WEAKLY KÄHLER HYPERBOLICITY IS BIRATIONAL

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Abstract. We show that a compact Kähler manifold bimeromorphic to a weakly Kähler hyperbolic manifold is weakly Kähler hyperbolic, providing an answer to a problem raised by J. Kollár [Kol95, Open Problem 18.7].

1. Introduction

In the chapter devoted to open problems of his book "Shafarevic maps and automorphic forms", J. Kollár speaking about Gromov's Kähler hyperbolicity says

"From the birational point of view, Kähler hyperbolicity is not natural since it depends on the birational model chosen. It would be desirable to have a birational variant developed. The natural choice seems to be to requires Gromov's condition not for Kähler form but for a degenerate Kähler form."

Let X be a compact Kähler manifold. In [BDET24] the authors introduce a notion called *weak Kähler hyperbolicity*, mainly in order to study desingularizations of subvarieties of Kähler hyperbolic manifold in the sense of

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Gromov [Gro91]. It turns out that many key features of Kähler hyperbolic manifolds generalize (in an appropriate way) to weakly Kähler hyperbolic manifolds and, moreover, weak Kähler hyperbolicity seems to be the right notion in order to have birational invariance.

In fact, it is proved in [BDET24] that if $\mu\colon X \to Y$ is a modification of compact Kähler manifolds or, more generally, a surjective holomorphic map between compact Kähler manifolds of the same dimension, then the weak Kähler hyperbolicity of Y implies that of X. In other words, weak Kähler hyperbolicity pulls-back along modifications.

In this note we prove the analogous for pushing-forwards.

Main Theorem. *Let $\mu\colon Z \to X$ be a modification of compact Kähler manifolds. If Z is weakly Kähler hyperbolic, then X is weakly Kähler hyperbolic, too.*

Following [BDET24], given a compact Kähler manifold X, define the subset $\mathcal{W}_X \subset H^{1,1}(X,\mathbb{R})$ to be the set of real cohomology $(1,1)$-classes which are big and nef, and moreover admits a smooth representative which is $\tilde{d}$-bounded. Here, $\tilde{d}$-bounded means that such a representative becomes d-exact once pulled-back to the universal covering of X and moreover admits a primitive which is bounded with respect to any metric coming from X. A compact Kähler manifold X is said to be weakly Kähler hyperbolic if $\mathcal{W}_X \neq \emptyset$.

To be more precise, what we show in our Main Theorem is that if $\alpha \in \mathcal{W}_Z$, then the push-forward $\mu_*\alpha$ belongs to $\mathcal{W}_X$.

Being able to push-forward through a modification the property of being weakly Kähler hyperbolic enables us to finally treat the behaviour of weakly Kähler hyperbolicity with respect to meromorphic mappings. The following corollary, for instance, thus provides an answer to [Kol95, Open Problem 18.7].

Corollary 1.1. *If X and Y are bimeromorphic compact Kähler manifolds, then X is weakly Kähler hyperbolic if and only if Y is weakly Kähler hyperbolic.*

Finally, this other corollary answers instead to [BDET24, Question 2.30].

Corollary 1.2. *Let $f\colon X \dashrightarrow Y$ be a generically finite meromorphic mapping between compact Kähler manifolds. If Y is weakly Kähler hyperbolic, then X is weakly Kähler hyperbolic.*

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2. Tools

For the reader convenience, we collect in this section the main tools for the proof of our Main Theorem.

2.1. **Push-forward and pull-back of big or nef classes.** First, we recall how the bigness of a general real $(1,1)$-class behaves under generically finite holomorphic maps.

Proposition 2.1 ([Bou02, Proposition 4.12])**.** *Let $f\colon Y \to X$ be a surjective holomorphic map between compact Kähler manifolds of the same dimension. Let α and β be real $(1,1)$-cohomology classes on X and Y respectively. Then,*

(i) *the class α is big if and only if $f^*\alpha$ is big,*
(ii) *the class $f_*\beta$ is big if β is.*

Next, we (to be removed) The following proposition is stated for compact Kähler manifolds, whereas the original statement is in Bott–Chern cohomology for general compact complex manifolds.

