Acta Mech
https://doi.org/10.1007/s00707-024-03961-8

ORIGINAL PAPER

Carmine Di Nucci · Simone Michele · Marcello Di Risio

Decomposition of the mechanical stress tensor: from the compressible Navier–Stokes equation to a turbulent potential flow model

Received: 2 November 2023 / Revised: 26 February 2024 / Accepted: 24 April 2024

Abstract We frame the mechanical stress tensor decomposition in a general procedure which involves the Helmholtz–Hodge decomposition. We highlight the impact of the mechanical stress tensor decomposition on the Navier–Stokes equation, with emphasis on the dissipation function. For fluids with low compressibility, we draw some insights on the Reynolds Averaged Navier–Stokes equations, and on the Reynolds stress tensor decomposition. We derive a turbulent potential flow model, and investigate the transition from viscous potential flow to turbulent potential flow. Under low Mach number approximation, we apply the turbulent potential flow model to one-dimensional propagation of large amplitude pressure waves in liquid-filled pipe.

1 Introduction

Within the framework of the Reynolds Averaged Navier–Stokes equations (RANS equations), we explore the possibility of modeling the turbulent flow in terms of potential flow. Turbulent flow is unsteady, irregular, random and chaotic; consequently, this model should be considered to analyze turbulent flow when vorticity effects are negligible. To build the potential flow model and to identify a possible field of application, our analysis starts from the forces acting on a continuum system. In classical continuum mechanics, the forces are classified as volume and surface forces [1]: the firsts are external forces acting at a distance; the seconds are contact forces. According to the action and reaction principle, only the external forces are responsible for the rate of change of momentum [2]. Assuming that the surface forces are defined by the symmetric Cauchy stress tensor, we split the Cauchy stress tensor into the thermodynamic tensor, and the mechanical stress tensor [3]. Consistent with the decomposition quoted in [4–7], we identify the part of the mechanical stress tensor that does not alter the momentum and the kinetic energy. We denote this tensor as mechanical internal stress tensor; the counterparty, which contribute to the rate of change of momentum, is the mechanical external stress tensor. We frame this decomposition in a general decomposition procedure for a second-order tensor which involves the Helmholtz–Hodge decomposition. As an application of this general procedure, we highlight two insights. On the one hand, we formally express the momentum equation in the form of Helmholtz–Hodge decomposition;

Simone Michele and Marcello Di Risio have contributed equally to this work.

C. Di Nucci (✉) · M. Di. Risio
Environmental and Maritime Hydraulic Laboratory (LIam), Civil, Construction-Architectural and Environmental Engineering Department (DICEAA), University of L'Aquila, Piazzale Ernesto Pontieri 1, Monteluco di Roio, L'Aquila 67100, Italy
E-mail: carmine.dinucci@univaq.it

M. Di. Risio
E-mail: marcello.dirisio@univaq.it

S. Michele
School of Engineering Computing and Mathematics, University of Plymouth, Drake Circus, Plymouth PL4 8AA, UK
E-mail: simone.michele@plymouth.ac.uk

Published online: 25 May 2024

on the other hand, we show the thermomechanical role played by mechanical stress tensors. For Navier–Stokes fluids, the findings lead to the analysis of the Stokes hypothesis [8–11], and the viscous potential flow approach [12]. Subsequently, we focus on the RANS equation for fluids with low compressibility. We transpose the decomposition of the Cauchy stress tensor to the Reynolds stress tensor. As a consequence, we split the Reynolds stress tensor into thermodynamic and mechanical stress tensor, and we apply the general procedure to decompose the mechanical Reynolds stress tensor into internal and external parts. The obtained insights allow us to formulate few considerations about the constitutive equation for the Reynolds stress tensor, and to propose a turbulent potential flow model. All these issues are discussed in Sect. 2. An in-depth analysis on the RANS equations for fluids with low compressibility is given in Appendix 1. In Sect. 3, through dimensional analysis, we identify the dimensionless number which governs the transition from viscous potential flow to turbulent potential flow. In Sect. 4 we show the application of the turbulent potential flow model to one-dimensional propagation of large amplitude pressure waves in liquid-filled pipe. Details about the proposed analytical solution to simulate the pressure waves are summarized in Appendix 2. In Sect. 5 we outline the main features of the paper.

2 Decomposition of the mechanical stress tensor

We assume that the generic field function (scalar b, vector $\boldsymbol{b}$, or second order tensor $\underline{\boldsymbol{B}}$) is a spatial field which depends on the Eulerian coordinates $\boldsymbol{r}$ and t, where $\boldsymbol{r}$ is space coordinate, and t is time coordinate.

2.1 Decomposition of a second-order tensor

We decompose a second order tensor $\underline{\boldsymbol{B}}$ in the form

$$\underline{\boldsymbol{B}} = \underline{\boldsymbol{B}}_1 + \underline{\boldsymbol{B}}_2, \tag{1}$$

$$\nabla \cdot \underline{\boldsymbol{B}}_2 = 0. \tag{2}$$

To do this, we can use a general procedure that starts from the Helmholtz–Hodge decomposition of the divergence of the tensor $\underline{\boldsymbol{B}}$, i.e., on the existence of a scalar field φ and a vector field $\boldsymbol{\psi}$ such that

$$\nabla \cdot \underline{\boldsymbol{B}} = \nabla\varphi - \nabla \times \boldsymbol{\psi}, \tag{3}$$

with $\nabla \cdot \boldsymbol{\psi} = 0$ without loss of generality.

We introduce an antisymmetric stress tensor $\underline{\boldsymbol{\Psi}}$ defined as

$$\nabla \cdot \underline{\boldsymbol{\Psi}} = -\nabla \times \boldsymbol{\psi}, \tag{4}$$

and obtain

$$\begin{aligned} \nabla \cdot \underline{\boldsymbol{B}} &= \nabla\varphi + \nabla \cdot \underline{\boldsymbol{\Psi}} \\ &= \nabla \cdot \left(\varphi \underline{\mathbf{1}} + \underline{\boldsymbol{\Psi}}\right), \end{aligned} \tag{5}$$

In agreement with Eq. (5), we get

$$\underline{\boldsymbol{B}}_1 = \varphi \underline{\mathbf{1}} + \underline{\boldsymbol{\Psi}}, \tag{6}$$

$$\underline{\boldsymbol{B}}_2 = \underline{\boldsymbol{B}} - \left(\varphi \underline{\mathbf{1}} + \underline{\boldsymbol{\Psi}}\right), \tag{7}$$

where $\underline{\mathbf{1}}$ is the unit tensor.

When a direct manipulation is not possible, the functions φ and $\boldsymbol{\psi}$ can be found by taking the divergence and the curl of Eq. (3), so that

$$\nabla \cdot \left(\nabla \cdot \underline{\boldsymbol{B}}\right) = \nabla^2\varphi, \tag{8}$$

$$\nabla \times \left(\nabla \cdot \underline{\boldsymbol{B}}\right) = -\nabla^2\boldsymbol{\psi}. \tag{9}$$

Even when Eqs. (8)–(9) cannot be solved analytically, the properties of the solution of the Poisson equations describe the qualitative properties of φ and $\boldsymbol{\psi}$.

2.2 Cauchy equation

We consider the momentum equation

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} - \rho \boldsymbol{f} = -\nabla \cdot \left(\underline{\boldsymbol{T}} + \underline{\boldsymbol{Q}}\right), \tag{10}$$

where ρ is the density; $\boldsymbol{v}$ the velocity; $\boldsymbol{f}$ the mass force per unit mass, $\boldsymbol{f} = \boldsymbol{g} = -g\hat{\boldsymbol{e}}_z = -\nabla(gz)$, with $\boldsymbol{g}$ the gravitational acceleration, $g = |\boldsymbol{g}|$, $\hat{\boldsymbol{e}}_z$ the unit vector in vertical direction, z the elevation in gravitational field; $\underline{\boldsymbol{T}}$ the Cauchy stress tensor; $\underline{\boldsymbol{Q}} = \rho\boldsymbol{v} \otimes \boldsymbol{v}$ the momentum flux density tensor.

Equation (10) is obtained by coupling the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{11}$$

with the Cauchy equation

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \boldsymbol{v} \cdot \boldsymbol{v} = \rho \boldsymbol{f} - \nabla \cdot \underline{\boldsymbol{T}}. \tag{12}$$

We split the Cauchy stress tensor $\underline{\boldsymbol{T}}$ into the thermodynamic tensor $\underline{\boldsymbol{P}}$, and the mechanical stress tensor $\underline{\boldsymbol{M}}$ [3]

$$\underline{\boldsymbol{T}} = \underline{\boldsymbol{P}} + \underline{\boldsymbol{M}}, \tag{13}$$

where $\underline{\boldsymbol{P}} = p\underline{\mathbf{1}}$, with p the thermodynamic pressure [13]. Accordingly, Eq. (10) becomes

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} - \rho \boldsymbol{f} = -\nabla p - \nabla \cdot \left(\underline{\boldsymbol{M}} + \underline{\boldsymbol{Q}}\right). \tag{14}$$

By using the general procedure, we can formally write the momentum Eq. (14) in the form of a Helmholtz–Hodge decomposition

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} - \rho \boldsymbol{f} = -\nabla p - \nabla\left(\varphi_{\underline{\boldsymbol{M}}} + \varphi_{\underline{\boldsymbol{Q}}}\right) + \nabla \times \left(\boldsymbol{\psi}_{\underline{\boldsymbol{M}}} + \boldsymbol{\psi}_{\underline{\boldsymbol{Q}}}\right), \tag{15}$$

where

$$\nabla \cdot \underline{\boldsymbol{M}} = \nabla \varphi_{\underline{\boldsymbol{M}}} - \nabla \times \boldsymbol{\psi}_{\underline{\boldsymbol{M}}}, \tag{16}$$

$$\nabla \cdot \underline{\boldsymbol{Q}} = \nabla \varphi_{\underline{\boldsymbol{Q}}} - \nabla \times \boldsymbol{\psi}_{\underline{\boldsymbol{Q}}}. \tag{17}$$

On the other hand, in line with the decomposition

$$\underline{\boldsymbol{M}} = \underline{\boldsymbol{M}}_1 + \underline{\boldsymbol{M}}_2, \tag{18}$$

where

$$\nabla \cdot \underline{\boldsymbol{M}}_2 = \mathbf{0}, \tag{19}$$

we express the Cauchy equation as

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho \boldsymbol{f} - \nabla p - \nabla \cdot \underline{\boldsymbol{M}}_1, \tag{20}$$

In agreement with Eq. (20), we refer to $\underline{\boldsymbol{M}}_1$ as the mechanical external stress tensor, $\underline{\boldsymbol{M}}_2$ as the mechanical internal stress tensor; we identify $-\nabla \cdot \underline{\boldsymbol{M}}_1$ as the mechanical external surface force (per unit volume), $-\nabla \cdot \underline{\boldsymbol{M}}_2$ as mechanical internal surface force. With this meaning, Eq. (20) agrees with the action and reaction principle, according to which only the external forces are responsible for the rate of change of momentum [2]. This line of reasoning highlights the difference between surface forces and stresses: though $\underline{\boldsymbol{M}}_2 \neq \underline{\mathbf{0}}$, where $\underline{\mathbf{0}}$ is the zero tensor, we obtain $-\nabla \cdot \underline{\boldsymbol{M}}_2 = \mathbf{0}$ (i.e., though the mechanical internal stresses are different from zero, the mechanical internal surface forces vanish identically, where the mechanical internal stresses are the scalar components of $\underline{\boldsymbol{M}}_2$).

