p_j(t))$$

But

$$\partial_t g(q_j(t), p_j(t)) = \frac{d}{dt} q_j \cdot \nabla_x g + \frac{d}{dt} p_j \cdot \nabla_v g$$

so we can identify the coefficients of $\nabla_x g$ and $\nabla_v g$ respectively:

$$\frac{d}{dt} q_j(t) = p_j(t)$$

$$\frac{d}{dt} p_j(t) = -\frac{1}{N} \sum_{k=1}^{N} \nabla_x U(q_j(t) - q_k(t))$$

Those equations are simply Newton equations for N particles interacting via a potential U/N.

As for BE, VE can be derived from those Newton equations rigorously. Of course a limit procedure is required, specifically we have to let N go to infinity. That limit can be interpreted as a scale separation limit simply by imposing that the density stays constant. In fact

$$\rho \approx \frac{N}{|\Lambda|_{micr}}$$

where $|\Lambda|_{micr}$ is the volume of the container Λ in microscopic units. In macroscopic units

$$\rho \approx \frac{N}{\varepsilon^{-d} |\Lambda|}$$

Thus when $\varepsilon \to 0$ the number of particles has to diverge as $N \approx \varepsilon^{-d}$, being d the dimension of the space.

VE is widely used in plasma physics where it is coupled with Maxwell equations. We shall develop a model which inherits from VE the idea of replacing the actual interaction with an average of it. The potential used will be of Kac type, that is weak and long ranged.

1.2 Phase Segregation

We deal with nonlinear non-equilibrium phenomena describing the separation of the phases of a mixture of two or more components.

Consider for simplicity a mixture of two fluids or a binary alloy. When the system is suddenly quenched to a temperature below the coexistence curve, a coarsening starts which leads from the homogeneous phase to a situation where two phases coexist, each rich in one species, separated by an interface. But let us begin from the description of a simpler case, that is the coexistence of the phases of a single substance.

1.2.1 Equilibrium Of The Phases

A system in thermodynamic equilibrium can exhibit more than one phase, depending on the values of the temperature and the pressure, for example. A phase is a state of the body characterized by an order parameter or by a particular symmetry. The liquid and the vapour phases of water are distinguished by the values of the respective densities, but there is no difference as to spatial displacement of the molecules. On the contrary the solid state of water has a peculiar symmetry which allows to recognize that phase without any doubt. Actually we note in advance that the distinction between vapour and liquid is quite formal and beyond a certain temperature it looses any sense to speak about two different phases (existence of a critical point).

There are three kinds of conditions determining the coexistence of different phases at equilibrium. The first two are the equality of the temperature and the pressure. If we ask for only two coexisting phases, then let T_1, P_1 and T_2, P_2 be their temperatures and pressures. At equilibrium we have

$$T_1 = T_2$$

and

$$P_1 = P_2$$

The third type of condition involves the chemical potentials μ_i, i runs over the phases. At equilibrium also chemical potentials have to coincide:

$$\mu_1(T, P) = \mu_2(T, P)$$

where T and P are the values of temperature and pressure of the two coexisting phases. In general we can expect that a whole curve in the plane (T, P) corresponds to a situation of phase coexistence. It means that above that curve the system is homogeneous and all the volume is occupied by, say, phase 1. For values of temperature and pressure below the curve, it is phase 2 which fills all the available space. If (T, P) lies on the curve, then the volume is divided in two or more parts by interfaces between the two different phases.

Experimentally it can happen that a sample of a substance can stay in a certain phase also if the values of temperature and pressure pertain to a different phase. Of course those values have to be quite close to the coexistence curve. In that case the state of the system is not of equilibrium and it is then called metastable.

If we switch to the plane (v, T) of the specific volume and of the temperature, then the picture changes considerably because there is no more equality between the specific volumes of each phase, then there is no coexistence curve. The plane is divided in domains where the system stays in a particular phase; but there is also a region of values which do not correspond to any equilibrium state. Consider for simplicity a fixed temperature at which only two phases (e.g. liquid and vapour) can coexist at most. Then, keeping constant the total mass M (and the pressure) too, we increase by degrees the volume V. If at the beginning the system was in a liquid phase, it stays liquid until a precise volume V_a is reached. At that point the vapour phase appears; the respective fraction of volume and mass is initially very small, but the specific volume has already a well defined value. Also the specific volume of the liquid is now frozen to the value V_a/M. Progressively the liquid phase disappears and when a volume V_b is reached, the whole system is in the vapour phase.

There is a simple rule which links the mass of each phase and the value of the specific volume. Let M_i and V_i be the mass and the volume of phase i. Then

$$M_1 + M_2 = M, \quad V_1 + V_2 = V \tag{1.14}$$

Moreover, if $V_a \leq V \leq V_b$, then

$$\frac{V_1}{M_1} = v_a, \quad \frac{V_2}{M_2} = v_b$$

where $v_a = V_a/M$ and $v_b = V_b/M$ are constants depending only on the temperature (and the pressure). We can rewrite the second relation of (1.14) as

$$v_a M_1 + v_b M_2 = V$$

We have a system in the unknown M_1 and M_2. The solution is

$$M_1 = \frac{M v_b - V}{v_b - v_a}, \quad M_2 = \frac{V - M v_a}{v_b - v_a}$$

It follows that

$$\frac{M_1}{M_2} = \frac{M v_b - V}{V - M v_a} = \frac{v_b - \frac{V}{M}}{\frac{V}{M} - v_a}$$

which is equivalent to

$$M_1 \left(\frac{V}{M} - v_a \right) = M_2 \left(v_b - \frac{V}{M} \right)$$

known as the lever rule.

From the equalities of the chemical potentials μ_1 and μ_2 above, we have inferred the existence of a curve in the plane (T, P) whose points correspond to states of the system where two phases coexist. But there are points where three phases can appear at the same time. Indeed in principle if we require that

$$\mu_1(P,T) = \mu_2(P,T) = \mu_3(P,T)$$

we can still find a solution to that system. Of course it is not possible that more than three phases can coexist. The points where three phases are present are called triple points. They are the intersection of the coexistence curves of phase $(1,2)$, $(1,3)$ and $(2,3)$.

For any fixed value of the pressure, we can plot the chemical potentials of phases 1 and 2, say. The point of intersection at temperature T_0 specifies the phase transition, in the sense that if $T < T_0$ the system is in the phase with the lower potential and when $T > T_0$ the equilibrium is characterized by the other phase. The heat transferred from or to the body during the transition is linked to the derivatives of μ_1 and μ_2 in the point T_0. In fact it is given by $q = T_0(s_2 - s_1)$, where s_i is the entropy of phase i; moreover one has that $s_i = -\partial\mu_i/\partial T$, thus $q > 0$ (heat is absorbed from the ambient) if we are going in the direction of higher temperatures.

1.2.2 Critical Points

As mentioned above, phases can be distinguished by quantitative means by measuring, for example, their specific volumes or by qualitative considerations respective to their symmetry. In the former case it can happen that the coexistence curve stops in a point. For values of the temperature and of the pressure greater than those of that point, the two phases coincide, any distinction between them looses sense. Those points are called critical.

In presence of a critical point, in the plane (v, T) the domain of unstable states acquires a typical bell shape. At the vertex of the bell is located the critical point. There the specific volumes of the coexisting phases are the same and so there is no more quantitative distinction.

Following a line of reversible transformations surrounding the critical point, it is possible to go from a phase to another without crossing the coexistence curve, that is the two phases never come in contact.

If the phases have different symmetry, than a continuous path linking them can not exist. As a consequence there is no critical point and the coexistence curve goes to infinity or stops in the intersection with another curve.

1.2.3 A Simple Model Of Droplet Growth

When a sample is led below the coexistence curve by lowering the temperature, a germination of the second phase begins through the appearance of droplets. Just to give a first impression of the type of problem we are dealing with, we briefly report of a classical model of droplet growth, which although very simple is able to capture some interesting aspects of the system. Consider an Ising model with spins up and down. Let H be an external magnetic field. Suppose that the temperature T is below its critical value T_c: $T < T_c$. This is necessary to have metastable states, from which the nucleation process starts. When $H > 0$ the picture is well known: the system is composed by a sea of spins up and some isolated droplets of spins down, whose size is microscopic, that is essentially not visible at the scale of our observation. Let us denote with l the number of spins down in a droplet. On average we can think that there is a typical size R_l associated to cluster of l spins. Then we can study the distribution of the clusters respective to the number of spins from which they are constituted and we can then deduce the behaviour of R_l. Let n_l be the number of droplets built by l spins. At the equilibrium we have

$$n_l = Ne^{-\beta\varepsilon_l}$$

where ε_l is a suitable free energy. The natural guess for ε_l is

$$\varepsilon_l = 2Hl + \sigma l^{\frac{d-1}{d}}$$

We recognize two terms. The first on the left hand side is a bulk term which corresponds to the flipping of l spins. It is the cost to be paid to have l spin down in the magnetic field H. The second term is characterized by the surface tension $\sigma > 0$ and it is proportional to the interaction between the spins on the surface of the droplet and those surrounding it.

Until H is positive, when l increases n_l decreases. It means that the size of the clusters of spins down is microscopic. If we switch the sign of the magnetic field, $H < 0$, then a competition between bulk and surface term is primed. It yields a value l_c such that if $l < l_c$ then n_l is small, but if $l > l_c$ the size of the cluster becomes macroscopic. All of these are equilibrium considerations, but we can imagine that during the out of equilibrium process that leads to the formation of spins down droplets, the small ones shrink while the big ones grow. Indeed if almost all the spins of a droplet stay on its surface, then the interaction with the surrounding is able to destroy the droplet. On the contrary a stable droplet has a huge bulk, which is not sensible to what happens outside.

1.2.4 Multiple Components

Let us generalize the above considerations to a mixture of two or more components. We suppose that no chemical reaction takes place at equilibrium. Thus we can describe the

system using the concentrations $c_i,\ i = 1, \ldots, n$, of each component. Of course we have to remember the constraint

$$\sum_{i=1}^{n} c_i = 1$$

which amounts to say that only $n - 1$ concentrations are independent. The system can exhibit several phases containing one or more components. From the conditions determining the thermodynamical equilibrium of the mixture we can deduce how many phases can coexist. Indeed, as above, the temperature and the pressure have to be the same for all the phases of each component. Now let us denote the chemical potential respective to component i in the phase α with μ_i^α, then for any $i = 1, \ldots, n$

$$\mu_i^1 = \mu_i^2 = \ldots = \mu_i^r$$

where r is the number of different phases. Those chemical potentials are function of P and T but also of $n - 1$ independent concentrations for each phase. Thus we have $2 + r(n - 1)$ unknowns and $n(r - 1)$ conditions. The system is solvable only if we require that

$$n(r - 1) \leq 2 + r(n - 1) \iff r \leq n + 2$$

that is at equilibrium it is not possible to find more than $n + 2$ coexisting phases. That condition is known as Gibbs rule of the phases. We see at once that if $n = 1$, at most the system can present three phases in mutual contact; we know that this happens at the triple point and nowhere else.

If $n = 2$ and there is only one phase, then to describe completely the system we have to specify the pressure, the temperature and the concentration of one of the components in that phase. In fact if we call $f = n + 2 - r$ the number of thermodynamical degrees of freedom, in the above setting we get $f = 3$, that is there are three free parameters to choose. But if we want two phases, then $f = 2$ and giving the temperature and the pressure is sufficient to establish the equilibrium values of the concentrations of all the components in each phase.

From a different point of view, we can think to change the concentration, for example, keeping constant T and P and observe the system undergoing a transition from an homogeneous state to a situation in which two phases coexist.

To fix the ideas, let n be equal to 2. We can represent the state of the system with a point in a three-dimensional reference of coordinates: the pressure P, the temperature T and the chemical potential μ of one of the components. Because of the rule of the phases, the points corresponding to the coexistence of two phases lie on a surface. Three phases are present at the same time along a line (triple points line) and four phases can coexist only in isolated points.

If we replace the chemical potential with the concentration c, then the picture changes slightly. Now a surface is present too, but it divides the space in domains where equilibrium

is possible and others where a segregation of phases takes place. If we keep constant the pressure, we obtain from the intersection with the above surface a curve (equilibrium curve). Suppose to start from a homogeneous state and to increase the concentration c. Then two things may happen: a) the system evolves from a phase to another continuously; b) at a certain point a new phase appears, where all components have well defined concentrations. In the latter case we stay below the critical point.

We remark that under the equilibrium curve the system is not stable. If from an homogeneous state outside the not equilibrium region we suddenly quench the system into it, then it evolves in such a way that the concentrations converge more or less rapidly to the values obtained by intersecting the equilibrium curve with an horizontal line at the temperature of the cooling. Thus if the sample is quenched deeply below the equilibrium curve, spatially we observe regions where the concentration of the first component is low and that of the other is high and regions where the opposite happens. We decide to study that system at a scale where the layer separating the phases is reduced to a surface called interface. In this situation the evolution of the system towards equilibrium can be accounted of simply by describing the geometrical motion of the interface.

1.2.5 Nucleation And Spinodal Decomposition

Till now we dealt with equilibrium states, but the process which lead the sample from an homogeneous single phase condition to a two phases non homogeneous one goes through a sequence of not equilibrium states characterized by complicated structures that grow finally into droplets.

We specialize to the case where a critical point is present. In that situation the coexistence curve has a typical bell shape and the critical point is located at its vertex [GSS].

We also anticipate that because the initial and the final states are of thermodynamical equilibrium, a free energy function does exist and it provides the driving force of the whole phenomenon. If the final states was not of equilibrium, it would not be obvious to write a proper Lyapunov functional.

A classical approach to phase segregation distinguishes below the coexistence curve a region of unstable states from one of metastable states. If the system is cooled inside the metastable region, it begins to develop finite amplitude fluctuations like droplets. This process is known as nucleation. On the other hand, if we quench the system into the unstable region, infinitesimal fluctuations appear that yield macroscopic pattern. That is called spinodal decomposition. In late stages of phase segregation, the two process tend to coincide.

Also if nowadays it is not believed that a sharp distinction between nucleation and spinodal decomposition exists, we can formally define a line separating the unstable region from the metastable one by requiring that a suitable susceptibility diverges on its points.

In a mean field frame, the graph of the chemical potential as a function of the concentration exhibits a Van der Waals loop: stable states correspond to points where the first derivative is positive; unstable states are associated to negative first derivative; the classical spinodal points are those where the first derivative vanishes. Indeed the susceptibility can be defined as the inverse of $\partial\mu/\partial c$.

1.2.6 Germination Of The New Phase

The first stage of phase segregation is characterized only by stability criteria, in the sense that the growth or the disappearance of a concentration fluctuation rely on energy balance considerations.

In the framework of a simple mean field theory, we expect that a fluctuation of the concentration in a point r is favored or not according to the sign of the susceptibility in r. Indeed that sign corresponds to that of the curvature of the free energy as a function of the concentration.

In the metastable region, the susceptibility is positive. As said above droplets start to grow; but their size has to be such that the gain in free energy is greater than the loss due to surface tension. Thus a critical size exists and if droplets are smaller than that, they are going to disappear.

Below the spinodal curve, the susceptibility is negative, then in principle all fluctuations may develop. But now also, if the size of the new structure is too small, it is not energetically favored. On the other hand, large fluctuations have a very low rate of growth. So a typical length arises here too, which corresponds to the fastest growing patterns.

1.2.7 Experimental Techniques

There are two main methods to study experimentally phase segregation and structure formation: direct optical observation and small angle scattering of X-rays or neutrons. The former gives interesting information on the spatial distribution of droplets and interconnected patterns arising during nucleation and spinodal decomposition. The latter provides an easy measure of the mean amplitude of the fluctuations which destroy the unstable or metastable states, yielding to the thermodynamical equilibrium of two phases.

During an optical observation, one sees small droplets growing if initially the sample was quenched in the metastable region. The interface between different phases is initially diffuse but it sharpens more and more.

With scattering techniques the Fourier transform of the average of the product of the concentration in two different points is measured. It presents a maximum corresponding to the inverse length which characterizes the size of the droplets. It is so simple to estimate the temporal dependence of the dimensions of the growing structures. We will see that a power

law is expected and the exponents can be found by some properties of scale invariance of the system.

1.2.8 Scale Invariance And Power Laws

As said above, nucleation and spinodal decomposition become indistinguishable in late stages of phase segregation. Indeed droplets originated by nucleation can merge, thus building up more complex structures characteristic of the spinodal decomposition.

Moreover the thickness of the interfaces vanishes and the only length scale of the system is given by the average domain size, call it R. One can assume a space time invariance of the evolution and then a power law dependence of R is expected. In other words, the dynamics is such that if we get a shot of the concentrations, for example, at time t and then another one at time $b^{1/a}t$ and we zoom out the latter by br, then they can be superimposed.

In general it may be necessary to scale the observable also by b^{-c}. So let $F(r,t)$ be a function such as the concentration; then we suppose that

$$F(r,t) = b^{-c}F(br, b^{\frac{1}{a}}t)$$

All the information is brought by F at time 1, say. So we can put

$$b^{\frac{1}{a}}t = 1$$

and write

$$F(r,t) = t^{ac}F(rt^{-a},1) = R^c\tilde{F}\left(\frac{r}{R}\right), \quad \tilde{F}(\cdot) = F(\cdot,1)$$

where we introduced the length scale

$$R = t^a$$

We are interested in the exponent a, because it can be measured experimentally. There are large classes of system exhibiting the same exponents.

By our scale invariance assumption, using a simple model of a dissipative system with a conserved quantity, it is possible, but not shown here, to establish the following formula for a:

$$a = \frac{1}{d+2+\zeta-h}$$

where d is the dimension, ζ and h are the scale exponents of a mobility function and of an energy function respectively. Their values are determined by the physical specifications of the system. If the surface mobility, that is the individual motion of atoms on the interfaces, is predominant, then $\zeta = 1$; if, on the other hand the bulk mobility, that is the individual motion of atoms inside the domains, is the most relevant, then $\zeta = 0$. For fluids

the convective motion of the droplets determines the value $\zeta = -2$. As to h, if thermal fluctuations are effective then $h = 0$; if surface tension dominates then $h = d - 1$.

In a solid (alloy) the bulk mobility and the surface tension are the most crucial effects and then

$$a = \frac{1}{3}$$

For a liquid the surface tension is effective and $\zeta = -2$, so

$$a = 1$$

Of course there are situations where those exponents are not obtained exactly.

A peculiar case is represented by highly viscous fluids or by polymer blends. In fact they are liquid but they are expected to be ruled by dynamics of Mullins-Sekerka type, which provides a growth law as $R \propto t^{1/3}$.

1.2.9 Reducing Gravity Effects

Any experiment on earth has to face the presence of the gravity field. Of course in a great variety of situations its effects can be neglected, because with respect to other forces involved gravity may be often considered weak. But it is obvious that in phase segregating system the ultimate spatial configuration of the domains is ruled by gravity. Indeed the lighter phase ends to stay on top of the heavier one. Convection flows arise that modify the growth process of domains. In order to isolate the intrinsic characteristics of the process of phase segregation it is useful to establish experimental methods able to reduce or even to cancel gravity influences [BGP].

It turns out that the relevance of all known and remarkable effects of gravity is measured by the product

$$g\delta\rho$$

of the intensity g of the gravity field and of the difference $\delta\rho$ between the densities of each components. Then there are two ways: a) microgravity experiments, b) matching of the densities through the modification of the proportion between different isotopes of one of the components.

The validity of the approach b) has to be tested by comparing its results with those of the experiments carried out in microgravity conditions. It has been done for mixture of cyclohexane (C) and methanol (M). These fluids have densities very close to each other:

$$\rho_C \approx 0.77 g/cm^3$$
$$\rho_M \approx 0.79 g/cm^3$$

In order to match more accurately the values of the densities it is possible to modify the isotopic concentration of cyclohexane. Indeed a deuterated version of C, call it C^*, has

density

$$\rho_{C^*} \approx 0.89 g/cm^3$$

Denoting with M_C, V_C and M_{C^*}, V_{C^*} the masses and the volumes of C and C^* respectively, we require that

$$\frac{M_C + M_{C^*}}{V_C + V_{C^*}} = \rho_M$$

But

$$V_C = \frac{M_C}{\rho_C}, \quad V_{C^*} = \frac{M_{C^*}}{\rho_{C^*}}$$

so

$$\frac{M_C + M_{C^*}}{\frac{M_C}{\rho_C} + \frac{M_{C^*}}{\rho_{C^*}}} = \rho_M \tag{1.15}$$

Now let x be the concentration of C^*:

$$x = \frac{M_{C^*}}{M_C + M_{C^*}}, \quad 1 - x = \frac{M_C}{M_C + M_{C^*}}$$

then, if in (1.15) we divide numerator and denominator on the left hand side by $M_C + M_{C^*}$, we obtain

$$\frac{1}{\frac{1-x}{\rho_C} + \frac{x}{\rho_{C^*}}} = \rho_M$$

Solving with respect to x yields

$$x = \frac{\rho_{C^*}(\rho_M - \rho_C)}{\rho_M(\rho_{C^*} - \rho_C)}$$

It means that in order to match the densities of methanol and cyclohexane a deuteration ratio $x \approx 0.19$ is expected to be necessary.

A subtle objection can be put forward regarding the actual nature of the mixture of methanol, cyclohexane and its heavier isotope. Indeed formally it is a ternary mixture and no more a binary one. Thus it might happen that the shape and position of the coexistence curve and the value of the susceptibility are modified. Well, this is not the case. Accurate experiments were done proving the strict equivalence of CM and C^*CM (density matched) with respect to the process of phase segregation.

Moreover if some doubt still remains, a lucky circumstance makes the matched densities approach even more affordable. Indeed, though the densities of the components are matched, those of different phases may not. It is a fact that in order to compensate such a difference a smaller deuteration ratio is needed, its value being around $0.03 \div 0.04$.

As said above the validation of the measures carried on isodensity systems goes through microgravity experiments. They are performed on rockets able to reach altitudes where g is reduced by a factor of 10^{-4}. In such condition the duration of an experiment can not exceed a time of about 10 minutes. Then in order to observe a nearly complete phase segregation,

the sample has to be prepared over the critical point and then quenched below T_c of a small amount of about $10mK$. It is not easy to perform that task. So the first experiments gave results difficult to be understood, because no phase segregation was observed. Then it was shown that a deviation from the critical point of about 1% is able to destroy the fast linear in time growth process of domains and to slow the segregation, which spends some hours to complete.

Subsequent experiments were performed, which showed a precise equivalence with earth isodensity experiments. Indeed interconnected structures arise, whose pattern is identical to that observed in laboratory.

Just a comment upon the images obtained by mean of direct observation. The droplets seen are not the whole phase domains, but just a section of them along the thin layer next to the observer.

Our model disregards completely any external force field and the two species of particle involved have the same mass. The experiments described above legitimate our simplifications. Indeed, as we will show, we are able to reproduce the late stages behaviour of a binary fluid (with very high viscosity) as expected by observation.

Chapter 2

The Model

2.1 Preliminaries

We want to study phenomena of phase segregation in systems composed by two components. In particular we are interested in the late stages of the process, when well defined domains rich in each species are separated by surfaces called interfaces, whose width is vanishing. The expression sharp interface is used for this situation. Physically it means that the width of the interfaces is very small compared to the size of the container.

So two scales are present in the problem: a macroscopic one, that of the container, which is used to describe the bulk properties; a microscopic one that allows to zoom in the neighborhood of the interface and to see as regular and smooth the transition between the phases, that macroscopically appears sharp and not continuous.

Our starting point is the more fundamental that we are able to deal with. In other words we decide to put our model in a kinetic context, because the space and the time are continuous but we still manage to perform our computations without too much difficulties. This would not be possible starting, for example, from Newton equations.

Moreover, phase segregation of binary fluids is often driven by hydrodynamical flows, which are best described in certain regimes by Boltzmann equation. So it is believed that a proper coupling of Boltzmann and Vlasov equation can provide a complete description of the motion of domains along the lines of the velocity field.

The model we propose is a modification of the Vlasov-Boltzmann equation, where the collision kernel is replaced by a linear (Fokker-Planck) operator over the velocities. Thus there is no more short range (and hard) interaction between particles, but the system is put in contact with a reservoir at fixed temperature β^{-1}. Then energy and momentum are not conserved and they can not play a dominant role in the diffusive limit which links the kinetic scale to the macroscopic one. Only mass is conserved. The problem is easier, but still rich of properties and the approach to equilibrium is not trivial because it is

characterized by a rearrangement of the shapes of the domains, without the simplification provided by the disappearance of one of the phase, as in models where no quantity is conserved at all.

Probably we could admit a short range interaction, but it would have to be weak in such a way that each component is in a high temperature regime or however far from a vapour-liquid or a liquid-solid transition. Indeed we want that only two phases are present, while from a two component mixture we can expect up to four phases.

2.2 Kac Potentials

Particles of the same species do not interact; between different components a long range repulsive potential U_γ is active. It is modeled by a compact support smooth function U and its intensity and range are modulated by a parameter γ.

$$U_\gamma(r) = \gamma^d U(\gamma r)$$

In our case $d = 3$ and U depends only on $|r|$. We mean that if two particles of different species are at distance $|r|$ their interaction energy is $U_\gamma(r)$.

Kac potentials have been used with success to investigate the equilibrium properties of a great variety of systems. In fact they solve the inconsistencies introduced by mean field models. As it is well known, in the infinite volume limit, mean field gives a non convex free energy. It is thermodynamically incorrect. With Kac potentials, if after the thermodynamic limit, one performs the limit $\gamma \to 0$, then a convexification of the free energy is provided. The sequence of limits $L \to \infty$, L the size of the container, and $\gamma \to 0$ is called Lebowitz-Penrose limit. Of course inverting the order of the limits would give a free theory. Interesting results come from the possibility to take those limits together choosing, e.g., $L = \gamma^{-1}$.