Proposition 2.2 ([Pău98, Théorème 1])**.** *Let $f\colon Y \to X$ be a surjective holomorphic map between compact Kähler manifolds. Let α be a real $(1,1)$-cohomology class on X. Then, the class $f^*\alpha$ is nef if and only if α is.*

Remark 2.3. Observe that, in the equidimensional case, an analogous statement for push-forwards of nef classes is not true in general as soon as $\dim X = \dim Y \geq 3$ (it is indeed for surfaces). See Boucksom's PhD thesis. Complete describing briefly the example

2.2. **Brunnbauer, Kotschick, and Schönlinner result on hyperbolic classes.** Here we want state [BKS24, Theorem 2.4], which will be the crucial ingredient for the proof of birationality of weakly Kähler hyperbolicity. So, let us introduce some terminology following [BKS24].

First, for X a topological space, we define the aspherical subspace $V^k_{\mathrm{asph}}(X)$ of the real singular cohomology $H^k(X,\mathbb{R})$ to be the set of k-cohomology classes whose pull-back to any sphere is zero.

Next, for X a finite simplicial complex, define the hyperbolic subspace $V^k_{\mathrm{hyp}}(X)$ of the real singular cohomology $H^k(X,\mathbb{R})$ to be the set of k-cohomology classes for which there exists a representing closed k-form ω whose pull-back to the universal cover has a d-primitive which is bounded with respect to some lifted Riemannian metric. Here, we are using that there is a natural isomorphism $H^k_{\mathrm{dR}}(X,\mathbb{R}) \simeq H^k(X,\mathbb{R})$ between the de Rham cohomology and the singular cohomology.

Note that $V^k_{\mathrm{hyp}}(X) \subset V^k_{\mathrm{asph}}(X)$ as soon as $k \geq 2$, since every continuous map from a k-dimensional sphere $f\colon S^k \to X$ factorizes through the universal cover of X.

Now, take a finite simplicial complex X with fundamental group $G := \pi_1(X)$ and consider (a model of) the classifying space $EG \to BG$. Given a universal covering $\tilde{X} \to X$, there is unique (up to homotopy) classifying map of this universal covering

$$c_{\tilde{X},X}\colon X \to BG$$

such that $\tilde{X}$ is isomorphic to the pull-back $c^*_{\tilde{X},X}EG$ as G-principal bundles.

Finally, following [Gro91, Subsection 0.2.C], we want to define the hyperbolic subspace for BG which does not necessarily have any longer the homotopy type of a finite simplicial complex. So, the subspace $V^k_{\mathrm{hyp}}(BG)$ of the real singular cohomology $H^k(BG,\mathbb{R})$ will be the set of k-cohomology classes whose pull-back to any finite simplicial complex is hyperbolic.

We are now ready to state the following theorem, which is the key for the birationality of weak Kähler hyperbolicity.

Theorem 2.4 ([BKS24, Theorem 2.4], see also [Kęd09, Theorem 5.1])**.** *Let X,Y be two finite simplicial complex (e.g. compact complex manifolds), $c_{\tilde X,X}\colon X\to B\pi$ the classifying map of the universal covering $\tilde X\to X$, and $f\colon Y\to B\pi$ an arbitrary continuous map.*

If $w\in H^2(B\pi,\mathbb{R})$ is a cohomology class such that $c^*_{\tilde X,X}w\in V^2_{\mathrm{hyp}}(X)$, then $f^*w\in V^2_{\mathrm{hyp}}(Y)$.

Here is a fundamental consequence for us, which is mentioned in [BKS24], and for which we reproduce a proof here.

Corollary 2.5. *For the hyperbolic subspace in degree 2 of a finite simplicial complex X, we have*

$$c^*_{\tilde X,X}\big(V^2_{\mathrm{hyp}}(BG)\big)=V^2_{\mathrm{hyp}}(X).$$

In particular, we see that $V^2_{\mathrm{hyp}}(X)$ depends only on the fundamental group of X, since different (that is, relative to different G-principal bundle structures) classifying maps of the universal covering yields the same image.

During the proof we shall need the following fact, surely well-known to experts.

Lemma 2.6. *The pull-back $c^*_{\tilde X,X}\colon H^2(BG,\mathbb{R})\hookrightarrow H^2(X,\mathbb{R})$ is injective and, more importantly,*

$$c^*_{\tilde X,X}\big(H^2(BG,\mathbb{R})\big)=V^2_{\mathrm{asph}}(X).$$

Maybe I gave really too much details here... But for me singular homology/cohomology is always a huge mystery so I preferred to write down everything. Feel free to shorten it even drastically!