In accordance with Eq. (20), the kinetic energy equation reads as

$$\begin{aligned}\frac{\partial(\rho e_k)}{\partial t} + \nabla\cdot(\rho \boldsymbol{v} e_k) &= \rho \boldsymbol{f}\cdot\boldsymbol{v} - \nabla p\cdot\boldsymbol{v} - \left(\nabla\cdot\underline{\boldsymbol{M}}_1\right)\cdot\boldsymbol{v} \\ &= \rho \boldsymbol{f}\cdot\boldsymbol{v} - \nabla p\cdot\boldsymbol{v} - \nabla\cdot\left(\underline{\boldsymbol{M}}_1\cdot\boldsymbol{v}\right) - \phi,\end{aligned} \tag{21}$$

where $e_k = \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} = \frac{v^2}{2}$ is the specific kinetic energy, with $v = |\boldsymbol{v}|$; $\phi = -\underline{\boldsymbol{M}}_1 : \nabla\boldsymbol{v}$ denoting the dissipation function, i.e., the rate of dissipation of kinetic energy per unit volume. Equation (21) assures that only the external forces expend power on kinetic energy (the internal forces perform no work). Therefore, mechanical effects of the internal stresses vanish.

To show the thermodynamic role played by the mechanical stress tensors, we express the first principle of thermodynamics as

$$\frac{\partial}{\partial t}(\rho(u_i + e_k)) + \nabla\cdot(\rho\boldsymbol{v}(u_i + e_k)) = \rho\boldsymbol{f}\cdot\boldsymbol{v} - \nabla\cdot(p\cdot\boldsymbol{v}) - \nabla\cdot\left(\underline{\boldsymbol{M}}\cdot\boldsymbol{v}\right) - \nabla\cdot\boldsymbol{q}, \tag{22}$$

where u_i is the specific internal energy; $\boldsymbol{q}$ the heat flux vector. Accordingly, the internal energy equation reads as

$$\begin{aligned}\frac{\partial(\rho u_i)}{\partial t} + \nabla\cdot(\rho u_i \boldsymbol{v}) &= -p\nabla\cdot\boldsymbol{v} - \underline{\boldsymbol{M}} : \nabla\boldsymbol{v} - \nabla\cdot\boldsymbol{q} \\ &= -p\nabla\cdot\boldsymbol{v} + \phi - \underline{\boldsymbol{M}}_2 : \nabla\boldsymbol{v} - \nabla\cdot\boldsymbol{q},\end{aligned} \tag{23}$$

whilst, the entropy equation is given as

$$\begin{aligned}\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\boldsymbol{v}) &= \frac{\phi}{T} - \frac{1}{T}\underline{\boldsymbol{M}}_2 : \nabla\boldsymbol{v} - \frac{1}{T}\nabla\cdot\boldsymbol{q} \\ &= \frac{\phi}{T} - \frac{1}{T}\underline{\boldsymbol{M}}_2 : \nabla\boldsymbol{v} - \frac{\boldsymbol{q}}{T^2}\cdot\nabla T - \nabla\cdot\frac{\boldsymbol{q}}{T} \\ &= s_p + s_f,\end{aligned} \tag{24}$$

where s is the specific entropy; T the absolute temperature. In Eq. (24), s_p indicates the entropy production (per unit volume)

$$s_p = \frac{\phi}{T} - \frac{1}{T}\underline{\boldsymbol{M}}_2 : \nabla\boldsymbol{v} - \frac{\boldsymbol{q}}{T^2}\cdot\nabla T, \tag{25}$$

s_f a non-dissipative term which contributes to entropy flux

$$s_f = -\nabla\cdot\frac{\boldsymbol{q}}{T}. \tag{26}$$

Summarizing, the effect of the mechanical internal stress tensor is manifested in the stress state (the state of stress is defined by the Cauchy stress tensor $\underline{\boldsymbol{T}}$); it is not manifested in the rate of change of momentum, and of kinetic energy. As thermodynamic effect, the mechanical internal stress tensor contributes to the entropy production.

Within the framework of classical irreversible thermodynamics [14,15], the heat flux vector is given by the Fourier law

$$\boldsymbol{q} = -k\nabla T, \tag{27}$$

where k is thermal conductivity. Assuming $T = constant$, Eqs. (24)–(26) reduce to

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\boldsymbol{v}) = \frac{\phi}{T} - \nabla\cdot\left(\frac{1}{T}\underline{\boldsymbol{M}}_2\cdot\boldsymbol{v}\right), \tag{28}$$

$$s_p = \frac{\phi}{T}, \tag{29}$$

$$s_f = -\nabla\cdot\left(\frac{1}{T}\underline{\boldsymbol{M}}_2\cdot\boldsymbol{v}\right), \tag{30}$$

being $\underline{\boldsymbol{M}}_2 : \nabla\boldsymbol{v} = \nabla\cdot\left(\underline{\boldsymbol{M}}_2\cdot\boldsymbol{v}\right)$. Under this condition, $\underline{\boldsymbol{M}}_2$ does not contribute to entropy production.

2.3 Navier–Stokes equation

For linear viscous fluids, the mechanical stress tensor $\underline{\boldsymbol{M}}$ is expressed as

$$\underline{\boldsymbol{M}} = -2\mu_1 \underline{\boldsymbol{D}} - \mu_2 (\nabla \cdot \boldsymbol{v}) \underline{\mathbf{1}}, \tag{31}$$

where $\underline{\boldsymbol{D}} = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}\right)$ is the strain rate tensor; μ_1 is the first viscous coefficient; μ_2 the second viscous coefficient.

Within the framework of Navier–Stokes fluids (i.e., assuming $\mu_1 = constant$, and $\mu_2 = constant$), the momentum equation reads as [4]

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho \boldsymbol{f} - \nabla p + (2\mu_1 + \mu_2) \nabla (\nabla \cdot \boldsymbol{v}) - \mu_1 \nabla \times \boldsymbol{\omega}, \tag{32}$$

whilst, a direct management of the mechanical stress tensor $\underline{\boldsymbol{M}}$, that does not invoke the general procedure, provides [4–7]

$$\underline{\boldsymbol{M}}_1 = -(2\mu_1 + \mu_2)(\nabla \cdot \boldsymbol{v}) \underline{\mathbf{1}} - 2\mu_1 \underline{\boldsymbol{\Omega}}, \tag{33}$$

$$\underline{\boldsymbol{M}}_2 = 2\mu_1 (\nabla \cdot \boldsymbol{v}) \underline{\mathbf{1}} - 2\mu_1 \nabla \boldsymbol{v}^{\mathrm{T}}, \tag{34}$$

where $\underline{\boldsymbol{\Omega}} = \frac{1}{2}\left(\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{T}\right)$ is the vorticity tensor, with $\nabla \times \boldsymbol{\omega} = -2\nabla \cdot \underline{\boldsymbol{\Omega}}$, and $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ the vorticity. We stress that the differential identity $\nabla \cdot \left(\nabla \boldsymbol{v}^{\mathrm{T}}\right) = \nabla (\nabla \cdot \boldsymbol{v})$ assures that

$$\nabla \cdot \underline{\boldsymbol{M}}_2 = 2\mu_1 \nabla ((\nabla \cdot \boldsymbol{v})) - 2\mu_1 \nabla \cdot \left(\nabla \boldsymbol{v}^{T}\right) = \mathbf{0}. \tag{35}$$

In the Helmholtz–Hodge decomposition, the Navier–Stokes equation (32) formally reads as

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} - \rho \boldsymbol{f} = -\nabla p + \nabla \left((2\mu_1 + \mu_2)(\nabla \cdot \boldsymbol{v}) - \varphi_{\underline{\boldsymbol{Q}}}\right) - \nabla \times \left(\mu_1 \boldsymbol{\omega} - \boldsymbol{\psi}_{\underline{\boldsymbol{Q}}}\right). \tag{36}$$

In agreement with Eq. (33), the dissipation function is given as

$$\phi = (2\mu_1 + \mu_2)(\nabla \cdot \boldsymbol{v})^2 + 2\mu_1 \omega^2. \tag{37}$$

where ω^2 is the enstrophy, with $\omega = |\boldsymbol{\omega}|$ According to Eq. (37), for a fluid making a solid-like rotation, the dissipation function becomes

$$\phi = 2\mu_1 \omega^2. \tag{38}$$

To overcome this (apparent) paradox (for the problem under consideration, the dissipation function must be null), we observe that in the special case in which the fluid makes a solid-like rotation, we obtain $\underline{\boldsymbol{T}} = \underline{\boldsymbol{P}} = p\underline{\mathbf{1}}$, $\underline{\boldsymbol{M}} = \mathbf{0}$, $\nabla \cdot \underline{\boldsymbol{M}}_1 = \mathbf{0}$. Thus, according to Eq. (21), $\phi = 0$, and $\underline{\boldsymbol{M}}_1$ does not contribute to entropy production.

In agreement with Eq. (37), the term $(2\mu_1 + \mu_2)(\nabla \cdot \boldsymbol{v})^2$ relates to kinetic energy dissipation during isotropic dilatations. According to Stokes hypothesis [8–10], setting $\frac{2}{3}\mu_1 + \mu_2 = 0$ we obtain $(2\mu_1 + \mu_2) = \frac{4}{3}\mu_1$, and the dissipation function is still formally expressed by Eq. (37). This result disagrees with the idea according to which, under Stokes hypothesis, isotropic dilatations are reversible processes. The theoretical refutation of the Stokes hypothesis is given in [11].

