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Towards integrable perturbation of 2*d* CFT on de Sitter space

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Abstract
We describe a procedure to deform the dynamics of a two-dimensional conformal net to possibly obtain a Haag–Kastler net on the de Sitter spacetime. The new dynamics is given by adding a primary field smeared on the time-zero circle to the Lorentz generators of the conformal net. As an example, we take an extension of the chiral $\mathrm{U}(1)$-current net by a charged field with conformal dimension $d < \frac{1}{4}$. We show that the perturbing operators are defined on a dense domain.

Keywords Modular Hamiltonian · Geodesic KMS condition · de Sitter space · Primary fields · Conformal field theory · Integrable perturbation

Mathematics Subject Classification 81T05 · 81T40 · 46L60

1 Introduction

The first interacting quantum field theories, the $\mathscr{P}(\phi)_2$-models, have been constructed by starting with the free field on the Minkowski space, defining an interaction term, perturbing the dynamics by it locally, finding the interacting vacuum and changing the Hilbert space [11]. The $\mathscr{P}(\phi)_2$-models have been constructed also on the de Sitter space [10], then recently formulated into the operator-algebraic framework [5].

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Perturbing the dynamics on the de Sitter space has the advantage that one may construct interacting models on the same Hilbert space, as one can avoid Haag's theorem [24].

The procedure of perturbing the dynamics has been formulated in [13]: One starts with a Haag–Kastler net on the de Sitter space (in the sense of [3]), then alters the Lorentz boosts by defining the new ones as the modular groups for a rotation invariant, interacting vacuum vector. The above $\mathscr{P}(\phi)_2$-models fit in this programme. As the arguments do not depend on the properties of free fields, one may wish to find other examples. We propose such an example in this work, where the starting QFT is a two-dimensional conformal field theory and the perturbation is given by a primary field. Specifically, the conformal field theory is a two-dimensional extension of the chiral U(1)-current algebra, and we take the charge-carrying field as the interaction term. Such fields have been constructed recently as two-dimensional conformal Wightman fields [1]. Such a conformal field can be seen as a field on the de Sitter space through a conformal map.

The U(1)-current algebra is defined on the Hilbert space $\mathcal{H}_0$, and the two-dimensional extension contains two copies of it as the left and right chiral components. The chiral components have charged sectors $\mathcal{H}_\alpha$ parametrized by $\alpha \in \mathbb{R}$. For a fixed $\alpha_0 \in \mathbb{R}$, we take $\bigoplus_{j\in\mathbb{Z}} \mathcal{H}_{j\alpha_0} \otimes \mathcal{H}_{j\alpha_0}$ as the Hilbert space[1]. For $\alpha \in \alpha_0\mathbb{Z}$, there is a charged field $Y_\alpha(z)$ that maps $\mathcal{H}_\beta \otimes \mathcal{H}_\beta$ to $\mathcal{H}_{\beta+\alpha} \otimes \mathcal{H}_{\beta+\alpha}$, $\beta \in \alpha_0\mathbb{Z}$. This is the basis of our perturbing field. We show that the symmetric field $Y_\alpha(z) \otimes Y_\alpha(z^{-1}) + Y_\alpha(z)^* \otimes Y_\alpha(z^{-1})^*$ can be added to the Lorentz generators of the de Sitter space, and they still satisfy the Lorentz relations weakly.

From a physical point of view, perturbing the Hamiltonian of a CFT by a (relevant) field has been proposed to obtain massive integrable models in [26]. Depending on the initial CFT and the perturbing field, various integrable models should be obtained. While our results are specific to the U(1)-current, the proof of (weak) Lorentz relations depends essentially on the fact that we take a primary field that is commutative at the time-zero circle. Therefore, the idea should generalize to many CFTs and primary fields.

This paper is organized as follows. In Sect. 2, we briefly recall the algebraic framework on the de Sitter spacetime, how a two-dimensional CFT can be considered on the de Sitter spacetime and the perturbation of the dynamics by a local field. In Sect. 3, a family of two-dimensional extensions of the U(1)-current net and their charged fields are reviewed. In Sect. 4, we make estimates of the charged fields restricted to the time-zero circle and show that the restriction defines operators if the charge α satisfies $|\alpha| < \frac{1}{\sqrt{2}}$. Section 5 shows that the time-zero charged fields commute with each other. In Sect. 6, we show that the Lorentz generators perturbed by the charged field still satisfy the Lorentz relations weakly on a certain domain. In Sect. 7, we describe how this programme can be completed.

[1] From a general point of view, it is more natural to take $\bigoplus_{j\in\mathbb{Z}} \mathcal{H}_{j\alpha_0} \otimes \mathcal{H}_{-j\alpha_0}$ (cf. [16]). In this case, the resulting net is unitarily equivalent through the map $J(z) \to -J(z)$, which can be unitarily implemented. That is, by denoting θ the adjoint action by this unitary, one has $\rho_\alpha \circ \theta = \theta \circ \rho_{-\alpha}$. This should be distinguished from **unitary (non)equivalence** of sectors, $\mathrm{Ad}\, U \circ \rho_\alpha \neq \rho_{-\alpha}$ for any U if $\alpha \neq 0$. Note that $J(z)$ is defined in Sect. 3.1.

2 General strategy

2.1 Haag–Kastler nets on the de Sitter space

The two-dimensional de Sitter space dS^2 is embedded in the ambient three-dimensional Minkowski space $\mathbb{R}^{1+2}$ by the equation $x_0^2 - x_1^2 - x_2^2 = -r^2$, where $r > 0$. The isometry group of dS^2 is the (proper orthochronous) Lorentz group $L_+^\uparrow$ (the connected component of the stabilizer subgroup of the point $(0, 0, 0)$ in the three-dimensional Poincaré group), also called the de Sitter group. On this space, the causal structure and the metric can be introduced by restricting those of the ambient Minkowski space. The region $\{(x_0, x_1, x_2) \in \mathbb{R}^{1+2} \mid |x_0| < x_1\}$ is called the wedge in the x_1-direction. We denote its intersection with dS^2 by W_1. Any image of W_1 by a Lorentz transformation is called a **wedge** in dS^2. For any wedge W, there is a one-parameter group $\Lambda_W(t)$ of Lorentz boosts that fix W, which are referred to as the boosts associated with W.

The Haag–Kastler axioms, usually considered on the Minkowski space, can be also formulated on dS^2 [3], where the spectrum condition is replaced by the geodesic KMS property as below. A **Haag–Kastler net on** dS^2 is a triple $(\mathcal{A}, U, \Omega)$, where $\mathcal{A}$ is a family of von Neumann algebras on a Hilbert space $\mathcal{H}$ parametrized by open regions $O \subset \mathrm{dS}^2$, U is a unitary representation of $L_+^\uparrow$ on $\mathcal{H}$ (continuous in the strong-operator topology) and Ω is a vector in $\mathcal{H}$, such that

- (HK1) **Isotony:** $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ for $O_1 \subset O_2$;
- (HK2) **Locality:** If O_1 and O_2 are spacelike separated, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$;
- (HK3) **Lorentz covariance:** $\mathcal{A}(gO) = \mathrm{Ad}\, U(g)(\mathcal{A}(O))$ for $g \in L_+^\uparrow$;
- (HK4) **Cyclicity**: Ω is cyclic for each $\mathcal{A}(O)$;
- (HK5) **The geodesic KMS property:** For any wedge W, it holds that $U(\Lambda_W(2\pi t)) = \Delta_W^{-it}$, where Δ_W^{it} is the modular group of the algebra $\mathcal{A}(W)$ with respect to Ω.

The geodesic KMS property is equivalently stated by saying that $U(\Lambda_W(t))$ satisfies the KMS condition for $\mathcal{A}(W)$ with temperature 2π with respect to Ω [3].

2.2 Two-dimensional conformal net on the de Sitter space

Our starting point is a two-dimensional conformal field theory. A two-dimensional conformal field theory on the Minkowski space is a theory that is (locally) covariant with respect not only to the Poincaré group but also to the universal covering $\overline{\text{Möb}} \times \overline{\text{Möb}}$ of the Möbius group Möb × Möb including special conformal transformations, and often further to $\overline{\mathrm{Diff}_+(S^1)} \times \overline{\mathrm{Diff}_+(S^1)}$. It is known that any two-dimensional conformal (Haag–Kastler) net, *a priori* defined on the Minkowski space $\mathbb{R}^{1+1}$, extends to the Einstein cylinder [15] [17, Theorem A.5]. Then, the de Sitter space dS^2 can be embedded conformally in the Einstein cylinder, and we can restrict the given conformal net to this subset [12], see Fig. 1. Therefore, a two-dimensional CFT can be considered as a QFT on dS^2 in this natural sense. Let us briefly review how this is done.

