BETWEEN AN n-ARY AND AN $n+1$-ARY
NEAR-UNANIMITY TERM

PAOLO LIPPARINI

ABSTRACT. We devise a condition strictly between the existence of an n-ary and an $n+1$-ary near-unanimity term. We evaluate exactly the distributivity and modularity levels implied by such a condition.

1. INTRODUCTION

Varieties with a near-unanimity term form a distinguished class of congruence distributive varieties and have been studied from the 70's in the past century [3, 32]. More recently, a fundamental paper by Berman, Idziak, Marković, McKenzie, Valeriote, Willard [7] showed that, among congruence distributive varieties, near-unanimity terms play a very important role in tractability problems [16]. Recent results about near-unanimity terms include [4, 9, 10, 11, 28, 30, 33, 34].

A majority term is the simplest ternary case of a near-unanimity term and corresponds exactly to 2-distributivity, the first and strongest nontrivial level in the Jónsson hierarchy. Levels in this sense [14] are measured by the minimal number of Jónsson terms witnessing congruence distributivity, as recalled in Theorem 2.1(2) below. For near-unanimity terms of larger arity the conditions overlap no more and near-unanimity provides a condition strictly stronger than congruence distributivity [32]. For $n \geq 3$, the existence of an n-ary near-unanimity term, for short, an n-*near-unanimity term*, implies $2n-4$-distributivity [32, Theorem 2]. In [27, Theorem 3.6] we showed that the above result is sharp, even when restricted to locally finite varieties with a symmetric near-unanimity term.

Since, by the above results, the existence of an $n+1$-near-unanimity term implies $2n-2$-distributivity, the results suggest that there might possibly be a condition strictly between an n- and an $n+1$-near-unanimity term and which implies $2n-3$-distributivity. Henceforth, an "$n\frac{1}{2}$-near-unanimity term" is a suitable name for such a condition. A candidate for such a condition has been proposed in [27, Definition 4.7], involving an $n+2$-ary term. In Section 3 we show that the condition proposed in [27] does satisfy the required properties, hence actually deserves the name of an $n\frac{1}{2}$-*near-unanimity term*. More involved arguments in Section 4 show that the result about the distributivity level is optimal, namely, that an $n\frac{1}{2}$-near-unanimity term does not necessarily imply $2n-4$-distributivity. Corresponding results are proved for modularity levels in Section 5.

Under the above terminological conventions, a compact way to express the main results of the present note goes as follows, where parameters vary on integers and half-integers.

Theorem 1.1. *Suppose that $h, k \in \mathbb{N} \cup (\mathbb{N} + \frac{1}{2})$, $h, k \geq 3$.*

(1) *Suppose that $h < k$. Then every variety with an h-near-unanimity term has a k-near-unanimity term. Moreover, there is a locally finite variety with a k-near-unanimity term without an h-near-unanimity term.*

(2) *Every variety with an h-near-unanimity term is $2h{-}4$-distributive. There is a locally finite variety with an h-near-unanimity term which is not $2h{-}5$-distributive.*

(3) *Every variety with an h-near-unanimity term is $2h{-}3$-modular. If $h \geq 4$ there is a locally finite variety with an h-near-unanimity term which is not $2h{-}4$-modular.*

Details for the proof of Theorem 1.1(1) shall be given at the end of Section 3. The remaining items shall be proved in Section 5.

A few variations, still between an n- and an $n{+}1$-near-unanimity term, are presented in Section 6. Section 7 presents some problems.

2. Preliminaries

Throughout, n is a natural number ≥ 2, frequently, ≥ 3.

A *near-unanimity term* is a term u of arity ≥ 3 and such that all the equations of the form

$$u(x, x, \ldots, x, y, x, \ldots, x, x) = x$$

are satisfied (in some given algebra or variety), with just one occurrence of y in any possible position. For notational convenience, an n-ary near-unanimity term shall be sometimes called an *n-near-unanimity term*.

An n-ary term u is *symmetric* if the equations

$$u(x_1, \ldots, x_n) = u(x_{\tau(1)}, \ldots, x_{\tau(n)})$$

hold, for all permutations τ of $\{1, \ldots, n\}$.

The following theorem provides important Maltsev conditions characterizing congruence distributivity.

Theorem 2.1. *For every variety $\mathcal{V}$, the following conditions are equivalent.*

(1) *$\mathcal{V}$ is congruence distributive.*

(2) *(Jónsson [17]) For some natural number k, $\mathcal{V}$ has a sequence of Jónsson terms, that is, terms $t_0, \ldots, t_k$ satisfying*

(Eq. 2.1) $$x = t_0(x, y, z),$$

(Eq. 2.2) $$x = t_i(x, y, x), \qquad \text{for } 0 \leq i \leq k,$$

(Eq. 2.3)
$$t_i(x, x, z) = t_{i+1}(x, x, z), \quad \text{for even } i,\ 0 \leq i < k,$$
$$t_i(x, z, z) = t_{i+1}(x, z, z), \quad \text{for odd } i,\ 0 \leq i < k,$$

(Eq. 2.4) $$t_k(x, y, z) = z.$$

(3) *(Kazda, Kozik, McKenzie, Moore [19]) For some natural number k, $\mathcal{V}$ has a sequence of* directed Jónsson terms, *that is, terms satisfying* (Eq. 2.1), (Eq. 2.2), (Eq. 2.4) *and*

(Eq. 2.5) $$t_i(x, z, z) = t_{i+1}(x, x, z) \quad \text{for } 0 \leq i < k.$$

To the best of our knowledge, directed Jónsson terms first appeared implicitly in the proof of [31, Theorem 2.3] and explicitly (but unnamed) in [34, Theorem 4.1].

Definition 2.2. A variety $\mathcal{V}$ is said to be *k-distributive* if $\mathcal{V}$ has a sequence $t_0, \ldots, t_k$ of Jónsson terms. It is standard to see that a variety $\mathcal{V}$ is k-distributive if and only if $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ .^{k}..$ In this and similar identities juxtaposition denotes intersection and $.^{k}.$ means that we are considering k factors, namely, $k-1$ occurrences of $\circ$. If, say, k is even, then we might write $\alpha\beta \circ \alpha\gamma \circ .^{k}.\circ \alpha\gamma$ when we want to make clear that $\alpha\gamma$ is the last factor. Notice that an inclusion of the form $A \subseteq B$ is equivalent to the identity $AB = A$, hence we can always use the expression "identity".

Theorem 2.3. *(Mitschke [32, Theorem 2]) A variety with a near-unanimity term is congruence distributive.*

In more detail, for $n \geq 3$, a variety with an n-ary near-unanimity term is $2n-4$-distributive [32] and has a sequence $t_0, \ldots, t_{n-1}$ of directed Jónsson terms [6, Section 5.3.1].

Notice that the counting conventions about the number of directed Jónsson terms are not uniform through the literature (not even in the works by the present author).

A direct proof (not relying on Theorem 2.1) that a variety with a near-unanimity term is congruence distributive can be found in [18, Lemma 1.2.12]. The proof is credited to E. Fried. Compare also [26, Section 5].

We have showed in [27, Theorem 3.6 and Remark 4.5] that no parts of the second statement in Theorem 2.3 can be improved.

Theorem 2.4. [12] *For every variety $\mathcal{V}$, the following conditions are equivalent.*

    (1) $\mathcal{V}$ is congruence modular.

    (2) For some natural number k, $\mathcal{V}$ has a sequence of Day terms, *namely, 4-ary terms $t_0, \ldots, t_k$ satisfying*

$$x = t_i(x, y, y, x), \qquad \text{for } 0 \leq i \leq k,$$
$$x = t_0(x, y, z, w),$$
$$t_i(x, x, w, w) = t_{i+1}(x, x, w, w), \quad \text{for } i \text{ even, } 0 \leq i < k,$$
$$t_i(x, y, y, w) = t_{i+1}(x, y, y, w), \quad \text{for } i \text{ odd, } 0 \leq i < k,$$
$$t_k(x, y, z, w) = w.$$

A congruence modular variety $\mathcal{V}$ is *k-modular* if $\mathcal{V}$ has a sequence $t_0, \ldots, t_k$ of Day terms; this is equivalent to the congruence identity $\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ \alpha\gamma \circ .^{k}..$

3. BETWEEN AN n-ARY AND AN $n+1$-ARY NEAR-UNANIMITY TERM

Definition 3.1. [27, Definition 4.7] If $n \geq 2$, an $n\frac{1}{2}$-*near-unanimity term* is an $n+2$-ary term u such that the following equations hold.