Proof. First, every class in $H^k(BG,\mathbb{R})$, $k\geq 2$, is aspherical. This is because, for Z a (nice) topological space, a class in $H^k(Z,\mathbb{R})$ is aspherical if and only if its kernel as a linear functional on $H_k(Z,\mathbb{R})$ contains the image of the Hurewicz homomorphism $\pi_k(Z)\to H_k(Z,\mathbb{R})$, and by definition we have $\pi_k(BG)=0$ as soon as $k\geq 2$. Moreover, the image of an aspherical class belonging to $V^2_{\mathrm{asph}}(BG)=H^2(BG,\mathbb{R})$ via the pull-back is obviously aspherical.

Now, we can construct a model of BG by attaching cells of dimension 3 or higher to X to make the universal cover contractible without affecting $\pi_1(X)$. So, we can suppose that $X\subset BG$, the classifying map $c_{\tilde X,X}$ is the inclusion, and moreover X and BG share the same 2-skeleton. In particular, the relative homology $H_2(BG,X,\mathbb{R})$ vanishes and by duality so does $H^2(BG,X,\mathbb{R})$. We now form the long exact sequence of the pair $X\subset BG$

in cohomology, to get

$$0 \to H^2(BG,\mathbb{R}) \to H^2(X,\mathbb{R}) \xrightarrow{\partial} H^3(BG,X,\mathbb{R}),$$

where ∂ is the connecting homomorphism, which gives injectivity at once. It only remains to prove that if $w \in V^2_{\mathrm{asph}}(X)$, then $\partial(w) = 0$. This is seen directly, since $\partial(w) \in H_3(BG,X,\mathbb{R})^*$ acts on a relative 3-cycle $\gamma + C_3(X)$ by $\partial(w)\big(\gamma + C_3(X)\big) = w(\partial_3\gamma)$, where ∂_3 is the boundary operator and $\partial_3\gamma \in Z_2(X)$ is a 2-cycle in X which, if non trivial, comes from a 3-cell attached to X so that $\partial_3\gamma$ is homeomorphic to S^2. Therefore, $w(\partial_3\gamma) = 0$ since w is aspherical. $\square$

Proof of Corollary 2.5. We have by definition $c^*_{\tilde{X},X}\big(V^2_{\mathrm{hyp}}(BG)\big) \subset V^2_{\mathrm{hyp}}(X)$. For the other inclusion, if $\alpha \in V^2_{\mathrm{hyp}}(X)$, then $\alpha \in V^2_{\mathrm{asph}}(X)$, and thus by Lemma 2.6 there exists $w \in H^2(BG,\mathbb{R})$ such that $\alpha = c^*_{\tilde{X},X}w$, so that the pull-back $c^*_{\tilde{X},X}w$ is hyperbolic. But then, by Theorem 2.4, the pull-back of w to any finite simplicial complex is hyperbolic, *i.e.* $w \in V^2_{\mathrm{hyp}}(BG)$. $\square$

2.3. **Links between bounded cohomology and hyperbolic classes.** Before stating the next lemma, let us recall that the cohomology of a group G can be computed by means of the usual (inhomogeneous) cochains complex

$$C^k(G,\mathbb{R}) := \Big\{c : G^k \to \mathbb{R}\Big\}$$

together with the differential

$$\begin{aligned} &dc(g_1,\dots,g_{k+1}) := c(g_2,\dots,g_{k+1}) + \\ &\quad\sum_{i=1}^{k}(-1)^i c(g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_{k+1}) + (-1)^{k+1}c(g_1,\dots,g_k). \end{aligned}$$

The subspace of bounded cochains

$$C^k_b(G,\mathbb{R}) := \Big\{c : G^k \to \mathbb{R} \mid c \text{ is bounded}\Big\}$$

is stable under d and gives rise to the complex of bounded cochains; the corresponding cohomology groups

$$H^k_b(G,\mathbb{R}) := H^k\left(C^\bullet_b(G,\mathbb{R}), d\right)$$

are the so-called groups of bounded cohomology (see [Fri17] for a more in depth discussion). Let us finally observe that the natural inclusion of complexes

$$C^\bullet_b(G,\mathbb{R}) \hookrightarrow C^\bullet(G,\mathbb{R})$$

induces natural maps in cohomology

$$H^k_b(G,\mathbb{R}) \longrightarrow H^k(G,\mathbb{R}).$$

As above, we focus on the degree 2 case and state the following (well-known?) result.