In our framework, a conformal Haag–Kastler net can be described as follows. First, we consider $\mathbb{R}^{1+1}$ as the product of two lightrays. Each lightray $\mathbb{R}$ has S^1

as the one-point compactification, and the group $\mathrm{Diff}_+(S^1)$ acts on it. Therefore, $\overline{\mathrm{Diff}_+(S^1)} \times \overline{\mathrm{Diff}_+(S^1)}$ acts on $\mathbb{R}^{1+1}$ locally in the sense of [4]. Furthermore, by spacelike locality, this action factors through the subgroup $\mathfrak{R} := \{R_{2n\pi} \times R_{-2n\pi} : n \in \mathbb{Z}\}$ where $R_t \in \overline{\mathrm{Diff}_+(S^1)}$ is the lift of the rotation by t [15, Proposition 2.1] (see also [17, Theorem A.5]). We denote this group by $\mathscr{C}$. The Minkowski space $\mathbb{R}^{1+1}$ is conformally equivalent to the product $I_{2\pi} \times I_{2\pi}$ of open intervals of length 2π, and through the local action of $\mathscr{C}$, the Haag–Kastler net can be extended to $\mathbb{R} \times \mathbb{R}$ quotiented by the action of $\mathfrak{R}$, where $\mathbb{R}$ is the universal covering of S^1. This space is conformally equivalent to the Einstein cylinder $\mathcal{E} = S^1 \times \mathbb{R}$ (the product structure is different from the previous one). We say that two regions O_1, O_2 are spacelike separated if there is a diamond obtained by shifting the Minkowski space $I_{2\pi} \times I_{2\pi}$, which includes O_1, O_2 such that O_1 and O_2 are spacelike separated there.

To be precise, the axioms for conformal nets on $\mathcal{E}$ are the following: Let $\mathcal{A}$ be a family of von Neumann algebras on $\mathcal{H}$ parametrized by open regions in $\mathcal{E}$, let U be a unitary projective representation of the group $\mathscr{C}$ on $\mathcal{H}$ (note that the restriction of U to the subgroup $\overline{\text{Möb}} \times \overline{\text{Möb}}/\mathfrak{R}$ can be actually made into a true representation of $\overline{\text{Möb}} \times \overline{\text{Möb}}$ [2, Theorem 7.1], and the generators of one-parameter subgroups in $\overline{\text{Möb}} \times \overline{\text{Möb}}$ are uniquely defined) and let Ω be a vector in $\mathcal{H}$ such that

(CN1) **Isotony:** $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ for $O_1 \subset O_2$;
(CN2) **Locality:** If O_1 and O_2 are spacelike separated, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$;
(CN3) **Conformal covariance:** $\mathcal{A}(\gamma O) = \mathrm{Ad}\, U(\gamma)(\mathcal{A}(O))$ for $\gamma \in \mathscr{C}$, and if O is disjoint from $\mathrm{supp}\,\gamma$, then $\mathrm{Ad}\, U(\gamma)(x) = x$ for $x \in \mathcal{A}(O)$;
(CN4) **Positive energy**: The generators of the chiral rotation subgroups $R_t \times \iota, \iota \times R_t$, where $\iota \in \overline{\mathrm{Diff}_+(S^1)}$ is the unit element, are positive;
(CN5) **Vacuum**: Ω is cyclic for each $\mathcal{A}(O)$ and is a unique (up to scalar) vector such that $U(\gamma)\Omega = \Omega$ for $\gamma \in \overline{\text{Möb}} \times \overline{\text{Möb}}$ in the sense above.

In a conformal net, the Bisognano–Wichmann property holds automatically [4].

On the Einstein cylinder $\mathcal{E}$, the strip of temporal width π, see Fig. 1, is conformally equivalent to the de Sitter space [12] (for each radius r there are different de Sitter spaces, but we do not specify r, because the only point is that any of such de Sitter spaces is conformally equivalent to the same part of the cylinder). The time-zero circle S^1 ($x_0 = 0$) in the de Sitter space is the (compactified) time-zero line ($a_0 = 0$) on the cylinder, and space rotations act on it. Other Lorentz transformations are contained in the conformal group $\mathscr{C}$. Indeed, the spacelike rotations $R_t \times R_{-t}$ and Lorentz boosts (that is the product of lightlike dilations with opposite sign) generate the three-dimensional Lie group SO(2, 1) (the $(2+1)$-dimensional Lorentz group), also called the $2d$ de Sitter group. Therefore, by restricting a conformal net to the de Sitter space, it satisfies the axioms (HK1-5): the geodesic KMS property is satisfied because of the Bisognano–Wichmann property.

In general, a two-dimensional conformal net $\mathcal{A}$ contains **chiral components**, that are observables living on the lightrays and invariant under the action of $\iota \times \overline{\mathrm{Diff}_+(S^1)}$ ($\overline{\mathrm{Diff}_+(S^1)} \times \iota$, respectively). They can be regarded as Haag–Kastler nets on the lightrays $\mathbb{R}$. They extend to S^1 by conformal covariance and are called **conformal nets on** S^1. More precisely, a triple $(\mathcal{A}_0, U_0, \Omega_0)$, where $\mathcal{A}_0$ is a family of von Neumann algebras on a Hilbert space $\mathcal{H}_0$ parametrized by open connected nonempty nondense

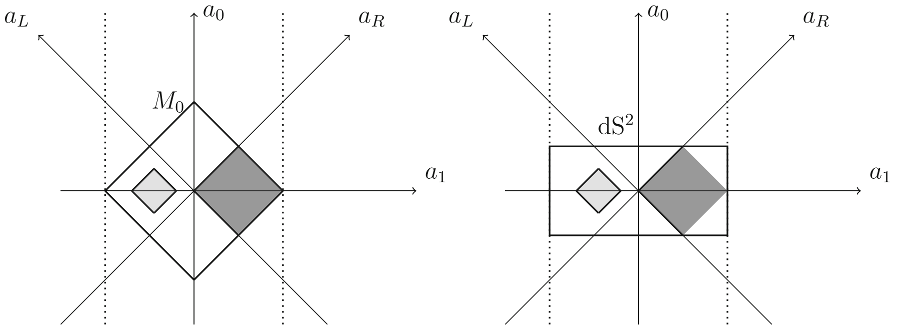

Fig. 1 The Minkowski space M_0 (cf. [1, Figure 1]) and the de Sitter space dS^2 conformally embedded in $\mathbb{R}^2$. The cylinder is obtained by identifying the dotted lines. The dark grey region is a wedge W and the light grey region is a double cone

intervals in S^1, U_0 is a unitary projective representation of $\mathrm{Diff}_+(S^1)$, and Ω_0 is a vector in $\mathcal{H}_0$, is called a **conformal net on** S^1 if it satisfies

(CNS1) **Isotony:** $\mathcal{A}_0(I_1) \subset \mathcal{A}_0(I_2)$ for $I_1 \subset I_2$;
(CNS2) **Locality:** If I_1 and I_2 are disjoint, then $\mathcal{A}_0(I_1) \subset \mathcal{A}_0(I_2)'$;
(CNS3) **Conformal covariance:** $\mathcal{A}_0(\gamma I) = \mathrm{Ad}\, U_0(\gamma)(\mathcal{A}_0(I))$ for $\gamma \in \mathrm{Diff}_+(S^1)$, and if I is disjoint from $\mathrm{supp}\,\gamma$, then $\mathrm{Ad}\, U_0(\gamma)(x) = x$ for $x \in \mathcal{A}_0(I)$;
(CNS4) **Positive energy**: The generator of the rotation subgroup in $\mathrm{Diff}_+(S^1)$ is positive;
(CNS5) **Vacuum**: Ω_0 is cyclic for each $\mathcal{A}_0(I)$ and is a unique (up to a scalar) vector such that $U_0(\gamma)\Omega_0 = \Omega_0$ for $\gamma \in$ Möb;

A two-dimensional conformal net $\mathcal{A}$ contains both left and right chiral components. Indeed, the operators $U(\gamma_0 \times \iota)$, $U(\iota \times \gamma_0)$ are such elements. In addition, left and right chiral components commute with each other [20].