2.3 The Equations

Let f_i denote a distribution function on the phase space of a single particle of species i, $i = 1, 2$. In the following we will write often equations where indexes i and j appear together; we always mean that they may assume the values 1 and 2 and that they are not equal, that is if $i = 1$ then $j = 2$ only and viceversa. The equations solved by f_i are

$$\partial_\tau f_i + v \cdot \nabla_x f_i + F_i \cdot \nabla_v f_i = L_\beta f_i \tag{2.1}$$

where L_β is a Fokker-Planck operator modeling the contact with a reservoir at temperature β^{-1}:

$$L_\beta f_i = \nabla_v \cdot \left(M_\beta \nabla_v \left(\frac{f_i}{M_\beta} \right) \right)$$

and M_β is a maxwellian with mean zero and variance β^{-1},

$$M_\beta = \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} e^{-\beta\frac{v^2}{2}}$$

The interaction between particles of different species is contained in F_i which is the auto-consistent Vlasov force

$$F_i = -\nabla_x \int dx' U_\gamma(x - x') \int dv f_j(x', v, \tau)$$

The potential U_γ is positive, that is the force is repulsive. Integrating over positions and velocities equation (2.1), we discover that the total mass of each component is conserved. We work initially on a torus to avoid boundary terms.

2.4 Free Energy And Gradient Flux Equation

Now we want to underline the deep relationship between our equations and a wide class of evolution equations for the density profiles of the system. The dynamics provided by such evolution equations is, in many cases, directly justified by a microscopic model on a lattice. Moreover it exactly captures the behaviour of the system at equilibrium. We are speaking about gradient flux equations, whose form is

$$\partial_t \rho = \nabla \cdot \left[\sigma(\rho)\nabla\left(\frac{\delta\mathcal{F}(\rho)}{\delta\rho}\right)\right]$$

where ρ is the density, σ is the mobility and $\mathcal{F}$ is a suitable free energy functional. $\mathcal{F}$ decreases along the trajectories of the gradient flux equation and equilibrium is reached when the density profile equals the minimizer of the free energy. It is well known that the thermodynamics arising from the Lebowitz-Penrose limit is equivalent to that generated by the minimization of a proper free energy over a well defined set of density profiles. This fact allows to go from a microscopic scale to a mesoscopic one where lattice is replaced by space continuum and powerful analytic tool become available. As to the non-equilibrium dynamics yielding the free energy minimizing densities, it was proved in several cases, through a scaling procedure, the convergence of the empirical densities over a lattice to the solutions of a gradient flux equation. Those links with microscopic processes make us confident in using gradient flux equations.

Our kinetic equations have got two points of contact with the gradient flux equations. First, we can define a functional $\mathcal{G}$ of the distribution functions which is decreasing in time if computed on the solutions of (2.1). Then, by scaling space as ε^{-1} and time as ε^{-2} and choosing $\gamma = \varepsilon$ and sending ε to zero, we obtain an equation that can be put in

gradient flux form. Moreover the corresponding free energy can be deduced by $\mathcal{G}$ when the dependence on the velocity of the distribution functions f_i is maxwellian.

Then let us introduce

$$\mathcal{G}(f_1, f_2) = \int dxdv(f_1 \ln f_1 + f_2 \ln f_2) + \frac{\beta}{2} \int dxdv(f_1 + f_2)v^2 + \\ + \beta \int dxdy U_\gamma(x-y) \int dv f_1(x,v) \int dv' f_2(y,v')$$

The derivative of $\mathcal{G}$ with respect to the time is

$$\frac{d}{dt}\mathcal{G}(f_1, f_2) = \sum_{i=1}^{2} \int dxdv \partial_t f_i \left[\ln f_i + 1 + \frac{\beta}{2} v^2 + \beta \int dy \gamma^3 U(\gamma|x-y|) \int dv' f_j \right]$$

We notice that

$$\ln M_\beta = -\frac{3}{2} \ln \frac{2\pi}{\beta} - \frac{\beta}{2} v^2$$

from which we can extract $\beta v^2/2$. Moreover we replace $\partial_t f_i$ with its expression from the equations of motion, so we have

$$\frac{d}{dt}\mathcal{G}(f_1, f_2) = \sum_{i=1}^{2} \int dxdv \left[\nabla_v \cdot \left(M_\beta \nabla_v \left(\frac{f_i}{M_\beta} \right) \right) - v \cdot \nabla_x f_i + \right.$$

$$\left. + \nabla_v f_i \cdot \nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv'' f_j \right] \left[\ln \frac{f_i}{M_\beta} + 1 - \frac{3}{2} \ln \frac{2\pi}{\beta} + \beta \int dy \gamma^3 U(\gamma|x-y|) \int dv' f_j \right]$$

We recall that we are working on a torus; thus we can put to zero all boundary terms coming from the use of the divergence theorem. Then it follows that

$$\int dxdv \nabla_v \cdot \left(M_\beta \nabla_v \left(\frac{f_i}{M_\beta} \right) \right) \left[1 - \frac{3}{2} \ln \frac{2\pi}{\beta} + \beta \int dy \gamma^3 Y(\gamma|x-y|) \int dv' f_j \right] = 0$$

$$\int dxdv \nabla_v f_i \cdot \nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv'' f_j \left[1 - \frac{3}{2} \ln \frac{2\pi}{\beta} + \beta \int dy \gamma^3 U(\gamma|x-y|) \int dv' f_j \right] = 0$$

$$\int dxdv(-v \cdot \nabla_x f_i) \left[1 - \frac{3}{2} \ln \frac{2\pi}{\beta} \right] = 0$$

Now let us define

$$I_1 = \int dxdv(-v \cdot \nabla_x f_i) \left[\ln \frac{f_i}{M_\beta} + \beta \int dy \gamma^3 U(\gamma|x-y|) \int dv' f_j \right]$$

$$I_2 = \int dxdv \left(\nabla_v f_i \cdot \nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv'' f_j \right) \ln \frac{f_i}{M_\beta}$$

Then

$$I_1 = \int dxdv f_i v \cdot \left[\nabla_x \ln \frac{f_i}{M_\beta} + \beta \nabla_x \int dy \gamma^3 U(\gamma|x-y|) \int dv' f_j \right]$$

and

$$I_2 = \int dxdv(-f_i)\nabla_v \ln \frac{f_i}{M_\beta} \cdot \nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv'' f_j =$$
$$= \int dxdv(-f_i) \left[\nabla_v \ln f_i + \beta v \right] \cdot \nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv'' f_j$$

So we have

$$I_1 + I_2 = \int dxdv \left[f_i v \cdot \nabla_x \ln f_i - f_i \nabla_v \ln f_i \cdot \nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv'' f_j \right] =$$
$$= \int dxdv \left[v \cdot \nabla_x f_i - \nabla_v f_i \cdot \nabla_x \int dx' \gamma^3 U(\gamma|x-x'|) \int dv'' f_j \right] = 0$$

Summarizing it survives only a term:

$$\sum_{i=1}^{2} \int dxdv \nabla_v \cdot \left(M_\beta \nabla_v \left(\frac{f_i}{M_\beta} \right) \right) \ln \frac{f_i}{M_\beta}$$

But then

$$\frac{d}{dt}\mathcal{G} = \sum_{i=1}^{2} \int dxdv \nabla_v \cdot \left(M_\beta \nabla_v \left(\frac{f_i}{M_\beta} \right) \right) \ln \frac{f_i}{M_\beta} = -\sum_{i=1}^{2} \int dxdv \frac{M_\beta^2}{f_i} \left[\nabla_v \left(\frac{f_i}{M_\beta} \right) \right]^2 \leq 0$$

It shows that $\mathcal{G}$ is a Lyapunov functional for the evolution of the segregating mixture. It also apparent that at equilibrium the solution has to be maxwellian in velocities, that is in the form $f_i = \rho_i M_\beta$, where $\rho_i = \int dv f_i$ is the density and it depends only on position.

If we fill (2.1) with $f_i = \rho_i M_\beta$ and we require that f_i is stationary, that is $\partial_t f_i = 0$, then we find the equation

$$\frac{1}{\beta} \ln \rho_i(x) + \int dx' U_\gamma(x-x') \rho_j(x') = C_i \tag{2.2}$$

The constants C_i are associated to the initial datum and they determine whether a solution exists and it is unique or not and thus if it can be non homogeneous. If we now compute $\mathcal{G}$ on functions of the form $\rho_i M_\beta$, we get the following free energy

$$\mathcal{F}_\gamma(\rho_1, \rho_2) = \int dx(\rho_1 \ln \rho_1 + \rho_2 + \ln \rho_2) + \beta \int dxdy U_\gamma(x-y)\rho_1(x)\rho_2(y)$$

where we disregarded some terms because we decided to work on a set of densities whose integral, that is the total mass, is fixed. In this framework the disregarded terms are

constants and they can not modify the behaviour of the free energy. It can be proved by a suitable choice of the masses and if the temperature is quite low that $\mathcal{F}$ is minimized by non homogeneous solutions of (2.2). In dimension $d = 1$, those solutions are called fronts and from now on they will be denoted by w_i. The fronts have good monotonicity properties, reaching their asymptotic values exponentially fast. They describe the best way to match the different densities of two coexisting phases. From a macroscopic viewpoint they appear as step functions. We have that

$$\lim_{z \to \pm\infty} w_i(z) = \rho_i^{\pm}$$

where $\rho_i^{\pm}$ is the equilibrium density of component i in the phase $+$ or $-$.

We will prove rigorously in a later chapter that, solutions of (2.1) converge to solutions of

$$\beta^2 \partial_t \rho_i = \triangle \rho_i + \beta \cdot (\rho_i \nabla U \star \rho_j) \tag{2.3}$$

in a proper scaling of time and space. The relevant observation is the possibility to rewrite the above equation in a gradient flux form

$$\partial_t \bar{\rho} = \nabla \cdot \left(\mathcal{M}(\bar{\rho}) \nabla \left(\frac{\delta \mathcal{F}}{\delta \bar{\rho}} \right) \right)$$

where

$$\bar{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \quad \mathcal{M}(\bar{\rho}) = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$$

and $\mathcal{F} = \mathcal{F}_1$. It means that the free energy which drives the evolution of the macroscopic density ρ_i coincides with that yielded by $\mathcal{G}$ when it is computed on distributions maxwellian in velocities, which are the equilibrium states expected.

2.5 Results

Our results, both rigorous and formal, are obtained through scaling procedures. That technique allows to exploit the separation between the different scales present in the problem. As we have already said in late stages of nucleation and spinodal decomposition, a big gap arises between the size of the container and the width of the interfaces. The latter in our model, using a non local potential, is determined by the range of the interaction between different components. Then we can depict several physical situations, simply by adopting a suitable scaling for γ.

2.5.1 The Macroscopic Equation

In order to converge to the gradient flux equation written above, we need to scale space as ε^{-1} and time as ε^{-2}. Moreover if we choose $\gamma = \varepsilon$ we obtain, at macroscopic scale, a width

of the interfaces which is finite and of order 1. Thus we avoid to loose information on early steps of the segregation process and we can trust the resulting equation as we would do with the Cahn-Hilliard equation.

We proved that for any ε sufficiently small, the solution of the scaled kinetic equation f_i^ε does not differ from $M_\beta \rho_i$, being ρ_i the solution of (2.3), more than a constant times ε, with respect to the following norm

$$||h||_-^2 = \int_{\mathbb{T}^d \times \mathbb{R}^d} dx dv M_\beta^{-1} \sum_{i=1,2} h_i^2$$

2.5.2 Stefan Problem And Mullins-Sekerka Flow From Macroscopic Equation

Starting from equation (2.3) we formally show that scaling space as ε^{-1} and time as ε^{-q} with $q = 2, 3$ we obtain well known dynamics for the densities and the interfaces. We are now investigating the late stages of phase segregation. Choosing $q = 2$ we get as limiting system a Stefan problem, that is a parabolic equation for the densities with boundary conditions on the interfaces moving with a velocity determined by the gradient of the densities:

$$\begin{cases} \partial_t \hat{\rho}_i^{(0)} = \dfrac{1}{\beta} \nabla \cdot (\hat{\rho}_i^{(0)} \nabla \hat{\mu}_i^{(0)}) & \text{in the bulk} \\ \lim\limits_{h \to 0^\pm} \hat{\rho}_i^{(0)}(r + h\nu) = \rho_i^\pm & \text{on the interface} \\ \hat{\rho}_i^{(0)}(r, 0) = \bar{\rho}_i(r) & \text{everywhere} \end{cases}$$

where $\bar{\rho}_i$ is a suitable initial datum and $\hat{\mu}_i^{(0)}$ is the chemical potential. The velocity of the interface is given by

$$V = \frac{[\hat{\rho}_i^{(0)} \nu \cdot \nabla \hat{\mu}_i^{(0)}]_-^+}{\beta [w_i]_{-\infty}^{+\infty}} \tag{2.4}$$

where $[\cdot]_-^+$ denotes the jump through the interface. Thanks to symmetry properties of the fronts the velocity is actually independent of the index i.

When $q = 3$, the density profiles have reached wherever the equilibrium values and the dynamics is reduced to the reshaping of the domains following a flow given by a combination of Mullins-Sekerka and Hele-Shaw motion. They both preserve the total area of the domains (Hele-Shaw of each domain) while promoting a surface reduction. Spheres are stationary solutions. We write the limiting equation in the subsection below.

We remark that in order to establish the limiting equation it is not necessary to consider a dependence of the interface from the small parameter ε. It becomes essential, instead, when we try to correctly determine the higher orders correction to the interface motion.

2.5.3 Mullins-Sekerka From Kinetic Equation

We succeeded, formally, in combining the macroscopic limit and the sharp interface limit by choosing $\gamma = 1$ and scaling space as ε^{-1} and time as ε^{-q}, $q = 2, 3$. In this notes we report only about the case $q = 3$. It yields a quasi-static motion of interfaces, which is a superposition of Mullins-Sekerka and Hele-Shaw motion. The limiting equations are organized in two coupled sets of relations involving linear combinations of the first correction in the small parameter ε to the chemical potential. They are

$$\begin{cases} \triangle\psi = 0 & \text{in the bulk} \\ \psi = \dfrac{KS}{[w_1]_{-\infty}^{+\infty}} & \text{on the interface} \\ V = \dfrac{1}{2\beta(\rho_1^+ - \rho_1^-)} \left[\dfrac{1}{\bar\rho}(\bar\rho^2 - \bar\varphi^2)[\nu \cdot \nabla\psi]_-^+ + \dfrac{1}{\bar\rho}[\bar\varphi\nu \cdot \nabla\zeta]_-^+ \right] & \text{on the interface} \end{cases}$$

and

$$\begin{cases} \triangle\zeta = 0 & \text{in the bulk} \\ [\zeta]_-^+ = \dfrac{2|\bar\varphi|KS}{[w_1]_{-\infty}^{+\infty}} & \text{on the interface} \\ 0 = [\nu \cdot \nabla\zeta]_-^+ & \text{on the interface} \end{cases}$$

All notation will be clear later. We say only that K is the curvature and S the surface tension.

Chapter 3

Stefan Problem And Mullins-Sekerka Flow From Macroscopic Equation

We start the presentation of our results with a formal computation. In the next chapter we will turn to a rigorous approach which will clarify the relationship between our kinetic model and the macroscopic equations used in this chapter.

Here we derive from the macroscopic equation arising from our kinetic model a limiting dynamics for the densities and the interfaces of the mixture under study. At this stage it is not necessary to consider a dependence of the interface on the scale separation parameter ε. It will turn to be important when in a later chapter we will compute the corrections of higher order in ε to the limiting equations.

3.1 The Scaling

The macroscopic evolution of the density profiles of the two components is given by the parabolic equation

$$\beta^2 \partial_\tau \rho_i = \triangle \rho_i + \beta \nabla \cdot (\rho_i \nabla U \star \rho_j)$$

We expect that the late stages of phase segregation are characterized by domains separated by very thin interface. In order to explore that regime we scale space and time as

$$\tau = \varepsilon^{-q} t, \quad x = \varepsilon^{-1} r$$

where $q = 2, 3$ discriminates between a situation where densities are not still equal everywhere to the thermodynamic equilibrium values and they keep evolving and one where in the bulk there is nothing moving and only interfaces perform an almost static deformation leading to the final equilibrium state.

From a macroscopic viewpoint, ε is the width of the interfaces with respect to the size of the container. Let us define scaled functions

$$\rho_i^{\varepsilon,q}(r,t) = \rho_i(\varepsilon^{-1}r, \varepsilon^{-q}t)$$

which satisfy

$$\beta^2\varepsilon^{q-2}\partial_t\rho_i^{\varepsilon,q} = \triangle\rho_i^{\varepsilon,q} + \beta\nabla\cdot(\rho_i^{\varepsilon,q}\nabla U^\varepsilon \star \rho_j^{\varepsilon,q}) \tag{3.1}$$

and

$$U^\varepsilon(r) = \varepsilon^{-d}U(\varepsilon^{-1}r)$$

where d is the space dimension; in our computations we set $d = 3$. If we introduce the so called chemical potential

$$\mu_i^{\varepsilon,q} = \frac{1}{\beta}\ln\rho_i^{\varepsilon,q} + U^\varepsilon \star \rho_j^{\varepsilon,q}$$

we can rewrite (3.1) as a continuity equation:

$$\varepsilon^{q-2}\partial_t\rho_i^{\varepsilon,q} = \frac{1}{\beta}\nabla\cdot(\rho_i^{\varepsilon,q}\nabla\mu_i^{\varepsilon,q})$$

Formally the scaled densities can be written as power series of the small parameter ε. Moreover we can imagine to use, just around the interfaces between domains, a special functional form for the coefficient of the powers of ε. Indeed we isolate the dependence on the distance $\phi(r)$ of the point r from an interface in a variable z, which is fast in the sense that a variation of order 1 in ϕ corresponds to a change of ε^{-1} in z.

Thus we propose two expansions for $\rho_i^{\varepsilon,q}$. The former is

$$\rho_i^{\varepsilon,q} = \sum_{i=0}^{\infty}\varepsilon^n\hat{\rho}_i^{(n)}$$

which is thought to be valid in the bulk. The latter is

$$\rho_i^{\varepsilon,q} = \tilde{\rho}_i^{\varepsilon,q} = \sum_{i=0}^{\infty}\varepsilon^n\tilde{\rho}_i^{(n)}$$

where $\tilde{\rho}_i^{\varepsilon,q} = \tilde{\rho}_i^{\varepsilon,q}(z,r,t)$ and, if we denote with $\bar{\nabla}$ the gradient computed keeping z fixed and with ν the unit vector in the direction of the normal to the interface, $\nu\cdot\bar{\nabla}\tilde{\rho}_i^{\varepsilon,q} = 0$. This expansion has to be used next to the interface.

3.1.1 Matching Expansions

Of course the two power series have to be matched in some point. It means that, for a generic function f which can represent the densities or the chemical potentials, we require that

$$(\hat{f}_i^{(0)} + \varepsilon\hat{f}_i^{(1)} + \varepsilon^2\hat{f}_i^{(2)} + \ldots)(x(r) + \varepsilon z\nu, t) \approx (\tilde{f}_i^{(0)} + \varepsilon\tilde{f}_i^{(1)} + \varepsilon^2\tilde{f}_i^{(2)} + \ldots)(z,r,t)$$

where $x(r)$ is the intersection with the interface of the normal to it through the point r. Equating same orders in ε we get

$$\begin{aligned}
\hat{f}_i^{(0)}(r,t) &= \tilde{f}_i^{(0)}(z,r,t) && \text{(for } r \text{ next to } \Gamma_t \text{ and } z \text{ very large)} \\
\hat{f}_i^{(1)} + z\nu\cdot\nabla\hat{f}_i^{(0)} &= \tilde{f}_i^{(1)} && (\quad '' \quad) \\
\hat{f}_i^{(2)} + z\nu\cdot\nabla\hat{f}_i^{(1)} + \frac{z^2}{2}(\nu\cdot\nabla)^2\hat{f}_i^{(0)} &= \tilde{f}_i^{(2)} && (\quad '' \quad) \\
&\cdots
\end{aligned}$$

3.2 The Hat Series

The expansion in ε for the densities induces a similar power series for the chemical potentials $\mu_i^{\varepsilon,q}$

$$\mu_i^{\varepsilon,q} = \sum_{n=0}^{\infty}\varepsilon^n\hat{\mu}_i^{(n)}, \quad \mu_i^{\varepsilon,q} = \tilde{\mu}_i^{\varepsilon,q} = \sum_{n=0}^{\infty}\varepsilon^n\tilde{\mu}_i^{(n)}$$

Filling (3.1) with those power series, a set of equations arises which can be solved step by step. We begin writing the equations for each order, discriminating between $q=2$ and $q=3$. For the hat series we have

$$\sum_{n=0}^{\infty}\varepsilon^{n+q-2}\partial_t\hat{\rho}_i^{(n)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varepsilon^{n+m}\frac{1}{\beta}\nabla\cdot(\hat{\rho}_i^{(n)}\nabla\hat{\mu}_i^{(m)})$$

$$\Longleftrightarrow$$

$$\sum_{n=q-2}\varepsilon^n\partial_t\hat{\rho}_i^{(n-q+2)} = \sum_{n=0}^{\infty}\varepsilon^n\sum_{\substack{l,l'\geq 0\\ l+l'=n}}\frac{1}{\beta}\nabla\cdot(\hat{\rho}_i^{(l)}\nabla\hat{\mu}_i^{(l')})$$

Thus if $q=2$, for any $n\geq 0$

$$\partial_t\hat{\rho}_i^{(n)} = \sum_{\substack{l,l'\geq 0\\ l+l'=n}}\frac{1}{\beta}\nabla\cdot(\hat{\rho}_i^{(l)}\nabla\hat{\mu}_i^{(l')})$$

On the other hand, if $q=3$, then for $n=0$

$$\frac{1}{\beta}\nabla\cdot(\hat{\rho}_i^{(0)}\nabla\hat{\mu}_i^{(0)}) = 0$$

and for $n\geq 1$

$$\partial_t\hat{\rho}_i^{(n-1)} = \sum_{\substack{l,l'\geq 0\\ l+l'=n}}\frac{1}{\beta}\nabla\cdot(\hat{\rho}_i^{(l)}\nabla\hat{\mu}_i^{(l')})$$

3.3 The Tilde Series

In order to write the equations for the tilde series, we need some formulas relating the differential operators in r to those in z. We choose $z = \varepsilon^{-1}\phi(r)$, noting that there is a temporal dependence too, implicit in ϕ because the interfaces are moving. From that the velocity V of the interfaces arises. We have

$$\nabla\tilde{f} = \frac{1}{\varepsilon}\nu\partial_z\tilde{f} + \bar{\nabla}\tilde{f}; \quad \partial_t\tilde{f} = \frac{1}{\varepsilon}V\partial_z\tilde{f} + \partial_t\tilde{f}; \quad \triangle\tilde{f} = \frac{1}{\varepsilon^2}\partial_z^2\tilde{f} + \frac{1}{\varepsilon}(\nabla\cdot\nu)\partial_z\tilde{f} + \bar{\triangle}\tilde{f}$$

Then from (3.1) that can be rewritten as

$$\varepsilon^{q-2}\partial_t\rho_i^{\varepsilon,q} = \frac{1}{\beta}\left[\nabla\rho_i^{\varepsilon,q}\cdot\nabla\mu_i^{\varepsilon,q} + \rho_i^{\varepsilon,q}\triangle\mu_i^{\varepsilon,q}\right]$$

we arrive to

$$\varepsilon^{q-2}(\varepsilon^{-1}V\partial_z\tilde{\rho}_i^{\varepsilon,q} + \partial_t\tilde{\rho}_i^{\varepsilon,q}) = \frac{1}{\beta}\left[(\varepsilon^{-1}\nu\partial_z\tilde{\rho}_i^{\varepsilon,q} + \bar{\nabla}\tilde{\rho}_i^{\varepsilon,q})\cdot(\varepsilon^{-1}\nu\partial_z\tilde{\mu}_i^{\varepsilon,q} + \bar{\nabla}\tilde{\mu}_i^{\varepsilon,q}) + \right.$$
$$\left. + \tilde{\rho}_i^{\varepsilon,q}(\varepsilon^{-2}\partial_z^2\tilde{\mu}_i^{\varepsilon,q} + \varepsilon^{-1}(\nabla\cdot\nu)\partial_z\tilde{\mu}_i^{\varepsilon,q} + \bar{\triangle}\tilde{\mu}_i^{\varepsilon,q})\right]$$

Replacing $\tilde{\rho}_i^{\varepsilon,q}$ with its power series we get

$$\sum_{n=q-3}^{\infty}\varepsilon^n V\partial_z\tilde{\rho}_i^{(n-q+3)} + \sum_{n=q-2}^{\infty}\varepsilon^n\partial_t\tilde{\rho}_i^{(n-q+2)} = \frac{1}{\beta}\left[\sum_{n=-2}^{\infty}\varepsilon^n \sum_{\substack{l,l'\geq 0\\ l+l'-2=n}} \partial_z\tilde{\rho}_i^{(l)}\partial_z\tilde{\mu}_i^{(l')} + \right.$$
$$+\sum_{n=0}^{\infty}\varepsilon^n \sum_{\substack{l,l'\geq 0\\ l+l'=n}} \bar{\nabla}\tilde{\rho}_i^{(l)}\cdot\bar{\nabla}\tilde{\mu}_i^{(l')} + \sum_{n=-2}^{\infty}\varepsilon^n \sum_{\substack{l,l'\geq 0\\ l+l'-2=n}} \tilde{\rho}_i^{(l)}\partial_z^2\tilde{\mu}_i^{(l')} +$$
$$\left. +\sum_{n=-1}^{\infty}\varepsilon^n \sum_{\substack{l,l'\geq 0\\ l+l'-1=n}} (\nabla\cdot\nu)\tilde{\rho}_i^{(l)}\partial_z\tilde{\mu}_i^{(l')} + \sum_{n=0}^{\infty}\varepsilon^n \sum_{\substack{l,l'\geq 0\\ l+l'=n}} \tilde{\rho}_i^{(l)}\bar{\triangle}\tilde{\mu}_i^{(l')}\right]$$

Now suppose that $q = 2$, then for $n = -2$ we have

$$\frac{1}{\beta}[\partial_z\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(0)} + \tilde{\rho}_i^{(0)}\partial_z^2\tilde{\mu}_i^{(0)}] = 0$$

