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A link between Guidobaldo dal Monte and Galileo Galilei in the study of conic sections? Some evidence from the manuscript UCLA 170/624

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Abstract

This article examines some constructions of conic sections found in a manuscript (UCLA, MS 170/624, fol. 85r) and attributed to Guidobaldo dal Monte. Following centuries of limited access to Greek sources in the Latin West, the sixteenth century saw a revival of interest in conics, spurred by translations of Apollonius and the work of figures such as as Commandino and Maurolico. Like other mathematicians of the period, Guidobaldo developed, or recovered from ancient sources, methods to draw conic sections. In doing so, he addressed both the theoretical needs of mathematics and the practical requirements of gnomonics, astronomy, and mechanics. The folio includes three geometrical constructions of the conic sections and a marginal note: "Del Galileo", suggesting a possible link with Galileo Galilei. The article analyzes these constructions, highlighting both the theoretical rigor and practical intent of Guidobaldo's approach. Particular focus is given to the construction of the hyperbola, which plays a key role in the solution of the Apollonian three circles problem, a topic Guidobaldo discussed in correspondence with Galileo. This problem is another example of the scientific relationship between Galileo and Guidobaldo, along with the study of the centers of gravity of solid figures, already described by the same authors in a paper published in 2022: "Galileo Galilei and the centers of gravity of solids: a reconstruction based on a newly discovered version of the conical frustum contained in manuscript UCLA 170/624".

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1 Introduction

The study of conic sections has been a pillar of mathematical inquiry since antiquity, bridging geometry, astronomy, and mechanics. During the Renaissance, the revival of ancient Greek mathematical texts, such as those of Apollonius, Archimedes, and Ptolemy, sparked a renewed interest in these curves, both for their theoretical elegance and their practical applications. However, during the Middle Ages the knowledge of conics was very scarce in the Latin West also due to the complexity of the subject matter. In this period, conic sections were primarily studied for their applications in parabolic burning mirrors, as evidenced in Alhazen's *De speculis comburentibus* (translated into Latin by Gerard of Cremona in the twelfth century) and in Witelo's *Perspectiva*, which includes several propositions on conics.[1]

The first complete Latin translation of the first four books of Apollonius' *Conics* was completed by Giambattista Memmo and published posthumously in 1537.[2] However, this translation was considered inadequate, leading mathematicians to seek alternative versions.

Mathematicians like Federico Commandino and Francesco Maurolico took up this challenge, producing translations or revisions of Apollonius's work, thus providing the Latin West with the opportunity to access a more reliable version of the *Conics*.[3]

But there is also another side of the question more related to constructions and practical needs, for example for sun clock design. Guidobaldo dal Monte explored innovative methods for constructing conic sections and applied their properties to problems in gnomonics, mechanics, and astronomy. These efforts not only demonstrated the enduring relevance of classical knowledge, but also paved the way for new developments in early modern science.

On this point, it may be worth briefly recalling that also Francesco Maurolico and Federico Commandino wrote important contributions to the theory of sundials. Their works on this topic help to clarify the relevance of conic sections in Renaissance sundial theory.

Maurolico's *De lineis horariis libri tres* was composed between 1553 and 1569 and published in his *Opuscula mathematica* (Venezia, *Apud Franciscum Franciscium Senensem*, 1575, pp. 161–285). In the third book, it contains a geometrical study of conic sections. It is interesting to note that Giovanni Alfonso Borelli, even in 1679, expressed his appreciation for this study: "in quo [de lineis horariis] egregias demonstrationes excogitavit linearum tangentium sectiones conicas".[4] Commandino, in turn, published in 1562 his *Liber de horologiorum descriptione*, a synthetic treatment of gnomonics rooted in classical geometry. As Ciocci wrote[5]:

[1] For a comprehensive analysis of the medieval Latin traditions of conic sections, see Clagett (1980).

[2] Apollonius (1537).

[3] On Commandino's contribution, see Argante (2023); on Maurolico's, see Roberta (1995).

[4] Borelli (1679), 2. See also Sinisgalli and Vastola (2000) and Amodeo (1908).

[5] "Sembrerebbe, pertanto, che lo scopo del libro di Commandino sia meramente tecnico-applicativo. Eppure nel bel mezzo della trattazione degli orologi orizzontali (pp. 56v–60r) inserisce raffinate dimostrazioni geometriche che mettono a frutto lo studio delle *Coniche*: di Apollonio." (Ciocci 2023, 111).

It would seem, therefore, that the purpose of Commandino's book is merely technical-applicative. Yet in the middle of his treatment of horizontal horologues (pp. 56v–60r), he inserts refined geometrical demonstrations that make use of his study of Apollonius' *Conics*.

The manuscript folio examined in this article (Los Angeles, Charles E. Young Research Library, University of California-Los Angeles, MS 170/624, fol. 85r) provides a unique glimpse into this dynamic intellectual context. It includes geometric constructions of the three conic sections, illustrated with diagrams. In the folio, there is a marginal note: "Del Galileo" that raises intriguing questions about the relationship between these notes and Galileo's work. While the attribution of the folio to Galileo remains uncertain, its content undoubtedly reflects a broader mathematical culture in which conic sections served as both theoretical objects of study and practical tools for solving applied problems.[6]

The article begins with an analysis of the letters between Guidobaldo and Galileo on the three circles problem (Sect. 2); then the geometric constructions themselves are presented and the techniques are contextualized within the broader mathematical practices of the Renaissance, in which point-by-point constructions served as an alternative to mechanical instruments and enabled greater theoretical insight and precision (Sect. 3).

The discussion then moves to Guidobaldo dal Monte's broader contributions to the study of conic sections, including his methods for constructing ellipses (Sect. 4) and hyperbolas (Sect. 5) in works such as the *Planisphaeriorum theorica* and the *Meditatiunculae de rebus mathematicis*.

By comparing UCLA manuscript's constructions to those found in Guidobaldo's published and unpublished works, the article highlights both the continuity and the innovation in his approach to conics. Particular emphasis is placed on Guidobaldo's theoretical rigor and his development of practical methods for constructing conics,

[6] This folio is part of the section of the manuscript mainly attributed to Guidobaldo, datable between 1580 and 1590 (see Bellé and Sisana 2022, 500–503). Several elements support this dating. First, a key *terminus post quem* is provided by the explicit reference, on folio 85r, to Guidobaldo's *Planisphaeriorum universalium theorica*, which was printed in 1579. This reference implies that folio 85r must have been written after 1579. Second, the folio is also closely linked to the Apollonian three circles problem, which appears in the *Meditatiunculae de rebus mathematicis*, Bibliothèque Nationale de France, *Par. Lat.* 10246, 37, 38, and 38bis. Frank (2013, 315) suggests that these pages could be related to Commandino's teachings, which Guidobaldo attended between 1568 and 1574, while Commandino was still alive (he died in 1575). Ciocci (2025, §6) agrees with this hypothesis. However, Tassora (2001, 19–51), in her in-depth analysis, proposes a later composition date for these folios of the *Meditatiunculae*, between 1586–87 and 1593. Despite the discrepancy, Frank himself suggests that Guidobaldo revisited some of these materials in detail years later, during his correspondence with Galileo (1587–1588). Some of these later developments seem to have been integrated into the UCLA manuscript. For instance, folio 90 contains a variant of the three circles problem; in September 1588, Galileo wrote to Guidobaldo asking for help with this problem. This exchange (see below pp. 5–6 for details) suggests that Guidobaldo had already studied the problem previously. It is plausible that Guidobaldo initially explored the problem during his collaboration with Commandino, and later refined his solutions during his collaboration with Galileo. Folio 85r, when considered within this broader timeline of scientific exchange, is likely one of these later elaborations. Although it is difficult to determine its precise date, the evidence strongly suggests the second half of the 1580s—certainly after 1579 (due to the reference to the *Planisphaeriorum theorica*) and probably around 1588, in connection with Galileo's request. This dating is consistent with the general chronology of the second part of the UCLA manuscript and the related sections of *Meditatiunculae*.

which were crucial for his resolution of the Apollonian three circles problem (Sect. 6). The construction of the hyperbola found in UCLA manuscript is fundamental in the solution of the three circles problem proposed by Guidobaldo in his *Meditatiunculae* (Sect. 7). In a recent article, Ciocci discussed the importance of this problem in shaping the approach of Guidobaldo to the reconstruction of ancient mathematics, following the path of his mentor and teacher Commandino.[7]

We then explore a possible connection between the manuscript's content and Galileo Galilei, taking into account the enigmatic marginal note "Del Galileo" and Galileo's well-documented interest in conic sections. His early work on the centers of gravity of solids, as well as his studies on projectile motion and parabolic trajectories, clearly demonstrate his familiarity with such curves. Notably, Galileo also showed an interest in the three circles problem, as it emerges from two letters addressed to him by Guidobaldo dal Monte. In the first, Guidobaldo replies to a presumed request from Galileo, promising to send him a solution of the problem, which he had previously solved and kept among his notes. In the second letter, Guidobaldo forwards the problem to Galileo, asking him to refine if he intended to send it to Flanders. However, the problem itself is not preserved in the extant correspondence.[8] The article examines the possibility that the constructions in folio 85r might reflect a shared intellectual environment between Guidobaldo and Galileo, or even a direct collaboration or influence.