On the other hand, if we consider potential flows, which arise from the kinematic assumption $\boldsymbol{\omega} = \mathbf{0}$, the dissipation function (37) reduces to

$$\phi = (2\mu_1 + \mu_2)(\nabla \cdot \boldsymbol{v})^2. \tag{39}$$

As a consequence of Eq. (39), the statement according to which the potential flow implies inviscid flows is incorrect. The readers are referred to [12], and the references therein, for a deeper insight into the viscous potential flow.

2.4 RANS equation

We consider the decompositions of the Reynolds stress tensor $\underline{\boldsymbol{T}}_{\mathrm{Re}}$. We recall that the Reynolds stress tensor arises in Reynolds Averaged Navier–Stokes (RANS) turbulence models [16]. We focus on the fluids with low compressibility for which the RANS equation can be written as (see Appendix 1)

$$\frac{\partial}{\partial t}\left(\rho\left[\boldsymbol{v}\right]\right)+\nabla\cdot\left(\rho\left[\boldsymbol{v}\right]\otimes\left[\boldsymbol{v}\right]\right)=\rho\boldsymbol{f}-\nabla\left[p\right]+\left(2\mu_1+\mu_2\right)\nabla\left(\nabla\cdot\left[\boldsymbol{v}\right]\right)-\mu_1\nabla\times\left[\boldsymbol{\omega}\right]-\nabla\cdot\underline{\boldsymbol{T}}_{\mathrm{Re}}, \quad (40)$$

where $[\ ]$ is the ensemble-mean operator; $\underline{\mathbf{T}}_{\mathrm{Re}}=\rho\left[\boldsymbol{v}'\otimes\boldsymbol{v}'\right]$, with $\boldsymbol{v}'$ denoting the turbulent fluctuation of $\boldsymbol{v}$. In agreement with the previous findings, we invoke the decompositions

$$\underline{\mathbf{T}}_{\mathrm{Re}}=\underline{\mathbf{P}}_{tur}+\underline{\mathbf{M}}_{tur}, \quad (41)$$

$$\underline{\boldsymbol{M}}_{tur}=\underline{\boldsymbol{M}}_{tur,1}+\underline{\boldsymbol{M}}_{tur,2}, \quad (42)$$

$$\nabla\cdot\underline{\boldsymbol{M}}_{tur,2}=\mathbf{0}, \quad (43)$$

where $\underline{\boldsymbol{P}}_{tur}$ is the homologous of the isotropic thermodynamic pressure tensor $\underline{\boldsymbol{P}}$; $\underline{\boldsymbol{M}}_{tur}$ the homologous of the mechanical stress tensor $\underline{\boldsymbol{M}}$; $\underline{\boldsymbol{M}}_{tur,1}$ and $\underline{\boldsymbol{M}}_{tur,2}$ the homologues of $\underline{\boldsymbol{M}}_1$ and $\underline{\boldsymbol{M}}_2$. According to the general procedure, we can introduce the field functions φ_{tur}, $\boldsymbol{\psi}_{tur}$, and $\underline{\boldsymbol{\Psi}}_{tur}$ such that

$$\underline{\boldsymbol{M}}_{tur,1}=\varphi_{tur}\underline{\mathbf{1}}+\underline{\boldsymbol{\Psi}}_{tur}, \quad (44)$$

$$\underline{\boldsymbol{M}}_{tur,2}=\underline{\boldsymbol{M}}_{tur}-\left(\varphi_{tur}\underline{\mathbf{1}}+\underline{\boldsymbol{\Psi}}_{tur}\right), \quad (45)$$

where the antisymmetric stress tensor $\underline{\boldsymbol{\Psi}}_{tur}$ is defined as

$$\nabla\cdot\underline{\boldsymbol{\Psi}}_{tur}=-\nabla\times\boldsymbol{\psi}_{tur}. \quad (46)$$

We stress that a turbulence model based on the Helmholtz–Hodge decomposition of the Reynolds stress tensor can be found in [17].

With this setting, the RANS momentum Eq. (40) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho\left[\boldsymbol{v}\right]\right)+\nabla\cdot\left(\rho\left[\boldsymbol{v}\right]\otimes\left[\boldsymbol{v}\right]\right)&=\rho\boldsymbol{f}-\nabla\left[p\right]+\left(2\mu_1+\mu_2\right)\nabla\left(\nabla\cdot\left[\boldsymbol{v}\right]\right)-\mu_1\nabla\times\left[\boldsymbol{\omega}\right]\\&\quad-\nabla\cdot\underline{\boldsymbol{P}}_{tur}+\nabla\times\boldsymbol{\psi}_{tur}-\nabla\varphi_{tur}.\end{aligned} \quad (47)$$

We underline that

- the term $\nabla\cdot\underline{\boldsymbol{P}}_{tur}$ takes into account the effects of turbulent fluctuations on the momentum flux (see Appendix 1);
- the tensor $\underline{\boldsymbol{M}}_{tur,1}$ is such that the additional friction forces (per unit volume) due to turbulent fluctuations are given as $-\nabla\cdot\underline{\boldsymbol{M}}_{tur,1}$.

Whitin the framework of Boussinesq formulation [16], we assume $\underline{\boldsymbol{T}}_{\mathrm{Re}}=\underline{\boldsymbol{T}}_{\mathrm{Re}}\left([\boldsymbol{v}]\right)$. Thus, we argue that

- the constitutive equation for $\underline{\boldsymbol{P}}_{tur}$ can be given as $\underline{\boldsymbol{P}}_{tur}=m_{tur}\left[v\right]^2\underline{\mathbf{1}}$, where m_{tur} is the coefficient of turbulent diffusivity of momentum;
- the implicit constitutive equation for φ_{tur} can be given as $\varphi_{tur}=\varphi_{tur}\left(\nabla\cdot[\boldsymbol{v}]\right)$;
- the implicit constitutive equation for $\underline{\boldsymbol{\Psi}}_{tur}$ can be expressed as $\underline{\boldsymbol{\Psi}}_{tur}=\underline{\boldsymbol{\Psi}}_{tur}\left([\boldsymbol{\omega}]\right)$.

Following the suggestion proposed in [18], we propose the ansatz

$$\varphi_{tur}=-\mu_{d,tur}\nabla\cdot\left[\boldsymbol{v}\right]. \quad (48)$$

We refer to $\mu_{d,tur}$ as the turbulent dilatational viscosity. We stress that turbulent dilatational viscosity shows a substantial similarity to the turbulent bulk viscosity [18]. On the other hand, we can introduce the tensor $\underline{\boldsymbol{C}}_{tur}$ such that

$$\boldsymbol{\psi}_{tur}=-\nabla\cdot\underline{\boldsymbol{C}}_{tur}, \quad (49)$$

where

$$\underline{\boldsymbol{C}}_{tur}=-\mu_{r,tur}\nabla\left[\boldsymbol{\omega}\right]. \quad (50)$$

We refer to $\mu_{r,tur}$ as the turbulent rotational viscosity. Accordingly, the RANS Eq. (47) can be expressed as

$$\frac{\partial}{\partial t}(\rho[\boldsymbol{v}]) + \nabla\cdot(\rho[\boldsymbol{v}]\otimes[\boldsymbol{v}]) = \rho\boldsymbol{f} - \nabla[p] + (2\mu_1+\mu_2)\nabla(\nabla\cdot[\boldsymbol{v}]) - \mu_1\nabla\times[\boldsymbol{\omega}] - \nabla\left(m_{tur}[v]^2\right) + \nabla\left(\mu_{d,tur}\nabla\cdot[\boldsymbol{v}]\right) + \nabla\times\left(\nabla\cdot\left(\mu_{r,tur}\nabla[\boldsymbol{\omega}]\right)\right). \tag{51}$$

In Helmholtz–Hodge decomposition, Eq. (51) reads as

$$\frac{\partial}{\partial t}(\rho[\boldsymbol{v}]) - \rho\boldsymbol{f} = -\nabla\left([p] + \left(m_{tur}[v]^2\right) + \varphi_{\left[\underline{\boldsymbol{Q}}\right]}\right) + \nabla\left(\left(2\mu_1+\mu_2+\mu_{d,tur}\right)\nabla\cdot[\boldsymbol{v}]\right) - \nabla\times\left(\mu_1[\omega] - \nabla\cdot\left(\mu_{r,tur}\nabla[\boldsymbol{\omega}]\right) - \boldsymbol{\psi}_{\left[\underline{\boldsymbol{Q}}\right]}\right), \tag{52}$$

where

$$\nabla\cdot(\rho[\boldsymbol{v}]\otimes[\boldsymbol{v}]) = \nabla\cdot\left[\underline{\boldsymbol{Q}}\right] = \nabla\varphi_{\left[\underline{\boldsymbol{Q}}\right]} - \nabla\times\boldsymbol{\psi}_{\left[\underline{\boldsymbol{Q}}\right]}. \tag{53}$$

In placing Eqs. (49)–(50), we are guided by the theory of second order fluids as developed in [19]. The connection between the second order fluids and turbulence is first noted in [20] (the readers are referred to [19,21] for more details). Assuming in first approximation $m_{tur} = constant$, $\mu_{d,tur} = constant$, $\mu_{r,tur} = constant$, and using the differential identity

$$\nabla\times\left(\nabla\cdot\left(\mu_{r,tur}\nabla[\boldsymbol{\omega}]\right)\right) = \mu_{r,tur}\nabla^2(\nabla\times[\boldsymbol{\omega}]), \tag{54}$$

Equation (51) becomes

$$\frac{\partial}{\partial t}(\rho[\boldsymbol{v}]) + \nabla\cdot(\rho[\boldsymbol{v}]\otimes[\boldsymbol{v}]) = \rho\boldsymbol{f} - \nabla[p] - \mu_1\nabla\times[\boldsymbol{\omega}] - m_{tur}\nabla[v]^2 + \left(2\mu_1+\mu_2+\mu_{d,tur}\right)\nabla(\nabla\cdot[\boldsymbol{v}]) + \mu_{r,tur}\nabla^2(\nabla\times[\boldsymbol{\omega}]). \tag{55}$$

For incompressible fluids, Eq. (55) reduces to

$$\frac{\partial}{\partial t}(\rho[\boldsymbol{v}]) + \nabla\cdot(\rho[\boldsymbol{v}]\otimes[\boldsymbol{v}]) = \rho\boldsymbol{f} - \nabla[p] - \mu_1\nabla\times[\boldsymbol{\omega}] - m_{tur}\nabla[v]^2 + \mu_{r,tur}\nabla^2(\nabla\times[\boldsymbol{\omega}]). \tag{56}$$

The numerical treatment of Eq. (56), with appropriate boundary conditions, can be found in [22].

