On the Bahri-Lions conjecture for elliptic equations with non-symmetric nonlinearities

Riccardo Molle
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica n. 1, 00133 Roma, Italy
molle@mat.uniroma2.it

Donato Passaseo
Dipartimento di Matematica "E. De Giorgi", Università di Lecce
P.O. Box 193, 73100 Lecce, Italy

August 5, 2022

Abstract

We prove the existence of infinitely many solutions for a class of elliptic Dirichlet problems with non-symmetric nonlinearities. In particular, this result gives a positive answer to a well known conjecture formulated by A. Bahri and P.L. Lions, at least when the domains are cubes of $\mathbb{R}^n$.
The proof is based on a minimization method which does not require the use of techniques of deformation from the symmetry. This method allows us to piece together solutions of Dirichlet problems in suitable subdomains, so we obtain infinitely many nodal solutions with a prescribed nodal structure.

MSC: 35J20, 58E05.

Keywords: nonlinear elliptic equations, multiplicity of solutions, critical point theory, non-symmetric problems.

1 Introduction

In this paper we are concerned with Dirichlet problems of the form

$$-\Delta u = |u|^{p-1}u + \psi \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with $n \geq 1$, $\psi \in L^2(\Omega)$, $p > 1$ and $p < \frac{n+2}{n-2}$ when $n \geq 3$. The solutions of problem (1.1) are the critical points of the energy functional $E_\psi : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$E_\psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx - \int_\Omega \psi u \, dx \tag{1.2}$$

where, under our assumptions, the exponent $p+1$ is less than the critical Sobolev exponent $2^* = \frac{2n}{n-2}$ for $n \geq 3$.

If $\psi \equiv 0$ in Ω, the functional E_ψ is even, so the equivariant Lusternik-Schnirelmann theory for $\mathbb{Z}_2$-symmetric sets may be applied and guarantees the existence of infinitely many solutions (see for instance [1, 3, 9, 18–20, 29, 30, 37, 39]).
A natural question, which goes back to the beginning of the eighties, is whether the infinite number of solutions still persists for $\psi \not\equiv 0$.
In particular, this question was raised to the attention by Rabinowitz in his monograph on minmax methods (see [39, Remark 10.58]). In [4] Bahri proved that, if $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, then there exists an open dense set of ψ in $L^2(\Omega)$ such that problem (1.1) admits infinitely many solutions. In [8] Bahri and Lions proved that, if $n \geq 3$ and $1 < p < \frac{n}{n-2}$, then problem (1.1) admits infinitely many solutions for every $\psi \in L^2(\Omega)$. These results suggest the following conjecture, proposed by Bahri and Lions in [8]: the multiplicity result obtained in [8] holds also under the more general assumption $1 < p < \frac{n+2}{n-2}$.
In the present paper we prove that, if the domain Ω is a cube of $\mathbb{R}^n$, then problem (1.1) has infinitely many solutions for every $\psi \in L^2(\Omega)$. Thus, for $n \geq 3$, our result shows that the Bahri-Lions conjecture is true at least when Ω is a cube of $\mathbb{R}^n$.
In order to show that the infinite number of solutions we have for $\psi \equiv 0$ persists under perturbations, a detailed analysis was originally carried on in [2, 3, 5–8, 26, 31, 32, 38, 41, 45] by Ambrosetti, Bahri, Beresticki, Ekeland, Ghoussoub, Krasnoselskyii, Lions, Marino, Prodi, Rabinowitz, Struwe and Tanaka by introducing new perturbation methods.
More recently, a new approach to tackle the break of symmetry in elliptic problems has been developed by Bolle, Chambers, Ghoussoub and Tehrani (see [10, 11, 17]). However, that approach (which works also for more general nonlinear problems) did not allow to solve the Bahri-Lions conjecture.
Related results can be found also in other, more recent, papers (see for example [40] and references therein).
In the present paper we develop a method introduced in [34] in order to construct infinitely many nodal solutions of problem (1.1), having a prescribed nodal structure.
The idea is to piece together the solutions of Dirichlet problems in suitable subdomains of Ω. A similar idea has been first used by Struwe in earlier papers (see [41–43] and references therein). We consider as nodal regions some subdomains of Ω that are deformations of cubes by suitable Lipschitz maps (so we obtain nodal solutions having a "check" nodal structure). Notice that Lipschitz conditions combined with the covering of $\mathbb{R}^n$ by cubes with vertices in $\mathbb{Z}^n$ have been also used in some recent papers by Rabinowitz and Byeon in order to construct solutions with a prescribed pattern for the Allen-Cahn model equation (see [14, 15] and references therein).
The main result of the present paper is stated in Theorem 2.1 (which is a direct consequence of Proposition 2.2) and says that if Ω is a cube of $\mathbb{R}^n$, $n \geq 1$, $p > 1$ and $p < \frac{n+2}{n-2}$ when $n > 2$, then for all $\psi \in L^2(\Omega)$ there exist infinitely many nodal solutions of problem (1.1), having as nodal structure suitable partitions of Ω in subdomains that are Lipschitz deformations of arbitrarily small cubes. More precisely, in Proposition 2.2 we prove that there exists $\bar{k} \in \mathbb{N}$ such that for all positive integer $k \geq \bar{k}$ there exist at least two solutions $u_k(x)$ and $v_k(x)$ of problem (1.1) such that the nodal regions of the functions $u_k\left(\frac{x}{k}\right)$ and $v_k\left(\frac{x}{k}\right)$, after translations, tend to the cube Ω as $k \to \infty$. Moreover, the number of nodal regions of u_k, v_k and their energy $E_\psi(u_k)$, $E_\psi(v_k)$ tend to infinity as $k \to \infty$, while the size of the nodal regions tends to zero.

Notice that, in dimension $n = 1$, the existence of infinitely many solutions for all ψ in $L^2(\Omega)$ follows from a result obtained by Ehrmann in [25] (see also [24, 28] for related results). However, the method used by Ehrmann relies on a shooting argument, typical of ordinary differential equations, combined with counting the oscillations of the solutions in the interval Ω. On the contrary, in the present paper we use a method which is more similar to the one introduced by Nehari in [35], that can be in a natural way extended to the case $n > 1$. In fact, Nehari's method was used by Coffman in [19, 20] and, independently, by Hempel in [29, 30] to study an analogous problem for partial differential equations.
More recently, Nehari's method has been used also by Conti, Terracini and Verzini to study optimal partition problems, existence of changing sign solutions etc. (see [21–23, 46]).
Let us remark that Nehari consider an odd differential operator (so the corresponding energy functional is even) and prove that for every positive integer k there exists a solution having exactly k nodal regions. On the contrary, as Ehrmann in [25], we find solutions with a large number of nodal regions. However, let us point out that our multiplicity result is sharp because, as we proved in [34, Proposition 3.5], ψ can be chosen in $L^2(\Omega)$ in such a way that problem (1.1) does not have solutions with a small number of nodal regions: more precisely, for every positive integer h there exists ψ_h in $L^2(\Omega)$ such that every solution of problem (1.1) with $\psi = \psi_h$ has at least h nodal regions.
Now, let us describe the method we use to prove our result. For every cube Ω of $\mathbb{R}^n$ and every positive integer k, let us consider the k^n cubic open subdomains $C_1^k, C_2^k, \dots, C_{k^n}^k$, having all the same size, such that $\overline{\Omega} = \cup_{i=1}^{k^n} \overline{C}_i^k$. So, these subdomains are pairwise disjoint and, for all $i \in \{1, \dots, k^n\}$ the cube kC_i^k is a translation of the cube Ω.
Moreover, for all $L \in]0, 1[$, let us consider the set D_L of all the deformations $T : \overline{\Omega} \to \overline{\Omega}$ such that T differs from the identity map in $\overline{\Omega}$ by a Lipschitz function with Lipschitz constant L, $T(\overline{\Omega}) = \overline{\Omega}$ and $T(F) = F$ for every face F of the cube $\overline{\Omega}$. Notice that, since $L \in]0, 1[$, every deformation $T \in D_L$ is a bilipschitz map in $\overline{\Omega}$. Then, for all T in D_L and k in $\mathbb{N}$, by using a Nehari type minmax argument in every subdomain $T(C_i^k)$ with $i \in \{1, \dots, k^n\}$, we construct two distinct nodal functions u_k^T and v_k^T in $H_0^1(\Omega)$ whose nodal regions are the subdomains $T(C_i^k)$, for $i = 1, \dots, k^n$, and such that, for k large enough, u_k^T and v_k^T satisfy equation (1.1) in each nodal region and are solutions of the Dirichlet problem (1.1) in Ω when, in addition, they satisfy a suitable stationary property. Moreover, the construction of u_k^T and v_k^T shows that v_k^T behaves as $-u_k^T$ when $k \to \infty$.
Now, for all $k \in \mathbb{N}$, we minimize the energy functional E_ψ in the set $\{u_k^T \ : \ T \in D_L\}$; moreover, we show that, if the minimum is achieved by a map T_k^L in D_{L_k} with $L_k \in]0, L[$, then the corresponding function $u_k^{T_k^L}$ satisfies the stationarity condition which allows us to conclude that it is a solution of problem (1.1) for k large enough.
Indeed, we show that there exists a sequence $(L_k)_k$ of positive numbers such that $\lim_{k\to\infty} L_k = 0$ and $T_k^L \in D_{L_k}$ $\forall k \in \mathbb{N}$, so $L_k \in]0, L[$ for k large enough and the solution $u_k = u_k^{T_k^L}$ satisfies all the assertions of Proposition 2.2 (in analogous way one can construct the solutions v_k that behaves as $-u_k$ when $k \to \infty$).
In particular, we obtain that T_k^L tends as $k \to \infty$ to the identity map in $\overline{\Omega}$ and that the rescaled nodal regions $kT_k^L(C_i^k)$, after translations, tend to the cube Ω as $k \to \infty$, uniformly with respect to $i \in \{1, \dots, k^n\}$.
The existence of such a sequence $(L_k)_k$, which plays a crucial role in the proof, is strictly

related to a minimality property of the cubes in $\mathbb{R}^n$. In fact, the functions $u_k^T\left(\frac{x}{k}\right)$, suitably rescaled, tend as $k \to \infty$ to solutions of the equation (1.1) with $\psi = 0$. Therefore, since the effect of the term ψ tends to vanish as $k \to \infty$, the rescaled nodal regions $kT_k^L(C_i^k)$, suitably translated, tend to polyhedra as $k \to \infty$. Among these polyhedra, the cubes of $\mathbb{R}^n$ are the unique minimizers of the "shape factor" $\varphi(\chi)$ defined by

$$\varphi(\chi) = m(\chi)|\chi|^{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)}, \tag{1.3}$$

where $|\chi|$ is the volume of χ and

$$m(\chi) = \min \left\{ \int_\chi |\nabla U|^2 dx \ : \ U \in H_0^1(\chi), \ \int_\chi |U|^{p+1} dx = 1 \right\} \tag{1.4}$$

(notice that $\varphi(\chi)$ depends only on the shape of χ and not on its size because it is invariant with respect to translations and rescaling of χ). Therefore, taking into account the asymptotic behaviour of $E_\psi(u_k^{T_k^L})$ as $k \to \infty$, the minimality of T_k^L implies that $\varphi(kT_k^L(C_i^k)) \to \overline{\varphi}$, where $\overline{\varphi}$ denotes the shape factor of every cube of $\mathbb{R}^n$, while the volumes $|kT_k^L(C_i^k)| \to 1$ as $k \to \infty$, uniformly with respect to $i \in \{1, \ldots, k^n\}$.
As a consequence, taking also into account the conditions of T_k^L on $\partial\Omega$, we infer that, after translations, the rescaled nodal regions $kT_k^L(C_i^k)$ tend to Ω as $k \to \infty$ and there exists a sequence $(L_k)_k$ having the desired properties.
It is clear that our method does not require techniques of deformations from the symmetry and may be applied to more general problems. For example, it may be easily adapted to deal with the case where in problem (1.1) the nonlinear term $|u|^{p-1}u$ is replaced by $c_+(u^+)^p - c_-(u^-)^p$ with c_+ and c_- two positive constants. Moreover, this method may be adapted to work even in case of nonlinear elliptic equations involving critical Sobolev exponents. For example, it allows us to obtain in this case a multiplicity result similar to Theorem 2.1, which is announced in Theorem 3.18.

2 Variational framework and statement of the main results

Our aim is to prove the following theorem.

Theorem 2.1. *Let Ω be a cube of $\mathbb{R}^n$ with $n \geq 1$, let $p > 1$ and $p < \frac{n+2}{n-2}$ when $n \geq 3$. Then, for every $\psi \in L^2(\Omega)$, problem* (1.1) *admits infinitely many solutions.*

Without any loss of generality, we can assume that

$$\Omega = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ 0 < x_i < 1 \ \text{ for } i = 1, \ldots, n\}. \tag{2.1}$$

For all positive integer k and for all $z \in \mathbb{Z}^n$, let us set

$$C_z^k = \frac{1}{k}(z + \Omega) \ \text{ and } \ \sigma(z) = (-1)^{\sum_{i=1}^n z_i} \tag{2.2}$$

(thus, in particular, we have $C_0^1 = \Omega$).

Notice that for all $k \in \mathbb{N}$ we have $C_z^k \subseteq \Omega$ if and only if $0 \le z_i \le k-1$ for $i = 1, \dots, n$; moreover, if we set

$$Z_k = \{z = (z_1, \dots, z_n) \in \mathbb{Z}^n \ : \ 0 \le z_i \le k-1 \ \text{ for } i = 1, \dots, n\}, \tag{2.3}$$

we have

$$\overline{\Omega} = \bigcup_{z \in Z_k} \overline{C}_z^k \ \text{ and } \ C_z^k \cap C_{z'}^k = \emptyset \ \text{ for } \ z \ne z' \ (z, z' \in Z_k). \tag{2.4}$$

Then, the following proposition holds (it obviously implies Theorem 2.1).

Proposition 2.2. *Under the assumptions of Theorem 2.1, if Ω is the cube* (2.1)*, for all $\psi \in L^2(\Omega)$ there exists $\bar{k} \in \mathbb{N}$ such that for every $k \ge \bar{k}$ problem* (1.1) *admits two solutions u_k and v_k having the following properties (here we consider u_k and v_k extended by the value zero in $\mathbb{R}^n \setminus \Omega$). For all $k \ge \bar{k}$ there exist two bilipschitz maps $T_{k,u}, T_{k,v} : \Omega \to \Omega$ (with Lipschitz constants independent of k) such that for every choice of z^k in Z_k the functions U_{z^k} and V_{z^k} defined by*

$$U_{z^k}(x) = \frac{\sigma(z^k)}{k^{\frac{2}{p-1}}} u_k \left[\frac{x}{k} + T_{k,u}\left(\frac{z^k}{k} \right) \right] \qquad \forall x \in \mathbb{R}^n, \ \forall k \ge \bar{k}, \tag{2.5}$$

$$V_{z^k}(x) = \frac{-\sigma(z^k)}{k^{\frac{2}{p-1}}} v_k \left[\frac{x}{k} + T_{k,v}\left(\frac{z^k}{k} \right) \right] \qquad \forall x \in \mathbb{R}^n, \ \forall k \ge \bar{k}, \tag{2.6}$$

restricted to Ω, both converge as $k \to \infty$ to a positive solution of problem (1.1) *with $\psi \equiv 0$ in Ω, satisfying*

$$E_0(U) = \min\{E_0(U) \ : \ U \in H_0^1(\Omega) \setminus \{0\}, \ E_0'(U)[U] = 0\}. \tag{2.7}$$

Moreover, the sequences $(T_{k,u})_k$ and $(T_{k,v})_k$ both converge to the identity map uniformly in Ω, while the domains $k \left[T_{k,u}\left(C_{z^k}^k\right) - T_{k,u}\left(\frac{z^k}{k}\right)\right]$ and $k \left[T_{k,v}\left(C_{z^k}^k\right) - T_{k,v}\left(\frac{z^k}{k}\right)\right]$ tend to Ω as $k \to \infty$ for every choice of z^k in Z_k.

The proof is reported in Section 3.

In order to prove Theorem 2.1 and Proposition 2.2, we proceed as follows. For every $t \in [0,1]$ and $i \in \{1, \dots, n\}$, let us consider the set

$$F_i^t = \{(x_1, \dots, x_n\} \in \Omega \ : \ x_i = t\} \tag{2.8}$$

(in particular, if $t = 0$ or $t = 1$, F_i^t is a face of the cube Ω).
Now, let us fix $L \in]0,1[$ and consider the set D_L of the admissible deformations of $\overline{\Omega}$ defined by

$$\begin{aligned} D_L = \{T : \overline{\Omega} \to \overline{\Omega} \ : T(\overline{\Omega}) = \overline{\Omega}, \ T(F_i^t) = F_i^t \ \text{ for } t = 0, 1, \ i = 1, \dots, n, \\ |T(x) - T(y) - x + y| \le L|x - y| \ \forall x, y \in \overline{\Omega}\}. \end{aligned} \tag{2.9}$$

Notice that for every deformation $T \in D_L$ one can write $T(x) = I(x) + S(x)$ where $I(x) = x$ $\forall x \in \overline{\Omega}$ and $S : \overline{\Omega} \to \mathbb{R}^n$ is a Lipschitz continuous function with Lipschitz constant L. Moreover, we have

$$(1 - L)|x - y| \le |T(x) - T(y)| \le (1 + L)|x - y| \qquad \forall x, y \in \overline{\Omega} \tag{2.10}$$

where $1 - L > 0$ because we assumed $L \in]0,1[$. Thus, T is invertible and both T and T^{-1} are Lipschitz continuous functions in $\overline{\Omega}$.
Other important consequences of the definition of D_L are presented in next proposition where we describe some geometrical properties of the deformations $T(F_i^t)$ of the sets F_i^t with respect to the straight lines orthogonal to F_i^t (these properties motivate the introduction of this class of admissible deformations).