(Eq. 3.1) $u(z, z, x, x, \ldots, x) = x,$

(Eq. 3.2) $u(x, \ldots, x, \underset{i}{z}, x, \ldots, x) = x,$ for $2 \leq i \leq n+2,$

(Eq. 3.3) $u(x, x, x, z, z, \ldots, z) = u(x, z, z, z, z, \ldots, z).$

As we mentioned in the introduction, the terminology comes from the fact that an $n\frac{1}{2}$-near-unanimity term is a notion strictly between an n-near-unanimity term

and an $n + 1$-near-unanimity term, as we will show in Theorem 3.4. In order to simplify some parts in the proof of Theorem 3.4 we need the following proposition of independent interest.

Proposition 3.2. *If $n \geq 3$, then every variety with an $n\frac{1}{2}$-near-unanimity term is $2n-3$-distributive.*

Proof. If u is an $n\frac{1}{2}$-near-unanimity term, define

$$\begin{aligned}
t_0(x, y, z) &= x, \\
t_1(x, y, z) &= u(x, x, x, x, x, x, \ldots, x, x, x, y, z), \\
t_2(x, y, z) &= u(x, x, x, x, x, x, \ldots, x, x, x, z, z), \\
t_3(x, y, z) &= u(x, x, x, x, x, x, \ldots, x, x, y, z, z), \\
t_4(x, y, z) &= u(x, x, x, x, x, x, \ldots, x, x, z, z, z), \\
t_5(x, y, z) &= u(x, x, x, x, x, x, \ldots, x, y, z, z, z), \\
&\quad \cdots \\
t_{2n-7}(x, y, z) &= u(x, x, x, x, y, z, \ldots, z, z, z, z, z), \\
t_{2n-6}(x, y, z) &= u(x, x, x, x, z, z, \ldots, z, z, z, z, z), \\
t_{2n-5}(x, y, z) &= u(x, x, x, y, z, z, \ldots, z, z, z, z, z), \\
t_{2n-4}(x, y, z) &= u(x, y, z, z, z, z, \ldots, z, z, z, z, z), \\
t_{2n-3}(x, y, z) &= z. \qquad \qquad \qquad \qquad \qquad \square
\end{aligned}$$

Remark 3.3. (a) Notice that the case $i = 3$ in equation (Eq. 3.2) has not been used in the proof of Proposition 3.2.

(b) With the only exception of the bottom lines, the proof of Proposition 3.2 implicitly uses directed Jónsson terms. Compare [6, Section 5.3.1] and [19, Observation 1.2].

In fact, the terms $t_0, t_1, t_3, t_5, \ldots, t_{2n-7}, t_{2n-5}$ above satisfy the equations (Eq. 2.1), (Eq. 2.2) and (Eq. 2.5) for directed Jónsson terms. At the end, the relations change, we have $t_{2n-5}(x, z, z) = t_{2n-4}(x, z, z)$ and $t_{2n-4}(x, x, z) = z$, instead. Thus, in the terminology from [19], the sequence $t_1, t_3, t_5, \ldots, t_{2n-7}, t_{2n-5}, t_{2n-4}$ is a sequence of *directed Gumm terms*. Notice that we do not need the case $i = 2$ in equation (Eq. 3.2) in order to get a sequence of directed Gumm terms.

Directed Gumm terms characterize congruence modularity, they do not necessarily imply congruence distributivity. However, using the case $i = 2$ in (Eq. 3.2), we have in addition $t_{2n-4}(x, y, x) = x$ and this further equation is enough to get congruence distributivity.

(c) In other words, relabeling the terms, while a sequence $d_1, d_2, \ldots, d_{m-2}, d_{m-1}$ of directed Jónsson terms implies $2m-2$-distributivity, under the counting convention from [19, Observation 1.2], a sequence $d_1, d_2, \ldots, d_{m-2}, q$ of directed Gumm terms implies $2m-3$-distributivity, *provided the term q satisfies the additional equation $q(x, y, x) = x$.*

General forms of similar equations have been studied in [20].

Lattice operations shall be denoted by $+$ and juxtaposition. Complement in Boolean algebras is denoted by $'$.

Theorem 3.4. *Let $n \geq 3$.*

(1) *Every variety with an $n\frac{1}{2}$-near-unanimity term has an $n+1$-ary near-unanimity term.*

(2) *There is a locally finite variety with an $n+1$-ary near-unanimity term but without an $n\frac{1}{2}$-near-unanimity term.*

(3) *Every variety with an n-ary near-unanimity term has an $n\frac{1}{2}$-near-unanimity term.*

(4) *There is a locally finite variety with an $n\frac{1}{2}$-near-unanimity term but without an n-ary near-unanimity term.*

Proof. (1) If u is an $n\frac{1}{2}$-near-unanimity term, then the $n+1$-ary term v defined by

$$v(x_1, x_2, x_3, \ldots, x_{n+1}) = u(x_1, x_1, x_2, x_3, \ldots, x_{n+1})$$

is a near-unanimity term, by (Eq. 3.1) and (Eq. 3.2).

Notice that we have not used (Eq. 3.3), and we do not need the case $i = 2$ in (Eq. 3.2). Notice further that the present argument works also in the case $n = 2$. This aspect shall be put in a clearer light in Remark 6.1 below.

(2) In [27, Definition 3.5] we constructed a locally finite variety $\mathcal{N}_{n+1}$ and we showed in [27, Theorem 3.6] that $\mathcal{N}_{n+1}$ has an $n+1$-ary near-unanimity term but is not $2n-3$-distributive. By Proposition 3.2, $\mathcal{N}_{n+1}$ witnesses (2).

(3) If w is an n-ary near-unanimity term, then, by adding two initial dummy variables, w becomes an $n\frac{1}{2}$-near-unanimity term.

(4) Let $\mathcal{V}$ be the term reduct of the variety of Boolean algebras obtained considering the term

$$\text{(Eq. 3.4)} \qquad u(x_1, x_2, x_3, \ldots, x_{n+2}) = (x_1 + x_2') \prod_{\substack{1 \le i < j \le n+2 \\ i \ne 2, j \ne 2}} (x_i + x_j).$$

Since $n \ge 3$, the term u satisfies (Eq. 3.1) - (Eq. 3.3) in Boolean algebras, thus u, as an operation, satisfies (Eq. 3.1) - (Eq. 3.3) in $\mathcal{V}$. Hence $\mathcal{V}$ has an $n\frac{1}{2}$-near-unanimity term. $\mathcal{V}$ is locally finite, being a term reduct of the locally finite variety of Boolean algebras.

Let $\mathbf{2}$ denote the two-element Boolean algebra with base set $\{0, 1\}$. Let $\mathbf{A}$ be the u term-reduct of the nth power of $\mathbf{2}$. We shall show that $B = A \setminus (1, 1, \ldots, 1)$ is the universe for an algebra in $\mathcal{V}$. Indeed, let $b_1, b_2, \ldots, b_{n+2} \in B$, thus, for every $i \le n+2$, at least one component of b_i is 0. Actually, we shall not use the assumption that $b_2 \in B$. Since we are working in a power with n components and the sequence $b_1, b_3, \ldots, b_{n+2}$ has length $n+1$, there are $i \ne j \le n+2$ with $i \ne 2$, $j \ne 2$ such that b_i and b_j have some 0 at the same component. Thus $b = u(b_1, b_2, \ldots, b_{n+2})$ has 0 at that component, hence $b \in B$.

Now it is standard to see that $\mathcal{V}$ has not an n-ary near-unanimity term. If, by contradiction, v is such a term, then in $\mathbf{A}$

$$v((0, 1, 1, \ldots, 1), (1, 0, 1, \ldots, 1), \ldots, (1, 1, 1, \ldots, 0)) =$$
$$(v(0, 1, 1, \ldots, 1), v(1, 0, 1, \ldots, 1), \ldots, v(1, 1, 1, \ldots, 0)) = (1, 1, 1, \ldots, 1),$$

contradicting the just-proved fact that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. $\qquad\square$

Problem 3.5. Is there a more direct proof of item (2) in Theorem 3.4 which does not rely on [27]? For example, is some variation on the arguments in the proof of 3.4(4) enough?

We are now in the position to give a proof for item (1) in Theorem 1.1. Parts (2) and (3) shall need more effort.