On the other hand, if we consider turbulent potential flows, putting $\boldsymbol{\omega} = [\boldsymbol{\omega}] = \boldsymbol{0}$, Eq. (55) reduces to

$$\rho\frac{\partial[\boldsymbol{v}]}{\partial t} + \rho\nabla\frac{[v]^2}{2} = \rho\boldsymbol{f} - \nabla[p] - m_{tur}\nabla[v]^2 + \left(2\mu_1+\mu_2+\mu_{d,tur}\right)\nabla(\nabla\cdot[\boldsymbol{v}]), \tag{57}$$

being

$$\frac{\partial}{\partial t}(\rho[\boldsymbol{v}]) + \nabla\cdot(\rho[\boldsymbol{v}]\otimes[\boldsymbol{v}]) = \rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho\nabla\boldsymbol{v}\cdot\boldsymbol{v} = \rho\frac{\partial[\boldsymbol{v}]}{\partial t} + \rho\nabla\frac{[v]^2}{2} - \rho[\boldsymbol{v}]\times[\boldsymbol{\omega}] = \frac{\partial[\boldsymbol{v}]}{\partial t} + \rho\nabla\frac{[v]^2}{2}. \tag{58}$$

We stress that the mean dissipation function linked to Eq. (57) is given as

$$[\phi] = \left(2\mu_1+\mu_2+\mu_{d,tur}\right)(\nabla\cdot[\boldsymbol{v}])^2. \tag{59}$$

3 Brief remarks on the transition from viscous potential flow to turbulent potential flow

We recall that, under potential flow conditions, the Navier–Stokes equation reads as

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \frac{v^2}{2} = -\rho g \hat{\boldsymbol{e}}_z - \nabla p + (2\mu_1 + \mu_2) \nabla (\nabla \cdot \boldsymbol{v}) . \tag{60}$$

Equation (60), can be rewritten as

$$\nabla \left(\frac{\partial \vartheta}{\partial t} + \frac{\nabla \vartheta \cdot \nabla \vartheta}{2} + gz \right) = -\frac{1}{\rho} \nabla p + \frac{(2\mu_1 + \mu_2)}{\rho} \nabla (\nabla \cdot \boldsymbol{v}) , \tag{61}$$

which implies

$$\nabla \times \left(-\frac{1}{\rho} \nabla p + \frac{(2\mu_1 + \mu_2)}{\rho} \nabla (\nabla \cdot \boldsymbol{v}) \right) = \mathbf{0}, \tag{62}$$

where ϑ is the potential velocity function, $\boldsymbol{v} = \nabla \vartheta$. According to Eq. (62), there exist a function Λ such that [23]

$$-\frac{1}{\rho} \nabla p + \frac{(2\mu_1 + \mu_2)}{\rho} \nabla (\nabla \cdot \boldsymbol{v}) = \nabla \Lambda. \tag{63}$$

According to Eq. (63), the fluid evolves in barotropic condition [24]. In line with the previous settings, we assume the isothermal condition at $T = constant$, and we express the barotropic pressure-density relation as follows [25]

$$\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}p}{\varepsilon}, \tag{64}$$

where ε is the fluid bulk modulus. Consistent with Eq. (64), the speed of sound is given as

$$c = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}} = \sqrt{\frac{\varepsilon}{\rho}}. \tag{65}$$

To investigate the transition from viscous potential flow to turbulent potential flow we rewrite Eq. (60) in dimensionless form. For this scope, in addition to the velocity scale v_s and the length scale ℓ_s, we introduce a pressure scale p_s defined as [26,27]

$$p_s = \rho_s v_s c_s, \tag{66}$$

where ρ_s is the density reference value, c_s the refence value of the speed of sound. Using c_s, we define the time scale t_s as [27]

$$t_s = \frac{\ell_s}{c_s}. \tag{67}$$

Setting the dimensionless terms in curly bracket, Eq. (60) becomes (for all details the readers are referred to [27])

$$\frac{1}{\mathrm{Ma}} \left\{ \frac{\partial \boldsymbol{v}}{\partial t} \right\} + \left\{ \nabla \frac{v^2}{2} \right\} = -\frac{1}{\mathrm{Fr}^2} \hat{\boldsymbol{e}}_z - \frac{1}{\mathrm{Ma}} \left\{ \frac{1}{\rho} \nabla p \right\} + \frac{(2\mu_1 + \mu_2)\, c_s}{p_s \ell_s} \left\{ \frac{1}{\rho} \nabla (\nabla \cdot \boldsymbol{v}) \right\} , \tag{68}$$

where $\mathrm{Ma} = \frac{v_s}{c_s}$ is the Mach number; $\mathrm{Fr} = \sqrt{\frac{v_s^2}{g \ell_s}}$ the Froude number. We stress that the dimensionless number $\frac{p_s \ell_s}{(2\mu_1+\mu_2) c_s}$ is very similar to the Russo Spena number [27]; therefore, we define $\frac{p_s \ell_s}{(2\mu_1+\mu_2) c_s}$ as the Russo Spena number under potential flow conditions, $\mathrm{RS}_p = \frac{p_s \ell_s}{(2\mu_1+\mu_2) c_s}$. The RS_p number is related to the viscous forces which arise from the volume variation. According to Eq. (68), we conjecture that the transition occurs when the RS_p exceed a critical value $\mathrm{RS}_{p,c}$: as $\mathrm{RS}_p > \mathrm{RS}_{p,c}$, the stabilizing effect due to viscous forces become evanescent, and the viscous potential flow turns into the turbulent potential flow. In agreement with the barotropic Eq. (64), the transition can be occurred when a pressure wave travels through the fluid, and therefore, a change in density happens; as a consequence, the transition is caused by the energy transport, and no mass transport is needed.

4 Pressure waves of large amplitude

To test the turbulent potential flow model, we consider the hydraulic transient in liquid-filled pipe. This unsteady flow arises as a result of a local liquid velocity variations due to valve maneuvers [28–32]; the distinct characteristic is the propagation of large amplitude pressure waves at low frequency. For 1D flow in a horizontal pipe, Eqs. (57) and (59) become

$$\rho\frac{\partial U}{\partial t}+\rho U\frac{\partial U}{\partial x}+2m_{tur}U\frac{\partial U}{\partial x}=-\frac{\partial \Pi}{\partial x}+\left(2\mu_1+\mu_2+\mu_{d,tur}\right)\frac{\partial^2 U}{\partial x^2}, \tag{69}$$

$$\phi_{1D}=\left(2\mu_1+\mu_2+\mu_{d,tur}\right)\left(\frac{\partial U}{\partial x}\right)^2, \tag{70}$$

where $U(x,t)$ denotes the velocity; $\Pi(x,t)$ the thermodynamic pressure; $\phi_{1D}(x,t)$ the dissipation function; x the coordinate along the flow direction.

Due to very important kinetic energy dissipation in propagation of large amplitude pressure waves [33,34], we conjecture that $\mu_{d,tur}\gg 2\mu_1+\mu_2$ (for water, under low frequency condition, we can assume $2\mu_1+\mu_2\approx 10^{-3}\,\frac{kg}{ms}$ [35]). Accordingly, Eqs. (69)–(70) reduce to

$$\rho\frac{\partial U}{\partial t}+\rho U\frac{\partial U}{\partial x}+2m_{tur}U\frac{\partial U}{\partial x}=-\frac{\partial \Pi}{\partial x}+\mu_{d,tur}\frac{\partial^2 U}{\partial x^2}, \tag{71}$$

$$\phi_{1D}=\mu_{d,tur}\left(\frac{\partial U}{\partial x}\right)^2. \tag{72}$$

In pipe of constant cross-section area, the continuity equation reads as (details about the continuity equation are given in Appendix 1)

$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial U}{\partial x}+U\frac{\partial \rho}{\partial x}=0. \tag{73}$$

The barotropic pressure-density relation (64) becomes

$$\frac{\mathrm{d}\rho}{\rho}=\frac{\mathrm{d}\Pi}{\tilde{\varepsilon}}, \tag{74}$$

where $\tilde{\varepsilon}$ is the bulk modulus of the pipe-liquid system [18,36–38]. According to Eq. (74), we express the wave celerity c by the relationship

$$c=\sqrt{\frac{\mathrm{d}\Pi}{\mathrm{d}\rho}}=\sqrt{\frac{\tilde{\varepsilon}}{\rho}}. \tag{75}$$

Setting $\tilde{\varepsilon}=constant$, Eq. (74) leads to

$$\rho=\rho_0\mathrm{e}^{\frac{(\Pi-\Pi_0)}{\tilde{\varepsilon}}}, \tag{76}$$

where $\rho_0=\rho(\Pi_0)$ is the density reference value at the pressure reference value Π_0, whilst the continuity equation (73) becomes

$$\frac{\rho}{\tilde{\varepsilon}}\frac{\partial \Pi}{\partial t}+\frac{\rho}{\tilde{\varepsilon}}U\frac{\partial \Pi}{\partial x}+\rho\frac{\partial U}{\partial x}=0. \tag{77}$$

Without loss of generality, we assume Π_0 as the relative atmospheric pressure, $\Pi_0=0$.

Under low Mach number approximation (very common in water hammer phenomenon [39]), $c\gg U$ (i.e., $\tilde{\varepsilon}\gg 1$); as a consequence, the inertial term is much larger than the convective one, i.e. $\frac{\partial(\,)}{\partial t}\gg U\frac{\partial(\,)}{\partial x}$ [28], $\rho\approx\rho_0$, $c\approx c_0=\sqrt{\frac{\tilde{\varepsilon}}{\rho_0}}$, and the system of equations can be approximated as

$$\rho_0\frac{\partial U}{\partial t}+\frac{\partial \Pi}{\partial x}=\mu_{d,tur}\frac{\partial^2 U}{\partial x^2}, \tag{78}$$

$$\frac{1}{\rho_0}\frac{\partial \Pi}{\partial t}+c_0^2\frac{\partial U}{\partial x}=0, \tag{79}$$

in the unknown functions $U(x,t)$, $\Pi(x,t)$. According to Eq. (78), $\mu_{d,tur}$ plays a similar role to that of the artificial viscosity in shock capturing methods [40].