Proposition 2.3. *Let $T \in D_L$ and $L \in]0,1[$. Then*

a) for all $t \in [0,1]$, $i \in \{1,\dots,n\}$ and $y \in \Omega$ there exists a unique $x \in F_i^t$ such that $P_i \circ T(x) = P_i(y)$, where P_i denotes the orthogonal projection of $\mathbb{R}^n$ on the subspace $\{x = (x_1,\dots,x_n) \in \mathbb{R}^n \ : \ x_i = 0\}$ (that is, every straight line orthogonal to F_i^t meets $T(F_i^t)$ in a unique point);

b) for all t', t'' in $[0,1]$ such that $t' < t''$ and for all $x' \in F_i^{t'}$ and $x'' \in F_i^{t''}$ such that $P_i \circ T(x') = P_i \circ T(x'')$, we have $T_i(x') < T_i(x'')$ (that is, the deformation $T(F_i^t)$ of the set F_i^t meets every straight line orthogonal to F_i^t in a unique point whose i^{th} coordinate increases as t increases).

Proof. In order to prove (a), first notice that, for all $t \in [0,1]$, $i \in \{1,\dots,n\}$ and $y \in \Omega$, there exists $x \in F_i^t$ such that $P_i \circ T(x) = P_i(y)$.
In fact, let us consider the function $P_i \circ T : F_i^t \to F_i^0$, which is a continuous function satisfying

$$P_i \circ T(F_i^t \cap F_j^0) \subseteq F_i^0 \cap F_j^0, \quad P_i \circ T(F_i^t \cap F_j^1) \subseteq F_i^0 \cap F_j^1 \qquad \forall j \in \{1,\dots,n\} \setminus \{i\}. \tag{2.11}$$

Therefore, since $P_i(y) \in F_i^0$, there exists $x \in F_i^t$ such that $P_i \circ T(x) = P_i(y)$ (as follows from [33]).
Now, let us prove that such a x is unique. Arguing by contradiction, assume that there exists another $\tilde{x}$ in F_i^t, $\tilde{x} \neq x$, such that $P_i \circ T(\tilde{x}) = P_i(y)$, which implies

$$[T(x) - T(\tilde{x})] \cdot (x - \tilde{x}) = 0. \tag{2.12}$$

Since $T \in D_L$ with $L \in]0,1[$, we infer that

$$|T(x) - T(\tilde{x}) + \tilde{x} - x| \leq L|x - \tilde{x}| \tag{2.13}$$

and, as a consequence,

$$|T(x) - T(\tilde{x}) + \tilde{x} - x|^2 < |x - \tilde{x}|^2 \tag{2.14}$$

because $x \neq \tilde{x}$. On the other hand, from (2.12) we obtain

$$\begin{aligned} |T(x) - T(\tilde{x}) + \tilde{x} - x|^2 &= |T(x) - T(\tilde{x})|^2 + |\tilde{x} - x|^2 - 2[T(x) - T(\tilde{x})] \cdot (x - \tilde{x}) \\ &= |T(x) - T(\tilde{x})|^2 + |\tilde{x} - x|^2 \\ &> |\tilde{x} - x|^2 \end{aligned} \tag{2.15}$$

in contradiction with (2.14).
Thus, (a) is completely proved.

In order to prove (b), we argue again by contradiction and assume that there exist t', t'' in $[0,1]$ such that $t' < t''$ and $x' \in F_i^{t'}$, $x'' \in F_i^{t''}$ such that

$$P_i \circ T(x') = P_i \circ T(x'') \quad \text{and} \quad T_i(x') \geq T_i(x''). \tag{2.16}$$

Notice that (2.16) implies

$$[T(x') - T(x'')] \cdot (x' - x'') \leq 0. \tag{2.17}$$

Therefore, we obtain

$$\begin{aligned} |T(x') - T(x'') + x'' - x'|^2 &= |T(x') - T(x'')|^2 + |x' - x''|^2 - 2[T(x') - T(x'')] \cdot (x' - x'') \\ &\geq |T(x') - T(x'')|^2 + |x' - x''|^2 \\ &\geq |x' - x''|^2. \end{aligned} \tag{2.18}$$

On the other hand, since $T \in D_L$ with $L \in]0,1[$ and $x' \neq x''$, we infer that

$$T(x') - T(x'') + |x'' - x'|^2 \leq L^2 |x' - x''|^2 < |x' - x''|^2 \tag{2.19}$$

in contradiction with (2.18).
Thus, we can conclude that, if $P_i \circ T(x') = P_i \circ T(x'')$ and $t' < t''$, then $T_i(x') < T_i(x'')$, so the proof is complete.

□

Now, we exploit the class of admissible deformations D_L in order to construct the solutions u_k and v_k. We first construct the solutions u_k (then one can proceed in a similar way to construct the solutions v_k). For all $k \in \mathbb{N}$, $z \in Z_k$ and $T \in D_L$ with $L \in]0,1[$, let us set

$$E_\psi(k, z, T) = \inf \left\{ E_\psi \ : \ u \in H_0^1(T(C_z^k)), \ \int_{T(C_z^k)} |u|^{p+1} dx = 1 \right\}. \tag{2.20}$$

Since $p < \frac{n+2}{n-2}$ when $n \geq 3$, one can easily verify that the infimum in (2.20) is achieved. Moreover, for all $k \in \mathbb{N}$ and $L \in]0,1[$, also the infimum

$$\inf\{E_\psi(k, z, T) \ : \ z \in Z_k, \ T \in D_L\} \tag{2.21}$$

is achieved, as one can prove by standard arguments using Ascoli-Arzelà Theorem.
For the construction of the functions u_k we need the following Lemmas.

Lemma 2.4. *For all $L \in]0,1[$ we have*

$$\lim_{k \to \infty} \min\{E_\psi(k, z, T) \ : \ z \in Z_k, \ T \in D_L\} = \infty \tag{2.22}$$

and there exists $k(L) \in \mathbb{N}$ such that, for all $k \geq k(L)$, $z \in Z_k$ and $T \in D_L$, the infimum

$$\inf \left\{ E_\psi(u) \ : \ u \in H_0^1(T(C_z^k)), \ \int_{T(C_z^k)} |u|^{p+1} dx < 1 \right\} \tag{2.23}$$

is achieved by a unique minimizing function $\tilde{u}_{k,z}^T$. Moreover, we have

$$\lim_{k \to \infty} \sup \left\{ \int_{T(C_z^k)} |\nabla \tilde{u}_{k,z}^T|^2 dx \ : \ z \in Z_k, \ T \in D_L \right\} = 0. \tag{2.24}$$

Proof. For all $k \in \mathbb{N}$, let us consider $z^k \in Z_k$ and $T_k \in D_L$ realizing the minimum (2.22) and $\bar{u}_k \in H^1_0(T(C^k_{z^k}))$ realizing the minimum $E_\psi(k, z^k, T_k)$.
Let us extend the function $\bar{u}_k$ in all of Ω by the value zero in $\Omega \setminus C^k_{z^k}$. Since $T_k \in D_L$ $\forall k \in \mathbb{N}$, taking into account the second inequality in (2.10), we obtain

$$\lim_{k\to\infty} \operatorname{meas} T_k(C^k_{z^k}) = 0, \tag{2.25}$$

so (up to a subsequence) $\bar{u}_k \to 0$ almost everywhere in Ω. It follows that

$$\lim_{k\to\infty} \int_\Omega |\nabla \bar{u}_k|^2 dx = \infty \tag{2.26}$$

otherwise, since $p < \frac{n+2}{n-2}$ for $n \geq 3$, $\bar{u}_k \to 0$ also in $L^{p+1}(\Omega)$, which is impossible because $\int_\Omega |\bar{u}_k|^{p+1} dx = 1$ $\forall k \in \mathbb{N}$. As a consequence, since

$$E_\psi(k, z_k, T^k) = \frac{1}{2} \int_\Omega |\nabla \bar{u}_k|^2 dx - \frac{1}{p+1} - \int_\Omega \bar{u}_k \psi \, dx, \tag{2.27}$$

we obtain (2.22).
Notice that (2.22) implies that for all $L \in]0,1[$ there exists $k(L) \in \mathbb{N}$ satisfying

$$0 < \min \left\{ E_\psi(u) \ : \ u \in H^1_0(T(C^k_z)), \ \int_{T(C^k_z)} |u|^{p+1} dx = 1 \right\} \quad \forall k \geq k(L), \ \forall z \in Z_k, \ \forall T \in D_L. \tag{2.28}$$

Since $E_\psi(0) = 0$, it follows by standard arguments that for all $k \geq k(L)$, $z \in Z_k$ and $T \in D_L$ there exists $\tilde{u}^T_{k,z} \in H^1_0(T(C^k_z))$ such that

$$E_\psi(\tilde{u}^T_{k,z}) = \min \left\{ E_\psi(u) \ : \ u \in H^1_0(T(C^k_z)), \ \int_{T(C^k_z)} |u|^{p+1} dx < 1 \right\}. \tag{2.29}$$

Taking into account that

$$E_\psi(\tilde{u}^T_{k,z}) \leq E_\psi(0) = 0 \quad \forall k \geq k(L), \ \forall z \in Z_k, \ \forall T \in D_L, \tag{2.30}$$

it follows that

$$\sup \left\{ \int_{T(C^k_z)} |\nabla \tilde{u}^T_{k,z}|^2 dx \ : \ k \geq k(L), \ z \in Z_k, \ T \in D_L \right\} < \infty. \tag{2.31}$$

In order to prove (2.24), we argue by contradiction and assume that for all $k \geq k(L)$ there exist $z^k \in Z_k$ and $T_k \in D_L$ such that

$$\liminf_{k\to\infty} \int_{T_k(C^k_{z^k})} |\nabla \tilde{u}^{T_k}_{k,z^k}|^2 dx > 0. \tag{2.32}$$

From (2.31) we infer that the sequence $(\tilde{u}^{T_k}_{k,z^k})_k$ (with $\tilde{u}^{T_k}_{k,z^k}$ extended by the value zero outside $T_k(C^k_{z^k})$) is bounded in $H^1_0(\Omega)$. Moreover, up to a subsequence, $\tilde{u}^{T_k}_{k,z^k} \to 0$ as $k \to \infty$ almost everywhere in Ω because $T_k \in D_L$, so $\text{meas}(T_k(C^k_{z^k})) \to 0$ as $k \to \infty$. Therefore, $\tilde{u}^{T_k}_{k,z^k} \to 0$ as $k \to \infty$ also in $L^{p+1}(\Omega)$. Then, from $E_\psi(\tilde{u}^{T_k}_{k,z^k}) \leq 0 \ \forall k \in \mathbb{N}$ it follows easily that

$$\limsup_{k\to\infty} \int_{T_k(C^k_{z^k})} |\nabla \tilde{u}^{T_k}_{k,z^k}|^2 dx = 0 \tag{2.33}$$

in contradiction with (2.32). Thus, we can conclude that (2.24) holds.
Finally, notice that the functional E_ψ is strictly convex in a suitable neighborhood of zero. Therefore, for k large enough, $\tilde{u}^{T_k}_{k,z^k}$ is the unique minimizing function for (2.23) for all $z \in Z_k$ and $T \in D_L$. So the proof is complete.

□

Lemma 2.5. *For all $k \geq k(L)$, $z \in Z_k$ and $T \in D_L$, there exists a function $u^T_{k,z}$ in $H^1_0(T(C^k_z))$ such that $u^T_{k,z} \not\equiv \tilde{u}^T_{k,z}$, $\sigma(z)[u^T_{k,z} - \tilde{u}^T_{k,z}] \geq 0$ in $T(C^k_z)$ and*

$$\begin{aligned} E_\psi(u^T_{k,z}) &= M_\psi(u^T_{k,z}) \\ &= \min\{M_\psi(u) \ : \ u \in H^1_0(T(C^k_z)),\ u \not\equiv \tilde{u}^T_{k,z},\ \sigma(z)[u - \tilde{u}^T_{k,z}] \geq 0 \ \ in\ T(C^k_z)\} \end{aligned} \tag{2.34}$$

where, for all $u \in H^1_0(T(C^k_z))$, $M_\psi(u)$ is defined by

$$M_\psi(u) = \max\{E_\psi(\tilde{u}^T_{k,z} + t(u - \tilde{u}^T_{k,z})) \ : \ t \geq 0\}. \tag{2.35}$$

Proof. First notice that the maximum in (2.35) is achieved for all $u \in H^1_0(T(C^k_z))$ because $p > 1$. Now, let us consider a sequence $(u_i)_i$ in $H^1_0(T(C^k_z))$ such that $u_i \not\equiv \tilde{u}^T_{k,z}$, $\sigma(z)[u_i - \tilde{u}^T_{k,z}] \geq 0$ in $T(C^k_z)$ $\forall i \in \mathbb{N}$ and

$$\lim_{i\to\infty} M_\psi(u_i) = \inf\{M_\psi(u) \ : \ u \in H^1_0(T(C^k_z)),\ u \not\equiv \tilde{u}^T_{k,z^k},\ \sigma(z)[u - \tilde{u}^T_{k,z}] \geq 0 \text{ in } T(C^k_z)\}. \tag{2.36}$$

Then, let us set $w_i = \|u_i - \tilde{u}^T_{k,z}\|^{-1}_{L^{p+1}}(u_i - \tilde{u}^T_{k,z})$ and notice that, obviously, $M_\psi(u_i) = M_\psi(\tilde{u}^T_{k,z} + w_i)$. Moreover notice that, since the sequence $(w_i)_i$ is bounded in L^{p+1}, (2.36) implies that it is bounded also in H^1_0. Since $p < \frac{n+2}{n-2}$ when $n \geq 3$, it follows that (up to a subsequence) $(w_i)_i$ converges weakly in H^1_0, in L^{p+1} and almost everywhere to a function $\hat{w} \in H^1_0(T(C^k_z))$. As a consequence, $\|\hat{w}\|_{L^{p+1}} = 1$ and $\sigma(z)\hat{w} \geq 0$ in $T(C^k_z)$. Indeed, $w_i \to \hat{w}$ as $i \to \infty$ strongly in $H^1_0(T(C^k_z))$. In fact, since we have the weak convergence, arguing by contradiction assume that $\|w_i\|^2_{H^1_0}$ does not converge to $\|\hat{w}\|^2_{H^1_0}$ as $i \to \infty$, that is

$$\int_{T(C^k_z)} |\nabla \hat{w}|^2 dx < \lim_{i\to\infty} \int_{T(C^k_z)} |\nabla w_i|^2 dx, \tag{2.37}$$

which, combined with the weak convergence, implies $M_\psi(\tilde{u} + \hat{w}) < \lim_{i\to\infty} M_\psi(\tilde{u} + w_i)$. Therefore, we obtain a contradiction because $\hat{w} \not\equiv 0$ and, as a consequence, $\lim_{i\to\infty} M_\psi(\tilde{u} + w_i) \leq M_\psi(\tilde{u} + \hat{w})$ because of (2.36). Thus, we can conclude that $w_i \to \hat{w}$ in $H^1_0(T(C^k_z))$ as $i \to \infty$, which imples $\lim_{i\to\infty} M_\psi(\tilde{u} + w_i) = M_\psi(\tilde{u} + \hat{w})$.
Moreover, since $p > 1$, there exists $\hat{t} > 0$ such that $E_\psi(\tilde{u}^T_{k,z} + \hat{t}\hat{w}) = M_\psi(\tilde{u}^T_{k,z} + \hat{t}\hat{w})$, so all the assertions in Lemma 2.5 hold with $u^T_{k,z} = \tilde{u}^T_{k,z} + \hat{t}\hat{w}$.

□

Remark 2.6. Notice that the function $u^T_{k,z}$ given by Lemma 2.5, for k large enough, satisfies $E_\psi(u^T_{k,z}) \geq E_\psi(k,z,T)$ because $u^T_{k,z} \not\equiv \tilde{u}^T_{k,z}$ in $T(C^k_z)$.
Thus, by (2.22) we get

$$\lim_{k\to\infty} \min\{E_\psi(u^T_{k,z}) \ : \ z \in Z_k, \ T \in D_L\} = \infty. \tag{2.38}$$

Now, we extend every function $u^T_{k,z}$ in all of Ω by the value zero outside $T(C^k_z)$ and we consider the function $u^T_k \in H^1_0(\Omega)$ defined by $u^T_k = \sum_{z\in Z_k} u^T_{k,z}$. Using Ascoli-Arzelà Theorem, one can verify that for all $k \geq k(L)$ there exists an admissible deformation $T^L_k \in D_L$ such that

$$E_\psi(u^{T^L_k}_k) = \min\{E_\psi(u^T_k) \ : \ T \in D_L\}. \tag{2.39}$$

In next section we show that $u^{T^L_k}_k$ is a solution of problem (1.1) for k large enough and that Proposition 2.2 holds with $u_k = u^{T^L_k}_k$ and $T_{k,u} = T^L_k$. In order to construct the solutions v_k, we proceed in analogous way. In fact, as in Lemma 2.5, for all $k \geq k(L)$, $z \in Z_k$ and $T \in D_L$, there exists also a function $v^T_{k,z}$ in $H^1_0(T(C^k_z))$ such that $v^T_{k,z} \not\equiv \tilde{u}^T_{k,z}$, $\sigma(z)[v^T_{k,z} - \tilde{u}^T_{k,z}] \leq 0$ in $T(C^k_z)$ and

$$E_\psi(v^T_{k,z}) = M_\psi(v^T_{k,z}) = \min\{M_\psi(v) \ : \ v \in H^1_0(T(C^k_z)), \ v \not\equiv \tilde{u}^T_{k,z}, \ \sigma(z)[v - \tilde{u}^T_{k,z}] \leq 0 \text{ in } T(C^k_z)\}. \tag{2.40}$$

Then, we set $v^T_k = \sum_{z\in Z_k} v^T_{k,z}$ (where $v^T_{k,z}$ is extended in Ω by the value zero outside $T(C^k_z)$) and, using Ascoli-Arzelà Theorem, we minimize $E_\psi(v^T_k)$ with respect to T in D_L. If $T_{k,v} \in D_L$ is a minimizing admissible deformation, the function $v^{T_{k,v}}_k$ is a solution of problem (1.1) for k large enough and Proposition 2.2 holds with $v_k = v^{T_{k,v}}_k$, as we show in next section.

