Journal of Functional Analysis 286 (2024) 110239

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Convergence in Total Variation for nonlinear functionals of random hyperspherical harmonics

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A R T I C L E I N F O

Article history:
Received 7 November 2022
Accepted 26 October 2023
Available online 11 November 2023
Communicated by Pietro Caputo

MSC:
60G60
60B10
42C10
60H07

Keywords:
Gaussian eigenfunctions
High energy asymptotics
Malliavin calculus
Total Variation distance

A B S T R A C T

Random hyperspherical harmonics are Gaussian Laplace eigenfunctions on the unit d-dimensional sphere ($d \geq 2$). We study the convergence in Total Variation distance for their nonlinear statistics in the high energy limit, i.e., for diverging sequences of Laplace eigenvalues. Our approach takes advantage of a recent result by Bally, Caramellino and Poly (2020): combining the Central Limit Theorem in Wasserstein distance obtained by Marinucci and Rossi (2015) for Hermite-rank 2 functionals with new results on the asymptotic behavior of their Malliavin-Sobolev norms, we are able to establish second order Gaussian fluctuations in this stronger probability metric as soon as the functional is regular enough. Our argument requires some novel estimates on moments of products of Gegenbauer polynomials that may be of independent interest, which we prove via the link between graph theory and diagram formulas.

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https://doi.org/10.1016/j.jfa.2023.110239

Contents

1. Introduction

Random hyperspherical harmonics $\{T_\ell\}_{\ell\in\mathbb{N}}$ are Gaussian Laplace eigenfunctions on the unit d-dimensional sphere $\mathbb{S}^d$ ($d \geq 2$). They are the Fourier components of isotropic Gaussian spherical random fields, therefore used in a wide range of disciplines; in particular, for $d = 2$ they play a key role in cosmology – in connection with the analysis of the Cosmic Microwave Background radiation data – as well as in medical imaging and atmospheric sciences, see [13, Chapter 1] for more details. For these reasons, in the last years the investigation of their geometry received a great attention, in particular the asymptotic behavior, for large eigenvalues (as $\ell \to +\infty$), of their nonlinear statistics $\{\tilde{X}_\ell\}_{\ell\in\mathbb{N}}$, see [17,18,15,9,6,24,16] and the references therein. The main goal of most of these papers is to study first and second order fluctuations for $\tilde{X}_\ell$ to be some geometric functional of the excursion sets of T_ℓ, such as the so-called Lipschitz-Killing curvatures [1, Section 6.3] that in dimension 2 are the area, the boundary length and the Euler-Poincaré characteristic. Hence it is clear that $\tilde{X}_\ell$ may be a function of the sole T_ℓ (in the case of the excursion measure for instance) or a function of T_ℓ and its derivatives.

The above mentioned references take advantage of Wiener-Itô theory, the random variables $\{\tilde{X}_\ell\}_{\ell\in\mathbb{N}}$ being square integrable functionals of Gaussian fields. In this framework, the techniques developed allow one to establish Central Limit Theorems (CLTs) via a powerful combination of chaotic expansions and fourth moment theory by Nourdin and Peccati [19]. It is well known that the link between Malliavin calculus and Stein's method established by these two authors permits to get estimates on the rate of convergence to the limiting Gaussian law in various probability metrics [19, Appendix C.2], at least when a finite number of chaoses are involved. For general functionals instead, the so-called second order Poincaré inequality [20] may be evoked, even in its improved version [26].

However, the existing results in the literature for the above mentioned geometric functionals $\{\tilde{X}_\ell\}_{\ell\in\mathbb{N}}$ (which do have an infinite chaos expansion) of random hyperspherical harmonics $\{T_\ell\}_{\ell\in\mathbb{N}}$ only deal with the Wasserstein distance, see e.g. [15,6,22]. The typical situation is a single chaotic component dominating the whole series expansion, entailing the Wasserstein distance to be controlled by the square root of the fourth cumulant of this leading term plus the $L^2(\mathbb{P})$-norm of the series tail. Moreover, generally there are no information on the optimal speed of convergence.

A natural question is whether or not these results could be upgraded to stronger probability metrics. Here we address this issue, indeed we are interested in quantitative CLTs in Total Variation distance [19, Section C.2] for nonlinear statistics $\{\tilde{X}_\ell\}_{\ell\in\mathbb{N}}$ of random hyperspherical harmonics $\{T_\ell\}_{\ell\in\mathbb{N}}$ in the high energy limit (as $\ell \to +\infty$). We are able to solve the problem for integral functionals of the sole T_ℓ, that are regular enough in the Malliavin sense, by taking advantage of a recent result in [4]. In this paper, the authors prove some regularization lemmas that enable one to upgrade the distance of convergence from smooth Wasserstein to Total Variation (in a quantitative way) for any sequence of random variables which are smooth and non-degenerate in some sense. The price to pay is to control the smooth Wasserstein distance between the sequence of their Malliavin covariance matrices and its limit, that however does *not* need to be the Malliavin covariance matrix of the limit. Remarkably, this technique requires neither the sequence of random variables of interest to be functionals of a Gaussian field nor the limit law to be Normal, situations that naturally occur since the underlying randomness may be not Gaussian [3,8] or related functionals may show non-Normal second order fluctuations [14].

Let us write down explicitly our functional of interest: we consider

$$\tilde{X}_\ell = \frac{X_\ell - \mathbb{E}[X_\ell]}{\sqrt{\mathrm{Var}(X_\ell)}} \quad \text{where} \quad X_\ell := \int_{\mathbb{S}^d} \varphi(T_\ell(x))dx,$$

$\varphi : \mathbb{R} \to \mathbb{R}$ being square integrable w.r.t. the Gaussian density. In [15], the authors prove that, under mild assumptions, the above functional $\tilde{X}_\ell$ converges in Wasserstein distance towards a Gaussian random variable as $\ell \to +\infty$; in order to strengthen this result, in light of [4], we need to investigate the asymptotic behavior of the Malliavin covariance of $\tilde{X}_\ell$, that we denote by σ_ℓ. Under some additional regularity properties on the function φ which are needed to ensure the existence of Malliavin derivatives of $\tilde{X}_\ell$, we are able to prove the convergence in Wasserstein distance of σ_ℓ towards a non-degenerate deterministic limit, that together with the uniform boundedness of Malliavin-Sobolev norms of $\tilde{X}_\ell$ guarantees the convergence in Total Variation distance for $\tilde{X}_\ell$. To the best of our knowledge, ours is the first quantitative Limit Theorem in Total Variation distance for nonlinear functionals of random hyperspherical harmonics having an infinite chaotic expansion, generalizing in particular the work [15].

As a bonus, we gain some new results on the asymptotic behavior of Malliavin derivatives of these functionals, and some novel estimates on the moments of products of powers of Gegenbauer polynomials (the latter describing the covariance structure of the random hyperspherical harmonics $\{T_\ell\}_{\ell\in\mathbb{N}}$) thus extending some formulas in [12,22] (see Lemma 5.6 and Lemma 5.7). For our investigation we also exploit an explicit link between the diagram formula for moments of Hermite polynomials and the graph theory, inspired by [12] (see Lemma 5.3). In particular, we extrapolate a graph from each of these diagrams and use the fact that every connected graph can be covered by a tree, eventually studying only the contribution coming from these trees. In order to make the reading pleasant and smooth, we collect the proofs of these key results on Gegenbauer integrals in Appendix A.1 and Appendix A.2.

Finally, it is worth stressing that in the context of Gaussian approximations for random variables that are functionals of an underlying Gaussian field, the second order Poincaré inequality by Vidotto [26] has led to quantitative CLTs for nonlinear functionals of stationary Gaussian fields related to the Breuer-Major theorem, with presumably optimal rates of convergence in Total Variation distance. However, we choose to exploit the technique developed in [4] with a view to a subsequent generalization of our result to the interesting case of random eigenfunctions of the standard flat torus (arithmetic random waves), where the attainable limit laws include linear combinations of independent chi-square distributed random variables [14,5]. Moreover, it turns out that in order to obtain fruitful bounds via the second order Poincaré inequality for the Gaussian approximation of our functional of interest $\tilde{X}_\ell$, the estimates on moments of products of powers of Gegenbauer polynomials should be much finer than those required by the approach developed in [4] (the one that we follow).

1.1. Notation

Throughout this manuscript we denote with ν the standard Gaussian law on $\mathbb{R}$ and with $Z \sim \mathcal{N}(0,1)$ a standard Gaussian random variable (r.v.). When we will speak about Malliavin calculus and chaos expansion based on Z, we just intend the classical one dimensional approach in the space $L^2(\nu) := L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \nu)$ (see e.g. [19, Chapter 1]), where $\mathscr{B}(\mathbb{R})$ denotes the Borel-σ field on the real line. In particular, we will denote by L and D^k (for integers $k \geq 1$) the Ornstein-Uhlenbeck operator and the k-th order Malliavin derivative, respectively. As usual, $Dom(L)$ and $\mathbb{D}^{k,p}$ (for $p \geq 1$) will stand, respectively, for the set of random variables measurable w.r.t. $\sigma(Z)$ on which L is well defined and that are derivable in Malliavin sense up to order k, whose derivatives all belong to $L^p(\mathbb{P}) := L^p(\Omega, \mathcal{F}, \mathbb{P})$. Here and in what follows $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space and without loss of generality we may assume the random objects in this paper are defined on this common probability space.

Conventions. In this paper we set $\mathbb{N} := \{0, 1, 2 \dots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Given two sequences of positive numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, $a_n = O(b_n)$ if $\{\frac{a_n}{b_n}\}_n$ is asymptotically bounded and $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Acknowledgments

The authors would like to thank Domenico Marinucci for useful discussions. L.C. and G.G. acknowledge support from the MIUR *Excellence* Project MatMod@TOV awarded to the Department of Mathematics, University of Rome "Tor Vergata", CUP E83C23000330006, and the Project ASPRO from University of Rome "Tor Vergata", CUP E83C22001780005. The research of M.R. has been supported by the ANR-17-CE40-0008 Project *Unirandom*, funded by the French National Research Agency. L.C. and M.R. acknowledge support from the Beyond Borders Project AMP CUP E89C20000680005, of the University of Rome "Tor Vergata".

2. Motivations and main results

For an integer $d \geq 2$, we denote by $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ the d-dimensional unit sphere. Accordingly, we set $\mathscr{B}(\mathbb{S}^d)$ as the Borel σ-field and we write $\mathrm{Leb}(dx) = dx$ for the Lebesgue measure on $(\mathbb{S}^d, \mathscr{B}(\mathbb{S}^d))$. It is known that $\int_{\mathbb{S}^d} dx = 2\pi^{\frac{d+1}{2}}/\Gamma(\frac{d+1}{2}) =: \mu_d$, Γ being the Gamma function. For $f : \mathbb{S}^d \to \mathbb{R}$ and $\lambda \geq 0$, we consider the Helmotz equation

$$\Delta_{\mathbb{S}^d} f = -\lambda f, \tag{2.1}$$

where $\Delta_{\mathbb{S}^d}$ denote the Laplacian operator on $\mathbb{S}^d$. The eigenvalues are of the form $-\lambda = -\lambda_{\ell;d} = -\ell(\ell + d - 1)$ for $\ell \in \mathbb{N}$, and the dimension of the ℓ-th eigenspace is

$$n_{0;d} = 1 \quad \text{and} \quad n_{\ell;d} = \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1}, \quad \ell \in \mathbb{N}^*.$$

Notice that

$$n_{\ell;2} = 2\ell + 1 \quad \text{and} \quad n_{\ell;d} \sim \frac{2}{(d-1)!} \ell^{d-1} \quad \text{as} \quad \ell \to +\infty. \tag{2.2}$$

We choose the family of real-valued hyperspherical harmonics [27, Section 9.3] $(Y_{\ell,m;d})_{m=1}^{n_{\ell;d}}$ as orthonormal system of the ℓ-th eigenspace. We recall that $Y_{\ell,m;d}$, $m = 1, \dots, n_{\ell;d}$, are the restriction to $\mathbb{S}^d$ of harmonic polynomials of degree $\ell \in \mathbb{N}$ in $d + 1$ variables.