Then for $n=-1$

$$V\partial_z\tilde{\rho}_i^{(0)} = \frac{1}{\beta}\Big[\partial_z\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(1)} + \partial_z\tilde{\rho}_i^{(1)}\partial_z\tilde{\mu}_i^{(0)} + \\ +\tilde{\rho}_i^{(0)}\partial_z^2\tilde{\mu}_i^{(1)} + \tilde{\rho}_i^{(1)}\partial_z^2\tilde{\mu}_i^{(0)} + (\nabla\cdot\nu)\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(0)}\Big]$$

Finally for any $n \geq 0$ the equations are

$$V\partial_z\tilde{\rho}_i^{(n+1)} + \partial_t\tilde{\rho}_i^{(n)} = \frac{1}{\beta}\Bigg[\sum_{\substack{l,l'\geq 0\\ l+l'-2=n}} \partial_z\tilde{\rho}_i^{(l)}\partial_z\tilde{\mu}_i^{(l')} + \tilde{\rho}_i^{(l)}\partial_z^2\tilde{\mu}_i^{(l')} + \\ + \sum_{\substack{l,l'\geq 0\\ l+l'-1=n}} (\nabla\cdot\nu)\tilde{\rho}_i^{(l)}\partial_z\tilde{\mu}_i^{(l')} + \sum_{\substack{l,l'\geq 0\\ l+l'=n}} \bar{\nabla}\tilde{\rho}_i^{(l)}\cdot\bar{\nabla}\tilde{\mu}_i^{(l')} + \tilde{\rho}_i^{(l)}\bar{\triangle}\tilde{\mu}_i^{(l')}\Bigg]$$

Now set $q=3$. For $n=-2$ we obtain again

$$\frac{1}{\beta}[\partial_z\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(0)} + \tilde{\rho}_i^{(0)}\partial_z^2\tilde{\mu}_i^{(0)}] = 0$$

Let us choose $n=-1$:

$$\frac{1}{\beta}\Big[\partial_z\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(1)} + \partial_z\tilde{\rho}_i^{(1)}\partial_z\tilde{\mu}_i^{(0)} + \tilde{\rho}_i^{(0)}\partial_z^2\tilde{\mu}_i^{(1)} + \\ +\tilde{\rho}_i^{(1)}\partial_z^2\tilde{\mu}_i^{(0)} + (\nabla\cdot\nu)\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(0)}\Big] = 0$$

For $n=0$ we have

$$V\partial_z\tilde{\rho}_i^{(0)} = \frac{1}{\beta}\Big[\partial_z\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(2)} + \partial_z\tilde{\rho}_i^{(1)}\partial_z\tilde{\mu}_i^{(1)} + \partial_z\tilde{\rho}_i^{(2)}\partial_z\tilde{\mu}_i^{(0)} + \\ +\bar{\nabla}\tilde{\rho}_i^{(0)}\cdot\bar{\nabla}\tilde{\mu}_i^{(0)} + \tilde{\rho}_i^{(0)}\partial_z^2\tilde{\mu}_i^{(2)} + \tilde{\rho}_i^{(1)}\partial_z^2\tilde{\mu}_i^{(1)} + \tilde{\rho}_i^{(2)}\partial_z^2\tilde{\mu}_i^{(0)} + \\ +(\nabla\cdot\nu)\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(1)} + (\nabla\cdot\nu)\tilde{\rho}_i^{(1)}\partial_z\tilde{\mu}_i^{(0)} + \tilde{\rho}_i^{(0)}\bar{\triangle}\tilde{\mu}_i^{(0)}\Big]$$

And at the end for any $n \geq 1$

$$V\partial_z\tilde{\rho}_i^{(n)} + \partial_t\tilde{\rho}_i^{(n-1)} = \frac{1}{\beta}\left[\sum_{\substack{l,l' \geq 0 \\ l+l'-2=n}} \partial_z\tilde{\rho}_i^{(l)}\partial_z\tilde{\mu}_i^{(l')} + \tilde{\rho}_i^{(l)}\partial_z^2\tilde{\mu}_i^{(l')} + \right.$$

$$\left. + \sum_{\substack{l,l' \geq 0 \\ l+l'-1=n}} (\nabla\cdot\nu)\tilde{\rho}_i^{(l)}\partial_z\tilde{\mu}_i^{(l')} + \sum_{\substack{l,l' \geq 0 \\ l+l'=n}} \bar{\nabla}\tilde{\rho}_i^{(l)}\cdot\bar{\nabla}\tilde{\mu}_i^{(l')} + \tilde{\rho}_i^{(l)}\bar{\triangle}\tilde{\mu}_i^{(l')}\right]$$

3.4 The Case $q = 2$: Stefan Problem

The late stages behaviour of the two species melt can now be established by looking at the above equations. We start from the case $q = 2$ because the result is more transparent and it is easier to get it. From the hat series we deduce at the lowest order in ε the following evolution equation

$$\partial_t\hat{\rho}_i^{(0)} = \frac{1}{\beta}\nabla\cdot(\hat{\rho}_i^{(0)}\nabla\hat{\mu}_i^{(0)})$$

where the order zero chemical potential is

$$\hat{\mu}_i^{(0)} = \frac{1}{\beta}\ln\hat{\rho}_i^{(0)} + \hat{U}\hat{\rho}_j^{(0)}$$

and $\hat{U} = \int drU(r)$. Provided a suitable initial datum, we still need to specify the values of $hat\rho_i^{(0)}$ on the interface. In fact we can consider the limiting problem as a free boundary evolution, where the velocity of the interface is related to the derivatives of the densities. Then we rewrite the lowest order equation for the tilde series

$$\partial_z[\tilde{\rho}_i^{(0)}\partial_z\tilde{\mu}_i^{(0)}] = 0$$

it is equivalent to

$$\tilde{\rho}_i^{(0)} = C_i$$

C_i a constant. Well, it can be proved that this equation admits a unique solution. But then

$$\tilde{\rho}_i^{(0)} = w_i$$

where w_i is the front, solution of

$$\partial_z\tilde{\mu}_i^{(0)} = 0$$

As a byproduct we see that $C_i = 0$. The front has got a well defined behaviour at infinity, where it reaches exponentially fast the values of the densities of each component in each phase at equilibrium. In other terms

$$\lim_{z \to \pm\infty} w_i(z) = \rho_i^{\pm}$$

where ρ_i^+ is the density of component i in the phase $+$ and analogously for ρ_i^-.

From the matching conditions we deduce that

$$\lim_{h \to 0^{\pm}} \hat{\rho}_i^{(0)}(r + h\nu) = \rho_i^{\pm}$$

with r on the interface. The velocity V appear for the first time in the order ε^{-1} equation of the tilde series:

$$\begin{aligned} V \partial_z \tilde{\rho}_i^{(0)} &= \frac{1}{\beta}[\partial_z \tilde{\rho}_i^{(0)} \partial_z \tilde{\mu}_i^{(1)} + \tilde{\rho}_i^{(0)} \partial_z^2 \tilde{\mu}_i^{(1)}] \\ &\Longleftrightarrow \\ V \partial_z w_i &= \frac{1}{\beta} \partial_z (w_i \partial_z \tilde{\mu}_i^{(1)}) \end{aligned}$$

because w_i and as a consequence $\tilde{\mu}_i^{(0)} = T \ln w_i + \tilde{u} \star w_j$ are functions only of z. $\tilde{U}$ is the potential integrated over all variables but one.

Integrating in z we get

$$V[w_i]_{-\infty}^{+\infty} = \frac{1}{\beta}[w_i \partial_z \tilde{\mu}_i^{(1)}]_{-\infty}^{+\infty}$$

and by using the matching conditions

$$V = \frac{[\hat{\rho}_i^{(0)} \nu \cdot \nabla \hat{\mu}_i^{(0)}]_-^+}{\beta [w_i]_{-\infty}^{+\infty}} \tag{3.2}$$

where $[\cdot]_-^+$ denotes the jump through the interface. Thanks to symmetry properties of the fronts the velocity is actually independent of the index i.

The limiting problem is then

$$\begin{cases} \partial_t \hat{\rho}_i^{(0)} = \dfrac{1}{\beta} \nabla \cdot (\hat{\rho}_i^{(0)} \nabla \hat{\mu}_i^{(0)}) & \text{in the bulk} \\ \lim\limits_{h \to 0^{\pm}} \hat{\rho}_i^{(0)}(r + h\nu) = \rho_i^{\pm} & \text{on the interface} \\ \hat{\rho}_i^{(0)}(r, 0) = \bar{\rho}_i(r) & \text{everywhere} \end{cases}$$

where $\bar{\rho}_i$ is a suitable initial datum. The velocity of the interface is given by (3.2).

3.5 The Case $q = 3$: Mullins-Sekerka Flow

We now turn to the case $q = 3$. We anticipate that the final equations will involve intricate combinations of the first order corrections to the densities. Indeed the order zero densities are simply constant because of the very late stages of phase segregation we are dealing with.

The very initial datum is chosen constant and equating the values given by equilibrium in each phase. From the lowest order equation in the hat series

$$\frac{1}{\beta}\nabla \cdot (\hat{\rho}_i^{(0)} \nabla \hat{\mu}_i^{(0)}) = 0 \tag{3.3}$$

we deduce that $\hat{\rho}_i^{(0)}$ has to be constant and then to coincide with the initial datum. This follows from the fact that equation (3.3) admits constants as solutions, with the property that if the value of the constant is in the unstable region of the phase diagram, then it in also an unstable solution of the equation. On the contrary if, as in our case, thanks to the choice of the initial data, the value corresponds to a stable region of the phase diagram, then the solution is stable and it can not vary also at later times.

As to the lowest order in the tilde series, it is the same as in the case $q = 2$. Then we can easily conclude that

$$\tilde{\rho}_i^{(0)} = w_i$$

For $n = 1$ in the bulk we have

$$\partial_t \hat{\rho}_i^{(0)} = \frac{1}{\beta}\left[\nabla \cdot (\hat{\rho}_i^{(0)} \nabla \hat{\mu}_i^{(1)}) + \nabla \cdot (\hat{\rho}_i^{(1)} \nabla \hat{\mu}_i^{(0)})\right] \tag{3.4}$$

but $\hat{\rho}_i^{(0)}$ and thus $\hat{\mu}_i^{(0)}$ are constants, then (3.4) implies

$$\triangle \hat{\mu}_i^{(1)} = 0$$

This equation is a fundamental ingredient in building the limiting problem. On the tilde series side we write down the next order equation

$$\frac{1}{\beta}\partial_z (w_i \partial_z \tilde{\mu}_i^{(1)}) = 0$$

which is equivalent to

$$w_i \partial_z \tilde{\mu}_i^{(1)} = C_i$$

By the matching conditions we know that

$$\partial_z \tilde{\mu}_i^{(1)} \to \nu \cdot \nabla \hat{\mu}_i^{(0)}$$

as $z \to \infty$. But $\nabla \hat{\mu}_i^{(0)} = 0$ because as seen above the densities at the lowest order are constants. It follows that $C_i = 0$. Thus

$$\partial_z \tilde{\mu}_i^{(1)} = 0$$

It means that also $\tilde{\mu}_i^{(1)}$ is a constant. Now let us introduce some auxiliary functions. We begin from

$$\bar{\rho}_i = 1_{z<0}\rho_i^- + 1_{z\geq 0}\rho_i^+$$

which is a step function through the interface. Then

$$\psi = \hat{\mu}_1^{(1)} - \hat{\mu}_2^{(1)}, \;\; \zeta = \bar{\rho}_1\hat{\mu}_1^{(1)} + \bar{\rho}_2\hat{\mu}_2^{(1)}$$

It clear at once that ψ and ζ are harmonic

$$\triangle\psi = 0, \;\; \triangle\zeta = 0$$

Moreover it can be proved that

$$\tilde{\mu}_1^{(1)}[w_1]_{-\infty}^{+\infty} + \tilde{\mu}_2^{(1)}[w_2]_{-\infty}^{+\infty} = KS$$

where K is the curvature and S the surface tension. But by matching conditions and symmetry properties of the fronts, it follows that

$$\tilde{\mu}_1^{(1)}[w_1]_{-\infty}^{+\infty} + \tilde{\mu}_2^{(1)}[w_2]_{-\infty}^{+\infty} = \psi[w_1]_{-\infty}^{+\infty}$$

so that

$$\psi[w_1]_{-\infty}^{+\infty} = KS$$

In order to determine the dependence of the velocity from the density fields we need a further order in the tilde series:

$$V\partial_z w_i = \frac{1}{\beta}[\partial_z w_i \partial_z \tilde{\mu}_i^{(2)} + w_i \partial_z^2 \tilde{\mu}_i^{(2)}]$$
$$\Longleftrightarrow$$
$$V\partial_z w_i = \frac{1}{\beta}\partial_z(w_i \partial_z \tilde{\mu}_i^{(2)})$$

Integrating over z and using matching conditions we get

$$V[w_i]_{-\infty}^{+\infty} = \frac{1}{\beta}[\bar{\rho}_i \nu \cdot \nabla \hat{\mu}_i^{(1)}]_-^+$$

If we introduce

$$\bar{\rho} = \frac{\bar{\rho}_1 + \bar{\rho}_2}{2}, \;\; \bar{\varphi} = \frac{\bar{\rho}_1 + \bar{\rho}_2}{2}$$

it easy to show, but we will do that in a later chapter, that

$$V = \frac{1}{2\beta(\rho_1^+ - \rho_1^-)} \left[\frac{1}{\bar{\rho}} (\bar{\rho}^2 - \bar{\varphi}^2)[\nu \cdot \nabla \psi]_-^+ + \frac{1}{\bar{\rho}} [\bar{\varphi} \nu \cdot \nabla \zeta]_-^+ \right]$$

and

$$[\zeta]_-^+ = \frac{2|\bar{\varphi}|KS}{[w_1]_{-\infty}^{+\infty}}, \quad [\nu \cdot \nabla \zeta]_-^+ = 0$$

At the end the limiting problem is given by the following two coupled system of equations

$$\begin{cases} \triangle \psi = 0 & \text{in the bulk} \\ \psi = \dfrac{KS}{[w_1]_{-\infty}^{+\infty}} & \text{on the interface} \\ V = \dfrac{1}{2\beta(\rho_1^+ - \rho_1^-)} \left[\dfrac{1}{\bar{\rho}} (\bar{\rho}^2 - \bar{\varphi}^2)[\nu \cdot \nabla \psi]_-^+ + \dfrac{1}{\bar{\rho}} [\bar{\varphi} \nu \cdot \nabla \zeta]_-^+ \right] & \text{on the interface} \end{cases}$$

and

$$\begin{cases} \triangle \zeta = 0 & \text{in the bulk} \\ [\zeta]_-^+ = \dfrac{2|\bar{\varphi}|KS}{[w_1]_{-\infty}^{+\infty}} & \text{on the interface} \\ 0 = [\nu \cdot \nabla \zeta]_-^+ & \text{on the interface} \end{cases}$$

More details will be given later.

Chapter 4

Macroscopic Equation

4.1 Expansion

In this section we begin the study of the hydrodynamical limit for the Vlasov-Fokker-Plank equations which constitute our kinetic model. We consider the diffusive scaling in which the space is scaled as ε^{-1} and time as ε^{-2} and $\gamma = \varepsilon$ so that the width of the interface is of order 1 on the macroscopic scale. Define

$$f^{\varepsilon}{}_i(x,v,t) := f_i(\varepsilon^{-1}x, v, \varepsilon^{-2}t), \qquad i = 1,2,\ x \in \mathbb{T}^d, v \in \mathbb{R}^d.$$

The equation for $f^{\varepsilon}{}_i$ is

$$\partial_t f_i^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_i^\varepsilon + \frac{1}{\varepsilon} F_i^\varepsilon \cdot \nabla_v f_i^\varepsilon = \frac{1}{\varepsilon^2} L_\beta f_i^\varepsilon \tag{4.1}$$

Here F_i^ε is the rescaled Vlasov force with $\gamma = \varepsilon$:

$$F_i^\varepsilon(x,t) = -\nabla_x \int_{\mathbb{T}^d} \mathrm{d}x' U(|x-x'|) \int_{\mathbb{R}^d} \mathrm{d}v f_j^\varepsilon(x',v,t) := -\nabla_x U \star \rho_j^\varepsilon$$

We substitute in (4.1) the formal power series for f_i^ε and F_i^ε

$$f_i^\varepsilon = \sum_{n=0}^\infty \varepsilon^n f_i^{(n)}, \qquad F_i^\varepsilon = \sum_{n=0}^\infty \varepsilon^n F_i^{(n)}$$

$$F_i^{(n)} = -\nabla_x U \star \int_{\mathbb{R}^d} \mathrm{d}v f_j^{(n)}(x',v,t)$$

We get

$$\frac{1}{\varepsilon^2} L_\beta f_i^{(0)} + \frac{1}{\varepsilon}\left\{ L_\beta f_i^{(1)} - v \cdot \nabla_x f_i^{(0)} - F_i^{(0)} \cdot \nabla_v f_i^{(0)} \right\} -$$

$$\sum_{n=0}^\infty \varepsilon^n \left\{ \partial_t f_i^{(n)} + v \cdot \nabla_x f_i^{(n+1)} + \sum_{l,l' \geq 0 : l+l'-1=n} \left(F_i^{(l)} \cdot \nabla_v f_i^{(l')} \right) - L_\beta f_i^{(n+2)} \right\} = 0$$

At each order in ε we get an equation. We write down here explicitly the first three orders:

ε^{-2}) $L_\beta f_i^{(0)} = 0$

ε^{-1}) $v \cdot \nabla_x f_i^{(0)} + F_i^{(0)} \cdot \nabla_v f_i^{(0)} = L_\beta f_i^{(1)}$

ε^{0}) $\partial_t f_i^{(0)} + v \cdot \nabla_x f_i^{(1)} + F_i^{(0)} \cdot \nabla_v f_i^{(1)} + F_i^{(1)} \cdot \nabla_v f_i^{(0)} = L_\beta f_i^{(2)}$

From ε^{-2}) we deduce from the properties of L_β that $f_i^{(0)}$ is the Maxwellian M_β multiplied by a density factor depending on x and t:

$$f_i^{(0)} = \rho_i(x,t) M_\beta \tag{4.2}$$

Replacing this expression in the second equation (order ε^{-1}) we get

$$M_\beta v \cdot (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)) = L_\beta f_i^{(1)}$$

So a solution has to be of the form:

$$f_i^{(1)} = M_\beta (A + B \cdot v)$$

where A will be fixed by the equations of the next orders and B is the vector

$$B = -\frac{1}{\beta}(\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j))$$

If we put these expressions of $f_i^{(0)}$ and $f_i^{(1)}$ in the ε^0) equation and integrate over v, remembering that

$$\int_{\mathbb{R}^3} dv M_\beta v_i v_j = \frac{1}{\beta}\delta_{ij},$$

we find the equations for the zero order densities

$$\partial_t \rho_i - \frac{1}{\beta^2}(\Delta_x \rho_i + \beta \nabla_x \cdot (\rho_i \nabla_x (U \star \rho_j))) = 0. \tag{4.3}$$

Now our aim is to show that a solution of equation (4.1) does exist and its limit as ε goes to zero is given by (4.2), with ρ_i satisfying (4.3). We try to solve (4.1) in terms of a truncated expansion

$$f_i^\varepsilon = \sum_{n=0}^{K} \varepsilon^n f_i^{(n)} + \varepsilon^m R_i \tag{4.4}$$

Replacing expression (4.4) for f_i^ε in equation (4.1) we get:
For any n between 0 and K

$$\partial_t f_i^{(n-2)} + v \cdot \nabla_x f_i^{(n-1)} + \sum_{l,l' \geq 0 : l+l'-1=n-2} \left[-\nabla_x U \star \int_{\mathbb{R}^3} dv f_j^{(l)} \right] \cdot \nabla_v f_i^{(l')} - L_\beta f_i^{(n)} = 0 \tag{4.5}$$

$f_i^{(s)} = 0, s \leq 0$ and for the remainder

$$\partial_t R_i + \frac{1}{\varepsilon}[v \cdot \nabla_x R_i + F_i^\varepsilon \cdot \nabla_v R_i + B_i \cdot \Gamma_i] = \frac{1}{\varepsilon^2} L_\beta R_i - \varepsilon^{K-m-1} A_i \tag{4.6}$$

where we defined

$$B_i = \sum_{n=0}^{K} \varepsilon^n \nabla_v f_i^{(n)}, \qquad \Gamma_i = -\nabla_x U \star \int_{\mathbb{R}^3} dv R_j$$

$$A_i = \partial_t f_i^{(K-1)} + v \cdot \nabla_x f_i^{(K)} + \varepsilon \partial_t f_i^{(K)} + \sum_{n=K-1}^{2K-1} \varepsilon^{n-K+1} \sum_{\substack{0 \leq l,l' \leq K \\ l+l'-1=n}} F_i^{(l)} \cdot \nabla_v f_i^{(l')}$$

We will find solutions $f_i^{(n)}$ to equations (4.5) in next section and we will study the equation for the remainder R_i in section 5. Here we state the results:

Denote by $(\cdot,\cdot)_-$ the following L_2 scalar product and with $||\cdot||_-$ the associated norm

$$(h,g)_- := \int_{\mathbb{T}^d \times \mathbb{R}^d} dx dv M_\beta^{-1}(v) \sum_{i=1,2} [h_i(x,v) g_i(x,v)].$$

Put $A = \{A_i\}_{i=1,2}$ and $R = \{R_i\}_{i=1,2}$. Then

Theorem 1 *Given a classical solution $\rho_i(x,t)$ of the macroscopic equations (4.3) in the time interval $[0,T]$, there is a constant C depending on T, such that a unique solution to (4.6) exists and satisfies the bounds*

$$\sup_{t \in [0,T]} ||R(\cdot,t)||_- \leq C \varepsilon^{K-1-m} ||A||_-$$

As a consequence,

Corollary 1 *Under the assumptions of Theorem 1 and $m \geq 1, K-1-m \geq 0$ there is a positive constant ε_0 such that for $\varepsilon < \varepsilon_0$ there is a smooth solution $f_i^\varepsilon(x,v,t)$ to the rescaled Vlasov-Fokker-Plank equations (4.1) satisfying for some constant C*

$$\sup_{t \in [0,T]} ||f_i^\varepsilon - M_\beta \rho_i||_- \leq C\varepsilon$$

4.2 Expansion Terms

In this section we show existence and regularity properties of $f_i^{(n)}$. For simplicity, we write down the proof only for $K = 2$, but the argument goes on for any K. The structure of equations (4.5) is very simple: they are of the form

$$L_\beta f = h \tag{4.7}$$

with h a given function. In the Hilbert space with scalar product

$$(h,g)_M = \int_{\mathbb{R}^d} dv h(\cdot,v) g(\cdot,v) M_\beta^{-1}$$

the kernel $\mathcal{N} = \ker(L_\beta)$ is made of constants in velocity multiplied by M_β. Hence this equation has a solution iff h is in the orthogonal to the kernel of L_β namely iff

$$\int dv h(v) = 0. \tag{4.8}$$

Moreover, the solution is determined but for a term in the kernel which is of the form a function of x,t times the Maxwellian. Starting from the lowest order, we will see that $h(x,v,t) = P(x,t,v)M_\beta$ with P a polynomial of the velocity with coefficients eventually depending on x,t. If P is a polynomial in v the equation (4.7) can be solved uniquely in the orthogonal to the null space of L_β. The solution is again a polynomial of the same degree of P multiplied by the maxwellian M_β. In other words, if $M_\beta P \in \mathcal{N}^\perp$ with P a polynomial, then there exists a unique $f \in \mathcal{N}^\perp$ such that (4.7) holds. This statement can be shown by finding explicitly solutions to the problem (4.7) for different choices of the polynomial P. We are interested to polynomials of degree up to the second.

For $n = 0, 1$ the equations 4.5 are of the form 4.7 with $h = 0$ and $h = b_i v_i$ respectively and have already been discussed in the previous section. We recall that $f^{(1)}$ can be found as $M_\beta(A_i + B_i v_i)$, with $B_i = -\frac{1}{\beta} b_i$. A_i would be determined by the compatibility condition at the order $n = 3$. Since we are truncating the expansion at $n = 2$ we can safely choose A_i equal to zero.