3 Asymptotic estimates and proof of the main results

In this section we describe the asymptotic behaviour as $k \to \infty$ of the functions u_k and v_k, arising in Proposition 2.2, we constructed in Section 2. Then, we show that these functions are solutions of problem (1.1) for k large enough and satisfy all the assertions of Proposition 2.2.
As follows from Proposition 2.3, for all $T \in D_L$, $i \in \{1,\dots,n\}$ and $t \in [0,1]$, the set $T(F^t_i)$ is the graph of a function $f^{t,T}_i : F^0_i \to \mathbb{R}$ and

$$T(C^k_z) = \{x \in \Omega \ : \ f_i^{\frac{z_i}{k},T} \circ P_i(x) < x_i < f_i^{\frac{z_i+1}{k},T} \circ P_i(x) \text{ for } i = 1,\dots,n\} \quad \forall k \in \mathbb{N}, \ \forall z \in Z_k. \tag{3.1}$$

In next lemma we prove that $f^{t,T}_i$ is a Lipschitz continuous function.

Lemma 3.1. *If $T \in D_L$ with $L \in]0,1[$, then for all $i \in \{1,\dots,n\}$ and $t \in [0,1]$ we have*

$$|f^{t,T}_i(x) - f^{t,T}_i(y)| \leq \frac{L}{1-L}|x-y| \qquad \forall x, y \in F^0_i. \tag{3.2}$$

Proof. For all x, y in F_i^0, there exist x^t, y^t in F_i^t such that $P_i \circ T(x^t) = x$, $P_i \circ T(y^t) = y$ and, as a consequence, $f_i^{t,T}(x) = T_i(x^t)$, $f_i^{t,T}(y) = T_i(y^t)$.
Thus, since $x_i^t = y_i^t = t$ and $T \in D_L$, we obtain

$$\begin{aligned} |f_i^{t,T}(x) - f_i^{t,T}(y)| &= |T_i(x^t) - T_i(y^t)| = |T_i(x^t) - T_i(y^t) - x_i^t + y_i^t| \\ &\leq |T(x^t) - T(y^t) - x^t + y^t| \\ &\leq L|x^t - y^t|. \end{aligned} \tag{3.3}$$

Moreover, since $L \in]0, 1[$, we obtain

$$\begin{aligned} |x - y| &= |P_i \circ T(x^t) - P_i \circ T(y^t)| = |P_i[T(x^t) - T(y^t) + y^t - x^t - (y^t - x^t)]| \\ &\geq |x^t - y^t| - |P_i[T(x^t) - T(y^t) + y^t - x^t]| \\ &\geq (1 - L)|x^t - y^t| \end{aligned} \tag{3.4}$$

which, combined with (3.3), implies (3.2).

□

Let us denote by $\mathrm{Lip}(f_i^{t,T})$ the best Lipschitz constant of the function $f_i^{t,T}$, that is

$$\mathrm{Lip}(f_i^{t,T}) = \sup\left\{ \frac{|f_i^{t,T}(x) - f_i^{t,T}(y)|}{|x - y|} \ : \ x, y \in P_i(\Omega), \ x \neq y \right\}. \tag{3.5}$$

Then, from (3.2) it follows that $\mathrm{Lip}(f_i^{t,T}) \to 0$ as $L \to 0$.
Corollary 3.3 shows, in some sense, that also the converse in true. Notice that, if we set $S_T(x) = T(x) - x \ \forall x \in \Omega$, then $T \in D_L$ if and only if

$$\mathrm{Lip}(S_T) := \sup\left\{ \frac{|S_T(x) - S_T(y)|}{|x - y|} \ : \ x, y \in \Omega, \ x \neq y \right\} \leq L. \tag{3.6}$$

Moreover, it is obvious that the set D_L may be also written as

$$D_L = \{T : \overline{\Omega} \to \overline{\Omega} \ : \ T(\overline{\Omega}) = \overline{\Omega}, \ T(F_i^t) = F_i^t \text{ for } t = 0, 1, \ i = 1, \dots, n, \ \mathrm{Lip}(S_T) \leq L\}. \tag{3.7}$$

Lemma 3.2. *Let $T \in D_L$ with $L \in]0, 1[$ and assume that there exists $\Lambda \in \left]0, \frac{1}{n}\right[$ such that*

$$\mathrm{Lip}(F_i^{t,T}) \leq \Lambda \qquad \forall t \in [0, 1], \ \forall i \in \{1, \dots, n\} \tag{3.8}$$

and

$$|f_i^{t_1,T}(x) - f_i^{t_2,T}(x) + t_2 - t_1| \leq \Lambda|t_1 - t_2| \qquad \forall t_1, t_2 \in [0, 1], \ \forall i \in \{1, \dots, n\}, \ \forall x \in F_i^0. \tag{3.9}$$

Then,

$$|T(x) - T(y) - x + y| \leq \frac{(n+1)\sqrt{n}\,\Lambda}{1 - n\Lambda}\,|x - y| \quad \forall x, y \in \overline{\Omega}, \tag{3.10}$$

that is $\mathrm{Lip}(S_T) \leq \frac{(n+1)\sqrt{n}\,\Lambda}{1 - n\Lambda}$*, so* $T \in D_{L(\Lambda)}$ *with* $L(\Lambda) = \frac{(n+1)\sqrt{n}\,\Lambda}{1 - n\Lambda}$.

Proof. Notice that $T_i(x) = f_i^{x_i,T}(P_i \circ T(x))$ for all $x \in \overline{\Omega}$ and $i \in \{1, \dots, n\}$.
Thus, for $x, y \in \Omega$ and $h = y - x$, we obtain

$$\begin{aligned} T_i(x+h) - T_i(x) - h_i &= f_i^{x_i+h_i,T}(P_i \circ T(x+h)) - f_i^{x_i,T}(P_i \circ T(x+h)) - h_i \\ &\quad + f_i^{x_i,T}(P_i \circ T(x+h)) - f_i^{x_i,T}(P_i \circ T(x)) \\ &= \mu_i h_i + \sum_{j=1}^{n} \nu_i^j [T_j(x+h) - T_j(x)] \end{aligned} \tag{3.11}$$

where, for all i and j in $\{1, \dots, n\}$, μ_i and ν_i^j are suitable numbers in $[-\Lambda, \Lambda]$ because of our assumptions on the functions $f_i^{t,T}$.
It follows that

$$|T_i(x+h) - T_i(x) - h_i| \leq \Lambda |h_i| + \Lambda \sum_{j=1}^{n} |T_j(x+h) - T_j(x) - h_j| + \Lambda \sum_{j=1}^{n} |h_j| \quad \forall i \in \{1, \dots, n\} \tag{3.12}$$

and, summing up,

$$\sum_{i=1}^{n} |T_i(x+h) - T_i(x) - h_i| \leq (1+n)\Lambda \sum_{i=1}^{n} |h_i| + n\Lambda \sum_{i=1}^{n} |T_i(x+h) - T_i(x) - h_i|. \tag{3.13}$$

Since $\Lambda < \frac{1}{n}$, we obtain

$$\sum_{i=1}^{n} |T_i(x+h) - T_i(x) - h_i| \leq \frac{(n+1)\Lambda}{1 - n\Lambda} \sum_{i=1}^{n} |h_i| \tag{3.14}$$

which implies

$$|T(x+h) - T(x) - h| \leq \sum_{i=1}^{n} |T_i(x+h) - T_i(x) - h_i| \leq \frac{(n+1)\Lambda}{1 - n\Lambda} \sum_{i=1}^{n} |h_i| \leq \frac{(n+1)\sqrt{n}\,\Lambda}{1 - n\Lambda} |h|. \tag{3.15}$$

So the proof is complete.

□

The following corollary is a direct consequence of Lemma 3.2.

Corollary 3.3. *Let $(T_k)_k$ be a sequence in D_L with $L \in]0, 1[$ and assume that, for a suitable sequence $(\Lambda_k)_k$ in $\left]0, \frac{1}{n}\right[$, the same conditions as in Lemma 3.2 are satisfied with T replaced by T_k and Λ by Λ_k for all $k \in \mathbb{N}$.*
Then, $\lim_{k\to\infty} \Lambda_k = 0$ implies $\lim_{k\to\infty} \mathrm{Lip}(S_{T_k}) = 0$.

Remark 3.4. Notice that, if $\mathrm{Lip}(S_{T_k}) \longrightarrow 0$ as $k \to \infty$, then S_{T_k} converges to a constant function S_∞ uniformly in Ω. Moreover, taking into account that $T_k \in D_L$ $\forall k \in \mathbb{N}$ so T_k must satisfy suitable conditions on $\partial\Omega$, we can say that $S_\infty \equiv 0$, that is T_k converges to the identity function in Ω.

Now, let us prove the assertions of Proposition 2.2 for the function $u_k = u_k^{T_k^L}$ (in a similar way one can proceed for the function $v_k = v_k^{T_k,v}$). First, we prove the following proposition (here we use the notation introduced in Lemmas 2.4 and 2.5).

Proposition 3.5. *For all $k \ge k(L)$ the function $u_k = u_k^{T_k^L}$ (extended to $\mathbb{R}^n$ by the value zero in $\mathbb{R}^n \setminus \Omega$) has the following asymptotic behaviour.*
For every choice of z^k in Z_k, there exists a function $\hat{T} : \overline{\Omega} \to \mathbb{R}^n$ such that

$$\hat{T}(0) = 0, \qquad (1-L)|x-y| \le |\hat{T}(x) - \hat{T}(y)| \le (1+L)|x-y| \qquad \forall x, y \in \overline{\Omega} \tag{3.16}$$

and, if we set $\chi := \hat{T}(\Omega)$, the function U_{z^k} defined by

$$U_{z^k}(x) = \sigma(z^k) k^{-\frac{2}{p-1}} u_k^{T_k^L}\left(\frac{x}{k} + T_k^L\left(\frac{z^k}{k}\right)\right) \qquad \forall x \in \mathbb{R}^n, \tag{3.17}$$

restricted to χ, as $k \to \infty$ converges in $H^1(\chi)$ to a positive solution U_χ of the Dirichlet problem

$$-\Delta U = |U|^{p-1}U \quad \text{in } \chi, \qquad U = 0 \quad \text{on } \partial\chi, \tag{3.18}$$

satisfying

$$\left(\int_\chi |U_\chi|^{p+1} dx\right)^{-\frac{2}{p-1}} \int_\chi |\nabla U_\chi|^2 dx = m(\chi), \tag{3.19}$$

where

$$m(\chi) := \min\left\{\int_\chi |\nabla U|^2 dx \ : \ U \in H_0^1(\chi), \ \int_\chi |U|^{p+1} dx = 1\right\}. \tag{3.20}$$

Moreover, we have

$$\begin{aligned} \lim_{k\to\infty} k^{\left(n-2\frac{p+1}{p-1}\right)} E_\psi(u_{k,z^k}^{T_k^L}) &= \lim_{k\to\infty}\left[\frac{1}{2}\int_\chi |\nabla U_{z^k}|^2 dx - \frac{1}{p+1}\int_\chi |U_{z^k}|^{p+1} dx\right] \\ &= \frac{1}{2}\int_\chi |\nabla U_\chi|^2 dx - \frac{1}{p+1}\int_\chi |U_\chi|^{p+1} dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)[m(\chi)]^{\frac{p+1}{p-1}}. \end{aligned} \tag{3.21}$$

Proof. For all $k \in \mathbb{N}$, let us rescale problem (1.1) by replacing every function $u \in H_0^1(\Omega)$ by the function $R^k u \in H_0^1(k\Omega)$ defined by

$$R^k u(x) = k^{-\frac{2}{p-1}} u\left(\frac{x}{k}\right) \qquad \forall x \in k\Omega \tag{3.22}$$

(here $R^k u$ is extended by the value zero outside $k\Omega$).
Then, our problem becomes

$$-\Delta U = |U|^{p-1}U + \psi_k \quad \text{in } k\Omega, \qquad U = 0 \quad \text{on } \partial(k\Omega) \tag{3.23}$$

where $\psi_k \in L^2(k\Omega)$ is defined by

$$\psi_k(x) = k^{-\frac{2p}{p-1}} \psi\left(\frac{x}{k}\right) \qquad \forall x \in k\Omega. \tag{3.24}$$

Moreover, the corresponding functional becomes

$$E^k(U) = \frac{1}{2}\int_{k\Omega} |\nabla U|^2 dx - \frac{1}{p+1}\int_{k\Omega} |U|^{p+1} dx - \int_{k\Omega} \psi_k U \, dx, \tag{3.25}$$

defined for all $u \in H_0^1(k\Omega)$.
Since $T_k^L \in D_L\ \forall k \in \mathbb{N}$, so in particular it satisfies (2.10), also the function $kT_k^L\left(\frac{x+z^k}{k}\right)$, defined for all $x \in \overline{\Omega}$, satisfies (2.10) and, as a consequence,

$$kT_k^L\left(\frac{x+z^k}{k}\right) \in B\left(kT_k^L\left(\frac{z^k}{k}\right), (1+L)\sqrt{n}\right) \qquad \forall x \in \overline{\Omega}. \tag{3.26}$$

Therefore, using Ascoli-Arzelà Theorem, we infer that (up to a subsequence) the function $kT_k^L\left(\frac{\cdot+z^k}{k}\right) - kT_k^L\left(\frac{z^k}{k}\right)$ converges as $k \to \infty$ to a function $\hat{T} : \overline{\Omega} \longrightarrow B(0, (1+L)\sqrt{n})$ uniformly in $\overline{\Omega}$. As a consequence, $\hat{T}$ satisfies (3.16) in $\overline{\Omega}$.
From Lemmas 2.4 and 2.5 we infer that $R^k \tilde{u}_{k,z^k}^{T_k^L}$ and $R^k u_{k,z^k}^{T_k^L}$ belong to $H_0^1(kC_{z^k}^k)$ and that

$$E^k(R^k \tilde{u}_{k,z^k}^{T_k^L}) = \min\left\{E^k(U)\ :\ U \in H_0^1(kT_k^L(C_{z^k}^k)),\ \int_{kT_k^L(C_{z^k}^k)} |U|^{p+1} < 1\right\} \qquad \forall k \in \mathbb{N}, \tag{3.27}$$

$$\lim_{k\to\infty} E^k(R^k \tilde{u}_{k,z^k}^{T_k^L}) = \lim_{k\to\infty} \int_{kT_k^L(C_{z^k}^k)} |\nabla R^k \tilde{u}_{k,z^k}^{T_k^L}|^2 dx = 0. \tag{3.28}$$

Moreover,

$$\begin{aligned} E^k(R^k u_{k,z^k}^{T_k^L}) = M^k(R^k u_{k,z^k}^{T_k^L}) = \min\{M^k(U)\ : U \in H_0^1(kT_k^L(C_{z^k}^k)),\ U \not\equiv R^k \tilde{u}_{k,z^k}^{T_k^L}, \\ \sigma(z^k)[U - R^k \tilde{u}_{k,z^k}^{T_k^L}] \geq 0 \text{ in } kT_k^L(C_{z^k}^k)\} \end{aligned} \tag{3.29}$$

where $M^k(U)$ is defined by

$$M^k(U) = \max\{E^k(R^k \tilde{u}_{k,z^k}^{T_k^L} + t(U - R^k \tilde{u}_{k,z^k}^{T_k^L})\ :\ t \geq 0\} \qquad \forall U \in H_0^1(kT_k^L(C_{z^k}^k)). \tag{3.30}$$

Notice that

$$\int_{k\Omega} \psi_k^2 dx = k^{n-\frac{4p}{p-1}} \int_\Omega \psi^2 dx \tag{3.31}$$

where $n < \frac{4p}{p-1}$ under our assumptions on p. In fact, for $n \leq 4$ it is obviously true because $p > 1$ while for $n > 4$ it is true because $1 < p < \frac{n+2}{n-2}$, as one can easily verify by direct computation (taking into account that $\frac{n+2}{n-2} < \frac{n}{n-4}$). As a consequence, we obtain in particular

$$\lim_{k\to\infty} \int_{kT_k^L(C_{z^k}^k)} \psi_k^2 dx = 0. \tag{3.32}$$

Therefore, we infer that the function U_{z^k} satisfies all the assertions in Proposition 3.5, that is its restriction to χ converges to a positive solution U_χ of the asymptotic problem (3.18), satisfying the minimality condition (3.19).
In fact, (3.28) and (3.32) imply

$$\begin{aligned} 0 &< \liminf_{k\to\infty} \left[\frac{1}{2}\int_\chi |\nabla U_{z^k}|^2 dx - \frac{1}{p+1}\int_\chi |U_{z^k}|^{p+1} dx\right] \\ &\leq \limsup_{k\to\infty} \left[\frac{1}{2}\int_\chi |\nabla U_{z^k}|^2 dx - \frac{1}{p+1}\int_\chi |U_{z^k}|^{p+1} dx\right] < \infty \end{aligned} \tag{3.33}$$

and
$$\lim_{k\to\infty}\left[\int_\chi |\nabla U_{z^k}|^2 dx - \int_\chi |U_{z^k}|^{p+1} dx\right] = 0. \tag{3.34}$$
It follows that
$$\limsup_{k\to\infty} \frac{\int_\chi |\nabla U_{z^k}|^2 dx}{\left(\int_\chi |U_{z^k}|^{p+1} dx\right)^{\frac{2}{p+1}}} < \infty \tag{3.35}$$
so, up to a subsequence, $\left(\int_\chi |U_{z^k}|^{p+1} dx\right)^{-\frac{1}{p+1}} U_{z^k}$ converges as $k \to \infty$ to a positive function $\overline{U}_\chi \in H_0^1(\chi)$ almost everywhere in χ, strongly in $L^{p+1}(\chi)$ and weakly in $H^1(\chi)$.
Moreover, the minimality property of $u_{k,z^k}^{T_k^L}$ implies, by standard arguments, that
$$\int_\chi |\nabla \overline{U}_\chi|^2 dx = m(\chi) \tag{3.36}$$
and that, as $k \to \infty$, U_{z^k} converges strongly in $H^1(\chi)$ to the function $U_\chi = m(\chi)^{\frac{1}{p-1}} \overline{U}_\chi$, which is a positive solution of problem (3.18).
Therefore, taking also into account (3.24), we obtain
$$\begin{aligned}\lim_{k\to\infty} k^{\left(n-2\frac{p+1}{p-1}\right)} E_\psi\left(u_{k,z^k}^{T_k^L}\right) &= \lim_{k\to\infty} E^k\left(R^k u_{k,z^k}^{T_k^L}\right)\\ &= \lim_{k\to\infty}\left[\frac{1}{2}\int_\chi |\nabla U_{z^k}|^2 dx - \frac{1}{p+1}\int_\chi |U_{z^k}|^{p+1} dx\right]\\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)[m(\chi)]^{\frac{p+1}{p-1}}.\end{aligned} \tag{3.37}$$
So the proof is complete.