Proof of Theorem 1.1(1). Suppose that $h, k \in \mathbb{N} \cup (\mathbb{N}+\frac{1}{2})$ and $h, k \geq 3$. If h is a half-integer and $\mathcal{V}$ has an h-near-unanimity term, then $\mathcal{V}$ has an $h+\frac{1}{2}$-near-unanimity term by Theorem 3.4(1). If h is an integer and $\mathcal{V}$ has an h-near-unanimity term, then $\mathcal{V}$ has an $h+\frac{1}{2}$-near-unanimity term by Theorem 3.4(3).

By induction, we get that if $h \leq k$, then every variety with an h-near-unanimity term has a k-near-unanimity term. It then follows from Theorem 3.4(2)(4) that if $h < k$, then there is a locally finite variety with a k-near-unanimity term and without a $k-\frac{1}{2}$-near-unanimity term, hence without an h-near-unanimity term, since $h \leq k - \frac{1}{2}$. □

4. Distributivity levels

In this section we hint to a proof that Proposition 3.2 cannot be improved, namely, for every $n \geq 3$, there is a variety with an $n\frac{1}{2}$-near-unanimity term which is not $2n-4$-distributive. The proof relies heavily on [27]. We first need to establish a considerable amount of notation and conventions. In particular, we need to recast many notions from [27] in the setting where a dummy variable is added to a special set of lattice terms.

Throughout the present section n is a natural number ≥ 3 and $m = n + 1$. We first define the variety we shall use in order to provide the main counterexample.

Definition 4.1. Recall that $m \geq 4$ and suppose that $2 \leq j \leq m$.

(a) We define $u_{j,m}^{+}$ to be the $m+1$-ary lattice term

$$(\text{Eq. } 4.1) \qquad u_{j,m}^{+}(x_1, \ldots, x_{m+1}) = \prod_{|J|=j} \sum_{i \in J} x_i$$

where J varies on subsets of $\{1, 3, 4, 5, \ldots, m-1, m, m+1\}$. The definition is given modulo any fixed but otherwise arbitrary arrangement of summands and factors; in fact, we shall never use the actual term structure of $u_{j,m}^{+}$, we shall only deal with the way it is evaluated. The term $u_{j,m}^{+}$ corresponds to the term $u_{j,m}$ from [27, Definition 3.3] when a dummy variable is added at the second place. This is necessary for definiteness, since we shall combine lattice reducts endowed with operations given by $u_{j,m}^{+}$ and Boolean reducts endowed with an $m+1$-ary operation.

(b) Again with $m \geq 4$ and $2 \leq j \leq m$, we let $\mathbf{N}^{j,m,+}$ be the $u_{j,m}^{+}$-reduct of the two-element lattice with base set $\{0, 1\}$.

(c) For $n \geq 3$, we let $\mathbf{G}^{\mathrm{nu},n}$ denote the term reduct of the two-element Boolean algebra obtained by considering the $n+2$-ary term defined in (Eq. 3.4), which we recall below:

$$(\text{Eq. } 4.2) \qquad u(x_1, x_2, x_3, \ldots, x_{n+2}) = (x_1 + x_2') \prod_{\substack{1 \leq i < j \leq n+2 \\ i \neq 2, j \neq 2}} (x_i + x_j).$$

(d) Suppose that $n \geq 4$ and $m = n + 1$. Let $\ell = \frac{m+1}{2}$ if m is odd and $\ell = \frac{m}{2}$ if m is even. We define $\mathcal{N}_{n\frac{1}{2}}$ to be the variety generated by the algebras

$$\mathbf{G}^{\mathrm{nu},n}, \quad \mathbf{N}^{3,m,+}, \quad \mathbf{N}^{4,m,+}, \quad \ldots, \quad \mathbf{N}^{\ell,m,+}.$$

Notice that all the above algebras have an $n+2$-ary operation, since $m = n + 1$ and $u_{j,m}^{+}$ is $m+1$-ary. Hence the definition is correct.

(e) Under the assumptions in (d), we let $\mathcal{N}_m^{3,+}$ be the variety generated by the algebras
$$\mathbf{N}^{3,m,+}, \quad \mathbf{N}^{4,m,+}, \quad \ldots, \quad \mathbf{N}^{\ell,m,+}.$$

(f) In (d) we have defined $\mathcal{N}_{n\frac{1}{2}}$ for $n \geq 4$; however, we need to define $\mathcal{N}_{n\frac{1}{2}}$ also for $n = 3$. We let $\mathcal{N}_{3\frac{1}{2}}$ be the variety generated by $\mathbf{G}^{\mathrm{nu},3}$. The variety $\mathcal{N}_{3\frac{1}{2}}$ is term-equivalent to a variety we have called $\mathcal{I}_4^-$ in [27, Section 4, p. 15]. We now recall the definition of $\mathcal{I}_4^-$ from [27]. The variety $\mathcal{I}_4^-$ is generated by term reducts of Boolean algebras by considering both the terms $f(x, y, z) = x(y' + z)$ and $u_{2,4}(x_1, x_2, x_3, x_4) = \prod_{i<j\leq4}(x_i + x_j)$. We now notice that these terms are expressible as a function of u from (Eq. 4.2) in the case $n = 3$. Indeed, $f(x, y, z) = u(z, y, x, x, x)$ and $u_{2,4}(x_1, x_2, x_3, x_4) = u(x_1, x_1, x_2, x_3, x_4)$. Conversely, u can be expressed as $u(x_1, x_2, x_3, x_4, x_5) = f(u_{2,4}(x_1, x_3, x_4, x_5), x_2, x_1)$. Hence $\mathcal{N}_{3\frac{1}{2}}$ and $\mathcal{I}_4^-$ are term-equivalent.

Remark 4.2. If $m \geq 5$, then the operation in $\mathcal{N}_m^{3,+}$ has the property that, disregarding the second argument, if all the other arguments but two are given the same value, then the operation returns this value. In fact, this property holds in all the generating algebras, since the first upper indices in 4.1(e) are all ≥ 3, namely, j in (Eq. 4.1) is always ≥ 3. Moreover, since $m \geq 5$, we always have $m \geq j + 2$, for $j = 3, 4, \ldots, \ell$, because of the definition of ℓ.

It is convenient to extend the $.\overset{k}{.}.$ notation from the introduction. If R and S are reflexive binary relations, we let $R \circ S \circ .\overset{1}{.}.= R$ and $R \circ S \circ .\overset{0}{.}.= 0$, where 0 is the minimal congruence on the algebra under consideration. Moreover, R^k is $R \circ R \circ .\overset{k}{.}.$, in particular, $R^0 = 0$.

Lemma 4.3. *Let $n \geq 4$ and $m = n + 1$. Then there are an algebra $\mathbf{A}_3^+ \in \mathcal{N}_m^{3,+}$ and a subalgebra $\mathbf{F}$ of $\mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$ such that the congruence identity*

(Eq. 4.3) $$\tilde{\alpha}(\tilde{\beta} \circ \tilde{\gamma}) \subseteq (\tilde{\alpha}(\tilde{\gamma} \circ \tilde{\beta}))^{m-4}$$

fails in $\mathbf{F}$.

Moreover, the failure of (Eq. 4.3) can be witnessed by elements $(a, 1)$, $(d, 1)$, $(c, 0)$ and congruences $\tilde{\alpha}$, induced by 1×0, $\tilde{\beta}$, $\tilde{\gamma}$ of $\mathbf{F}$ such that $(a, 1) \; \tilde{\alpha} \; (d, 1)$, $(a, 1) \; \tilde{\beta} \; (c, 0) \; \tilde{\gamma} \; (d, 1)$ and $((a, 1), (d, 1)) \notin (\tilde{\alpha}(\tilde{\gamma} \circ \tilde{\beta}))^{m-4}$.

Proof. This is proved like the Claim and the Subclaim in the proof of Theorem 3.6 in [27] with $j = 3$ and $q = 2$. The only difference is that here the second argument of the operations need not be considered in the reasonings from [27]. The reasonings obviously work here in the case of algebras in $\mathcal{N}_m^{3,+}$, whose operation does not depend on the second argument and is otherwise the same as in the variety $\mathcal{N}_m^3$ from [27, p. 8]. On the other hand, the only property of the other algebra (called $\mathbf{N}^{2,m}$ there) used in the Claim in [27, p. 8] is that if two 0's appear in the arguments of the operation, then the outcome is 0. Not considering the second argument, this property is true of $\mathbf{G}^{\mathrm{nu},n}$, as well, since its operation is defined by (Eq. 4.2). $\square$

The next lemma is essentially a special case of [27, Lemma 2.2] for terms with an additional variable. We give a direct proof since it is simpler than establishing the notation necessary for exploiting the connection with [27, Lemma 2.2]. The "types" in the statement of the next lemma are marked with a σ in order to distinguish them

from similar types we have used in some other places, permuting the coordinates. The issue is discussed in [27, p. 4].