Lemma 2.7. *We have an inclusion for any finitely presented group G:*

$$\operatorname{im}\left(H_b^2(G,\mathbb{R}) \to H^2(G,\mathbb{R})\right) \subset H^2_{\mathrm{hyp}}(G,\mathbb{R}).$$

In particular, if G is Gromov-hyperbolic, we have:

$$H^2(G,\mathbb{R}) = H^2_{\mathrm{hyp}}(G,\mathbb{R}).$$

Proof. According to Corollary 2.5, we can pick any compact differentiable manifold X sch that $\pi_1(X) \simeq G$ to check that a bounded 2-class gives rise to a hyperbolic one. Let $c \in C_b^2(G,\mathbb{R})$ be a bounded cochain that is a cocycle; it satisfies:

$$\forall\,(g,h,k) \in G^3,\ c(g,h) = c(h,k) - c(gh,k) + c(g,hk).$$

We can rewrite it in the following form:

$$\forall\,(g,h,\gamma) \in G^3,\ c(\gamma^{-1}g, g^{-1}h) = c(g,g^{-1}h) - c(\gamma,\gamma^{-1}h) + c(\gamma,\gamma^{-1}g). \tag{1}$$

We now explain how to construct a 2-form α defined on the universal cover $\tilde{X}$ which has a bounded primitive and is invariant under the action of the group $G = \pi_1(X)$. These two points together ensure that α defines a hyperbolic class in $H^2(X,\mathbb{R})$.

To do so, let us fix a finite open cover $(V_i)_{i\in I}$ with V_i simply connected. It means that p is trivial over V_i where $p : \tilde{X} \to X$ is the universal covering map. We can thus choose one connected component U_i of $p^{-1}(V_i)$ and we have thus

$$p^{-1}(V_i) = \bigcup_{g\in G} g(U_i) = \bigcup_{g\in G} U_{i,g}.$$

Finally, let us denote by $(\chi_i)_{i\in I}$ a partition of unity subordinate to the covering $(V_i)_{i\in I}$; it induces a partition of $\tilde{X}$ to be denoted by $(\chi_{i,g})_{(i,g)\in I\times G}$: the function $\chi_{i,g}$ is supported in $U_{i,g}$ and is nothing but $\chi_i \circ p$. This family enjoys the obvious equivariance property:

$$\forall\,(\gamma,g) \in G^2, \forall\, i \in I,\ \gamma^*\chi_{i,g} = \chi_{i,\gamma g}. \tag{2}$$

With this at hand, let us consider

$$\begin{aligned}\alpha &:= \sum_{\substack{(i,j)\in I^2\\(g,h)\in G^2}} c(g,g^{-1}h)\, d\chi_{i,g} \wedge d\chi_{j,h}\\ &= d\left(\sum_{\substack{(i,j)\in I^2\\(g,h)\in G^2}} c(g,g^{-1}h)\,\chi_{i,g} \wedge d\chi_{j,h}\right).\end{aligned}$$

The last equality implies easily that α has a bounded primitive since c is a bounded cochain. Now, let us check that α is invariant under the action of

$\gamma \in G$:

$$\begin{aligned}
\gamma^*\alpha &= \sum_{\substack{(i,j)\in I^2\\(g,h)\in G^2}} c(g,g^{-1}h)\gamma^*\left(d\chi_{i,g}\wedge d\chi_{j,h}\right)\\
&= \sum_{\substack{(i,j)\in I^2\\(g,h)\in G^2}} c(g,g^{-1}h)d\chi_{i,\gamma g}\wedge d\chi_{j,\gamma h} \qquad (3)\\
&= \sum_{\substack{(i,j)\in I^2\\(g,h)\in G^2}} c(\gamma^{-1}g,g^{-1}h)d\chi_{i,g}\wedge d\chi_{j,h} \qquad (4)\\
&= \sum_{\substack{(i,j)\in I^2\\(g,h)\in G^2}} \big(c(g,g^{-1}h)-c(\gamma,\gamma^{-1}h)+c(\gamma,\gamma^{-1}g)\big)\, d\chi_{i,g}\wedge d\chi_{j,h}=\alpha. \qquad (5)
\end{aligned}$$