Let us now deal with a polynomial of degree two:

$$P(v) = a + b_i v_i + c_{ij} v_i v_j.$$

By gaussian integration, the condition (4.8) becomes

$$a + \frac{1}{\beta} c_{ii} = 0. \tag{4.9}$$

We look for a solution of (4.7) of the following type:

$$f(v) = M_\beta(A + B_i v_i + C_{ij} v_i v_j).$$

Plugging this ansatz in our equation we find

$$\partial_{v_k}(M_\beta \partial_{v_k}(A + B_i v_i + C_{ij} v_i v_j)) = M_\beta P(v).$$

Recall that $\partial_{v_k} v_i = \delta_{ki}$ and $\partial_{v_k} v_i v_j = \delta_{ki} v_j + \delta_{kj} v_i$; then the left hand side of the above equation simplifies to

$$\begin{gathered}
\partial_{v_k}(M_\beta(B_k + C_{kj} v_j + C_{ik} v_i)) = \\
M_\beta(-\beta v_k(B_k + C_{kj} v_j + C_{ik} v_i) + \delta_{kj} C_{kj} + \delta_{ik} C_{ik}) = \\
M_\beta(2C_{ii} - \beta B_i v_i - 2\beta C_{ij} v_i v_j)
\end{gathered}$$

and identifying the coefficients of the corresponding powers of v_i one gets

$$B_i = -\frac{1}{\beta} b_i \;\; C_{ij} = -\frac{1}{2\beta} c_{ij}$$

with the redundant relation $a = 2C_{ii}$ which is automatically verified, thanks to the compatibility condition (4.9). In order to fix the parameter A we impose the analog of (4.9): $A + C_{ii}/\beta = 0$, namely as before we are choosing equal to zero the projection on the null space of L_β. Thus

$$A = -\frac{a}{2\beta}.$$

In the context of our problem the known term is always in the form of a polynomial multiplied by a maxwellian and the coefficients of the v_i are functions of the position. In the case where only first powers of v appear, i.e. $P(v) = a_k^{(i)} v_k$, the $a_k^{(i)}$ are given by

$$a^{(i)} = \nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)$$

here $i, j = 1, 2$ and $i \neq j$. When $P(v) = a^{(i)} + b_k^{(i)} v_k + c_{hk}^{(i)} v_h v_k$ the coefficients are the following:

$$\begin{aligned} a^{(i)} &= \partial_t \rho_i + \nabla_x (U \star \rho_j) \cdot \tfrac{1}{\beta} (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)) \\ b_k^{(i)} &= 0 \\ c_{hk}^{(i)} &= -\tfrac{1}{\beta} \partial_{x_h} (\partial_{x_k} \rho_i + \beta \rho_i \partial_{x_k} (U \star \rho_j)) - \partial_{x_h} (U \star \rho_j)(\partial_{x_k} \rho_i + \beta \rho_i \partial_{x_k} (U \star \rho_j)) \end{aligned}$$

Summing up we denote by $f_i^{(k)}$, $k = 0, 1, 2$ the following functions of v and x:

$$\begin{aligned} f_i^{(0)} &= M_\beta \rho_i(x, t) \\ f_i^{(1)} &= -\tfrac{1}{\beta} M_\beta v \cdot (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)) \\ f_i^{(2)} &= -\tfrac{1}{2\beta} M_\beta \Big[\partial_t \rho_i + \nabla_x (U \star \rho_j) \cdot \tfrac{1}{\beta} (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j)) \\ &- v \cdot \tfrac{1}{\beta} \nabla_x (v \cdot (\nabla_x \rho_i + \beta \rho_i (\nabla_x (U \star \rho_j))) - v \cdot \nabla_x (U \star \rho_j) v \cdot (\nabla_x \rho_i + \beta \rho_i \nabla_x (U \star \rho_j))) \Big] \end{aligned}$$

where ρ_i is solution of

$$\partial_t \rho_i - \frac{1}{\beta^2} (\Delta_x \rho_i + \beta \nabla_x \cdot (\rho_i \nabla_x (U \star \rho_j))) = 0.$$

The known term A_i appearing in the equation for the remainder becomes

$$A_i = \partial_t f_i^{(1)} + v \cdot \nabla_x f_i^{(2)} + \varepsilon \partial_t f_i^{(2)} + \sum_{n=1}^{3} \varepsilon^{n-1} \sum_{0 \leq l, l' \leq 2 \; l + l' - 1 = n} F_i^{(l)} \cdot \nabla_v f_i^{(l')}$$

where we recall that

$$F_i^{(n)} = -\nabla_x \int_{\varepsilon\Omega} dx' U(|x-x'|) \int_{\mathbb{R}^3} dv' f_j^{(n)}$$

It is easy to show that the sum over l, l' is given by

$$-\nabla_x (U \star \rho_j) \cdot \nabla_v f_i^{(2)}$$

indeed $F_i^{(1)} = 0 = F_i^{(2)}$ because the functions $f_i^{(1)}$ and $f_i^{(2)}$ belong both to $\mathcal{N}^\perp$ and $F_i^{(0)} = -\nabla_x(U \star \rho_j)$.

In conclusion, the $f_i^{(n)}$ are always of the form M_β times a polynomial in v times a function of x, t which depends on the derivatives of $\rho_i(x,t)$ solution of the macroscopic equations. If we fix an initial datum for (4.3) in $C^2(\mathbb{T}^d)$ then the corresponding unique solution will be classical as shown in section 4.4 and the $f^{(n)} = \{f^{(n)}\}_i$ as well A_i will satisfy the regularity properties

$$||f^{(n)}||_- \leq C, \quad ||A||_- \leq C$$

4.3 Remainder

In this section we will find a solution to equation (4.6), which is a weakly non linear equation if $m \geq 1, K-1-m \geq 0$, by considering first the linear problem with the force term F_i^ε assumed given so that general results will grant the existence of this linear problem in a suitable space. Then, a fixed point argument applies by using ε as small parameter. From here on we will simplify notation by setting $M = M_\beta$.

Define $\tilde{f} = f/M$ and

$$\tilde{L}_\beta \tilde{f} = \frac{1}{M} \nabla_v \cdot (M \nabla_v (\tilde{f})).$$

Moreover, we introduce the Hilbert space associated to the L_2 scalar product $(\cdot,\cdot)_M$ weighted by the maxwellian and with $||\cdot||_M$ the associated norm. In this Hilbert space $\tilde{L}_\beta$ is self-adjoint and non positive:

$$(g_1, \tilde{L}_\beta g_2)_M = (\tilde{L}_\beta g_1, g_2)_M$$

$$(g, \tilde{L}_\beta g)_M = \int_{\mathbb{T}^d \times \mathbb{R}^d} dx dv M g \frac{1}{M} \nabla_v \cdot (M \nabla_v (g)) = -||\nabla_v g||_M^2.$$

If we put $R_i = \psi_i M$, the equation for the remainder becomes

$$\partial_t \psi_i + \varepsilon^{-1} \left[v \cdot \nabla_x \psi_i + \frac{F_i^\varepsilon \cdot \nabla_v (M\psi_i)}{M} + \frac{B_i \cdot \Gamma_i}{M} \right] = \varepsilon^{-2} \tilde{L}_\beta \psi_i - \varepsilon^{K-1-m} \frac{A_i}{M}. \tag{4.10}$$

In order to estimate $||\psi_i||_M$ one multiplies the above equation by $M\psi_i$ and integrates over x and v. So the first term on the left hand side becomes

$$\frac{1}{2}\partial_t||\psi_i||_M^2$$

while the gradient with respect to the position disappears because of the periodic boundary conditions.

We assume that the force terms F_i^ε are given functions that we will call $\hat{F}_i$ and are such that

$$||\hat{F}_i||_\infty \leq \alpha_{\hat{F}}$$

Hence

$$\left|\int dxdv\psi_i\hat{F}_i\cdot\nabla_v(M\psi_i)\right| = \left|\int dxdv\hat{F}_i\cdot(M^{\frac{1}{2}}M^{\frac{1}{2}}\psi_i\nabla_v\psi_i)\right| \leq ||\hat{F}_i||_\infty||\psi_i||_M||\nabla_v\psi_i||_M$$

where we integrated by parts ($\hat{F}_i$ depends only on x) and we used Schwartz inequality. Now the term with the convolution of the remainder with the gradient of the potential is estimated in the following way:

$$\begin{aligned}
& \left|\int dxdv\psi_i(x,v)\nabla_v(\rho(x)M(v))\cdot\int dx'\nabla_x U(|x-x'|)\int dv'M(v')\psi_j(x',v')\right| \\
= & \left|\int dxdv\rho M^{\frac{1}{2}}(v)M^{\frac{1}{2}}(v)\nabla_v\psi_i(x,v)\cdot\int dx'dv'\nabla_x U M^{\frac{1}{2}}(v')M^{\frac{1}{2}}(v')\psi_j(x',v')\right| \\
\leq & |\mathbb{T}^d|\ \sup_{\mathbb{T}^d}|\rho|\ \sup_{\mathbb{T}^d}|\nabla_x U|\ ||\psi_j||_M\ ||\nabla_v\psi_i||_M.
\end{aligned}$$

As before we first integrated by parts and then we applied Schwartz inequality twice. Here it has been considered only the lowest order in ε of the sum which constitutes B_i; the other terms are treated similarly. The last estimate is the one for A_i:

$$\int dxdv\psi_i A_i = \int dxdvM^{\frac{1}{2}}\psi_i\frac{A_i}{M_{\frac{1}{2}}} \leq$$

$$||\psi_i||_M\ ||M^{-1}A_i||_M \leq \frac{1}{2}(||\psi_i||_M^2 + ||M^{-1}A_i||_M^2).$$

Summing up, we have

$$\begin{aligned}
\frac{1}{2}\partial_t||\psi_i||_M^2 \leq -\varepsilon^{-2}||\nabla_v\psi_i||_M^2 + (c_1+\varepsilon^{-1}c_2)||\psi_j||_M||\nabla_v\psi_i||_M + \alpha_{\hat{F}_i}\varepsilon^{-1}||\psi_i||_M||\nabla_v\psi_i||_M \\
+\frac{\varepsilon^{K-1-m}}{2}(||\psi_i||_M^2 + ||M_{-1}A_i||_M^2).
\end{aligned}$$

Note that c_1 contains powers of ε greater than ε^{-1}. Now one exploits the inequality

$$-\frac{\varepsilon^{-2}x^2}{2} + (\sigma_1+\varepsilon^{-1}\sigma_2)xy \leq \frac{(\varepsilon\sigma_1+\sigma_2)^2}{2}y^2 \tag{4.11}$$

but first we need to introduce the norm $||\psi||_M^2 := ||\psi_1||_M^2 + ||\psi_2||_M^2$, so we have

$$\begin{aligned}
\frac{1}{2}\partial_t||\psi||_M^2 &\leq -\varepsilon^{-2}(||\nabla_v\psi_1||_M^2 + ||\nabla_v\psi_2||_M^2) + (c_1 + \varepsilon^{-1}c_2)\big(||\psi_2||_M||\nabla_v\psi_1||_M \\
&+ ||\psi_1||_M||\nabla_v\psi_2||_M) + \alpha_{\hat{F}}\varepsilon^{-1}(||\psi_1||_M||\nabla_v\psi_1||_M + ||\psi_2||_M||\nabla_v\psi_2||_M) \\
&+ \frac{\varepsilon^{K-1-m}}{2}[(||\psi_1||_M^2 + ||\psi_2||_M^2) + (||M_{-1}A_1||_M^2 + ||M_{-1}A_2||_M^2)] \\
&\leq \frac{(\varepsilon c_1 + c_2)^2}{2}(||\psi_1||_M^2 + ||\psi_2||_M^2) + \frac{\alpha_{\hat{F}}^2}{2}(||\psi_1||_M^2 + ||\psi_2||_M^2) \\
&+ \frac{\varepsilon^{K-1-m}}{2}[(||\psi_1||_M^2 + ||\psi_2||_M^2) + (||M_{-1}A_1||_M^2 + ||M_{-1}A_2||_M^2)]
\end{aligned}$$

where we used the inequality (4.11) in two different ways; in fact we divided the negative term in two halves and then once we chose $c_1 = \sigma_1$ and $c_2 = \sigma_2$ and once we put $c_1 = 0$ and $c_2 = \alpha_{\hat{F}}$.

Multiplying by 2 both members one gets

$$\partial_t||\psi||_M^2 \leq \lambda||\psi||_M^2 + d$$

where $\lambda = \lambda(\alpha_{\hat{F}}) = \alpha_{\hat{F}}^2 + (\varepsilon c_1 + c_2)^2 + \varepsilon^{K-1-m}$ and $d = \varepsilon^{K-1-m}(||M^{-1}A_1||_M^2 + ||M^{-1}A_2||_M^2)$. Integrating over the time, by the Gronwall inequality:

$$\begin{aligned}
f(t) \leq K(t) + \lambda\int_0^t d\tau f(\tau) \leq K(T) + \lambda\int_0^t d\tau f(\tau) \\
\implies \quad f(t) \leq K(T)e^{\lambda t} \leq K(T)e^{\lambda T}
\end{aligned}$$

where $f = ||\psi||_M^2$, $K(t) = \int_0^t d\tau d(\tau)$ is a non decreasing function of time and we used the initial condition $f(0) = 0$.

Now consider the sequence of forces

$$\hat{F}_i^{(k)} = -\nabla_x U \star \int_{\mathbb{R}^3} \mathrm{d}v \sum_{n=0}^{K} \varepsilon^n f_j^{(n)} - \varepsilon^m \nabla_x U \star \int_{\mathbb{R}^3} \mathrm{d}v R_j^{(k-1)}$$

with $k \geq 1$ and $R_i^{(0)} = 0$. Let $\alpha_k = \max\{||\hat{F}_1^{(k)}||_\infty, ||\hat{F}_2^{(k)}||_\infty\}$, then

$$\alpha_k \leq \bar{\alpha} + \varepsilon^m C \int \mathrm{d}x\mathrm{d}v|R_j^{(k-1)}|$$

where j is chosen such that it corresponds to the maximum in the definition of α_k and

$$\bar{\alpha} = \sup_{x\in\mathbb{T}^d}\left|\nabla_x U \star \int \mathrm{d}v \sum_{n=0}^{K} \varepsilon^n f_j^{(n)}\right|.$$

Write $\int dv|R_j^{(k-1)}| = \int \mathrm{d}v|M\psi_j^{(k-1)}| = \int \mathrm{d}vM^{\frac{1}{2}}|M^{\frac{1}{2}}\psi_j^{(k-1)}|$; using Schwartz inequality we get

$$\int \mathrm{d}x\mathrm{d}v|R_j^{(k-1)}| \leq |\mathbb{T}^d|^{\frac{1}{2}}||\psi_j^{(k-1)}||_M.$$

Thus, recalling the estimate for $||\psi||_M^2$, we can conclude that

$$\alpha_k \leq \bar{\alpha} + \varepsilon^m \mu(\alpha_{k-1})$$

where the non decreasing function μ is defined by $\mu(\alpha_k) = C(|\mathbb{T}^d|K(T)\exp(\lambda(\alpha_k)T))^{1/2}$. By induction on k we show that $\alpha_k \leq 2\bar{\alpha}\ \forall k$. In fact

$$\alpha_1 \leq \bar{\alpha} \leq 2\bar{\alpha}.$$

Then suppose $\alpha_{k-1} \leq 2\bar{\alpha}$; we have

$$\alpha_k \leq \bar{\alpha} + \varepsilon^m \mu(\alpha_{k-1}) \leq \bar{\alpha} + \varepsilon^m \mu(2\bar{\alpha}) \leq 2\bar{\alpha}$$

because we applied the inductive hypothesis, exploited the monotonicity of μ and chose ε so small that $\varepsilon^m \mu(2\bar{\alpha}) \leq \bar{\alpha}$.

Denote with $\delta\psi_i^{(k)}$ the difference $\psi_i^{(k)} - \psi_i^{(k-1)}$. The equation solved by $\delta\psi_i^{(k)}$ is

$$\partial_t(\delta\psi_i^{(k)}) + \varepsilon^{-1}\left[v \cdot \nabla_x(\delta\psi_i^{(k)}) + \frac{\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k)}) - \hat{F}_i^{(k-1)} \cdot \nabla_v(M\psi_i^{(k-1)})}{M} + \frac{B_i \cdot \Gamma_i}{M}\right]$$
$$= \varepsilon^{-2}\tilde{L}_\beta(\delta\psi_i^{(k)})$$

where is understood that Γ_i contains $\delta\psi_j^{(k)}$ and no more ψ_j. Summing and subtracting the quantity $\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k-1)})$ one has

$$\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k)}) - \hat{F}_i^{(k-1)} \cdot \nabla_v(M\psi_i^{(k-1)}) = \hat{F}_i^{(k)} \cdot \nabla_v(M\delta\psi_i^{(k)}) + \delta\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k-1)})$$

where

$$\delta\hat{F}_i^{(k)} = \hat{F}_i^{(k)} - \hat{F}_i^{(k-1)} = -\varepsilon^m \nabla_x U \star \int \mathrm{d}v' M\delta\psi_j^{(k-1)}.$$

If one multiplies the equation for $\delta\psi_i^{(k)}$ by $M\delta\psi_i^{(k)}$ and integrates in space and velocities, it is possible to replicate the above estimates for the norm of the remainder. Only one thing is worth noting: the known term with A_i is now replaced by the following quantity

$$\int \mathrm{d}x\mathrm{d}v\delta\psi_i^{(k)}\delta\hat{F}_i^{(k)} \cdot \nabla_v(M\psi_i^{(k-1)}) = -\int \mathrm{d}x\mathrm{d}vM\psi_i^{(k-1)}\delta\hat{F}_i^{(k)} \cdot \nabla_v(\delta\psi_i^{(k)})$$

which one estimates in this way:

$$\varepsilon^m \left| \int \mathrm{d}x\mathrm{d}v M(v)(\psi_i^{(k-1)}\nabla_v\delta\psi_i^{(k)})(x,v)\int dx'\nabla_x U(|x-x'|)\int \mathrm{d}v' M(v')\delta\psi_j^{(k-1)}(x',v')\right|$$
$$\leq \varepsilon^m \sup|\nabla_x U|\left(\int \mathrm{d}x\mathrm{d}v|M\psi_i^{(k-1)}\nabla_v\delta\psi_i^{(k)}|\right)\left(\int \mathrm{d}x'\mathrm{d}v' M|\delta\psi_j^{(k-1)}|\right)$$
$$\leq \varepsilon^m|\mathbb{T}^d|^{\frac{1}{2}}\sup|\nabla_x U|\ ||\psi_i^{(k-1)}||_M||\nabla_v\delta\psi_i^{(k)}||_M||\delta\psi_j^{(k-1)}||_M$$
$$\leq \varepsilon^m c||\nabla_v\delta\psi_i^{(k)}||_M||\delta\psi_j^{(k-1)}||_M \leq \frac{\varepsilon^m c}{2}(||\nabla_v\delta\psi_i^{(k)}||_M^2+||\delta\psi_j^{(k-1)}||_M^2).$$

In c the bound for $||\psi_i^{(k-1)}||_M$ is also present. In brief we have the following situation:

$$f_k' \leq Cf_k + \theta f_{k-1}$$

for some C; θ depends on ε and is small as we like if $m \geq 1$, and of course $f_k = ||\delta\psi^{(k)}||_M^2$ with the same notation as above. By integrating in time and using Gronwall inequality we obtain

$$f_k \leq \int_0^T \theta f_{k-1}e^{CT} \leq \int_0^T \theta e^{CT}\int_0^T \theta e^{CT} f_{k-2} \leq ... \leq const(\theta e^{CT}T)^k$$

thus, by a standard argument, we conclude that the sequence $\{\psi_i(k)\}$ is a Cauchy sequence and the limit ψ is the unique solution of (4.10) with bounded norm $||\psi||_M$.

4.4 Limiting equation

We follow a strategy similar to the one used in the previous section: we consider first a linear problem, prove existence for it and then use a fixed point argument to give the existence for the full non linear equation. Since we do not have at our disposal a small parameter we use compactness arguments and the Schauder Fixed Point Theorem. We seek for weak solutions in the following sense:

Let W be the Hilbert space

$$W(0,T;H^1,H^{-1}) := \{f : f \in L^2(0,T;H^1), \frac{df}{dt} \in L^2(0,T;H^{-1})\}.$$

$H^1(\mathbb{T}^d)$ and $H^{-1}(\mathbb{T}^d)$ Sobolev spaces on the torus with norms

$$|v|_2^2 = \int_{\mathbb{T}^d}|v|^2, \qquad ||v||_1^2 = |v|_2^2 + |\nabla v|_2^2$$

$$||v||_{-1} = \sup_{u\in H^1}[2(u,v) - ||u||_1^2)] = \int dk\frac{|\hat{v}(k)|^2}{1+k^2}$$

$(\cdot,\cdot)$ scalar product in L^2.

$$||v||^2_W = \int_0^T [||v(t)||^2_1 + ||v'(t)||^2_{-1}]dt$$

with $v' = dv/dt$. Let W_1 be the convex subset of W

$$W_1 = \{v \in W : \int_{\mathbb{T}^d} v(x,t) = 1 \quad \text{a.e in } [0,T]\}$$

We say that ρ is a weak solution of the linear problem (4.12) below if for $\bar{\rho} \in L^2(\mathbb{T}^d)$ and for all $v \in H^1(\mathbb{T}^d)$ and a.a. $0 \le t \le T$

$$\beta^2(v,\rho') + (\nabla v, \nabla\rho + \beta\rho\nabla U \star h) = 0$$

and $\rho(\cdot,0) = \bar{\rho}(\cdot)$.

We remark that since $\rho \in W$ implies $\rho \in C([0,T]; L^2(\mathbb{T}^d))$ we have that $\rho(0) \in L^2(\mathbb{T}^d)$.

Theorem 2 *For any $h \in L^1(\mathbb{T}^d)$ and $\bar{u} \in L^2(\mathbb{T}^d)$ there exists a unique solution in W_1 to the following Cauchy problem*

$$\begin{aligned} \beta^2\partial_t u &= \Delta u + \beta\nabla\cdot(u\nabla(U\star h)) \\ u(\cdot,0) &= \bar{u}(\cdot) \end{aligned} \tag{4.12}$$

Proof. Since $h \in L^1(\mathbb{T}^d)$ and ∇U as well as $\nabla^2 U$ are bounded we have $\nabla(U\star h)$ and $\nabla^2(U\star h)$ in $L^\infty([0,T]\times\mathbb{T}^d)$. Hence by standard arguments there exists a solution in W. Since the equation is in form of divergence, the total mass is conserved so that the solution is in W_1.

Moreover, we have some useful a priori estimates for the solution of (4.12) (indeed the proof of existence can be achieved by approximation methods and these a priori estimates). Denote by $|u|_2$ the norm in $L^2(\mathbb{T}^d)$:$|u|_2^2 = \int_{\mathbb{T}^d} dx|u|^2(x,t)$. We have that

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 = -\frac{1}{\beta^2}|\nabla u|_2^2 - \frac{1}{\beta}\int_{\mathbb{T}^d} \mathrm{d}x u(x,t)\nabla u(x,t)\nabla(U\star h)(x,t) \tag{4.13}$$

Since $h \in L^1(\mathbb{T}^d)$ and ∇U is bounded

$$\sup_{x,t} |\nabla(U\star h)(x,t)| \le \bar{c}$$

Then, for any $\delta > 0$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u|_2^2 &\le -\frac{1}{\beta^2}|\nabla u|_2^2 + \frac{\bar{c}}{\beta}|\nabla u|_2|u|_2 \\ &\le -(1-\delta)\frac{1}{\beta^2}|\nabla u|_2^2 + \frac{1}{4\delta}\bar{c}|u|_2^2 \end{aligned} \tag{4.14}$$

By Gronwall there exists a constant C such that

$$|u|_2^2 \leq |\bar{u}|_2^2 e^{Ct}$$

so that

$$\int_0^T \mathrm{d}t |u(t)|_2^2 \leq C|\bar{u}|_2^2, \quad \int_0^T |\nabla u|_2^2 \leq C|\bar{u}|_2^2$$

for some constant C. Here and below C denotes a running constant. Moreover,

$$||u'||_{-1} = \sup_{v \in H^1 : ||v||_1 = 1} \{ - \int_{\mathbb{T}^d} \nabla v [\frac{1}{\beta} \nabla U \star h) + \frac{1}{\beta^2} \nabla u \} \leq \frac{\bar{c}}{\beta} |u|_2 + \frac{1}{\beta^2} |\nabla u|_2$$

Hence

$$\int_0^T \mathrm{d}t ||u'(t)||_{-1}^2 \leq C|\bar{u}|_2^2$$

Consider now functions $u : \mathbb{T}^d \to R^2$. We define the Hilbert space W in this case as before, simply using as scalar product $(\cdot, \cdot)$ the scalar product in $L^2(\mathbb{T}^d; R^2)$. We use the same notation for W and W_1. We say that $\rho = (\rho_1, \rho_2)$ is a weak solution of (4.3) if for all $v \in H^1(\mathbb{T}^d; R^2)$ and a.e. $0 \leq t \leq T$

$$\beta^2 (v, \rho_i') + (\nabla v, \nabla \rho_i + \beta \rho_i \nabla U \star \rho_j) = 0$$

and $\rho(\cdot, 0) = \rho(\cdot)$.

Theorem 2 defines a map A from $L^2(0, T; L^2(\mathbb{T}^d; R^2))$ in itself by applying it to a set of two equations for $u_i, i = 1, 2$ with a given term depending on $g_i, i = 1, 2$ in the following way

$$\begin{aligned} \beta^2 \partial_t u_i &= \Delta u_i + \beta \nabla (u_i \nabla (U \star g_j)) \\ u_i(\cdot, 0) &= \bar{u}_i(\cdot) \\ i, j = 1, 2, \quad & i \neq j \end{aligned} \tag{4.15}$$

We use $g = (g_1, g_2)$ and $u = (u_1, u_2)$, $|g|_2^2 = \sum_{i=1,2} |g_i|_2^2$. Then, since the L^1 norm of g_j is bounded by a constant times the L^2 norm, namely $L^1([0, T], \mathbb{T}^d) \in L^2([0, T], \mathbb{T}^d)$, there exists a solution u in W and we can write

$$A(g) = u$$

$$||A(g)||_W^2 \leq C|\bar{u}|_2^2$$

We now prove the existence theorem for the non linear set of equations by proving that A is continuous and maps a closed convex set in a compact set.

Compactness. We consider the closed and convex set $X \in L^2([0,T],\mathbb{T}^d)$

$$X = \{h : ||h||^2_{L^2(0,T;L^2)} \leq k\}$$

Since $A(h)$ is in W and W is compactly imbedded in $L^2([0,T],\mathbb{T}^d)$ the image of X is compact.

Continuity. Consider $g, \tilde{g} \in L^2(0,T;L^2)$. Let $u = A(g)$ and $\tilde{u} = A(\tilde{g})$ the corresponding weak solutions. We have that, for $i \neq j$

$$\begin{aligned}(u_i - \tilde{u}_i, u_i' - \tilde{u}_i') = & - \frac{1}{\beta^2}\int_{\mathbb{T}^d} |\nabla(u_i - \tilde{u}_i)|^2 - \frac{1}{\beta}\int_{\mathbb{T}^d}(u_i - \tilde{u}_i)\nabla(u_i - \tilde{u}_i)\cdot\nabla(U \star g_j) \\ & - \frac{1}{\beta}\int_{\mathbb{T}^d}\tilde{u}_i\nabla(u_i - \tilde{u}_i)\cdot\nabla U \star (g_j - \tilde{g}_j) \end{aligned} \tag{4.16}$$

$$\frac{1}{2}\frac{d}{dt}|u_i - \tilde{u}_i|_2^2 \leq -C|\nabla(u_i - \tilde{u}_i)|_2^2 + c_1|u_i - \tilde{u}_i|_2^2 + c_2|g_j - \tilde{g}_j|_2^2$$

We have used that the L^1 norm of $(g - \tilde{g})$ is bounded by the L^2 norm. Therefore,

$$||u - \tilde{u}||_{L^2([0,T],L^2)} \leq C||g - \tilde{g}||_{L^2([0,T],L^2)}$$

which proves the continuity of A in $L^2([0,T],L^2)$.

By Schauder's theorem the map A has a fixed point in $L^2([0,T],L^2)$ which is the weak solution we were looking for.