As the authors of the present paper discussed in a previous article (Bellé and Sisana 2022), Galileo—especially in his early years and thanks to his solid mathematical training—nurtured genuine interest in mathematics that extended beyond its application to mechanical or physical problems. His theorems on the centers of gravity solids (some of which are preserved in the UCLA manuscript under discussion, ff. 75–76) clearly attest to his engagement with questions of pure geometry. Similarly, his active involvement in the three circles problem, as evidenced by his correspondence with Guidobaldo, further demonstrates his attention to the theoretical aspects of mathematics.

Finally, the article situates these constructions within the broader historical context of the Renaissance mathematical community. The revival of conic sections during this period was marked by a vibrant exchange of ideas among mathematicians across Europe, as seen in the works of Commandino, Clavius, Van Roomen, and Viète. The manuscript page, with its mix of classical references and original constructions, provides a valuable case study of how Renaissance mathematicians engaged with ancient texts while developing new methods and tools to address contemporary problems.

2 Guidobaldo and Galileo epistolary exchange on the three circles problem

In 1588, after a prolonged exchange on the centers of gravity of solid bodies, Guidobaldo dal Monte and Galileo Galilei shifted the focus of their correspondence

[7] See Ciocci (2025).

[8] See Galilei (1891-1909), X, 37–38.

to the so-called three circles problem.[9] Of this exchange, several crucial testimonies remain.

Guidobaldo's engagement with the problem took shape within a broader network of intellectual exchange that included prominent figures such as Federico Commandino. According to Frank and Ciocci, Guidobaldo's earliest investigations—recorded in the *Meditatiunculae*—likely date back to the early 1570s, during his collaboration with Commandino.[10] A decisive turning point, however, occurred more than a decade later, when Galileo explicitly asked Guidobaldo for a solution to the Apollonian problem.

Two surviving letters, dated 16 September and 7 October 1588, document this request. In the first, Guidobaldo explains that he had once been prompted by Pappus' text to tackle the problem, and that—after much effort—he had indeed solved it, although he was currently unable to find the relevant papers[11]:

> Pappus, in the tenth proposition of the fourth book, prompted me to attempt the problem, since he does not teach how to find it; and after much musing, I did find it, and I will send it to you, even if I still hope to use it one day in print. But you are so kind to me that I cannot refuse: I only need to find it again among my papers, which are in some disarray.

In a second letter, dated 7 October 1588, Guidobaldo sends Galileo the recovered material, with a remarkable degree of trust and deference[12]:

> I send you the problem you requested, and beg your pardon for the delay. If you intend to send it to Flanders, please adjust it as you see fit, because I am sending it to you just as I found it among my old papers. I would be glad to know if it pleases you.

Although Galileo's original request is now lost, the surviving correspondence provides valuable insight into their active collaboration and sheds light on the evolution of Guidobaldo's geometric constructions. Based on this exchange, and on internal evidence from the manuscripts, three points support the hypothesis of a direct link between the three circles problem, the hyperbola construction in the UCLA manuscript, and the mathematical dialogue between Guidobaldo and Galileo.

[9] The problem is a particular case of the work *Tangencies* attributed to Apollonius by Pappus in book VII of the *Collection*, see Jones' translation (Pappus 1986, 90–94), or Commandino's (Pappus 1588, 159r–160r). It consists in finding a circle tangent to three given circles. It appears also in book IV, proposition 10 of the same Pappus' work, see H. Sefrin-Weis's translation and commentary (Pappus 2010, xxiii and 98–99), and Commandino's edition (Pappus 1588, 44r).

[10] According to Ciocci (2025, § 6), these pages can be dated between 1569 and 1575, when Federico Commandino was still alive. Other dates proposed by Frank (2013) and Tassora (2001) are discussed in detail by Ciocci in note 72 of the same article.

[11] "Circa il problema propostoli delli tre circoli, Pappo ne, quarto libro, alla decima propositione, mi fece venir voglia di trovarlo, perché: Pappo non insegna di trovarlo; e così doppo molto fantasticare lo trovai, et lo mandarò a V.S., se ben io spero di servirmene un giorno in istampa; ma lei è tanto cortese verso di me, che non voglio mancare: ma non posso adesso, perché io l'ho fra certe mie carte, che Dio sa dove sono, per haver assai scombossalato il mio studio, essend'io stato fuori, dove mi bisognarà forse ritornare." (Galilei 1891-1909, X, 37).

[12] "Mand'a V.S. il problema che mi: adimandò e mi escusi se sono stato troppo a mandarglielo. Se lo mandarà in Fiandra, di gratia lo accomodi come gli parerà, perchè glie lo mando così come io l'ho trovato fra certe mie cartaccie. Haverò caro d'intendere se le sarà piaciuto." (Galilei 1891-1909, X, 38).

First, the intensity and specificity of Galileo's request likely prompted Guidobaldo to retrieve and refine a preexisting construction. This renewed engagement may have led to the insertion of page 38bis into the *Meditatiunculae* and to the improved version of the solution preserved in folio 90r of the UCLA manuscript.[13]

Second, the construction of the hyperbola on folio 85r of the UCLA manuscript appears just a few folios before the geometric solution to the three circles problem (folio 90r). This suggests Guidobaldo was developing or refining a constructive tool specifically adapted to that problem. The UCLA folios, therefore, may represent material closely related to the *Meditatiunculae* but never fully integrated into it; both likely formed part of the *Colibeto*, a personal notebook Guidobaldo is known to have compiled.[14]

Finally, the marginal note "Del Galileo" on the same folio that contains the conic constructions is an indication of Galileo's involvement. Though its precise origin is unclear, it signals that the material was at some point associated with Galileo—either as recipient, interlocutor, or intended user. In the context of the 1588 correspondence, this note is a clue of a shared interest in conic constructions during Galileo's formative years.

In this light, the three circles problem emerges not only as a classical geometric challenge, but also as a concrete case in which abstract theoretical tools—like the properties of conics—were translated into operational, pointwise constructions through the dialogue between Guidobaldo and Galileo.

It is precisely in this context that folio 85r of the UCLA manuscript acquires particular significance. In addition to the construction of the hyperbola, connected (as we will see in Sect. 6) to the solution of the Apollonian problem, the folio contains constructions of the parabola and the ellipse. The latter is especially noteworthy, as Guidobaldo had already encountered and addressed the construction of the ellipse in an earlier work, the *Planisphaeriorum theorica*, which will be examined in detail in the following sections. Together, these three conic sections—constructed point by point and preserved on a single folio—offer a unique insight into Guidobaldo's geometric toolkit and the concrete strategies he mobilized in response to Galileo's 1588 inquiry.

3 The construction of conics in the UCLA manuscript

The UCLA manuscript contains three folios in Guidobaldo's hand that are connected to Galileo. Folios 75–76 preserve Galileo's proof for the center of gravity of the conical frustum, a demonstration that would later appear as an appendix to Galileo's book *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* (Leiden, 1638).

[13] Ciocci writes "This fact would also explain the insertion of page 38bis in the manuscript of the *Meditatiunculae* where we find the revised version of A, which we have called version B. Guidobaldo added B to the original manuscript of his *Meditatiunculae* in a period after the one in which he wrote A. Maybe B dates back to September–October 1588 when Galileo asked him for help to search for a solution to the problem of the three circles." (Ciocci 2025, § 7).

[14] See Bellé and Sisana (2022), 473.

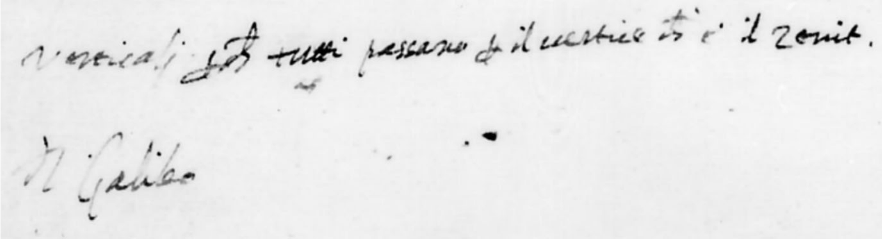


Fig. 1 Note at f. 85r

Folio 85 also refers to Galileo, albeit in a more indirect way, and we consider it useful to provide an edition of it here.[15]

This folio contains three geometric constructions of conics and, at the end of the page, a brief note that appears to be unrelated to conics: "verticali perché tutti passano per il vertice che è il Zenit" and on another line: "Del Galileo" (Fig. 1).