Indeed, the equality (3) is just the equivariance property (2), (4) is a change of variables and the equality (5) is nothing but the cocycle property (1). To see that the remaining terms of the sum vanishes, it suffices to observe that we can rewrite the corresponding sum as

$$\begin{aligned}
&\sum_{\substack{(i,j)\in I^2\\(g,h)\in G^2}} c(\gamma,\gamma^{-1}h)d\chi_{i,g}\wedge d\chi_{j,h}\\
&\qquad\qquad = -\sum_{(j,h)\in I\times G} c(\gamma,\gamma^{-1}h)d\chi_{j,h}\wedge d\left(\sum_{(i,g)\in I\times G}\chi_{i,g}\right)=0
\end{aligned}$$

since the sum between the parenthesis is constant (equals 1).

The last assertion concerning Gromov-hyperbolic group is a consequence of [Min01, Theorem 11]: for such a group the natural maps

$$H^k_b(G,\mathbb{R}) \longrightarrow H^k(G,\mathbb{R})$$

are surjective for any $k \geq 2$. □

3. Proof of Main Theorem

Let $\mu\colon Z \to X$ be a modification of compact Kähler manifolds, and take $\alpha \in \mathcal{W}_Z$. We want to show that $\mu_*\alpha$ is in $\mathcal{W}_X$.

First of all, since α is a big class and μ is surjective between equidimensional manifolds, Proposition 2.1 gives that $\mu_*\alpha$ is a big class.

To show the nefness of $\mu_*\alpha$ is subtler, since in general it is false as we observed in Remark 2.3. This will be achieved together with $\tilde{d}$-boundedness of $\mu_*\alpha$ thanks to the topological description of $\tilde{d}$-bounded classes by Brunnbauer, Kotschick, and Schönlinner, as follows.

Let $G = \pi_1(X)$, and fix a universal covering $\tilde{X}$ of X. Consider the diagram

$$\begin{tikzcd} & & BG \\ Z \arrow[urr, "c_{\tilde{X},X} \circ \mu"] \arrow[dr, "\mu"'] & & \\ & X. \arrow[uur, "c_{\tilde{X},X}"'] & \end{tikzcd}$$

The map μ is a modification between smooth manifold, and thus $\mu_* \colon \pi_1(Z) \to \pi_1(X) = G$ is an isomorphism. Therefore, pulling-back via μ the universal covering $\tilde{X}$ of X gives a universal covering of $\tilde{Z} \to Z$. By construction, $\tilde{Z}$ is the pull-back of EG via $c_{\tilde{X},X} \circ \mu$, so that

Do we need a reference here?

$$c_{\tilde{X},X} \circ \mu = c_{\tilde{Z},Z}$$

is the classifying map of the universal covering $\tilde{Z}$ as a G-principal bundle on Z.

Since $\alpha \in V^2_{\mathrm{hyp}}(Z)$, by Corollary 2.5, there exists a (unique) class $w \in V^2_{\mathrm{hyp}}(BG)$ such that $c^*_{\tilde{Z},Z} w = \alpha$. Call $\beta := c^*_{\tilde{X},X} w \in V^2_{\mathrm{hyp}}(X)$. Then, we have $\alpha = \mu^*\beta$ by construction and, since μ is a modification, we also have

$$\mu_*\alpha = \mu_*\mu^*\beta = \beta,$$

so that β is a real $(1,1)$-class which is, by definition of $V^2_{\mathrm{hyp}}(BG)$, $\tilde{d}$-bounded. What we have gained is that $\alpha = \mu^*\beta$ is now a pull-back by the surjective holomorphic map μ. Then, we can apply Proposition 2.2 to obtain that β is nef, since α is nef, and Proposition 2.1 to obtain that β is big, since α is big and X and Z have the same dimension.

Summing up, $\beta \in \mathcal{W}_X$, as desired.

4. Proofs of Corollaries

In this section we give the proofs of the corollaries of our Main Theorem.