The system of Eqs. (78)–(79) becomes closed if auxiliary expressions for $\mu_{d,tur}$ and c_0 are specified. On the other hand, we can calibrate the parameters $\mu_{d,tur}$ and c_0 matching numerical results to experimental data. The experimental data used herein consist of thermodynamic pressure time series collected at the downstream section of a steel pipe (in $x = L$, where $L = 72\ m$, with $D = 42\ mm$), during the water hammer phenomenon ($\rho_0 = 1000\ \frac{kg}{m^3}$). The experiments are carried out using standard laboratory facilities (reservoir-pump-pressurized tank-pipe-valve): the pipe connects the upstream pressurized tank to a downstream maneuver valve; the piezometric signals are measured employing piezoresistive transducers, the steady-state discharge by electromagnetic flowmeters. Once the initial steady-state is established, the water hammer is generated by an instantaneous closure of the valve. The readers are referred to [41] for all details about experimental set-ups and procedures. The initial conditions refer to the normal formulas for circular pipe

$$\Pi(x,0) = \Pi(L,0) + \rho_0 f \frac{U_0^2}{2D}(L-x), \tag{80}$$

$$U(x,0) = U_0, \tag{81}$$

where $\Pi(L,0) = 5.04 \cdot 10^5\ Pa$; $U_0 = 0.405\ \frac{m}{s}$. The Darcy friction factor f is computed by the Blasius formula [42]

$$f = \frac{0.3164}{Re^{0.25}}, \tag{82}$$

with $Re = \frac{\rho_0 U_0 D}{\mu_1} = 17010$ being Reynolds number. The boundary conditions are defined as

$$U(L,t) = U_0 \mathcal{H}(-t), \tag{83}$$

$$\Pi(0,t) = constant, \tag{84}$$

where $\mathcal{H}(-t)$ is the Heaviside step function, whereas $\Pi(0,t) = \Pi(L,0) + \rho_0 f \frac{U_0^2}{2D} L = 5.08 \cdot 10^5\ Pa$. We stress that Eq. (83) simulates the valve instantaneous closure.

We deduce the linear damped equation

$$\frac{\mu_{d,tur}}{\rho_0}\frac{\partial^3 U}{\partial x^2 \partial t} + c_0^2 \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = 0, \tag{85}$$

by combining Eqs. (78)–(79). According to Eqs. (80)–(81) and (83)–(84), Eq. (85) is subject to the initial conditions

$$U(x,0) = U_0, \tag{86}$$

$$\frac{\partial U}{\partial t}(x,0) = 0, \tag{87}$$

and the boundary conditions

$$\frac{\partial U}{\partial x}(0,t) = 0, \tag{88}$$

$$U(L,t) = U_0 \mathcal{H}(-t). \tag{89}$$

As shown in Appendix 2, the analytical solution can be expressed in the following explicit form

$$U = U_0 \mathcal{H}(-t) + \sum_{n=0}^{\infty} \cos(\lambda_n x) \frac{4U_0 (-1)^n e^{-\frac{\kappa_n t}{2}}}{\pi(1+2n)\sqrt{\kappa_n^2 - 4\Theta_n^2}} \left(\sqrt{\kappa_n^2 - 4\Theta_n^2} \cosh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right) - \kappa_n \sinh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)\right), \tag{90}$$

where

$$\lambda_n = \frac{\pi}{L}\left(n + \frac{1}{2}\right), \tag{91}$$

$$\Theta_n = c_0 \lambda_n, \tag{92}$$
$$\kappa_n = \frac{\mu_{d,tur}}{\rho_0} \lambda_n^2. \tag{93}$$

At $x = L$, the pressure field reads as

$$\Pi(L,t) = \Pi(L,0) + \frac{4U_0 \rho_0 c_0^2}{L} \sum_{n=0}^{\infty} \frac{e^{-\frac{\kappa_n t}{2}} \sinh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)}{\sqrt{\kappa_n^2 - 4\Theta_n^2}}. \tag{94}$$

By using a standard test-and-try procedure, the comparison between experimental and analytical results allows to find the values $\mu_{d,tur} = 2.84 \cdot 10^6 \frac{kg}{ms}$, $c_0 = 1228.8 \frac{m}{s}$. Figure 1 shows the comparison in the range $0 < t \leq 3\ s$. Our results show good agreement against experimental results in terms of attenuation, and dispersion of wave fronts; the phase shift is also suitably described, i.e., the wave celerity is properly reproduced. Figure 2 shows the evolution of the averaged dissipation function $\Phi(t)$ defined as

$$\begin{aligned}\Phi(t) &= \frac{1}{L}\int_0^L \phi_{1D}(x,t)\,\mathrm{d}x = \frac{1}{L}\int_0^L \mu_{d,tur}\left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x \\ &= 2\frac{\mu_{d,tur} U_0^2}{\rho_0 L^2} \sum_{n=0}^{\infty}\left(\frac{e^{-\frac{\kappa_n t}{2}}}{\sqrt{\kappa_n^2 - 4\Theta_n^2}}\left(\sqrt{\kappa_n^2 - 4\Theta_n^2}\cosh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)\right.\right. \\ &\qquad \left.\left. -\kappa_n \sinh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)\right)\right)^2. \end{aligned} \tag{95}$$

The averaged dissipation function tends to infinity for $t \to 0$ because of the instantaneous closure at the valve. Figure 3 shows the monotonous increase of the function $\Sigma(t)$ defined as

$$\begin{aligned}\Sigma(t) &= \int_0^t \Phi(t)\,\mathrm{d}t = \int_0^t \left(\frac{1}{L}\int_0^L \frac{\mu_{d,tur}}{\rho_0}\left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right)\mathrm{d}t \\ &= \frac{\mu_{d,tur} U_0^2}{\rho_0 L^2}\sum_{n=0}^{\infty} \frac{e^{-\kappa_n t}}{\kappa_n\left(\kappa_n^2 - 4\Theta_n^2\right)}\left(4\Theta_n^2 + e^{-\frac{\kappa_n t}{2}}\left(\kappa_n^2 - 4\Theta_n^2\right)\right. \\ &\qquad \left. -\kappa_n^2 \cosh\left(t\sqrt{\kappa_n^2 - 4\Theta_n^2}\right) + \kappa_n\sqrt{\kappa_n^2 - 4\Theta_n^2}\sinh\left(t\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)\right). \end{aligned} \tag{96}$$

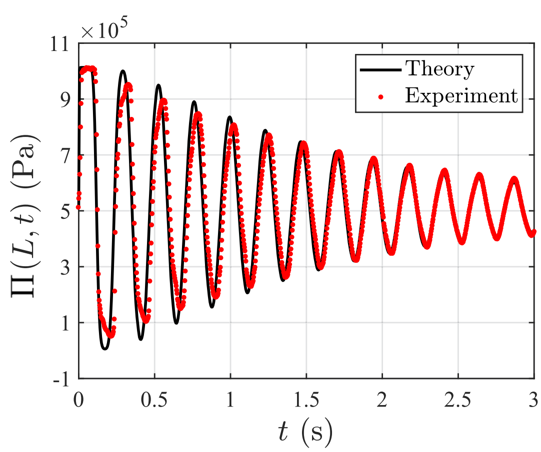

Fig. 1 Time-series of $\Pi(L,t)$; — analytical results; • experimental results

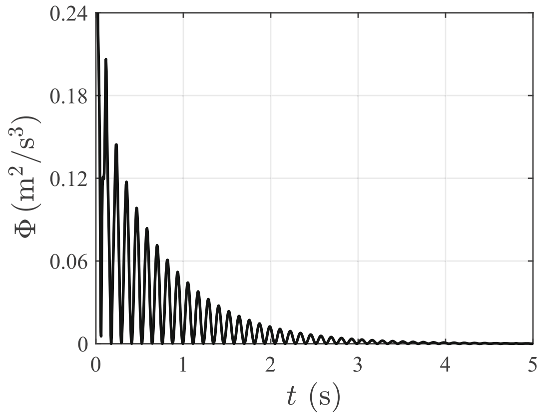

Fig. 2 Time evolution of Φ

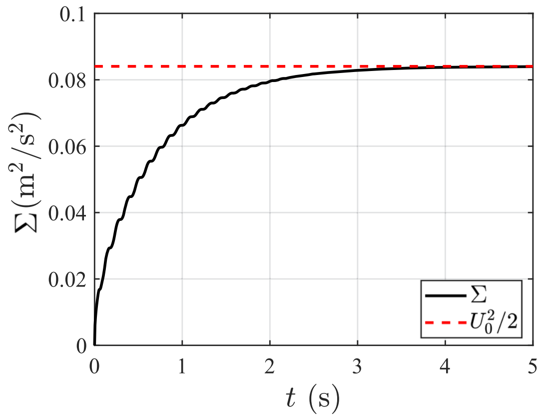

Fig. 3 Time evolution of Σ

According to the energy conservation law, we deduce $\Sigma\,(t) \to \rho_0 \frac{U_0^2}{2}$ for $t \to \infty$

$$\lim_{t\to\infty} \Sigma\,(t) = \frac{\mu_{d,tur} U_0^2}{\rho_0 L^2} \sum_{n=0}^{\infty} \frac{1}{\kappa_n} = \frac{\mu_{d,tur} U_0^2}{\rho_0 L^2} \sum_{n=0}^{\infty} \frac{4\rho_0 L^2}{\eta_{tur}\pi^2} \frac{1}{(2n+1)^2}$$
$$= \frac{4U_0^2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{U_0^2}{2}. \tag{97}$$

5 Conclusions

We have explored the decomposition of the mechanical stress tensor into internal and external stress tensors, and we have shown the thermomechanical roles played by them. We have framed the mechanical stress tensor decomposition in a general decomposition procedure; as a result, we have formally expressed the momentum equation (Cauchy equation, Navier–Stokes equation, RANS equation) in the Helmholtz–Hodge decomposition. For fluids with low compressibility, the findings have been used to develop a turbulent potential flow model. Through dimensional analysis, we have investigated the transition from viscous potential flow to turbulent potential flow. In low Mach number approximation, we have tested the turbulent potential flow model to one-dimensional propagation of large amplitude pressure waves in liquid-filled pipe. We have shown that the model is in agreement in a reasonable way with the experimental data available in literature.