Lemma 4.4. *Suppose that $n \geq 4$ and $m = n + 1$. Let $\mathbf{4}$ denote the four-element Boolean algebra with base set $\{0, e, e', 1\}$ and let $\mathbf{A}$ be the u-term reduct of $\mathbf{4}$, where u is the term defined in equation* (Eq. 4.2) *in Definition 4.1(c).*

Let $\mathbf{A}_3^+ \in \mathcal{N}_m^{3,+}$, $\mathbf{F} \subseteq \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$, $a, d \in A_3^+$ and let B be the subset of $A \times A \times F$ consisting of those elements which have at least one of the following types (modulo the natural identification of a triple containing a subpair with a quadruple)

Type I^σ	Type II^σ	Type III^σ	Type IV^σ
$(-, 0, a, -)$	$(0, 0, -, -)$,	$(0, -, d, -)$	$(-, -, -, 0)$,

where dashed places can be filled with arbitrary elements from the corresponding algebras, under the provision that each 4-uple actually belongs to $A \times A \times F$, namely, that the pair consisting of the last two coordinates belongs to F.

Then B is the base set for a subalgebra $\mathbf{B}$ of $\mathbf{A} \times \mathbf{A} \times \mathbf{F}$, hence also a subalgebra of $\mathbf{A} \times \mathbf{A} \times \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$.

Proof. First, notice that B is nonempty, for example, $(0, 0, f_1, f_2) \in B$, if $(f_1, f_2) \in F$. Let $b_1, b_2, \ldots, b_{n+2} \in B$. We have to show that $b = u(b_1, b_2, \ldots b_{n+2})$, as computed in $\mathbf{A} \times \mathbf{A} \times \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$, belongs to B. Since the last two coordinates of each b_i form a pair in F, then the last two coordinates of b form a pair belonging to F, since $\mathbf{F}$ is a subalgebra of $\mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$. It remains to show that b has one of the types I^σ - IV^σ.

In the following considerations we shall deal with the $n+1$-element set $\{b_1, b_3, b_4, \ldots b_{n+2}\}$; the element b_2 shall play no role in the discussion. If two or more among the b_i's (not considering b_2) have 0 at the fourth component, then $u(b_1, b_2, \ldots b_{n+2})$ has 0 at the fourth component, since the fourth component is evaluated in $\mathbf{G}^{\mathrm{nu},n}$. Here we have used the fact, already mentioned in the proof of Lemma 4.3, that 0 in $\mathbf{G}^{\mathrm{nu},n}$ is "2-absorbing disregarding the second argument" for the term defined in equation (Eq. 4.2). In this case b has type IV^σ, hence we are done.

Therefore we may suppose that at most one b_i has 0 at the fourth component. If two or more among the b_i's have 0 at the first component and two or more among the b_i's have 0 at the second component (again, in both cases, not considering b_2), then b has 0 both at the first and at the second component, again by the absorbing properties of 0, thus b has type II^σ, hence $b \in B$ and we are done in this case, too.

Otherwise, again not considering b_2, there are, say, at least $n - 1$ many b_i's with the first component different from 0, hence of type I^σ, since we have excluded type IV^σ and we are dealing with elements having at least one type from I^σ - IV^σ. Such b_i's have thus a at the third component, hence b has a at the third component, by Remark 4.2, to the effect that the operation on the third coordinate is "$n-1$-majority disregarding the second argument". Thus b has type I^σ, hence b belongs to B.

Similarly, if there are at least $n-1$ many b_i's with the second component different from 0, then they are of type III^σ and then b, too, has type III^σ, thus $b \in B$. $\square$

Theorem 4.5. *For every $n \geq 3$, the variety $\mathcal{N}_{n\frac{1}{2}}$ introduced in Definition 4.1(d)(f) has an $n\frac{1}{2}$-near-unanimity term and is not $2n-4$-distributive.*

More generally, the congruence identity

(Eq. 4.4) $\qquad\qquad\qquad \alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ (\alpha(\gamma \circ \beta))^{n-3} \circ \alpha\gamma$

fails in $\mathcal{N}_{n\frac{1}{2}}$.

Proof. We first deal with the special case $n = 3$. The first paragraph in the proof of Theorem 3.4(4) shows that $\mathcal{N}_{3\frac{1}{2}}$ has a $3\frac{1}{2}$-near-unanimity term. On the other hand, $\mathcal{N}_{3\frac{1}{2}}$ is not 2-distributive. Indeed, in Definition 4.1(f) we have showed that $\mathcal{N}_{3\frac{1}{2}}$ is term-equivalent to the variety $\mathcal{I}_4^-$ from [27, Section 4, p. 15]. Moreover, [27, Proposition 4.4] shows that $\mathcal{I}_4^-$ is not 2-distributive, hence $\mathcal{N}_{3\frac{1}{2}}$ is not 2-distributive, since, by definition, 2-distributivity is preserved under term-equivalence. It follows that (Eq. 4.4) fails for $\mathcal{N}_{3\frac{1}{2}}$ when $n = 3$, since, by our conventions, if $n = 3$, then (Eq. 4.4) reads $\alpha(\beta\circ\gamma) \subseteq \alpha\beta\circ\alpha\gamma$, and this identity is equivalent to 2-distributivity, by the comment in Definition 2.2. We have proved the theorem in the case $n = 3$.

So let $n \geq 4$. The variety $\mathcal{N}_{n\frac{1}{2}}$ has an $n\frac{1}{2}$-near-unanimity term since the operation in each generating algebra is $n\frac{1}{2}$-near-unanimity. Indeed, as already mentioned in the proof of Theorem 3.4(4), the term introduced in (Eq. 3.4) and recalled in (Eq. 4.2) induces an $n\frac{1}{2}$-near-unanimity operation. Moreover, for $3 \leq j \leq \ell$, the operation on the algebras $\mathbf{N}^{j,m,+}$ introduced in Definition 4.1(b) is $n\frac{1}{2}$-near-unanimity. In fact, disregarding the second variable, the term $u_{j,m}^+$ from equation (Eq. 4.1) is a near-unanimity term, hence equations (Eq. 3.2) hold. Since $u_{j,m}^+$ does not depend on the second variable, (Eq. 3.1) holds, as well. Finally, (Eq. 3.3) holds in view of Remark 4.2 and since, again, $u_{j,m}^+$ does not depend on the second variable. Notice that the assumption that $3 \leq j \leq \ell$ is used only in order to get equation (Eq. 3.3).

We now show that the second statement of the theorem implies the failure of $2n-4$-distributivity. Indeed, since $\alpha\gamma \circ \alpha\beta \subseteq \alpha(\gamma \circ \beta)$, if equation (Eq. 4.4) fails, then also $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ \overset{2n-4}{\cdots} \circ \alpha\gamma$ fails. By the comment in Definition 2.2, this means exactly that $2n-4$-distributivity fails.

It remains to show that (Eq. 4.4) fails in $\mathcal{N}_{n\frac{1}{2}}$. Let $\mathbf{A}_3^+ \in \mathcal{N}_m^{3,+}$, $\mathbf{F} \subseteq \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$ and $a, d \in A_3^+$ be given by Lemma 4.3. Correspondingly, let $\mathbf{B}$ be the algebra given by Lemma 4.4. Then $\mathbf{B} \in \mathcal{N}_{n\frac{1}{2}}$, since $\mathbf{B}$ is a subalgebra of $\mathbf{A} \times \mathbf{A} \times \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$, $\mathbf{A}_3^+ \in \mathcal{N}_m^{3,+}$, $\mathcal{N}_m^{3,+}$ is a subvariety of $\mathcal{N}_{n\frac{1}{2}}$, $\mathbf{G}^{\mathrm{nu},n} \in \mathcal{N}_{n\frac{1}{2}}$ by construction and $\mathbf{A}$ belongs to the variety generated by $\mathbf{G}^{\mathrm{nu},n}$, hence $\mathbf{A} \in \mathcal{N}_{n\frac{1}{2}}$.

Define the following congruences on $\mathbf{A}$. The congruence β^* is the congruence induced by the partition $\{\{1, e\}, \{e', 0\}\}$ and γ^* is the congruence induced by the partition $\{\{1, e'\}, \{e, 0\}\}$. Since each of the above partitions induces a congruence on the four-element Boolean algebra, then each partition induces a congruence on any term reduct.