2.1. Random hyperspherical harmonics

For $\ell \in \mathbb{N}^*$, we define the ℓ-th random hyperspherical harmonic T_ℓ on $\mathbb{S}^d$ through

$$T_\ell(x) = \sqrt{\frac{\mu_d}{n_{\ell;d}}} \sum_{m=1}^{n_{\ell;d}} a_{\ell,m} Y_{\ell,m;d}(x), \quad x \in \mathbb{S}^d, \tag{2.3}$$

where $(a_{\ell,m})_{m=1}^{n_{\ell;d}}$ are standard Gaussian i.i.d. random variables in $\mathbb{R}$ and $\sqrt{\mu_d/n_{\ell;d}}$ is a normalizing factor. Then $x \mapsto T_\ell(x)$ is a *random* eigenfunction of the Helmotz equation (2.1), with eigenvalue $-\lambda_{\ell;d}$. Moreover, T_ℓ is an isotropic and centered Gaussian random field on $\mathbb{S}^d$ with covariance kernel (see [2, Section 9.6])

$$\mathrm{Cov}\,(T_\ell(x), T_\ell(y)) = \frac{\mu_d}{n_{\ell;d}} \sum_{m=1}^{n_{\ell;d}} Y_{\ell,m;d}(x) Y_{\ell,m;d}(y) = G_{\ell;d}(\langle x, y\rangle), \quad x, y \in \mathbb{S}^d. \tag{2.4}$$

Hereafter $G_{\ell;d}$ denotes the ℓ-th Gegenbauer polynomial [23, §4.7] (for $d = 2$, $G_{\ell;2} \equiv P_\ell$, that is, the Legendre polynomial of degree ℓ) and $\langle x, y\rangle = \cos d(x, y)$, where $d(x, y)$ is the geodesic distance between $x, y \in \mathbb{S}^d$. Recall that Gegenbauer polynomials are orthogonal on $[-1, 1]$ w.r.t. the weight $(1 - t^2)^{(d-2)/2}$. We take $G_{\ell;d}(1) = 1$, so $\mathrm{Var}(T_\ell(x)) = 1$, $x \in \mathbb{S}^d$.

2.2. Statistics of random hyperspherical harmonics

We are interested in functionals of random hyperspherical harmonics of the type

$$X_\ell := \int_{\mathbb{S}^d} \varphi(T_\ell(x)) dx, \tag{2.5}$$

where $\varphi \in L^2(\nu)$. In particular, we study the asymptotic behavior of the sequence of random variables $\{X_\ell\}_{\ell \in \mathbb{N}}$ as $\ell \to +\infty$ by means of chaotic decompositions [19, §2.2]: if $Z \sim \mathcal{N}(0, 1)$, then $\varphi(Z)$ can be written as an orthogonal series in $L^2(\mathbb{P})$ as follows

$$\varphi(Z) = \sum_{q \geq 0} \frac{b_q}{q!} H_q(Z), \quad \text{where} \quad b_q := \mathbb{E}[\varphi(Z) H_q(Z)], \tag{2.6}$$

where, from now on, H_q denotes the Hermite polynomial in $\mathbb{R}$ of order q (see e.g. [19, §1.4]). Substituting (2.6) into (2.5) gives the chaotic expansion for X_ℓ:

$$X_\ell = X_\ell[0] + \sum_{q \geq 2} X_\ell[q] \quad \text{where} \quad X_\ell[q] := \frac{b_q}{q!} \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx \tag{2.7}$$

(the term corresponding to $q = 1$ is null because of the orthogonality of hyperspherical harmonics). By standard properties of Hermite polynomials [19, §2.2] one gets

$$\mathbb{E}[X_\ell] = X_\ell[0] = \mathbb{E}[\varphi(Z)]\mu_d, \quad \mathrm{Var}(X_\ell) = \sum_{q\geq 2} \frac{b_q^2}{q!} \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x,y\rangle)^q \, dxdy. \tag{2.8}$$

Remark 2.1. The asymptotic behavior of the q-th moment of Gegenbauer polynomials is recalled in Proposition 4.1. We just notice here that $G_{\ell;d}(t) = (-1)^\ell G_{\ell;d}(-t)$, so if both ℓ and q are odd the q-th moment of $G_{\ell;d}$ vanishes. Hence we take only even ℓ: by $\ell \to +\infty$ we mean *as* ℓ *goes to infinity along even* ℓ.

We now define the standardized statistic

$$\tilde{X}_\ell = \frac{X_\ell - \mathbb{E}[X_\ell]}{\sqrt{\mathrm{Var}(X_\ell)}}, \tag{2.9}$$

that is the r.v. whose asymptotic behavior we are interested in. Notice that $\tilde{X}_\ell$ has Hermite rank 2 if and only if $b_2 \neq 0$ (see (2.6)). Let us first recall the well known central limit theorem stated in [15], which is proved under the Wasserstein distance

$$d_{\mathrm{W}}(X,Y) := \sup_{h\in \mathrm{Lip}(1)} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]| \,. \tag{2.10}$$

Here, X and Y are random variables, Lip(1) denotes the space of functions $h : \mathbb{R} \to \mathbb{R}$ such that $|h(x) - h(y)| \leq |x - y|$, $\forall x, y \in \mathbb{R}$.

Theorem 2.2 *(Theorem 1.7 in [15]). Let* φ *be as in* (2.6) *such that* $b_2 \neq 0$*. Then, as* $\ell \to +\infty$*,*

$$\mathrm{Var}(X_\ell) \sim \frac{b_2^2}{2} \frac{(\mu_d)^2}{n_{\ell;d}}, \tag{2.11}$$

and moreover

$$d_{\mathrm{W}}(\tilde{X}_\ell, Z) = O\big(\ell^{-\frac{1}{2}}\big). \tag{2.12}$$

The main goal of this paper is to strengthen and upgrade Theorem 2.2 from Wasserstein to Total Variation distance, which is defined as follows: for random variables X and Y,

$$d_{\mathrm{TV}}(X,Y) := \sup_{A\in \mathscr{B}(\mathbb{R})} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \,. \tag{2.13}$$

2.3. Statement of the main result

The assumptions on φ in Theorem 2.2 are rather weak: it suffices that φ is a square integrable function w.r.t. the Gaussian measure ν and $b_2 \neq 0$. In order to investigate the convergence for X_ℓ towards the Gaussian law in Total Variation distance, we need

$\varphi(Z)$ to satisfy some additional regularity properties in the Malliavin sense. These are summarized in the following condition.

Assumption 2.3. *Let $\varphi(Z)$ fulfill (2.6). We assume that $b_2 \neq 0$. Moreover, $\varphi(Z) \in Dom(L)$ and $\varphi(Z), L\varphi(Z) \in \cap_{k\geq 0} \cap_{p\geq 2} \mathbb{D}^{k,p}$, that is, for every $k \in \mathbb{N}$ and $p \geq 2$ the k-th order Malliavin derivative of $\varphi(Z)$ and of $L\varphi(Z)$, given by*

$$D^k\varphi(Z) = \sum_{q\geq k} \frac{b_q}{(q-k)!} H_{q-k}(Z) \quad \text{and} \quad D^k L\varphi(Z) = -\sum_{q\geq k} q\, \frac{b_q}{(q-k)!} H_{q-k}(Z), \tag{2.14}$$

exist and belong to $L^p(\mathbb{P})$. Furthermore, the same properties are satisfied by the function $\phi \in L^2(\nu)$ defined by

$$\phi(z) := \sum_{q\geq 2} \frac{|b_q|}{q!} H_q(z), \tag{2.15}$$

that is, $\phi(Z) \in Dom(L)$ and $\phi(Z), L\phi(Z) \in \cap_{k\geq 0} \cap_{p\geq 2} \mathbb{D}^{k,p}$: for $k \in \mathbb{N}$ and $p \geq 2$,

$$D^k\phi(Z) = \sum_{q\geq 2\vee k} \frac{|b_q|}{(q-k)!} H_{q-k}(Z) \quad \text{and} \quad D^k L\phi(Z) = -\sum_{q\geq 2\vee k} q \frac{|b_q|}{(q-k)!} H_{q-k}(Z) \tag{2.16}$$

both belong to $L^p(\mathbb{P})$.

From now on we assume that Assumption 2.3 holds. The requested Malliavin regularity will not be really surprising once the mathematical tools we are going to use will become clear (namely, the use of Proposition 3.1). As a meaningful example, take $\varphi(z) = \mathrm{e}^{tz}$, where $t \in \mathbb{R}$ denote a parameter. Then φ satisfies the well known representation

$$\mathrm{e}^{tz} = \mathrm{e}^{\frac{t^2}{2}} \sum_{q\geq 0} \frac{t^q}{q!} H_q(z), \quad z \in \mathbb{R},$$

it does not live in a finite number of Wiener chaoses, and it satisfies Assumption 2.3. Let us now give right away a sufficient condition for φ to satisfy Assumption 2.3.

Proposition 2.4. *Suppose that there exist $C, R > 0$ such that $|b_q| \leq CR^q$ for every $q \geq 0$ in (2.6). Then Assumption 2.3 holds.*

The proof of Proposition 2.4 is a consequence of the multiplier theorem for Wiener chaos series and Meyer's inequality (see [21, §1.4.3 and §1.5]), hence we omit the details for brevity sake.

We are now in a position to state the main result of this paper.

Theorem 2.5. *Let φ satisfy Assumption 2.3, then, for any $0 < \varepsilon < 1$, as $\ell \to +\infty$,*

$$d_{\mathrm{TV}}(\tilde{X}_\ell, Z) = O_\varepsilon\big(\ell^{-\frac{1-\varepsilon}{2}}\big) \tag{2.17}$$

where O_ε means that the constants involved in the O-notation depend on ε.

It is worth noticing that, *conditionally on Theorem 2.2 and the use of Proposition 3.1 below*, our result, i.e., the upper bound in (2.17) for the Total Variation distance, cannot be improved. Indeed, the upper bound given in Proposition 3.1 for $d_{\mathrm{TV}}(\tilde{X}_\ell, Z)$ cannot be smaller than $d_W(\tilde{X}_\ell, Z)^{1-\varepsilon}$, and $d_W(\tilde{X}_\ell, Z) = O(\ell^{-\frac{1}{2}})$.

Theorem 2.5 is the first result on the convergence of statistics of random hyperspherical harmonics (in particular having an infinite chaos expansion) in Total Variation distance. For $\varphi = H_q$ the q-th Hermite polynomial with $q \geq 2$, or for φ equal to a linear combination of such Hermite polynomials, bounding from above $d_{\mathrm{TV}}(\tilde{X}_\ell, Z)$ is an application of the fourth moment theorem by Nourdin and Peccati, see [18] for results in the two-dimensional case and [15,22] for higher dimensions.

An intermediate key step to prove our main result relies on the investigation of the asymptotic behavior of the sequence of Malliavin derivatives of X_ℓ; we stress that this analysis leads to some results of independent interest, see Proposition 3.2 and Proposition 3.3 for more details.

Besides the case of higher Hermite rank functionals, that we do believe it can be dealt with by using the same approach as the one developed for the proof of Theorem 2.5 though involving heavier computations, we leave as a topic for future research the interesting case of the indicator function: for $u \in \mathbb{R}$,

$$\varphi(z) = \mathbb{1}_{[u,+\infty)}(z), \quad z \in \mathbb{R},$$

thus X_ℓ is the so-called excursion area at level u, see [18]. Indeed, $\varphi(Z)$ is *not* derivable in Malliavin sense, and the Assumption 2.3 is not satisfied.

3. Proof of the main result

In this Section we explain the main ideas behind our argument, eventually giving the proof of our main result.

3.1. Proof strategy

To show the main ideas of the proof of Theorem 2.5 and of the results that we are going to use, we need to introduce some properties associated with the Malliavin regularity of the random variables at hands. We give here a result developed in [4], holding in a *purely abstract* Malliavin calculus setting (see [4, §2.1]), that is, based on a random noise that does not need to be Gaussian (see e.g. the one used in [3]). Let us resume it here briefly. First of all, it is assumed that the following ingredients are given:

- a set $\mathcal{E} \subset \cap_{p\geq 2} L^p(\Omega)$ such that for every $n \in \mathbb{N}^*$, $f \in C_p^\infty(\mathbb{R}^n)$ and $F = (F_1,\dots,F_n) \in \mathcal{E}^n$ then $f(F) \in \mathcal{E}$ (so, $\mathcal{E}$ is an algebra);
- a Hilbert space $\mathcal{H}$, whose inner product and associated norm will be denoted by $\langle \cdot,\cdot\rangle_{\mathcal{H}}$ and $|\cdot|_{\mathcal{H}}$ respectively; we let $L^p(\Omega;\mathcal{H})$ stand for the set of the r.v.'s taking values in $\mathcal{H}$ whose norm has moment of order p.

Hereafter $C_p^\infty(\mathbb{R}^n)$ denotes the set of functions $f : \mathbb{R}^n \to R$ that are continuously differentiable up to any order and all derivatives have polynomial growth.

In this environment, it is assumed that there exist two linear operators

$$D : \mathcal{E} \to \cap_{p\geq 2} L^p(\Omega;\mathcal{H}) \quad \text{and} \quad L : \mathcal{E} \to \mathcal{E}$$

such that

(M1) for every $F \in \mathcal{E}$ and $h \in \mathcal{H}$, $D_h F := \langle DF, h\rangle_{\mathcal{H}} \in \mathcal{E}$;
(M2) for every $n \in \mathbb{N}^*$, $f \in C_p^\infty(\mathbb{R}^n)$ and $F = (F_1,\dots,F_n) \in \mathcal{E}^n$ one has

$$Df(F) = \sum_{i=1}^n \partial_{x_i} f(F) DF_i \in \mathcal{E};$$

(M3) for every $F, G \in \mathcal{E}$ one has $\mathbb{E}[LF\, G] = -\mathbb{E}[\langle DF, DG\rangle_{\mathcal{H}}] = \mathbb{E}[F\, LG]$.