Uniqueness. The proof is standard.

Summarizing, we have proved the following

Theorem 3 *There exists a unique weak solution in W_1 to the following Cauchy problem*

$$\begin{aligned} \beta^2\partial_t\rho_i &= \Delta\rho_i + \beta\nabla(\rho_i\nabla(U\star\rho_j)), \\ \rho_i(\cdot,0) &= \bar{\rho}_i(\cdot) \\ i,j = 1,2, & \quad i \neq j \end{aligned} \tag{4.17}$$

Regularity. If $\nabla U \star \rho \in C^0([0,T];C^1)$ and $\bar{\rho} \in C^2(\mathbb{T}^d)$ then the linear equation has a classical solution. Since the weak solution ρ is also in $C^0([0,T];L^2)$ we have that indeed $\nabla U \star \rho \in C^0([0,T];C^1)$ and therefore the weak solution ρ corresponding to an initial datum in $C^2(\mathbb{T}^d)$ is a classical solution.

Chapter 5

Mullins-Sekerka From Kinetic Equation

5.1 Sharp interface limit

In this section we study the solutions of (2.1) in the sharp interface limit in a 3-d torus Ω. We introduce again the scale separation parameter ε, which has the meaning of ratio between the kinetic and macroscopic scales. Then, we scale position and time as ε^{-1} and ε^{-3}, respectively, while keeping fixed (equal to 1) γ. The width of the interface on the macroscopic scale is then of order ε, so that in the limit $\varepsilon \to 0$ the interface becomes sharp. The rescaled density distributions $f_i^\varepsilon(r,v,t) = f_i(\varepsilon^{-1}r, v, \varepsilon^{-3}t)$, are solutions of

$$\partial_t f_i^\varepsilon + \varepsilon^{-2} v \cdot \nabla_r f_i^\varepsilon + \varepsilon^{-2} F_i^\varepsilon \cdot \nabla_v f_i^\varepsilon = \varepsilon^{-3} L_\beta f_i^\varepsilon . \tag{5.1}$$

$$F_i^\varepsilon(r,t) = -\nabla_r \int \mathrm{d}r' \varepsilon^{-3} U(\varepsilon^{-1}|r-r'|) \int \mathrm{d}v' f_j^\varepsilon(r',v',t) =: -\nabla_r g_i^\varepsilon .$$

In this section F_i^ε depends on ε through the function f_j^ε but also through the potential since we are keeping fixed γ. We consider a situation in which initially an interface is present. Since the stationary non homogeneous solutions of (2.1) are given by the Maxwellian multiplied by the front density profiles we let our system start initially close to those stationary solutions and choose as initial datum $f_i^\varepsilon(r,v) = M_\beta(v)\rho_i^\varepsilon$, where the density profiles are very close to a profile such that in the bulk its values are $\rho_i^\pm$, the values of the densities in the two pure phases at temperature T, and the interpolation between them on the interface is realized along the normal direction in each point by the fronts. We put $\rho_1^\pm = \bar\rho^\pm$ and use the symmetry properties of the segregation phase transition giving $\rho_2^\mp = \bar\rho^\pm$. Consider a smooth surface $\Gamma_0 \subset \Omega$. Let $d(r,\Gamma_0)$ be the signed distance of the point $r \in \Omega$ from the interface. Consider an initial profile for the densities ρ_i^ε of the following type: at distance greater than $O(\varepsilon)$ from the interface (in the bulk) the density

profiles $\rho_i^\varepsilon(r)$ are almost constant equal to $\rho_i^\pm$; at distance $O(\varepsilon)$ (near the interface) we choose

$$\rho_i^\varepsilon(r) = w_i(\varepsilon^{-1} d(r, \Gamma_0)) + O(\varepsilon) \tag{5.2}$$

where $w_i(z)$ are the fronts, which are one dimensional stationary solutions of (2.3) with asymptotic values $\rho_i^\pm$. Since these solutions are unique up to a translation we fix a solution by imposing that $w_1(0) = w_2(0)$.

Let Γ_t^ε be an interface at time t defined by

$$\Gamma_t^\varepsilon = \{r \in \Omega : \rho_1^\varepsilon(r,t) = \rho_2^\varepsilon(r,t)\}$$

and T be such that Γ_t^ε is regular for $t \in [0,T]$. Let $d^\varepsilon(r,t)$ be the signed distance $d(r, \Gamma_t^\varepsilon)$ of $r \in \Omega$ from the interface Γ_t^ε, such that $d^\varepsilon > 0$ in $\Omega_t^{\varepsilon,+}$ and $d^\varepsilon < 0$ in $\Omega_t^{\varepsilon,-}$, where $\Omega = \Gamma_t^\varepsilon \cup \Omega_t^{\varepsilon,+} \cup \Omega_t^{\varepsilon,-}$. For sake of simplicity we drop from now on the apex ε. For any r such that $|d(r,t)| < \frac{1}{k(\Gamma_t)}$, $k(\Gamma_t) = \sup_{x \in \Gamma_t} k(x)$ with $k(x)$ the maximum of the principal curvatures in x, there exists $s(r) \in \Gamma_t$ such that

$$\nu(s(r)) d(r,t) + s(r) = r$$

where $\nu(s(r))$ is the normal to the surface Γ_t in $s(r)$. Hence,

$$\nu(s(r)) = \nabla d(r,t), \quad r \in \Gamma_t, \quad r \in \Gamma_t.$$

Define the normal velocity of the interface as

$$V(s(r)) = \partial_t d(r,t).$$

The curvature K (the sum of the principal curvatures) is given by $K = \Delta_r d(r,t), \quad r \in \Gamma_t..$ Define, for ε^0 small enough,

$$\mathcal{N}(\delta) := \{r : |d(r,t)| < \delta\}$$

where $\delta = \frac{1}{m}, \quad m = \max_{t \in [0,T], 0 \le \varepsilon \le \varepsilon^0} k(\Gamma_t)$.

We follow the approach based on the truncated Hilbert expansions introduced by Caflish [C]. This method, which has been used in the previous chapter to prove the hydrodynamic limit for the Vlasov-Fokker-Planck equation, has been improved by including boundary layer expansions in [ELM], to prove the hydrodynamic limit for the Boltzmann equation in a slab. Here we try to adapt the arguments in [ELM] to the fact that the boundary is not given a priori and has to be found as a result of the expansion. The Hilbert expansion is nothing but a power expansion in ε for the solution of the kinetic equation

$$f^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n f^{(n)}. \tag{5.3}$$

Since we expect that the behavior of the solution will be different in the bulk and near the interface, we decompose $f^{(n)}$ in two parts: the bulk part $\hat{f}^{(n)}(r,t)$ and boundary terms $\tilde{f}^{(n)}$ which will be fast varying functions close to the interface, namely they depend on r,t in the following way

$$\tilde{f}^{(n)} = \tilde{f}^{(n)}(\varepsilon^{-1}d(r,t), r, t)$$

while $\hat{f}^{(n)}(r,t)$ are slowly varying functions on the microscopic scale. More precisely, a fast varying function $h(r,t)$ for $r \in \mathcal{N}$ can be represented as a function $h(z,r,t)$, $z = \varepsilon^{-1}d(r,t)$, with the condition $h(z, r+\ell\nu(s(r)), t) = h(z,r,t), \forall \ell$ small enough. Hence in $\mathcal{N}$ we can write

$$\nabla_r h = \frac{1}{\varepsilon}\nu\partial_z h + \overline{\nabla}_r h; \quad \partial_t h = \frac{1}{\varepsilon}V\partial_z h + \partial_t h; \quad \Delta_r h = \frac{1}{\varepsilon^2}\partial_z^2 h + \frac{1}{\varepsilon}(\nabla_r \cdot \nu)\partial_z h + \overline{\Delta}_r h \quad (5.4)$$

where the bar on the derivative operators means derivatives with respect to r, keeping fixed the other variables. Note that $\nu \cdot \overline{\nabla}_r h(z,r,t) = 0$.

To write the expansion for the force term F_i^ε we introduce $U^\varepsilon \star \sum_{n=0}^\infty \varepsilon^n \rho_j^{(n)} = \sum_{n=0}^\infty \varepsilon^n g_i^{(n)}$ and $F_i^{(n)} = -\nabla_r g_i^{(n)}$.

We expand also the signed distance

$$d(r,t) = \sum_{i=0}^\infty \varepsilon^n d^{(n)}(r,t) \quad (5.5)$$

We will denote by $\nu^{(n)}$ the gradient $\nabla_r d^{(n)}$, with $\bar{\nu} := \nu^{(0)}$. The condition $|\nabla_r d|^2 = 1$ is equivalent to:

$$|\nabla_r d^{(0)}|^2 = 1, \quad \nabla_r d^{(0)}\nabla_r d^{(1)} = 0, \quad \nabla_r d^{(0)}\nabla_r d^{(j)} = -\frac{1}{2}\sum_{i=1}^{j-1} \nabla_r d^{(i)}\nabla_r d^{(j-1)}, \quad j \geq 2$$

so that $d^{(0)}$ can be interpreted as a signed distance from an interface that we denote by $\bar{\Gamma}_t$. As a consequence of (5.5) the velocity of the interface Γ_t has the form

$$\sum_{i=0}^\infty \varepsilon^i V^{(i)}, \quad \bar{V} := V^{(0)}.$$

We remark that giving the velocity V determines the curve evolving with it. The velocity $\bar{V}$ will generate an order zero interface $\bar{\Gamma}_t$. The interface generated by $\sum_i \varepsilon^i V^{(i)}$ will be a deformation, small for small ε, of $\bar{\Gamma}_t$. We define

$$\mathcal{N}^0(m) := \{r : |d^{(0)}(r,t)| < m\}, \bar{\Gamma}_t := \{r : |d^{(0)}(r,t)| = 0\}, \Omega^{+(-)} := \{r : |d^{(0)}(r,t)| > (<)0\}$$

and fix m so that $\mathcal{N}^0(m) \subset \mathcal{N}(\delta)$.

We assume that in $\Omega \setminus \mathcal{N}^0(\frac{m}{2})$

$$f^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \hat{f}^{(n)}. \tag{5.6}$$

and that in $\mathcal{N}^0(m)$, the solution is of the form

$$f^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \tilde{f}^{(n)} \tag{5.7}$$

In $\mathcal{N}^0(m) \setminus \mathcal{N}^0(\frac{m}{2})$ the inner and outer expansions have to match. Hence, we require as $|z| \to \infty$ [CF]

$$\begin{aligned}
\tilde{f}_i^{(0)} &= \hat{f}_i^{(0)} + O(e^{-\alpha|z|}) \\
\tilde{f}_i^{(1)} &= \hat{f}_i^{(1)} + \nabla_r \hat{f}_i^{(0)} \cdot (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)}) + O(e^{-\alpha|z|}) \\
\tilde{f}_i^{(2)} &= \hat{f}_i^{(2)} + \nabla_r \hat{f}_i^{(1)} \cdot (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)}) + \\
&+ \nabla_r \hat{f}_i^{(0)} \cdot (-\nu^{(0)} d^{(2)} + \nu^{(1)}(z - d^{(1)}) - \nu^{(2)} d^{(0)}) + \\
&+ \frac{1}{2} \partial_{r_h} \partial_{r_k} \hat{f}_i^{(0)} (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_h (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_k + O(e^{-\alpha|z|}) \\
\tilde{f}_i^{(3)} &= \hat{f}_i^{(3)} + \nabla_r \hat{f}_i^{(2)} \cdot (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)}) + \\
&+ \nabla_r \hat{f}_i^{(1)} \cdot (-\nu^{(0)} d^{(2)} + \nu^{(1)}(z - d^{(1)}) - \nu^{(2)} d^{(0)}) + \\
&+ \frac{1}{2} \partial_{r_h} \partial_{r_k} \hat{f}_i^{(1)} (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_h (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_k + \\
&+ \nabla_r \hat{f}_i^{(0)} \cdot (-\nu^{(0)} d^{(3)} - \nu^{(1)} d^{(2)} + \nu^{(2)}(z - d^{(1)}) - \nu^{(3)} d^{(0)}) + \\
&+ \partial_{r_h} \partial_{r_k} \hat{f}_i^{(0)} (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_h (-\nu^{(0)} d^{(2)} + \nu^{(1)}(z - d^{(1)}) - \nu^{(2)} d^{(0)})_k + \\
&+ \frac{1}{6} \partial_{r_h} \partial_{r_k} \partial_{r_l} \hat{f}_i^{(0)} (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_h (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_k (\nu^{(0)}(z - d^{(1)}) - \nu^{(1)} d^{(0)})_l + \\
&+ O(e^{-\alpha|z|}) \\
&\dots
\end{aligned}$$

The hat functions are computed on the interface, from the side + when $z \to \infty$ and from the side − when $z \to -\infty$. Moreover, we require that the previous condition is satisfied in $\mathcal{N}^0(m)$. For that, it is necessary to define $\hat{f}_i^{(n)}$ also in $\mathcal{N}^0(m)$. We take functions satisfying (5.8) below in $\Omega^\pm \setminus \bar{\Gamma}_t$ and extend them smoothly beyond the interface. We denote by $(\hat{f}_i^{(n)})^\pm$ the smooth extensions of $\hat{f}_i^{(n)}$, in $\Omega^\pm \setminus \bar{\Gamma}_t$. We replace (5.6) and (5.7) in the equations and equate terms of the same order in ε separately in $\Omega^\pm \setminus \mathcal{N}^0(\frac{m}{2})$ and $\mathcal{N}^0(m)$. We will use the notation $\rho_i^{(n)} = \int dv f_i^{(n)}$, and we denote by $\hat{h}, \tilde{h}$ a function $h(f_i^{(n)})$ whenever is evaluated on $\hat{f}_i^{(n)}$, $\tilde{f}_i^{(n)}$.

Outer expansion

In $\Omega^{\pm} \setminus \mathcal{N}^0(\frac{m}{2})$, $n \geq 0$

$$\partial_t \hat{f}_i^{(n-3)} + v \cdot \nabla_r \hat{f}_i^{(n-1)} + \sum_{l,l' \geq 0 : l+l'=n-1} \hat{F}_i^{(l)} \cdot \nabla_v \hat{f}_i^{(l')} = L_\beta \hat{f}_i^{(n)}, \tag{5.8}$$

with $\hat{f}_i^{\alpha} = 0, \alpha < 0$.

Inner expansion

In $\mathcal{N}^0(m)$ $n \geq 0$ we have

$$\begin{aligned} &\sum_{l,l' \geq 0 : l+l'=n-2} V^{(l')} \partial_z \tilde{f}_i^{(l)} + \sum_{k+k'=n} \nu^{(k)} \cdot v \partial_z \tilde{f}_i^{(k')} + v \cdot \overline{\nabla}_r \tilde{f}_i^{(n-1)} + \partial_t \tilde{f}_i^{(n-3)} \\ &- \sum_{l,l',l'' \geq 0 : l+l'+l''=n} \partial_z \tilde{g}_i^{(l)} \nu^{(l')} \cdot \nabla_v \tilde{f}_i^{(l'')} + \sum_{l,l' \geq 0 : l+l'=n-1} \overline{\nabla}_r \tilde{g}_i^{(l)} \cdot \nabla_v \tilde{f}_i^{(l')} = L_\beta \tilde{f}_i^{(n)}, \end{aligned} \tag{5.9}$$

with $\tilde{f}_i^{\alpha} = 0, \alpha < 0$. We will look for solutions $\tilde{f}_i^{(n)}$ of the differential equations decaying exponentially in z at infinity, because we want boundary layer corrections sensibly different from zero only close to the interface. Finally, we stress that the terms $f_i^{(n)}$ of the expansion do not depend on ε but for being computed on z, which depends on ε because of the rescaling and also because the interface at time t still depends on ε. The latter is a new feature in the framework of the Hilbert expansion due to the fact that the boundary is not fixed but is itself unknown.

The strategy for a rigorous proof is to construct, once the functions $f_i^{(n)}$ have been determined, the solution in terms of a truncated Hilbert expansion as

$$f^{\varepsilon} = \sum_{n=0}^{N} \varepsilon^n f^{(n)} + \varepsilon^m R \tag{5.10}$$

where the functions are evaluated in $z = \varepsilon^{-1} d^N(r,t)$, with $d^N(r,t) = \sum_{i=0}^{N-2} \varepsilon^n d^{(n)}(r,t)$ and then write a weakly non linear equation for the remainder. In this approach it is essential to have enough smoothness for the terms of the expansion. On the contrary, they would be discontinuous on the border of $\mathcal{N}^0(m)$ since $\tilde{f}^{(n)}$ are not exactly equal to $\hat{f}^{(n)}$ there but differ for terms exponentially small in ε. One can modify the expansion terms by interpolating in a smooth way between the outside and the inside getting smooth terms which do not satisfy the equations for terms exponentially small in ε, that can be put in the remainder. With this in mind, we did not put in the equations the terms coming from the force such that in the convolution r is in $\mathcal{N}^0(m)$ and r' in $\Omega \setminus \mathcal{N}^0(m)$. That is possible because the potential is of finite range.

In this section we show how to construct the terms $f_i^{(n)}$. The argument is formal because we do not prove boundedness of the remainder nor the regularity properties of the terms of the expansion. We plan to report on that in the future.

Now we go back to the Hilbert power series and start examining the equations order by order. We will find explicitly only the first three terms in the expansion to explain the procedure.

Outer expansion

At the lowest order ε^{-3} $(n = 0)$:

$$L_\beta \hat{f}_i^{(0)} = 0$$

which implies that $\hat{f}_i^{(0)}$ has to be Maxwellian in velocity with variance T times a function $\hat{\rho}_i^{(0)}(r,t)$. The latter is found by looking at the equations at the next two orders. At order ε^{-2} $(n = 1)$:

$$v \cdot \nabla_r \hat{f}_i^{(0)} + \hat{F}_i^{(0)} \cdot \nabla_v \hat{f}_i^{(0)} = L_\beta \hat{f}_i^{(1)}. \tag{5.11}$$

The solution is of the form

$$\hat{f}_i^{(1)} = \hat{\rho}_i^{(1)} M_\beta - M_\beta \hat{\rho}_i^{(0)} v \cdot \nabla_r \hat{\mu}_i^{(0)} \tag{5.12}$$

where $\mu_i^\varepsilon(\rho^\varepsilon) = T \ln \rho_i^\varepsilon + U^\varepsilon \star \rho_j^\varepsilon$ and $\mu_i^\varepsilon = \sum_{n=0}^\infty \varepsilon^n \mu_i^{(n)}$.

The order ε^{-1} equation $(n = 2)$ is

$$v \cdot \nabla_r \hat{f}_i^{(1)} + \hat{F}_i^{(0)} \cdot \nabla_v \hat{f}_i^{(1)} + \hat{F}_i^{(1)} \cdot \nabla_v \hat{f}_i^{(0)} = L_\beta \hat{f}_i^{(2)}. \tag{5.13}$$

The solvability condition for this equation says that the integral on the velocity of the l.h.s. has to be zero. By integrating over the velocity and using the explicit expression for $\hat{f}_i^{(1)}$ we get

$$-T \nabla_r \cdot (\hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(0)}) = 0.$$

Hence the solvability condition for the equation $n = 2$ gives the equation determining $\hat{\rho}_i^{(0)}$. The choice of the initial data implies that the only solution of that equation is the constant one, with values $\rho_i^\pm$ in $\Omega^\pm$. We look at next order $n = 3$ to find $\hat{\rho}_i^{(1)}$ by the solvability condition. By integrating over v the equation $n = 3$ and taking into account that $\hat{f}_i^{(0)}$ is Maxwellian in velocity, we get the following condition on $\hat{u}^{(2)}$, where $u_i^{(n)} = \int dv v f_i^{(n)}$,

$$\nabla_r \cdot \hat{u}_i^{(2)} = 0. \tag{5.14}$$

Then, $\hat{f}_i^{(2)}$ is determined, by replacing (5.12) in equation (5.13), as

$$\hat{f}_i^{(2)} = -M_\beta \hat{\rho}_i^{(0)} v \cdot \nabla_r \hat{\mu}_i^{(1)} + \hat{\rho}_i^{(2)} M_\beta. \tag{5.15}$$

where $\hat{\mu}_i^{(1)} = T \hat{\rho}_i^{(1)} / \hat{\rho}_i^{(0)} + \hat{g}_i^{(1)}$.

We use $\hat{f}_i^{(2)}$ as given by (5.15) to get $\hat{u}_i^2 = -T \hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(1)}$ and plug it in (5.14) to get the equation for $\hat{\mu}_i^{(1)}$

$$\Delta_r \hat{\mu}_i^{(1)} = 0.$$

We consider equation (5.8) for $n = 3$

$$\partial_t \hat{f}_i^{(0)} + v \cdot \nabla_r \hat{f}_i^{(2)} + \sum_{l,l' \geq 0: l+l'=2} \hat{F}_i^{(l)} \cdot \nabla_v \hat{f}_i^{(l')} = L_\beta \hat{f}_i^{(3)}, \tag{5.16}$$

whose solution is

$$\hat{f}_i^{(3)} = -M_\beta v \cdot [\hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(2)} + \hat{\rho}_i^{(1)} \nabla_r \hat{\mu}_i^{(1)}] + M_\beta \frac{T}{2} \hat{\rho}_i^{(0)} (v \cdot \nabla_r)(v \cdot \nabla_r) \hat{\mu}_i^{(1)} + \hat{\rho}_i^{(3)} M_\beta. \tag{5.17}$$

The equation for $\hat{\mu}_i^{(2)} = T\hat{\rho}_i^{(2)}/\hat{\rho}_i^{(0)} - T/2(\hat{\rho}_i^{(1)}/\hat{\rho}_i^{(0)})^2 + \hat{g}_i^{(2)}$ comes from the equation for $n = 4$

$$\partial_t \hat{f}_i^{(1)} + v \cdot \nabla_r \hat{f}_i^{(3)} - \sum_{l,l' \geq 0,\, l+l'=3} \nabla_r \hat{g}_i^{(l)} \cdot \nabla_v \hat{f}_i^{(l')} = L_\beta \hat{f}_i^{(4)}$$

which gives as solvability condition $\nabla_r \cdot \hat{u}_i^{(3)} = -\partial_t \hat{\rho}_i^{(1)}$ where $\hat{u}_i^{(3)} = \int dv \hat{f}_i^{(3)}$. By using (5.17) we get

$$\Delta \hat{\mu}_i^{(2)} = \frac{1}{T\hat{\rho}_i^{(0)}} \partial_t \hat{\rho}_i^{(1)} - \frac{\nabla_r \hat{\rho}_i^{(1)} \cdot \nabla_r \hat{\mu}_i^{(1)}}{\hat{\rho}_i^{(0)}} := \frac{S_i}{\hat{\rho}_i^{(0)}}, \quad S_i = \beta \partial_t \hat{\rho}_i^{(1)} - \nabla_r \hat{\rho}_i^{(1)} \cdot \nabla_r \hat{\mu}_i^{(1)}$$

Inner expansion

At the lowest order ($n = 0$)

$$v \cdot \bar{\nu} \partial_z \tilde{f}_i^{(0)} - \bar{\nu} \cdot \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(0)}.$$

In Appendix B it is proved that any solution of this equation has the form $M_\beta(v)\tilde{\rho}_i^{(0)}$, with $\tilde{\rho}_i^{(0)}$ a function of z. Plugging back in the equation we have

$$\partial_z \tilde{\rho}_i^{(0)} + \beta \tilde{\rho}_i^{(0)} \partial_z (\tilde{U} \star \tilde{\rho}_j^{(0)}) = 0 \iff \partial_z \tilde{\mu}_i^{(0)} = 0, \tag{5.18}$$

where $\tilde{U}$ is the potential U integrated over all coordinates but one. We solve this equation with the conditions at infinity $\rho_i^{\pm}$, given by the matching conditions, and call w_i this front solution. The exponential decay of w_i has been proved for the one-component case [DOPT] and the same argument should provide the proof also in this case. We can conclude that in Ω

$$f_i^{(0)}(r,t) = M_\beta[w(\frac{d(r,t)}{\varepsilon})\chi_m + (1 - \chi_m)\hat{\rho}_i^{(0)}],$$

with χ_m the characteristic function of $\mathcal{N}^0(m)$. This solution differs from the front solution w_i in Ω for terms which are exponentially small in ε and has the disadvantage of not being continuous on the border of $\mathcal{N}^0$. As explained before, it has to be modified as

$$f_i^{(0)}(r,t) = M_\beta[w(\frac{d(r,t)}{\varepsilon})h(d(r,t)) + (1 - h(d(r,t)))\hat{\rho}_i^{(0)}(r,t)]$$

with h a smooth version of χ_m.