It is unclear if "Del Galileo" refers to the brief note concerning vertical circles and the zenith, the geometrical constructions of conics, or none of these. Nevertheless, this page is of interest as it offers insights into the study of conics by Guidobaldo and his contemporaries and correspondents, including Galileo.

There are three constructions, each illustrated by a corresponding geometrical diagram: to draw an ellipse given the axes, a hyperbola given the focal distance and a vertex, and a parabola given the axis and the latus rectum.[16] The points on the conical section are found by the intersections of straight lines and circumferences.

3.1 Ellipse

Given the axes AB and CD, intersecting in E, and taking two points on the major axis, F and G (with $FD = AE$), the ellipse is determined in this way[17]:

1. choose any point (for example H) on the major axis;
2. trace the circumference with radius AH and center F and the circumference with radius BH and center G;
3. the points of intersection (for example K) of these circumferences are on the ellipse.

No explicit proof is provided, but there is a reference to the *Planisphaeriorum theorica* ("che si cava dal Planispherio"), published by Guidobaldo in 1579.[18] In this work, Guidobaldo discusses constructions of the ellipse, although the two methods presented

[15] For an analysis of the studies on the centers of gravity by Galileo and a description of the manuscript, see Bellé and Sisana (2022).

[16] The construction of the hyperbola is partially covered by a leaflet attached to the folio; for the parabola, two almost identical versions are given. See the description in Sect. 11.1.

[17] Guidobaldo explicitly provides a method for finding only the position of F, stating that F should be selected such that $FD = AE$. No instruction is given for G, but it is understood that $GD = BE$. F and G are the foci of the ellipse; however, Guidobaldo never mentions this fact.

[18] The construction is based on Apollonius' proposition III.52, even if Guidobaldo did not refer explicitly to Apollonius. We can today express the proposition as follows: the sum of the distances of the point of intersection K from the foci F and G is equal to the major axis. The term "focus" does not appear

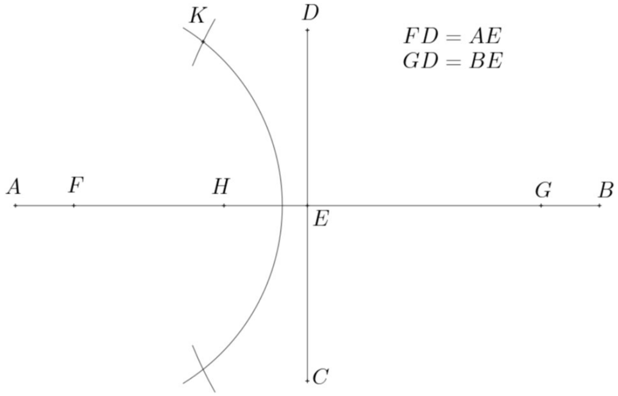


Fig. 2 The construction of the ellipse

there differ from the one proposed in this manuscript. However, the *Planisphaeriorum theorica* includes a method for determining the foci, which could be applied here to find F and G (Fig. 2).[19]

3.2 Hyperbola

The construction of the hyperbola is similar to that of the ellipse, but even more sketched. A line AB is given and a point C ("dove deve esser il vertice dell'hyperbola" in Guidobaldo own words) is marked on it.[20] The construction then proceeds as follows:

1. take another line on which points E, F, and D are placed, in this order and such that $AC = DE$ and $BC = DF$; take any point (e.g., G) beyond point D;
2. trace the circumference with radius EG and center A and the circumference with radius FG and center B;
3. the points of intersection of these circumferences lie on the hyperbola.

In perfect analogy to the case of the ellipse, the points lie on the hyperbola by proposition III.51 of Apollonius' *Conics*: if A and B are the foci, a point K is on the curve if $KA - KB = EG - FG = EF = AC - BC$.[21] The auxiliary lines EG and FG

Footnote 18 conunied
in Apollonius, see Heiberg's edition of the *Conics* (Apollonius 1974, I, 436–439), English translation in Apollonius (1988), 258–259: "If in an ellipse a rectangle equal to the fourth part of the figure is applied from both sides to the major axis and deficient by a square figure, and from the points resulting from the application [the foci] straight lines are deflected to the line of the section, then they will be equal to the axis".

[19] See dal Monte (1579), 103–104 (note that p. 103 is mistakenly printed as 107). More details on *Planisphaeriorum theorica* will be given later, in Sect. 4, p. 11.

[20] In this case, the foci are A and B, although Guidobaldo did not express it explicitly.

[21] In the drawing, the point C is nearer to B than to A. See Heiberg's edition of the *Conics* (Apollonius 1974, I, 434–437), English translation in Apollonius (1988), 257–258: "If a rectangle equal to the fourth part of the figure is applied from both sides to the axis of a hyperbola or opposite seciont and exceeding

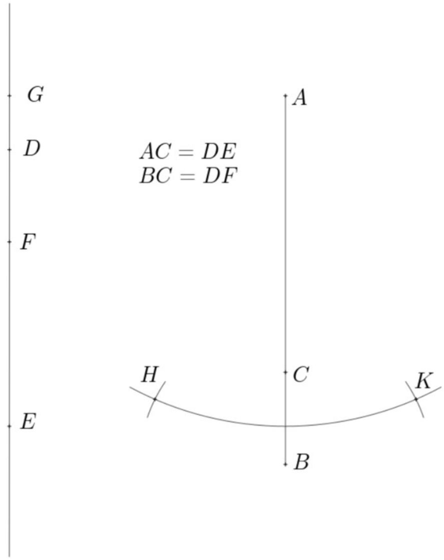


Fig. 3 The construction of the hyperbola

are used to graphically represent the difference between the distances of the point K on the hyperbola from the foci B and A (Fig. 3).

Guidobaldo treated other two constructions of the hyperbola in his *Meditatiunculae rebus mathematicis* which remained in manuscript form.[22]

3.3 Parabola

In the case of the parabola, two similar versions of the same construction are given.[23] This construction is based on the fundamental property of the parabola: the ordinate is the proportional mean between the latus rectum and the abscissa. To find a point on the parabola, the following steps are taken:

1. take a point E on the axis of the parabola AB;
2. take another point H on the axis AB, below E, such that EH equals the latus rectum AG;
3. trace the perpendicular line to the axis passing through E;
4. trace the circumference on the diameter AH[24];

Footnote 21 continued
by a square figure, and straight lines are deflected from the resulting points of application [the foci] to either one of the sections, then the greater of the two straight lines exceeds the less by exactly as much as the axis".

[22] The instrument used to construct hyperbolas, described in detail in Sect. 5 and analyzed by Tassora (2001, 156–162), is situated in a historical context of great interest for the study of conic sections. The similarities with the instrument described by Muzio Oddi in his *Degli horologi* (Oddi 1614, 187–190) underscore the importance of these devices in the scientific practice of the seventeenth century.

[23] On these two versions, see Sect. 11.1.

[24] The first version of the construction only produces half of the parabola tracing a semicircle.

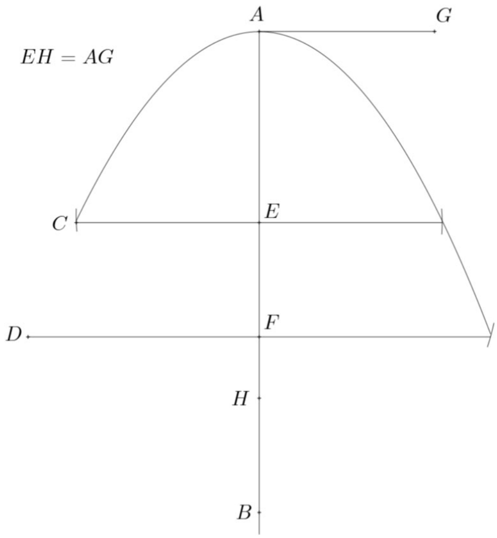


Fig. 4 The first construction of the parabola

5. the points of intersection (although in the drawing only one of them is marked with a letter) between the perpendicular line and the circumference are on the parabola.

In this case, a very brief proof is provided, referencing Apollonius' *Conics* I.11, introduced with the words: "La dimostrazione è questa" (Fig. 4).