Since $\mathbf{B}$ is a subalgebra of $\mathbf{A} \times \mathbf{A} \times \mathbf{F}$, then every congruence on $\mathbf{A} \times \mathbf{A} \times \mathbf{F}$ induces a congruence on $\mathbf{B}$. Let β and γ be, respectively, the congruences induced on $\mathbf{B}$ by the congruences $\beta^* \times \beta^* \times \tilde{\beta}$ and $\gamma^* \times \gamma^* \times \tilde{\gamma}$ on $\mathbf{A} \times \mathbf{A} \times \mathbf{F}$, where $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are given by Lemma 4.3. Similarly, let α be induced by $1 \times 1 \times \tilde{\alpha}$. Here, as in Lemma 4.4, we are identifying a triple containing a subpair with a quadruple.

Let a, d and c be given by Lemma 4.3 and consider the elements $\bar{a} = (1, 0, a, 1)$ and $\bar{d} = (0, 1, d, 1)$ in B of types, respectively, I^σ and III^σ. Clearly, $\bar{a} \; \alpha \; \bar{d}$, recalling from Lemma 4.3 that $(a, 1) \; \tilde{\alpha} \; (d, 1)$. The element $\bar{c} = (e, e', c, 0)$ of type IV^σ witnesses that $(\bar{a}, \bar{d}) \in \alpha(\beta \circ \gamma)$, since $\bar{a} \; \beta \; \bar{c} \; \gamma \; \bar{d}$. We are going to show that $(\bar{a}, \bar{d}) \notin \alpha\beta \circ (\alpha(\gamma \circ \beta))^{n-3} \circ \alpha\gamma$, hence (Eq. 4.4) fails in an algebra in $\mathcal{N}_{n\frac{1}{2}}$.

Suppose by contradiction that $(\bar{a}, \bar{d}) \in \alpha\beta \circ (\alpha(\gamma \circ \beta))^{n-3} \circ \alpha\gamma$, hence there are elements $\bar{g}, \bar{h} \in B$ such that $\bar{a}\ \alpha\beta\ \bar{g}$, $(\bar{g}, \bar{h}) \in (\alpha(\gamma \circ \beta))^{n-3}$ and $\bar{h}\ \alpha\gamma\ \bar{d}$. By β-equivalence, the first component of $\bar{g}$ is either 1 or e, in any case, not 0. By α-equivalence, the last component of $\bar{g}$ is 1, since $\tilde{\alpha}$ is induced by 1×0 on $\mathbf{F} \subseteq \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu}, n}$, by Lemma 4.3. Since $\bar{g}$ must be in B, then $\bar{g}$ has type I^σ, hence its third component is a. Symmetrically, the second component of $\bar{h}$ is not 0, the last component of $\bar{h}$ is 1, $\bar{h}$ has type III^σ and its third component is d. From $(\bar{g}, \bar{h}) \in (\alpha(\gamma \circ \beta))^{n-3}$ we then get $((a, 1), (d, 1)) \in (\tilde{\alpha}(\tilde{\gamma} \circ \tilde{\beta}))^{n-3}$, contradicting Lemma 4.3 (recall that $m = n + 1$). $\qquad\square$

Remark 4.6. As a corollary of Theorem 4.5, we get another proof of Theorem 3.4(4). Indeed, $\mathcal{N}_{n\frac{1}{2}}$ is locally finite, being the join of locally finite varieties, $\mathcal{N}_{n\frac{1}{2}}$ has an $n\frac{1}{2}$-near-unanimity term, but $\mathcal{N}_{n\frac{1}{2}}$ has not an n-near-unanimity term, otherwise it would be $2n-4$-distributive, by Theorem 2.3. This contradicts Theorem 4.5.

5. Modularity levels and more identities

Proposition 5.1. *If $n \geq 3$, then every variety $\mathcal{V}$ with an $n\frac{1}{2}$-near-unanimity term is $2n-2$-modular.*

Proof. If u is an $n\frac{1}{2}$-near-unanimity term, define

$$t_0(x, y, w, z) = x,$$
$$t_1(x, y, w, z) = u(x, x, x, x, x, x, \ldots, x, x, x, y, z),$$
$$t_2(x, y, w, z) = u(x, x, x, x, x, x, \ldots, x, x, x, w, z),$$
$$t_3(x, y, w, z) = u(x, x, x, x, x, x, \ldots, x, x, y, z, z),$$
$$t_4(x, y, w, z) = u(x, x, x, x, x, x, \ldots, x, x, w, z, z),$$
$$t_5(x, y, w, z) = u(x, x, x, x, x, x, \ldots, x, y, z, z, z),$$

(Eq. 5.1) $\qquad\qquad\qquad\qquad \cdots$

$$t_{2n-7}(x, y, w, z) = u(x, x, x, x, y, z, \ldots, z, z, z, z, z),$$
$$t_{2n-6}(x, y, w, z) = u(x, x, x, x, w, z, \ldots, z, z, z, z, z),$$
$$t_{2n-5}(x, y, w, z) = u(x, x, x, y, z, z, \ldots, z, z, z, z, z),$$
$$t_{2n-4}(x, y, w, z) = u(x, x, x, w, z, z, \ldots, z, z, z, z, z),$$
$$t_{2n-3}(x, y, w, z) = u(y, w, z, z, z, z, \ldots, z, z, z, z, z),$$
$$t_{2n-2}(x, y, w, z) = z.$$

The terms $t_0, \ldots, t_{2n-2}$ satisfy the equations in Theorem 2.4(2), hence $\mathcal{V}$ is $2n-2$-modular. $\qquad\square$

Except for the penultimate line, the construction of the terms in (Eq. 5.1) is identical with [33, Theorem 3.19]. From another point of view, the proof of Proposition 5.1 exploits the fact mentioned in Remark 3.3(b) that a variety with an $n\frac{1}{2}$-near-unanimity term has directed Gumm terms; then classical arguments from [12, 15, 25] can be used in order to get Day terms from a sequence of ternary terms.

Recall that item (1) in Theorem 1.1 has been proved at the end of Section 3. We can now complete the proof.

Proof of Theorem 1.1 (continued). (2) When h is an integer, the first statement in (2) is Mitschke's Theorem [32, Theorem 2], reported here in Theorem 2.3. When h

is a half-integer, the first statement in (2) is Proposition 3.2. Indeed, in the latter case $n = h - \frac{1}{2}$, hence $2n - 3 = 2h - 4$.

If h is an integer, the last statement in (2) is Theorem 3.6(1) in [27]. If h is a half-integer, then a counterexample is the variety $\mathcal{N}_{n\frac{1}{2}}$ introduced in Definition 4.1(d)(f), as shown in Theorem 4.5.

(3) When h is an integer the first part follows from [33, Theorem 3.19]. When h is a half-integer it follows from Proposition 5.1.

As for the second part, if h is an integer, this follows from [27, Theorem 3.6(3)].

$\square$ to be continued

In order to deal with the remaining case, we need the 3-dimensional analogue of Lemma 4.3.

Lemma 5.2. *Let $n \geq 4$ and $m = n + 1$. Then there are an algebra $\mathbf{A}_3^+ \in \mathcal{N}_m^{3,+}$ and a subalgebra $\mathbf{F}$ of $\mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$ such that the congruence identity*

(Eq. 5.2) $$\tilde{\alpha}(\tilde{\beta} \circ \tilde{\alpha}\tilde{\gamma} \circ \tilde{\beta}) \subseteq (\tilde{\alpha}(\tilde{\gamma} \circ \tilde{\beta} \circ \tilde{\gamma}))^{m-4}$$

fails in $\mathbf{F}$.