4.1. **Proof of Corollary 1.1.** Let $f \colon X \dashrightarrow Y$ be a bimeromorphic mapping. Then, by resolving the singularities of the closure of the graph of f in $X \times Y$, we get a compact Kähler manifold Z together with modifications $p \colon Z \to X$ and $q \colon Z \to Y$ as follows

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"'] & & Y. \end{tikzcd}$$

Now, take $\alpha \in \mathcal{W}_X$. We want to show that $\beta := q_* p^* \alpha$ gives a class in $\mathcal{W}_Y$. By [BDET24, Proposition 2.29] $\tilde{\alpha} := p^*\alpha$ is a weakly Kähler hyperbolic class in $\mathcal{W}_Z$, therefore $q_*\tilde{\alpha}$ is a weakly Kähler hyperbolic class in $\mathcal{W}_Y$, by our Main Theorem.

4.2. **Proof of Corollary 1.2.** Let $f\colon X \dashrightarrow Y$ be a generically finite meromorphic mapping. Then, by resolving the singularities of the closure of the graph of f in $X \times Y$, we get a compact Kähler manifold Z together with a modification $p\colon Z \to X$ and this time a generically finite holomorphic map $q\colon Z \to Y$ as follows

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"'] & & Y. \end{tikzcd}$$

Now, take $\alpha \in \mathcal{W}_Y$. Here, we want to show that $\beta := p_* q^* \alpha$ gives a class in $\mathcal{W}_X$. By [BDET24, Proposition 2.29] $\tilde{\alpha} := q^*\alpha$ is a weakly Kähler hyperbolic class in $\mathcal{W}_Z$, therefore $p_*\tilde{\alpha}$ is a weakly Kähler hyperbolic class in $\mathcal{W}_Y$, by our Main Theorem.

5. Toroidal compactifications of ball quotients

In this section, we exhibit a new family of natural examples of weakly Kähler hyperbolic manifolds. To do so, let $G < \mathrm{PU}(n, 1)$ be a non-uniform lattice (assumed to be torsion-free to simplify the exposition). The manifold $X^\circ := G\backslash\mathbb{B}^n$ is thus a quasi-projective variety that can be compactified in various ways:

(1) The Baily–Borel compactification $X^\circ \subset X_{\mathrm{BB}}$ is the projective variety obtained from X° by adding points (*closing the cusps* from the differential geometric viewpoint).
(2) The toroidal compactification $X^\circ \subset X_{\mathrm{tor}}$ is a smooth projective variety; the boundary $X_{\mathrm{tor}} \setminus X^\circ$ is a finite union of disjoint hypersurfaces that are abelian varieties.

Here we should cite [AMRT10, HS96, Sar23]. I'm not sure of a convincing reference for the points compactification X_{BB}.

Both compactifications can be compared in the following way: there is a birational morphism

$$\pi\colon X_{\mathrm{tor}} \longrightarrow X_{\mathrm{BB}}$$

which contracts exactly the abelian varieties alluded to (and is an isomorphism over X°).

Theorem 5.1. *Up to replacing G with a finite index subgroup, the manifold X_{tor} is weakly Kähler hyperbolic ; it is not the blow-up of a Kähler hyperbolic manifold.*

Proof. According to [Zhu24, Lemma 3.2], we know that X_{BB} is aspherical and that $\pi_1(X_{\mathrm{BB}})$ is Gromov-hyperbolic (this is only achieved after replacing G with a finite index subgroup). Asphericity implies that

$$H^2(X_{\mathrm{BB}}, \mathbb{R}) = H^2(\pi_1(X_{\mathrm{BB}}), \mathbb{R})$$

and, according to Lemma 2.7 and Gromov-hyperbolicity of $\pi_1(X_{\mathrm{BB}})$, we infer that

$$H^2(X_{\mathrm{BB}}, \mathbb{R}) = H^2_{\mathrm{hyp}}(\pi_1(X_{\mathrm{BB}}), \mathbb{R}).$$

Since any class is hyperbolic, we can pick an ample class α coming from an embedding $X_{\mathrm{BB}} \hookrightarrow \mathbb{P}^N$ and finally it is enough to consider

$$\omega = \pi^*(\alpha) \in H^2(X_{\mathrm{tor}}, \mathbb{R}).$$

This class is obviously big and nef and it is also hyperbolic since α is so. $\square$

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