Thus, we recognize that these are settings and properties typically fulfilled in Malliavin calculus (but not in *any* Malliavin calculus framework - for example this is not in the case of jump processes, where the chain rule **(M2)** does not hold in general, see e.g. the discussion and the references quoted in [4, §1]). Hence, we call D the Malliavin derivative and L the Ornstein-Uhlenbeck operator. The higher order Malliavin derivatives can be defined straightforwardly: for $k \geq 2$,

$$D^k : \mathcal{E} \to \cap_{p\geq 2} L^p(\Omega;\mathcal{H}^{\otimes k})$$

is the multilinear operator such that for every $h_1,\dots,h_k \in \mathcal{H}$ and $F \in \mathcal{E}$,

$$D^k_{h_1,\dots,h_k} F := \langle D^k F, h_1 \otimes \dots \otimes h_k\rangle_{\mathcal{H}^{\otimes k}} = D_{h_k} D^{k-1}_{h_1,\dots,h_{k-1}} F.$$

Notice that, when dealing with a concrete Malliavin calculus, one can choose $\mathcal{E}$ either the set of the simple functionals or the set $\mathbb{D}^\infty$ of the r.v.'s whose Malliavin derivative of any order does exist and has finite moment of any power.

In order to introduce the result in [4] that we are going to use, we first need to define the involved Malliavin-Sobolev norms: for $F = (F_1,\dots,F_n) \in \mathcal{E}^n$, we set

$$|F|_{1,q} = \sum_{k=1}^q \sum_{i=1}^n |D^k F_i|_{\mathcal{H}^{\otimes k}}, \quad |F|_q = |F| + |F|_{1,q}, \quad \|F\|_{k,p} = \||F|_k\|_p, \tag{3.1}$$

where $\|\cdot\|_p$ is the standard norm in $L^p(\Omega)$. Then, for $k \in \mathbb{N}^*$ and $p \geq 2$, we set

$$\mathbb{D}^{k,p} = \overline{\mathcal{E}}^{\|\cdot\|_{k,p}} \quad \text{and} \quad \mathbb{D}^{k,\infty} = \cap_{p\geq 2}\mathbb{D}^{k,p}.$$

We also extend the operator L in the usual way: for $F = (F_1, \dots, F_n) \in \mathcal{E}^n$, we set $LF = (LF_1, \dots, LF_n)$ and $\|F\|_{\mathrm{OU}} = \|F\|_2 + \|LF\|_2$. And we define $Dom(L) = \overline{\mathcal{E}}^{\|\cdot\|_{\mathrm{OU}}}$.

Now, fix $q \in \mathbb{N}$ and $F = (F_1, \dots, F_n) \in (\mathbb{D}^{q+1,\infty})^n$. If $F = (F_1, \dots, F_n) \in (Dom(L))^n$ and $LF = (LF_1, \dots, LF_n) \in (\mathbb{D}^{q,\infty})^n$, the following quantities are well posed:

$$\begin{aligned}
&\mathcal{C}_q(F) = \big(|F|_{1,q+1} + |LF|_q\big)^q\big(1 + |F|_{1,q+1}\big)^{4nq},\\
&\mathcal{C}_{q,p}(F) = \|\mathcal{C}_q(F)\|_p,\\
&\mathcal{Q}_q(F) = \mathcal{C}_{q,2}(F)\|(\det\sigma_F)^{-1}\|_{2q}^q,
\end{aligned} \tag{3.2}$$

in which $p \geq 2$ and σ_F is the Malliavin covariance matrix of F, that is,

$$(\sigma_F)_{i,j} = \langle DF_i, DF_j\rangle_{\mathcal{H}}, \quad i,j = 1,\dots,n. \tag{3.3}$$

Notice that the quantity $\mathcal{C}_{q,p}(F)$, respectively $\mathcal{Q}_q(F)$, in (3.2) is in principle well posed whenever $F_i \in \mathbb{D}^{q+1,\bar{p}}$ for a suitable $\bar{p} \geq p$, respectively $\bar{p} \geq 2$.

We are now ready to state the result in [4] on which our asymptotic analysis will be based:

Proposition 3.1. *Let F and G be random vectors in $\mathbb{R}^n$ such that*

$$M_q(F,G) := \mathcal{C}_{q,1}(F) + \mathcal{Q}_q(G) < \infty,$$

for every $q \geq 1$. Let $U > 0$ be a real random variable such that $\|U^{-1}\|_q < \infty$ for every $q \geq 1$. Then for every $\varepsilon > 0$ there exist $C_\varepsilon > 0$ and $q_\varepsilon > 1$ such that

$$d_{\mathrm{TV}}(F,G) \leq C_\varepsilon\big(M_{q_\varepsilon}(F,G) + \|U^{-1}\|_{2/\varepsilon}\big)\big(d_{\mathrm{W}}(F,G) + d_{\mathrm{W}}(\det\sigma_F, U)\big)^{1-\varepsilon}.$$

This is actually [4, Proposition 3.12], see in particular (3.30), with the choice $p = p' = 1$ (remark that, as it immediately and clearly follows from the proof, there is a misprint in the requests therein: it is erroneously asked that $\mathcal{C}_{q,1}(G), \mathcal{Q}_q(F) < \infty$ instead of $\mathcal{C}_{q,1}(F), \mathcal{Q}_q(G) < \infty$).

Our plan is to use Proposition 3.1 with $F = \tilde{X}_\ell$ and $G = Z \sim \mathcal{N}(0,1)$. Indeed, in our framework, the underlying Hilbert space is $H = L^2(\mathbb{S}^d, \mathscr{B}(\mathbb{S}^d), \mathrm{Leb})$, the random eigenfunction T_ℓ admitting the isonormal representation (4.1). Thus

$$\sigma_\ell = \sigma_{\tilde{X}_\ell} = \int_{\mathbb{S}^d} |D_y\tilde{X}_\ell|^2\,dy. \tag{3.4}$$

First, Assumption 2.3 will guarantee that all the involved Malliavin functionals are well defined (we will give more details about Malliavin calculus for Gaussian random fields in

§ 4.2). Theorem 2.2 already ensures that $d_{\mathrm{W}}(\tilde{X}_\ell, Z) \to 0$ (giving also an estimation of the speed of convergence). Therefore, we obtain the stronger convergence $d_{\mathrm{TV}}(\tilde{X}_\ell, Z) \to 0$ (together with a useful upper bound on the rate), once we prove that:

(H1) there exists a *deterministic* $U > 0$ such that $d_{\mathrm{W}}(\sigma_\ell, U) \to 0$ with some speed,
(H2) for every $q \geq 1$, $\sup_\ell M_q(\tilde{X}_\ell, Z) < \infty$, where $M_q(\tilde{X}_\ell, Z)$ is defined in Proposition 3.1.

3.2. Proof of Theorem 2.5

Concerning **(H1)**, we will prove the following key result.

Theorem 3.2. *Let σ_ℓ be the Malliavin covariance of $\tilde{X}_\ell$. Under Assumption 2.3, we have*

$$|\mathbb{E}[\sigma_\ell] - 2| = O\left(\eta_{\ell;d}\right) \quad \text{and} \quad \mathrm{Var}(\sigma_\ell) = O\left(\ell^{-1}\mathbb{1}_{d=2} + \ell^{-(d-1)/2}\mathbb{1}_{d\geq 3}\right),$$

where

$$\eta_{\ell;d} = \mathbb{1}_{d=2}\left(\mathbb{1}_{b_4\neq 0}\frac{\log \ell}{\ell} + \mathbb{1}_{b_4=0}\frac{1}{\ell}\right) + \frac{1}{\ell}\mathbb{1}_{d\geq 3}.$$

As for **(H2)**, it is enough to prove that, uniformly in ℓ, all the moments of the main Malliavin operators involved in $M_q(\tilde{X}_\ell, Z)$ are bounded. This is why we will prove the following result.

Proposition 3.3. *Under Assumption 2.3, for every $k \in \mathbb{N}$ and $n \geq 1$, there exists $\tilde{C}_{n,k,d} > 0$ such that*

$$\sup_{\ell \text{ even}} \mathbb{E}[|D^{(k)}\tilde{X}_\ell|^n_{\mathcal{H}^{\otimes k}}] \leq \tilde{C}_{n,k,d} \quad \text{and} \quad \sup_{\ell \text{ even}} \mathbb{E}[|D^{(k)}L\tilde{X}_\ell|^n_{\mathcal{H}^{\otimes k}}] \leq \tilde{C}_{n,k,d}.$$

We postpone the proofs of Proposition 3.2 and of Proposition 3.3 to Sections constructed ad hoc (see § 5 and § 6 respectively). Based on such results, the proof of the CLT in Total Variation distance (Theorem 2.5) follows.

Proof of Theorem 2.5 assuming Propositions 3.2 and 3.3. We use Proposition 3.1 with $F = \tilde{X}_\ell$, $G = Z$ and $U = 2$. We have

$$d_{\mathrm{W}}(\sigma_\ell, 2) \leq \|\sigma_\ell - 2\|_1 \leq \|\sigma_\ell - 2\|_2 \leq \mathrm{Var}(\sigma_\ell)^{1/2} + |\mathbb{E}[\sigma_\ell] - 2| \to 0$$

and then, recalling the asymptotic behavior of σ_ℓ in Proposition 3.2 we obtain

$$d_{\mathrm{W}}(\sigma_\ell, 2) = \begin{cases} O(\ell^{-1/2}) & d = 2, 3 \\ O(\ell^{-3/4}) & d = 4 \\ O(\ell^{-1}) & d \geq 5 \end{cases} \tag{3.5}$$

Since $G = Z \sim \mathcal{N}(0,1)$, $DG = 1$, that gives $\sigma_G = 1$, $D^kG = 0$ for every $k \geq 2$ and $LG = -G$, so that (see (3.1)-(3.2)) $\mathcal{Q}_q(G) = \mathcal{Q}_1(G) < \infty$ for every $q \geq 1$. As for $\mathcal{C}_{q,1}(\tilde{X}_\ell)$, we have

$$\begin{aligned}\mathcal{C}_{q,1}(\tilde{X}_\ell) = \|\mathcal{C}_q(\tilde{X}_\ell)\|_1 &= \mathbb{E}[\big(|\tilde{X}_\ell|_{1,q+1} + |L\tilde{X}_\ell|_q\big)^q\big(1+|\tilde{X}_\ell|_{1,q+1}\big)^{4q}] \\ &\leq \mathbb{E}[\big(|\tilde{X}_\ell|_{1,q+1} + |L\tilde{X}_\ell|_q\big)^{2q}]^{\frac{1}{2}}\mathbb{E}[\big(1+|\tilde{X}_\ell|_{1,q+1}\big)^{8q}]^{\frac{1}{2}} \\ &\leq (\mathbb{E}[|\tilde{X}_\ell|^{2q}_{1,q+1}]^{\frac{1}{2}} + \mathbb{E}[|L\tilde{X}_\ell|^{2q}]^{\frac{1}{2}})(1+\mathbb{E}[|\tilde{X}_\ell|^{8q}_{1,q+1}]^{\frac{1}{2}})\end{aligned}$$

and Proposition 3.3 allows one to check that $\sup_\ell \mathcal{C}_{q,1}(\tilde{X}_\ell) < \infty$ for every $q \geq 1$. Then, Theorem 2.5 ensures that, for $\varepsilon > 0$,

$$d_{\mathrm{TV}}(\tilde{X}_\ell, Z) \leq \mathrm{C}_\varepsilon\big(d_{\mathrm{W}}(\tilde{X}_\ell, Z) + d_{\mathrm{W}}(\det\sigma_\ell, 2)\big)^{1-\varepsilon}.$$

Now, combining the above estimate on $d_{\mathrm{W}}(\sigma_\ell, 2)$ and the result on $d_{\mathrm{W}}(\tilde{X}_\ell, Z)$ in Theorem 2.2, we conclude the proof. □

Comparing (2.12) and (3.5), when applying Proposition 3.1 the presence of $d_{\mathrm{W}}(\sigma_\ell, 2)$ does not worsen the quantitative convergence rate for $d_{\mathrm{TV}}(\tilde{X}_\ell, Z)$: in fact, whenever $d \geq 2$ we obtain that $d_{\mathrm{TV}}(\tilde{X}_\ell, Z) = O_\varepsilon(d_{\mathrm{W}}(\tilde{X}_\ell, Z)^{1-\varepsilon})$, for any $\varepsilon > 0$ close to 0. In other words, the term coming from the Malliavin covariance does not slow down the convergence speed.

4. Background on Gaussian random fields

In this Section we recall the isonormal representation for random hyperspherical harmonics along with the Wiener-Itô chaos theory, finally we deal with Malliavin calculus for Gaussian fields.