Moreover, the failure of (Eq. 5.2) *can be witnessed by elements $(a,1)$, $(d,1)$, $(c_1,0)$, $(c_2,0)$ and congruences $\tilde{\alpha}$, induced by 1×0, $\tilde{\beta}$, $\tilde{\gamma}$ of $\mathbf{F}$ such that $(a,1)$ $\tilde{\alpha}$ $(d,1)$, $(a,1)$ $\tilde{\beta}$ $(c_1,0)$ $\tilde{\alpha}\tilde{\gamma}$ $(c_2,0)$ $\tilde{\beta}$ $(d,1)$ and $((a,1),(d,1)) \notin (\tilde{\alpha}(\tilde{\gamma} \circ \tilde{\beta} \circ \tilde{\gamma}))^{m-4}$.*

Proof. This corresponds to the case $j = 3$ and $q = 3$ in the Claim in [27] and is proved using the remarks in the proof of Lemma 4.3 here. Notice that we have not indicated the dependency on q in [27], thus the algebra $\mathbf{A}_3^+$ here is not the same algebra as in Lemma 4.3. $\square$

Proof of Theorem 1.1 (continued). The remaining case to be proved is the last statement in (3) when $h > 4$ is a half integer. So let $h = n\frac{1}{2}$, $n \geq 4$ and $m = n + 1 = h + \frac{1}{2}$. Consider the algebras and elements given by Lemma 5.2. Since (Eq. 5.2) fails in $\mathbf{F}$, then (Eq. 5.2) fails when we consider $\tilde{\alpha}\tilde{\gamma}$ in place of $\tilde{\gamma}$. Since $\tilde{\alpha}(\tilde{\alpha}\tilde{\gamma} \circ \tilde{\beta} \circ \tilde{\alpha}\tilde{\gamma}) = \tilde{\alpha}\tilde{\gamma} \circ \tilde{\alpha}\tilde{\beta} \circ \tilde{\alpha}\tilde{\gamma}$, we get that

(Eq. 5.3) $$\tilde{\alpha}(\tilde{\beta} \circ \tilde{\alpha}\tilde{\gamma} \circ \tilde{\beta}) \subseteq \tilde{\alpha}\tilde{\gamma} \circ \tilde{\alpha}\tilde{\beta} \circ \overset{2m-7}{\dots} \circ \tilde{\alpha}\tilde{\gamma}$$

fails in $\mathbf{F}$. We have used the fact that $m - 5$ pairs of factors of the form $\tilde{\alpha}\tilde{\gamma}$ mutually absorb, when computing $(\tilde{\alpha}\tilde{\gamma} \circ \tilde{\alpha}\tilde{\beta} \circ \tilde{\alpha}\tilde{\gamma})^{m-4}$.

Now the proof of Theorem 4.5 carries over with no essential modification in order to show that

(Eq. 5.4) $$\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{2m-7}{\dots} \circ \alpha\gamma) \circ \alpha\beta, \quad \text{that is,}$$

(Eq. 5.5) $$\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ \alpha\gamma \circ \overset{2m-5}{\dots} \circ \alpha\beta$$

fail in $\mathcal{N}_{n\frac{1}{2}}$.

In more detail, let $\mathbf{A}_3^+ \in \mathcal{N}_m^{3,+}$, $\mathbf{F} \subseteq \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$ and $a, d \in A_3^+$ be given by Lemma 5.2 and let $\mathbf{B}$ be the subalgebra of $\mathbf{A} \times \mathbf{A} \times \mathbf{A}_3^+ \times \mathbf{G}^{\mathrm{nu},n}$ given by Lemma 4.4. Let α, β, γ, $\bar{a} = (1, 0, a, 1)$ and $\bar{d} = (0, 1, d, 1)$ be defined as in the proof of 4.5. The elements $\bar{c}_1 = (e, 0, c_1, 0)$ and $\bar{c}_2 = (0, e, c_2, 0)$ witness that $(\bar{a}, \bar{d}) \in \alpha(\beta \circ \alpha\gamma \circ \beta)$. We claim that, on the other hand, $(\bar{a}, \bar{d})$ does not belong to the right-hand side of (Eq. 5.4). In comparison with the proof of Theorem 4.5, here we need to add a factor of the form $\alpha\beta$, rather than $\alpha\gamma$, at the right outer edge of (Eq. 5.3), as in (Eq. 5.4), compare (Eq. 4.4). This involves no change in the proof. Indeed, since

$\bar{d} = (0, 1, d, 1)$, if $\bar{h}\ \alpha\beta\ \bar{d}$, then the second component of $\bar{h}$ is distinct from 0 by β-equivalence. The last component of $\bar{h}$ is 1 by α-equivalence, hence $\bar{h}$ has type III^σ, thus it has d as the third component. All the rest is identical to the proof of 4.5.

Since $m = h + \frac{1}{2}$, then $2m - 5 = 2h - 4$, hence the failure of (Eq. 5.5) shows that $2h-4$-modularity fails in $\mathcal{N}_{n\frac{1}{2}}$, in virtue of the comment after Theorem 2.4. $\qquad\square$

Remark 5.3. The above arguments together with the proof of Theorem 3.6(4) in [27] show that if $n \geq 4$, then the following congruence identity fails in $\mathcal{N}_{n\frac{1}{2}}$

(Eq. 5.6) $\quad \alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{.\,.\,.} \circ \alpha\beta^\bullet) \circ \gamma^\bullet) \subseteq \alpha\beta \circ (\alpha(\gamma \circ \beta \circ \overset{q}{.\,.} \circ \beta^\bullet))^{n-3} \circ \alpha\gamma^\bullet,$

for every $q \geq 2$, where $\beta^\bullet = \beta$, $\gamma^\bullet = \gamma$ if q is even and $\beta^\bullet = \gamma$, $\gamma^\bullet = \beta$ if q is odd.

6. Some variations

Remark 6.1. The notion of an $n\frac{1}{2}$-near-unanimity term makes sense for $n = 2$; in this case we get a $\frac{5}{2}$-near-unanimity term. Such a term is required to satisfy

(Eq. 6.1)
$$u(z, z, x, x) = x, \quad u(x, z, x, x) = x, \quad u(x, x, z, x) = x,$$
$$u(x, x, x, z) = x, \quad u(x, z, z, z) = x,$$

where the last equation follows from the fact that in the case $n = 2$ (Eq. 3.3) reads $u(x, x, x, z) = u(x, z, z, z)$.

We shall show that the existence of a $\frac{5}{2}$-near-unanimity term is equivalent to the existence of a Pixley term. Indeed, if we let $t(x, y, z) = u(x, y, z, z)$, then t is a Pixley term witnessing arithmeticity. Notice that we do not need the third and fourth equations in (Eq. 6.1). Conversely, if t is a Pixley term, then $u(x, y, z, w) = t(x, t(y, z, w), w)$ is a $\frac{5}{2}$-near-unanimity term. In this respect, the proof of Theorem 3.4(1) generalizes the fact that if t is a Pixley term, then $s(x, z, w) = t(x, t(x, z, w), w)$ is a majority term.

The existence of a 2-near-unanimity term might be interpreted as a condition implying that we are in a trivial variety; in this sense, a $\frac{5}{2}$-near-unanimity term is a condition strictly between a 2-near-unanimity term and a 3-near-unanimity term, extending Theorem 3.4. On the other hand, a $\frac{5}{2}$-near-unanimity term, equivalently, a Pixley term, does not imply $2n - 4 = 1$-distributivity (= being a trivial variety), hence the assumption $n \geq 3$ is necessary in Proposition 3.2.

Remark 6.2. The term u defined by equation (Eq. 3.4) for $n \geq 3$ in the proof of Theorem 3.4(4) satisfies many more equations in Boolean algebras, besides the equations (Eq. 3.1)-(Eq. 3.3) defining an $n\frac{1}{2}$-near-unanimity term. For example, u satisfies

(Eq. 6.2) $\quad u(x, z, x, \ldots, x, \underset{i}{y}, x, \ldots, x) = x,$

(Eq. 6.3) $\quad u(x, y, x_3, x_4, \ldots, \underset{i}{x}, \ldots, x_{n+2}) = u(x, z, x_3, x_4, \ldots, \underset{i}{x}, \ldots, x_{n+2}),$

(Eq. 6.4) $\quad u(x, x, z, z, \ldots, z, \underset{i}{x}, z, \ldots, z) = u(x, z, z, z, \ldots, z, z, z, \ldots, z),$

(Eq. 6.5) $\quad u(x_1, x_2, x_3, x_4, \ldots, x_{n+2}) = u(x_1, x_2, x_{\tau(3)}, x_{\tau(4)}, \ldots, x_{\tau(n+2)}),$

where in (Eq. 6.2) - (Eq. 6.4) i varies with $3 \leq i \leq n + 2$ and in (Eq. 6.5) τ is a permutation of the set $\{3, 4, \ldots, n + 2\}$.

Corollary 6.3. *Theorems 1.1 and 3.4 hold if in the definition of an $n\frac{1}{2}$-near-unanimity term we add some or all of the equations* (Eq. 6.2) - (Eq. 6.4).

Theorem 4.5 holds if in the definition of an $n\frac{1}{2}$-near-unanimity term we add some or all of the equations (Eq. 6.2) - (Eq. 6.5).

If some variety $\mathcal{V}$ has a symmetric n-near-unanimity term, then $\mathcal{V}$ has an $n\frac{1}{2}$-near-unanimity term satisfying (Eq. 6.2) - (Eq. 6.5).