3.4 Sources

These methods are based on the well-known properties of the conics and it is possible to find comparable constructions in contemporary mathematical texts. A broad examination of geometrical and mechanical constructions of conics can be found in E. Ulivi "Le fonti di Bonaventura Cavalieri: la costruzione delle coniche fino allo Specchio ustorio".[25]

One of the methods proposed by Clavius for the hyperbola and the ellipse in his *Gnomonices libri octo* (1581) is similar to Guidobaldo's construction.[26] Moreover, the fact that Guidobaldo refers to the *Planisphaeriorum theorica* can suggest that his interest arose in dealing with gnomonics and stereographic projections. The same method (but only for the ellipse) is also found in Fabrizio Mordente's work *Il compasso con altri istromenti mathematici ritrovati da Gasparo suo fratello* (1581).[27]

For the parabola, an analogous method is found, for example, at the end of Marino Ghetaldi's work *Nonnullae propositiones de parabola* (1603), *Problema II. Propos. VII*: *Parabolam ad constructionem speculi propositum intervallum comburentis in plano describere*.[28]

[25] See Ulivi (1987).

[26] See Clavius (1581), 32 and 34; and Ulivi (1987), 155–156.

[27] See Mordente (1581), 61–63.

[28] See Ghetaldi (1603), 18–19. Clavius in the *Gnomonices libri octo* adopted a different method for the parabola. See Ulivi (1987), 155 (Clavius's method) and 143 (Ghetaldi's method).

It is also worth noting that page 115^5v of Guidobaldo's *Meditatiunculae* contains three sketched constructions of conic sections by mechanical means. These diagrams, representing instruments to trace parabola, hyperbola, and ellipse (in that order), are accompanied by brief captions. Tassora hypothesizes that these constructions are based on propositions I.20 and I.21 of Apollonius' *Conics*, particularly as interpreted in Eutocius' commentary, where similar methods are described. These rely on the principle that the ordinates are mean proportionals between two given segments. Although the fundamental idea behind the instruments depicted in page 115^5v can be understood from the drawings, it remains impossible to determine exactly how Guidobaldo intended to implement them in practice. That these sketches refer to actual instruments rather than theoretical constructions is suggested by the presence of guiding tracks, apparently designed to accommodate sliding rods—features that clearly go beyond classical geometric construction with straightedge and compass.[29]

4 Guidobaldo's methods for constructing ellipses

The ellipse, among the conic sections, holds a prominent place in the work of Guidobaldo dal Monte. In particular, in his efforts to continue Commandino's research on the construction of sundials, the ellipse became a central object of study. Building on Commandino's investigations into the analemma and the geometric principles underlying gnomonic projection, Guidobaldo explored the ellipse's properties with exceptional rigor. His interest in this curve stemmed not only from its theoretical significance, but also from the practical challenges it posed—especially in relation to the accurate drawing of meridian and hour lines. These investigations laid the groundwork for his innovative contributions to the construction of ellipses.

4.1 The *Planisphaeriorum theorica*

Guidobaldo dal Monte made a groundbreaking discovery: he proved that the meridian lines and time lines, essential for the accurate construction of sundials and astrolabes, are arcs of ellipses when orthographically projected onto a plane. Proudly, in the second book of his *Planisphaeriorum theorica*, he remarked: "It will be easy for us to show, even according to their [Gemma Frisius, Juan de Rojas and other astronomers] construction (though they may not know what they are doing), that they are ellipses."[30] This insight created the need for precise methods to draw these curves.

[29] For further details, see *Par. Lat.* 10246, 115^5v and Tassora (2001), 159–161.

[30] "Iam ad meridianos horariosque circulos deveniamus; et quid sint in astrolabo demonstremus. Nam eos nonnulli circulos nuncupant, alii lineas curvas anomalas; quae neque circuli sunt neque certa designatione constitutae; sed tantum per puncta adsignata manu diligenti traductae, ut Gemma Frisius. Alii relinquunt eos innominatos, ut ipsemet Ioannes de Roias. Nobis vero facile erit ostendere, etiam secundum ipsorum constructionem (quamvis, quid faciant, ignorent) ellipses esse. His tamen prius demonstratis." (dal Monte 1579), 79, wrongly printed 77.

Among those earlier authors, Juan de Rojas[31] had played a prominent role in disseminating the orthographic projection of the celestial sphere, especially through his influential *Commentariorum in astrolabium* (Paris 1551). Yet, as Guidobaldo noted, Rojas—despite his pioneering work—did not identify these curves as ellipses. By contrast, Guidobaldo's explicit classification marked a conceptual leap: what had remained vague or unnamed in the Flemish tradition was now clearly interpreted through the lens of Apollonian geometry.

Emphasizing the need for rigorous methods, Guidobaldo cited Gemma Frisius's *De astrolabo catholico liber* (1556). Frisius acknowledged the difficulty of tracing meridian lines accurately: "The meridians themselves are described uncertainly, point-by-point, with an uneven line; and as this is not within the reach of every craftsman, it often happens that mistakes are made, both in the description and in the use."[32]

Earlier mathematicians, including those who drew generic curves, relied on the careful plotting of individual points ("per puncta adsignata manu diligenti"). Guidobaldo recognized the limitations of this approach and stressed that ellipses are difficult to draw precisely, even for skilled craftsmen: "it seems that [ellipses], since they are not within the reach of every craftsman, can hardly be described with great accuracy, and perhaps not correctly at all".[33] To address these challenges, he proposed two main approaches:

1. Rigorous point-by-point construction—a detailed, precise method of drawing ellipses in the tradition of earlier mathematicians.
2. Innovative instrumental methods—the development of new instruments and techniques to streamline the process of drawing ellipses.

Before presenting his own solutions, Guidobaldo first recalled notable examples of the first approach, citing Eutocius, Commandino, and Dürer.[34] He closed this review by remarking: "Therefore, it will not be useless to show a method of drawing an ellipse,

[31] Juan de Rojas y Sarmiento (ca. 1510–1573), a member of the Castilian nobility, studied with Reiner Gemma Frisius in Louvain and later followed the imperial court to Flanders. He became known across Europe for his *Commentariorum in astrolabium libri duodecim* (Paris 1551), which introduced a new form of astrolabe based on orthographic projection. Although the edition had some resonance in Paris, it mainly reflected the innovations of the Flemish school of instrument making, notably the work of Hugo Helt, a pupil of Frisius. See Maddison (1966).

[32] "Nam Gemma Frisius eodem in loco, libro scilicet de astrolabo catholico capite primo, dum huius instrumenti incommoda commemorat, inquit: ≪Ipsi meridiani incerta designatione per puncta inaequali ductu describuntur: idque cum non sit cuiuslibet artificis, fit, ut saepe contingat hallucinari, cum in descriptione, tum in usu quoque≫." (dal Monte 1579, 99).

[33] "Videtur itaque, cum non sit cuiuslibet artificis, ut vix maximaque cum difficultate et forsan minime recte describi possint. Et quaquam nos supra ellipses hos esse demonstravimus, eadem tamen incommoda in ellipsi describenda contingere multis fortasse videbitur." (dal Monte 1579, 99).

[34] "Cum ellipsim quoque non nisi per puncta vel diligenti manu lineare sit necesse, aut enim per puncta (ut supra dictum est) inventa, aut quemadmodum Eutocius in commentario in XXI primi Conicorum Apollonii docet, vel ut Federicus Commandinus in libro de horologiorum descriptione, sive ut Albertus Durerus in sua Geometria, vel aliis quibuscunque modis [describatur]." (dal Monte 1579, 99). In Proposition I.21 of the *Conics*, Apollonius proves a fundamental property of hyperbolas and ellipses, concerning the ratio between the squares of the ordinates and the rectangles constructed on the segments of the transverse axis intersected by the ordinates themselves. Eutocius, commenting on this proposition, proposes a constructive method for drawing the hyperbola, and by analogy the ellipse, using the very property proved by Apollonius (see (Apollonius 1566, 19rv)). This method, based on the construction of rectangles of known area, is taken

not by points, but by some instrument that can draw an elliptical continued line."[35] This statement does not yet introduce his innovative instruments, but rather serves to preface his first method, still rooted in the point-by-point tradition: the well-known gardener's method, based on Apollonius' *Conics* III.52[36]:

> The first method, then, although it had an author, is nevertheless attributed to no one (even though I myself wrote it), since it is clearly derived from the 52nd proposition of the third book of Apollonius's *Conics*, and is of great usefulness to mechanics. Indeed, artisans, especially masons engaged in building houses (as we see daily), very often describe an ellipse in this way using a string, when preparing the beams for the construction of vaulted ceilings.

This technique, common among craftsmen such as bricklayers shaping room beams, consists of fixing a thread of a defined length to two points (the foci) and tracing the curve with a stylus. However, it was seldom used for astrolabes, which often required ellipses defined by given axes rather than known foci.