4.1. Isonormal representation and Wiener chaos expansion

Let us recall the equivalent way to introduce random hyperspherical harmonics as isonormal Gaussian random fields (for details, see [19, Chapter 2]). We denote $H = L^2(\mathbb{S}^d, \mathscr{B}(\mathbb{S}^d), \mathrm{Leb})$ the real separable Hilbert space of square integrable functions on $\mathbb{S}^d$ w.r.t. the Lebesgue measure, with inner product $\langle f, g\rangle_H = \int_{\mathbb{S}^d} f(x)g(x)dx$. Let W denote a Gaussian white noise on $\mathbb{S}^d$. Then the Gaussian field T_ℓ in (2.3) can be represented (in law) as

$$T_\ell(x) = \int_{\mathbb{S}^d} \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\langle x, y\rangle) W(dy), \quad x \in \mathbb{S}^d. \tag{4.1}$$

Using (4.1), the covariance function of the random field T_ℓ is given by

$$\mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell;d}(\langle x, y\rangle), \quad x, y \in \mathbb{S}^d, \tag{4.2}$$

in fact (4.2) is an immediate consequence of the following reproducing property: for every $x, y \in \mathbb{S}^d$,

$$\int_{\mathbb{S}^d} G_{\ell;d}(\langle x,z\rangle) G_{\ell;d}(\langle z,y\rangle) dz = \frac{\mu_d}{n_{\ell;d}} G_{\ell;d}(\langle x,y\rangle). \tag{4.3}$$

For later use, we recall the asymptotic behavior as $\ell \to \infty$ of the moments of the Gegenbauer polynomials, that we resume as follows.

Proposition 4.1. *For $q \in \mathbb{N}$, $q \geq 2$, set*

$$c_{q;d} = \begin{cases} \left(2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!\right)^q \int_0^\infty J_{\frac{d}{2}-1}(u)^q u^{-q(\frac{d}{2}-1)+d-1} du & \text{if } q \geq 3, \\ \frac{(d-1)!\mu_d}{4\mu_{d-1}} & \text{if } q = 2, \end{cases} \tag{4.4}$$

where $J_{\frac{d}{2}-1}$ is the Bessel function of order $\frac{d}{2}-1$.

For $q \geq 2$ and $d \geq 2$, the function $\mathbb{S}^d \ni y \mapsto \int_{\mathbb{S}^d} G_{\ell;d}(\langle x,y\rangle)^q dx$ is constant. Moreover, the following properties hold.

For $d \geq 2$ one has $\int_{\mathbb{S}^d} G_{\ell;d}(\langle x,y\rangle)^2 dx = \frac{\mu_d}{n_{\ell;d}}$ and, as $\ell \to \infty$,

$$\int_{\mathbb{S}^d} G_{\ell;d}(\langle x,y\rangle)^2 dx = 2\mu_{d-1} \frac{c_{2;d}}{\ell^{d-1}}(1 + o_{2;d}(1)). \tag{4.5}$$

Set now $q \geq 3$. Then

- *if $d \geq 3$, then*

$$\int_{\mathbb{S}^d} G_{\ell;d}(\langle x,y\rangle)^q dx = 2\mu_{d-1} \frac{c_{q;d}}{\ell^d}(1 + o_{q;d}(1)); \tag{4.6}$$

- *if $d = 2$, the behavior differs according to $q \neq 4$ (being as in (4.6)) and $q = 4$:*

$$\int_{\mathbb{S}^2} G_{\ell;2}(\langle x,y\rangle)^q dx \equiv \int_{\mathbb{S}^2} P_\ell(\langle x,y\rangle)^q dx = \begin{cases} \frac{12 \log \ell}{\pi \ell^2}(1 + o_{4;2}(1)) & q = 4 \\ \frac{4\pi c_{q;2}}{\ell^2}(1 + o_{q;2}(1)) & q = 3 \text{ or } q \geq 5. \end{cases} \tag{4.7}$$

Proof details can be found in [15, Proposition 1.1]), see also [17,18]. It is worth noticing that constants $c_{q;d}$ in (4.4) are *strictly positive* for every q and d; in particular, for odd q this result is highly non trivial, see [10].

Let us now briefly recall the notion of Wiener chaos. Let W denote the Gaussian noise as in (4.1) and set $\mathscr{F} = \sigma(\int_{\mathbb{S}^d} f(x) W(dx) : f \in H)$, where $H = L^2(\mathbb{S}^d, \mathscr{B}(\mathbb{S}^d), \text{Leb})$. Then every random variable $F \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ admits the Wiener chaos expansion

$$F=\sum_{q\geq 0} J_q(F),$$

where J_q is the orthogonal projection operator on the q-th chaos, which is the closure in $L^2(\Omega,\mathscr{F},\mathbb{P})$ of $\mathrm{Span}(H_q(T(f))\,:\, f\in H, \|f\|_H=1)$, where $T(f)=\int_{\mathbb{S}^d} f(x)W(dx)$ and H_q denotes the Hermite polynomial in $\mathbb{R}$ of degree q (see [21, Chapter 1]). Once the Wiener chaos expansion is defined, we can introduce the Ornstein-Uhlenbeck operator L, which will play an important role in our approach: for $F\in L^2(\Omega,\mathscr{F},\mathbb{P})$, one says that $F\in Dom(L)$ if and only if $\sum_{q\geq 1} q^2\mathbb{E}[J_q(F)^2]<\infty$ and in such a case,

$$LF=-\sum_{q\geq 1} qJ_q(F). \tag{4.8}$$

4.2. Malliavin calculus for Gaussian random fields

In this section we show that the standard Malliavin calculus for Gaussian random fields fulfills the requests in § 3.1. All details can be found fully explained in [19, §2.3] or [21].

Let W be the Gaussian noise in (4.1) and let $\mathcal{S}$ denote the set of the simple functionals, defined as follows: $F\in\mathcal{S}$ if there exist $m\geq 1$, $f\in C_p^\infty(\mathbb{R}^m)$ and $g_1,\dots,g_m\in H$ such that

$$F=f\big(T(g_1),\dots,T(g_m)\big),\quad \text{with}\quad T(g_i)=\int_{\mathbb{S}^d} g_i(x)W(dx), \tag{4.9}$$

We recall that $\mathcal{S}$ is dense in $L^p(\Omega):=L^p(\Omega,\mathscr{F},\mathbb{P})$.

Given $k\in\mathbb{N}$, we denote with $H^{\otimes k}$ and $H^{\odot k}$, respectively, the k-th tensor product and the k-th symmetric tensor product. Let $F\in\mathcal{S}$ be given by (4.9) and $k\in\mathbb{N}$. The k-th *Malliavin derivative* is the element of $L^2(\Omega;H^{\odot k})$ defined by

$$D^{(k)}F=\sum_{i_1,\dots,i_k=1}^{m}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(T(g_1),\dots,T(g_m))\,g_{i_1}\otimes\cdots\otimes g_{i_m}.$$

For $k\in\mathbb{N}$ and $p\geq 1$, the space $\mathbb{D}^{k,p}$ is defined as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{\mathbb{D}^{k,p}}=\big(\mathbb{E}[|F|^p]+\mathbb{E}[\|F\|_H^p]+\ldots+\mathbb{E}[\|D^{(k)}F\|^p_{H^{\otimes k}}]\big)^{\frac{1}{p}}$$

and the Malliavin derivative can be extended to the set $\mathbb{D}^{k,p}$, being the *domain* of $D^{(k)}$ in $L^p(\Omega;\mathbb{R})$. In particular, the space $\mathbb{D}^{k,2}$ is a Hilbert space with respect to the inner product

$$\langle F,G\rangle_{\mathbb{D}^{k,2}}=\mathbb{E}[FG]+\sum_{r=1}^{k}\mathbb{E}[\langle D^{(r)}F,D^{(r)}G\rangle_{H^{\odot r}}].$$

Moreover, the chain rule property does hold: for every $\phi \in C_b^1(\mathbb{R}^m)$ and $F = (F_1, \dots, F_m)$ with $F_i \in \mathbb{D}^{1,p}$, $i = 1, \dots, m$, for some $p \geq 1$, then $\phi(F) \in \mathbb{D}^{1,p}$ and

$$D\phi(F) = \sum_{r=1}^{k} \frac{\partial \phi}{\partial x_r}(F) DF_r. \tag{4.10}$$

It is well known that such a Malliavin calculus framework satisfies the abstract hypotheses required in [4, §2.1] and resumed here in §3.1 (see e.g. [21] or [19]): just take $\mathcal{E} = \mathcal{S}$, $\mathcal{H} = H = L^2(\mathbb{S}^d, \mathscr{B}(\mathbb{S}^d), \mathrm{Leb})$ and L as the Ornstein-Uhlenbeck operator defined in (4.8). In particular, the duality relationship **(M3)** does hold for $F, G \in \mathbb{D}^{2,2}$ and therefore, it holds true on $\mathcal{E}$.

To conclude, we give some formulas that will be used in the sequel. Let T_ℓ be the Gaussian random field in (4.1) and let H_q denote the Hermite polynomial in $\mathbb{R}$ of degree $q \in \mathbb{N}$. As an immediate consequence of the chain rule (4.10), for $q \in \mathbb{N}$ and $p \geq 1$ then $H_q(T_\ell(x)) \in \mathbb{D}^{1,p}$ and, from (4.9), (4.1) and $H_q' = qH_{q-1}$,

$$D_y H_q(T_\ell(x)) = qH_{q-1}(T_\ell(x)) D_y T_\ell(x) = qH_{q-1}(T_\ell(x)) \sqrt{\frac{n_{\ell;d}}{\mu_d}} G_{\ell;d}(\langle x, y \rangle).$$

Iterating the argument, $H_q(T_\ell(x)) \in \mathbb{D}^{k,p}$ for every $k \in \mathbb{N}$ and

$$D^{(k)}_{y_1,\dots,y_k} H_q(T_\ell(x)) = \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^{\frac{k}{2}} \frac{q!}{(q-k)!} H_{q-k}(T_\ell(x)) \prod_{r=1}^{k} G_{\ell;d}(\langle x, y_r \rangle). \tag{4.11}$$

Moreover, by developing standard density arguments, (4.11) gives $\int_{\mathbb{S}^d} H_q(T_\ell(x))dx \in \mathbb{D}^{k,p}$ and

$$D^{(k)}_{y_1,\dots,y_k} \int_{\mathbb{S}^d} H_q(T_\ell(x))dx = \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^{\frac{k}{2}} \frac{q!}{(q-k)!} \int_{\mathbb{S}^d} H_{q-k}(T_\ell(x)) \prod_{r=1}^{k} G_{\ell;d}(\langle x, y_r \rangle)dx. \tag{4.12}$$

5. Convergence of Malliavin covariances

In this section we prove Lemma 3.2. Recalling the (finite dimensional) chaos expansion for $\varphi(Z)$ in (2.6) and substituting it in (2.5), we obtain the following expansion for X_ℓ:

$$X_\ell = m_d + \sum_{q \geq 2} \frac{b_q}{q!} \int_{\mathbb{S}^d} H_q(T_\ell(x))dx, \tag{5.1}$$

where $m_d = \mathbb{E}[X_\ell] = \mathbb{E}[\varphi(Z)]\mu_d$. Notice also that (5.1) says that the projection on the chaos of order 1 is null, as an immediate consequence of the fact that $\int_{\mathbb{S}^d} T_\ell(x)dx = 0$. Following (5.1), the chaos expansion of the normalized r.v. $\tilde{X}_\ell$ is given by

$$\tilde{X}_\ell = \frac{1}{v_{\ell;d}} \sum_{q \geq 2} \frac{b_q}{q!} \int_{\mathbb{S}^d} H_q(T_\ell(x))dx, \text{ where } v_{\ell;d}^2 = \mathrm{Var}(X_\ell). \tag{5.2}$$

We stress that, by (2.11) from Theorem 2.2 and (2.2),

$$v_{\ell;d} \sim b_2 c_d \ell^{-\frac{d-1}{2}}. \tag{5.3}$$

5.1. On the diagram formula and cross moments of Gegenbauer polynomials

Now we introduce some notation and results that are useful to prove Lemma 3.2. We first provide a reformulation of the well known diagram formula for Hermite polynomials [13, Proposition 4.15], equivalently of a particular case of the standard Feynman diagram representation of moments of Wick products [11, Theorem 3.12].

Definition 5.1. *For $q_1, \dots, q_n \in \mathbb{N}$, we define $\mathcal{A}_{q_1,\dots,q_n}$ as the set given by the indexes $\{k_{ij}\}_{i,j=1}^n$ such that for every $i, j = 1, \dots, n$,*

$$k_{i,j} \in \mathbb{N},\ k_{ii} = 0,\ k_{ij} = k_{ji} \quad \text{and} \quad \sum_{j=1}^n k_{ij} = q_i. \tag{5.4}$$

Lemma 5.2. *Let $n \geq 2$ and let $(Z_1, \dots, Z_n)$ be a n-dimensional centered Gaussian vector. For $q_1, \dots, q_n \in \mathbb{N}$, consider $\mathcal{A}_{q_1,\dots,q_n}$ as in Definition 5.1. Then,*

$$\mathbb{E}[\prod_{r=1}^n H_{q_r}(Z_r)] = \prod_{r=1}^n q_r! \times \sum_{\{k_{i,j}\}_{i,j=1}^n \in \mathcal{A}_{q_1,\dots,q_n}} \prod_{\substack{i,j=1\\ i<j}}^n \frac{\mathbb{E}[Z_i Z_j]^{k_{ij}}}{k_{ij}!}. \tag{5.5}$$

In particular, taking $Z_1 = \dots = Z_n = Z \sim \mathcal{N}(0,1)$, one has

$$\mathbb{E}[\prod_{r=1}^n H_{q_r}(Z)] = \prod_{r=1}^n q_r! \times \sum_{\{k_{i,j}\}_{i,j=1}^n \in \mathcal{A}_{q_1,\dots,q_n}} \prod_{\substack{i,j=1\\ i<j}}^n \frac{1}{k_{ij}!}. \tag{5.6}$$

We remark that (5.5) is tailored for our purposes. Besides, as it does not involve diagrams but merely an explicit set of indexes (see $\mathcal{A}_{q_1,\dots,q_n}$), it appears more friendly than the usual diagram formula (see (A.1)). Its proof makes a strong use of non trivial combinatorics arguments. Since combinatorial tools are limited to this special case, for the sake of readability we postpone the proof of (5.5) to Appendix A.1. Let us see now how we apply such result.