The positive-energy representation U_0 is associated with a positive-energy representation of the Virasoro algebra $\{L_m\}$ [7, Appendix]:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n},$$

for a certain value $c > 0$ (by an abuse of notations, we use the symbols $\{L_m\}$ both for abstract Lie algebra elements and for unbounded operators satisfying the above relations). The self-adjoint operators $\mathfrak{t} = \frac{1}{2}L_0 - \frac{1}{4}(L_1 + L_{-1})$, $\mathfrak{d} = \frac{i}{2}(L_{-1} - L_1)$ are the generators of translations and dilations of $\mathbb{R} \subset S^1$ (in the sense that $\mathbb{R}$ is embedded in S^1 by the stereographic projection), respectively [23, Appendix A].

The two-dimensional conformal group $\mathscr{C}$ has the tensor product $\{L_m \otimes \mathbb{1}, \mathbb{1} \otimes L_n\}$ as the Lie algebra. The Lorentz boosts are generated by $\mathfrak{k}_1 := \mathfrak{d} \otimes \mathbb{1} - \mathbb{1} \otimes \mathfrak{d}$, while the rotations of the time-zero circle of the Einstein cylinder (identified with that of the de Sitter space) are generated by $\mathfrak{k}_0 := L_0 \otimes \mathbb{1} - \mathbb{1} \otimes L_0$. Note that the second boost is given by $\mathfrak{k}_2 := i[\mathfrak{k}_1, \mathfrak{k}_0]$.

For $z \in S^1$, one can consider the operator-valued distribution $T(z) = \sum_n L_n z^{-n}$, called the Virasoro field (the convention of the exponent is the one such that

$T(z)^* = T(z)$ [9], different from $L(z) = \sum_n L_n z^{-n-2}$ in vertex algebras [14]). The two-dimensional stress–energy tensor $T^{\mu\nu}$ has four components, and the fields $T(z) \otimes \mathbb{1} + \mathbb{1} \otimes T(z^{-1})$ and $T(z) \otimes \mathbb{1} - \mathbb{1} \otimes T(z^{-1})$ correspond to the components of the stress–energy tensor T^{01} and T^{00} [19].

2.3 Perturbation by a local field

As we saw above, a two-dimensional conformal net can be considered as a Haag–Kastler net on dS^2 and hence as a starting point for a new construction in the sense of [13]. We take a starting Haag–Kastler net $\mathcal{A}$ on dS^2, a unitary representation U of the $(2+1)$-dimensional Lorentz group and a vacuum vector Ω. We call "time-zero wedges" wedge regions whose end points reside on S^1.

The general strategy of [13, Theorem 4.1] goes as follows:

- Assume there is a rotation-invariant vector $\widetilde{\Omega}$ cyclic for $\mathcal{A}(W_1)$ in the natural positive cone $\mathcal{P}(\mathcal{A}(W_1), \Omega)$ associated with the pair $\mathcal{A}(W_1)$ and the free vacuum vector Ω. The rotation-invariance of $\widetilde{\Omega}$ implies that $\widetilde{\Omega} \in \mathcal{P}(\mathcal{A}(W), \Omega_0)$ for all time-zero wedges W.
- Assume there exists a new (interacting) representation $\widetilde{U}$ of the Lorentz group such that
  - its restriction to the rotation subgroup coincides with that of U;
  - its (interacting) Lorentz boost associated with the wedge W_1 is implemented by the modular group for the pair $\mathcal{A}(W_1)$ and (the interacting vacuum vector) $\widetilde{\Omega}$;
  - it satisfies the finite speed of light condition [13, Definition 3.3], which roughly says that the action of $\widetilde{U}$ on the time-zero algebras $\mathcal{A}(O_I)$, with O_I a double cone given by the intersection of time-zero wedges, preserves locality.

  The last two assumptions should be satisfied automatically if the new representation $\widetilde{U}$ is generated from a local field, as below.[2].
- The Lorentz covariance of the new net is given by $\widetilde{U}$. For any wedge region W, $\widetilde{\mathcal{A}}(W) := \mathrm{Ad}\,\widetilde{U}(g)(\mathcal{A}(W_1))$ (by definition), where g is such that $gW_1 = W$ (this is well defined by finite speed of propagation, in particular, the free time-zero wedge algebras are preserved by the new boosts). Any double cone is written as $O = \bigcap_{W \supset O} W$, and accordingly we define $\widetilde{\mathcal{A}}(O) = \bigcap_{W \supset O} \widetilde{\mathcal{A}}(W)$. This satisfies locality again by finite speed of propagation.

Then, $(\widetilde{\mathcal{A}}, \widetilde{U}, \widetilde{\Omega})$ is a new Haag–Kastler net on dS^2. This construction avoids Haag's theorem [24], because the spacetime is compact and there is no dilation covariance that pushes a double cone to infinity.

Assume that the net $\mathcal{A}$ is generated by a conformal Wightman field ψ and it has a well-defined restriction to the $x_0 = 0$ circle, which we denote by $\psi(0, \theta)$. The field $\psi(0, \theta)$ smeared by a test function f on the circle is denoted by $\psi(0, f)$. Let $\mathfrak{e}_n(\theta) = e^{in\theta}$. We find interesting candidates for such $\widetilde{U}$ by adding $\psi(0, \mathfrak{e}_1)$, $\psi(0, \mathfrak{e}_{-1})$

[2] For this implication, Trotter's product formula was used in [5, Theorem 10.1.1]. For this, it is necessary that the new generators are essentially self-adjoint on the same domain. Otherwise, new techniques would be needed.

to the building blocks $\mathfrak{l}_1 := L_1 \otimes \mathbb{1} + \mathbb{1} \otimes L_{-1}, \mathfrak{l}_{-1} := L_{-1} \otimes \mathbb{1} + \mathbb{1} \otimes L_1$ of the generators

$$\mathfrak{k}_1 = \frac{1}{2}(\mathfrak{l}_1 + \mathfrak{l}_{-1}), \qquad \mathfrak{k}_2 = \frac{1}{2i}(\mathfrak{l}_1 - \mathfrak{l}_{-1}),$$

of the Lorentz boosts and $\psi(0, \mathfrak{e}_0)$ to the generator of the rotations $L_0 \otimes \mathbb{1} - \mathbb{1} \otimes L_0$ leaving the $x_0 = 0$ circle invariant. By definition, $\mathfrak{l}_1 = \mathfrak{k}_1 + i\mathfrak{k}_2$ and $\mathfrak{l}_{-1} = \mathfrak{k}_1 - i\mathfrak{k}_2$, and consequently, $\mathfrak{l}_1^* = \mathfrak{l}_{-1}$.

Below we take concrete examples of two-dimensional CFTs and a candidate for $\widetilde{U}$ using the charged fields in it.

3 The U(1)-current and its two-dimensional extension

3.1 Chiral components

To make the programme concrete, we consider a two-dimensional CFT, whose chiral components are the U(1)-current nets on S^1 [6]. It is generated by the current (the derivative of the massless free field) $J(z) = \sum_n J_n z^{-n-1}$, where $z \in S^1$, and its Fourier coefficients J_n satisfy the commutation relations

$$[J_m, J_n] = m\delta_{m,-n}. \tag{1}$$

There is a representation of this algebra with a unique **vacuum vector** Ω_0 such that $J_n\Omega_0 = 0$ for $n \geq 0$. The Hilbert space $\mathcal{H}_0$ is spanned by the vectors of the form

$$J_{-n_1} \cdots J_{-n_k}\Omega_0,$$

where $n_1 \geq n_2 \geq \cdots \geq n_k$. We denote by $\mathcal{H}_0^{\mathrm{fin}}$ the linear span of these vectors. $\mathcal{H}_0$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ with respect to which it holds that $J_n^* = J_{-n}$.

This current can be smeared by a smooth function f and gives an unbounded operator $J(f) = \sum_n f_n J_n$, where $f_n = \frac{1}{2\pi}\int e^{-in\theta} f(\theta)d\theta$ are the Fourier components. The exponential $W(f) = e^{iJ(f)}$ is called a Weyl operator. One can construct a conformal net on S^1 by $\mathcal{A}_0(I) = \{e^{iJ(f)} : \operatorname{supp} f \subset I\}''$. A representation of the Virasoro algebra $\{L_n\}$ is given by the Sugawara formula $L_n = \frac{1}{2}\sum_k : J_{n-k}J_k :$, and it integrates to a projective unitary representation U_0 of $\mathrm{Diff}_+(S^1)$. With this U_0, $(\mathcal{A}_0, U_0, \Omega_0)$ is called the U(1)**-current net**.