Proof. Since Clauses (1) and (2) in Theorem 3.4 hold for $n\frac{1}{2}$-near-unanimity terms, these clauses still hold if we replace "$n\frac{1}{2}$-near-unanimity term" with a stronger notion. The argument in the proof of 3.4(3) carries over for (Eq. 6.2) - (Eq. 6.4), since the first two variables are dummy. The term u defined by equation (Eq. 3.4) satisfies (Eq. 6.2) - (Eq. 6.5), hence the proof of 3.4(4) carries over, too. Thus Theorem 3.4 holds for the stronger notions when we add some equations from (Eq. 6.2) - (Eq. 6.4).

In order to prove that Theorem 4.5 holds in the generalized setting, we just check that (Eq. 6.2) - (Eq. 6.5) are satisfied by the operation of $\mathcal{N}_{n\frac{1}{2}}$. It suffices to show that this holds for all the algebras generating $\mathcal{N}_{n\frac{1}{2}}$. We have just mentioned that the operation in the algebra $\mathbf{G}^{\mathrm{nu},n}$ introduced in Definition 4.1(c) satisfies (Eq. 6.2) - (Eq. 6.5). The term $u_{j,m}^{+}$ introduced in Definition 4.1 does not depend on the second variable, hence satisfies (Eq. 6.3). Moreover, $u_{j,m}^{+}$ is near-unanimity with regard to the remaining variables, hence it satisfies (Eq. 6.2). It satisfies (Eq. 6.5) since it is symmetric, disregarding the second variable. If $m \geq 5$ and $m \geq j + 2$, then $u_{j,m}^{+}$ satisfies (Eq. 6.4), since both members evaluate as z, by Remark 4.2. In conclusion, all the algebras generating $\mathcal{N}_{n\frac{1}{2}}$ satisfy (Eq. 6.2) - (Eq. 6.5), hence such equations hold in $\mathcal{N}_{n\frac{1}{2}}$. This proves the generalized version of Theorem 4.5, then the general version of Theorem 1.1 follows, noticing that in the final part of the proof of 1.1(3) we use again the variety $\mathcal{N}_{n\frac{1}{2}}$.

Finally, if w is a symmetric n-near-unanimity term, then, adding two dummy variables at the first two places, equation (Eq. 6.5) holds. Equations (Eq. 6.2) - (Eq. 6.4) have been already taken care of. Thus the last statement holds. □

In particular, the equations (Eq. 3.1) - (Eq. 3.3), (Eq. 6.2) - (Eq. 6.5) together neither imply $2n-4$-distributivity, nor imply the existence of an n-near-unanimity term.

7. Some problems

We are not claiming that the problems below are difficult.

Problem 7.1. Study the notion of a *weak $n\frac{1}{2}$-near-unanimity term*, that is, an idempotent $n+2$-ary term satisfying (Eq. 3.3) from Definition 3.1, as well as

$$u(z, z, x, x, \ldots, x) = u(x, \ldots, x, \underset{i}{z}, x, \ldots, x), \qquad \text{for } 2 \leq i \leq n + 2.$$

Compare [29]. One might possibly add some equations of the form (Eq. 6.3) - (Eq. 6.5) and

$$u(z, z, x, x, \ldots, x) = u(x, z, x \ldots, x, \underset{i}{y}, x, \ldots, x), \qquad \text{for } 3 \leq i \leq n + 2$$

(compare (Eq. 6.2)).

Edge terms are an important generalization of near-unanimity terms and, possibly in equivalent formulations, play a chief role in many computational problems, e. g. [1, 2, 5, 7, 8, 16, 21, 22, 23, 24] and further references in the quoted papers. A variety with an edge term is congruence modular [7, Theorem 4.2]; moreover, for $k \geq 3$, a congruence distributive variety $\mathcal{V}$ has a k-edge term if and only if $\mathcal{V}$ has a k-near-unanimity term [7, Theorem 4.4]. Henceforth the following problem suggests itself naturally.

Problem 7.2. Is there a notion strictly between a k- and a $k+1$-edge term?

A variety with a k-edge term is $2k-3$-modular: just forget about the last term in the proof of [7, Theorem 4.2], or [26, Proposition 5.3]. Hence a candidate for a "$k\frac{1}{2}$-edge term" should imply $2k-2$-modularity but not $2k-3$-modularity.

In connection with Problem 7.2, it is probably interesting to study the following notion weaker than an $n\frac{1}{2}$-near-unanimity term.

Definition 7.3. For $n \geq 1$, a *skew-edge term* is an $n+2$-ary term satisfying equations (Eq. 3.1), (Eq. 3.3) from Definition 3.1, as well as equations (Eq. 3.2) for $4 \leq i \leq n+2$.

An equivalent characterization of varieties with a skew-edge term is given in Remark 7.8 below. A skew-edge term implies congruence modularity, but does not imply congruence distributivity.

Proposition 7.4. *If $n \geq 3$, then every variety with an $n+2$-ary skew-edge term is $2n-2$-modular.*

Every congruence permutable variety has a skew-edge term.

Proof. Only the equations satisfied by a skew-edge term are used in the proof of Proposition 5.1. Compare also Remark 3.3(a)(b).

A 3-ary term is a Maltsev term if and only if it is a skew-edge term. In any case, adding dummy variables after the third, we get a skew-edge term of arbitrary arity. $\square$

Problem 7.5. Is there a variety $\mathcal{V}$ with a $3\frac{1}{2}$-near-unanimity term and such that $\mathcal{V}$ is not 3-modular?

Problem 7.6. Is there a notion equivalent (for varieties) to the existence of an $n\frac{1}{2}$-near-unanimity term and whose definition involves a term (or, anyway, a set of terms) of smaller arity? (that is, of arity $< n+2$)

Is there a notion strictly between an n-near-unanimity term and an $n+1$-near-unanimity term, satisfying the analogue of Theorem 1.1 and whose definition involves a single term of arity $n+1$?

Remark 7.7. For $n \geq 3$, the condition

($\Diamond_n$) there is an $n+1$-ary near-unanimity term $+$ $2n-3$-distributivity

involves only terms of arity $\leq n+1$. The condition $\Diamond_n$ follows from the existence of an $n\frac{1}{2}$-near-unanimity term, by Proposition 3.2 and Theorem 3.4(1). Thus, by Theorem 3.4(4), $\Diamond_n$ does not imply the existence of an n-near-unanimity term. The condition $\Diamond_n$ does not imply $2n-4$-distributivity, by Theorem 4.5.

Is there a $2n-3$-distributive variety with an $n+1$-near-unanimity term but without an $n\frac{1}{2}$-near-unanimity term?

A 3-distributive variety with an $n+1$-near-unanimity term but without an n-near-unanimity term has been constructed in [27, Proposition 4.4].

Remark 7.8. For $n \geq 3$ and $\mathcal{V}$ a variety, the following conditions are equivalent.
  (i) $\mathcal{V}$ has an $n+2$-ary term u satisfying (Eq. 3.1), (Eq. 3.3) and (Eq. 3.2) for $[i = 2$ and] $4 \leq i \leq n + 2$,
  (ii) $\mathcal{V}$ has an n-ary term v and a ternary term t such that

$$v(x, \ldots, x, \underset{i}{z}, x, \ldots, x) = x, \qquad \text{for } 2 \leq i \leq n,$$

(Eq. 7.1) $\qquad\qquad v(x, z, z, z, \ldots, z, z) = t(x, z, z)$

$$t(x, x, z) = z, \qquad [t(x, y, x) = x]$$

Indeed, if u satisfies (i), then $v(x_1, x_2, x_3, \ldots) = u(x_1, x_1, x_1, x_2, x_3, \ldots)$ and $t(x, y, z) = u(x, y, z, z, z, \ldots)$ satisfy (Eq. 7.1). Conversely, if v and t are given by (Eq. 7.1), then $u(x_1, x_2, x_3, x_4, \ldots) = t(x_1, x_2, v(x_3, x_4, \ldots))$ satisfies (i).

As we mentioned in Remark 3.3(a), the case $i = 3$ in equation (Eq. 3.2) is not necessary in order to get $2n-3$-distributivity. As we mentioned in the proof of Proposition 7.4, the cases $i = 2$ and $i = 3$ in (Eq. 3.2) are not necessary in order to get $2n-2$-modularity. However, the case $i = 3$ seems necessary in order to get that the existence of an $n\frac{1}{2}$-near-unanimity term implies the existence of an $n+1$-near-unanimity term.