For fixed $n \geq 2$ and $x_1, \dots, x_n \in \mathbb{S}^d$, the random vector $(T_\ell(x_1), \dots, T_\ell(x_n))$ is a centered Gaussian random vector whose covariances are given by (see (4.2))

$$\mathbb{E}[T_\ell(x_i)T_\ell(x_j)] = G_{\ell;d}(\langle x_i, x_j \rangle), \quad i, j = 1, \dots, n.$$

When dealing with our proofs, we often need to compute and/or estimate quantities of the type

$$\int_{(\mathbb{S}^d)^n} \mathbb{E}[\prod_{r=1}^n H_{q_r}(T_\ell(x_r))]dx.$$

By using (5.5), we have

$$\int_{(\mathbb{S}^d)^n} \mathbb{E}[\prod_{r=1}^n H_{q_r}(T_\ell(x_r))]dx = \prod_{r=1}^n q_r! \times \sum_{\{k_{i,j}\}_{i,j=1}^n \in \mathcal{A}_{q_1,\dots,q_n}} \int_{(\mathbb{S}^d)^n} \prod_{\substack{i,j=1 \\ i<j}}^n \frac{G_{\ell;d}(\langle x_i, x_j\rangle)^{k_{ij}}}{k_{ij}!} dx. \tag{5.7}$$

Therefore, it would be very useful to get a good estimate for the integrals appearing in the r.h.s. of (5.7), that is, for cross moments of Gegenbauer polynomials. To this purpose we need to introduce the concept of *extrapolated graph* from a given $\kappa = \{k_{ij}\}_{i,j=1}^n \in \mathcal{A}_{q_1,\dots,q_n}$. Such graph is defined as the pair $\mathfrak{G}_\kappa = (V, E_\kappa)$ in which the set of the nodes is given by $V = \{1, \dots, n\}$ and the set of the edges is given as follows: the edge (i, j) does exist iff $k_{ij} \neq 0$ (notice that, since $k_{ii} = 0$, there are no self-loops). The use of graphs is a key point in our approach, that is why in Appendix A.2 we recall the main definitions and properties.

Lemma 5.3. *For $n \in \mathbb{N}^*$, let $\kappa = \{k_{ij}\}_{i,j=1}^n \in \mathcal{A}_{q_1,\dots,q_n}$ be fixed. Let $\mathfrak{G}_\kappa$ denote the extrapolated graph from κ and N_κ denote the number of connected components of $\mathfrak{G}_\kappa$. Then,*

$$\int_{(\mathbb{S}^d)^n} \prod_{\substack{i,j=1 \\ i<j}}^n G_{\ell;d}(\langle x_i, x_j\rangle)^{2k_{ij}} dx \le \frac{C_d(N_\kappa)}{\ell^{(d-1)(n-N_\kappa)}} \tag{5.8}$$

where $C_d(N_\kappa) = (8\mu_d\mu_{d-1}c_{2;d})^{n-N_\kappa}\mu_d^{N_\kappa}$, $c_{2;d}$ being given in (4.4). As a consequence, for $n = 2p$,

$$\int_{(\mathbb{S}^d)^{2p}} \prod_{\substack{i,j=1 \\ i<j}}^{2p} G_{\ell;d}(\langle x_i, x_j\rangle)^{2k_{ij}} dx \le \frac{C_{d;p}}{\ell^{(d-1)p}}, \tag{5.9}$$

where $C_{d;p} = (2(d-1)!\mu_d^2)^{2p}\mu_d^p$.

The proof of Lemma 5.3 relies on an accurate study based on a rewriting of the integrals in the l.h.s. of (5.8) in terms of special connected graphs. As these arguments are developed exclusively for Lemma 5.3, we postpone the proof in an appendix ad hoc (see Appendix A.2).

We remark that, in principle, (5.9) might be useful to get some estimates on concatenated sums of products of so-called Clebsch-Gordan coefficients $\{C^{L,M}_{\ell_1,m_1,\ell_2,m_2}\}$ that we define by

$$Y_{\ell_1,m_1;d}(x)Y_{\ell_2,m_2;d}(x) = \sum_{L=0}^{\ell_1+\ell_2} \sum_{M=1}^{n_{L;d}} C^{L,M}_{\ell_1,m_1,\ell_2,m_2;d} Y_{L,M;d}(x), \quad x \in \mathbb{S}^d.$$

This is because there exists a precise link between such quantities and moments of Gegenbauer polynomials which can be established via the addition formula (2.4). However, it is not clear whether it is actually possible to obtain optimal or novel estimates, even if in dimension $d > 2$ a little is known about these coefficients. (See [13, §3.5] for a complete discussion in the case of the 2-sphere).

5.2. Proof of Theorem 3.2

We are now in a position to prove Theorem 3.2, that is the main result on the convergence in Wasserstein distance for the Malliavin covariances of $\tilde{X}_\ell$, as $\ell \to +\infty$. Let us anticipate that the proof requires a finer different method for the case $d = 2$ than $d \geq 3$. Therefore, as it will be clear from reading the proof, we will have to split in two different approaches.

Proof of Theorem 3.2. By using (4.12) (with $k = 1$) and classical density arguments, the Malliavin derivative $D\tilde{X}_\ell : \Omega \to H$ is given by

$$D_y\tilde{X}_\ell = \frac{1}{v_{\ell;d}}\sqrt{\frac{n_{\ell;d}}{\mu_d}} \sum_{q\geq 2} \frac{b_q}{(q-1)!} \int_{\mathbb{S}^d} H_{q-1}(T_\ell(x)) G_{\ell;d}(\langle x, y\rangle)dx.$$

Following (3.3), with $n = 1$ and $\mathcal{H} = H = L^2(\mathbb{S}^d, \mathscr{B}(\mathbb{S}^d), \text{Leb})$, we can write down the Malliavin covariance σ_ℓ of $\tilde{X}_\ell$:

$$\sigma_\ell = \int_{\mathbb{S}^d} |D_y\tilde{X}_\ell|^2 dy = \frac{1}{v^2_{\ell;d}} \frac{n_{\ell;d}}{\mu_d} \sum_{q,p\geq 2} \frac{b_q b_p}{(q-1)!(p-1)!} \times \int_{\mathbb{S}^d} \int_{(\mathbb{S}^d)^2} H_{q-1}(T_\ell(x)) H_{p-1}(T_\ell(z)) G_{\ell;d}(\langle x, y\rangle) G_{\ell;d}(\langle z, y\rangle) dxdzdy.$$

By using the duplication formula (4.3), we obtain

$$\sigma_\ell = \frac{1}{v^2_{\ell;d}} \sum_{q,p\geq 2} \frac{b_q b_p}{(q-1)!(p-1)!} \int_{(\mathbb{S}^d)^2} H_{q-1}(T_\ell(x)) H_{p-1}(T_\ell(z)) G_{\ell;d}(\langle x, z\rangle) dxdz. \quad (5.10)$$

Therefore, by (4.2),

$$
\begin{aligned}
\mathbb{E}[\sigma_\ell] &= \frac{1}{v_{\ell;d}^2} \sum_{q,p\geq 2} \frac{b_q b_p}{(q-1)!(p-1)!} \int_{(\mathbb{S}^d)^2} \mathbb{E}[H_{q-1}(T_\ell(x))H_{p-1}(T_\ell(z))] G_{\ell;d}(\langle x, z\rangle) dxdz \\
&= \frac{1}{v_{\ell;d}^2} \sum_{q\geq 2} \frac{b_q^2}{(q-1)!} \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x, z\rangle)^q dxdz.
\end{aligned}
$$

Then, from the asymptotics for moment of Gegenbauer polynomials in Proposition 4.1 and from (5.3), we have that

$$
\mathbb{E}[\sigma_\ell] - 2 = O\left(\mathbb{1}_{d\geq 3}\frac{1}{\ell} + \mathbb{1}_{d=2}\left(\mathbb{1}_{b_4\neq 0}\frac{\log \ell}{\ell} + \mathbb{1}_{b_4=0}\frac{1}{\ell}\right)\right)
$$

as $\ell \to \infty$. In the above result we underline that the difference between $d = 2$ and $d \geq 3$ changes the asymptotic behavior when $b_4 \neq 0$.

Now we study the variance of σ_ℓ. Denoting with $dx := dx_1dx_2dx_3dx_4$, we have that

$$
\begin{aligned}
\mathbb{E}[\sigma_\ell^2] = \frac{1}{v_{\ell;d}^4} &\sum_{q_1,q_2,q_3,q_4\geq 2} \Big(\prod_{j=1}^4 \frac{b_{q_j}}{(q_j-1)!}\Big) \\
&\times \int_{(\mathbb{S}^d)^4} \mathbb{E}[\prod_{i=1}^4 H_{q_i-1}(T_\ell(x_i))] G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_3, x_4\rangle) dx.
\end{aligned}
$$

By using Lemma 5.2, we have

$$
\begin{aligned}
\mathbb{E}[\sigma_\ell^2] = \frac{1}{v_{\ell;d}^4} &\sum_{q_1,q_2,q_3,q_4\geq 2} \Big(\prod_{j=1}^4 \frac{b_{q_j}}{(q_j-1)!}\Big) \prod_{r=1}^4 (q_r-1)! \sum_{\{k_{i,j}\}_{i,j=1}^4 \in \mathcal{A}_{q_1-1,\dots,q_4-1}} \prod_{\substack{i,j=1\\ i<j}}^4 \frac{1}{k_{ij}!} \\
&\times \int_{(\mathbb{S}^d)^4} \prod_{\substack{i,j=1\\ i<j}}^4 G_{\ell;d}(\langle x_i, x_j\rangle)^{k_{ij}} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_3, x_4\rangle) dx.
\end{aligned}
$$

Let us first study the case $q_1 = q_2$ and $q_3 = q_4$ and $k_{13} = k_{14} = k_{23} = k_{24} = 0$, so that $k_{12} = q_1 - 1$ and $k_{34} = q_3 - 1$. Then we have to deal with

$$
\sum_{q_1,q_3\geq 2} \frac{b_{q_1}^2 b_{q_3}^2}{(q_1-1)!(q_3-1)!} \int_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_2\rangle)^{q_1} G_{\ell;d}(\langle x_3, x_4\rangle)^{q_3} dx
$$

and this is exactly $v_{\ell;d}^4\,\mathbb{E}[\sigma_\ell]^2$. Then we define $\mathcal{N}_{q_1-1,q_2-1,q_3-1,q_4-1}$ as the set of $\kappa = \{k_{ij}\}_{i,j=1}^4$ such that

$$
k_{12} = q_1 - 1 = q_2 - 1,\ k_{34} = q_3 - 1 = q_4 - 1,\ k_{13} = k_{14} = k_{23} = k_{24} = 0
$$

and we set

$$\mathcal{C}_{q_1-1,q_2-1,q_3-1,q_4-1} = \mathcal{A}_{q_1-1,\dots,q_4-1} \setminus \mathcal{N}_{q_1-1,q_2-1,q_3-1,q_4-1}. \tag{5.11}$$

Hence we obtain

$$\mathrm{Var}(\sigma_\ell) = \mathbb{E}[\sigma_\ell^2] - \mathbb{E}[\sigma_\ell]^2 = \frac{1}{v_{\ell;d}^4} \sum_{q_1,q_2,q_3,q_4\geq 2} \prod_{i=1}^4 \frac{b_{q_i}}{(q_i-1)!} \prod_{r=1}^4 (q_r-1)!$$

$$\times \sum_{\{k_{i,j}\}_{i,j=1}^4 \in \mathcal{C}_{q_1-1,\dots,q_4-1}} \prod_{\substack{i,j=1\\ i<j}}^4 \frac{1}{k_{ij}!} \int_{(\mathbb{S}^d)^4} \prod_{\substack{i,j=1\\ i<j}}^4 G_{\ell;d}(\langle x_i,x_j\rangle)^{k_{ij}} G_{\ell;d}(\langle x_1,x_2\rangle) G_{\ell;d}(\langle x_3,x_4\rangle)dx,$$

so that

$$\mathrm{Var}(\sigma_\ell) \leq \frac{1}{v_{\ell;d}^4} \sum_{q_1,q_2,q_3,q_4\geq 2} \prod_{i=1}^4 \frac{|b_{q_i}|}{(q_i-1)!} \prod_{r=1}^4 (q_r-1)!$$

$$\times \sum_{\{k_{i,j}\}_{i,j=1}^4 \in \mathcal{C}_{q_1-1,\dots,q_4-1}} \prod_{\substack{i,j=1\\ i<j}}^4 \frac{1}{k_{ij}!} \Big| \int_{(\mathbb{S}^d)^4} \prod_{\substack{i,j=1\\ i<j}}^4 G_{\ell;d}(\langle x_i,x_j\rangle)^{k_{ij}} G_{\ell;d}(\langle x_1,x_2\rangle) G_{\ell;d}(\langle x_3,x_4\rangle)dx\Big| \tag{5.12}$$