A **representation** of a conformal net $\mathcal{A}_0$ on S^1 is a family of isomorphisms $\{\rho_I\}$ of local algebras $\{\mathcal{A}_0(I)\}$ to von Neumann algebras on a certain Hilbert space $\mathcal{H}_\rho$ that satisfy the compatibility condition

$$\rho_{I_1}(a) = \rho_{I_2}(a) \qquad \text{for } a \in \mathcal{A}(I_1), \quad I_1 \subset I_2.$$

For each $\alpha \in \mathbb{R}$, the U(1)-current net admits a representation ρ_α. This representation ρ_α can be realized on the same Hilbert space $\mathcal{H}_0$ as the vacuum representation, but to distinguish them we denote it by $\mathcal{H}_\alpha$ and the lowest weight vector by Ω_α. The assignment $W(f) \mapsto \rho_\alpha(W(f)) = e^{i\alpha\int f(\theta)d\theta}W(f)$ gives the representation. This is an **automorphism** of each local algebra $\mathcal{A}_0(I)$. In terms of generators, it amounts to replacing J_0 (which acts as the zero operator in the vacuum representation) by the scalar α. We denote the generator of the current algebra (1) on this space by $J_{\alpha,n}$. By the Sugawara formula $L_{\alpha,n} = \frac{1}{2}\sum_k : J_{\alpha,n-k}J_{\alpha,k} :$, there is also a representation of the Virasoro algebra with the same central charge $c = 1$. We denote by $\mathcal{H}_\alpha^{\mathrm{fin}}$ the subspace spanned by vectors of the form $J_{\alpha,-n_1}\cdots J_{\alpha,-n_k}\Omega_\alpha$.

Two such automorphisms $\rho_{\alpha_1}, \rho_{\alpha_2}$ can be composed, and yield a new automorphism (representation) $\rho_{\alpha_1+\alpha_2}$. This composition law is called the fusion rule for the U(1)-current. For a fixed $\alpha_0 \in \mathbb{R}$, the family $\{\rho_{j\alpha_0}\}_{j\in\mathbb{Z}}$ on $\{\mathcal{H}_{j\alpha_0}\}$ is closed under fusion, indeed, $\rho_{j_1\alpha_0}\rho_{j_2\alpha_0} = \rho_{(j_1+j_2)\alpha_0}$.

3.2 Charged primary fields

Let us fix $\alpha_0 \in \mathbb{R}$ with $|\alpha_0| \le 1, \alpha_0 \neq 0$. We now construct[3] for $\alpha \in \alpha_0\mathbb{Z}$, on a dense domain in the Hilbert space $\hat{\mathcal{H}} = \bigoplus_{j\in\mathbb{Z}}\mathcal{H}_{j\alpha_0}$, a formal series[4] by

$$Y_\alpha(z) = \sum_{s\in\mathbb{R}} Y_{\alpha,s}z^{-s-d}, \tag{2}$$

where $d = \frac{\alpha^2}{2}$, as follows. On $\hat{\mathcal{H}}$, the operators

$$\hat{J}_n = \bigoplus_{j\in\mathbb{Z}} J_{j\alpha_0,n}\,, \qquad \hat{L}_n = \bigoplus_{j\in\mathbb{Z}} L_{j\alpha_0,n}$$

can be defined naturally (depending on α_0, they act on a Hilbert space $\hat{\mathcal{H}} = \bigoplus_{j\in\mathbb{Z}}\mathcal{H}_{j\alpha_0}$ which also depends implicitly on α_0). Let c_α be the unitary charge shift operator $\mathcal{H}_\beta \to \mathcal{H}_{\beta+\alpha}$ defined by $c_\alpha\hat{J}_{-n_1}\cdots\hat{J}_{-n_k}\Omega_\beta = \hat{J}_{-n_1}\cdots\hat{J}_{-n_k}\Omega_{\beta+\alpha}, n_j > 0$. We can regard c_α as an operator on $\hat{\mathcal{H}} = \bigoplus_{j\in\mathbb{Z}}\mathcal{H}_{j\alpha_0}$. Following [22], we define

$$E^\pm(\alpha, z) = \exp\Bigl(\mp\sum_{n>0}\frac{\alpha\hat{J}_{\pm n}}{n}z^{\mp n}\Bigr). \tag{3}$$

The operators $Y_\alpha(z)$ are now specified by

$$Y_\alpha(z) = c_\alpha E^-(\alpha, z)E^+(\alpha, z)z^{\alpha J_0}, \tag{4}$$

where $z^{\alpha J_0}$ means $z^{\alpha\beta}$ on $\mathcal{H}_\beta$. Each coefficient $Y_{\alpha,s}$ is the direct sum of maps $\mathcal{H}_\beta \to \mathcal{H}_{\beta+\alpha}$ and on each $\mathcal{H}_\beta$, $\beta \in \alpha_0\mathbb{Z}$, only $Y_{\alpha,s}$ with $s \in \mathbb{Z} - \alpha\beta - \frac{\alpha^2}{2} = \mathbb{Z} - \alpha\beta - d$ are nonzero.

[3] The case $\alpha = 0$ is possible but uninteresting because $Y_0(z) = \mathbb{1}$, thus we do not consider it although not explicitly excluded.

[4] Recall [1] that we use formal series $\sum_{s\in\mathbb{R}} A_s z^s$ for a family of operators $\{A_s\}_{s\in\mathbb{R}}$ (actually the formal series is just the parametrized family $\{A_s\}$ itself, but certain operations on them are implicit).

The operators $\hat{L}_n$ generate a projective unitary representation $\hat{U}_0$ of $\overline{\mathrm{Diff}_+(S^1)}$. We know from [1] (see [22] and [21] for the original references) that, if $|\alpha| \leq 1$, then Y_α satisfies, with $d = \frac{\alpha^2}{2}$ as above,

$$[\hat{L}_m, Y_{\alpha,s}] = ((d-1)m - s)Y_{\alpha,m+s}, \tag{5}$$

$$\|Y_{\alpha,s}\| \leq 1. \tag{6}$$

By (6), we can smear the field by a test function supported in $\mathbb{R}$ and obtain a bounded operator $Y_\alpha(f)$ (even for $|\alpha| > 1$ one can define $Y_\alpha(f)$, but they are unbounded). By (5), it is conformally covariant with respect to $\hat{U}$ with the conformal dimension d. The formal series $Y_\alpha(z) = \sum_{s\in\mathbb{R}} Y_{\alpha,s} z^{-s-d}$ is not local with itself, but satisfies a braiding relation.

From the construction (4), it is easy to see the commutation relation $[\hat{J}_m, Y_\alpha(z)] = \alpha Y_\alpha(z) z^m$, or equivalently,

$$[\hat{J}_m, Y_{\alpha,s}] = \alpha Y_{\alpha,m+s}. \tag{7}$$

This implies that $Y_\alpha(z)$ is relatively local to $J(w)$.

3.3 Two-dimensional Wightman field

From two copies of a conformal nets $\mathcal{A}_0$ on S^1, one can construct a two-dimensional conformal net by tensor product: $\mathcal{A}_0(I_+ \times I_-) := \mathcal{A}_0(I_+) \otimes \mathcal{A}_0(I_-)$. The unitary representation of $\mathscr{C}$ is given by $U_0(\gamma_+) \otimes U_0(\gamma_-)$ and the vacuum by $\Omega_0 \otimes \Omega_0$. For $\alpha_0 \in \mathbb{R}$, $|\alpha_0|$ fixed above, there is an extension of this net on the space $\bigoplus_{j\in\mathbb{Z}} \mathcal{H}_{j\alpha_0} \otimes \mathcal{H}_{j\alpha_0}$ [1, 18]: on each direct summand $\mathcal{H}_{j\alpha_0} \otimes \mathcal{H}_{j\alpha_0}$, the tensor product net $\mathcal{A}_0$ acts by the representation $\rho_{j\alpha_0} \otimes \rho_{j\alpha_0}$.