□

In next lemma we describe other properties of the function $\hat{T}$ and of the domain $\hat{T}(\Omega)$ arising in Proposition 3.5.

Lemma 3.6. *Let $(z^k)_k$, $\hat{T}$ and χ be as in Proposition 3.5. Then, the function $S_{\hat{T}} : \overline{\Omega} \to \mathbb{R}^n$ defined by $S_{\hat{T}}(x) = \hat{T}(x) - x\ \forall x \in \overline{\Omega}$, satisfies the Lipschitz condition*
$$|S_{\hat{T}}(x) - S_{\hat{T}}(y)| \le L|x-y| \qquad \forall x, y \in \overline{\Omega}. \tag{3.38}$$
Moreover, for every $i \in \{1, \dots, n\}$ there exist two functions $f_i^0, f_i^1 : P_i(\chi) \to \mathbb{R}$, Lipschitz continuous with Lipschitz constant $\frac{L}{1-L}$, such that $f_i^0 \circ P_i(0) = 0$, $f_i^0 \circ P_i(x) < f_i^1 \circ P_i(x)$ $\forall x \in \chi$ and
$$\chi = \{x \in \mathbb{R}^n\ :\ P_i(x) \in P_i(\chi),\ f_i^0 \circ P_i(x) < x_i < f_i^1 \circ P_i(x)\ \text{ for } i = 1, \dots, n\}. \tag{3.39}$$
Proof. Notice that, as the functions $S_{T_k^L} : \overline{\Omega} \to \mathbb{R}^n$ defined by $S_{T_k^L}(x) = T_k^L(x) - x\ \forall x \in \Omega$, also the functions $k\left[T_k^L\left(\frac{x+z^k}{k}\right) - T_k^L\left(\frac{z^k}{k}\right) - \frac{x}{k}\right]$ are Lipschitz continuous with Lipschitz constant

L for all $k \in \mathbb{N}$. Therefore, as $k \to \infty$, we infer that the function $S_{\hat{T}}$ satisfies (3.38). In order to obtain the functions f_i^0 and f_i^1, we use Lemma 3.1. From (3.1) it follows that

$$T_k^L(C^k_{z^k}) = \{x \in \Omega \ : \ f_i^{\frac{z_i^k}{k}, T_k^L} \circ P_i(x) < x_i < f_i^{\frac{z_i^k+1}{k}, T_k^L} \circ P_i(x) \ \text{ for } i = 1, \dots, n\}. \tag{3.40}$$

Now, notice that, as the functions $f_i^{\frac{z_i^k}{k}, T_k^L}$ and $f_i^{\frac{z_i^k+1}{k}, T_k^L}$, also the functions $f^0_{i,k}$ and $f^1_{i,k}$ defined by

$$f^0_{i,k}(x) = k\left[f_i^{\frac{z_i^k}{k}, T_k^L}\left(\frac{x}{k} + P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right) - f_i^{\frac{z_i^k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right] \tag{3.41}$$

and

$$f^1_{i,k}(x) = k\left[f_i^{\frac{z_i^k+1}{k}, T_k^L}\left(\frac{x}{k} + P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right) - f_i^{\frac{z_i^k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right] \tag{3.42}$$

are both Lipschitz continuous with Lipschitz constant $\frac{L}{1-L}$. Moreover, for all $k \in \mathbb{N}$ we have

$$f^0_{i,k}(0) = 0 \quad \forall i \in \{1, \dots, n\}, \qquad k\left|T_k^L\left(\frac{x + z^k}{k}\right) - T_k^L\left(\frac{z^k}{k}\right)\right| \le (1+L)\sqrt{n} \quad \forall x \in \overline{\Omega} \tag{3.43}$$

and

$$f^1_{i,k}(x) - f^0_{i,k}(x) \ge (1-L)|x^1_{i,k} - x^0_{i,k}| \ge (1-L) > 0 \quad \forall x \in P_i(\overline{\chi}),\ \forall i \in \{1, \dots, n\},\ \forall k \in \mathbb{N}, \tag{3.44}$$

where $x^1_{i,k}$ and $x^0_{i,k}$ are the points in $\overline{\Omega}$ such that

$$k\left[T_k^L\left(\frac{x^1_{i,k} + z^k}{k}\right) - T_k^L\left(\frac{z^k}{k}\right)\right] = (x, f^1_{i,k}(x)) \tag{3.45}$$

and

$$k\left[T_k^L\left(\frac{x^0_{i,k} + z^k}{k}\right) - T_k^L\left(\frac{z^k}{k}\right)\right] = (x, f^0_{i,k}(x)), \tag{3.46}$$

which implies $|x^1_{i,k} - x^0_{i,k}| \ge 1$.
Therefore, by Ascoli-Arzelà Theorem we can say that, up to a subsequence, the functions $f^1_{i,k}$ and $f^0_{i,k}$ converge as $k \to \infty$ uniformly in $P_i(\overline{\chi})$ respectively to functions f_i^1 and f_i^0 satisfying all the assertions in Lemma 3.6.

$\square$

Lemma 3.7. *Let $(z^k)_k$ and χ be as in Proposition 3.5. Then, for every choice of z^k in Z_k, the domain χ is a cube of $\mathbb{R}^n$ having a vertex in the origin and the sides of lenght 1. Moreover, we have*

$$\lim_{k\to\infty} \max\{\left|k^n|T_k^L(C_z^k)| - 1\right| \ : \ z \in Z_k\} = 0 \tag{3.47}$$

(where $|T_k^L(C_z^k)|$ denotes the volume of $T_k^L(C_z^k)$) and

$$\lim_{k\to\infty} \max\left\{\left|k^{\left(n-2\,\frac{p+1}{p-1}\right)} E_\psi(u^{T_k^L}_{k,z}) - \left(\frac{1}{2} - \frac{1}{p+1}\right)\bar{m}^{\frac{p+1}{p-1}}\right| \ : \ z \in Z_k\right\} = 0 \tag{3.48}$$

where $\bar{m} = m(\Omega)$.

Proof. Notice that, as we pointed out in the proof of Proposition 3.5, the effect of the term ψ in problem (1.1) tends to vanish as $k \to \infty$ because in the rescaled problem (3.23) ψ is replaced by the function ψ_k defined by (3.31) and, since $n < \frac{4p}{p-1}$,

$$\lim_{k\to\infty} \int_{k\Omega} \psi_k^2 dx = \lim_{k\to\infty} k^{n-\frac{4p}{p-1}} \int_{\Omega} \psi^2 dx = 0. \tag{3.49}$$

As a consequence, taking into account the minimality of T_k^L, the interfaces between the domains $kT_k^L(C_z^k)$, with $z \in Z_k$, tend to be flat, so these domains tend as $k \to \infty$ to polyhedra with $2n$ faces, having minimality properties inherited by the analogous properties of the domains $kT_k^L(C_z^k)$, related to the minimality of T_k^L.
In particular, arguing as in the proof of Proposition 3.5, one can show in addition that, for every $\rho > 0$, the function $k\left[T_k^L\left(\frac{x+z^k}{k}\right) - T_k^L\left(\frac{z^k}{k}\right)\right]$ (up to a subsequence) converges as $k \to \infty$ to a function $\hat{T}$ uniformly in the domain $\Omega_\rho = \cup_{z\in\mathbb{Z}_\rho} \overline{C}_z^1$, where

$$\mathbb{Z}_\rho = \{z \in \mathbb{Z}^n \ : \ |z| < \rho, \ z^k + z \in Z_k \ \forall k \in \mathbb{N}\} \tag{3.50}$$

(not only in $\overline{\Omega}$, which is strictly enclosed in Ω_ρ for $\rho > \sqrt{n}$).
Moreover, if we set $\chi_z := \hat{T}(C_z^1)\ \forall z \in \mathbb{Z}_\rho$, the number $\sum_{z\in\mathbb{Z}_\rho}[m(\chi_z)]^{\frac{p+1}{p-1}}$ has to be as small as possible for all $\rho > 0$. By symmetry reasons, among this polyhedra χ, the cubes of $\mathbb{R}^n$ are the unique minimizers of the value $\varphi(\chi) := m(\chi)|\chi|^{2\left(\frac{1}{p+1}-\frac{1}{2^*}\right)}$ (where $|\chi|$ is the volume of χ and $m(\chi)$ is defined in (3.20)) that is, if we set $\bar{\varphi} = \varphi(\Omega)$, $\varphi(\chi) = \bar{\varphi}$ if χ is a cube and $\varphi(\chi) > \bar{\varphi}$ otherwise (notice that $\varphi(\chi)$ depends only on the shape of χ and not on its size because it is invariant with respect to translations and rescaling of χ).
By Proposition 3.5, for every choice of z^k in Z_k, the corresponding limit domain χ satisfies

$$\lim_{k\to\infty} k^n |T_k^L(C_{z^k}^k| = |\chi| \tag{3.51}$$

and

$$\begin{aligned}\lim_{k\to\infty} k^{n-2\frac{p+1}{p-1}} E_\psi(u_{k,z^k}^{T_k^L})[k^n|T_k^L(C_{z^k}^k)|]^{2\left(\frac{1}{p+1}-\frac{1}{2^*}\right)\frac{p+1}{p-1}} &= \left(\frac{1}{2} - \frac{1}{p+1}\right)[m(\chi)]^{\frac{p+1}{p-1}}|\chi|^{2\left(\frac{1}{p+1}-\frac{1}{2^*}\right)\frac{p+1}{p-1}} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)[\varphi(\chi)]^{\frac{p+1}{p-1}},\end{aligned} \tag{3.52}$$

which implies

$$\lim_{k\to\infty} k^{\left(n-2\frac{p+1}{p-1}\right)} E_\psi(u_{k,z^k}^{T_k^L}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)[\varphi(\chi)]^{\frac{p+1}{p-1}}|\chi|^{2\left(\frac{1}{2^*}-\frac{1}{p+1}\right)\frac{p+1}{p-1}}. \tag{3.53}$$

Therefore, taking into account the minimality of T_k^L, it follows that, for every choice of z^k in Z_k, the limit domain χ is a cube, that is $\varphi(\chi) = \bar{\varphi}$. As a consequence, since it is true for every choice of z^k in Z_k, we can say that

$$\lim_{k\to\infty} \max\left\{\left|k^{n-2\frac{p+1}{p-1}} E_\psi(u_{k,z}^{T_k^L})[k^n|T_k^L(C_z^k)|]^{2\left(\frac{1}{p+1}-\frac{1}{2^*}\right)\frac{p+1}{p-1}} - \left(\frac{1}{2} - \frac{1}{p+1}\right)\bar{\varphi}^{\frac{p+1}{p-1}}\right| \ : z \in Z_k\right\} = 0. \tag{3.54}$$

Now, let us set

$$\varphi_k(z) = \left[k^{n-2\frac{p+1}{p-1}} \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} E_\psi(u_{k,z}^{T_k^L})\right]^{\frac{p-1}{p+1}} [k^n |T_k^L(C_z^k)|]^{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)}. \tag{3.55}$$

Then, we have

$$\begin{aligned} k^{n-2\frac{p+1}{p-1}} E_\psi(u_k^{T_k^L}) &= k^{n-2\frac{p+1}{p-1}} \sum_{z \in Z_k} E_\psi(u_{k,z}^{T_k^L}) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{z \in Z_k} [\varphi_k(z)]^{\frac{p+1}{p-1}} [k^n |T_k^L(C_z^k)|]^{-2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)\frac{p+1}{p-1}}. \end{aligned} \tag{3.56}$$

Taking into account the minimality of T_k^L, since $\sum_{z \in Z_k} |T_k^L(C_z^k)| = 1$, we obtain

$$[\varphi_k(z)]^{\frac{p+1}{p-1}} = \mu_k [k^n |T_k^L(C_z^k)|]^{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)\frac{p+1}{p-1}+1} \qquad \forall k \in \mathbb{N},\ \forall z \in Z_k \tag{3.57}$$

where $\mu_k > 0$ is a suitable Lagrange multiplier. It follows that

$$|T_k^L(C_z^k)| = \frac{1}{k^n} \left(\frac{[\varphi_k(z)]^{\frac{p+1}{p-1}}}{\mu_k}\right)^{\frac{1}{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)\frac{p+1}{p-1}+1}} \qquad \forall k \in \mathbb{N},\ \forall z \in Z_k \tag{3.58}$$

which, summing up, yields

$$\mu_k^{\frac{1}{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)\frac{p+1}{p-1}+1}} = \frac{1}{k^n} \sum_{z \in Z_k} [\varphi_k(z)]^{\frac{1}{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)+\frac{p-1}{p+1}}}. \tag{3.59}$$

Since

$$\lim_{k \to \infty} \max\{|\varphi_k(z) - \bar{\varphi}| \ : \ z \in Z_k\} = 0 \tag{3.60}$$

(because of (3.54)), we obtain

$$\lim_{k \to \infty} \frac{1}{k^n} \sum_{z \in Z_k} [\varphi_k(z)]^{\frac{1}{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)+\frac{p-1}{p+1}}} = \bar{\varphi}^{\frac{1}{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)+\frac{p-1}{p+1}}} \tag{3.61}$$

which implies

$$\lim_{k \to \infty} \mu_k = \bar{\varphi}^{\frac{p+1}{p-1}}. \tag{3.62}$$

From (3.57) we get

$$k^n |T_k^L(C_z^k)| = \left[\frac{\varphi_k(z)]^{\frac{p+1}{p-1}}}{\mu_k}\right]^{\frac{1}{2\left(\frac{1}{p+1} - \frac{1}{2^*}\right)\frac{p+1}{p-1}+1}}. \tag{3.63}$$

Thus, we can easily obtain (3.47) from (3.60) and (3.62) and then (3.48) from (3.47) and (3.54). Finally, we can say that, for every choice of z^k in $\mathbb{Z}_k$, the corresponding limit domain χ is a cube of $\mathbb{R}^n$ with sides of lenght 1. Moreover, the construction of χ shows that this cube has a vertex in the origin, so the proof is complete.