Acknowledgements We are very grateful to Prof. R. Szymkiewicz (Gdansk University of Technology, Poland), and to Prof. M. Mitosek (Warsaw University of Technology, Poland) for providing the experimental data.

Funding Open access funding provided by Università degli Studi dell'Aquila within the CRUI-CARE Agreement.

Data availibility Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A RANS equations for fluids with low compressibility

Under isothermal conditions at $T = constant$, the balance equations for Navier–Stokes fluids read as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{A1}$$

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \boldsymbol{v} \cdot \boldsymbol{v} = \rho \boldsymbol{f} - \nabla p - \nabla \cdot \underline{\boldsymbol{M}}, \tag{A2}$$

$$\frac{\partial}{\partial t}(\rho e_m) + \nabla \cdot \left(\rho e_m \boldsymbol{v} + p \boldsymbol{v} + \underline{\boldsymbol{M}} \cdot \boldsymbol{v}\right) = p \nabla \cdot \boldsymbol{v} + \underline{\boldsymbol{M}} : \nabla \boldsymbol{v}, \tag{A3}$$

$$\frac{\partial}{\partial t}(\rho e_t) + \nabla \cdot \left(\rho e_t \boldsymbol{v} + p \boldsymbol{v} + \underline{\boldsymbol{M}} \cdot \boldsymbol{v}\right) = 0, \tag{A4}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla \cdot (\rho u_i \boldsymbol{v}) = -p \nabla \cdot \boldsymbol{v} - \underline{\boldsymbol{M}} : \nabla \boldsymbol{v}, \tag{A5}$$

$$\frac{\partial}{\partial t}(\rho T s) + \nabla \cdot (\rho T s \boldsymbol{v}) = -\underline{\boldsymbol{M}} : \nabla \boldsymbol{v}, \tag{A6}$$

$$\frac{\partial}{\partial t}\left(\rho e_p\right) + \nabla \cdot \left(\rho e_p \boldsymbol{v}\right) = -p \nabla \cdot \boldsymbol{v}, \tag{A7}$$

where $e_m = e_k + gz$ is the specific mechanical energy; $e_t = u_i + e_m$ the specific total energy; e_p the specific elastic potential energy. We stress that Eqs. (A5)–(A7) are in agreement with the Gibbs relationship

$$\mathrm{d}u_i = T \mathrm{d}s + \frac{p}{\rho^2} \mathrm{d}\rho = T \mathrm{d}s + \mathrm{d}e_p. \tag{A8}$$

Setting $\varepsilon = constant$ in Eq. (64), the relationship between the density and the thermodynamic pressure is given as

$$\rho = \rho_0 \mathrm{e}^{\frac{(p-p_0)}{\varepsilon}} = \rho_0 \mathrm{e}^{\frac{p}{\varepsilon}}, \tag{A9}$$

where ρ_0 is the density reference value at the relative atmospheric pressure $p_0 = 0$. By using the Reynolds decomposition

$$b = [b] + b', \tag{A10}$$

where $[b]$ is the mean-value of the generic field b, and b' the turbulent fluctuation, we conjecture that for fluids with low compressibility, i.e., fluids in the liquid state for which $\varepsilon \gg 1$, the turbulent density fluctuations are evanescent [18]. Accordingly

$$\rho = [\rho] = \rho_0 \mathrm{e}^{\frac{[p]}{\varepsilon}}, \tag{A11}$$

$$\rho' = 0. \tag{A12}$$

Indeed, as $\varepsilon \gg 1$, $\frac{p'}{\varepsilon} \ll 1$, $\mathrm{e}^{\frac{p'}{\varepsilon}} \approx 1$, and $\rho = \rho_0 \mathrm{e}^{\frac{([p]+p')}{\varepsilon}} \approx \rho_0 \mathrm{e}^{\frac{[p]}{\varepsilon}}$. With this setting, by using the conventional RANS procedure [16], we obtain the following equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho [\boldsymbol{v}]) = 0, \tag{A13}$$

$$\frac{\partial}{\partial t} (\rho [\boldsymbol{v}]) + \nabla \cdot (\rho [\boldsymbol{v}] \otimes [\boldsymbol{v}]) = -\rho \frac{\partial}{\partial t} [\boldsymbol{v}] + \rho \nabla [\boldsymbol{v}] \cdot [\boldsymbol{v}] = \rho g \hat{\boldsymbol{e}}_z - \nabla [p] - \nabla \cdot \left[\underline{\boldsymbol{M}}\right] - \nabla \cdot \underline{\boldsymbol{T}}_{Re}, \tag{A14}$$

$$\begin{aligned} \frac{\partial}{\partial t} (\rho [e_m]) + \nabla \cdot \left(\rho [e_m] [\boldsymbol{v}] + [p] [\boldsymbol{v}] + \left[\underline{\boldsymbol{M}}\right] \cdot [\boldsymbol{v}] + \rho \left[e'_m \boldsymbol{v}'\right] + \left[p' \boldsymbol{v}'\right] + \left[\underline{\boldsymbol{M}}' \cdot \boldsymbol{v}'\right]\right) \\ = -[p] \nabla \cdot [\boldsymbol{v}] - [\boldsymbol{M}] : \nabla [\boldsymbol{v}] - \left[p' \nabla \cdot \boldsymbol{v}'\right] - \left[\underline{\boldsymbol{M}}' : \nabla \boldsymbol{v}'\right], \end{aligned} \tag{A15}$$

$$\frac{\partial}{\partial t} (\rho [e_t]) + \nabla \cdot \left(\rho [e_t] [\boldsymbol{v}] + [p] [\boldsymbol{v}] + \left[\underline{\boldsymbol{M}}\right] \cdot [\boldsymbol{v}] + \rho \left[e'_t \boldsymbol{v}'\right] + \left[p' \boldsymbol{v}'\right] + \left[\underline{\boldsymbol{M}}' \cdot \boldsymbol{v}'\right]\right) = 0, \tag{A16}$$

$$\frac{\partial}{\partial t} (\rho [u_i]) + \nabla \cdot \left(\rho [u_i] [\boldsymbol{v}] + \rho \left[u'_i \boldsymbol{v}'\right]\right) = -[p] \nabla \cdot [\boldsymbol{v}] - \left[\underline{\boldsymbol{M}}\right] : \nabla [\boldsymbol{v}] - \left[p' \nabla \cdot \boldsymbol{v}'\right] - \left[\underline{\boldsymbol{M}}' : \nabla \boldsymbol{v}'\right], \tag{A17}$$

$$\frac{\partial}{\partial t} (\rho T [s]) + \nabla \cdot \left(\rho T [s] [\boldsymbol{v}] + \rho T \left[s' \boldsymbol{v}'\right]\right) = -\left[\underline{\boldsymbol{M}}\right] : \nabla [\boldsymbol{v}] - \left[\underline{\boldsymbol{M}}' : \nabla \boldsymbol{v}'\right], \tag{A18}$$

$$\frac{\partial}{\partial t} \left(\rho \left[e_p\right]\right) + \nabla \cdot \left(\rho \left[e_p\right] [\boldsymbol{v}] + \rho \left[e'_p \boldsymbol{v}'\right]\right) = -[p] \nabla \cdot [\boldsymbol{v}] - \left[p' \nabla \cdot \boldsymbol{v}'\right]. \tag{A19}$$

We stress that in 1D contest, Eq. (A13) reduces to Eq. (73). By using the decomposition (41), Eq. (A14) can be written as

$$\frac{\partial}{\partial t} (\rho [\boldsymbol{v}]) + \nabla \cdot (\rho [\boldsymbol{v}] \otimes [\boldsymbol{v}]) = \rho g \hat{\boldsymbol{e}}_z - \nabla [p] - \nabla \cdot \underline{\boldsymbol{P}}_{tur} - \nabla \cdot \underline{\boldsymbol{G}} = \rho g \hat{\boldsymbol{e}}_z - \nabla [p] - \nabla \cdot \underline{\boldsymbol{P}}_{tur} - \nabla \cdot \underline{\boldsymbol{G}}, \tag{A20}$$

where

$$\underline{\boldsymbol{G}} = \left[\underline{\boldsymbol{M}}\right] + \underline{\boldsymbol{M}}_{tur}. \tag{A21}$$

The physical insights about $\underline{\boldsymbol{P}}_{tur}$ and $\underline{\boldsymbol{M}}_{tur}$ can be highlighted by using an alternative form of the RANS equations. For this purpose, we deduce the mean mechanical energy equation from Eq. (A14) by scalar product by $[\boldsymbol{v}]$

$$\begin{aligned} \frac{\partial}{\partial t} (\rho [e_m]) + \nabla \cdot \left(\rho [e_m] [\boldsymbol{v}] + [p] [\boldsymbol{v}] + \underline{\boldsymbol{P}}_{tur} \cdot [\boldsymbol{v}] + \underline{\boldsymbol{G}} \cdot [\boldsymbol{v}]\right) \\ = [p] \nabla \cdot [\boldsymbol{v}] + \underline{\boldsymbol{P}}_{tur} : \nabla [\boldsymbol{v}] + \underline{\boldsymbol{G}} : \nabla [\boldsymbol{v}]. \end{aligned} \tag{A22}$$

According to Eq. (A22), the mean total energy equation can be given as

$$\frac{\partial}{\partial t} (\rho [e_t]) + \nabla \cdot \left(\rho [e_t] [\boldsymbol{v}] + [p] [\boldsymbol{v}] + \underline{\boldsymbol{P}}_{tur} \cdot [\boldsymbol{v}] + \underline{\boldsymbol{G}} \cdot [\boldsymbol{v}]\right) = 0, \tag{A23}$$

whilst the mean internal energy can be expressed as

$$\frac{\partial}{\partial t} (\rho [u_i]) + \nabla \cdot (\rho [u_i] [\boldsymbol{v}]) = -[p] \nabla \cdot [\boldsymbol{v}] - \underline{\boldsymbol{P}}_{tur} : \nabla [\boldsymbol{v}] - \underline{\boldsymbol{G}} : \nabla [\boldsymbol{v}]. \tag{A24}$$