Furthermore, for $\alpha \in \alpha_0\mathbb{Z}$, let $Y_{\alpha,s}$ as above on $\hat{\mathcal{H}} = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}_{j\alpha_0}$. The components $Y_{\alpha,s} \otimes \mathbb{1}$ of the charged field $Y_\alpha(z) \otimes \mathbb{1}$ of the left chiral component act on $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$ trivially on the right chiral component. Similarly, the components $\mathbb{1} \otimes Y_{\alpha,s}$ of the charged field $\mathbb{1} \otimes Y_\alpha(z)$ of the right chiral component act on $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$, trivially on the left chiral component.

For $\alpha \in \alpha_0\mathbb{Z}$, we consider the combined charged field

$$\widetilde{\psi}^\alpha(w, z) = Y_\alpha(w) \otimes Y_\alpha(z) + (Y_\alpha(w) \otimes Y_\alpha(z))^*,$$

which is a formal series whose coefficients are operators acting on $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$. Actually, as the components of $Y_\alpha(w) \otimes Y_\alpha(z)$ raise (respectively the components of $(Y_\alpha(w) \otimes Y_\alpha(z))^*$ lower) the left and right charges by α at the same time, they restrict to $\widetilde{\mathcal{H}} = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}_{j\alpha_0} \otimes \mathcal{H}_{j\alpha_0}$. As we take $|\alpha| \leq 1$, by [1, Theorem 5.9], the field $\widetilde{\psi}^\alpha(w, z)$ is a two-dimensional conformal Wightman field that generates a two-dimensional Haag–Kastler net. Let us call this net $\widetilde{\mathcal{A}}$.

4 Estimates for the charged fields

As we wish to perturb the conformal net by a charged field, we are interested in the time-zero restriction of $\widetilde{\psi}^{\alpha}(w,z) = Y_\alpha(w)\otimes Y_\alpha(z) + (Y_\alpha(w)\otimes Y_\alpha(z))^*$. This amounts to taking $z = w^{-1}$. However, it is *a priori* unclear whether this is possible, because taking $z = w^{-1}$ should give a formal series of z alone, but each component is an infinite sum of components of $\widetilde{\psi}^{\alpha}(w,z)$:

$$Y_\alpha(w)\otimes Y_\alpha(z) = \sum_{s\in\mathbb{R}} Y_{\alpha,s} w^{-s-d} \otimes \sum_{t\in\mathbb{R}} Y_{\alpha,t} z^{-t-d}$$

Therefore, by substituting $z = w^{-1}$,

$$\begin{aligned} Y_\alpha(w)\otimes Y_\alpha(w^{-1}) &= \sum_{s\in\mathbb{R}} Y_{\alpha,s} w^{-s-d} \otimes \sum_{t\in\mathbb{R}} Y_{\alpha,t} w^{t+d} \\ &= \sum_{s\in\mathbb{R}}\sum_{t\in\mathbb{R}} Y_{\alpha,t}\otimes Y_{\alpha,t-s} w^{-s} \end{aligned}$$

and we have to make sure that the sum $\sum_{t\in\mathbb{R}} Y_{\alpha,t}\otimes Y_{\alpha,t-s}$ (the sum is countable on each $\mathcal{H}_{j\alpha_0}\otimes\mathcal{H}_{j\alpha_0}$) gives a finite result on a certain dense domain. As the dense domain, we take $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0}$ where $\bigoplus_{j,\mathrm{alg}}$ denotes the algebraic direct sum and $\otimes_{\mathrm{alg}}$ denotes the algebraic tensor product. We show that the above sum is convergent if $|\alpha| < \frac{1}{\sqrt{2}}$.

For this purpose, we need a general result on primary fields with conformal weight d [8, Appendix B (141)]. This is proven for fields with integer d, but it is straightforward to generalize it (because one only needs the primarity). It states that

$$\|Y_{\alpha,-n-d}\Omega\|^2 = \binom{2d+n-1}{n} = \frac{\Gamma(2d+n)}{\Gamma(n+1)\Gamma(2d)} \sim n^{2d-1}, \tag{8}$$

where the last asymptotic follows from the Stirling's approximation of the Gamma function $\Gamma(x)$ with complex variable [25, 12.33].

We first observe that, for $|\alpha| \geq \frac{1}{\sqrt{2}}$, $d = \frac{\alpha^2}{2} \geq \frac{1}{4}$, thus $\sum_{n\in\mathbb{Z}} Y_{\alpha,-n}\otimes Y_{\alpha,-n-s}$ does not converge on $\Omega_0\otimes\Omega_0$. Indeed, $\|Y_{\alpha,-n-d}\otimes Y_{\alpha,-n-d-s}\cdot\Omega_0\otimes\Omega_0\|^2 = \binom{2d+n-1}{n}\binom{2d+n+s-1}{n+s} \sim n^{4d-2}$ (for a fixed s, as $n\to\infty$) and these vectors are orthogonal to each other, hence the sum $\sum_{n\in\mathbb{Z}} Y_{\alpha,-n-d}\otimes Y_{\alpha,-n-d-s}$ diverges on $\Omega_0\otimes\Omega_0$. This means that, if $|\alpha| \geq \frac{1}{\sqrt{2}}$, there is no hope to define an operator of the form $\sum_n Y_{\alpha,-n-d}\otimes Y_{\alpha,-n-d-s}$ on a domain containing $\Omega_0\otimes\Omega_0$.

On the other hand, if $d < \frac{1}{4}$, there is still hope that we can carry through the general programme of Sect. 2.3.

Theorem 4.1 *Let*[5] $|\alpha| < \frac{1}{\sqrt{2}}$*. Then, each coefficient of* w^s *in the formal series* $\widetilde{\psi}^\alpha(w, w^{-1})$*, applied to any vector in* $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{fin}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{fin}_{j\alpha_0}$*, is convergent.*

Proof We note first that a general vector in one tensor component is a linear combination of $c_{j\alpha}\hat{J}_{-m_1}\cdots\hat{J}_{-m_k}\Omega_0$, where $m_\ell > 0$ and $c_{j\alpha}$ commutes with $\hat{J}_m, m \neq 0$. Therefore, it is enough to prove the convergence on vectors in $\mathcal{H}^{\mathrm{fin}}_0 \otimes_{\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_0$.

We claim that $\|Y_{\alpha,-n-d}\hat{J}_{-m_1}\cdots\hat{J}_{-m_k}\Omega_0\|^2 \leq C_1(n+C_2)^{4d-2}$ where C_1, C_2 depend on the vector but not on n. This is clear for $k = 0$. To prove the claim by induction on k, let us observe that $Y_{\alpha,-n-d}\hat{J}_{-m_1}\cdots\hat{J}_{-m_k}\Omega_0$ can be reduced, using the commutation relations (7), $[Y_{\alpha,-n-d}, \hat{J}_m] = -\alpha Y_{\alpha,-n+m-d}$, as follows:

$$\begin{aligned}Y_{\alpha,-n-d}\hat{J}_{-m_1}\cdots\hat{J}_{-m_k}\Omega_0 &= ([Y_{\alpha,-n-d}, \hat{J}_{-m_1}] + \hat{J}_{-m_1}Y_{\alpha,-n-d})\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\\ &= (-\alpha Y_{\alpha,-n+m_1-d} + \hat{J}_{-m_1}Y_{\alpha,-n-d})\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\,.\end{aligned}$$

Using $\|\Psi_1 + \Psi_2\|^2 \leq 2(\|\Psi_1\|^2 + \|\Psi_2\|^2)$, it is enough to show that the norm of each term decays as desired. The first term decays by the induction hypothesis. Let us calculate the norm of the second term:

$$\begin{aligned}&\|\hat{J}_{-m_1}Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2\\ &= \langle Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0, ([\hat{J}_{m_1}, \hat{J}_{-m_1}] + \hat{J}_{-m_1}\hat{J}_{m_1})Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\rangle\\ &= \langle Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0, (m_1 + \hat{J}_{-m_1}\hat{J}_{m_1})Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\rangle\\ &= m_1\|Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2 + \|\hat{J}_{m_1}Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2\\ &= m_1\|Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2\\ &\quad + \|(\alpha Y_{\alpha,-n+m_1-d} + Y_{\alpha,-n-d}\hat{J}_{m_1})\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2\\ &\leq m_1\|Y_{\alpha,-n-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2\\ &\quad + 2\|\alpha Y_{\alpha,-n+m_1-d}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2 + 2\|Y_{\alpha,-n-d}\hat{J}_{m_1}\hat{J}_{-m_2}\cdots\hat{J}_{-m_k}\Omega_0\|^2\,.\end{aligned}$$

The last term can be reduced, using $[\hat{J}_m, \hat{J}_n] = m\delta_{m,-n}$ and $\hat{J}_m\Omega_0$ (as $m > 0$), to a sum of norms of vectors of the above form. This completes the induction. That is, the norm of $Y_{\alpha,-n-d}\hat{J}_{-m_1}\cdots\hat{J}_{-m_k}\Omega_0$ is a linear combination of terms that decay like $(n+C_2)^{2d-1}$.