We now prove that, thanks to Lemma 5.3, there exists $c > 0$ such that for every $\{k_{i,j}\}_{i,j=1}^n \in \mathcal{C}_{q_1-1,\dots,q_4-1}$,

$$\Big| \int_{(\mathbb{S}^d)^4} \prod_{\substack{i,j=1\\ i<j}}^4 G_{\ell;d}(\langle x_i,x_j\rangle)^{k_{ij}} G_{\ell;d}(\langle x_1,x_2\rangle) G_{\ell;d}(\langle x_3,x_4\rangle)dx\Big| \leq \frac{c_d}{\ell^{2d-2+\frac{d-1}{2}}}. \tag{5.13}$$

For a fixed $\kappa = \{k_{ij}\}_{i,j=1}^4 \in \mathcal{C}_{q_1-1,\dots,q_4-1}$, let N_κ be the number of the connected components of the extrapolated graph $\mathfrak{G}_\kappa$. We observe that $N_\kappa \in \{1,2\}$, recall in fact that by (5.1) for any i there exists at least an index $j \neq i$ such that $k_{ij} > 0$ (here, $q_i - 1 \geq 1$ for every i). So, we split our reasoning according to $N_\kappa = 1$ and $N_\kappa = 2$.

Case 1: $N_\kappa = 1$**.** By using the Cauchy-Schwarz inequality we have

$$\Big| \int_{(\mathbb{S}^d)^4} \prod_{\substack{i,j=1\\ i<j}}^4 G_{\ell;d}(\langle x_i,x_j\rangle)^{k_{ij}} G_{\ell;d}(\langle x_1,x_2\rangle) G_{\ell;d}(\langle x_3,x_4\rangle)dx\Big|$$

$$\leq \Big(\int_{(\mathbb{S}^d)^4} \prod_{\substack{i,j=1\\ i<j}}^4 G_{\ell;d}(\langle x_i,x_j\rangle)^{2k_{ij}}dx\Big)^{1/2} \Big(\int_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1,x_2\rangle)^2 G_{\ell;d}(\langle x_3,x_4\rangle)^2 dx\Big)^{1/2}$$

Estimating the first factor by means of (5.8) with $N_\kappa = 1$ and computing the second factor by means of (4.5), straightforward computations give (5.13)

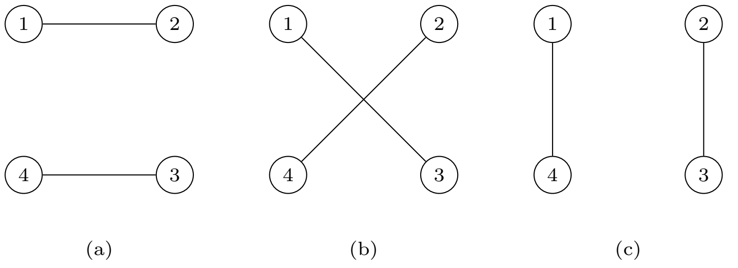

Fig. 1. All possible extrapolated graphs $\mathfrak{G}_\kappa$ from $\kappa = \{k_{ij}\}_{i,j=1}^4$ having exactly two connected components.

Case 2: $N_\kappa = 2$**.** Fig. 1 shows all possible extrapolated graphs having 2 connected components. We notice that the graph in (a) is extrapolated by an element $\kappa = \{k_{ij}\}_{i,j=1}^4$ belonging to $\mathcal{N}_{q_1-1,\dots,q_4-1}$. Such indexes have been already deleted, so we study the cases shown in (b) and in (c).

As for case (b), we have

$$\Big| \int\limits_{(\mathbb{S}^d)^4} \prod_{\substack{i,j=1\\ i<j}}^4 G_{\ell;d}(\langle x_i, x_j\rangle)^{k_{ij}} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_3, x_4\rangle) dx \Big|$$

$$= \Big| \int\limits_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_1, x_3\rangle)^{q_1-1} G_{\ell;d}(\langle x_2, x_4\rangle)^{q_2-1} G_{\ell;d}(\langle x_3, x_4\rangle) dx \Big|.$$

Assume that $q_1 = 2$ or $q_2 = 2$. W.l.g. we set $q_1 = 2$. Then, using (4.3), we have

$$\Big| \int\limits_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_1, x_3\rangle) G_{\ell;d}(\langle x_2, x_4\rangle)^{q_2-1} G_{\ell;d}(\langle x_3, x_4\rangle) dx_1 dx_2 dx_3 dx_4 \Big|$$

$$= \Big| \frac{\mu_d}{n_{\ell;d}} \int\limits_{(\mathbb{S}^d)^3} G_{\ell;d}(\langle x_2, x_3\rangle) G_{\ell;d}(\langle x_2, x_4\rangle)^{q_2-1} G_{\ell;d}(\langle x_3, x_4\rangle) dx_2 dx_3 dx_4 \Big|$$

$$= \Big| \frac{\mu_d^2}{n_{\ell;d}^2} \int\limits_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x_2, x_4\rangle)^{q_2} dx_2 dx_4 \Big|$$

$$\le \frac{\mu_d^2}{n_{\ell;d}^2} \int\limits_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x_2, x_4\rangle)^2 dx_2 dx_4 \le \frac{c_d}{\ell^{3d-3}},$$

the last inequality following from (4.5). If instead $q_1 \ge 3$ and $q_2 \ge 3$,

$$\Big| \int\limits_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_1, x_3\rangle)^{q_1-1} G_{\ell;d}(\langle x_2, x_4\rangle)^{q_2-1} G_{\ell;d}(\langle x_3, x_4\rangle) dx \Big|$$

$$\leq \int\limits_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_3\rangle)^2 G_{\ell;d}(\langle x_2, x_4\rangle)^2 |G_{\ell;d}(\langle x_3, x_4\rangle)| dx$$

By integrating first w.r.t. x_1, then w.r.t. x_2 and by using (4.5), we get

$$\Big| \int\limits_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_1, x_3\rangle)^{q_1-1} G_{\ell;d}(\langle x_2, x_4\rangle)^{q_2-1} G_{\ell;d}(\langle x_3, x_4\rangle) dx\Big|$$

$$\leq \frac{c_d}{\ell^{2d-2}} \int\limits_{(\mathbb{S}^d)^2} |G_{\ell;d}(\langle x_3, x_4\rangle)| dx_3 dx_4.$$

Now we use the Cauchy-Schwarz inequality and again apply (4.5). We finally obtain (5.13).

Case (c) in Fig. 1 can be treated analogously, so (5.13) finally holds.

Coming back to the study of $\mathrm{Var}(\sigma_\ell)$, we use (5.3), we insert the estimates (5.13) in (5.12) and we have

$$\mathrm{Var}(\sigma_\ell) \leq \frac{c_d\, \ell^{2d-2}}{\ell^{2d-2+\frac{d-1}{2}}} \sum_{q_1,q_2,q_3,q_4 \geq 2} \prod_{i=1}^4 \frac{|b_{q_i}|}{(q_i-1)!} \prod_{r=1}^4 (q_r-1)! \sum_{\{k_{i,j}\}_{i,j=1}^4 \in \mathcal{A}_{q_1-1,\dots,q_4-1}} \prod_{\substack{i,j=1\\ i<j}}^4 \frac{1}{k_{ij}!}$$

We now use (5.6) and Assumption 2.3: for $Z \sim \mathcal{N}(0,1)$,

$$\mathrm{Var}(\sigma_\ell) \leq \frac{c_d\, \ell^{2d-2}}{\ell^{2d-2+\frac{d-1}{2}}} \sum_{q_1,q_2,q_3,q_4 \geq 2} \prod_{i=1}^4 \frac{|b_{q_i}|}{(q_i-1)!} \mathbb{E}\Big[\prod_{r=1}^4 H_{q_r-1}(Z)\Big]$$

$$= \frac{c_d}{\ell^{\frac{d-1}{2}}} \mathbb{E}\Big[\Big(\sum_{q\geq 2} \frac{|b_q|}{(q-1)!} H_{q-1}(Z)\Big)^4\Big] = \frac{c_d}{\ell^{\frac{d-1}{2}}} \mathbb{E}[|D\phi(Z)|^4] = O(\ell^{-\frac{d-1}{2}}),$$

This concludes the proof for the case $d \geq 3$. If $d = 2$, the above estimate gives $\mathrm{Var}(\sigma_\ell) = O(\ell^{-\frac{1}{2}})$, which is not enough as it would give, in Theorem 2.5, $O_\varepsilon(\ell^{-\frac{1-\varepsilon}{4}})$ instead of $O_\varepsilon(\ell^{-\frac{1-\varepsilon}{2}})$. This, in turn, would imply that the convergence speed in Total Variation distance depends on the dimension. However, in the case $d = 2$ we can actually prove that $\mathrm{Var}(\sigma_\ell) = O(\ell^{-1})$, allowing us to reach the optimal bound $O_\varepsilon(\ell^{-\frac{1-\varepsilon}{2}})$ in Theorem 2.5. So, when $d = 2$ we need to improve the estimates of the integrals in (5.12). The proof strategy is different and needs a long analysis, in what follows we give the main steps, leaving the technical details to a supplementary file (see also [7, Appendix B]).

Let us come back to (5.12) for $d = 2$, in particular $G_{\ell;2} = P_\ell$ the ℓ-th Legendre polynomial. By recalling (5.3), by applying the estimate in Proposition 5.4 below, by using (5.6) and Assumption 2.3, we get

$$\mathrm{Var}(\sigma_\ell) \le C\ell^2 \times \frac{1}{\ell^3} \sum_{q_1,q_2,q_3,q_4 \ge 2} \prod_{i=1}^{4} \frac{|b_{q_i}|}{(q_i-1)!} \mathbb{E}\Big[\prod_{r=1}^{4} H_{q_r-1}(Z)\Big]$$
$$\le \frac{C}{\ell}\,\mathbb{E}\Big[\Big(\sum_{q\ge 2} \frac{|b_q|}{(q-1)!} H_{q-1}(Z)\Big)^4\Big] = \frac{C}{\ell}\,\mathbb{E}[|D\phi(Z)|^4] = O(\ell^{-1}),$$

hence the proof is concluded. □

Proposition 5.4. *There exists $C > 0$ such that for every $q_1, \ldots, q_4 \ge 2$ and $\kappa = \{k_{ij}\}_{i,j=1}^4 \in \mathcal{C}_{q_1-1,\ldots,q_4-1}$ (see (5.11)) one has*

$$\left| \int_{(\mathbb{S}^2)^4} P_\ell(\langle x_1, x_2\rangle)^{k_{12}+1} \prod_{i<j, i<3} P_\ell(\langle x_i, x_j\rangle)^{k_{ij}}\, P_\ell(\langle x_3, x_4\rangle)^{k_{34}+1} dx \right| \le \frac{C}{\ell^3}. \tag{5.14}$$

The proof of Proposition 5.4 is technical, so for the brevity sake it is fully collected in §SM2 of the supplementary file (see also [7, Appendix B.2]). In what follows we just give a key result which is of independent interest, i.e., Lemma 5.6, and its consequences in terms of cross moments of Gegenbauer polynomials (Lemma 5.7).

5.2.1. Concatenated sums of Gaunt integrals

Definition 5.5. *For $d \ge 2$, $q \in \mathbb{N}$ and $n_1, \ldots, n_q \in \{0, \ldots, n_{\ell;d}\}$, the generalized Gaunt integral on $\mathbb{S}^d$ is:*

$$\mathcal{G}_{\ell n_1,\ldots,\ell n_q} = \int_{\mathbb{S}^d} \prod_{i=1}^{q} Y_{\ell n_i}(x) dx.$$

The following result, which extends Lemma 1.5 in [22], gives a useful representation of convolutions of Gaunt integrals on $\mathbb{S}^d$. This is of interest in itself, in particular for the analysis of some random functionals on $\mathbb{S}^d$ which are beyond the scopes of this paper.