To address this limitation, in the following proposition, Guidobaldo showed how to determine the foci when the axes are given.[37] This proposition anticipates the construction described in the manuscript, which, as noted above (see Sect. 3.1), requires knowing the position of the two points today called foci.

Yet this method had a flaw: the thread is always somewhat flexible, and it is very difficult—practically impossible—to stretch it in exactly the same way each time. As a result, the line produced is only approximately an ellipse. While this may suffice for a bricklayer or gardener, it is not acceptable for a scientific instrument such as an astrolabe.

Guidobaldo's second method, as we shall see in the next section, offered a more accurate approach, particularly for planispheres, as it avoided the mechanical inaccuracies inherent in other techniques.[38]

Footnote 34 continued
up and developed by Commandino in *De horologiorum descriptione* (Ptolemy 1562, 58v–59r), where it is applied to the construction of sundials. Dürer, too, in his *Underweysung der Messung mit dem Zirkel und Richtsceyt* (1525), translated into Latin as *Albertus Dürerus Institutionum geometricarum libri quatuor* by Joachim Camerarius in 1532, devotes considerable space to the construction of conic sections, drawing inspiration from Apollonius. In particular, in the first book of the work, Dürer analyzes the construction of the three conic sections and presents a method of construction based on the same fundamental property proved by Apollonius (Dürer 1532, 29–34).

35 "Quapropter non inutile erit si ellipsis describere modum ostenderimus, non sane per puncta verum instrumento aliquo quod ellipsis lineam describat." (dal Monte 1579, 99–100).

36 "Primus itaque modus, quamvis auctorem habuerit, nemini tamen (quod ipse scriverim) ascribitur, quippe qui ex quinquagesimasecuna tertii libri Conicorum Apollonii aperte elicitur ac mechanicis mangopere usui est. Et enim artifices, praecipue vero caementarii domibus aedificandis (ut cotidie cernimus), dum ligna ad camera construenda parant, quam saepissime filo ellipsim hoc pacto describunt." (dal Monte 1579, 101–102).

37 "Quia vero in astrolabio ellipsis describendae semper dati sunt axes, ut ex iis quae diximus manifeste apparet: proinde puncta sane C D [foci] invenire facillimum erit hoc pacto." (dal Monte 1579, 102).

38 "Alium idcirco a nobis excogitatum afferemus modum, qui (ni fallor) ad ellipsim describendam forsan valde utilis, et ad hoc praecipue planisphaerium lineandum apprime accomodus erit." (dal Monte 1579, 104).

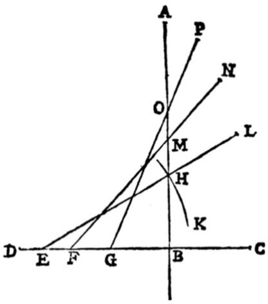


Fig. 5 dal Monte, *Planisphaeriorum theorica* (1579), 118

4.2 The geometrical method: point-by-point construction

The description of the device for drawing ellipses more precisely (see Sect. 4.3 and dal Monte 1579, 124–128) is preceded by a geometrical proof of the pointwise construction that guarantees its correct functioning.[39]

4.2.1 The construction

Let AB and CB be, respectively, the major and minor semiaxes of the ellipse (see Fig. 5), then Guidobaldo's procedure can be summarized as follows[40]:

1. extend CB to D such that $CD = AB$;
2. draw the circumference with center E (any point on BD) and radius BD; it will intersect AB in H;
3. extend EH to a point L such that $HL = BC$; L lies on the ellipse.

The correctness of this construction relies on the following theorem[41]:

> Given the axes AB and CD of an ellipse, meeting orthogonally at point E, let FG be a segment equal in length to the semi-major axis AE, with one endpoint F placed on the minor axis. If the distance GH—with H the point where the segment intersects the major axis—is equal to the semi-minor axis CE, then the other endpoint G lies on the ellipse.

To prove this theorem, Guidobaldo used two propositions that the reader of *Planisphaerium theorica* could find on the previous pages:

Lemma 1 *Given two diameters AB and EG of an ellipse AGC, if from points A and G two lines (AM and NG) are drawn, intersecting the diameters outside the ellipse*

[39] "Antequam autem instrumenti huius operationem ostendamus, primum ea, quae ad ipsius demonstrationem pertinent, ostendere oportunum videtur. Ut, cum eius operationem afferemus, statim operatio ipsa per se manifestissima reddatur." (dal Monte 1579, 110).

[40] See dal Monte (1579), 118.

[41] "Datis axibus ellipsis, si recta linea dimidio maioris axis aequalis maiorem axem secet; alterumque ipsius extrmum in recta linea minoris axis existat; alterum vero a puncto, ubi linea maiorem axem secat, quantitate dimidii axis minoris sit distans; erit punctum hoc in ellipsi." (dal Monte 1579, 114–115). Reference to Fig. 6.

Fig. 6 Theorem and Lemma 1

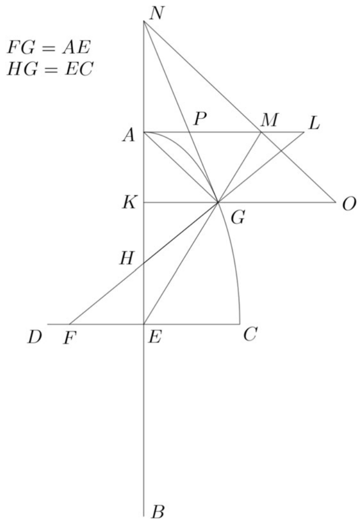
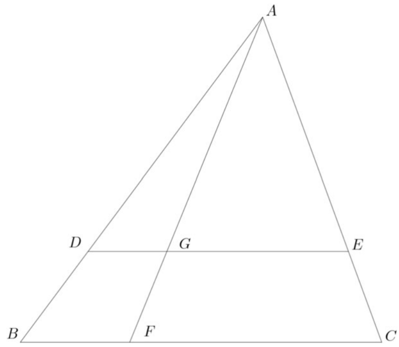


Fig. 7 Lemma 2



in M and N, in such a way that the triangles ANP and PGM have equal area, then, if one of the two lines (AM) is tangent to the ellipse, the other line (NG) will also be tangent.[42]

Lemma 2 *In the triangle ABC, let DE be drawn parallel to BC, then the line AF divides the parallels in the same ratio,* $DG : GE = BF : FC$.[43]

Guidobaldo then proved that the point G, as constructed in the theorem, belongs to the ellipse.[44]

42 Reference again to Fig. 6. "Si duae ellipsis diametri sectionem secent, et ab intersectionum punctis lineae extra sectionem cum diametris conveniant, quae quidem se invicem secent; triangulaque ad verticem facta inter se sint aequalia; harumque linearum una sectionem contingat, et altera quoque sectionem continget." (dal Monte 1579, 110–113). The proof of this lemma is based on Apollonius' *Conics* III.1, explicitly cited by Guidobaldo on page 112.

43 Reference to Fig. 7. "Si intra triangulum uni lateri aequidistans ducatur ab opposito autem angulo intra triangulum quoque recta ducatur linea, aequidistantes lineas in eadem proportione dispescet." (dal Monte 1579, 113). This theorem is easily proven using Euclid's *Elements* VI.2 and VI.4.

44 See Fig. 6; for a more detailed demonstration, see dal Monte (1579), 115–117.

1. From points A and G, draw perpendiculars to AE, namely AL and GK; extend FG to meet AL at L and draw EG, extended to meet AL at M; extend EA and set $AN = GL$; draw NM, intersecting GK (extended) at O; finally, draw NG, intersecting AM at P;
2. triangles FGE and LGM are similar: $FG : LG = GE : GM$; since $FG = AE$ and $GL = AN$, it follows that $AE : AN = GE : GM$;
3. considering triangle NME, since $AE : AN = GE : GM$, it follows that AG is parallel to NM (by Euclid's *Elements* VI.2); similarly, considering triangle NKO, we obtain $NA : AK = OG : GK$, since AG is parallel to NO and again by Euclid's *Elements* VI.2;
4. moreover, $NA : AK = OG : GK = MP : PA$, from Lemma 2 ($AM$ is parallel to KO by construction); and $NA : AK = NP : PG$ (because AP is parallel to KG); hence, $NP : PG = MP : PA$, and triangles ANP and PMG have equal area by Euclid's *Elements* VI.15;
5. the line AM is tangent to the ellipse at point A, since it is perpendicular to the major axis at an endpoint.