□

Lemma 3.8. *For all $k \in \mathbb{N}$ and $L \in]0,1[$, let $T_k^L \in D_L$ be a minimizing deformation as in Section 2. Then, for all $i \in \{1, \ldots, n\}$, we have*

$$\lim_{k\to\infty} \sup \left\{ \frac{\left| f_i^{\frac{h}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{z}{k}\right) - f_i^{\frac{h}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\zeta}{k}\right) \right|}{\left| P_i \circ T_k^L\left(\frac{z}{k}\right) - P_i \circ T_k^L\left(\frac{\zeta}{k}\right) \right|} : h \in \{0,1,\ldots,k\}, \right. \\ \left. z, \zeta \in Z_k,\ z \neq \zeta,\ z_i = \zeta_i = h \right\} = 0. \tag{3.64}$$

Proof. Arguing by contradiction, assume that for some $i \in \{1, \ldots, n\}$ there exist sequences $(h_k)_k$, $(z^k)_k$, $(\zeta^k)_k$ such that $h_k \in \{0,1,\ldots,k\}$, $z^k \in Z_k$, $\zeta^k \in Z_k$, $z^k \neq \zeta^k$, $z_i^k = \zeta_i^k = h_k$ $\forall k \in \mathbb{N}$ and (up to a subsequence)

$$\lim_{k\to\infty} \frac{\left| f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right) - f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right) \right|}{\left| P_i \circ T_k^L\left(\frac{z^k}{k}\right) - P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right) \right|} > 0. \tag{3.65}$$

We say that there exist two sequences $(\hat{z}^k)_k$ and $(\hat{\zeta}^k)_k$ in Z_k such that $\hat{z}_i^k = \hat{\zeta}_i^k = h_k$, $|\hat{z}^k - \hat{\zeta}^k| = 1$ $\forall k \in \mathbb{N}$ and

$$\liminf_{k\to\infty} \frac{\left| f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\hat{z}^k}{k}\right) - f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\hat{\zeta}^k}{k}\right) \right|}{\left| P_i \circ T_k^L\left(\frac{\hat{z}^k}{k}\right) - P_i \circ T_k^L\left(\frac{\hat{\zeta}^k}{k}\right) \right|} > 0. \tag{3.66}$$

In fact, if $\limsup_{k\to\infty} |z^k - \zeta^k| < 2$, it is obvious because in this case $|z^k - \zeta^k| = 1$ for k large enough (so we can set $\hat{z}^k = z^k$ and $\hat{\zeta}^k = \zeta^k$). On the contrary, if $\limsup_{k\to\infty} |z^k - \zeta^k| \geq 2$, let us set $\nu^k = \sum_{j=1}^n |z_i^k - \zeta_i^k|$. Then, one can choose $\nu_k + 1$ points $\pi_0, \pi_1, \ldots, \pi_{\nu^k}$ in Z_k such that

$$\pi_0 = z^k, \quad \pi_{\nu_k} = \zeta^k, \quad |\pi_j - \pi_{j-1}| = 1 \quad \forall j \in \{1, \ldots, \nu_k\} \tag{3.67}$$

(notice that all the points $\frac{\pi_0}{k}, \frac{\pi_1}{k}, \ldots, \frac{\pi_{\nu_k}}{k}$ must belong to $F_i^{\frac{h_k}{k}}$ because of the choice of ν_k). Therefore, we obtain

$$\begin{aligned} &\left| f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right) - f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right) \right| \\ &\qquad \leq \sum_{j=1}^{\nu_k} \left| f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\pi_j}{k}\right) - f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\pi_{j-1}}{k}\right) \right| \\ &\qquad \leq \max_{1\leq j\leq \nu_k} \frac{\left| f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\pi_j}{k}\right) - f_i^{\frac{h_k}{k}, T_k^L} \circ P_i \circ T_k^L\left(\frac{\pi_{j-1}}{k}\right) \right|}{\left| P_i \circ T_k^L\left(\frac{\pi_j}{k}\right) - P_i \circ T_k^L\left(\frac{\pi_{j-1}}{k}\right) \right|} \cdot \\ &\qquad\qquad \cdot \sum_{j=1}^{\nu_k} \left| P_i \circ T_k^L\left(\frac{\pi_j}{k}\right) - P_i \circ T_k^L\left(\frac{\pi_{j-1}}{k}\right) \right| \end{aligned} \tag{3.68}$$

where

$$\left|P_i\circ T_k^L\left(\frac{\pi_j}{k}\right)-P_i\circ T_k^L\left(\frac{\pi_{j-1}}{k}\right)\right|\le\left|T_k^L\left(\frac{\pi_j}{k}\right)-T_k^L\left(\frac{\pi_{j-1}}{k}\right)\right|\le(1+L)\left|\frac{\pi_j}{k}-\frac{\pi_{j-1}}{k}\right| \qquad \forall j\in\{1,\dots,\nu_k\}, \tag{3.69}$$

which implies

$$\begin{aligned}\sum_{j=1}^{\nu_k}\left|P_i\circ T_k^L\left(\frac{\pi_j}{k}\right)-P_i\circ T_k^L\left(\frac{\pi_{j-1}}{k}\right)\right| &\le (1+L)\sum_{j=1}^{\nu_k}\left|\frac{\pi_j}{k}-\frac{\pi_{j-1}}{k}\right|\\ &\le (1+L)\sqrt{n-1}\left|\frac{z^k}{k}-\frac{\zeta_k}{k}\right|\\ &\le \sqrt{n-1}\frac{1+L}{1-L}\left|T_k^L\left(\frac{z^k}{k}\right)-T_k^L\left(\frac{\zeta^k}{k}\right)\right|\end{aligned} \tag{3.70}$$

where

$$\begin{aligned}&\left|T_k^L\left(\frac{z^k}{k}\right)-T_k^L\left(\frac{\zeta^k}{k}\right)\right|\\ &\quad\le\left|P_i\circ T_k^L\left(\frac{z^k}{k}\right)-P_i\circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|+\left|f_i^{\frac{h_k}{k},T_k^L}\circ P_i\circ T_k^L\left(\frac{z^k}{k}\right)-f_i^{\frac{h_k}{k},T_k^L}\circ P_i\circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|\end{aligned} \tag{3.71}$$

and, by Lemma 3.1

$$\left|f_i^{\frac{h_k}{k},T_k^L}\circ P_i\circ T_k^L\left(\frac{z^k}{k}\right)-f_i^{\frac{h_k}{k},T_k^L}\circ P_i\circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|\le\frac{L}{1-L}\left|P_i\circ T_k^L\left(\frac{z^k}{k}\right)-P_i\circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|. \tag{3.72}$$

Therefore, (3.65) implies

$$\liminf_{k\to\infty}\max_{1\le j\le\nu_k}\frac{\left|f_i^{\frac{h_k}{k},T_k^L}\circ P_i\circ T_k^L\left(\frac{\pi_j}{k}\right)-f_i^{\frac{h_k}{k},T_k^L}\circ P_i\circ T_k^L\left(\frac{\pi_{j-1}}{k}\right)\right|}{\left|P_i\circ T_k^L\left(\frac{\pi_j}{k}\right)-P_i\circ T_k^L\left(\frac{\pi_{j-1}}{k}\right)\right|}>0. \tag{3.73}$$

So, if the maximum in (3.73) is achieved for $j=j_k$, our assertion (3.66) holds for $\hat{z}^k=\pi_{j_k}$ and $\hat{\zeta}^k=\pi_{j_k-1}$.
Now, for all $i\in\{1,\dots,n\}$ let us consider the vector $e^i=(e_1^i,\dots,e_n^i)\in\mathbb{R}^n$ such that $e_i^i=1$, $e_j^i=0$ for $j\neq i$, $i,j\in\{1,\dots,n\}$ and the function $\delta_i^k:\Omega\to\mathbb{R}^n$ defined by

$$\begin{aligned}&\delta_i^k(x)=k\left[T_k^L\left(\frac{z}{k}+\frac{e^i}{k}\right)-T_k^L\left(\frac{z}{k}\right)\right]\\ &\forall x\in\Omega\ \text{ such that } \frac{z_j}{k}\le x_j<\frac{z_j+1}{k}\quad\forall j\in\{1,\dots,n\}\ \text{ with } z\in Z_k.\end{aligned} \tag{3.74}$$

Notice that the set $Z_\Omega=\cup_{k\in\mathbb{N}}\frac{1}{k}Z_k$ is a subset of $\overline{\Omega}$, $\overline{Z}_\Omega=\overline{\Omega}$ and, for all $i\in\{1,\dots,n\}$, the sequence of functions $\delta^k_{i\,|Z_\Omega}$, up to a subsequence, converges as $k\to\infty$ to a function $\delta_i:Z_\Omega\to\mathbb{R}^n$.

Taking into account Lemma 3.7, for all $x \in Z_\Omega$ we have

$$\delta_i(x)\cdot\delta_i(x) = 1, \qquad \delta_i(x)\cdot\delta_j(x) = 0 \ \text{ for } i \neq j, \qquad \forall x \in Z_\Omega,\ \forall i,j \in \{1,\dots,n\}. \tag{3.75}$$

Moreover, we infer that

$$\liminf_{k\to\infty} \left|\delta_i^k\left(\frac{\hat z^k}{k}\right) - e^i\right| > 0 \tag{3.76}$$

because of (3.66), while

$$\lim_{k\to\infty} \delta_i^k\left[P_i\left(\frac{\hat z^k}{k}\right)\right] = e_i \tag{3.77}$$

because of the conditions satisfied by T_k^L on the boundary of Ω.
Therefore, since

$$\lim_{k\to\infty} \left|\delta_i\left(\frac{\hat z^k}{k}\right) - \delta_i^k\left(\frac{\hat z^k}{k}\right)\right| = 0 \tag{3.78}$$

and $\delta_i\left[P_i\left(\frac{\hat z^k}{k}\right)\right] = e^i\ \forall k \in \mathbb{N}$, it follows that

$$\liminf_{k\to\infty} \left|\delta_i\left(\frac{\hat z^k}{k}\right) - \delta_i\left[P_i\left(\frac{\hat z^k}{k}\right)\right]\right| > 0. \tag{3.79}$$

On the other hand, since $\hat z_i^k \leq k$, we obtain

$$\begin{aligned}\left|\delta_i\left(\frac{\hat z^k}{k}\right) - \delta_i\left[P_i\left(\frac{\hat z^k}{k}\right)\right]\right| &\leq \sum_{j=1}^{\hat z_i^k}\left|\delta_i\left(\frac{\hat z^k - je^i}{k}\right) - \delta_i\left(\frac{\hat z^k-(j-1)e^i}{k}\right)\right| \\ &\leq k \max_{1\leq j\leq \hat z_i^k}\left|\delta_i\left(\frac{\hat z^k - je^i}{k}\right) - \delta_i\left(\frac{\hat z^k-(j-1)e^i}{k}\right)\right|.\end{aligned} \tag{3.80}$$

We say that the last term tends to zero as $k \to \infty$ that is, if the maximum in (3.80) is achieved for $j = j_k$, we have

$$\lim_{k\to\infty} k\left|\delta_i\left(\frac{\hat z^k - j_k e^i}{k}\right) - \delta_i\left(\frac{\hat z^k-(j_k-1)e^i}{k}\right)\right| = 0. \tag{3.81}$$

In fact, assume that (up to a subsequence) the sequence $k\left[\delta_i\left(\frac{\hat z^k-(j_k-1)e^i}{k}\right) - \delta_i\left(\frac{\hat z^k - j_k e^i}{k}\right)\right]$ converges as $k \to \infty$ to a vector in $\mathbb{R}^n$, we denote by $D_i\hat\delta_i$, and the sequences $\delta_i\left(\frac{\hat z^k-(j_k-1)e^i}{k}\right)$ converge to some vectors $\hat\delta_i$ in $\mathbb{R}^n$. Then, we have that $\hat\delta_i\cdot\hat\delta_j = 0$ for $i \neq j$ and $\hat\delta_i\cdot\hat\delta_i = 1$ for $i,j \in \{1,\dots,n\}$ (because of Lemma 3.7). Therefore, in order to prove that $D_i\hat\delta_i = 0$, it suffices to prove that $D_i\hat\delta_i\cdot\hat\delta_j = 0\ \forall j \in \{1,\dots,n\}$.
First, notice that $D_i\hat\delta_i\cdot\hat\delta_i = 0$. In fact, we have

$$D_i\hat\delta_i\cdot\hat\delta_i = \frac{1}{2}\lim_{k\to\infty} k\left[\delta_i\cdot\delta_i\left(\frac{\hat z^k-(j_k-1)e^i}{k}\right) - \delta_i\cdot\delta_i\left(\frac{\hat z^k - j_k e^i}{k}\right)\right], \tag{3.82}$$

where the limit is equal to zero because

$$\delta_i \cdot \delta_i \left(\frac{\hat{z}^k - j_k e^i}{k} \right) = \delta_i \cdot \delta_i \left(\frac{\hat{z}^k - (j_k - 1)e^i}{k} \right) = 1 \qquad \forall k \in \mathbb{N}. \tag{3.83}$$

In order to prove that $D_i\hat{\delta}_i \cdot \hat{\delta}_j = 0$ for $j \neq i$, notice that, since $\delta_i \cdot \delta_j \equiv 0$ in Z_Ω, we have

$$0 = \lim_{k\to\infty} k \left[\delta_i \cdot \delta_j \left(\frac{\hat{z}^k - (j_k - 1)e^i}{k} \right) - \delta_i \cdot \delta_j \left(\frac{\hat{z}^k - j_k e^i}{k} \right) \right] = D_i\hat{\delta}_i \cdot \hat{\delta}_j + \hat{\delta}_i \cdot D_i\hat{\delta}_j. \tag{3.84}$$

Moreover, we have

$$\begin{aligned} D_i\hat{\delta}_j = D_j\hat{\delta}_i = \lim_{k\to\infty} k^2 \Bigg[& T_k^L(\hat{z}^k - (j_k - 1)e^i + e^j) - T_k^L \left(\frac{\hat{z}^k - (j_k - 1)e^i}{k} \right) \\ & - T_k^L \left(\frac{\hat{z}^k - j_k e^i + e^j}{k} \right) + T_k^L \left(\frac{\hat{z}^k - j_k e^i}{k} \right) \Bigg] \end{aligned} \tag{3.85}$$

and, as a consequence,

$$\hat{\delta}_i \cdot D_i\hat{\delta}_j = \hat{\delta}_i \cdot D_j\hat{\delta}_i = \frac{1}{2} \lim_{k\to\infty} k \left[\delta_i \cdot \delta_i \left(\frac{\hat{z}^k - j_k e^i + e^j}{k} \right) - \delta_i \cdot \delta_i \left(\frac{\hat{z}^k - j_k e^i}{k} \right) \right] = 0 \tag{3.86}$$

because $\delta_i \cdot \delta_i \equiv 1$ in Z_Ω. Therefore, from (3.84) we obtain $D_i\hat{\delta}_i \cdot \delta_j = 0$ also for $i \neq j$, so $D_i\hat{\delta}_i = 0$.
Then, from (3.80) we infer that

$$\lim_{k\to\infty} \left| \delta_i \left(\frac{\hat{z}^k}{k} \right) - \delta_i \left[P_i \left(\frac{\hat{z}^k}{k} \right) \right] \right| = 0 \tag{3.87}$$

in contradiction with (3.79).
Thus, we can conclude that (3.65) cannot hold and (3.64) is true. So the proof is complete. □

Indeed, the minimality of T_k^L allows us to prove a stronger result, stated in the following corollary.

Corollary 3.9. *Under the same assumptions of Lemma 3.8, for all $i \in \{1, \dots, n\}$ we have*

$$\lim_{k\to\infty} \sup \left\{ \frac{|f_i^{\frac{h}{k},T_k^L}(x) - f_i^{\frac{h}{k},T_k^L}(y)|}{|x-y|} \ : \ h \in \{0,1,\dots,k\}, \ x,y \in F_i^0, \ x \neq y \right\} = 0. \tag{3.88}$$

Proof. Since, under our assumptions on the values of T_k^L on $\partial\Omega$, $P_i \circ T_k^L$ is a one-to-one map between $F_0^{\frac{h}{k}}$ and F_i^0, (3.88) is equivalent to

$$\begin{aligned} \lim_{k\to\infty} \sup \Bigg\{ \frac{|f_i^{\frac{h}{k},T_k^L} \circ P_i \circ T_k^L(x) - f_i^{\frac{h}{k},T_k^L} \circ P_i \circ T_k^L(y)|}{|P_i \circ T_k^L(x) - P_i \circ T_k^L(y)|} \ & : h \in \{0,1,\dots,k\}, \\ & x,y \in F_i^{\frac{h}{k}}, \ x \neq y \Bigg\} = 0. \end{aligned} \tag{3.89}$$

Arguing by contradiction, assume that there exist sequences $(h_k)_k$, $(x^k)_k$, $(y^k)_k$ such that $h_k \in \{0, 1, \ldots, k\}$, x^k and y^k belong to $F_i^{\frac{h}{k}}$, $x^k \neq y^k$ $\forall k \in \mathbb{N}$ and (up to a subsequence)

$$\lim_{k\to\infty} \frac{|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(x^k) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(y^k)|}{|P_i \circ T_k^L(x^k) - P_i \circ T_k^L(y^k)|} > 0. \tag{3.90}$$

Notice that the interfaces between the domains $kT_k^L(C_z^k)$, $z \in Z_k$, tend to be flat because of the minimality of the admissible deformation T_k^L and, as follows from Lemmas 3.7 and 3.8, up to translations these domains tend to Ω that is, for every choice of z^k in Z_k, the domain $k\left[T_k^L(C_z^k) - T_k^L\left(\frac{z^k}{k}\right)\right]$ tends to $C_0^1 = \Omega$.
Therefore, (3.90) is possible only if $\lim_{k\to\infty} k|x^k - y^k| = \infty$ (otherwise, up to a subsequence, the segment $\{x^k + t(y^k - x^k) \ : \ t \in [0,1]\}$ meets only a finite number of subdomains $\overline{C}_z^k$ with $z \in Z_k$). In this case, if $x^k \in C_{z^k}^k$ and $y^k \in C_{\zeta^k}^k$ for suitable z^k and ζ^k in Z_k, we have

$$|kx^k - z^k| \leq \sqrt{n}, \quad |ky^k - \zeta^k| \leq \sqrt{n} \qquad \forall k \in \mathbb{N}, \qquad \lim_{k\to\infty} |z^k - \zeta^k| = \infty \tag{3.91}$$

and

$$\begin{aligned}
&\left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(x^k) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(y^k)\right| \\
&\qquad \leq \left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(x^k) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right| \\
&\qquad + \left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right)\right| \\
&\qquad + \left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(y^k)\right|
\end{aligned} \tag{3.92}$$

where

$$\lim_{k\to\infty} \frac{\left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|}{\left|P_i \circ T_k^L\left(\frac{z^k}{k}\right) - P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|} = 0 \tag{3.93}$$

(as follows from Lemma 3.8) and

$$\lim_{k\to\infty} \frac{\left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(x^k\right) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right|}{\left|P_i \circ T_k^L\left(x^k\right) - P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right|} = 0, \tag{3.94}$$

$$\lim_{k\to\infty} \frac{\left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(y^k\right) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|}{\left|P_i \circ T_k^L\left(y^k\right) - P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right)\right|} = 0 \tag{3.95}$$

because the segments $\left\{x^k + t\left(\frac{z^k}{k} - x^k\right) \ : \ t \in [0,1]\right\}$ and $\left\{y^k + t\left(\frac{\zeta^k}{k} - y^k\right) \ : \ t \in [0,1]\right\}$ are respectively enclosed in the subdomains $\overline{C}_{z^k}^k$ and $\overline{C}_{\zeta^k}^k$.