Lemma 5.6. *For $q \in \mathbb{N}$ and $n, n_1, \ldots, n_q \in \{0, \ldots, n_{\ell;d}\}$ one has*

$$\sum_{m_1,\ldots,m_r}^{n_{\ell;d}} \mathcal{G}_{\ell m_1,\ldots,\ell m_r,\ell n}\, \mathcal{G}_{\ell m_1,\ldots,\ell m_r,\ell n_1,\ldots,\ell n_q} = \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^{r-1} \hat{\gamma}_{\ell;r}\; \mathcal{G}_{\ell n,\ell n_1,\ldots,\ell n_q}$$

where

$$\hat{\gamma}_{\ell;r} = n_{\ell;d} \frac{\mu_{d-1}}{\mu_d} \int_{-1}^{1} G_{\ell;d}(t)^{r+1} \big(\sqrt{1-t^2}\big)^{d-2} dt. \tag{5.15}$$

Proof.. From (2.4) we have

$$\sum_{m_1,\dots,m_r} \mathcal{G}_{\ell m_1,\dots,\ell m_r,\ell n}\,\mathcal{G}_{\ell m_1,\dots,\ell m_r,\ell n_1,\dots,n_q} =$$
$$= \sum_{m_1,\dots,m_r} \int_{(\mathbb{S}^d)^2} \prod_{i=1}^{r} Y_{\ell m_i}(x)Y_{\ell m_i}(y)Y_{\ell n}(x)\prod_{j=1}^{q} Y_{\ell n_j}(y)dxdy$$
$$= \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^r \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x,y\rangle)^r\; Y_{\ell n}(x)\prod_{j=1}^{q} Y_{\ell n_j}(y)dxdy.$$

Since $\left(\left(\frac{\mu_{d-1}n_{\ell;d}}{\mu_d}\right)^{\frac12} G_{j;d}\right)_{j=0}^{r\ell}$ is an orthonormal system on $[-1,1]$ with the weight function $(1-t^2)^{d/2-1}$ (see [23]), we can write

$$G_{\ell;d}(t)^r = \sum_{j=0}^{r\ell} \gamma_{j,\ell;r} G_{j;d}(t) \tag{5.16}$$

where, for $j = 0, 1, \dots, r\ell$

$$\gamma_{j,\ell;r} = n_{j;d}\frac{\mu_{d-1}}{\mu_d}\int_{-1}^{1} G_{\ell;d}(t)^r G_{j;d}(t)\big(\sqrt{1-t^2}\big)^{d-2}dt. \tag{5.17}$$

Substituting (5.16), we have

$$\sum_{m_1,\dots,m_r} \mathcal{G}_{\ell m_1,\dots,\ell m_r,\ell n}\,\mathcal{G}_{\ell m_1,\dots,\ell m_r,\ell n_1,\dots,n_q}$$
$$= \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^r \int_{(\mathbb{S}^d)^2} \sum_{j=0}^{r\ell} \gamma_{j,\ell;r} G_{j;d}(\langle x,y\rangle)\; Y_{\ell n}(x)\prod_{i=1}^{q} Y_{\ell n_i}(y)dxdy$$
$$= \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^{r-1} \sum_{j=0}^{r\ell}\gamma_{j,\ell;r} \int_{(\mathbb{S}^d)^2} \sum_{h=0}^{n_{\ell;d}} Y_{jh}(x)Y_{jh}(y)\; Y_{\ell n}(x)\prod_{i=1}^{q} Y_{\ell n_i}(y)dxdy$$
$$= \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^{r-1} \sum_{j=0}^{r\ell}\gamma_{j,\ell;r} \sum_{h=0}^{n_{\ell;d}} \underbrace{\int_{\mathbb{S}^d} Y_{jh}(x)Y_{\ell n}(x)dx}_{=\mathbb{1}_{j=\ell}\mathbb{1}_{h=n}} \int_{\mathbb{S}^d} Y_{jh}(y)\prod_{i=1}^{q} Y_{\ell n_i}(y)dy$$
$$= \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^{r-1} \gamma_{\ell,\ell;r}\int_{\mathbb{S}^d} Y_{\ell n}(y)\prod_{i=1}^{q} Y_{\ell n_i}(y)dy$$
$$= \Big(\frac{n_{\ell;d}}{\mu_d}\Big)^{r-1} \gamma_{\ell,\ell;r}\,\mathcal{G}_{\ell n,\ell n_1,\dots,\ell n_q}.$$

Since $\hat{\gamma}_{\ell;r} = \gamma_{\ell,\ell;r}$, the statement follows. □

As a consequence of Lemma 5.6, we state the following properties for cross moments of Gegenbauer polynomials (which is of independent interest).

Lemma 5.7. *For $d \geq 2$, the following statements hold:*

1. *for $q_1, q_2 \geq 2$ there exists a positive constant $c_{d,1}$ such that*

$$\begin{aligned}&\int_{(\mathbb{S}^d)^3} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_1, x_4\rangle)^{q_1} G_{\ell;d}(\langle x_2, x_4\rangle)^{q_2} dx_1 dx_2 dx_4 \\ &\quad = c_{d,1} \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x, y\rangle)^{q_1+1} dx dy \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x, y\rangle)^{q_2+1} dx dy;\end{aligned} \tag{5.18}$$

2. *for $q_1, q_2 \geq 2$ and $q_3 \geq 1$ there exists a positive constant $c_{d,2}$ such that*

$$\begin{aligned}&\int_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_1, x_4\rangle)^{q_1} G_{\ell;d}(\langle x_2, x_3\rangle)^{q_2} G_{\ell;d}(\langle x_3, x_4\rangle)^{q_3} dx \\ &\quad = c_{d,2} \prod_{i=1}^3 \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x, y\rangle)^{q_i+1} dx dy;\end{aligned} \tag{5.19}$$

3. *for $q_1, q_2 \geq 2$ and $q_3 \geq 0$ there exists a positive constant $c_{d,3}$ such that*

$$\begin{aligned}&\int_{(\mathbb{S}^d)^4} G_{\ell;d}(\langle x_1, x_2\rangle) G_{\ell;d}(\langle x_1, x_4\rangle)^{q_1} G_{\ell;d}(\langle x_2, x_3\rangle)^{q_2} G_{\ell;d}(\langle x_2, x_4\rangle)^{q_3} G_{\ell;d}(\langle x_3, x_4\rangle) dx \\ &\quad = c_{d,3} \prod_{i=1}^3 \int_{(\mathbb{S}^d)^2} G_{\ell;d}(\langle x, y\rangle)^{q_i+1} dx dy.\end{aligned} \tag{5.20}$$

The proof of Lemma 5.7 can be found in §SM1 (see also [7, Appendix B.1]).

6. Uniform boundedness of Malliavin-Sobolev norms

This section is devoted to the proof of Proposition 3.3, that is, all moments of $|D^{(k)} \tilde{X}_\ell|_{\mathcal{H}^{\otimes k}}$ and of $|D^{(k)} L\tilde{X}_\ell|_{\mathcal{H}^{\otimes k}}$ are uniformly bounded in ℓ.

Proof of Proposition 3.3. Without loss of generality, we can assume that n is even. So, we fix $k \in \mathbb{N}$ and $n = 2p$, $p \in \mathbb{N}$. We first prove that $\sup_{\ell \text{ even}} \mathbb{E}[|D^{(k)} \tilde{X}_\ell|^{2p}_{\mathcal{H}^{\otimes k}}]$ is finite.

Recall the chaos expansion (5.2) and by using (4.12), it easily follows that

$$D^{(k)}_{y_1,\dots y_k} \tilde{X}_\ell = \frac{1}{v_{\ell;d}} \left(\frac{n_{\ell;d}}{\mu_d}\right)^{\frac{k}{2}} \sum_{q \geq 2 \vee k} \frac{b_q}{(q-k)!} \int_{\mathbb{S}^d} H_{q-k}(T_\ell(x)) \prod_{i=1}^k G_{\ell;d}(\langle x, y_i\rangle) dx.$$

Thus,

$$\mathbb{E}[|D^{(k)} \tilde{X}_\ell|^{2p}_{\mathcal{H}^{\otimes k}}] = \mathbb{E}\Big[\Big(\int_{(\mathbb{S}^d)^k} |D^{(k)}_{y_1,\dots,y_k} \tilde{X}_\ell|^2 dy_1 dy_2 \dots dy_k\Big)^p\Big]$$

$$= \frac{1}{v_{\ell;d}^{2p}} \mathbb{E}\Big[\Big(\sum_{q_1,q_2 \geq 2\vee k} \frac{b_{q_1} b_{q_2}}{(q_1-k)!(q_2-k)!} \times \int_{(\mathbb{S}^d)^2} H_{q_1-k}(T_\ell(y)) H_{q_2-k}(T_\ell(z)) G_{\ell;d}(\langle y,z\rangle)^k dydz\Big)^p\Big]$$

$$= \frac{1}{v_{\ell;d}^{2p}} \sum_{q_i \geq 2\vee k, i=1,\dots,2p} \prod_{i=1}^{2p} \frac{b_{q_i}}{(q_i-k)!} \prod_{r=1}^{2p} (q_r-k)! \sum_{\{k_{i,j}\}_{\substack{i,j=1\\i<j}}^{2p} \in A_{q_1-k,\dots,q_{2p}-k}} \prod_{\substack{i,j=1\\i<j}}^{2p} \frac{1}{k_{i,j}!} \times \int_{(\mathbb{S}^d)^{2p}} \prod_{\substack{i,j=1\\i<j}}^{2p} G_{\ell;d}(\langle x_i,x_j\rangle)^{k_{i,j}} \prod_{s=1}^{p} G_{\ell;d}(\langle x_s,x_{s+p}\rangle)^k dx,$$

in which we have used (5.5). We start to estimate the integrals in the above r.h.s. Using the Cauchy Schwarz inequality and (5.9), it follows that

$$\Big| \int_{(\mathbb{S}^d)^{2p}} \prod_{\substack{i,j=1\\i<j}}^{2p} G_{\ell;d}(\langle x_i,x_j\rangle)^{k_{i,j}} \prod_{s=1}^{p} G_{\ell;d}(\langle x_s,x_{s+p}\rangle)^k dx \Big|$$

$$\leq \Big(\int_{(\mathbb{S}^d)^{2p}} \prod_{\substack{i,j=1\\i<j}}^{2p} G_{\ell;d}(\langle x_i,x_j\rangle)^{2k_{i,j}} dx \int_{(\mathbb{S}^d)^{2p}} \prod_{s=1}^{p} G_{\ell;d}(\langle x_s,x_{s+p}\rangle)^{2k} dx\Big)^{\frac{1}{2}} \leq \frac{C_{d;p}}{\ell^{(d-1)p}}.$$

We insert the above estimate and, by using the asymptotics of $v_{\ell;d}$ in (5.3) and the representation (5.6), it follows that

$$\mathbb{E}[|D^{(k)}\tilde{X}_\ell|^{2p}_{\mathcal{H}^{\otimes k}}] \leq \frac{1}{v_{\ell;d}^{2p}} \frac{C_{d;p}}{\ell^{(d-1)p}} \times \sum_{q_i \geq 2\vee k, i=1,\dots,2p} \Big(\prod_{i=1}^{2p} \frac{|b_{q_i}|}{(q_i-k)!} \Big) \prod_{r=1}^{2p} (q_r-k)! \sum_{\{k_{i,j}\}_{\substack{i,j=1\\i<j}}^{2p} \in A_{q_1-k,\dots,q_{2p}-k}} \prod_{\substack{i,j=1\\i<j}}^{2p} \frac{1}{k_{i,j}!}$$

$$\leq \frac{\mathrm{Const}\big(\ell^{(d-1)p} + o(\frac{1}{\ell^{(d-1)p}})\big)}{\ell^{(d-1)p}} \times \sum_{q_i \geq 2\vee k, i=1,\dots,2p} \Big(\prod_{i=1}^{2p} \frac{|b_{q_i}|}{(q_i-k)!} \Big) \mathbb{E}[\prod_{r=1}^{2p} H_{q_r-k}(Z)]$$

$$= \mathrm{Const}(1+o(1)) \mathbb{E}\Big[\Big| \sum_{q \geq 2\vee k} \frac{|b_q|}{(q-k)!} H_{q-k}(Z)\Big|^{2p}\Big].$$

Hereafter Const denotes a positive constant, possibly changing from a line to another and possibly depending on d and p but independent of ℓ. Now, by (2.16) in Assumption 2.3, we obtain

$$\sup_\ell \mathbb{E}[|D^{(k)}\tilde{X}_\ell|^{2p}_{\mathcal{H}^{\otimes k}}] \leq \mathrm{Const}\, \mathbb{E}[|D^k \phi(Z)|^{2p}] < \infty.$$

Concerning the study of $\mathbb{E}[|D^{(k)}L\tilde{X}_\ell|^{2p}_{\mathcal{H}^{\otimes k}}]$, by using (4.8) we have

$$L\tilde{X}_\ell = -\frac{1}{v_{\ell;d}}\sum_{q\geq 2}\frac{b_q}{(q-1)!}\int_{\mathbb{S}^d} H_q(T_\ell(x))dx.$$

By comparing this expansion with (5.2), one deduces that one can repeat the same computations with b_q replaced by $-qb_q$. Hence,

$$\sup_\ell \mathbb{E}[|D^{(k)}L\tilde{X}_\ell|^{2p}_{\mathcal{H}^{\otimes k}}] \leq \mathrm{Const}\,\mathbb{E}\Big[\Big|\sum_{q\geq 2\vee k}\frac{q|b_q|}{(q-k)!}H_{q-k}(Z)\Big|^{2p}\Big] = \mathrm{Const}\,\mathbb{E}[|D^k L\phi(Z)|^{2p}]$$

and this is finite again because of (2.16) in Assumption 2.3. This concludes the proof. □

Data availability

No data was used for the research described in the article.