At this point, Guidobaldo invoked Lemma 1 to deduce that, if the line GN is tangent to the ellipse, then the point G lies on the curve. However, this reasoning is flawed, because Lemma 1 requires that the point G already belongs to the ellipse—which is precisely what Guidobaldo needs to prove.[45]

Some of Guidobaldo's contemporaries had already noticed the error. For example, Adriaan Van Roomen (Leuven, 1561–1615) in two letters addressed to Christoph Clavius highlighed the circularity of the reasoning in the *Planisphaeriorum theorica*.[46] In the first letter, there is only a brief note,[47]:

> I have read the two books of the *Planisphaeriorum* by Guidobaldo …I have noticed an error in the demonstration of the ellipse construction, an error that I would point out to the author, if I were certain whether he is still alive or not. …Since, however, I understand from your *Gnomonics* that you are on familiar terms with him, it would be of great service to me if you could let me know whether he is still living or not.

[45] Only the proof is flawed, the construction is correct, as Clavius 20 years later proved in his *Astrolabium*; see below (Sect. 4.2.1) for details.

[46] For the correspondence of Van Roomen, see Bockstaele (1976a, b). The letters to Clavius are cited from Baldini and Napolitani (1992), II, 171–172 and 181–182.

[47] "Legi duos planisphaeriorum libros Guidiubaldi …unum me animadvertisse errorem memini in demonstratione constructionis ellypsis, quem authori indicarem, si mihi constaret, num adhuc in vivis sit vel non. …Cum autem ex Gnomonice tua intelligam magnam tibi cum eo esse familiaritatem, rem mihi praestiteris gratissimam si significaris num superstes adhuc sit vel secus." (Baldini and Napolitani 1992, II, 171–172).

while in the second letter, maybe following an explicit request,[48] a detailed explanation was given to Clavius[49]

> What I recently wrote about the insufficiency of Guidobaldo's demonstration of the ellipse, I now show by analyzing his proof in the following way: "If two ellipses …coincide, etc." This is the proposition which he presents as a lemma, in order to demonstrate what he intends to prove. It is demonstrated on folio 133 of the Cologne edition. …Here, in order to prove that the line NG is tangent to the ellipse at point G, he assumes that the line EG intersects the ellipse at G—which is not proved.

Clavius, for his part, referred twice to Guidobaldo's ellipse construction, the first time in the *Gnomonices libri octo*:[50]

> But since the string is always somewhat tightened and slackened, the ellipse will be drawn much more precisely—though not as easily—by means of the device described by Guidobaldo Marquis dal Monte, a man in our time as renowned for his brilliance and learning as for the nobility of his lineage, in his *Planisphaeriorum universalium theorica*, recently published by him, where he has both invented it and demonstrated it with great acumen.

After some years, in 1593, Clavius in his *Astrolabium* provided again the same construction, with a correct proof completely different from the one Guidobaldo had shown in his *Planisphaeriorum*. He also added a comment on Guidobaldo's proof, with a tone somewhat different from his previous judgment:[51]

> The first part of this theorem was demonstrated in another and indeed longer way by the most learned man Guidobaldo, Marquis dal Monte, at the end of Book II of the *Planisphaeriorum universalium*.

Clavius's judgment thus shifted from the earlier "acutissime demonstravit" to the more veiledly critical "alio modo et quidem longiore". Perhaps this change in tone was influenced by the observations he had by then received from Van Roomen.

48 Unfortunately, the letters to van Roomen "were lost when the Louvain University Library burnt down in 1914". For more information see Bockstaele (1976a), 85–87. A detailed discussion on this proof is in Bockstaele (1976a), 96–97. Van Roomen referred to the 1581 edition of *Planisphaeriorum universalium theorica* printed in Cologne.

49 "Quod nuper scripsi de insufficientia demonstrationis ellipsis a Guidiubaldo propositae, eam resoIuta eiusdemonstratione hoc pacto ostendo: *Si duae ellipsis …conveniunt etc.*: Haec est propositio eius quam tanquam lemmma praemittit ad demonstrandum id quod intendit. Demonstraturque ea folio 133 editionis Coloniensis. Cui talem subiungat minorem necesse est. …Hic ut probet rectam NG tangere ellipsin in G, assumit rectam EG secare ellipsin in G, quod non est probatum." (Baldini and Napolitani 1992, II, 181).

50 "Sed quoniam filum intenditur semper et remittitur aliquantulum, multo accuratius ellipsis describetur, licet non tam facile, instrumento quod Guidus Ubaldus e Marchionibus Montis, vir hac aetate non minus ingenii atque doctrinae praetantia quam generis nobilitate clarissimus, in *Theorica planisphaeriorum :universalium* quae nuper ab eo edita est, invenit et acutissime demonstravit." (Clavius 1581, 37).

51 "Theorematis huius prior pars alio modo et quidem longiore demonstrata fuit ab eruditissimo viro Guido Ubaldo e Marchionibus Montis ad finem libri 2 *Planisphaeriorum :universalium*." (Clavius 1593, 168).

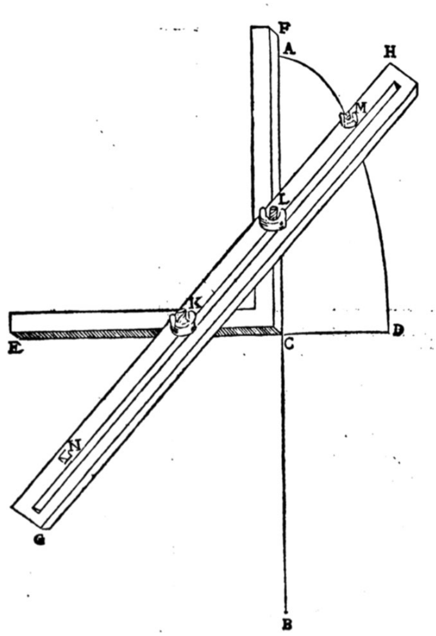


Fig. 8 dal Monte, *Planisphaeriorum theorica* (1579), 125

4.3 The instrument for tracing the ellipse

Guidobaldo concluded the treatise with a statement that marks a significant transition in the text.[52] Having laid the theoretical foundations, Guidobaldo now turned to the practical application of these principles. The focus shifts from the theoretical part, where the geometric principles underlying the construction of the ellipse were presented, to the practical part, where the application of these principles to the actual construction of the ellipse using a specific instrument is shown. Guidobaldo meticulously described the methodology of employing the instrument (a ruler with cursors and a square) to draw an ellipse.

A square FCE (with its right angle at C) is placed in C, the intersection of the axes. A ruler GH is employed, with cursors K, L, and M on it. A stylus is then laid in the point M, and the distance between the stylus and K is set equal to the semi-major axis AC, while the distance between L and M is set equal to the semi-minor axis CD (see Fig. 8).

Initially, the cursor K is positioned at C, the intersection of the axes. As a consequence, the stylus (point M) is located at A, one endpoint of the major axis. As cursor K moves along the side EC of the square, cursor L (constrained to move along the side of the square FC by the design of the instrument) shifts accordingly. This coordinated movement of the two cursors continues until cursor L reaches point C, the intersection of the axes; at that point, the stylus will reach point D, one endpoint

52 "His demonstratis, quomodo datis axibus ellipsis ellipsim lineare possimus, facile erit cognoscere; quodquidem, et regula cum suis cursoribus, veluti supra expositum est, atque norma, hoc modo assequemur" (dal Monte 1579, 124).

of the minor axis. It is thus evident that, through the coordinated movement of these two cursors, the stylus (placed in M) will trace one quadrant of the ellipse.

Guidobaldo concluded with practical recommendations: the importance of the square's appropriate thickness and the suitability of the materials used. He suggested using bronze or iron for the square and the ruler, and tempered iron or steel for the stylus, to preserve their form and ensure ellipses can be drawn easily and safely on any planisphere support, irrespective of its material composition.

4.4 The *Planisphaeriorum theorica* and the UCLA manuscript: a comparison of methods

A thorough examination of the *Planisphaeriorum theorica* reveals a profound interest in the ellipse and its properties on the part of Guidobaldo. Not satisfied with the traditional gardener's method, which relies on the identification of the foci, Guidobaldo undertook a rigorous theoretical analysis. This theoretical analysis led him not only to a pointwise construction of the ellipse, but also to the theoretical justification for his curve-drawing instrument. Consequently, it is evident that by 1579, Guidobaldo was familiar with several methods for rigorously drawing an ellipse. These include the gardener's method (based on the foci), a point-by-point construction (derived from his theoretical analysis), and continuous line drawing using a dedicated instrument (the outcome of his intellectual efforts). This range of approaches demonstrates his deep understanding and pursuit of various methods for precisely constructing ellipses for planispheres.

Given these various approaches, it is natural to ask why Guidobaldo continued his investigations. Having already devised a theoretical approach for constructing ellipses point by point and an instrument for continuous ellipse drawing, he continued his investigations, eventually documenting another method in his notes. This subsequent method was founded on the focal property of the ellipse.