In [9] Campanella, Conley, Valeriote proved that if $\mathcal{V}$ and $\mathcal{W}$ are idempotent varieties of the same type and with, respectively, an n-near-unanimity term and an m-near-unanimity term, then both the join and the Maltsev product [13] of $\mathcal{V}$ and $\mathcal{W}$ have an $n+m-1$-near-unanimity term. Moreover, they show by a counterexample that the result is the best possible.

Problem 7.9. Do the results from [9] hold also when one or both n and m are half-integers? Possibly, one needs to modify the definition of a $n\frac{1}{2}$-near-unanimity term as in Remark 6.2 and Corollary 6.3.

Remark 7.10. Let $n \geq 3$ and $\mathcal{V}$ be the variety with a single n-ary operation u satisfying the near-unanimity equations and no other equation (except, of course, for those equations which logically follow from the near-unanimity rule). We claim that $\mathcal{V}$ has no symmetric near-unanimity term, actually, no symmetric term of arity ≥ 2.

Indeed, every term in $\mathcal{V}$ has a normal form, obtained by applying the near-unanimity rule whenever possible (in particular, this shows that $\mathcal{V}$ is not locally finite). Suppose by contradiction that $\mathcal{V}$ has an m-ary symmetric term $s(x_1, \ldots, x_m)$ for some $m \geq 2$ and choose such an s in normal form of minimal complexity. Since s is symmetric and $m \geq 2$, then s cannot be a variable, hence s is written as

$$s(x_1, \ldots, x_m) = u(t_1(x_1, \ldots, x_m), \ldots, t_n(x_1, \ldots, x_m)),$$

for certain terms $t_1, t_2, \ldots, t_n$. Since s is symmetric, we have

$$u(t_1(x_1, \ldots, x_m), \ldots) = s(x_1, \ldots, x_m) = s(x_{\sigma 1}, \ldots, x_{\sigma m}) = u(t_1(x_{\sigma 1}, \ldots, x_{\sigma m}), \ldots)$$

for every permutation σ of $\{1, 2, \ldots, m\}$, hence

$$t_1(x_1, \ldots, x_m) = t_1(x_{\sigma 1}, \ldots, x_{\sigma m}),$$

for every σ, since we are dealing with normal forms. Thus t_1 is symmetric of complexity less than s, a contradiction.

Problem 7.11. Prove or disprove. A locally finite variety with a near-unanimity term has a symmetric near-unanimity term.

In this connection we just point out that if $\mathcal{V}$ is a variety (not necessarily locally finite) with an n-ary near-unanimity term u and with a symmetric $n!$-ary idempotent term t, then $\mathcal{V}$ has a symmetric n-ary near-unanimity term v. Just let $v(x_1, \ldots, x_n) = t(\ldots, u(x_{\tau(1)}, \ldots, x_{\tau(n)}), \ldots)$, where τ varies among all permutations of $\{1, \ldots, n\}$ and "different arguments of t are filled using different permutations".

References

[1] E. Aichinger, P. Mayr, Independence of algebras with edge term, Internat. J. Algebra Comput. **25**, 1145–1157 (2015)

[2] E. Aichinger, P. Mayr, R. McKenzie, On the number of finite algebraic structures, J. Eur. Math. Soc. (JEMS) **16**, 1673–1686 (2014)

[3] K. A. Baker, A. F. Pixley, Polynomial interpolation and the Chinese remainder theorem for algebraic systems, Math. Z. **143**, 165–174 (1975)

[4] L. Barto, Finitely related algebras in congruence distributive varieties have near unanimity terms, Canad. J. Math. **65**, 3–21 (2013)

[5] L. Barto, Finitely related algebras in congruence modular varieties have few subpowers, J. Eur. Math. Soc. (JEMS) **20**, 1439–1471 (2018)

[6] L. Barto, M. Kozik, Absorption in universal algebra and CSPV, in: The Constraint Satisfaction Problem: Complexity and Approximability, Dagstuhl Follow-Ups, 7 (Schloss Dagstuhl–Leibniz Zentrum für Informatik, Wadern, 2017), 45–77

[7] J. Berman, P. Idziak, P. Marković, R. McKenzie, M. Valeriote, R. Willard, Varieties with few subalgebras of powers, Trans. Amer. Math. Soc. **362**, 1445–1473 (2010)

[8] A. Bulatov, P. Mayr, Á. Szendrei, The subpower membership problem for finite algebras with cube terms, Log. Methods Comput. Sci. **15**, Paper No. 11, 48 (2019)

[9] M. Campanella, S. Conley, M. Valeriote, Preserving near unanimity terms under products, Algebra Universalis **76**, 293–300 (2016)

[10] V. Dalmau, Generalized majority-minority operations are tractable, Log. Methods Comput. Sci., **2**, 4:1, 14 (2006)

[11] B. A. Davey, L. Heindorf, R. McKenzie, Near unanimity: an obstacle to general duality theory, Algebra Universalis **33**, 428–439 (1995)

[12] A. Day, A characterization of modularity for congruence lattices of algebras, Canad. Math. Bull. **12**, 167–173 (1969)

[13] R. Freese, R. McKenzie, Maltsev families of varieties closed under join or Maltsev product, Algebra Universalis **77**, 29–50 (2017)

[14] R. Freese, M. A. Valeriote, On the complexity of some Maltsev conditions, Internat. J. Algebra Comput. **19**, 41–77 (2009)

[15] H.-P. Gumm, Geometrical methods in congruence modular algebras, Mem. Amer. Math. Soc. **45** (1983)

[16] P. Idziak, P. Marković, R. McKenzie, M. Valeriote, R. Willard, Tractability and learnability arising from algebras with few subpowers, SIAM J. Comput. **39**, 3023–3037 (2010)

[17] B. Jónsson, Algebras whose congruence lattices are distributive, Math. Scand. **21**, 110–121 (1967)

[18] K. Kaarli, A. F. Pixley, Polynomial Completeness in Algebraic Systems (Chapman & Hall/CRC, Boca Raton, FL, 2001)

[19] A. Kazda, M. Kozik, R. McKenzie, M. Moore, Absorption and directed Jónsson terms, in: J. Czelakowski (ed.), Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science, Outstanding Contributions to Logic **16**, Springer, Cham, 203–220 (2018)

[20] A. Kazda, M. Valeriote, Deciding some Maltsev conditions in finite idempotent algebras, J. Symb. Logic **85**, 539–562 (2020)

[21] A. Kazda, D. Zhuk, Existence of cube terms in finite algebras, Algebra Universalis **82**, Paper No. 11, 29 (2021)

[22] K. A. Kearnes, E. W. Kiss, Á. Szendrei, Growth rates of algebras, III: finite solvable algebras, Algebra Universalis **76**, 199–222 (2016)

[23] K. A. Kearnes, Á. Szendrei, Clones of algebras with parallelogram terms, Internat. J. Algebra Comput. **22**, 1250005, 30 (2012)

[24] K. A. Kearnes, Á. Szendrei, Cube term blockers without finiteness, Algebra Universalis **78**, 437–459 (2017)

[25] H. Lakser, W. Taylor, S. T. Tschantz, A new proof of Gumm's theorem, Algebra Universalis **20**, 115–122 (1985)

[26] P. Lipparini, The distributivity spectrum of Baker's variety, J. Aust. Math. Soc. **110**, 119–144 (2021)

[27] P. Lipparini, Mitschke's Theorem is sharp, Algebra Universalis **83**, 7, 1–20 (2022)

[28] M. Maróti, The existence of a near-unanimity term in a finite algebra is decidable, J. Symbolic Logic **74**, 1001–1014 (2009)

[29] M. Maróti, R. McKenzie, Existence theorems for weakly symmetric operations, Algebra Universalis **59**, 463–489 (2008)

[30] M. Maróti, L. Zádori, Reflexive digraphs with near unanimity polymorphisms, Discrete Math. **312**, 2316–2328 (2012)

[31] R. McKenzie, Monotone clones, residual smallness and congruence distributivity, Bull. Aust. Math. **41**, 283–300 (1990)

[32] A. Mitschke, Near unanimity identities and congruence distributivity in equational classes, Algebra Universalis **8**, 29–32 (1978)

[33] L. Sequeira, Near-unanimity is decomposable, Algebra Universalis **50**, 157–164 (2003)

[34] L. Zádori, Monotone Jónsson operations and near unanimity functions, Algebra Universalis **33**, 216–236 (1995)

Dipartimento di Matematica, Viale della Ricerca Intermedia, Università di Roma "Tor Vergata", I-00133 ROME ITALY

URL: http://www.mat.uniroma2.it/~lipparin