Appendix A

A.1. Proof of Lemma 5.2

This section is devoted to the proof of Lemma 5.2, which is based on the following diagram formula [13, Proposition 4.15]: for $n\geq 1$, for a centered Gaussian vector $(Z_1,\dots,Z_n)$ in $\mathbb{R}^n$ and for every $q_1,\dots,q_n\in\mathbb{N}$ one has

$$\mathbb{E}[\prod_{r=1}^n H_{q_r}(Z_r)] = \sum_{G\in\Gamma_{\overline{F}}(q_1,\dots,q_n)}\prod_{1\leq i<j\leq n}\mathbb{E}(Z_iZ_j)^{k_{ij}(G)}, \tag{A.1}$$

where $\Gamma_{\overline{F}}(q_1,\dots,q_n)$ is the set of no-flat diagram of order $(q_1,\dots,q_n)$ and $k_{ij}(G)$ is the number of edges from row i to row j of the diagram. Let us recall (see [13, §4.3.1], in particular the figure at page 97) that a diagram G of order $(q_1,\dots,q_n)$ is a set of points $\{(i,h): 1\leq i\leq n, 1\leq h\leq q_i\}$ called *vertices* and a partition of these points into pairs

$$\{((i,h),(j,k)): 1\leq i\leq j\leq n;\;\; 1\leq h\leq q_i,\;\; 1\leq k\leq q_j\}$$

called *edges*, such that $(i,h)\neq(j,k)$ (self loops are not allowed) and moreover, every vertex of the diagram is linked to one and only one vertex through an edge. One can graphically represent G by a set of n rows, where the i-th row contains q_i dots. The h-th dot (from left to right) of the i-th row represents the point (i,h). The edges of the diagram are represented as lines connecting the two corresponding dots. A diagram is *no-flat* if for all edges $((i,h),(j,k))$ we have $i\neq j$. It graphically means that we can connect only dots that are in two different rows.

Proof of Lemma 5.2. We start from the diagram formula (A.1). For a diagram in $\Gamma_{\overline{F}}(q_1, \dots, q_n)$, let R_i denote its i-th row, $i = 1, \dots, n$. Consider the first row R_1. In R_1 we have q_1 dots; we fix a partition of q_1 dots in $n-1$ groups of dots. We order the groups and denote them R_{1j}, $j = 2, \dots, n$: R_{1j} is the group of dots in R_1 that are linked with dots in the j-th row. We denote with k_{1j} the number of dots in R_{1j}, that coincides with the number of edges connecting row 1 with row j. We fix $k_{12} \in \{0, \dots, q_1\}$. There are $\binom{q_1}{k_{12}}$ choices for k_{12} dots in the first row. In general for $j = 3, \dots n$, we fix $k_{1j} = 0, \dots, (q_1 - \sum_{h=2}^{j-1} k_{1h})$ to have that

$$\sum_{j=2}^{n} k_{1j} = q_1.$$

For $j = 3, \dots, n$ there are $\binom{q_1 - \sum_{h=2}^{j-1} k_{1h}}{k_{1j}}$ choices for k_{1j}. Then, the number of choices of $\{k_{1,j}\}_{j=2}^n$ according to the above condition is

$$\prod_{j=1}^{n} \binom{q_1 - \sum_{r=1}^{j-1} k_{1r}}{k_{1j}} = \frac{q_1!}{\prod_{j=1}^{n} k_{1j}!}.$$

We recall that $k_{ii} = 0$ for $i = 1, \dots, n$ because we are considering no-flat diagrams. In practice we have computed the number of partitions of q_1 dots in $n-1$ groups. We can do the same for the other rows. And so we have that the number of partition of q_i that is

$$\prod_{j=1}^{n} \binom{q_i - \sum_{r=1}^{j-1} k_{ir}}{k_{ij}} = \frac{q_i!}{\prod_{j=1}^{n} k_{ij}!}.$$

Notice that $k_{ij} = k_{ji}$. Now we are able to compute the number of diagrams for fixed $\{k_{ij}\}_{i,j=1}^n$. We recall that k_{ij} represent the number of dots of the i-th row and of the j-th row that are linked. There are $k_{ij}!$ way to match the dots. Then the number of no-flat diagrams for a fixed $\{k_{ij}\}_{i,j=1}^n$ is

$$\prod_{i=1}^{n} \frac{q_i!}{\prod_{j=1}^{n} k_{ij}!} \prod_{\substack{r,s=1 \\ r<s}}^{n} k_{rs}! = \prod_{r=1}^{n} q_r! \prod_{\substack{i,j=1 \\ i<j}}^{n} \frac{1}{k_{ij}}.$$

In order to conclude, it remains to determine the set of all admissible $\{k_{ij}\}_{i,j=1}^n$. Recalling that, for a fixed no-flat diagram, k_{ij} is the number of edges connecting row i with row j, then of course $k_{ij} = k_{ji}$. Moreover, $k_{ii} = 0$ because the diagram is no-flat and $\sum_{j=1}^n k_{ij} = q_i$ for every i, as every vertex belongs to a unique edge. This means that $\{k_{ij}\}_{i,j=1}^n \in \mathcal{A}_{q_1,\dots,q_n}$ (see Definition 5.1). The statement now follows. □

A.2. Proof of Lemma 5.3

Before presenting the proof of Lemma 5.3, we start by recalling some elementary concepts of graph theory [25].

A *graph* is a set of point called *nodes* linked together by lines called *edges*. Formally, a graph is a pair $\mathfrak{G} = (V, E)$ of sets, where V is the set of nodes and E is the set of edges. We can identify E with a subset of $V \times V$. Precisely, if $V = \{x_1, \dots, x_n\}$ and there exists an edge between x_i and x_j, then the pair $(x_i, x_j) \in E$. A *subgraph* of $\mathfrak{G} = (V, E)$ is a graph $\mathfrak{G}' = (V', E')$ where $V' \subset V$ and E' is the set of all the edges of E that link only nodes in V'. We say that a node x has degree m if there are m edges that are incident to x, the case $m = 0$ meaning that the node is isolated.

A *path* between two nodes x, y of $\mathfrak{G}$ is a sequence of edges connecting x with y and joining a sequence of distinct nodes, so, in particular, all edges of the path are distinct. We say that two nodes x, y of a graph $\mathfrak{G}$ are *connected* if $\mathfrak{G}$ contains a path between x and y. A graph is said to be *connected* if every pair of nodes in the graph is connected. A *connected component* of a graph $\mathfrak{G}$ is connected subgraph of the graph that is maximal. We can consider a graph as the union of its connected components.

In our treatment we are interested in a particular class of connected graphs: the trees. A *tree* is a connected graph where each pair of nodes is connected by exactly one path. We first observe that in a tree there exists a non-empty subset of nodes with degree 1. In fact, equivalently, a tree is a connected graph in which every subgraph (and in particular the graph itself) contains at least one node with degree 1. Hence when we delete some of 1 degree nodes, the subgraph that we obtain is also a tree, that has again a subset of new 1 degree nodes. If we progressively delete the 1 degree nodes, we finally obtain a empty graph.

The last and most important property (for our treatment) of connected graphs is the following: a connected graph $\mathfrak{G}$ always contains a *spanning tree*, i.e. a subgraph of $\mathfrak{G}$ that is a tree and contains all nodes of $\mathfrak{G}$. Now we prove Lemma 5.3.

Proof of Lemma 5.3. Let $\kappa = \{k_{ij}\}_{i,j=1}^n \in \mathcal{A}_{q_1-1,\dots,q_n-1}$ (see (5.5)). We extrapolate from κ the graph $\mathfrak{G} = (V, E)$ with $V = \{x_1, \dots, x_n\}$ and $(x_i, x_j) \in E$ iff $k_{ij} \neq 0$. We recall that for every $i = 1, \dots, n$, $k_{ii} = 0$, then there are no self loops in $\mathfrak{G}$.

Let N be the number of the connected components of $\mathfrak{G}$ and we denote with $\mathfrak{G}_h$, $h = 1, \dots, N$ these components. We denote with m_h the number of nodes in $\mathfrak{G}_h$. Then $\sum_{h=1}^N m_h = n$. We observe that if x_i is a node of $\mathfrak{G}_{h_1}$, x_j is a node of $\mathfrak{G}_{h_2}$ and $h_1 \neq h_2$ then $k_{ij} = 0$. This justifies the following equality:

$$\begin{aligned}&\int_{(\mathbb{S}^d)^n} \prod_{\substack{i,j=1\\ i<j}}^n G_{\ell;d}(\langle x_i, x_j \rangle)^{2k_{ij}}\, dx_1 \dots dx_n \\ &= \prod_{h=1}^N \int_{(\mathbb{S}^d)^{m_h}} \prod_{x_{i_r}, x_{i_s} \in \mathfrak{G}_h} G_{\ell;d}(\langle x_{i_r}, x_{i_s} \rangle)^{2k_{i_r i_s}}\, dx_{i_1} \dots dx_{i_{m_h}}.\end{aligned} \tag{A.2}$$

Now we observe that if $\mathfrak{G}_h$ is a tree, the 1 degree nodes of $\mathfrak{G}_h$ are the variables x_{i_r} for which there exists one and only one i_s such that $k_{i_r i_s} \neq 0$. Hence there is one and only one polynomial $G_{\ell;d}$ in the variable x_{i_r} in the integral (A.2). We identify the action of deleting 1 degree nodes with that of integrating the polynomial $G_{\ell;d}(\langle x_{i_r}, x_{i_s}\rangle)$ in the variable x_{i_r}. Our connected components are not always trees, but we know that there always exists the spanning tree. So for all $\mathfrak{G}_h$, we consider the spanning tree $\tilde{\mathfrak{G}}_h$, and delete the edges of $\mathfrak{G}_h$ that are not in $\tilde{\mathfrak{G}}_h$. This deleting operation corresponds, when studying the integral in the r.h.s. of (A.2), with the estimate $|G_{\ell;d}(\langle x_{i_r}, x_{i_s}\rangle)| \leq 1$ for each pair (x_{i_r}, x_{i_s}) giving the deleted edge. Since in a tree with m_h nodes there are $m_h - 1$ edges, the resulting estimate consists in integrating $m_h - 1$ polynomials.

It follows that

$$\begin{aligned}\int_{(\mathbb{S}^d)^n} \prod_{\substack{i,j=1\\ i<j}}^{n} G_{\ell;d}(\langle x_i, x_j\rangle)^{2k_{ij}} dx &= \prod_{h=1}^{N} \int_{(\mathbb{S}^d)^{m_h}} \prod_{x_{i_r}, x_{i_s}\in\mathfrak{G}_h} G_{\ell;d}(\langle x_{i_r}, x_{i_s}\rangle)^{2k_{i_r i_s}} dx_{i_1}\dots dx_{i_{m_h}}\\ &\leq \prod_{h=1}^{N} \int_{(\mathbb{S}^d)^{m_h}} \prod_{x_{i_r}, x_{i_s}\in\tilde{\mathfrak{G}}_h} G_{\ell;d}(\langle x_{i_r}, x_{i_s}\rangle)^{2k_{i_r i_s}} dx_{i_1}\dots dx_{i_{m_h}}\\ &\leq \prod_{h=1}^{N} \int_{(\mathbb{S}^d)^{m_h}} \prod_{x_{i_r}, x_{i_s}\in\tilde{\mathfrak{G}}_h} G_{\ell;d}(\langle x_{i_r}, x_{i_s}\rangle)^{2} dx_{i_1}\dots dx_{i_{m_h}}\\ &\leq \prod_{h=1}^{N} \frac{(8\mu_d\mu_{d-1}c_{2;d})^{m_h}\mu_d^N}{\ell^{(d-1)(m_h-1)}} = \frac{(8\mu_d\mu_{d-1}c_{2;d})^{n-N}\mu_d^N}{\ell^{(d-1)(\sum_{h=1}^N m_h - N)}}\\ &= \frac{(8\mu_d\mu_{d-1}c_{2;d})^{n-N}\mu_d^N}{\ell^{(d-1)(n-N)}}.\end{aligned}$$

We end by observing that the maximum number N of connected components in a graph that contains $n = 2p$ nodes is p, when there aren't 0 degree nodes. Moreover there are exactly p connected components when all subgraph contains exactly 2 nodes. Then, being $8\mu_d\mu_{d-1}c_{2;d} > 1$, we have

$$\int_{(\mathbb{S}^d)^{2p}} \prod_{\substack{i,j=1\\ i<j}}^{2p} G_{\ell;d}(\langle x_i, x_j\rangle)^{2k_{ij}} dx \leq \frac{C_{d;p}}{\ell^{(d-1)p}} \tag{A.3}$$

where $C_{d;p} = (2(d-1)!\mu_d^2)^{2p}\mu_d^p$, thus concluding the proof. □

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jfa.2023.110239.

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