By using Eq. (A21), the comparison between Eqs. (A17) and (A24) leads to the relationship

$$\nabla \cdot \left(\rho \left[u'_i \boldsymbol{v}'\right]\right) + \left[p' \nabla \cdot \boldsymbol{v}'\right] + \left[\underline{\boldsymbol{M}}' : \nabla \boldsymbol{v}'\right] = \underline{\boldsymbol{P}}_{tur} : \nabla [\boldsymbol{v}] + \underline{\boldsymbol{M}}_{tur} : \nabla [\boldsymbol{v}]. \tag{A25}$$

We conjecture that

$$\nabla \cdot \left(\rho \left[u'_i \boldsymbol{v}'\right]\right) + \left[p' \nabla \cdot \boldsymbol{v}'\right] = \underline{\boldsymbol{P}}_{tur} : \nabla [\boldsymbol{v}], \tag{A26}$$

$$\left[\underline{\boldsymbol{M}}' : \nabla \boldsymbol{v}'\right] = \underline{\boldsymbol{M}}_{tur} : \nabla [\boldsymbol{v}]. \tag{A27}$$

By using Eqs. (A21) and (A27), the mean entropy Eq. (A18) becomes

$$\frac{\partial}{\partial t} (\rho T [s]) + \nabla \cdot \left(\rho T [s] [\boldsymbol{v}] + \rho T \left[s' \boldsymbol{v}'\right]\right) = -\underline{\boldsymbol{G}} : \nabla [\boldsymbol{v}]. \tag{A28}$$

Accordingly, we can split the entropy production in two parts

$$-\underline{\boldsymbol{G}} : \nabla [\boldsymbol{v}] = -\left[\underline{\boldsymbol{M}}\right] : \nabla [\boldsymbol{v}] - \underline{\boldsymbol{M}}_{tur} : \nabla [\boldsymbol{v}] , \tag{A29}$$

where $-\left[\underline{\boldsymbol{M}}\right] : \nabla [\boldsymbol{v}]$ is the contribution due to the viscosity; $-\underline{\boldsymbol{M}}_{tur} : \nabla [\boldsymbol{v}]$ is the contribution due the turbulent fluctuation. In agreement with this line of reasoning, the tensor field $\underline{\boldsymbol{P}}_{tur}$ is not responsible for entropy production; the term $\nabla \cdot \underline{\boldsymbol{P}}_{tur}$ takes into account the effects of turbulent fluctuations on the momentum flux; the tensor field $\underline{\boldsymbol{M}}_{tur}$ is such that the additional friction forces (per unit volume) due to turbulent fluctuations are given as $-\nabla \cdot \underline{\boldsymbol{M}}_{tur}$. Thus, identifying $\underline{\boldsymbol{P}}_{tur}$ as a "convective" term, and $\underline{\boldsymbol{M}}_{tur}$ as a "dissipative" term, the momentum equations can be written as

$$\frac{\partial}{\partial t} (\rho [\boldsymbol{v}]) + \nabla \cdot \left(\rho [\boldsymbol{v}] \otimes [\boldsymbol{v}] + \underline{\boldsymbol{P}}_{tur}\right) = -\rho g \hat{\boldsymbol{e}}_z - \nabla [p] - \nabla \cdot \underline{\boldsymbol{G}}. \tag{A30}$$

From Eq. (A8), putting

$$\frac{\partial}{\partial t} (\rho [u_i]) + \nabla \cdot (\rho [u_i] [\boldsymbol{v}]) - \frac{\partial}{\partial t} (\rho T [s]) + \nabla \cdot (T \rho [s] [\boldsymbol{v}]) = \frac{\partial}{\partial t} \left(\rho \left[e_p\right]\right) + \nabla \cdot \left(\rho \left[e_p\right] [\boldsymbol{v}]\right), \tag{A31}$$

and using Eqs. (A24) and (A28), we deduce

$$\frac{\partial}{\partial t} \left(\rho \left[e_p\right]\right) + \nabla \cdot \left(\rho \left[e_p\right] [\boldsymbol{v}]\right) = -[p] \nabla \cdot [\boldsymbol{v}] - \underline{\boldsymbol{P}}_{tur} : \nabla [\boldsymbol{v}] + \nabla \cdot \left(\rho T \left[s' \boldsymbol{v}'\right]\right). \tag{A32}$$

The comparison between Eqs. (A19) and (A32) provides the relationship

$$\nabla \cdot \left(\rho \left[e'_p \boldsymbol{v}'\right]\right) + \left[p' \nabla \cdot \boldsymbol{v}'\right] + \nabla \cdot \left(\rho T \left[s' \boldsymbol{v}'\right]\right) = \underline{\boldsymbol{P}}_{tur} : \nabla [\boldsymbol{v}] , \tag{A33}$$

whilst Eqs. (A26) and (A33) lead to the equation

$$\nabla \cdot \left(\rho \left[u'_i \boldsymbol{v}'\right]\right) - \nabla \cdot \left(\rho T \left[s' \boldsymbol{v}'\right]\right) = \nabla \cdot \left(\rho \left[e'_p \boldsymbol{v}'\right]\right). \tag{A34}$$

Appendix B Remarks on the analytical solution

We consider the governing equation for U in $x \in [0, L]$

$$\alpha \frac{\partial^3 U}{\partial x^2 \partial t} + \beta \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = 0, \tag{B35}$$

subject to the initial conditions

$$U(x, 0) = U_0, \tag{B36}$$

$$\frac{\partial U}{\partial t}(x, 0) = 0, \tag{B37}$$

and the boundary conditions

$$\frac{\partial U}{\partial x}(0, t) = 0, \tag{B38}$$

$$U(L, t) = U_0 \gamma(t), \tag{B39}$$

where $\alpha = \frac{\mu_{d,tur}}{\rho_0}$; $\beta = c_0^2$; $\gamma(t)$ the valve closing law. By using the separation of variables method as proposed in [43], we set

$$U = W(x, t) + U_0 \gamma(t), \tag{B40}$$

where W satisfies the boundary value problem

$$\alpha \frac{\partial^3 W}{\partial x^2 \partial t} + \beta \frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial t^2} = U_0 \frac{\mathrm{d}^2 \gamma}{\mathrm{d} t^2}, \tag{B41}$$

$$\frac{\partial W}{\partial x}(0,t)=0, \tag{B42}$$

$$W(L,t)=0, \tag{B43}$$

$$W(x,0)=U_0(1-\gamma(0)), \tag{B44}$$

$$\frac{\partial W}{\partial t}(x,0)=-U_0\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0). \tag{B45}$$

We assume the following eigenfunction expansion

$$W=\sum_{n=0}^{\infty}T_n\cos\left(\frac{\pi x}{L}\left(n+\frac{1}{2}\right)\right)=\sum_{n=0}^{\infty}T_n\cos(\lambda_n x), \tag{B46}$$

where $T_n=T_n(t)$ are the unknown expansion coefficients, and $\lambda_n=\frac{\pi}{L}\left(n+\frac{1}{2}\right)$ are the eigenvalues. Equation (B46) satisfies Eqs. (B42)–(B43) and the x-dependence of the homogeneous counterpart of Eq. (B41). By using the orthogonality property of the cosine eigenfunctions, we obtain the following non-homogeneous differential equation

$$\frac{\mathrm{d}^2T_n}{\mathrm{d}t^2}+\alpha\lambda_n^2\frac{\mathrm{d}T_n}{\mathrm{d}t}+\beta\lambda_n^2T_n=-\frac{2U_0}{L}\int_0^L\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\cos(\lambda_n x)\,\mathrm{d}x=Q_n, \tag{B47}$$

and the corresponding initial conditions for T_n

$$T_n(0)=-\frac{2U_0}{L}\int_0^L\gamma(0)\cos(\lambda_n x)\,\mathrm{d}x=F_n, \tag{B48}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}T_n(0)=-\frac{2U_0}{L}\int_0^L\frac{\mathrm{d}}{\mathrm{d}t}\gamma(0)\cos(\lambda_n x)\,\mathrm{d}x=G_n. \tag{B49}$$

By using the variation of the parameters method [44], the solution of the Cauchy problem (B47)–(B49) is given by

$$\begin{aligned}T_n=&\,\mathrm{e}^{-\frac{\alpha\lambda_n^2 t}{2}}\left(F_n\cosh\left(\frac{t\lambda_n}{2}\sqrt{\alpha^2\lambda_n^2-4\beta}\right)+\frac{2G_n+F_n\alpha\lambda_n^2}{\lambda_n\sqrt{\alpha^2\lambda_n^2-4\beta}}\sinh\left(\frac{t\lambda_n}{2}\sqrt{\alpha^2\lambda_n^2-4\beta}\right)\right)\\&+\frac{2}{\lambda_n\sqrt{\alpha^2\lambda_n^2-4\beta}}\int_0^t\mathrm{e}^{-\frac{\alpha\lambda_n^2(t-\tau)}{2}}\sinh\left(\frac{(t-\tau)\lambda_n}{2}\sqrt{\alpha^2\lambda_n^2-4\beta}\right)Q_n\mathrm{d}\tau.\end{aligned} \tag{B50}$$

The solution for U finally reads as

$$\begin{aligned}U=&\,U_0\gamma+\sum_{n=0}^{\infty}\cos(\lambda_n x)\left(\frac{2}{\lambda_n\sqrt{\alpha^2\lambda_n^2-4\beta}}\int_0^t\mathrm{e}^{-\frac{\alpha\lambda_n^2(t-\tau)}{2}}\sinh\left(\frac{(t-\tau)\lambda_n}{2}\sqrt{\alpha^2\lambda_n^2-4\beta}\right)Q_n\mathrm{d}\tau\right.\\&\left.+\mathrm{e}^{-\frac{\alpha\lambda_n^2 t}{2}}\left(F_n\cosh\left(\frac{t\lambda_n}{2}\sqrt{\alpha^2\lambda_n^2-4\beta}\right)+\frac{2G_n+F_n\alpha\lambda_n^2}{\lambda_n\sqrt{\alpha^2\lambda_n^2-4\beta}}\sinh\left(\frac{t\lambda_n}{2}\sqrt{\alpha^2\lambda_n^2-4\beta}\right)\right)\right).\end{aligned} \tag{B51}$$