We now find $\tilde{f}_i^{(1)}$ by examining the ε^{-2} order (n=1)

$$v \cdot \bar{\nu} \partial_z \tilde{f}_i^{(1)} - \bar{\nu} \cdot \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(1)} - \bar{\nu} \cdot \nabla_v \tilde{f}_i^{(1)} \partial_z \tilde{g}_i^{(0)} = L_\beta \tilde{f}_i^{(1)}. \tag{5.19}$$

The term involving $\nu^{(1)}$, $\nu^{(1)} \cdot (v \partial_z \tilde{f}_i^{(0)} - \nabla_v \tilde{f}_i^{(0)} \partial_z \tilde{g}_i^{(0)}) = \beta v \cdot \nu^{(1)} M_\beta \tilde{\rho}_i^{(0)} \partial_z \tilde{\mu}_i^{(0)} = 0$, because $\tilde{f}_i^{(0)}$ is solution of the lowest order equation and the bar operators vanish because $\tilde{\rho}_i^{(0)}$ is function of z only. In Appendix B we show that the solution is necessarily Maxwellian in velocity so that we can write $\tilde{f}_i^{(1)} = \tilde{\rho}_i^{(1)} M_\beta$ with $\tilde{\rho}_i^{(1)}$ to be determined by the following equation

$$\partial_z \tilde{\rho}_i^{(1)} + \beta \tilde{\rho}_i^{(0)} \partial_z \tilde{g}_i^{(1)} + \beta \tilde{\rho}_i^{(1)} \tilde{U} \star \partial_z \tilde{\rho}_j^{(0)} = 0. \tag{5.20}$$

Taking into account that $-\beta \tilde{U} \star \partial_z \tilde{\rho}_j^{(0)} = \partial_z \ln w_i$, from the equation for the front, we get

$$\partial_z \left(T \tilde{\rho}_i^{(1)} (w_i)^{-1} + \tilde{g}_i^{(1)} \right) = 0 \Longleftrightarrow \quad \partial_z \tilde{\mu}_i^{(1)} = 0. \tag{5.21}$$

We now find the value of $\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)}$ in $z = 0$ which is enough to find $\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)}$ for any z. From

$$\tilde{\mu}_i^{(1)} = T(\tilde{\rho}_i^{(1)})(w_i)^{-1} + \tilde{U} \star \tilde{\rho}_j^{(1)} + \bar{K} \int \mathrm{d}z' (z - z') \tilde{U}(z - z') w_j(z'),$$

where $\bar{K} = \Delta_r d^{(0)}(r, t)$ is the curvature of the interface $\bar{\Gamma}_t$ (see Appendix D), we want to find $\tilde{\rho}_i^{(1)}$ as determined by $\tilde{\mu}_i^{(1)}$. We define the operator $\mathcal{L}$ as $(\mathcal{L}h)_i = T h_i (w_i)^{-1} + \tilde{U} \star h_j$. The previous relation reads as

$$(\mathcal{L} \tilde{\rho}^{(1)})_i = \tilde{\mu}_i^{(1)} - \bar{K} \int dz' (z - z') \tilde{U}(z - z') w_j(z'). \tag{5.22}$$

The operator $\mathcal{L}$ has a zero mode since $\mathcal{L} w' = 0$, so that the equation $(\mathcal{L} \tilde{\rho}^{(1)})_i = h_i$ has a solution only if

$$\sum_{i=1,2} \int \mathrm{d}z h_i(z) w_i'(z) = 0.$$

Hence (5.22) implies

$$\sum_{i=1,2} \int \mathrm{d}z \tilde{\mu}_i^{(1)} w_i'(z) = \bar{K} \sum_{i=1,2} \int \mathrm{d}z \mathrm{d}z' w_i'(z) (z - z') \tilde{U}(z - z') w_j(z'). \tag{5.23}$$

which implies because $\tilde{\mu}_i^{(1)}$ are constant

$$\tilde{\mu}_1^{(1)}(0, r, t)[w_1]_{-\infty}^{+\infty} + \tilde{\mu}_2^{(1)}(0, r, t)[w_2]_{-\infty}^{+\infty} = \bar{K}(r, t) \sum_{i, i \neq j} \int \mathrm{d}z \mathrm{d}z' w_i'(z) (z - z') \tilde{U}(z - z') w_j(z').$$

In Appendix C it is shown that the sum in the right hand side is the surface tension S for this model, so we have (since $[w_1]_{-\infty}^{+\infty} = -[w_2]_{-\infty}^{+\infty}$)

$$(\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)})(0, r, t)[w_1]_{-\infty}^{+\infty} = \bar{K}(r, t)S. \tag{5.24}$$

The matching conditions impose that $\tilde{\mu}_1^{(1)} - \tilde{\mu}_2^{(1)} \to (\hat{\mu}_1^{(1)})^{\pm} - (\hat{\mu}_2^{(1)})^{\pm}$ for $z \to \pm\infty$, so that for $r \in \bar{\Gamma}_t$

$$[(\hat{\mu}_1^{(1)})^{\pm} - (\hat{\mu}_2^{(1)})^{\pm}][w_1]_{-\infty}^{+\infty} = \bar{K}(r, t)S. \tag{5.25}$$

The conservation law for the equation at the order ε^{-1} ($n = 2$) will give the velocity of the interface. By integrating over the velocity this equation

$$w_i'\bar{V} + \partial_z(\bar{\nu} \cdot \tilde{u}_i^{(2)}) = 0 \tag{5.26}$$

where the maxwellian dependence of $tilde f_i^{(0)}$ and $\tilde{f}_i^{(1)}$ is crucial for several cancellations. By integrating over z

$$-\bar{V}[w_i]_{-\infty}^{+\infty} = [\bar{\nu} \cdot \tilde{u}_i^{(2)}]_{-\infty}^{+\infty}. \tag{5.27}$$

By the matching conditions $\tilde{u}_i^{(2)} \to (\hat{u}_i^{(2)})^{\pm}$ at $\pm\infty$, so that for $r \in \bar{\Gamma}_t$

$$-\bar{V}[w_i]_{-\infty}^{+\infty} = (\bar{\nu} \cdot \hat{u}_i^{(2)})^{+} - (\bar{\nu} \cdot \hat{u}_i^{(2)})^{-} \qquad r \in \bar{\Gamma}_t. \tag{5.28}$$

Summarizing what we got so far: we have constructed functions $\hat{\mu}_i^{(1)}$ harmonic in $\Omega^{\pm}$. Their smooth extensions $(\hat{\mu}_i^{(1)})^{\pm}$ are harmonic and satisfy (5.25) and (5.28). We define now functions $\bar{\mu}_i^{(1)}$ in Ω, not necessarily smooth, equal to $(\hat{\mu}_i^{(1)})^{\pm}$ in $\Omega \setminus \bar{\Gamma}_t$ and such that $\lim_{d^{(0)}(r,t)\to 0^{\pm}} \bar{\mu}_i^{(1)} = (\hat{\mu}_i^{(1)})^{\pm}|_{\bar{\Gamma}_t}$ and the same for the derivatives. From now on all the quantities overlined will refer to these functions. This means that $\bar{\mu}_i^{(1)}$ satisfy:

$$\begin{gathered}
\Delta\bar{\mu}_i^{(1)} = 0, \quad r \in \Omega \setminus \bar{\Gamma}_t, \\
[\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)}]_{-}^{+}[\bar{\rho}^{+} - \bar{\rho}^{-}] = \bar{K}(r, t)S, \quad r \in \Omega \setminus \bar{\Gamma}_t, \\
\bar{V}[w_i]_{-\infty}^{+\infty} = [T\bar{\rho}_i\bar{\nu} \cdot \nabla_r\bar{\mu}_i^{(1)}]_{-}^{+} \qquad r \in \bar{\Gamma}_t
\end{gathered} \tag{5.29}$$

where $\bar{\rho}^{\pm} = w_1(\pm\infty)$. Let us write the last equation as

$$\bar{V}\beta[\bar{\rho}^{+} - \bar{\rho}^{-}] = [(\bar{\rho} + \bar{\varphi})\bar{\nu} \cdot \nabla_r\bar{\mu}_1^{(1)}]_{-}^{+} = [(\bar{\varphi} - \bar{\rho})\bar{\nu} \cdot \nabla_r\bar{\mu}_2^{(1)}]_{-}^{+} \tag{5.30}$$

and

$$\bar{\rho}(r) = \frac{\bar{\rho}_1(r) + \bar{\rho}_2(r)}{2}, \quad \bar{\varphi}(r) = \frac{\bar{\rho}_1(r) - \bar{\rho}_2(r)}{2}.$$

with $\bar{\rho}_i(r)$ the step functions $\bar{\rho}_i(r) := \bar{\rho}_i^{+}\chi^{+} + \bar{\rho}_i^{-}\chi^{-}$, $\chi^{\pm}$ the characteristic functions of the sets $d^{(0)}(r, t) > 0$, $d^{(0)}(r, t) < 0$ respectively. We know, because of the symmetry of

the phase transition, that $\bar{\rho}$ is constant while $\bar{\varphi}$ is discontinuous in 0 and $\bar{\varphi}(r) = \pm|\bar{\phi}|$ for $r \in \bar{\Omega}^{\pm}$. The previous equation implies

$$2\bar{V}\beta[\bar{\rho}^{+} - \bar{\rho}^{-}] = \bar{\rho}[\bar{\nu} \cdot \nabla_r(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)})]_{-}^{+} + [\bar{\varphi}\nu \cdot \nabla_r(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})]_{-}^{+} \tag{5.31}$$

$$0 = \bar{\rho}[\bar{\nu} \cdot \nabla_r(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})]_{-}^{+} + [\bar{\varphi}\bar{\nu} \cdot \nabla_r(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)})]_{-}^{+}. \tag{5.32}$$

We introduce the function $\zeta(r,t) = (\bar{\rho}_1\bar{\mu}_1^{(1)} + \bar{\rho}_2\bar{\mu}_2^{(1)})(r,t) = \bar{\rho}(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)}) + \bar{\varphi}(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)})$ so that $\Delta_r\zeta(r,t) = 0$ in $\Omega \setminus \bar{\Gamma}_t$ and (5.32) gives $[\bar{\nu} \cdot \nabla_r\zeta]_{-}^{+} = 0$. Moreover, it is discontinuous on $\bar{\Gamma}_t$ because of the function $\bar{\varphi}$. The jump is

$$\zeta^{+}(r,t) - \zeta^{-}(r,t) = 2|\bar{\varphi}|(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)}), \quad r \in \bar{\Gamma}_t$$

In conclusion, ζ satisfies

$$\begin{cases} \Delta_r\zeta(r,t) = 0 & r \in \Omega \setminus \bar{\Gamma}_t \\ [\zeta]_{-}^{+} = 2|\bar{\varphi}|S\bar{K}(r,t)/[w_1]_{-\infty}^{+\infty} & r \in \bar{\Gamma}_t \\ 0 = [\bar{\nu} \cdot \nabla_r\zeta]_{-}^{+} & r \in \bar{\Gamma}_t \end{cases} \tag{5.33}$$

It is possible to show by using the Green identity that this problem for a given function $\bar{K}(r,t)$, has the unique solution

$$\zeta(r,t) = \int_{\bar{\Gamma}_t} \mathrm{d}s(\zeta^{+} - \zeta^{-})(s,t)\nu \cdot \nabla G(r,s) = \frac{2S|\bar{\varphi}|}{[w_1]_{-\infty}^{+\infty}} \int_{\bar{\Gamma}_t} \mathrm{d}s\bar{K}(s,t)\nu \cdot \nabla G(r,s), \quad r \in \Omega \setminus \bar{\Gamma}_t$$

$$\frac{(\zeta^{+} + \zeta^{-})}{2}(r,t) = \frac{2S|\bar{\varphi}|}{[w_1]_{-\infty}^{+\infty}} \int_{\bar{\Gamma}_t} \mathrm{d}s\bar{K}(s,t)\bar{\nu} \cdot \nabla G(r,s), \quad r \in \bar{\Gamma}_t$$

where G is the Green function in Ω. We notice that $(\zeta^{+} + \zeta^{-}) = 2\bar{\rho}(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})$.

We consider now the function $\xi(r,t) = (\bar{\rho}_1\bar{\mu}_1^{(1)} - \bar{\rho}_2\bar{\mu}_2^{(1)})(r,t) = \bar{\rho}(\bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)}) + \bar{\varphi}(\bar{\mu}_1^{(1)} + \bar{\mu}_2^{(1)})$, which is discontinuous on $\bar{\Gamma}_t$ and satisfies

$$\begin{cases} \Delta_r\xi(r,t) = 0 & r \in \Omega \setminus \bar{\Gamma}_t \\ [\xi]_{-}^{+} = \dfrac{|\bar{\varphi}|}{\bar{\rho}}(\zeta^{+} + \zeta^{-}) & r \in \bar{\Gamma}_t \\ \bar{V} = \frac{T}{2}\frac{[\bar{\nu} \cdot \nabla_r\xi]_{-}^{+}}{[\bar{\rho}^{+} - \bar{\rho}^{-}]} & r \in \bar{\Gamma}_t \end{cases} \tag{5.34}$$

The problem is well posed because given the current configuration of the front the problem has a unique solution and this solution in turn determines the velocity of the front.

In conclusion we have determined $\bar{\mu}_1^{(1)}$ and $\bar{\mu}_2^{(1)}$. In $\mathcal{N}^0(m)$ $\tilde{\mu}_i^{(1)}$ is constant equal to the value $\bar{\mu}_i^{(1)}(r,t), \quad r \in \bar{\Gamma}_t$, which is determined by solving the limiting equation. Hence $\tilde{\mu}_i^{(1)}$ and $\hat{\mu}_i^{(1)}$ are known at this stage. As a consequence, $\hat{\rho}_i^{(1)}$ are known through the relation $\hat{\mu}_i^{(1)} = T\frac{\hat{\rho}_i^{(1)}}{w_i} + \hat{U}\hat{\rho}_j^{(1)}$ while $\tilde{\rho}_i^{(1)}$ are found as solutions of (5.22) with the r.h.s. decaying to

a constant as $z \to \pm\infty$ and the decay is exponential if w_i do so. Then, a modification of the argument in [CCO1] leads to the exponential decay of $\tilde{\rho}_i^{(1)}$. We notice that $\tilde{\rho}_i^{(1)}$ is determined by (5.22) but for a term $\alpha w_i'$ which is in the null of $\mathcal{L}$, with α independent of z. To fix α it is enough to put the condition $\tilde{\rho}_1^{(1)}(0,r,t)) = \tilde{\rho}_2^{(1)}(0,r,t), \quad r \in \mathcal{N}^0(m)$. Since we have fixed $\rho_1^\varepsilon = \rho_2^\varepsilon$ on Γ^ε we are allowed to choose $\tilde{\rho}_1^{(k)}(0,r,t)) = \tilde{\rho}_2^{(k)}(0,r,t), \quad r \in \mathcal{N}^0(m)$ for any k.

We proceed now constructing the higher orders of the expansion. For $n = 2$:

$$\begin{aligned}\bar{V}\partial_z \tilde{f}_i^{(0)} + \bar{\nu}\cdot v\partial_z \tilde{f}_i^{(2)} + v\cdot\overline{\nabla}_r \tilde{f}_i^{(1)} - \partial_z \tilde{g}_i^{(2)}\bar{\nu}\cdot\nabla_v \tilde{f}_i^{(0)} - \partial_z \tilde{g}_i^{(0)}\bar{\nu}\cdot\nabla_v \tilde{f}_i^{(2)} \\ -\partial_z \tilde{g}_i^{(1)}\bar{\nu}\cdot\nabla_v \tilde{f}_i^{(1)} + \overline{\nabla}_r \tilde{g}_i^{(1)}\cdot\nabla_v \tilde{f}_i^{(0)} = L_\beta \tilde{f}_i^{(2)} \quad , \end{aligned} \tag{5.35}$$

Again, the terms involving $\nu^{(2)}$ and $\nu^{(1)}$ are zero thanks to the previous equations. The matching conditions require for z large

$$\tilde{f}_i^{(2)}(\pm|z|,r,t) = (\hat{f}_i^{(2)})^\pm(r,t) + \nabla_r \hat{f}_i^{(1)}\cdot(\nu^{(0)}(z-d^{(1)}) - \nu^{(1)}d^{(0)}) + O(e^{-\alpha|z|}), \quad r \in \mathcal{N}^0(m).$$

Hence, we have to solve a stationary problem on the real line with given conditions at infinity. We replace in (5.35) $\tilde{f}_i^{(2)} = \tilde{q}_i^{(2)} + \tilde{\rho}_i^{(2)} M_\beta$ where $\tilde{q}_i^{(2)}$ is in the orthogonal to the kernel of L_β, namely $\int dv \tilde{q}_i^{(2)} = 0$; the scalar product is

$$(f,g)_{M_\beta} = \int dv M_\beta^{-1} f g$$

We have

$$\begin{aligned}M_\beta[\bar{V}\partial_z w_i + \bar{\nu}\cdot v\partial_z \tilde{\rho}_i^{(2)} + v\cdot\overline{\nabla}_r \tilde{\rho}_i^{(1)} + \beta\partial_z \tilde{g}_i^{(2)}\bar{\nu}\cdot v w_i + \beta\partial_z \tilde{g}_i^{(0)}\bar{\nu}\cdot v\tilde{\rho}_i^{(2)} + \beta\partial_z \tilde{g}_i^{(1)}\bar{\nu}\cdot v\tilde{\rho}_i^{(1)} \\ -\beta\overline{\nabla}_r \tilde{g}_i^{(1)}\cdot v w_i] = L_\beta \tilde{q}_i^{(2)} - \bar{\nu}\cdot v\partial_z \tilde{q}_i^{(2)} + \partial_z \tilde{g}_i^{(0)}\bar{\nu}\cdot\nabla_v \tilde{q}_i^{(2)}. \end{aligned} \tag{5.36}$$

By using the equation for the front w_i and the fact that $\partial_z \tilde{\mu}_i^{(1)} = 0$ together with $\tilde{\mu}_i^{(2)} = \frac{T}{w_i}\tilde{\rho}_i^{(2)} - \frac{T}{2}(\frac{\tilde{\rho}_i^{(1)}}{w_i})^2 + \tilde{g}_i^{(2)}$ we get

$$M_\beta \bar{\nu}\cdot v[\partial_z \tilde{\rho}_i^{(2)} + \beta w_i \partial_z \tilde{g}_i^{(2)} + \beta\tilde{\rho}_i^{(2)}\partial_z \tilde{g}_i^{(0)} + \beta\partial_z \tilde{g}_i^{(1)}\tilde{\rho}_i^{(1)}]$$

$$= M_\beta \bar{\nu}\cdot v w_i \partial_z[\frac{\tilde{\rho}_i^{(2)}}{w_i} + \beta\tilde{g}_i^{(2)} - \frac{\beta}{2}(\frac{\tilde{\rho}_i^{(1)}}{w_i})^2] = \beta M_\beta \bar{\nu}\cdot v w_i \partial_z \tilde{\mu}_i^{(2)}$$

Hence we can write the equation (5.36) in the form

$$\begin{aligned}\beta M_\beta \bar{\nu}\cdot v w_i \partial_z \tilde{\mu}_i^{(2)} &= L_\beta \tilde{q}_i^{(2)} - \bar{\nu}\cdot v\partial_z \tilde{q}_i^{(2)} + \partial_z \tilde{g}_i^{(0)}\bar{\nu}\cdot\nabla_v \tilde{q}_i^{(2)} + \beta M_\beta w_i v\cdot\overline{\nabla}_r \tilde{g}_i^{(1)} - \\ &- M_\beta v\cdot\overline{\nabla}_r \tilde{\rho}_i^{(1)} - M_\beta \bar{V}\partial_z w_i \end{aligned} \tag{5.37}$$

From (5.15) the conditions at infinity are:

$$\tilde{f}_i^{(2)}(\pm|z|r,t) = M_\beta\left[(\hat{\rho}_i^{(2)})^\pm - (\hat{\rho}_i^{(0)} v\cdot\nabla_r \hat{\mu}_i^{(1)})^\pm + (\nabla_r \hat{\rho}_i^{(1)}\cdot(\nu^{(0)}(z-d^{(1)}) - \nu^{(1)}d^{(0)}))^\pm\right] + O(e^{-\alpha|z|})$$

$$\tilde{\rho}_i^{(2)}(\pm|z|r,t) = (\hat{\rho}_i^{(2)})^{\pm}(r,t) + (\nabla_r \hat{\rho}_i^{(1)} \cdot (\nu^{(0)}(z-d^{(1)}) - \nu^{(1)}d^{(0)}))^{\pm} + O(e^{-\alpha|z|})$$

$$\lim_{z\to\pm\infty} \int dv P(v) \tilde{q}_i^{(2)}(z,r,t) = - < vP(v) >_\beta \cdot (\nabla_r \hat{\mu}_i^{(1)})^{\pm} (\hat{\rho}_i^{(0)})^{\pm}$$

where $P(v)$ is a polynomial in the velocity and $< \cdot >_\beta$ are the moments of the Maxwellian M_β: $< vP(v) >_\beta = \int dv v P(v) M_\beta$. From the definition (5.39) of $\tilde{\mu}_i^{(2)}$ we deduce its asymptotic behaviour:

$$\tilde{\mu}_i^{(2)} \sim \hat{\mu}_i^{(2)} + B_1(\hat{\rho}_i^{(1)}) + B_2(\hat{\rho}_i^{(0)}) +$$
$$+ \frac{T}{\hat{\rho}_i^{(0)}} \nabla_r \hat{\rho}_i^{(1)} \cdot (\nu^{(0)}(z-d^{(1)}) - \nu^{(1)}d^{(0)}) + \tilde{U} \star (\nabla_r \hat{\rho}_j^{(1)} \cdot (\nu^{(0)}(z-d^{(1)}) - \nu^{(1)}d^{(0)}))$$

where z is very large and r next to the interface, from one side or another depending on the sign of z and B_1, B_2 are defined in Appendix D. If

$$C_i = B_1(\hat{\rho}_i^{(1)}) + B_2(\hat{\rho}_i^{(0)}) + \frac{T}{\hat{\rho}_i^{(0)}} \nabla_r \hat{\rho}_i^{(1)} \cdot (\nu^{(0)}(z-d^{(1)}) - \nu^{(1)}d^{(0)}) + \tilde{U} \star (\nabla_r \hat{\rho}_j^{(1)} \cdot (\nu^{(0)}(z-d^{(1)}) - \nu^{(1)}d^{(0)}))$$

let us define

$$l_i = 1_{z<0} C^- + 1_{z>0} C^+$$

where $C^{\pm}$ is C_i with all the hat quantities computed on $\pm$ side of the interface. Now it follows that

$$w_i(+\infty)(\hat{\mu}_i^{(2)})^+ - w_i(-\infty)(\hat{\mu}_i^{(2)})^- = [w_i(\tilde{\mu}_i^{(2)} - l_i)]_{-\infty}^{+\infty}$$

The left hand side can be written as

$$\int dz \partial_z [w_i(\tilde{\mu}_i^{(2)} - l_i)] = \int dz [w_i \partial_z (\tilde{\mu}_i^{(2)} - l_i) + w_i'(\tilde{\mu}_i^{(2)} - l_i)]$$

We need to eliminate the dependence on $\tilde{\mu}_i^{(2)}$. Then we multiply (5.37) by $v_z = \bar{\nu} \cdot v$ and integrate over v

$$w_i \partial_z \tilde{\mu}_i^{(2)} = -\partial_z \int dv v_z^2 \tilde{q}_i^{(2)} - \beta \tilde{u}_i^{(2)} \cdot \bar{\nu} \tag{5.38}$$

We used $\tilde{q}_i^{(2)} \in [\mathrm{Ker} L_\beta]^\perp$ and $\bar{\nu} \cdot \bar{\nabla}(\cdot) = 0$ and also $L_\beta \tilde{q}_i^{(2)} = L_\beta \tilde{f}_i^{(2)}$. By matching conditions we have that for z large

$$\beta \tilde{u}_i^{(2)} \cdot \bar{\nu}(\pm|z|r,t) = -(\hat{\rho}_i^{(0)} \bar{\nu} \cdot \nabla_r \hat{\mu}_i^{(1)})^{\pm} + O(e^{-\alpha|z|})$$

Introduce the following function:

$$j_i = 1_{z<0} (\bar{\nu} \cdot \nabla_r \hat{\mu}_i^{(1)})^- + 1_{z>0} (\bar{\nu} \cdot \nabla_r \hat{\mu}_i^{(1)})^+$$

and note that

$$\partial_z l_i = j_i$$

Hence, we get

$$w_i\partial_z(\tilde{\mu}_i^{(2)} - l_i) = -\partial_z \int dv v_z^2 \tilde{q}_i^{(2)} - \beta\bar{\nu}\cdot\tilde{u}_i^{(2)} - w_i j_i$$

Then

$$w_i(+\infty)(\hat{\mu}_i^{(2)})^+ - w_i(-\infty)(\hat{\mu}_i^{(2)})^- = -\int dz[\beta\bar{\nu}\cdot\tilde{u}_i^{(2)} + w_i j_i] + \int dz w_i'\tilde{\mu}_i^{(2)} - \int dz w_i' l_i$$

because the matching conditions force $\int dv v_z^2 \tilde{q}_i^{(2)}$ to vanish at infinity. Moreover the function $\bar{\nu}\cdot\tilde{u}_i^{(2)}$ is known because

$$\bar{V}w_i' + \partial_z\bar{\nu}\cdot\tilde{u}_i^{(2)} = 0$$

It follows that

$$\bar{\nu}\cdot\tilde{u}_i^{(2)} = \frac{w_i - \bar{\rho}^-}{\bar{\rho}^+ - \bar{\rho}^-}\left[(\frac{\hat{\rho}_i^{(0)}}{\beta}\bar{\nu}\cdot\nabla_r\hat{\mu}_i^{(1)})^- - (\frac{\hat{\rho}_i^{(0)}}{\beta}\bar{\nu}\cdot\nabla_r\hat{\mu}_i^{(1)})^+\right] - (\frac{\hat{\rho}_i^{(0)}}{\beta}\bar{\nu}\cdot\nabla_r\hat{\mu}_i^{(1)})^-$$

So only the last term involving $\tilde{\mu}_i^{(2)}$ needs to be computed. But we know that

$$\tilde{\mu}_i^{(2)} = \frac{T}{w_i}\tilde{\rho}_i^{(2)} - \frac{T}{2}(\frac{\tilde{\rho}_i^{(1)}}{w_i})^2 + \tilde{g}_i^{(2)} = \mathcal{L}\tilde{\rho}_i^{(2)} + D_i(\tilde{\rho}_i^{(1)}, \tilde{\rho}_i^{(0)}). \tag{5.39}$$

The solvability condition is

$$\sum_i \int_{-\infty}^{+\infty} dz[\tilde{\mu}_i^{(2)} - D_i]w_i' = 0. \tag{5.40}$$

Summing on $i = 1, 2$

$$\begin{array}{c}\sum_i \int dz w_i'\tilde{\mu}_i^{(2)} = \sum_i \int_{-\infty}^{+\infty} dz D_i w_i' = \\ = \sum_i \int_{-\infty}^{+\infty} dz w_i'(B_1(\tilde{\rho}_i^{(1)}) + B_2(\tilde{\rho}_i^{(0)}))\end{array} \tag{5.41}$$

We introduce functions $\bar{\mu}_i^{(2)}$ defined as explained after (5.28) in terms of $(\mu_i^{(2)})^{\pm}$. We have

$$[\bar{\rho}_1\bar{\mu}_1^{(2)} + \bar{\rho}_2\bar{\mu}_2^{(2)}]_-^+ = -\sum_i \int dz[\beta\bar{\nu}\cdot\tilde{u}_i^{(2)} + w_i j_i] + \sum_i \int dz[B_1(\tilde{\rho}_i^{(1)}) + B_2(\tilde{\rho}_i^{(0)})] - \sum_i \int dz w_i' l_i =: H \tag{5.42}$$