The UCLA manuscript's construction, while requiring prior determination of the foci, is more immediate and intuitive than the point-by-point method described in *Planisphaeriorum theorica*. Whereas the latter relies on complex and likely obscure theorems, the manuscript emphasizes a fundamental property of the ellipse: the constant sum of the distances from the foci—a property that is both well known and easier to apply.

Guidobaldo himself noted the connection between the manuscript and the *Planisphaeriorum theorica*, a connection that remains significant regardless of whether he or Galileo authored the manuscript's construction. The comparative analysis of the *Planisphaeriorum theorica* and the UCLA manuscript thus highlights the complexity of Guidobaldo's thought, his multifaceted approach to the ellipse, and his ability to bridge theory and practice.

5 Guidobaldo's methods for constructing hyperbolas

Guidobaldo dal Monte explored hyperbola construction methods in the *Meditatiunculae de rebus mathematicis* with two distinct approaches, both based on Apollonius' *Conics* III.51[53]:

> Two methods for describing the hyperbola (in addition to those mentioned by Eutocius, Albrecht Dr̈er, and Commandino): one point-by-point, the other continuous.

The first is a mechanical method employing two different rulers. The longer ruler is fixed at point A, and the other at point B (the foci, though Guidobaldo uses Apollonius' terminology). The longer ruler exceeds the shorter one by a length AL (see Fig. 9) equal to the hyperbola's transverse axis, CG. Moving the rulers, keeping the remaining portions equal ($LD = DB$, $L'E = EB$ and so on), their intersection traces points on the hyperbola. The second approach, inspired by the gardener's method for ellipses, uses a string and a stylus. Fixing the string at points A and B such that the difference between AD and DB (or AE and EB, etc.) is equal to CG, and tracing with the stylus, Guidobaldo offered an intuitive, elegant solution for constructing a hyperbola. Neither approach includes a formal geometric proof.

While these methods demonstrate Guidobaldo's acumen in exploring different construction techniques beyond those of mathematicians like Eutocius, Dürer, and Commandino, their reliance on rulers, string, and stylus makes them impractical for contexts in which more precision is needed.

In the UCLA manuscript (f. 85r), a different method is described, more suitable, in our opinion, for the solution of the three circles problem (as we will show in the next section). Notably, a few folios later (f. 90r), there is a text extending *Meditatiunculae*'s discussion of the problem.

The presence of additional Galilean material within the UCLA manuscript, such as the proof concerning the centers of gravity of conical frustum (ff. 75–76), further contextualizes the significance of folio 85r. Galileo's correspondence with Guidobaldo confirms his interest in the solution to the three circles problem, thereby highlighting the importance of the UCLA manuscript's construction as a more effective approach to this geometric challenge.

[53] "Doi modi di descriver l'hiperbola (oltre a quelli che ha detto Eutocio, et Alberto Durero, et il Comandino) l'uno per punti, l'altro continuamente." *Par. Lat.*: 10246, 7–8. In a marginal note Apollonius' *Conics* I.21, Dürer's *Geometria* and Commandino's *De horologiorum descriptione* are cited as sources for his methods. Notably, these are the same references he cited in the *Planisphaeriorum theorica* when discussing the ellipse.

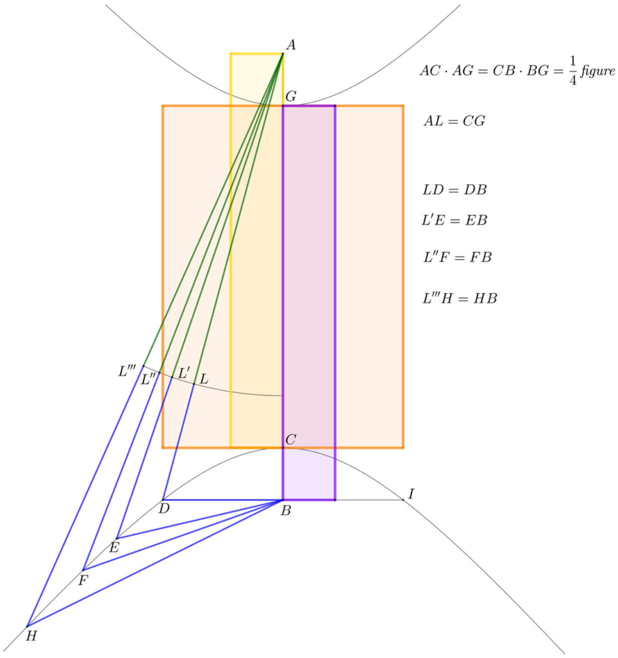


Fig. 9 The hyperbola in the *Meditatiunculae*

6 Constructing hyperbolas to solve the three circles problem

Guidobaldo dal Monte applied his knowledge of hyperbolas to solve a particular case of the more general Apollonian problem, known as three circles problem.[54] The particular problem, which involves constructing a fourth circle tangent to three given circles, received attention by Federico Commandino's Latin translation and commentary.[55] Guidobaldo's innovative solution involved the construction of two hyperbolas. By carefully analyzing the geometric properties of these curves and their intersections, he was able to determine the center of the desired fourth circle. This approach demonstrates Guidobaldo's deep understanding of conic sections and his ability to apply this knowledge to specific geometric challenges.

54 For a recent account, see Ciocci (2025). As said above (n. 9), the general problem is presented by Pappus in Book VII of his *Mathematical Collection*. It consists in determining, given in position any three elements among points, straight lines, and circles, a circle that passes through each of the given points (if any) and is tangent to each of the given lines or circles (see the description of the problem in (Pappus 1986, 91–93)). The variety of possible combinations among the three given elements (three points; three lines; two points and a line; etc.) gives rise to ten distinct propositions. Among these, the three circles problem consists of finding a circle tangent to three given circles, and, as Ciocci clearly shows, it is also connected to some propositions in Book IV of the *Mathematical Collection*.

55 "Sint tres circuli inaequales, qui sese contingant, et datas habeant diametros, quorum centra abc: et circa ipsos sit circulus contingens def, cuius oporteat diametrum invenire." (Pappus 1588, 44r).

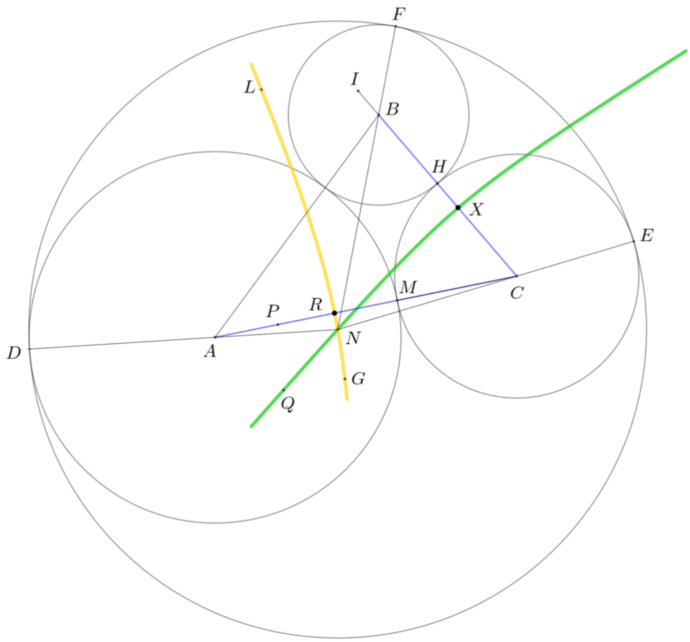


Fig. 10 Reconstructed figure of Apollonius' problem

6.1 Guidobaldo and the three circles problem

Guidobaldo's engagement with this problem was probably influenced by his correspondence with Commandino, who consulted him on the solution to the tenth proposition of the fourth book of the Pappus' *Collection*.[56] Guidobaldo examined the three circles problem in the *Meditatiunculae de rebus mathematicis* and in the UCLA manuscript.[57]

Our point is to show that the construction of the hyperbola of the UCLA manuscript was closely linked to the solution of the Apollonian problem proposed by Guidobaldo.

Consider three circles with centers A (the largest), B (the smallest), and C (see Fig. 10). The circles are given as mutually tangent, and so are the points of tangency H and M, where the circles centered at B and C, and at A and C, respectively, touch.

56 Ciocci (2025) gives a wider panorama of this problem in the Urbino school, showing the manuscript source and printed editions used by Federico Commandino and Guidobaldo dal Monte to solve the Apollonian problem.