When the operator $Y_{\alpha,-n-d}\otimes Y_{\alpha,-n-d-s}$ for a fixed s is applied to a vector that is the tensor product of two such vectors, the norm decays as $C_1(n+C_2)^{4d-2}$, which is summable in n. Therefore, this operator is defined on $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0}$. □

5 Commutativity of the time-zero charged field

Now we know that, for $|\alpha| < \frac{1}{\sqrt{2}}$, $Y_\alpha(w)\otimes Y_\alpha(w^{-1})$ makes sense as a formal series whose coefficients are (unbounded) operators on the dense domain $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0} \otimes$

[5] As $\alpha \in \alpha_0\mathbb{Z}$, we can take such an α if $|\alpha_0| < \frac{1}{\sqrt{2}}$.

$\mathcal{H}^{\mathrm{fin}}_{j\alpha_0}$. Next we show that it is not only local but also commutative, and moreover, $Y_\alpha(w) \otimes Y_\alpha(w^{-1})$ and $Y_\beta(z) \otimes Y_\beta(z^{-1})$ commute for possibly different $\alpha, \beta \in \alpha_0\mathbb{Z}$. More precisely, we have the following result.

Theorem 5.1 *As formal series, it holds that* $(Y_\alpha(w) \otimes Y_\alpha(w^{-1})^* = Y_{-\alpha}(w) \otimes Y_{-\alpha}(w^{-1})$ *under the convention* $w^* = w^{-1}$ *and* $Y_\alpha(w) \otimes Y_\alpha(w^{-1})$ *and* $Y_\beta(z) \otimes Y_\beta(z^{-1})$ *commute on the domain* $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{fin}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{fin}_{j\alpha_0}$ *weakly, that is, as sesquilinear forms.*

Proof The first claim follows easily from the definitions (3)(4) and $c^*_\alpha = c_{-\alpha}$.

Let us denote by $\hat{J}_m \otimes \mathbb{1}$ the operator on $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0}$ that acts as $\hat{J}_{j\alpha,m} \otimes \mathbb{1}$ on each component $\mathcal{H}^{\mathrm{fin}}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0}$. We calculate the commutator as a sesquilinear form, that is, we apply the operators to a vector and take a scalar product with another vector, but we omit them. We have $[\hat{J}_m, Y_{\alpha,s}] = \alpha Y_{\alpha,m+s}$. In terms of formal series, this amounts to $[Y_\alpha(w), \hat{J}_m] = -\alpha Y_\alpha(w) w^m$. Thus, it holds that, by the derivation property of the commutator with the generator,

$$\begin{aligned}
-[[Y_\alpha(w) \otimes Y_\alpha(w^{-1}), Y_\beta(z) \otimes Y_\beta(z^{-1})], \hat{J}_m \otimes \mathbb{1}]
&= [[\hat{J}_m \otimes \mathbb{1}, Y_\alpha(w) \otimes Y_\alpha(w^{-1})], Y_\beta(z) \otimes Y_\beta(z^{-1})] \\
&\quad + [[Y_\beta(z) \otimes Y_\beta(z^{-1}), \hat{J}_m \otimes \mathbb{1}], Y_\alpha(w) \otimes Y_\alpha(w^{-1})] \\
&= (\alpha w^m + \beta z^m)[Y_\alpha(w) \otimes Y_\alpha(w^{-1}), Y_\beta(z) \otimes Y_\beta(z^{-1})],
\end{aligned}$$

and similarly, with $\mathbb{1} \otimes \hat{J}_m$ that acts as $\mathbb{1} \otimes \hat{J}_m$ on each component $\mathcal{H}^{\mathrm{fin}}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0}$,

$$\begin{aligned}
&-[[Y_\alpha(w) \otimes Y_\alpha(w^{-1}), Y_\beta(z) \otimes Y_\beta(z^{-1})], \mathbb{1} \otimes \hat{J}_m] \\
&\quad = (\alpha w^{-m} + \beta z^{-m})[Y_\alpha(w) \otimes Y_\alpha(w^{-1}), Y_\alpha(z) \otimes Y_\alpha(z^{-1})].
\end{aligned}$$

We observe that, upon commuting with $\hat{J}_m \otimes \mathbb{1}$ or $\mathbb{1} \otimes \hat{J}_m$, we obtain the same operator $[Y_\alpha(w) \otimes Y_\alpha(w^{-1}), Y_\alpha(z) \otimes Y_\alpha(z^{-1})]$ multiplied by a scalar.

Now, to show that the commutator $[Y_\alpha(w) \otimes Y_\alpha(w^{-1}), Y_\alpha(z) \otimes Y_\alpha(z^{-1})]$ vanishes, we only have to check that the matrix element vanishes. It is easy to check that $Y_\alpha(w) \otimes Y_\alpha(w^{-1})$ commutes with $c_\alpha \otimes c_\alpha$; therefore, we only have to consider pairs of vectors in $\mathcal{H}_0 \otimes \mathcal{H}_0$ and $\mathcal{H}_\alpha \otimes \mathcal{H}_\alpha$. Furthermore, due to the above commutation relations, the linear functional

$$\langle \hat{J}_{-m_1} \cdots \hat{J}_{-m_k} \Omega_{2\alpha} \otimes \hat{J}_{-n_1} \cdots \hat{J}_{-n_\ell} \Omega_{2\alpha}, \quad \cdot \quad \hat{J}_{-m'_1} \cdots \hat{J}_{-m'_{k'}} \Omega_0 \otimes \hat{J}_{-n'_1} \cdots \hat{J}_{-n'_{\ell'}} \Omega_0 \rangle$$

can be reduced to the case $\langle \Omega_{2\alpha} \otimes \Omega_{2\alpha}, \cdot\, \Omega_0 \otimes \Omega_0 \rangle$.

Let us put $\underline{Y}_\alpha(z) = E^-(\alpha, z) E^+(\alpha, z)$ and $\underline{\widetilde{Y}}_\alpha(z) = \underline{Y}_\alpha(z) \otimes \underline{Y}_\alpha(z^{-1})$. As we have $E^+(\alpha, z)^* = E^-(-\alpha, z)$ (with the convention that $z^* = z^{-1}$), it follows that $\underline{Y}_\alpha(z)^* = \underline{Y}_{-\alpha}(z)$ and $\underline{\widetilde{Y}}_\alpha(z)^* = \underline{\widetilde{Y}}_{-\alpha}(z)$, or equivalently $\underline{\widetilde{Y}}^\dagger_{m,\alpha} = \underline{\widetilde{Y}}_{-m,-\alpha}$.

Expand $\underline{\widetilde{Y}}_\alpha(z)$ as $\underline{\widetilde{Y}}_\alpha(z) = \sum_{m\in\mathbb{Z}} \underline{\widetilde{Y}}_{\alpha,m} z^{-m} = \sum_m \sum_k \underline{Y}_{\alpha,k} \otimes \underline{Y}_{\alpha,k-m} z^{-m}$. Then, we have

$$\begin{aligned} Y_\alpha(z) \otimes Y_\alpha(z^{-1}) &= (c_\alpha \otimes c_\alpha) z^{\alpha J_0} z^{-\alpha J_0} \underline{Y}_\alpha(z) \otimes \underline{Y}_\alpha(z^{-1}) \\ &= (c_\alpha \otimes c_\alpha) \underline{\widetilde{Y}}_\alpha(z). \end{aligned}$$

Therefore, the question is further reduced to $\langle \Omega_0 \otimes \Omega_0, [\underline{\widetilde{Y}}_{\alpha,m}, \underline{\widetilde{Y}}_{\beta,n}]\Omega_0 \otimes \Omega_0 \rangle = 0$ for all m, n.