Moreover, for k large enough, we have

$$\left|P_i \circ T_k^L(x^k) - P_i \circ T_k^L\left(\frac{z^k}{k}\right)\right| \le \left|T_k^L(x^k) - T_k^L\left(\frac{z^k}{k}\right)\right| \le (1+L)\left(x^k - \frac{z^k}{k}\right)$$
$$\le (1+L)|x^k - y^k| \le \frac{1+L}{1-L}\,|T_k^L(x^k) - T_k^L(y^k)| \tag{3.96}$$

and, analogously,

$$\left|P_i \circ T_k^L(y^k) - P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right)\right| \le (1+L)|x^k - y^k| \le \frac{1+L}{1-L}\,|T_k^L(x^k) - T_k^L(y^k)| \tag{3.97}$$

with

$$|T_k^L(x^k) - T_k^L(y^k)| \le |P_i \circ T_k^L(x^k) - P_i \circ T_k^L(y^k)| + \left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(x^k) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(y^k)\right| \tag{3.98}$$

where

$$\left|f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(x^k) - f_i^{\frac{h_k}{k},T_k^L} \circ P_i \circ T_k^L(y^k)\right| \le \frac{L}{1-L}\,|P_i \circ T_k^L(x^k) - P_i \circ T_k^L(y^k)| \tag{3.99}$$

because of Lemma 3.1.
In a similar way we obtain

$$\left|P_i \circ T_k^L\left(\frac{z^k}{k}\right) - P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right)\right| \le \left|P_i \circ T_k^L\left(x^k\right) - P_i \circ T_k^L\left(y^k\right)\right|$$
$$+\left|P_i \circ T_k^L\left(\frac{z^k}{k}\right) - P_i \circ T_k^L\left(x^k\right)\right| + \left|P_i \circ T_k^L\left(\frac{\zeta^k}{k}\right) - P_i \circ T_k^L\left(y^k\right)\right|. \tag{3.100}$$

Therefore, it follows easily that (3.90) cannot be true.
Thus, we have a contradiction, so (3.88) holds and the proof is complete.

□

Lemma 3.10. *Under the same assumptions as in Lemma 3.8, for all $i \in \{1, \dots, n\}$ we have also*

$$\lim_{k\to\infty} \sup\left\{|k\,[f_i^{\frac{h}{k},T_k^L}(x) - f_i^{\frac{h-1}{k},T_k^L}(x)] - 1| \;:\; h \in \{1,\dots,k\},\; x \in F_i^0\right\} = 0. \tag{3.101}$$

Proof. Arguing by contradiction, assume that for all $k \in \mathbb{N}$ there exist $h_k \in \{1, \dots, k\}$ and $x^k \in F_i^0$ such that

$$\lim_{k\to\infty} k\left[f_i^{\frac{h_k}{k},T_k^L}(x^k) - f_i^{\frac{h_k-1}{k},T_k^L}(x^k)\right] \ne 1 \tag{3.102}$$

for some $i \in \{1, \dots, n\}$.
For all $k \in \mathbb{N}$, let us choose $y^k \in \Omega$ and $z^k \in Z_k$ such that

$$y^k \in C_{z^k}^k,\; P_i(y^k) = x^k \text{ and } f_i^{\frac{h_k-1}{k},T_k^L}(x^k) \le y_i^k \le f_i^{\frac{h_k}{k},T_k^L}(x^k) \qquad \forall k \in \mathbb{N}. \tag{3.103}$$

Therefore, we have

$$T_k^L(\overline{C}^k_{z^k}) = \left\{ y\in\Omega \ : \ f_i^{\frac{h_k-1}{k},T_k^L}\circ P_i(y) \le y_i \le f_i^{\frac{h_k}{k},T_k^L}\circ P_i(y) \text{ for } i=1,\dots,n \right\} \qquad \forall k\in\mathbb{N}. \tag{3.104}$$

Taking into account Lemmas 3.7, 3.8 and Corollary 3.9, the domain $k\left[T_L^k(\overline{C}^k_{z^k}) - T_k^L\left(\frac{z^k}{k}\right)\right]$ tends to $\overline{C}^1_0 = \overline{\Omega}$ as $k\to\infty$.
On the other hand, this convergence is not possible if (3.102) holds for some $i\in\{1,\dots,n\}$. Thus, we have a contradiction, (3.102) cannot hold for any $i\in\{1,\dots,n\}$ and (3.101) is true. So the proof is complete.

□

Now, for all $i\in\{1,\dots,n\}$, $t\in[0,1]$ and $k\in\mathbb{N}$, let us consider the function $\tilde f_i^{t,k,L} : F_i^0\to[0,1]$ defined by

$$\begin{aligned} \tilde f_i^{t,k,L}(x) = f_i^{\frac{h-1}{k},T_k^L}(x) + (kt-h+1)[f_i^{\frac{h}{k},T_k^L}(x) - f_i^{\frac{h-1}{k},T_k^L}(x)] \\ \forall x\in F_i^0,\ \forall t\in\left[\frac{h-1}{k},\frac{h}{k}\right],\ h\in\{1,\dots,k\}. \end{aligned} \tag{3.105}$$

Proposition 3.11. *For all $i\in\{1,\dots,n\}$, $t\in[0,1]$ and $k\in\mathbb{N}$, the functions $\tilde f_i^{t,k,L}$ defined by* (3.105) *have the following properties:*

$$\lim_{k\to\infty}\sup\left\{\mathrm{Lip}(\tilde f_i^{t,k,L}) \ : \ i\in\{1,\dots,n\},\ t\in[0,1]\right\} = 0 \tag{3.106}$$

$$\begin{aligned} \lim_{k\to\infty}\sup\Bigg\{ \frac{1}{|t_1-t_2|}\,|\tilde f_i^{t_1,k,L}(x) - \tilde f_i^{t_2,k,L}(x) - t_1 + t_2| \ : \\ i\in\{1,\dots,n\},\ t_1,t_2\in[0,1],\ t_1\neq t_2,\ x\in F_i^0 \Bigg\} = 0. \end{aligned} \tag{3.107}$$

Proof. Taking into account the definition of $\tilde f_i^{t,k,L}$, properties (3.106) and (3.107) follow by direct computation respectively from Corollary 3.9 and Lemma 3.10.

□

Lemma 3.12. *Let $\tilde f_i^{t,k,L}$ be the functions defined in* (3.105)*. Then, for all $x\in\overline{\Omega}$ there exists $y\in\overline{\Omega}$ such that $\tilde f_i^{x_i,k,L}\circ P_i(y) = y_i$ $\forall i\in\{1,\dots,n\}$ (that is, y belongs to the graph of $\tilde f_i^{x_i,k,L}$ for $i=1,\dots,n$).*

Proof. From Proposition 2.3 and (3.105) we infer that $\tilde f_i^{t,k,L}\circ P_i(y)$ is strictly increasing with respect to t in the interval $[0,1]$ for all $i\in\{1,\dots,n\}$, $k\in\mathbb{N}$, $y\in\overline{\Omega}$. Moreover, we have

$$0 = \tilde f_i^{0,k,L}\circ P_i(y) \le y_i \le \tilde f_i^{1,k,L}\circ P_i(y) = 1 \tag{3.108}$$

so for all $y\in\overline{\Omega}$ there exists a unique $t_i(y)\in[0,1]$ such that $\tilde f_i^{t_i(y),k,L}\circ P_i(y) = y_i$. Let us set $t(y) = (t_1(y),\dots,t_n(y))$. Then, the function $t(y)$, defined for all $y\in\overline{\Omega}$, is continuous in $\overline{\Omega}$ and satisfies

$$t(F_i^0)\subseteq F_i^0 \quad \text{and} \quad t(F_i^1)\subseteq F_i^1 \qquad \forall i\in\{1,\dots,n\}. \tag{3.109}$$

Therefore, from [33] we infer that for all $x \in \overline{\Omega}$ there exists at least one $y \in \overline{\Omega}$ such that $t(y) = x$, that is $t_i(y) = x_i$ for all $i \in \{1, \ldots, n\}$.
Thus, since $t_i(y) = x_i$ is equivalent to $\tilde{f}_i^{x_i,k,L} \circ P_i(y) = y_i$, the proof is complete.

□

Lemma 3.13. *Let $\tilde{f}_i^{t,k,L}$ be the functions defined in* (3.105)*. Then, there exists $\tilde{k}_1 \in \mathbb{N}$ such that for all $k \geq \tilde{k}_1$ the following property holds: for all $x \in \overline{\Omega}$ there exists a unique $y \in \overline{\Omega}$ such that*

$$\tilde{f}_i^{x_i,k,L} \circ P_i(y) = y_i \qquad \forall i \in \{1, \ldots, n\}. \tag{3.110}$$

Proof. In Lemma 3.12 we proved that for all $k \in \mathbb{N}$ and for all $x \in \overline{\Omega}$ there exists at least one $y \in \overline{\Omega}$ satisfying (3.110). Now, we have to prove that for k large enough such a y is unique.
For all $\mathcal{L} \geq 0$ let us set $C_i(\mathcal{L}) = \{x \in \mathbb{R}^n \ : \ |x_i| \leq \mathcal{L}|P_i(x)|\}$ and notice that the graph of $\tilde{f}_i^{x_i,k,L}$ is enclosed in $y + C_i(\mathrm{Lip}(\tilde{f}_i^{x_i,k,L}))$.
One can verify by direct computation that $\cap_{1 \leq i \leq n} C_i(\mathcal{L}_i) = \{0\}$ when $\mathcal{L}_i < (n-1)^{-\frac{1}{2}}$ $\forall i \in \{1, \ldots, n\}$.
Therefore, if $\mathrm{Lip}(\tilde{f}_i^{x_i,k,L}) < (n-1)^{-\frac{1}{2}}$ $\forall i \in \{1, \ldots, n\}$, y is the unique point in $\mathbb{R}^n$ satisfying (3.110). On the other hand, taking into account (3.106) of Proposition 3.11, we infer that there exists $\tilde{k}_1 \in \mathbb{N}$ such that

$$\mathrm{Lip}(\tilde{f}_i^{t,k,L}) < (n-1)^{-\frac{1}{2}} \qquad \forall k \geq \tilde{k}_1,\ \forall i \in \{1, \ldots, n\},\ \forall t \in [0,1]. \tag{3.111}$$

Thus, the assertion of Lemma 3.13 holds for such a $\tilde{k}_1$, so the proof is complete.

□

Definition 3.14. Taking into account Lemma 3.13, for all $k \geq \tilde{k}_1$ we can define a function $\widetilde{T}_k^L : \overline{\Omega} \to \overline{\Omega}$ in the following way. For all $x \in \overline{\Omega}$ we set $\widetilde{T}_k^L(x) = y$ where y is the unique point in $\overline{\Omega}$ satisfying (3.110), given by Lemma 3.13
Taking into account the properties of the function $\tilde{f}_i^{t,k,L}$ defined by (3.105) one can verify by standard arguments that $\widetilde{T}_k^L$ is a one-to-one continuous function and that $(\widetilde{T}_k^L)^{-1}$ is continuous too. Moreover, for all $i \in \{1, \ldots, n\}$ and $t \in [0,1]$, $\widetilde{T}_k^L(F_i^t)$ is the graph of the function $\tilde{f}_i^{t,k,L}$ (that is $f_i^{t,\widetilde{T}_k^L} = \tilde{f}_i^{t,k,L}$), $\widetilde{T}_k^L(F_i^t) = F_i^t$ for $t \in \{0,1\}$ and

$$\widetilde{T}_k^L(C_z^k) = T_k^L(C_z^k) \qquad \forall k \geq \tilde{k}_1,\ \forall z \in Z_k. \tag{3.112}$$

Proposition 3.15. *For all $k \geq \tilde{k}_1$ and $L \in]0,1[$, let $\widetilde{T}_k^L$ be the function introduced in Definition 3.14. Then, there exists $\tilde{k}_2 \in \mathbb{N}$ such that $\widetilde{T}_k^L \in D_L$ $\forall k \geq \tilde{k}_2$. Moreover, there exists a sequence of positive numbers $(L_k)_k$ such that $\lim_{k\to\infty} L_k = 0$ and $\widetilde{T}_k^L \subseteq D_{L_k}$ $\forall k \geq \tilde{k}_2$.*

Proof. From Proposition 3.11 we infer that there exists a sequence of positive numbers $(\Lambda_k)_k$ such that $\lim_{k\to\infty} \Lambda_k = 0$ and

$$\mathrm{Lip}(\tilde{f}_i^{t,k,L}) \leq \Lambda_k \qquad \forall k \in \mathbb{N},\ \forall i \in \{1, \ldots, n\},\ \forall t \in [0,1], \tag{3.113}$$

$$|\tilde{f}_i^{t_1,k,L}(x) - \tilde{f}_i^{t_2,k,L}(x) - t_1 + t_2| \leq \Lambda_k |t_1 - t_2| \qquad \forall k \in \mathbb{N},\ \forall i \in \{1, \ldots, n\},\ \forall x \in F_i^0. \tag{3.114}$$

Since $\Lambda_k \to 0$, we can choose $k_n \in \mathbb{N}$ such that $\Lambda_k < \frac{1}{n}$ $\forall k \geq k_n$. Hence, taking into account that $\tilde{f}_i^{t,k,L} = f_i^{t,\widetilde{T}_k^L}$, from Lemma 3.2 we infer that, for all $k \geq k_n$, $\widetilde{T}_k^L \in D_{L_k}$ where

$L_k = \frac{(n+1)\sqrt{n}\,\Lambda_k}{1-n\Lambda_k}$. Since $L_k \to 0$ as $k \to \infty$, we can choose $\tilde{k}_2$ such that $L_k \le L\ \forall k \ge \tilde{k}_2$. So the proof is complete.

$\square$

Now, we can show that the function $u_k = u_k^{T_k^L}$ is a solution of problem (1.1) for k large enough and satisfies all the assertions of Proposition 2.2 (in a similar way one can argue for the function v_k).

Notice that $u_k^{T_k^L} = u_k^{\widetilde{T}_k^L}$, where $\widetilde{T}_k^L \in D_L$ is the function introduced in Definition 3.14, because $T_k^L(C_z^k) = \widetilde{T}_k^L(C_z^L)\ \forall z \in Z_k$.

First, we prove that $u_k^{\widetilde{T}_k^L}$ is a solution of the Dirichlet problem in every subdomain $\widetilde{T}_k^L(C_z^k)$ for all $z \in Z_k$ and then we show that it satisfies a suitable stationarity condition which allows us to prove that, indeed, it is a solution of the Dirichlet problem (1.1) in the domain Ω.

Lemma 3.16. *There exists $k_1(L) \in \mathbb{N}$ such that, for all $k \ge k_1(L)$ and $z \in Z_k$, the function $u_{k,z}^{\widetilde{T}_k^L}$ is a solution of the Dirichlet problem*

$$-\Delta u = |u|^{p-1}u + \psi \quad \text{in } \widetilde{T}_k^L(C_z^k), \qquad u = 0 \quad \text{on } \partial\widetilde{T}_k^L(C_z^k). \tag{3.115}$$

Proof. For all $k \in \mathbb{N}$ and $z \in Z_k$, let us consider the function $G_{k,z}^{\widetilde{T}_k^L} : \widetilde{T}_k^L(C_z^k) \times \mathbb{R} \to \mathbb{R}$ such that $G_{k,z}^{\widetilde{T}_k^L}(x,\cdot) \in \mathcal{C}^2(\mathbb{R})\ \forall x \in \widetilde{T}_k^L(C_z^k)$ and

$$G_{k,z}^{\widetilde{T}_k^L}(x,t) \;=\; \frac{|t|^{p+1}}{p+1} + \psi(x)t \qquad \text{if } \sigma(z)[t - \tilde{u}_{k,z}^{\widetilde{T}_k^L}(x)] \ge 0, \tag{3.116}$$

$$\frac{\partial^2 G_{k,z}^{\widetilde{T}_k^L}(x,t)}{\partial t^2} \;=\; \frac{\partial^2 G_{k,z}^{\widetilde{T}_k^L}}{\partial t^2}(x, \tilde{u}_{k,z}^{\widetilde{T}_k^L}(x)) \quad \text{if } \sigma(z)[t - \tilde{u}_{k,z}^{\widetilde{T}_k^L}(x)] \le 0, \tag{3.117}$$

where $\tilde{u}_{k,z}^{\widetilde{T}_k^L}$ is the function given by Lemma 2.4 which, for k large enough, is a solution of problem (3.115) because it is a local minimum of the functional E_ψ in $H_0^1(\widetilde{T}_k^L(C_z^k))$.