We are now in position to find the first correction to the velocity of the interface, $V^{(1)}$. This is given by the solvability condition for the boundary equation for $n = 3$:

$$\begin{array}{c}\sum_{l,l'\geq 0: l+l'=1} V^{(l')}\partial_z\tilde{f}_i^{(l)} + \sum_{l,l'\geq 0: l+l'=3} v\cdot\nu^{(l)}\partial_z\tilde{f}_i^{(l')} + v\cdot\overline{\nabla}_r\tilde{f}_i^{(2)} + \partial_t\tilde{f}_i^{(0)} - \\ - \sum_{l,l',l''\geq 0: l+l'=3} \partial_z\tilde{g}_i^{(l)}\nu^{(l'')}\cdot\nabla_v\tilde{f}_i^{(l')} - \sum_{l,l'\geq 0: l+l'=2} \overline{\nabla}_r\tilde{g}_i^{(l)}\cdot\nabla_v\tilde{f}_i^{(l')} = L_\beta\tilde{f}_i^{(3)}\end{array}$$

After integration on v we get

$$\overline{\nabla}_r \cdot \tilde{u}_i^{(2)} + \bar{V}\partial_z \tilde{\rho}_i^{(1)} + V^{(1)}\partial_z w_i + \partial_z(\bar{\nu} \cdot \tilde{u}_i^{(3)}) + \partial_z(\nu^{(1)} \cdot \tilde{u}_i^{(2)}) = 0$$

Taking into account that

$$\overline{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z \bar{\nu} \cdot \tilde{u}_i^{(3)} = \overline{\nabla}_r \cdot \hat{u}_i^{(2)} + \bar{\nu} \cdot \nabla_r(\bar{\nu} \cdot \hat{u}_i^{(2)}) + O(e^{-\alpha|z|}) = \nabla_r \cdot \hat{u}_i^{(2)} + O(e^{-\alpha|z|}) = O(e^{-\alpha|z|}), \quad z \quad \text{large}$$

because $\nabla_r \cdot \hat{u}_i^{(2)} = 0$, we can integrate over z:

$$\bar{V}[\tilde{\rho}_i^{(1)}]_{-\infty}^{+\infty} + V^{(1)}[w_i]_{-\infty}^{+\infty} = -\int dz[\overline{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z(\bar{\nu} \cdot \tilde{u}_i^{(3)})] - [\nu^{(1)} \cdot \tilde{u}_i^{(2)}]_{-\infty}^{+\infty}. \tag{5.43}$$

By the matching conditions we have

$$\tilde{u}_i^{(3)} \sim \hat{u}_i^{(3)} + [(\bar{\nu}(z - d^{(1)}) - \nu^{(1)} d^{(0)}) \cdot \nabla_r] \hat{u}_i^{(2)}$$

Let us introduce the functions

$$n_i = 1_{z<0} \left[[(\bar{\nu}(z - d^{(1)}) - \nu^{(1)} d^{(0)}) \cdot \nabla_r] \bar{\nu} \cdot \hat{u}_i^{(2)} \right]^- + 1_{z>0} \left[[(\bar{\nu}(z - d^{(1)}) - \nu^{(1)} d^{(0)}) \cdot \nabla_r] \bar{\nu} \cdot \hat{u}_i^{(2)} \right]^+$$

and

$$m_i = \partial_z n_i = 1_{z<0}[\bar{\nu} \cdot \nabla_r(\bar{\nu} \cdot \hat{u}_i^{(2)})]^- + 1_{z>0}[\bar{\nu} \cdot \nabla_r(\bar{\nu} \cdot \hat{u}_i^{(2)})]^+$$

Thus

$$\int dz[\overline{\nabla}_r \cdot \tilde{u}_i^{(2)} + \partial_z(\bar{\nu} \cdot \tilde{u}_i^{(3)})] = intdz[\overline{\nabla}_r \cdot \tilde{u}_i^{(2)} + m_i + \partial_z(\bar{\nu} \cdot \tilde{u}_i^{(3)} - n_i)] =$$

$$= \int dz[\overline{\nabla}_r \cdot \tilde{u}_i^{(2)} + m_i] + [\bar{\nu} \cdot \hat{u}_i^{(3)}]_-^+ = A_i + [\bar{\nu} \cdot \hat{u}_i^{(3)}]_-^+$$

where we still have to compute A_i. Essentially we need to know $\overline{\nabla}_r \cdot \tilde{u}_i^{(2)}$. It can be derived from (5.37) multiplying for $\nu_\perp \cdot v$, where $\nu_\perp$ denotes one of the two directions orthogonal to $\bar{\nu}$, and integrating in v. After some computations we have

$$\beta \nu_\perp \cdot \tilde{u}_i^{(2)} = -\partial_z \int dv(\nu_\perp \cdot v)(\bar{\nu} \cdot v)\tilde{q}_i^{(2)} + w_i \nu_\perp \cdot \overline{\nabla}_r \tilde{g}_i^{(1)} + \frac{1}{\beta} \nu_\perp \cdot \overline{\nabla}_r \tilde{\rho}_i^{(1)}$$

In order to obtain the overlined divergence of $\tilde{u}_i^{(2)}$ we have to sum over the two orthogonal directions; then remembering the asymptotic behaviour of $\int dv P(v)\tilde{q}_i^{(2)}$, we can conclude that

$$A_i = \int dz \left[\frac{1}{\beta} w_i \overline{\nabla}_r \cdot \overline{\nabla}_r \tilde{g}_i^{(1)} + \frac{1}{\beta^2} \overline{\nabla}_r \cdot \overline{\nabla}_r \tilde{\rho}_i^{(1)} + m_i \right]$$

Here we recall that $\hat{u}_i^{(2)}$ depends only on quantities of the previous order in ε, which are known. Moreover by (5.17) we have also

$$\bar{u}_i^{(3)} = -T[\bar{\rho}_i \bar{\nu} \cdot \nabla_r \bar{\mu}_i^{(2)} + \bar{\rho}_i^{(1)} \bar{\nu} \cdot \nabla_r \bar{\mu}_i^{(1)}]$$

so that

$$\frac{1}{\beta}[\hat{\rho}_i^{(0)}\bar{\nu}\cdot\nabla_r\hat{\mu}_i^{(2)}]_-^+ = \bar{V}[\hat{\rho}_i^{(1)}]_-^+ + V^{(1)}[\hat{\rho}_i^{(0)}]_-^+ - \frac{1}{\beta}[\hat{\rho}_i^{(1)}\bar{\nu}\cdot\nabla_r\hat{\mu}_i^{(1)}]_-^+ + A_i - \frac{1}{\beta}[\hat{\rho}_i^{(0)}\nu^{(1)}\cdot\nabla_r\hat{\mu}_i^{(1)}]_-^+ \tag{5.44}$$

We consider the functions $\zeta^{(2)} = \bar{\rho}_1\bar{\mu}_1^{(2)} + \bar{\rho}_2\bar{\mu}_2^{(2)}$ and $\xi^{(2)} = \bar{\rho}_1\bar{\mu}_1^{(2)} - \bar{\rho}_2\bar{\mu}_2^{(2)}$. We have from (5.42)

$$[\zeta^{(2)}]_-^+ = H(r,t)$$

and by summing in equation (5.44)

$$[\bar{\nu}\cdot\nabla_r\zeta^{(2)}]_-^+ = \beta\bar{V}[\hat{\rho}_1^{(1)} + \hat{\rho}_2^{(1)}]_-^+ + \sum_i \left\{-[\hat{\rho}_i^{(1)}\bar{\nu}\cdot\nabla_r\hat{\mu}_i^{(1)}]_-^+ + \beta A_i - [\hat{\rho}_i^{(0)}\nu^{(1)}\cdot\nabla_r\hat{\mu}_i^{(1)}]_-^+\right\} := P$$

Moreover, we have the identity $[\xi^{(2)}]_-^+ = \zeta^{(2)+} + \zeta^{(2)-}$.

We get the velocity $V^{(1)}$ by taking the difference in (5.44) on the index i

$$\begin{aligned}2[\bar{\rho}^+ - \bar{\rho}^-]V^{(1)} = \frac{1}{\beta}[\bar{\nu}\cdot\nabla_r\xi^{(2)}]_-^+ - \bar{V}[\hat{\rho}_1^{(1)} - \hat{\rho}_2^{(1)}]_-^+ + \frac{1}{\beta}[\hat{\rho}_1^{(1)}\bar{\nu}\cdot\nabla_r\hat{\mu}_1^{(1)}]_-^+ - \\ -\frac{1}{\beta}[\hat{\rho}_2^{(1)}\bar{\nu}\cdot\nabla_r\hat{\mu}_2^{(1)}]_-^+ - A_1 + A_2 + \frac{1}{\beta}[\hat{\rho}_1^{(0)}\nu^{(1)}\cdot\nabla_r\hat{\mu}_1^{(1)}]_-^+ - \\ -\frac{1}{\beta}[\hat{\rho}_2^{(0)}\nu^{(1)}\cdot\nabla_r\hat{\mu}_2^{(1)}]_-^+ := 2[\bar{\rho}^+ - \bar{\rho}^-]Q\end{aligned}$$

Notice that $\int_{\Gamma_t} V^{(1)}$ is not necessarily zero as was $\int_{\Gamma_t}\bar{V}$. This implies that the volume enclosed by the interface $\Gamma_t^{(1)}$ evolving with $V^{(0)} + \varepsilon V^{(1)}$ is not conserved.

In conclusion, $\xi^{(2)}$ and $\zeta^{(2)}$ are solutions of

$$\begin{cases}\Delta_r\xi^{(2)}(r,t) = S_1 - S_2 & r\in\Omega\setminus\bar{\Gamma}_t\\ [\xi^{(2)}]_-^+(r,t) = \frac{|\bar{\phi}|}{\bar{\rho}}(\zeta^{(2)+} + \zeta^{(2)-}) & r\in\bar{\Gamma}_t\\ V^{(1)} = Q & r\in\bar{\Gamma}_t\end{cases} \tag{5.45}$$

and

$$\begin{cases}\Delta_r\zeta^{(2)}(r,t) = S_1 + S_2 & r\in\Omega\setminus\bar{\Gamma}_t\\ [\zeta^{(2)}]_-^+ = H(r,t) & r\in\bar{\Gamma}_t\\ \left[\bar{\nu}\cdot\nabla_r\zeta^{(2)}\right]_-^+ = P & r\in\bar{\Gamma}_t\end{cases} \tag{5.46}$$

S_i, Q have been determined before. The terms H and P depend on $d^{(0)}$ which has been already found and also on $d^{(1)}$ which is unknown and has to be determined by $V^{(1)} = \partial_t d^{(1)}$. These equations are different from the first order equations because the surface $\bar{\Gamma}_t$ is given, so that we are not facing a free boundary problem. In this sense they are "linearized" even if the equations remain non linear. The problem is well posed because given $d^{(1)}$ on $\bar{\Gamma}_t$ the problem has a unique solution and this solution in turn determines the velocity $V^{(1)}$. Then,

$d^{(1)}$ is found in $\mathcal{N}^0(m)$ through the condition $\nabla d^{(1)} \nabla d^{(0)} = 0$. The argument is analogous to the one in [ABC].

Once $\bar{\mu}_i^{(2)}$ are found as solutions of these equations we can find $\hat{\rho}_i^{(2)}$ in terms of $\hat{\mu}_i^{(2)}$. We have now the asymptotic values needed to solve (5.37). If the solution exists, it decays exponentially at infinity, because the known terms have this property. This equation admits a solution if the conditions at infinity satisfy suitable conditions. The matching conditions require that the solution at infinity grows linearly. This is a problem analogous to the so-called Kramers problem in the half space [BCN]. It can be reduced to a Riemann problem with fixed asymptotic value at infinity in the following way:
Since the solution $\tilde{f}_i^{(2)}$ has to be approximately $A_i^{\pm} + zB_i^{\pm}$ at infinity the functions $A_i^{\pm} = M_\beta((\hat{\rho}_i^{(2)})^{\pm} - v \cdot (\hat{\rho}_i^{(0)} \nabla_r \hat{\mu}_i^{(1)})^{\pm})$ and $B_i^{\pm} = M_\beta(\nu \cdot \nabla_r \hat{\rho}_i^{(1)})^{\pm}$ have to satisfy

$$\nu \cdot v(B_i^{\pm} + M_\beta \beta \hat{U}(\hat{\rho}_i^{(0)} \bar{\nu} \cdot \nabla_r \hat{\rho}_i^{(1)})^{\pm}) + M_\beta v \cdot (\hat{\rho}_i^{(0)} \overline{\nabla}_r \hat{\mu}_i^{(1)})^{\pm} = L_\beta(A_i^{\pm} + zB_i^{\pm})$$

because all the other terms in (5.35) vanish at infinity. This is equivalent to

$$L_\beta(B_i^{\pm}) = 0, \qquad L_\beta A_i^{\pm} = \bar{\nu} \cdot v(B_i^{\pm} + M_\beta \beta \hat{U}(\hat{\rho}_i^{(0)} \nu \cdot \nabla_r \hat{\rho}_i^{(1)})^{\pm}) + M_\beta v \cdot (\hat{\rho}_i^{(0)} \overline{\nabla}_r \hat{\mu}_i^{(1)})^{\pm}.$$

This is true by direct inspection. Then, the problem is reduced to a well posed problem of finding a solution h_i to eq. (5.35) decaying to $M_\beta(\hat{\rho}_i^{(2)})^{\pm}$ at infinity.

Similar arguments lead to the computation of higher order terms.

We conclude this section by remarking that our expansion is different from the one in [CCO1] which is more similar to a Chapmann-Enskog expansion because the terms of their expansion $f_i^{(n)}$ still depend on ε and are determined by equations which are nonlinear in the interface at every order in the sense that they are free-boundary problems determining for any n an interface $\Gamma^{(n)}$ moving with velocity $\sum_{i=0}^{n} \varepsilon^i V^{(i)}$. Our approach is intermediate between [ABC] and [Yu], where it is proven the hydrodynamic limit for the Boltzmann equation in presence of shocks.

5.2 Interface motion

In this section we discuss the equations for the interface motion. We start by rewriting them in terms of the quantities ζ and $\psi = \bar{\mu}_1^{(1)} - \bar{\mu}_2^{(1)}$. The equation for ψ is similar to the Mullins-Sekerka equation but for the fact that there is an extra term determining the velocity

$$\begin{cases} \Delta_r \psi(r,t) = 0 & r \in \Omega \setminus \Gamma_t \\ \psi(r,t) = \dfrac{S\bar{K}(r,t)}{\bar{\rho}^+ - \bar{\rho}^-} & r \in \Gamma_t \\ V = \frac{T}{2[\bar{\rho}^+ - \bar{\rho}^-]} \left[\frac{1}{\bar{\rho}} (\bar{\rho}^2 - |\bar{\varphi}|^2) [\bar{\nu} \cdot \nabla_r \psi]_-^+ + \frac{1}{\bar{\rho}} [\bar{\varphi} \bar{\nu} \cdot \nabla_r \zeta]_-^+ \right] & r \in \Gamma_t \end{cases} \tag{5.47}$$

The jump of $\bar{\varphi}\bar{\nu}\cdot\nabla_r\zeta$ in the last term on the r.h.s. is indeed $2|\bar{\varphi}|\nu\cdot\nabla_r\zeta(r,t), r\in\Gamma_t$ and

$$\begin{cases} \Delta_r\zeta(r,t)=0 & r\in\Omega\setminus\Gamma_t \\ [\zeta]^+_-=2|\bar{\varphi}|S\bar{K}(r,t)/[w_1]^{+\infty}_{-\infty} & r\in\Gamma_t \\ 0=[\bar{\nu}\cdot\nabla_r\zeta]^+_- & r\in\Gamma_t \end{cases} \tag{5.48}$$

Hence there are two contributions to the velocity of the interface: V_{MS}, the velocity of an interface in the Mullins-Sekerka motion, and V_{HS}, the velocity of an interface in the two-phases Hele-Shaw motion (5.48). The latter describes the motion of a bubble of gas expanding into a fluid in a radial Hele-Shaw cell and is a free-boundary problem for the pressure P

$$\begin{cases} \Delta_r P(r,t)=0 & r\in\Omega\setminus\Gamma_t \\ [p]^+_-=CK(r,t)/[w_1]^{+\infty}_{-\infty} & r\in\Gamma_t \\ V=\nu\cdot\nabla_r P & r\in\Gamma_t \end{cases} \tag{5.49}$$

Equations (5.47) and (5.48) are identical to the equations in [OE], describing the sharp interface motion for the dynamics of incompressible fluid mixtures driven by thermodynamic forces, modeling a polymer blend. In this paper the macroscopic equation is a modification of the Cahn-Hilliard equation for a mixture of two fluids that includes a lagrangian multiplier p ("pressure") to take into account the constraint of constant total density

$$\begin{aligned} &\partial_t\rho_i=\nabla\cdot(\rho_i\nabla(\mu_i+p)) \quad i=1,2 \\ &\rho_1+\rho_2=1. \end{aligned} \tag{5.50}$$

This produces in the macroscopic equation for the concentration a convective term which in turn gives rise to the Hele-Shaw contribution V_{HS} to the interface motion. The macroscopic equations (2.1) with $\mu_i=\delta\mathcal{F}/\delta\rho_i$ differs from the ones above for the constraint and hence for the pressure term. It is easy to see that the formal sharp interface limit is the same for both equations with $\nabla\zeta$ in the bulk a divergence-free field appearing as a velocity field in the equation for the total density which is constant in the bulk at the first order. Moreover, thermodynamic relations give that $\nabla\zeta=\nabla p^{(1)}$ with $p^{(1)}$ the first correction to the effective pressure. Hence, the role of $\nabla\zeta$ is exactly the same as the lagrangian multiplier p in [OE].

We refer to [OE] for the discussion on the behavior of the interface as given in (5.47) and (5.48). Here we want just to remark that the Hele-shaw motion has more conserved quantities than the Mullins-Sekerka motion. In fact, the former conserves the volume of each connected component of both phases, while the latter conserves only the total volume as we can easily see by starting from

$$\frac{d}{dt}|\Omega^+_\Gamma|=\int_\Gamma V$$

where Ω_Γ^+ is the region enclosed in the surface Γ. We consider now a situation in which there are N closed curves Γ_i dividing Ω in N connected components $\Omega_{\Gamma_I}^+$. In the Mullins-Sekerka problem the velocity is proportional to the jump of the normal derivative of the harmonic function f and this implies by using the divergence theorem

$$\sum_i \frac{d}{dt}|\Omega_{\Gamma_i}^+| = \sum_i \int_{\Gamma_i} [\nu \cdot \nabla f]_-^+ = \sum_i \int_{\Omega_{\Gamma_i}^+} \Delta f + \int_{\Omega^-} \Delta f = 0$$

where Ω^- is the complement of $\cup_i(\Omega_{\Gamma_i}^+ \cup \Gamma_i)$. In the Hele-Shaw problem the velocity is proportional to the normal derivative of f and

$$\frac{d}{dt}|\Omega_{\Gamma_i}^+| = \int_{\Gamma_i} \nu \cdot \nabla f = \int_{\Omega_{\Gamma_i}^+} \Delta f = 0$$

In the problem (5.47), (5.48) this fact has consequences on the evolution of the droplets of the two phases. The relative importance of the two contributions V_{HS} and V_{MS} is ruled by the coefficients: if $(\bar{\rho}^-)^{-1} - (\bar{\rho}^+)^{-1} << 1$ (near the critical point) the V_{MS} term dominates, while for deep quenches the V_{HS} term prevails.

Chapter 6

Conclusions

The phase segregation process is a challenging non equilibrium phenomenon, whose understanding and modeling are far from being complete. Multi-component mixtures can have very complex phase diagrams. Each model has to face two kinds of limitations: it is expected to be reliable in a certain set of values of the thermodynamical parameters and it reproduces the behaviour of the system in some, but not all, stages of its evolution.

We focus on the late stages of phase segregation. Our model is able to provide the same limiting dynamics given by a totally phenomenological equation such as the Cahn-Hilliard. It has a kinetic origin and is deeply related to models directly arising from microscopic systems on the lattice.

We proceeded mostly formally, but in such a way to make easier a future rigorous approach. It goes through the proof of all the properties stated for the functions introduced, starting from the fronts. Then a precise control of the remainder is needed. The choice of the right norm is related to the stability properties of the stationary solutions of our equations.

The main difficulty we faced to was to clarify the procedure to get the corrections to the limiting motion. Indeed at order zero some peculiarities of the problem still not arise. For example no dependence on the orthogonal coordinates to z is relevant till the first correction, when one has to find the way to compute it.

Matching conditions are a very natural choice, but with deep consequences. They allow to determine the value of the bulk quantities on the moving interface in terms of the asymptotic behaviour of the layer functions. In other words they are essential in order to make the double expansion approach meaningful.

Several extension of the model are possible. The most interesting is provided by the replacing of the Fokker-Planck operator with the Boltzmann one. In this case energy and momentum are conserved and a complicated hydrodynamics is expected.

Appendix A

Geometric Curvature

We want to recall some properties of the curvature in a point of a plain curve, which is assumed quite smooth. Curvature can be defined as the inverse of the ray of the osculating circle, that is the circle which best approximates the curve in the point in consideration. Alternatively let A and B be two point of the curve and call $\Delta\theta$ the angle between the direction of the tangent vectors in the points A and B; then, if Δs is the length of the arc joining A and B, the curvature in A is

$$\lim_{B\to A} \frac{\Delta\theta}{\Delta s} \tag{A.1}$$

Since the cartesian equation of a circle depends on three parameters, we can ask at most for an approximation of the second order. In other words if we substitute to the curve the osculating circle, we make an error which vanishes faster then $(\Delta s)^2$ when B goes to A, but not then $(\Delta s)^3$ in general. Consider a reference system where the curve can be locally parameterized around A by the coordinate x and a function f: $(x, f(x))$; and such that $(0, f(0)) = (0, 0)$ and $(1, f'(0)) = (1, 0)$. Suppose also for definiteness that $f''(0) > 0$. If $f''(0) = 0$ the curvature is by definition put equal to 0 and one says that the ray of curvature is infinite. A generic circle is

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

By geometric intuition we can see that y_0 has to be positive and that the interesting semi-circumference is

$$y = y_0 - \sqrt{r^2 - (x - x_0)^2}$$

We impose that

$$\lim_{x\to 0} \frac{f(x) - y_0 + \sqrt{r^2 - (x - x_0)^2}}{x^2} = 0$$

from which we deduce the first condition

$$-y_0 + \sqrt{r^2 - (x - x_0)^2} = 0 \tag{A.2}$$

because the numerator has to vanish in 0. Now we can apply De L'Hopital:

$$\lim_{x\to 0} \frac{f'(x) - \frac{x-x_0}{\sqrt{r^2-(x-x_0)^2}}}{2x} = 0$$

and write down the second condition

$$\frac{x_0}{\sqrt{r^2 - x_0^2}} = 0 \tag{A.3}$$

Repeating again the above procedure we get

$$\lim_{x\to 0} \frac{f''(x) - \frac{1}{\sqrt{r^2-(x-x_0)^2}} - \frac{(x-x_0)^2}{(\sqrt{r^2-(x-x_0)^2})^3}}{2} = 0$$

and the third condition

$$f''(0) - \frac{1}{\sqrt{r^2 - x_0^2}} \left(1 + \frac{x_0^2}{(r^2 - x_0^2)^2}\right) = 0 \tag{A.4}$$

From conditions (A.2), (A.3) and (A.4) we can compute all the parameters of the circumference:

$$x_0 = 0, \quad y_0 = \frac{1}{f''(0)}, \quad r = \frac{1}{f''(0)}$$

From the above calculations it appears clearly that we can not expect a better approximation. We learnt that in that system of reference

$$f(x) = \frac{1}{f''(0)} - \sqrt{\frac{1}{[f''(0)]^2} - x^2} + o(x^2)$$

and that the curvature, call it κ, is

$$\kappa = f''(0)$$

The same result can be obtained from the definition of curvature given in (A.1). In fact in the previous system of reference, $\Delta\theta$ is simply the angle α between the tangent in $(x, f(x))$ and the x axis. Thus $\tan(\alpha) = f'(x)$ and we can consider x as a function of α and compute the first derivative of x with respect to α:

$$\frac{dx}{d\alpha}(\alpha) = \frac{1 + \tan^2 \alpha}{f''(x(\alpha))}$$

In 0 we have

$$\frac{dx}{d\alpha}(0) = \frac{1}{f''(0)} \tag{A.5}$$

In order to conclude it suffices to show that Δs and x differ by a quantity that is of order superior to α. Indeed from (A.5) we deduce that x and α have the same order and if $\Delta s = x + o(x)$ then

$$\lim_{\alpha\to 0}\frac{\Delta s}{\alpha} = \lim_{\alpha\to 0}\frac{x+o(\alpha)}{\alpha} = \frac{1}{f''(0)}$$

We know that $f'(x) = f''(0)x + o(x)$ and that

$$\Delta s = \int_0^x dt\sqrt{1+[f'(t)]^2}$$

then actually

$$\Delta s = x + o(x^2)$$

If the curve is a circumference, the equivalence of the two definitions is obvious because $\Delta\theta$ coincides with the angle between the vector rays joining A and B to the center and Δs is just $r\Delta\theta$.