Note that, with F the flip operator between the left and right tensor components in $\mathcal{H}_0 \otimes \mathcal{H}_0$ (which is a unitary operator),

$$\begin{aligned} \underline{\widetilde{Y}}_{\alpha,m} \Omega_0 \otimes \Omega_0 &= \sum_k \underline{Y}_{\alpha,k} \otimes \underline{Y}_{\alpha,k-m} \Omega_0 \otimes \Omega_0 = F \cdot \sum_k \underline{Y}_{\alpha,k-m} \otimes \underline{Y}_{\alpha,k} \Omega_0 \otimes \Omega_0 \\ &= F \cdot \sum_k \underline{Y}_{\alpha,k} \otimes \underline{Y}_{\alpha,k+m} \Omega_0 \otimes \Omega_0 = F \cdot \underline{\widetilde{Y}}_{\alpha,-m} \Omega_0 \otimes \Omega_0 \,. \end{aligned}$$

This implies that

$$\begin{aligned} \langle \underline{\widetilde{Y}}_{\alpha,m} \Omega_0 \otimes \Omega_0, \underline{\widetilde{Y}}_{\beta,n} \Omega_0 \otimes \Omega_0 \rangle &= \langle F \cdot \underline{\widetilde{Y}}_{\alpha,-m} \Omega_0 \otimes \Omega_0,\ F \cdot \underline{\widetilde{Y}}_{\beta,-n} \Omega_0 \otimes \Omega_0 \rangle \\ &= \langle \underline{\widetilde{Y}}_{\alpha,-m} \Omega_0 \otimes \Omega_0,\ \underline{\widetilde{Y}}_{\beta,-n} \Omega_0 \otimes \Omega_0 \rangle \,. \end{aligned}$$

Moreover, note that the map $J_m \mapsto -J_m$ is a vacuum-preserving automorphism implemented by a unitary (the multiplication by $(-1)^k$ on the k-particle space), and its tensor product maps

$$\begin{aligned} \underline{\widetilde{Y}}_\alpha(z) &= E^-(\alpha, z) E^+(\alpha, z) \otimes E^-(\alpha, z^{-1}) E^+(\alpha, z^{-1}) \\ &\longmapsto E^-(-\alpha, z) E^+(-\alpha, z) \otimes E^-(-\alpha, z^{-1}) E^+(-\alpha, z^{-1}) = \underline{\widetilde{Y}}_{-\alpha}(z) \,. \end{aligned}$$

That is, $\underline{\widetilde{Y}}_{\alpha,m}$ is mapped to $\underline{\widetilde{Y}}_{-\alpha,m}$, therefore, by the invariance of $\Omega_0 \otimes \Omega_0$ by this unitary,

$$\langle \underline{\widetilde{Y}}_{\alpha,m} \Omega_0 \otimes \Omega_0,\ \underline{\widetilde{Y}}_{\beta,n} \Omega_0 \otimes \Omega_0 \rangle = \langle \underline{\widetilde{Y}}_{-\alpha,m} \Omega_0 \otimes \Omega_0,\ \underline{\widetilde{Y}}_{-\beta,n} \Omega_0 \otimes \Omega_0 \rangle \,.$$

From this, we can compute the commutator (as a sesquilinear form)

$$\begin{aligned} &\langle \Omega_0 \otimes \Omega_0, [\underline{\widetilde{Y}}_{\alpha,m}, \underline{\widetilde{Y}}_{\beta,n}] \Omega_0 \otimes \Omega_0 \rangle \\ &= \langle \Omega_0 \otimes \Omega_0, (\underline{\widetilde{Y}}_{\alpha,m} \underline{\widetilde{Y}}_{\beta,n} - \underline{\widetilde{Y}}_{\beta,n} \underline{\widetilde{Y}}_{\alpha,m}) \Omega_0 \otimes \Omega_0 \rangle \\ &= \langle \underline{\widetilde{Y}}_{-\alpha,-m} \Omega_0 \otimes \Omega_0, \underline{\widetilde{Y}}_{\beta,n} \Omega_0 \otimes \Omega_0 \rangle - \langle \underline{\widetilde{Y}}_{-\beta,-n} \Omega_0 \otimes \Omega_0, \underline{\widetilde{Y}}_{\alpha,m} \Omega_0 \otimes \Omega_0 \rangle \\ &= \langle \underline{\widetilde{Y}}_{-\alpha,m} \Omega_0 \otimes \Omega_0, \underline{\widetilde{Y}}_{\beta,-n} \Omega_0 \otimes \Omega_0 \rangle - \langle \underline{\widetilde{Y}}_{-\beta,-n} \Omega_0 \otimes \Omega_0, \underline{\widetilde{Y}}_{\alpha,m} \Omega_0 \otimes \Omega_0 \rangle \\ &= \langle \underline{\widetilde{Y}}_{\alpha,m} \Omega_0 \otimes \Omega_0, \underline{\widetilde{Y}}_{-\beta,-n} \Omega_0 \otimes \Omega_0 \rangle - \langle \underline{\widetilde{Y}}_{-\beta,-n} \Omega_0 \otimes \Omega_0, \underline{\widetilde{Y}}_{\alpha,m} \Omega_0 \otimes \Omega_0 \rangle. \end{aligned}$$

Furthermore, by construction (4) of $\underline{\widetilde{Y}}_m = \sum_{k\in\mathbb{R}} \underline{Y}_{\alpha,k} \otimes \underline{Y}_{\alpha,k-m}$, these expectation values give only real numbers. Therefore, by hermitianity of the scalar product, this commutator vanishes on the vacuum state. □

6 Perturbation by charged fields

According to the general idea of Sect. 2, we wish to perturb the net $\widetilde{\mathcal{A}}$ by a field by the methods of Barata-Jäkel-Mund [5]. That is, while keeping the T^{11} component of the stress–energy tensor, we add a smeared local field to the T^{00} component on the time-zero circle S^1.

The necessary condition for it to work is that the new operators satisfy the Lorentz relations (the generators of the Lorentz group are complex linear combinations of $\mathfrak{l}_m, m = -1, 0, 1$, cf. Sect. 2.3):

$$[\mathfrak{l}_m, \mathfrak{l}_n] = (m-n)\mathfrak{l}_{m+n}, \quad m, n = -1, 0, 1.$$

As we do not know whether the smeared field can be multiplied on the domain of the old generators, we consider the weak commutation relation: for two vectors Ψ_1, Ψ_2, we compute $\langle A^*\Psi_1, B\Psi_2\rangle - \langle B^*\Psi_1, A\Psi_2\rangle$. Obviously, if the commutator $[A, B]$ can be defined on the domain and calculated, then it implies the weak commutation relation.

We pick the symmetric field

$$\begin{aligned}\widetilde{\psi}^\alpha(w, z) &= Y_\alpha(w) \otimes Y_\alpha(z) + (Y_\alpha(w) \otimes Y_\alpha(z))^* \\ &= Y_\alpha(w) \otimes Y_\alpha(z) + Y_{-\alpha}(w) \otimes Y_{-\alpha}(z),\end{aligned}$$

restrict it to the time-zero circle $z = w^{-1}$ and smear it with $\mathfrak{e}_j(\theta), n = -1, 0, 1$. Correspondingly, we consider the coefficients of z^{-n}:

$$\sum_{k\in\mathbb{R}} Y_{\alpha,k} \otimes Y_{\alpha,-n+k} + \sum_{s\in\mathbb{R}} Y_{-\alpha,s} \otimes Y_{-\alpha,-n+s} = \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-n+k}.$$

Theorem 6.1 *The Lorentz relations are weakly satisfied for*

$$\begin{aligned}&\hat{L}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_{-1} + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-1+k}\,, \\ &\hat{L}_0 \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_0\,, \\ &\hat{L}_{-1} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_1 + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,1+k}\,,\end{aligned}$$

where k runs in $\mathbb{R}$, but there are only countable nonzero terms on each $\mathcal{H}_\beta \otimes \mathcal{H}_\beta$, on the domain $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{fin}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{fin}_{j\alpha_0}$.

Proof It is clear that, if $\lambda = 0$, these are the old generators and they satisfy the Lorentz relations on the domain $\bigoplus_{j,\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0} \otimes_{\mathrm{alg}} \mathcal{H}^{\mathrm{fin}}_{j\alpha_0}$. The new terms commute with each other in the weak sense as we have seen in Sect. 5; hence, we only have to check the commutation relations between the old terms and the new terms.