Moreover, let us set $g_{k,z}^{\widetilde{T}_k^L}(x,t) = \frac{\partial G_{k,z}^{\widetilde{T}_k^L}}{\partial t}(x,t)$. Then, let us consider the functional $E_{k,z,\widetilde{T}_k^L} : H_0^1(\widetilde{T}_k^L(C_z^k)) \to \mathbb{R}$ defined by

$$E_{k,z,\widetilde{T}_k^L}(u) = \frac{1}{2}\int_{\widetilde{T}_k^L(C_z^k)} |\nabla u|^2 dx - \int_{\widetilde{T}_k^L(C_z^k)} G_{k,z}^{\widetilde{T}_k^L}(x,u)\,dx. \tag{3.118}$$

Since $p > 1$, for k large enough one can verify that for all $u \not\equiv \tilde{u}_{k,z}^{\widetilde{T}_k^L}$ there exists $t_u > 0$ such that

$$E'_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t_u(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big)[u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}] = 0 \tag{3.119}$$

if and only if $[\sigma(z)[u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}(x)] \vee 0 \not\equiv 0$; in this case such a t_u is unique; in the other case we have

$$E'_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big)[u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}] > 0 \quad \forall t > 0 \tag{3.120}$$

and

$$\lim_{t\to\infty} E_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big) = \infty. \tag{3.121}$$

When $[\sigma(z)[u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}(x)] \vee 0 \not\equiv 0$, one can verify by direct computation that $E'_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big)[u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}]$ is positive for $t \in]0, t_u[$ and negative for $t > t_u$, so

$$E_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t_u(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big) = \max\{E_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big) \ : \ t > 0\}. \tag{3.122}$$

Moreover, one can verify that

$$E''_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t_u(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big)[u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}, u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}] < 0. \tag{3.123}$$

Assume, for example, that $\sigma(z) = 1$ (in a similar way one can argue when $\sigma(z) = -1$). In this case, we have

$$\begin{aligned} E_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big) =& E_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t[(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \vee 0]\big) + tE'_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L}\big)[(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0] \\ &+ \frac{t^2}{2} \int_{\widetilde{T}_k^L(C_z^k)} |\nabla[(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0]|^2 dx \\ &- \frac{t^2}{2}\, p \int_{\widetilde{T}_k^L(C_z^k)} |\tilde{u}_{k,z}^{\widetilde{T}_k^L}|^{p-1} [(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0]^2 dx \qquad \forall t > 0, \end{aligned} \tag{3.124}$$

where

$$E'_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L}\big)[(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0] = 0 \tag{3.125}$$

because $\tilde{u}_{k,z}^{\widetilde{T}_k^L}$ is a solution of problem (3.115).
Notice that

$$\begin{aligned} &\int_{\widetilde{T}_k^L(C_z^k)} |\tilde{u}_{k,z}^{\widetilde{T}_k^L}|^{p-1} [(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0]^2 dx \\ &\qquad \le \left(\int_{\widetilde{T}_k^L(C_z^k)} |\tilde{u}_{k,z}^{\widetilde{T}_k^L}|^{p+1} dx \right)^{\frac{p-1}{p+1}} \left(\int_{\widetilde{T}_k^L(C_z^k)} |(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0|^{p+1} dx \right)^{\frac{2}{p+1}} \end{aligned} \tag{3.126}$$

where, as follows from Lemma 2.4,

$$\lim_{k\to\infty} \max\left\{ \int_{\widetilde{T}_k^L(C_z^k)} |\tilde{u}_{k,z}^{\widetilde{T}_k^L}|^{p+1} dx \ : \ z \in Z_k \right\} = 0. \tag{3.127}$$

Moreover, we have

$$\int_{\widetilde{T}_k^L(C_z^k)} |\nabla[(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0]|^2 dx \ge \lambda_k \left(\int_{\widetilde{T}_k^L(C_z^k)} |(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \wedge 0|^{p+1} dx \right)^{\frac{2}{p+1}} \qquad \forall k \in \mathbb{N}, \ \forall z \in Z_k \tag{3.128}$$

where, for all $k \in \mathbb{N}$,

$$\lambda_k = \min \left\{ \int_{\widetilde{T}_k^L(C_z^k)} |\nabla v|^2 dx \ : \ z \in Z_k, \ v \in H_0^1\big(\widetilde{T}_k^L(C_z^k)\big), \ \int_{\widetilde{T}_k^L(C_z^k)} |v|^{p+1} dx = 1 \right\}. \quad (3.129)$$

We say that $\lim_{k\to\infty} \lambda_k = \infty$. In fact, otherwise, there exist suitable sequences $(z^k)_k$ in $\mathbb{R}^n$, and $(v_k)_k$ in $H_0^1(\Omega)$ such that $z^k \in Z_k$, $v_k \equiv 0$ in $\Omega \setminus C_{z^k}^k$, $\int_\Omega |v_k|^{p+1} dx = 1$ $\forall k \in \mathbb{N}$ and (up to a subsequence) $\lim_{k\to\infty} \int_\Omega |\nabla v_k|^2 dx < \infty$.
As a consequence, since $p < \frac{n+2}{n-2}$ when $n \geq 3$, there exists $\bar{v} \in H_0^1(\Omega)$ such that (up to a subsequence) $v_k \to \bar{v}$ as $k \to \infty$ weakly in $H_0^1(\Omega)$, in $L^{p+1}(\Omega)$ and almost everywhere in Ω.
Taking into account that $\lim_{k\to\infty} \operatorname{meas} \widetilde{T}_k^L(C_z^k) = 0$, the almost everywhere convegence implies that $\bar{v} \equiv 0$ in Ω, in contradiction with the convergence in $L^{p+1}(\Omega)$ because $\int_{\widetilde{T}_k^L(C_z^k)} |v_k|^{p+1} dx = 1$ $\forall k \in \mathbb{N}$. Thus, we can conclude that $\lim_{k\to\infty} \lambda_k = \infty$.
It follows that, for k large enough,

$$E_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t[(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}) \vee 0]\big) \leq E_{k,z,\widetilde{T}_k^L}\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} + t(u - \tilde{u}_{k,z}^{\widetilde{T}_k^L})\big) \qquad \forall t > 0, \ \forall u \in H_0^1(C_z^k). \ (3.130)$$

As a consequence, if we set

$$\Gamma = \{u \in H_0^1(\widetilde{T}_k^L(C_z^k)) \ : \ u \not\equiv \tilde{u}_{k,z}^{\widetilde{T}_k^L}, \ E'_{k,z,\widetilde{T}_k^L}(u)[u - \tilde{u}_{k,z}^{\widetilde{T}_k^L}] = 0\}, \quad (3.131)$$

we have $u_{k,z}^{\widetilde{T}_k^L} \in \Gamma$ and

$$E_\psi(u_{k,z}^{\widetilde{T}_k^L}) = E_{k,z,\widetilde{T}_k^L}(u_{k,z}^{\widetilde{T}_k^L}) = \min_\Gamma E_{k,z,\widetilde{T}_k^L}. \quad (3.132)$$

Therefore, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$E'_{k,z,\widetilde{T}_k^L}(u_{k,z}^{\widetilde{T}_k^L})[v] = \mu \left\{ E''_{k,z,\widetilde{T}_k^L}(u_{k,z}^{\widetilde{T}_k^L})[u_{k,z}^{\widetilde{T}_k^L} - \tilde{u}_{k,z}^{\widetilde{T}_k^L}, v] + E'_{k,z,\widetilde{T}_k^L}(u_{k,z}^{\widetilde{T}_k^L})[v] \right\} \qquad \forall v \in H_0^1(C_z^k). \quad (3.133)$$

In particular, if we choose $v = u_{k,z}^{\widetilde{T}_k^L} - \tilde{u}_{k,z}^{\widetilde{T}_k^L}$, we obtain $\mu = 0$ because

$$E'_{k,z,\widetilde{T}_k^L}(u_{k,z}^{\widetilde{T}_k^L})[u_{k,z}^{\widetilde{T}_k^L} - \tilde{u}_{k,z}^{\widetilde{T}_k^L}] = 0 \quad (3.134)$$

while

$$E''_{k,z,\widetilde{T}_k^L}(u_{k,z}^{\widetilde{T}_k^L})[u_{k,z}^{\widetilde{T}_k^L} - \tilde{u}_{k,z}^{\widetilde{T}_k^L}, u_{k,z}^{\widetilde{T}_k^L} - \tilde{u}_{k,z}^{\widetilde{T}_k^L}] \neq 0. \quad (3.135)$$

Thus, $u_{k,z}^{\widetilde{T}_k^L}$ is a weak solution of the Dirichlet problem

$$-\Delta u_{k,z}^{\widetilde{T}_k^L} = g(x, u_{k,z}^{\widetilde{T}_k^L}) \quad \text{ in } \widetilde{T}_k^L(C_z^k), \qquad u = 0 \quad \text{ on } \partial \widetilde{T}_k^L(C_z^k). \quad (3.136)$$

On the other hand, since $u_{k,z}^{\widetilde{T}_k^L} \geq \tilde{u}_{k,z}^{\widetilde{T}_k^L}$ in $\widetilde{T}_k^L(C_z^k)$, we have

$$g\big(x, u_{k,z}^{\widetilde{T}_k^L}(x)\big) = |u_{k,z}^{\widetilde{T}_k^L}(x)|^{p-1} u_{k,z}^{\widetilde{T}_k^L}(x) + \psi(x) \qquad \forall x \in \widetilde{T}_k^L(C_z^k). \quad (3.137)$$

So $u_{k,z}^{\widetilde{T}_k^L}$ is a solution of problem (3.115) and the proof is complete.

□

Proposition 3.17. *Under the assumptions of Proposition 2.2, there exists $\bar{k} \in \mathbb{N}$ such that the function $u_k^{\widetilde{T}_k^L} = \sum_{z\in Z_k} u_{k,z}^{\widetilde{T}_k^L}$ is a solution of problem* (1.1) *for all $k \geq \bar{k}$.*

Proof. From Lemma 3.16 we infer that, for a suitable $k_1(L) \in \mathbb{N}$, $E'_\psi(u_k^{\widetilde{T}_k^L})[v] = 0$ for all $v \in H_0^1(\widetilde{T}_k^L(C_z^k))$, $z \in Z_k$, $k \geq k_1(L)$.
Now, we have to prove that $E'_\psi(u_k^{\widetilde{T}_k^L})[v] = 0$ $\forall v \in H_0^1(\Omega)$. Taking into account Lemma 3.16, we obtain

$$\begin{aligned} E'_\psi(u_k^{\widetilde{T}_k^L})[v] &= \int_\Omega [\nabla u_k^{\widetilde{T}_k^L} \cdot \nabla v - |u_k^{\widetilde{T}_k^L}|^{p-1} u_k^{\widetilde{T}_k^L} v - \psi v]\, dx \\ &= \sum_{k\in Z_k} \int_{\widetilde{T}_k^L(C_z^k)} [\nabla u_k^{\widetilde{T}_k^L} \cdot \nabla v - |u_k^{\widetilde{T}_k^L}|^{p-1} u_k^{\widetilde{T}_k^L} v - \psi v]\, dx \\ &= \sum_{k\in Z_k} \int_{\partial\widetilde{T}_k^L(C_z^k)} v\, (\nabla u_k^{\widetilde{T}_k^L} \cdot \nu_{k,z})\, d\sigma \end{aligned} \tag{3.138}$$

wher $\nu_{k,z}$ denotes the outward normal on $\partial\widetilde{T}_k^L(C_z^k)$. Thus, in order to obtain $E'_\psi(u_k^{\widetilde{T}_k^L})[v] = 0$, we prove that

$$\nabla u_{k,z_1}^{\widetilde{T}_k^L}(x) = \nabla u_{k,z_2}^{\widetilde{T}_k^L}(x) \qquad \forall x \in \partial\widetilde{T}_k^L(C_{z_1}^k) \cap \partial\widetilde{T}_k^L(C_{z_2}^k) \tag{3.139}$$

for all $z_1, z_2 \in Z_k$ such that $|z_1 - z_2| = 1$ (that is when $\widetilde{T}_k^L(C_{z_1}^k)$ and $\widetilde{T}_k^L(C_{z_2}^k)$ are adjacent subdomains of Ω).
Notice that, since for all $k \geq k_1(L)$ and $z \in Z_k$ the function $u_{k,z}^{\widetilde{T}_k^L}$ is a solution of problem (3.115), for all vector field $\Phi \in \mathcal{C}_0^1(\overline{\Omega}, \mathbb{R}^n)$ we obtain

$$\begin{aligned} E'_\psi(u_k^{\widetilde{T}_k^L})[\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}] &= \int_\Omega [\nabla u_k^{\widetilde{T}_k^L} \cdot \nabla(\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}) - |u_k^{\widetilde{T}_k^L}|^{p-1} u_k^{\widetilde{T}_k^L} (\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}) - \psi(\Phi \cdot \nabla u_k^{\widetilde{T}_k^L})]\, dx \\ &= \sum_{z\in Z_k} \int_{\widetilde{T}_k^L(C_z^k)} [\nabla u_k^{\widetilde{T}_k^L} \cdot \nabla(\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}) - |u_k^{\widetilde{T}_k^L}|^{p-1} u_k^{\widetilde{T}_k^L} (\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}) - \psi(\Phi \cdot u_k^{\widetilde{T}_k^L})]\, dx \\ &= \sum_{z\in Z_k} \int_{\partial\widetilde{T}_k^L(C_z^k)} (\nabla u_k^{\widetilde{T}_k^L} \cdot \nu_{k,z})^2 (\Phi \cdot \nu_{k,z})\, d\sigma. \end{aligned} \tag{3.140}$$

Thus, it is easy to verify that in order to prove (3.139) it suffices to show that there exists $\bar{k}$ such that

$$E'_\psi(u_k^{\widetilde{T}_k^L})[\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}] = 0 \qquad \forall \Phi \in \mathcal{C}_0^1(\overline{\Omega}, \mathbb{R}^n),\ \forall k \geq \bar{k}. \tag{3.141}$$

From Proposition 3.15 we infer that there exists $\bar{k} \geq k_1(L)$ such that $\widetilde{T}_k^L \in D_{L/2}$ $\forall k \geq \bar{k}$.
Now, for all $\tau \in \mathbb{R}$ and $\Phi \in \mathcal{C}_0^1(\overline{\Omega}, \mathbb{R}^n)$, let us consider the function $T_{\tau,\Phi} : \overline{\Omega} \to \overline{\Omega}$ defined by the Cauchy problem

$$\frac{\partial T_{\tau,\Phi}(x)}{\partial \tau} = \Phi \circ T_{\tau,\Phi}(x), \qquad T_{0,\Phi}(x) = x \quad \forall \tau \in \mathbb{R},\ \forall x \in \overline{\Omega}. \tag{3.142}$$

One can verify by standard arguments that for all $\Phi \in \mathcal{C}_0^1(\overline{\Omega}, \mathbb{R}^n)$ there exists $\bar\tau_\Phi > 0$ such that $T_{\tau,\Phi} \circ \widetilde{T}_k^L \in D_L$ $\forall \tau \in [-\bar\tau_\Phi, \bar\tau_\Phi]$. It follows that

$$E_\psi(u_k^{\widetilde{T}_k^L}) = \sum_{z\in Z_k} E_\psi(u_{k,z}^{\widetilde{T}_k^L}) \le \sum_{z\in Z_k} E_\psi(u_{k,z}^{T_{\tau,\Phi}\circ\widetilde{T}_k^L}) = E_\psi(u_k^{T_{\tau,\Phi}\circ\widetilde{T}_k^L}) \qquad \forall \tau \in [-\bar\tau_\Phi, \bar\tau_\Phi] \quad (3.143)$$

because of the minimality of $\widetilde{T}_k^L$. Moreover, notice that

$$\frac{d}{d\tau} E_\psi(u_k^{\widetilde{T}_k^L} \circ T_{\tau,\Phi}^{-1})_{|\tau=0} = -E'_\psi(u_k^{\widetilde{T}_k^L})[\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}], \tag{3.144}$$

so we have to prove that

$$\frac{d}{d\tau} E_\psi(u_k^{\widetilde{T}_k^L} \circ T_{\tau,\Phi}^{-1})_{|\tau=0} = 0. \tag{3.145}$$

Arguing by contradiction, assume that (3.145) does not hold. We can assume, for example, that

$$\frac{d}{d\tau} E_\psi(u_k^{\widetilde{T}_k^L} \circ T_{\tau,\Phi}^{-1})_{|\tau=0} < 0 \tag{3.146}$$

(otherwise we replace Φ by $-\Phi$). Therefore, there exists a sequence of positive numbers $(\tau_i)_i$ such that $\lim_{i\to\infty} \tau_i = 0$ and

$$E_\psi(u_k^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1}) < E_\psi(\tilde u_k^{\widetilde{T}_k^L}) \qquad \forall i \in \mathbb{N}. \tag{3.147}$$

We say that, as a consequence of (3.146), for i large enough we have

$$\max\left\{ \sum_{z\in Z_k} E_\psi\big(\tilde u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} + t_z(u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} - \tilde u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1})\big) \ : \ t_z \ge 0 \ \forall z \in Z_k \right\} < E_\psi(u_k^{\widetilde{T}_k^L}). \tag{3.148}$$

In fact, arguing by contradiction, assume that (up to a subsequence still denoted by $(\tau_i)_i$) the inequality (3.148) does not hold.
Then, for all $i \in \mathbb{N}$ and $z \in Z_k$, there exists $t_{z,i} \ge 0$ such that

$$\sum_{z\in Z_k} E_\psi\big(\tilde u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} + t_{z,i}(u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} - \tilde u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1})\big) \ge E_\psi(u_k^{\widetilde{T}_k^L}) \qquad \forall i \in \mathbb{N}. \tag{3.149}$$

Since $p > 1$, the sequence $(t_{z,i})_i$ is bounded $\forall z \in Z_k$. Moreover, taking into account that

$$E_\psi(\tilde u_{k,z}^{\widetilde{T}_k^L} + t(u_{k,z}^{\widetilde{T}_k^L} - \tilde u_{k,z}^{\widetilde{T}_k^L})\big) < E_\psi(u_{k,z}^{\widetilde{T}_k^L}) \qquad \forall t \ne 1,\ \forall z \in Z_k, \tag{3.150}$$

we infer that $\lim_{i\to\infty} t_{z,i} = 1$ $\forall z \in Z_k$ and

$$\begin{aligned} \sum_{z\in Z_k} E_\psi\big(\tilde u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} + t_{z,i}(u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} - \tilde u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1})\big) &\ge E_\psi(u_k^{\widetilde{T}_k^L}) \\ &\ge \sum_{z\in Z_k} E_\psi\big(\tilde u_{k,z}^{\widetilde{T}_k^L} + t_{z,i}(u_{k,z}^{\widetilde{T}_k^L} - \tilde u_{k,z}^{\widetilde{T}_k^L})\big) \qquad \forall i \in \mathbb{N}. \end{aligned} \tag{3.151}$$

As a consequence, for all $i \in \mathbb{N}$ there exists $\tau_i' \in]0, \tau_i[$ such that

$$\frac{d}{d\tau} \sum_{z \in Z_k} E_\psi\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau,\Phi}^{-1} + t_{z,i}(u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau,\Phi}^{-1} - \tilde{u}_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau,\Phi}^{-1})\big)_{|_{\tau=\tau_i'}} \ge 0 \qquad \forall i \in \mathbb{N} \tag{3.152}$$

which, as $i \to \infty$, implies

$$\frac{d}{d\tau} E_\psi(u_k^{\widetilde{T}_k^L} \circ T_{\tau,\Phi}^{-1})_{|\tau=0} \ge 0, \tag{3.153}$$

in contradiction with (3.146). Thus, (3.148) holds. From Lemma 2.4 we infer that, if we choose $\bar{k}$ large enough, for all $k \ge \bar{k}$, $z \in Z_k$ and $i \in \mathbb{N}$ there exists a unique minimizing function $\tilde{u}_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}$. Moreover, $\tilde{u}_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L} \to \tilde{u}_{k,z}^{\widetilde{T}_k^L}$ in $H_0^1(\Omega)$, as $i \to \infty$, $\forall k \ge \bar{k}$, $\forall z \in Z_k$.
Then, using the functions $\tilde{u}_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}$ and arguing as in the proof of Lemma 3.16, for i large enough we obtain the functions $u_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}$.
The construction of the functions $\tilde{u}_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}$ and $u_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}$ shows also that

$$E_\psi(\tilde{u}_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}) \le E_\psi(\tilde{u}_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1}) \tag{3.154}$$

and

$$E_\psi(u_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}) \le \max\left\{ E_\psi(\tilde{u}_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} + t\big(u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} - \tilde{u}_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1})\big) \ : \ t \ge 0 \right\} \qquad \forall z \in Z_k. \tag{3.155}$$

Therefore, from (3.148) and (3.155) we obtain

$$\begin{aligned} E_\psi(u_k^{\widetilde{T}_k^L}) &> \max\left\{ \sum_{z \in Z_k} E_\psi\big(\tilde{u}_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} + t_z(u_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} - \tilde{u}_{k,z}^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1})\big) : \ t_z \ge 0 \ \forall z \in Z_k \right\} \\ &\ge \sum_{z \in Z_k} E_\psi\big(u_{k,z}^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}\big) = E_\psi\big(u_k^{T_{\tau_i,\Phi} \circ \widetilde{T}_k^L}\big) \end{aligned} \tag{3.156}$$

for i large enough, in contradiction with (3.143).
So we can conclude that $\frac{d}{d\tau} E_\psi\left(u_k^{\widetilde{T}_k^L} \circ T_{\tau_i,\Phi}^{-1} \right) = 0$, that is $E_\psi'(u_k^{\widetilde{T}_k^L})[\Phi \cdot \nabla u_k^{\widetilde{T}_k^L}] = 0$ for all vector field $\Phi \in \mathcal{C}_0^1(\overline{\Omega}, \mathbb{R}^n)$.
Thus, $u_k^{\widetilde{T}_k^L}$ is a solution of problem (1.1) for all $k \ge \bar{k}$.