Now we deduce an expression for the curvature of a generic curve described by $(x, g(x))$ in a point $(x_0, g(x_0))$. It is just a change of coordinates. Define the normal versors $\hat{v}$ and ν to be

$$\hat{v} = \frac{(1, g'(x_0))}{\sqrt{1+[g'(x_0)]^2}}, \quad \nu = \frac{(-g'(x_0), 1)}{\sqrt{1+[g'(x_0)]^2}}$$

then

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{\sqrt{1+[g'(x_0)]^2}}\begin{pmatrix} 1 & -g'(x_0) \\ g'(x_0) & 1 \end{pmatrix}\begin{pmatrix} \hat{v} \\ \nu \end{pmatrix}$$

Then the new coordinates $(X, f(X))$ of the points on the curve are such that

$$X\hat{v} + f(X)\nu = (x - x_0)e_1 + (g(x) - g(x_0))e_2$$

from which we get

$$\begin{aligned} X &= \frac{x - x_0 + (g(x) - g(x_0))g'(x_0)}{\sqrt{1+[g'(x_0)]^2}} \\ f(X) &= \frac{g(x) - g(x_0) - (x - x_0)g'(x_0)}{\sqrt{1+[g'(x_0)]^2}} \end{aligned} \tag{A.6}$$

The variable x can be regarded as a function of X implicitly defined by the former of the above relations. So we can compute $f''(X)$:

$$f''(X) = \frac{1}{\sqrt{1+[g'(x_0)]^2}}\left[g''(x)\left(\frac{dx}{dX}\right)^2 + (g'(x) - g'(x_0))\frac{d^2x}{dX^2}\right]$$

and in $X = 0$ which correspond to $x = x_0$

$$f''(0) = \frac{g''(x_0)}{\sqrt{1+[g'(x_0)]^2}}\left(\left.\frac{dx}{dX}\right|_{X=0}\right)^2$$

By deriving (A.6) we find

$$\frac{dx}{dX} = \frac{\sqrt{1+[g'(x_0)]^2}}{1+g'(x)g'(x_0)}$$

and

$$\left.\frac{dx}{dX}\right|_{X=0} = \frac{1}{\sqrt{1+[g'(x_0)]^2}}$$

Thus at the end

$$\kappa = f''(0) = \frac{g''(x_0)}{(1+[g'(x_0)]^2)^{\frac{3}{2}}} \tag{A.7}$$

Now we work in a more general setting. Let $x(s) = (x_1(s), \ldots, x_n(s))$ be the parametric representation of a curve in $\mathbb{R}^n$ with respect to its curvilinear coordinate s. The angle $\Delta\theta$ is function of s and it can be computed starting from

$$\cos(\Delta\theta) = \frac{x(s)\cdot x(s+\Delta s)}{\|x(s)\|\|x(s+\Delta s)\|}$$

Indeed we can introduce

$$\hat{v}(s) = \frac{x(s)}{\|x(s)\|}$$

and note that $\hat{v}\cdot\hat{v}' = 0$ because $2\hat{v}\cdot\hat{v}' = D\hat{v}^2 = D1 = 0$. Then, if we Taylor expand $x(s+\Delta s)$ and $\cos(\Delta\theta)$ we obtain

$$\begin{aligned} 1 - \frac{(\Delta\theta)^2}{2} &= \hat{v}(s)\cdot[\hat{v}(s) + \hat{v}'(s)\Delta s + \frac{1}{2}\hat{v}''(s)(\Delta s)^2] + o((\Delta s)^2) \\ &= 1 + \frac{1}{2}\hat{v}\cdot\hat{v}''(\Delta s)^2 + o((\Delta s)^2) \end{aligned}$$

thus

$$\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s} = \pm\sqrt{-\hat{v}\cdot\hat{v}''}$$

where we can adopt a suitable convention to choose the sign. For example we can assume that the angle $\Delta\theta$ is the acute one and we choose the plus sign if $x'(s+\Delta s)$ goes to $x'(s)$ rotating clockwise. By computing the second derivative of $\hat{v}$ we get

$$-\hat{v}\cdot\hat{v}'' = \frac{\|x'\|^2\|x''\|^2 - (x'\cdot x'')^2}{\|x'\|^4}$$

That expression can be greatly simplified recalling that s is the curvilinear coordinate. Indeed let $X(t)$ be a generic parametric representation of the curve, then

$$s = \int_{t_0}^{t}\|X'(\tau)\|d\tau \tag{A.8}$$

and $s = s(t)$ with $X(t) = x(s(t))$. If now we derive twice this relation we can solve with respect to x' and x'':

$$X' = x's', \quad X'' = x''(s')^2 + x's''$$

and

$$x' = \frac{X'}{s'}, \quad x'' = \frac{1}{(s')^2}\left(X'' - X'\frac{s''}{s'}\right)$$

But from (A.8) we have

$$s' = \|X'\|, \quad s'' = \frac{X' \cdot X''}{\|X'\|}$$

so that

$$x' = \frac{X'}{\|X'\|}, \quad x'' = \frac{1}{\|X'\|^2}\left(X'' - (X' \cdot X'')\frac{X'}{\|X'\|^2}\right)$$

We learn that if s is the curvilinear coordinate, then x' is a versor orthogonal to x'':

$$\|x'\| = 1, \quad x' \cdot x'' = 0$$

We can also evaluate the norm of x'', thus, at the end:

$$\kappa = \pm\sqrt{-\hat{v} \cdot \hat{v}''} = \pm\|x''\| = \pm\frac{\sqrt{\|X'\|^2\|X''\|^2 - (X' \cdot X'')^2}}{\|X'\|^3}$$

If $X = (x, g(x))$ we easily reproduce formula (A.7).

We now turn to a new topic. We want to show that the curvature can be computed as the second derivative of the function which measure the distance of a point from the curve. We put ourselves in the simplest system of reference, as above, and let f be given by

$$f(x) = \frac{1}{2}f''(0)x^2 + o(x^2)$$

where $f''(0) > 0$. Fix $\bar{y} > 0$ quite small and call $\chi(\bar{x})$ the point where the function

$$F(\bar{x}, \bar{y}, x) = \sqrt{(\bar{x} - x)^2 + (\bar{y} - f(x))^2}$$

attains its minimum. $\chi(\bar{x})$ has to satisfy $\partial_x F(\bar{x}, \bar{y}, \chi(\bar{x})) = 0$, then the relation defining it is the following:

$$\bar{x} - \chi(\bar{x}) + (\bar{y} - f(\chi(\bar{x})))f'(\chi(\bar{x})) = 0 \tag{A.9}$$

By geometric arguments we know that if $\bar{y}$ is sufficiently small, then $\chi(0) = 0$. The existence and well posedness of the function χ is guaranteed by the implicit function theorem when, see (A.9),

$$-1 + \bar{y}f''(0) \neq 0 \tag{A.10}$$

and this is the case if $\bar{y}$ is quite small. Now we write χ as a power series of $\bar{x}$:

$$\chi(\bar{x}) = \chi'(0)\bar{x} + o(\bar{x})$$

Thus $\chi(\bar{x})$ has the same order of $\bar{x}$. Of course, from the definition of f, one has $f'(x) = f''(0)x + o(x)$. Filling (A.9) with those expansions one obtains

$$\bar{x} + (\bar{y}f''(0) - 1)\chi'(0)\bar{x} + o(\bar{x}) = 0$$

and ,dividing by $\bar{x}$ and taking the limit $\bar{x} \to 0$,

$$\chi'(0) = \frac{1}{1 - \bar{y}f''(0)} \tag{A.11}$$

Introduce the function

$$\phi(\bar{x}) = F(\bar{x}, \bar{y}, \chi(\bar{x}))$$

Note that we are not writing explicitly the parametric dependence on $\bar{y}$. We are interested in evaluating the second derivative of ϕ in $(\bar{x}, \bar{y}) = (0, 0)$. First of all

$$\frac{d^2}{d\bar{x}^2}\phi = \partial_{\bar{x}\bar{x}}F + 2\partial_{x\bar{x}}F\chi' + \partial_{xx}F(\chi')^2 + \partial_x F\chi''$$

but the last term vanishes thanks to (A.9). The other derivatives are

$$\partial_{\bar{x}\bar{x}}F(\bar{x}, \bar{y}, \chi(\bar{x})) = \frac{1}{F} - \frac{(\bar{x} - \chi(\bar{x}))^2}{F^3}$$

$$\partial_{x\bar{x}}F(\bar{x}, \bar{y}, \chi(\bar{x})) = -\frac{1}{F}$$

$$\partial_{xx}F(\bar{x}, \bar{y}, \chi(\bar{x})) = -\frac{1}{F}(-1 - (f'(\chi(\bar{x})))^2 + (\bar{y} - f(\chi(\bar{x})))f''(\chi(\bar{x})))$$

Using these expressions and (A.11) we get

$$\left.\frac{d^2}{d\bar{x}^2}\phi\right|_{(\bar{x},\bar{y})=(0,\bar{y})} = \frac{f''(0)}{1 - \bar{y}f''(0)}$$

Thus

$$\left.\frac{d^2}{d\bar{x}^2}\phi\right|_{(\bar{x},\bar{y})=(0,0)} = f''(0)$$

but in our system of reference the right hand side is just the curvature κ. So condition (A.10) can be reformulated by saying that $\bar{y}$ has to be smaller than the ray of curvature.

If $\hat{v}$ is a versor, the double sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{v}_i(\partial_{r_i}\partial_{r_j}f)\hat{v}_j$$

is not but the second derivative of f in the direction of $\hat{v}$. Indeed it is a straight consequence of the equality

$$\frac{df}{d\hat{v}} = v \cdot \nabla f$$

Given a surface of codimension one, we can construct a function ϕ as the distance of a point from the surface. The matrix $\partial_{r_i}\partial_{r_j}\phi$ defined by continuity and evaluated on points of the surface can be diagonalized, an eigenvector of eigenvalue 0 being the outward normal. The other eigenvalues are, because of what seen above, the curvatures of the curves given by the intersection of the surface with the planes which contain each eigenvector and the outward normal. The trace of the matrix is $n-1$ times the mean curvature.

Appendix B

Uniqueness

In this Appendix we prove that the equations for the $\tilde{f}^{(n)}(z,r,t)$ we examined in section 5 have solutions whose dependence on the velocity is necessarily gaussian. We will omit for simplicity the dependence on the other variables.

We consider the following set of equations for $h_i(z,v), \quad i=1,2, \quad (z,v)\in\mathbb{R}\times\mathbb{R}^d$

$$v_z\partial_z h_i + F_i\partial_{v_z}h_i = L_\beta h_i \tag{B.1}$$

where $F_i = -\partial_z\int dz' U(|z-z'|)\rho_j(z') := -\partial_z V_i, \quad i\neq j,\ \rho_i(z)=\int dv h_i(z,v)$, with the conditions at infinity

$$h_i(\pm\infty,v) = M(v)\rho_i^{\pm} \tag{B.2}$$

and show that it has only a solution of the form $M(v)\rho_i(z)$.

Put $h_i=\psi_i(z,v)M(v)e^{-\beta V_i}$. V_i is bounded due to the assumptions on U. Then,

$$v_z\partial_z\psi_i + F_i\partial_{v_z}\psi_i = \tilde{L}_i\psi_i \tag{B.3}$$

where

$$\tilde{L}\psi_i = \frac{1}{M_\beta}\nabla_v\cdot(M_\beta\nabla_v\psi_i)$$

with the conditions at infinity

$$\psi_i(\pm\infty,v) = e^{+\beta V_i(\pm\infty)}\rho_i^{\pm}$$

Multiply by $M_\beta\psi_i$ and integrate over v

$$\frac{1}{2}\partial_z(v_z\psi_i,\psi_i)_{M_\beta} + F_i(\psi_i,\frac{d}{dv_z}\psi_i)_{M_\beta} = -(\psi_i,\tilde{L}\psi_i)_{M_\beta} \tag{B.4}$$

where $(h,g)_{M_\beta}=\int dv h(v)g(v)M_\beta(v)$. We have

$$\frac{1}{2}\frac{d}{dz}(v_z\psi_i,\psi_i)_{M_\beta} - \frac{\beta}{2}F_i(v_z\psi_i,\psi_i)_{M_\beta} = -e^{-\beta V_i}(\nabla_v\psi_i,\nabla_v\psi_i)_{M_\beta} \tag{B.5}$$

that we write as

$$\frac{d}{dz}[(v_z\psi_i,\psi_i)_{M_\beta}e^{-\beta V_i}] = -2(\nabla_v\psi_i,\nabla_v\psi_i)_{M_\beta}e^{\beta V_i} \tag{B.6}$$

We notice that $(v_z\psi_i,\psi_i)(\pm\infty) = 0$ because of the boundary conditions. Hence, by integrating over z we get

$$\int_{-\infty}^{+\infty} \mathrm{d}z e^{-\beta V_i}||\nabla_v\psi_i||_\beta^2 = 0$$

which implies $\nabla_v\psi_i = 0$ a.e. since V_i is bounded, so that $\psi_i = g(z)$, a function only of the position. Going back to the original equation we see that $g(z)$ has to be the front solution.

Next order equation.

We discuss now equation (5.19). We consider boundary conditions such that the function decays to zero at infinity. A solution has been explicitly found as a Maxwellian times the density $\tilde{\rho}^{(1)}$. Suppose that there are two different solutions h and h' such that $\rho_h = \rho_{h'}$. Then, the equation for the difference is of the form investigated above, so that $h - h' = 0$. This means that there is a unique solution of the form $M_\beta(v)\rho(z)$ in the class of solutions with fixed density ρ. Then, putting this expression back in the equation we determine ρ. The next order equations for $\tilde{f}_i^{(n)}$ have a similar form, but the solutions are not anymore of the form Maxwellian times a polynomial. The existence and uniqueness have to be proved by a different argument.

Appendix C

Surface Tension

The surface tension can be defined as the difference between the free energy of an equilibrium state with an interface and a homogeneous one [B]. The free energy for this model is

$$\Phi(n_1, n_2) = \int \mathrm{d}x f(n_1(x), n_2(x))$$

$$f(n_1, n_2) = T(n_1 \log n_1 + n_2 \log n_2) + \frac{1}{2} n_1 U \star n_2 + \frac{1}{2} n_2 U \star n_1.$$

We have that [B]

$$\sigma = \lim_{L\to\infty} \frac{1}{(2L)^{d-1}} \lim_{M\to\infty} \int_{-L}^{L} \mathrm{d}y_1 \ldots \int_{-L}^{L} \mathrm{d}y_{d-1} \int_{-M}^{M} dy_d [f(w_1, w_2) - f(\rho_1^+, \rho_2^-)]$$

where $\rho_1^{\pm}, \rho_2^{\pm}$ are the equilibrium values of the densities in presence of a phase transition and $w_i(q)$ are the front solutions, smooth functions satisfying the equations

$$T_0 \log w_i(q) + \int_{\mathbb{R}} \mathrm{d}q' \tilde{U}(|q - q'|) w_j(q') = C_i \tag{C.1}$$

where $\tilde{U}(q) = \int_{\mathbb{R}^2} \mathrm{d}y U(\sqrt{q^2 + y^2})$ and C_i are constants determined by the conditions at infinity $\rho_i^{\pm}$. Notice that $f(\rho_1^+, \rho_2^-) = f(\rho_1^-, \rho_2^+)$ since $\rho_1^{\pm} = \rho_2^{\mp}$.

We rewrite the surface tension by using integration by part and the condition at infinity

$$\sigma = \int_{-\infty}^{+\infty} \mathrm{d}z [f(w_1, w_2) - f(n_1^+, n_2^-)] = -\int_{-\infty}^{+\infty} \mathrm{d}z z \frac{d}{dz} f(w_1(z), w_2(z)).$$

We have

$$\frac{d}{dz} f(w_1, w_2) = T[(\log w_1 + 1) w_1' + (\log w_2 + 1) w_2'] + \frac{1}{2}[w_1' \tilde{U} \star w_2 + w_2' \tilde{U} \star w_1 + w_1 \tilde{U} \star w_2' + w_2 \tilde{U} \star w_1'].$$

By using (C.1) we get

$$\frac{d}{dz} f(w_1, w_2) = \frac{1}{2}[-w_1' \tilde{U} \star w_2 - w_2' \tilde{U} \star w_1 + w_1 \tilde{U} \star w_2' + w_2 \tilde{U} \star w_1'] + C_1 w_1' + C_2 w_2' + T(w_1' + w_2')$$

and for the surface tension

$$-\frac{1}{2}\int \mathrm{d}z\mathrm{d}z'\; z\sum_{i\neq j}[-w_i'(z)\tilde{U}(z-z')w_j(z')+w_i(z)\tilde{U}(z-z')w_j'(z')]-\frac{C+T}{2}\int_{-\infty}^{+\infty}\mathrm{d}z z[w_1'+w_2']$$

where C is the common value of C_1 and C_2. This is a consequence of the symmetry $\rho_1^{\pm}=\rho_2^{\mp}$ and

$$C_1 = T\log\rho_1^+ + \hat{V}\ \rho_2^+, \quad C_2 = T\log\rho_2^- + \hat{V}\ \rho_1^-.$$

We show now that the last term vanishes. The fronts w_i satisfy the following property

$$\int \mathrm{d}z(w_1(z)-\bar{\rho}_1)+(w_2(z)-\bar{\rho}_2)=0,$$

where $\bar{\rho}_i(z)=\rho_i^+, z>0$, $\bar{\rho}_i(z)=\rho_i^-, z<0$. This condition amounts to saying that the mass on the surface is zero. By integrating by parts, the previous condition is equivalent to the vanishing of the last integral in the expression of the surface tension. In conclusion,

$$\sigma=\frac{1}{2}\int dzdz'(z-z')\sum_{i\neq j}[w_i'(z)\tilde{U}(z-z')w_j(z')].$$

Appendix D

Forces

We show how to compute the terms $\hat{g}_i^{(n)}$ and $\tilde{g}_i^{(n)}$ up to order 3. The procedure can be easily extended at any order.

For a slowly varying function $h(r,t)$ we have that

$$\begin{aligned} U^\varepsilon \star h(r,t) &= \int_{\mathbb{R}^3} \varepsilon^{-3} U(\varepsilon^{-1}|r-r'|)h(r',t)dr' \\ &= \int_{\mathbb{R}^3} U(|x-x'|)[h(\varepsilon x',t)-h(\varepsilon x,t)]dx' + h(r,t)\int_{\mathbb{R}^3} U(|x-x'|)dx' \\ &= \int_{\mathbb{R}^3} U(|x-x'|)\Big[\varepsilon(x-x')\cdot\nabla_r h(r,t) \\ &+ \frac{\varepsilon^2}{2}\sum_{i,j}(x-x')_i(x-x')_j\frac{\partial^2}{\partial r_i \partial r_j}h(r,t) + \varepsilon^4 R_h(x,x')\Big]dx' + h(r,t)\hat{U} \\ &= h(r,t)\hat{U} + \varepsilon^2 \Delta_r h(r,t)\bar{U} + \varepsilon^4 U \star R_h \end{aligned} \tag{D.1}$$

where $\hat{U} = \int U(r)dr$, $\quad \bar{U} = \frac{1}{2}\int r^2 U(r)dr$. We have used the isotropy of U. Hence we have

$$\hat{g}_i^{(n)} = \hat{U}\hat{\rho}_j^{(n)},\ n=0,1; \quad \hat{g}_i^{(2)} = \hat{U}\hat{\rho}_j^{(2)} + \bar{U}\Delta_r\hat{\rho}_j^{(0)}, \quad \hat{g}_i^{(3)} = \hat{U}\hat{\rho}_j^{(3)} + \bar{U}\Delta_r\hat{\rho}_j^{(1)}.$$

To compute the expansion of $U^\varepsilon \star h$ for a fast varying function $h(z,r,t)$ it is more convenient to use a local system of coordinates. For a given curve Γ and for any point $s \in \Gamma$ we choose a reference frame centered in s with the axes $1,2$ along the directions of principal curvatures k_i and 3 in the direction of the normal. Consider two points r and r' and choose the reference frame centered in $s(r) : r = s(r) + \varepsilon z \nu(r)$. We denote by y_i and y_i' the microscopic coordinate of r and r' in this new frame. Then $y_1 = y_2 = 0, y_3 = z$ and $q_i' = \varepsilon^{-1} r_i'$, the microscopic coordinates of r', are related to y_i' by a linear transformation $q_i' = A_{i\ell} y_\ell'$. Moreover, z' is given in terms of y_i' by ([GL])

$$\begin{aligned} z'(\{y_i'\}) &= y_3' + \textstyle\sum_{i=1,2}\frac{1}{2}[\varepsilon k_i y_i'^2 - 2\varepsilon^2 k_i^2 y_i'^2 y_3'] + \frac{1}{2}\varepsilon^3(\sum_i k_i^2 y_i'^2)^2 \\ &-\varepsilon^3 \tfrac{1}{4}[\textstyle\sum_{ij}(\partial_i^2 k_j y_i'^2 y_j'^2(4k_i(1-\delta_{ij})+3-2\delta_{ij})] + O(\varepsilon^4), \end{aligned} \tag{D.2}$$

where $z' : r' = s(r') + \varepsilon\nu(r')z'$.

We denote by $\check{h}(y'_1, y'_2, y'_3, t)$ the function $h(\{\varepsilon A_{i\ell} y'_\ell\}, z'(\{y'_i\}), t)$. We have

$$\begin{aligned}(U^\varepsilon \star h)(z,r,t) &= \int_{\mathbb{R}^3} dy' U(|y-y'|)\check{h}(y'_1, y'_2, , y'_3, t) \\ &= \int_{\mathbb{R}^3} dy' U(|y-y'|)\check{h}(0,0,y'_3,t) + \frac{1}{2}\sum_{i=1,2}(\tilde{U}_{1,i} \star \frac{\partial^2 \check{h}}{\partial y'^2_i})(z,r,t) \\ &+ \frac{1}{4}\sum_{i=1,2}(\tilde{U}_{2,i} \star \frac{\partial^4 \check{h}}{\partial y'^4_i})(z,r,t)) + \frac{1}{4}\sum_{i,j=1,2, i\neq j} \tilde{U}_{2,ij} \star \frac{\partial^4 \check{h}}{\partial y'^2_i y'^2_j} + Q,\end{aligned}$$

where

$$\tilde{U}_{s,i}(|y_3 - y'_3|) = \int_{\mathbb{R}^2} dy'_1 dy'_2 U(\sqrt{|y_3 - y'_3|^2 + |y'_1|^2 + |y'_2|^2})|y'_i|^{2s}$$

$$\tilde{U}_{2,ij} = \int_{\mathbb{R}^2} dy'_1 dy'_2 U(\sqrt{|y_3 - y'_3|^2 + |y'_1|^2 + |y'_2|^2})|y'_i|^2 |y'_j|^2.$$

We have

$$\frac{\partial \check{h}}{\partial y'_i} = \varepsilon \sum_j \bar{\nabla}_j h A_{ji} + \frac{\partial h}{\partial z}\frac{\partial z'}{\partial y'_i}$$

$$\frac{\partial^2 \check{h}}{\partial y'^2_k} = \varepsilon^2 \sum_{j\ell} A_{jk} A_{\ell k} \bar{\nabla}^2_{j\ell} h + \frac{\partial z'}{\partial y'_k}[\varepsilon A_{jk}\bar{\nabla}_j \frac{\partial h}{\partial z} + \frac{\partial z'}{\partial y'_k}\frac{\partial^2 h}{\partial z^2}] + \frac{\partial h}{\partial z}\frac{\partial^2 z'}{\partial y'^2_k}.$$

By using the relation between z' and y'_3 D.2 we see that the second term equals to

$$\frac{\partial h}{\partial z}(0,0,y'_3,t)(\varepsilon k_k - \varepsilon^2 2k_k^2 y'_3).$$

It is true that [B]

$$\int_{\mathbb{R}^3} dy' U(|y-y'|)\frac{\partial h}{\partial z}(0,0,y'_3,t)\sum_{i=1}^{d-1}\frac{k_i^2 y'^2_i}{2} = \frac{K}{2}\int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)h(z',r,t). \quad \text{(D.3)}$$

To compute the contributions at different order in ε we go back to the specific curve Γ^ε_t and use the expansion $d^\varepsilon(r,t) = \sum_n \varepsilon^n d^{(n)}(r,t)$ which implies $k^\varepsilon_i = \sum_n \varepsilon^n k^{(n)}_i$ and $A^\varepsilon_{ij} = \sum_n \varepsilon^n A^{(n)}_{ij}$. In conclusion,

$$\begin{aligned}&(U^\varepsilon \star h)(z,r) = (\tilde{U} \star h)(z,r) + \varepsilon\frac{\bar{K}}{2}\int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)h(z',r) \\ &+ \sum_{i=1,2}\Big[\varepsilon^2\big(\tilde{U}_{1,i} \star D_{1,i}(h) + \tilde{U}_{2,i} \star D_{2,i}(h) + \sum_{j\neq i}\tilde{U}_{2,ij} \star D_{2,ij}(h) + \frac{K^{(1)}}{2}C(h)\big) + \varepsilon^3 B_3\Big] \\ &:= (\tilde{U} \star h)(z,r) + \sum_{n=1}^{3}\varepsilon^n B_n(h) + \tilde{R}_h, \quad \text{(D.4)}\end{aligned}$$

where $\tilde{R}_h$ is of order ε^4 and

$$D_{1,i}(h) = \frac{1}{2}\sum_{j\ell} A^{(0)} jk A^{(0)}_{\ell i} \bar{\nabla}^2_{j\ell} h - \frac{\partial h}{\partial z} 2(k_i^{(0)})^2 y_3'; \quad D_{2,i}(h) = \frac{3-6y_3'}{4}(k_i^{(0)})^2 \frac{\partial^2 h}{\partial z^2};$$

$$C(h) = \int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)h(z',r), \quad D_{2,ij}(h) = \frac{1}{4}k_i^{(0)} k_j^{(0)} \frac{\partial^2 h}{\partial z^2}.$$

We do not write explicitly the long and uninteresting formula for B_3. Hence we have

$$\begin{aligned}
\tilde{g}_i^{(0)} &= \tilde{U} \star \tilde{\rho}_j^{(0)}, \\
\tilde{g}_i^{(1)} &= \tilde{U} \star \tilde{\rho}_j^{(1)} + B_1(\tilde{\rho}_i^{(0)})(z,r) \\
\tilde{g}_i^{(2)} &= \tilde{U} \star \tilde{\rho}_j^{(2)} + B_1(\tilde{\rho}_i^{(1)}) + B_2(\tilde{\rho}_i^{(0)}) \\
\tilde{g}_i^{(3)} &= \tilde{U} \star \tilde{\rho}_j^{(3)} + B_1(\tilde{\rho}_i^{(2)}) + B_2(\tilde{\rho}_i^{(1)}) + B_3(\tilde{\rho}_i^{(0)}).
\end{aligned} \tag{D.5}$$

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