One new term can be applied to a vector in the domain and gives a convergent series; hence, we can compute the weak commutator term by term. For a primary field $Y_\alpha(z)$, it holds that $[\hat{L}_m, Y_{\alpha,n}] = ((d-1)m - n)Y_{\alpha,m+n}$ in the operator sense, and hence also in the weak sense. Therefore, we have the following commutation relation in the weak sense:

$$
\begin{aligned}
&\left[\hat{L}_m \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_{-m}\,,\ \sum_{k\in\mathbb{R}} Y_{\alpha,k} \otimes Y_{\alpha,-n+k}\right] \\
&\quad = \sum_{k\in\mathbb{R}} \Big(((d-1)m - k)Y_{\alpha,k+m} \otimes Y_{\alpha,-n+k} \\
&\qquad + ((d-1)(-m) - (-n+k))Y_{\alpha,k} \otimes Y_{\alpha,-m-n+k} \Big) \\
&\quad = \sum_{k\in\mathbb{R}} \Big(\Big((d-1)m - (k-m)\Big) Y_{\alpha,k} \otimes Y_{\alpha,-m-n+k} \\
&\qquad\quad + ((d-1)(-m) - (-n+k))Y_{\alpha,k} \otimes Y_{\alpha,-m-n+k} \Big) \\
&\quad = \sum_{k\in\mathbb{R}} (m+n-2k) Y_{\alpha,k} \otimes Y_{\alpha,-m-n+k}\,.
\end{aligned}
$$

As α is arbitrary, this holds even if α is replaced by $-\alpha$. Furthermore, $\sum_{k\in\mathbb{R}} Y_{\alpha,k} \otimes Y_{\alpha,-n+k}$ and $\sum_{k\in\mathbb{R}} Y_{-\alpha,k} \otimes Y_{-\alpha,-n+k}$ commute by Theorem 5.1. Altogether,

$$
\begin{aligned}
&\Bigg[\hat{L}_m \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_{-m} + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm 1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k}, \\
&\qquad\qquad \hat{L}_n \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_{-n} + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm 1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-n+k} \Bigg] \\
&= (m-n)\hat{L}_{m+n} \otimes \mathbb{1} - (m-n)\mathbb{1} \otimes \hat{L}_{-m-n} \\
&\quad + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm 1} (m+n-2k) Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k} \\
&\quad - \lambda \sum_{k\in\mathbb{R},\epsilon=\pm 1} (n+m-2k) Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-n-m+k} \\
&= (m-n)\hat{L}_{m+n} \otimes \mathbb{1} - (m-n)\mathbb{1} \otimes \hat{L}_{-m-n}\,,
\end{aligned}
$$

and for $m = 1, n = -1$, this is $2(\hat{L}_0 \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_0)$.

On the other hand,

$$\left[\hat{L}_0 \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_0,\ \hat{L}_m \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_{-m} + \lambda \sum_{k \in \mathbb{R}, \epsilon = \pm 1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k}\right]$$

$$= (-m)\hat{L}_m \otimes \mathbb{1} - (m)\mathbb{1} \otimes \hat{L}_{-m}$$
$$+ \lambda \sum_{k \in \mathbb{R}, \epsilon = \pm 1} \Big((-k)Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k} - (-(-m+k))Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k}\Big)$$
$$= (-m)\left(\hat{L}_m \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_{-m} + \lambda \sum_{k \in \mathbb{R}, \epsilon = \pm 1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k}\right).$$

For $m = 1, -1$, we obtain the right commutation relations between $\mathfrak{l}_0$ and $\mathfrak{l}_m$. □

Note that the Lorentz relations do not extend beyond $m = 1, 0, -1$, that is, they do not satisfy the Virasoro relations.

In order to implement the perturbation by this commutative field, we need to solve the following problems: show that the above generators are self-adjoint on a certain domain and generate a dynamics that satisfies finite speed of propagation.

On the other hand, on the time-zero circle, there is a new representation of the Virasoro algebra (with non-positive energy) with $c = 0$, or the Witt algebra.

Proposition 6.2 *The Virasoro relations are weakly satisfied with $c = 0$ for*

$$\hat{L}_m \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m} + i\lambda m \sum_{k \in \mathbb{R}, \epsilon = \pm 1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k},$$

where k runs in $\mathbb{R}$, but there are only countable nonzero terms on each $\mathcal{H}_\beta \otimes \mathcal{H}_\beta$.

Proof As before, we compute the commutation relations weakly. First, with $d = \frac{\alpha^2}{2} = \frac{(-\alpha)^2}{2}$,

$$\left[\hat{L}_m \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m}, \sum_{k \in \mathbb{R}, \epsilon = \pm 1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-n+k}\right]$$
$$= \sum_{k \in \mathbb{R}, \epsilon = \pm 1} \Big(((d-1)m - k)Y_{\epsilon\alpha,k+m} \otimes Y_{\epsilon\alpha,-n+k}$$
$$- ((d-1)(-m) - (-n+k))Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}\Big)$$
$$= \sum_{k \in \mathbb{R}, \epsilon = \pm 1} \Big(((d-1)m - (k-m))Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}$$
$$- ((d-1)(-m) - (-n+k))Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}\Big)$$
$$= \sum_{k \in \mathbb{R}, \epsilon = \pm 1} ((2d-1)m - n)Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}$$

and by plugging this into the full expressions,

$$\begin{aligned}
&\Bigg[\hat{L}_m \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m} + i\lambda m \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k}, \\
&\qquad \hat{L}_n \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-n} + i\lambda n \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-n+k}\Bigg] \\
&= (m-n)\hat{L}_{m+n} \otimes \mathbb{1} + (-m-n)\mathbb{1} \otimes \hat{L}_{-m+n} \\
&\quad + i\lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} (n((2d-1)m-n) - m((2d-1)n-m)) Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k} \\
&= (m-n)\left(\hat{L}_{m+n} \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m-n} + i\lambda(m+n) \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}\right).
\end{aligned}$$

□

The combination $i\lambda m$ means that we are taking the derivative $-i\partial_\theta \widetilde{\psi}^\alpha(e^{i\theta}, e^{-i\theta})$. The operators $\hat{L}_m \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m}$ are the generators of the time-zero Virasoro (Witt) algebra, and Proposition 6.2 tells that there are different actions of the Virasoro algebra with $c = 0$.

In addition, a formal calculation shows that, for $d = \frac{1}{2}$ (and only for this case), there is another set of expressions having similar relations. That is, for

$$\hat{L}_m \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m} + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k},$$

we calculate formally the commutators (this is only formal because for $d = \frac{1}{2}$ we do not have the convergence for the product of two such expressions evaluated in a pair of vectors). In Proposition 6.2, we have seen that

$$\begin{aligned}
&\left[\hat{L}_m \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m}, \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-n+k}\right] \\
&= \sum_{k\in\mathbb{R},\epsilon=\pm1} ((2d-1)m-n) Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}
\end{aligned}$$

and the full commutators are now

$$\begin{aligned}
&\Bigg[\hat{L}_m \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m} + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m+k}, \\
&\qquad \hat{L}_n \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-n} + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-n+k}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= (m-n)\hat{L}_{m+n} \otimes \mathbb{1} + (-m-n) \otimes \hat{L}_{-m+n} \\
&\quad + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} (((2d-1)m-n) - ((2d-1)n-m)) Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k} \\
&= (m-n)\Big(\hat{L}_{m+n} \otimes \mathbb{1} - \otimes \hat{L}_{-m-n} + 2d\lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}\Big).
\end{aligned}
$$

The last expression in the bracket coincides with $\hat{L}_{m+n} \otimes \mathbb{1} - \mathbb{1} \otimes \hat{L}_{-m-n} + \lambda \sum_{k\in\mathbb{R},\epsilon=\pm1} Y_{\epsilon\alpha,k} \otimes Y_{\epsilon\alpha,-m-n+k}$ if and only if $d = \frac{1}{2}$. The case $d = \frac{1}{2}$ is related to free fermions. This might indicate a hidden symmetry for free fermions.

7 Outlook

We need that the above generators are self-adjoint on a certain domain. This is open. To show that they are essentially self-adjoint on our domain, one way would be to use the analytic vector theorem, but it is unclear whether even $\Omega_0 \otimes \Omega_0$ is an analytic vector. Therefore, we need better estimates of the time-zero restriction $Y_\alpha(w) \otimes Y_\alpha(w^{-1})$. Such estimates will be needed also to show that the perturbed Lorentz generators do generate a new representation of the Lorentz group, that we can construct a new Haag–Kastler net on dS^2 and to find the interacting vacuum. For this purpose, studying the Euclidean models of these two-dimensional CFT might help.

There are many two-dimensional CFTs and some of the charged primary fields have been relatively well understood. It might be a good idea to take other models where charged fields allow better control.

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Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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