□

Proof of Proposition 2.2 (conclusion). If Ω is the cube (2.1), all the assertions of Proposition 2.2 hold for k large enough if we set $u_k = u_k^{\widetilde{T}_k^L}$ and $T_{k,u} = T_k^L$ (or $T_{k,u} = \widetilde{T}_k^L$) where the function $u_k^{T_k^L}$ and the admissible deformation T_k^L are obtained by the minimizing method described in Section 2 (the functions $v_k = v_k^{T_{k,v}}$ are obtained in a similar way: it suffices to replace $\sigma(z)$ by $\sigma(z)+1$). Notice that we have $u_k^{T_k^L} = u_k^{\widetilde{T}_k^L}$ (where $\widetilde{T}_k^L : \overline{\Omega} \to \overline{\Omega}$ is the function introduced in Definition 3.14) because $T_k^L(C_z^k) = \widetilde{T}_k^L(C_z^k)$ $\forall z \in Z_k$.

In fact, Proposition 3.17 guarantees that there exists $\bar{k} \in \mathbb{N}$ such that $u_k^{\widetilde{T}_k^L}$ is a solution of problem (1.1) for all $k \geq \bar{k}$.
The asymptotic behaviour of u_k as $k \to \infty$ is described by Proposition 3.5, Lemma 3.7 and Proposition 3.15. In fact, Proposition 3.5 shows that for every choice of z^k in Z_k, up to a subsequence, the function U_{z^k}, as $k \to \infty$ converges in $H^1(\chi)$ to a positive solution U_χ of the Dirichlet problem $-\Delta U = |U|^{p-1}U$ in χ, $U = 0$ on $\partial\chi$, satisfying

$$\left(\int_\chi |U_\chi|^{p+1}dx\right)^{-\frac{2}{p+1}} \int_\chi |\nabla U_\chi|^2 dx = \min\left\{\int_\chi |\nabla U|^2 dx \ : \ U \in H_0^1(\chi), \ \int_\chi |U|^{p+1}dx = 1\right\} \tag{3.157}$$

where χ is a bounded domain of $\mathbb{R}^n$. Lemma 3.7 says that the minimality of the admissible deformation T_k^L implies that χ must be a cube of $\mathbb{R}^n$ having a vertex in the origin and the sides of length 1 and finally Proposition 3.15 guarantees that $\widetilde{T}_k^L \in D_{L_k}$ for a suitable sequence $(L_k)_k$ in $]0,1[$ such that $\lim_{k\to\infty} L_k = 0$ so, as a consequence, $\chi = C_0^1 = \Omega$ and $\widetilde{T}_k^L$ converges as $k \to \infty$ to the identity function uniformly in Ω (as pointed out in Remark 3.4).
Notice that $T_k^L \in D_L \ \forall k \in \mathbb{N}$ but, unlike $\widetilde{T}_k^L$, we cannot say that $T_k^L \in D_{L_k} \ \forall k \in \mathbb{N}$. However, we can say that also T_k^L converges to the identity function uniformly in Ω because, taking into account the definition of $\widetilde{T}_k^L$, we have

$$T_k^L(x) \in \widetilde{T}_k^L(C_z^k) \qquad \forall x \in C_z^k, \ \forall z \in Z_k \tag{3.158}$$

so, as a consequence,

$$\sup\{|T_k^L(x) - \widetilde{T}_k^L(x)| \ : \ x \in \Omega\} \leq (1+L)\frac{\sqrt{n}}{k}. \tag{3.159}$$

Therefore, all the assertions in Proposition 2.2 hold for $T_{k,u} = \widetilde{T}_k^L$ and also for $T_{k,u} = T_k^L$, so the proof is complete.

□

Theorem 2.1 is a direct consequence of Proposition 2.2.

Notice that our method to construct solutions having this checked nodal structure does not require any technique of deformation from the symmetry and it works in case of more general nonlinearities, even when they are not perturbations of symmetric nonlinearities by lower order terms.
For example, it works when in problem (1.1) the term $|u|^{p-1}u + \psi$ is replaced by $c_+(u^+)^p - c_-(u^-)^p + \psi$ with $c_+ > 0$, $c_- > 0$ and $c_+ \neq c_-$.
Moreover, notice that our method works also when the nonlinear term has critical growth. For example, for $n > 2$ and $\lambda \in \mathbb{R}$ let us consider the Dirichlet problem

$$-\Delta u = |u|^{\frac{4}{n-2}}u + \lambda u + \psi \quad \text{ in } \Omega, \qquad u = 0 \quad \text{ on } \partial\Omega \tag{3.160}$$

whose solutions are critical points of the energy functional $\mathcal{F} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{n-2}{2n}\int_\Omega |u|^{\frac{2n}{n-2}}dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega \psi u \, dx. \tag{3.161}$$

It is well known that, if $\lambda = 0$ and $\psi \equiv 0$ in Ω, problem (3.160) has only the trivial solution $u \equiv 0$ for every bounded starshaped domain Ω, as a consequence of the Pohozaev identity (see [36]).
When $n \geq 4$, $\psi \equiv 0$ in Ω and λ is positive and strictly less than the first eigenvalue of the Laplace operator $-\Delta$ in $H_0^1(\Omega)$, there exists a positive solution that concentrates as a Dirach mass as $\lambda \to 0$ (see [12, 13] etc.); the existence of nodal solutions is studied for example in [16] etc..
Notice that also when $\psi \equiv 0$ in Ω, so that the functional $\mathcal{F}$ is even, the problem of finding infinitely many solutions is difficult because the well known Palais-Smale compactness condition is not satisfied, as a consequence of the presence of the critical Sobolev exponent (see [12, 13, 44] etc.).
When Ω is a cube of $\mathbb{R}^n$, our method, combined with some estimates as in [13], allows us to construct infinitely many solutions with many nodal regions and arbitrarily large energy level for all $\lambda > 0$ and $\psi \in L^2(\Omega)$.
In fact, as we prove in a paper in preparation, the following theorem holds (see also [27] for the particular case where Ω is a cube and $\psi \equiv 0$ in Ω).

Theorem 3.18. *Assume that Ω is a cube of $\mathbb{R}^n$ with $n \geq 4$ and $\lambda > 0$. Then, for every $\psi \in L^2(\Omega)$, problem (3.160) admits infinitely many solutions.*
More precisely, if Ω is for example the cube (2.1), for all $\psi \in L^2(\Omega)$ there exists $\bar{k} \in \mathbb{N}$ such that, for every $k \geq \bar{k}$, problem (3.160) admits a solution u_k having the following properties.
For all $k \geq \bar{k}$ there exists $T_k \in D_L$ such that, for every choice of z^k in Z_k, the function $u_{k,z^k} := u_k|_{T_k(C^k_{z^k})}$ belongs to $H_0^1\big(T_k(C^k_{z^k})\big)$ (here we consider u_{k,z^k} extended by the value zero in $\mathbb{R}^n \setminus \Omega$).
Moreover, there exist $\varepsilon_k > 0$ and $m_k \in C_0^1 = \Omega$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and the function U_k defined by

$$U_k(x) = \sigma(z^k) \left(\frac{\varepsilon_k}{k}\right)^{\frac{n-2}{2}} u_{k,z^k} \left[\frac{\varepsilon_k}{k}\, x + T_k \left(\frac{z^k + m_k}{k}\right)\right] \qquad \forall x \in \mathbb{R}^n,\ \forall k \in \mathbb{N} \tag{3.162}$$

converges as $k \to \infty$ to a function $\overline{U} \in \mathcal{D}(\mathbb{R}^n)$ such that

$$-\Delta \overline{U} = \overline{U}^{\frac{n+2}{n-2}}, \qquad \overline{U} > 0 \ \text{ in } \mathbb{R}^n, \qquad \overline{U}(0) = \max_{\mathbb{R}^n} \overline{U}. \tag{3.163}$$

The sequence $(T_k)_k$ converges to the identity map uniformly in Ω while the domains $k\Big[T_k(C^k_{z^k}) - T_k\left(\frac{z^k}{k}\right)\Big]$ tend to the cube Ω as $k \to \infty$ for every choice of z^k in Z_k.
Furthermore, for all $k \geq \bar{k}$ there exists also another solution v_k of problem (3.160) such that the function $-v_k$ presents an asymptotic behaviour as u_k when $k \to \infty$.

Acknowledgments: The authors have been supported by the INdAM-GNAMPA group; R.M. acknowledges also the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome “Tor Vergata”, CUP E83C18000100006.

References

[1] A. Ambrosetti. On the existence of multiple solutions for a class of nonlinear boundary value problems. *Rend. Sem. Mat. Univ. Padova*, 49:195–204, 1973.

[2] A. Ambrosetti. A perturbation theorem for superlinear boundary value problems. Tech. Sum. Report 1446, Univ. Wisconsin-Madison Math. Res. Center, 1974.

[3] A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Functional Analysis*, 14:349–381, 1973.

[4] A. Bahri. Topological results on a certain class of functionals and application. *J. Functional Analysis*, 41(3):397–427, 1981.

[5] A. Bahri and H. Berestycki. Points critiques de perturbations de fonctionnelles paires et applications. *C. R. Acad. Sci. Paris Sér. A-B*, 291(3):A189–A192, 1980.

[6] A. Bahri and H. Berestycki. A perturbation method in critical point theory and applications. *Trans. Amer. Math. Soc.*, 267(1):1–32, 1981.

[7] A. Bahri and H. Berestycki. Forced vibrations of superquadratic Hamiltonian systems. *Acta Math.*, 152(3-4):143–197, 1984.

[8] A. Bahri and P.-L. Lions. Morse index of some min-max critical points. I. Application to multiplicity results. *Comm. Pure Appl. Math.*, 41(8):1027–1037, 1988.

[9] T. Bartsch and M. Clapp. Critical point theory for indefinite functionals with symmetries. *J. Funct. Anal.*, 138(1):107–136, 1996.

[10] P. Bolle. On the Bolza problem. *J. Differential Equations*, 152(2):274–288, 1999.

[11] P. Bolle, N. Ghoussoub, and H. Tehrani. The multiplicity of solutions in non-homogeneous boundary value problems. *Manuscripta Math.*, 101(3):325–350, 2000.

[12] H. Brezis. Elliptic equations with limiting Sobolev exponents—the impact of topology. volume 39, pages S17–S39. 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985).

[13] H. Brézis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[14] J. Byeon and P. H. Rabinowitz. On a phase transition model. *Calc. Var. Partial Differential Equations*, 47(1-2):1–23, 2013.

[15] J. Byeon and P. H. Rabinowitz. Unbounded solutions for a periodic phase transition model. *J. Differential Equations*, 260(2):1126–1153, 2016.

[16] G. Cerami, S. Solimini, and M. Struwe. Some existence results for superlinear elliptic boundary value problems involving critical exponents. *J. Funct. Anal.*, 69(3):289–306, 1986.

[17] C. Chambers and N. Ghoussoub. Deformation from symmetry and multiplicity of solutions in non-homogeneous problems. *Discrete Contin. Dyn. Syst.*, 8(1):267–281, 2002.

[18] M. Clapp and D. Puppe. Critical point theory with symmetries. *J. Reine Angew. Math.*, 418:1–29, 1991.

[19] C. V. Coffman. A minimum-maximum principle for a class of non-linear integral equations. *J. Analyse Math.*, 22:391–419, 1969.

[20] C. V. Coffman. On a class of non-linear elliptic boundary value problems. *J. Math. Mech.*, 19:351–356, 1969/1970.

[21] M. Conti, S. Terracini, and G. Verzini. Nodal solutions to a class of nonstandard superlinear equations on $\mathbb{R}^N$. *Adv. Differential Equations*, 7(3):297–318, 2002.

[22] M. Conti, S. Terracini, and G. Verzini. An optimal partition problem related to nonlinear eigenvalues. *J. Funct. Anal.*, 198(1):160–196, 2003.

[23] M. Conti, S. Terracini, and G. Verzini. On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae. *Calc. Var. Partial Differential Equations*, 22(1):45–72, 2005.

[24] H. Ehrmann. Nachweis periodischer Lösungen bei gewissen nichtlinearen Schwingungsdifferentialgleichungen. *Arch. Rational Mech. Anal.*, 1:124–138, 1957.

[25] H. Ehrmann. Über die Existenz der Lösungen von Randwertaufgaben bei gewöhnlichen nichtlinearen Differentialgleichungen zweiter Ordnung. *Math. Ann.*, 134:167–194, 1957.

[26] I. Ekeland and N. Ghoussoub. Z_2-equivariant Ljusternik-Schnirelman theory for non-even functionals. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 15(3):341–370, 1998.

[27] D. Fortunato and E. Jannelli. Infinitely many solutions for some nonlinear elliptic problems in symmetrical domains. *Proc. Roy. Soc. Edinburgh Sect. A*, 105:205–213, 1987.

[28] S. Fučík and V. Lovicar. Periodic solutions of the equation $x''(t) + g(x(t)) = p(t)$. *Časopis Pěst. Mat.*, 100(2):160–175, 1975.

[29] J. A. Hempel. *Superlinear variational boundary value problems and nonuniqueness.* Ph.d. thesis, University of New England, 1970.

[30] J. A. Hempel. Multiple solutions for a class of nonlinear boundary value problems. *Indiana Univ. Math. J.*, 20:983–996, 1970/71.

[31] M. A. Krasnosel'skii. *Topological methods in the theory of nonlinear integral equations.* A Pergamon Press Book. The Macmillan Company, New York, 1964. Translated by A. H. Armstrong; translation edited by J. Burlak.

[32] A. Marino and G. Prodi. Metodi perturbativi nella teoria di Morse. *Boll. Un. Mat. Ital. (4)*, 11(3, suppl.):1–32, 1975. Collection of articles dedicated to Giovanni Sansone on the occasion of his eighty-fifth birthday.

[33] C. Miranda. Un'osservazione su un teorema di Brouwer. *Boll. Un. Mat. Ital. (2)*, 3:5–7, 1940.

[34] R. Molle and D. Passaseo. Infinitely many solutions for elliptic equations with non-symmetric nonlinearities. *Calc. Var. Partial Differential Equations*, 61(3):Paper No. 115, 2022.

[35] Z. Nehari. Characteristic values associated with a class of non-linear second-order differential equations. *Acta Math.*, 105:141–175, 1961.

[36] S. I. Pohožaev. On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Soviet Math. Dokl.*, 6:1408–1411, 1965.

[37] P. H. Rabinowitz. Variational methods for nonlinear eigenvalue problems. In *Eigenvalues of non-linear problems (Centro Internaz. Mat. Estivo (C.I.M.E.), III Ciclo, Varenna, 1974)*, pages 139–195. 1974.

[38] P. H. Rabinowitz. Multiple critical points of perturbed symmetric functionals. *Trans. Amer. Math. Soc.*, 272(2):753–769, 1982.

[39] P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[40] M. Ramos, H. Tavares, and W. Zou. A Bahri-Lions theorem revisited. *Adv. Math.*, 222(6):2173–2195, 2009.

[41] M. Struwe. Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems. *Manuscripta Math.*, 32(3-4):335–364, 1980.

[42] M. Struwe. Infinitely many solutions of superlinear boundary value problems with rotational symmetry. *Arch. Math. (Basel)*, 36(4):360–369, 1981.

[43] M. Struwe. Superlinear elliptic boundary value problems with rotational symmetry. *Arch. Math. (Basel)*, 39(3):233–240, 1982.

[44] M. Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187(4):511–517, 1984.

[45] K. Tanaka. Morse indices at critical points related to the symmetric mountain pass theorem and applications. *Comm. Partial Differential Equations*, 14(1):99–128, 1989.

[46] S. Terracini and G. Verzini. Oscillating solutions to second-order ODEs with indefinite superlinear nonlinearities. *Nonlinearity*, 13(5):1501–1514, 2000.