Equation (B51) can be also written in the following form

$$\begin{aligned}U=&\,U_0\gamma+\sum_{n=0}^{\infty}\cos(\lambda_n x)\left(\frac{2}{\sqrt{\kappa_n^2-4\Theta_n^2}}\int_0^t\mathrm{e}^{-\frac{\kappa_n(t-\tau)}{2}}\sinh\left(\frac{t-\tau}{2}\sqrt{\kappa_n^2-4\Theta_n^2}\right)Q_n\mathrm{d}\tau\right.\\&\left.+\mathrm{e}^{-\frac{\kappa_n t}{2}}\left(F_n\cosh\left(\frac{t}{2}\sqrt{\kappa_n^2-4\Theta_n^2}\right)+\frac{2G_n+F_n\kappa_n}{\sqrt{\kappa_n^2-4\Theta_n^2}}\sinh\left(\frac{t}{2}\sqrt{\kappa_n^2-4\Theta_n^2}\right)\right)\right),\end{aligned} \tag{B52}$$

where

$$\Theta_n=\sqrt{\beta}\lambda_n=\sqrt{\beta}\frac{\pi}{L}\left(n+\frac{1}{2}\right), \tag{B53}$$

$$\kappa_n=\alpha\lambda_n^2 \tag{B54}$$

represent the eigenfrequency and the damping rate of the n-th mode, respectively. In the case of instantaneous closure, $\gamma = \mathcal{H}(-t)$. By noting that $\frac{d^2\mathcal{H}(-t)}{dt^2} = -\frac{d\delta}{dt}$, where $\delta = \delta(t)$ is the Dirac Delta function, the analytical solution (B52) reduces to

$$U = U_0\mathcal{H}(-t) + \sum_{n=0}^{\infty} \cos(\lambda_n x) \frac{4U_0(-1)^n e^{-\frac{\kappa_n t}{2}}}{\pi(1+2n)\sqrt{\kappa_n^2 - 4\Theta_n^2}} \left(\sqrt{\kappa_n^2 - 4\Theta_n^2}\cosh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right) - \kappa_n \sinh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)\right). \tag{B55}$$

Accordingly, we obtain the following explicit expressions

$$\frac{\partial U}{\partial x}(L,t) = -\sum_{n=0}^{\infty} \frac{2U_0 e^{-\frac{\kappa_n t}{2}}}{L\sqrt{\kappa_n^2 - 4\Theta_n^2}} \left(\sqrt{\kappa_n^2 - 4\Theta_n^2}\cosh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right) - \kappa_n \sinh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)\right), \tag{B56}$$

$$\Pi(L,t) = \Pi(L,0) + \frac{4U_0\rho_0 c_0^2}{L} \sum_{n=0}^{\infty} \frac{e^{-\frac{\kappa_n t}{2}} \sinh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)}{\sqrt{\kappa_n^2 - 4\Theta_n^2}}. \tag{B57}$$

By using the orthogonality property of cosine eigenfunctions, the averaged dissipation function $\Phi(t)$ reads as

$$\begin{aligned}\Phi(t) &= \frac{1}{L}\int_0^L \mu_{d,tur}\left(\frac{\partial U}{\partial x}\right)^2 dx \\ &= 2\frac{\mu_{d,tur}U_0^2}{\rho_0 L^2} \sum_{n=0}^{\infty} \left(\frac{e^{-\frac{\kappa_n t}{2}}}{\sqrt{\kappa_n^2 - 4\Theta_n^2}} \left(\sqrt{\kappa_n^2 - 4\Theta_n^2}\cosh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right) - \kappa_n \sinh\left(\frac{t}{2}\sqrt{\kappa_n^2 - 4\Theta_n^2}\right)\right)\right)^2.\end{aligned} \tag{B58}$$

References

1. Batchelor, G.K.: An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge (1967)
2. Massey, B.S., Ward-Smith, J.: Mechanics of Fluids, vol. 1. Crc Press, Boca Ranton (1998)
3. Tzou, D.Y.: An engineering assessment to the relaxation time in thermal wave propagation. Int. J. Heat Mass Transf. **36**(7), 1845–1851 (1993)
4. Wu, J.-Z., Ma, H.-Y., Zhou, M.-D.: Vorticity and Vortex Dynamics. Springer, Berlin (2007)
5. Wu, J.-Z.: A theory of three-dimensional interfacial vorticity dynamics. Phys. Fluids **7**(10), 2375–2395 (1995)
6. Wu, J., Wu, J.: Vorticity dynamics on boundaries. Adv. Appl. Mech. **32**, 119–275 (1996)
7. Wu, J.-Z., Zhou, Y., Fan, M.: A note on kinetic energy, dissipation and enstrophy. Phys. Fluids **11**(2), 503–505 (1999)
8. Schlichting, H., Gersten, K.: Boundary-Layer Theory. Springer, Berlin (2016)
9. Gad-el-Hak, M.: Stokes' hypothesis for a newtonian, isotropic fluid. J. Fluids Eng. **117**(1), 3–5 (1995)
10. Buresti, G.: A note on Stokes' hypothesis. Acta Mech. **226**, 3555–3559 (2015)
11. Rajagopal, K.: A new development and interpretation of the Navier–Stokes fluid which reveals why the "Stokes assumption" is inapt. Int. J. Non-Linear Mech. **50**, 141–151 (2013)
12. Joseph, D.D.: Potential flow of viscous fluids: historical notes. Int. J. Multiph. Flow **32**(3), 285–310 (2006)
13. Rajagopal, K.: Remarks on the notion of "pressure". Int. J. Non-Linear Mech. **71**, 165–172 (2015)
14. Di Nucci, C., Celli, D., Fischione, P., Pasquali, D.: Elementary scales and the lack of Fourier paradox for Fourier fluids. Meccanica 1–4 (2022)
15. Di Nucci, C., Celli, D., Fischione, P., Pasquali, D.: Classical irreversible thermodynamics versus extended irreversible thermodynamics. The role of the continuity equation. Arch. Thermodyn. **43**(2) (2022)
16. Pozrikidis, C.: Fluid Dynamics: Theory, Computation, and Numerical Simulation. Springer, Berlin (2009)
17. Perot, B.: Turbulence modeling using body force potentials. Phys. Fluids **11**(9), 2645–2656 (1999)
18. Di Nucci, C., Pasquali, D., Celli, D., Pasculli, A., Fischione, P., Di Risio, M.: Turbulent bulk viscosity. Eur. J. Mech.-B/Fluids **84**, 446–454 (2020)
19. Fried, E., Gurtin, M.E.: A continuum mechanical theory for turbulence: a generalized Navier–Stokes-α equation with boundary conditions. Theoret. Comput. Fluid Dyn. **22**, 433–470 (2008)
20. Rivlin, R.S.: The relation between the flow of non-newtonian fluids and turbulent newtonian fluids. Q. Appl. Math. **15**(2), 212–215 (1957)

21. Mohseni, K., Kosović, B., Shkoller, S., Marsden, J.E.: Numerical simulations of the Lagrangian averaged Navier–Stokes equations for homogeneous isotropic turbulence. Phys. Fluids **15**(2), 524–544 (2003)
22. Kim, T.-Y., Dolbow, J., Fried, E.: A numerical method for a second-gradient theory of incompressible fluid flow. J. Comput. Phys. **223**(2), 551–570 (2007)
23. Joseph, D.D., Liao, T.Y.: Potential flows of viscous and viscoelastic fluids. J. Fluid Mech. **265**, 1–23 (1994)
24. Nerney, S., Nerney, E.G.: Theory of irrotational flow. arXiv preprint arXiv:2001.05850 (2020)
25. Panton, R.L.: Incompressible Flow. Wiley, Hoboken (2024)
26. Jing, H., Zhang, D., Li, G.: Pressure variations of fluid transients in a pressurized pipeline. Fluid Dyn. Res. **50**(4), 045514 (2018)
27. Di Nucci, C., Celli, D., Pasquali, D., Di Risio, M.: New dimensionless number for the transition from viscous to turbulent flow. Fluids **7**(6), 202 (2022)
28. Ghidaoui, M.S., Zhao, M., McInnis, D.A., Axworthy, D.H.: A review of water hammer theory and practice. Appl. Mech. Rev. **58**(1), 49–76 (2005)
29. Wan, W., Huang, W.: Water hammer simulation of a series pipe system using the MacCormack time marching scheme. Acta Mech. **229**, 3143–3160 (2018)
30. Triki, A.: Water-hammer control in pressurized-pipe flow using an in-line polymeric short-section. Acta Mech. **227**(3), 777–793 (2016)
31. Triki, A.: Dual-technique-based inline design strategy for water-hammer control in pressurized pipe flow. Acta Mech. **229**(5), 2019–2039 (2018)
32. Wahba, E.: A computational study of viscous dissipation and entropy generation in unsteady pipe flow. Acta Mech. **216**(1–4), 75–86 (2011)
33. Liepmann, H., Roshko, A.: Elements of Gasdynamics. Dover Publications Inc, Mineola (2001)
34. Urbanowicz, K., Bergant, A., Stosiak, M., Karpenko, M., Bogdevičius, M.: Developments in analytical wall shear stress modelling for water hammer phenomena. J. Sound Vib. **562**, 117848 (2023)
35. Landau, L.D., Lifshitz, E.M.: Fluid Mechanics: Landau and Lifshitz: Course of Theoretical Physics, vol. 6. Elsevier, Oxford (2013)
36. Di Nucci, C., Russo Spena, A.: On the propagation of one-dimensional acoustic waves in liquids. Meccanica **48**, 15–21 (2013)
37. Di Nucci, C., Petrilli, M., Russo Spena, A.: Unsteady friction and visco-elasticity in pipe fluid transients. J. Hydraul. Res. **49**(3), 398–401 (2011)
38. Di Nucci, C., Russo Spena, A.: On transient liquid flow. Meccanica **51**(9), 2135–2143 (2016)
39. Ghidaoui, M.S.: On the fundamental equations of water hammer. Urban Water J. **1**(2), 71–83 (2004)
40. Mattsson, A.E., Rider, W.J.: Artificial viscosity: back to the basics. Int. J. Numer. Meth. Fluids **77**(7), 400–417 (2015)
41. Szymkiewicz, R., Mitosek, M.: Alternative convolution approach to friction in unsteady pipe flow. J. Fluids Eng. **136**(1), 011202 (2014)
42. Munson, B.R., Okiishi, T.H., Huebsch, W.W., Rothmayer, A.P.: Fluid Mechanics. Wiley, Singapore (2013)
43. Mei, C., Li, Y., Michele, S., Sammarco, P., McBeth, P.: Anchoring and migration of balloon in REBOA. J. Fluid Mech. **927**, 20 (2021)
44. Lakshmikantham, V., Deo, S.G.: Method of Variation of Parameters for Dynamic Systems. CRC Press, Boca Ranton (1998)