57 There are two versions of the problem: one in which the three circles given are tangent to each other (pages 37–38 and 38bis of the *Meditatiunculae*) and another in which they are not (folio 90r of the UCLA manuscript). Roberta Tassora has identified a two-stage development in Guidobaldo's approach to the three circles problem. The first, more structured but older version, is found on pages 37–38 of the *Meditatiunculae* and is characterized by a limited number of corrections and a well-executed reference figure, probably drawn with a compass and ruler. The second, less refined but a more recent version, is found on page 38bis and in the UCLA manuscript. Despite its less polished appearance, the latter offers a more comprehensive solution, suggesting a connection between folio 38bis and folio 90r of the Los Angeles manuscript. This hypothesis supports the interpretation of the second draft as the more developed and final version of Guidobaldo's thinking on the subject. See Tassora (2001), 53–71, for a fuller exposition of the problem.

The goal is to determine the fourth circle tangent to the three given ones, by locating its points of tangency with them—namely, F, E, and D.

To this end, Guidobaldo introduced two hyperbolas, constructed through specific geometric steps so that the area of the rectangle formed by certain distances from the unknown center of the fourth circle equals a quarter of the Apollonian *figure*.[58] The intersection point of these two hyperbolas yields the center of the desired circle.

Guidobaldo did not limit himself to prove the existence of the required hyperbolas, but gave a construction based on the data of the problem: the first hyperbola is constructed by taking a point I such $HI = CH$, and a point X such that $HX = BI$; hence, $BI = CH - HB$ and $XB = HX + BH = HX + XC = HC$.

These relations imply that

$$\text{rect}(XC, CH) = \text{rect}(XB, BH)$$

, which is equivalent to stating that it is possible to find a hyperbola having C and B as its foci.[59]

In conclusion, it is possible to construct a hyperbola XNQ with vertex at X, and axis along HX, such that the area of the rectangle XC, CH (or XB, BH) equals a quarter of the *figure*.

The second hyperbola $GNLR$ is obtained in the same way with the other two circles. The point N, intersection of the two hyperbolas, is the center of the circle tangent to the other three.[60]

7 Comparing hyperbola constructions: UCLA and the three circles problem

Among the various techniques for tracing hyperbolas explored by Guidobaldo, the one preserved on folio 85r of the UCLA manuscript stands out for its direct applicability to the classical problem. It is precisely this construction that plays a pivotal role in Guidobaldo's resolution of the three circles problem, as we will now examine.

7.1 Constructing hyperbolas: a comparative analysis

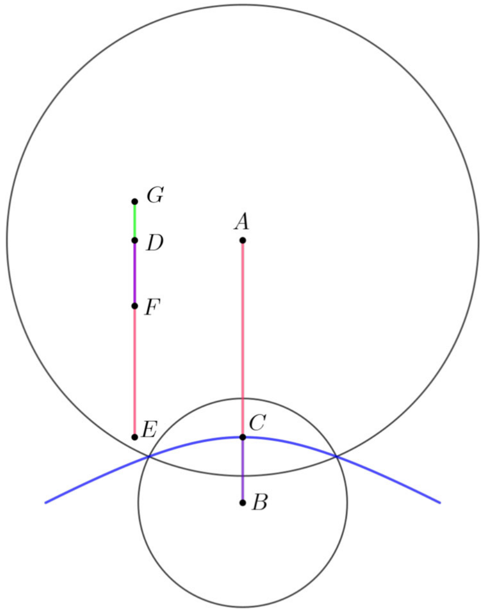
Guidobaldo's approach to the three circles problem shows clear affinities with the method outlined in the UCLA manuscript. In both cases, the hyperbola is obtained

58 In the Apollonian terminology, the *figure* (εἶδος) is the rectangle whose sides are the *latus transversum* and the *latus rectum*.

59 Guidobaldo, in this problem, also adopts Apollonius' terminology, without explicitly mentioning the foci.

60 For the proof, see *Par. Lat.* 10246, 37, 38, and 38bis, and UCLA 170/624, f. 90r. For an explanation of variations across different versions, see Tassora (2001), 56–67, and Ciocci (2025), §6. The key idea is to apply *Conics*, III.51, considering points C and B as the foci of one hyperbola, and C and A as foci of the other. Having located point N, it remains to prove that if lines are drawn from N to the centers of the three given circles (A, B, and C), and extended to intersect the respective circles at points D, F, and E, then the segments ND, NF, and NE are congruent. Consequently, the required circle is centered at N and has a radius equal to the length of any of these three segments.

Fig. 11 The construction of hyperbola (UCLA ms)



through pointwise construction, using the intersection of two circles. While the concept of focus is never explicitly invoked, the method is based on the Apollonian definition of the hyperbola, expressed through proportional relationships among segments and rectangles. A key difference lies in how the auxiliary segment is determined and how the excess added to the vertex–focus distance is handled. In the UCLA manuscript, the auxiliary segment EG is drawn independently, and the excess GD is freely chosen (Fig. 11). In the *Meditatiunculae*, by contrast, these elements are precisely determined based on the given conditions.

Consider, for example, the construction of the first hyperbola: the auxiliary segment is given by the distance between the centers of the two circles (C and B), while the excess BI, defined as the difference between the two radii ($BI = HC - HB$), is added to this segment (see Fig. 12). Similarly, for the second hyperbola, the auxiliary segment is the distance between the centers (A and C), and the excess (RM), the difference between the two radii ($RM = AM - MC$), is constructed directly on it (see Fig. 13).

In both cases, the points of the hyperbola are obtained as intersections of two appropriately defined circles. In the case of the hyperbola with vertex X, the points of the hyperbola correspond to the intersection of $\mathcal{C}(B, BX + BI)$, the circle with center B and radius $BX + BI$, and $\mathcal{C}(C, CX + BI)$; in the case of the hyperbola with vertex R, the points correspond to the intersection of the circles $\mathcal{C}(A, AR + RM)$ and $\mathcal{C}(C, CR + RM)$. Though the steps differ in formalization, the underlying geometric principle is shared and adapted to each context.

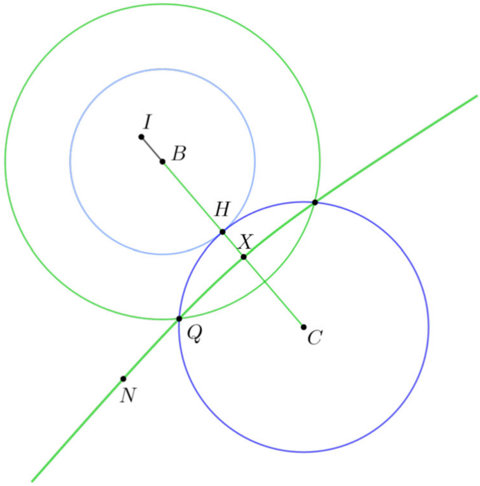


Fig. 12 The first hyperbola (*Meditatiunculae*)

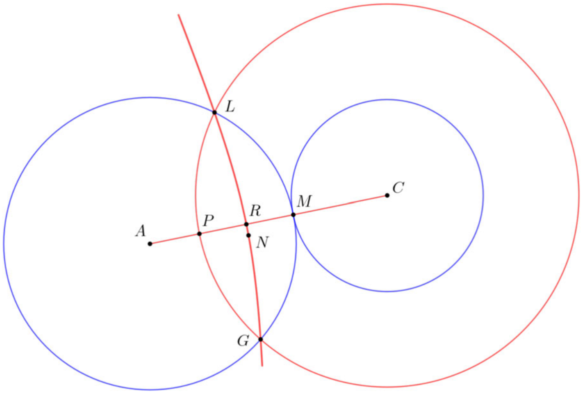


Fig. 13 The second hyperbola (*Meditatiunculae*)

7.2 Reinterpreting the UCLA construction through the lens of the Apollonian problem

Despite their formal differences, the UCLA construction and Guidobaldo's solution to the three circles problem share a common geometric foundation. Both define the hyperbola by a vertex, two foci, and a fixed excess added to the vertex–focus segment.

If we reinterpret the UCLA construction with point C as the vertex and the difference $AC - BC$ as the axis, choosing G so that GD equals the *excessus* (i.e., the distance $AC - BC$, which in the three circles problem corresponds to $BI = HX = XC - BH$), we obtain a structure that mirrors the one employed in the problem.

In both cases, the construction is based on the intersection of two circles centered at the foci. In the UCLA folio, the radii of these circles are defined as the vertex–focus distance (AC or BC) increased by the excess GD; in the problem, the radii are given by the vertex–focus distance (CX or BX) increased by the excess $BI = HX$.

The main difference lies in the greater explicitness of the *symptoma* (the condition that defines the hyperbola)[61] in the *Meditatiunculae*. Additionaly, the two texts differ in their initial conditions: in the *Meditatiunculae*, one of the two circles used to construct the hyperbola is already given in the problem's hypotheses. This significantly simplifies the construction, as only one additional circle must be drawn to determine each point of the curve. By contrast, in the UCLA manuscript, both circles are constructed from independently chosen data, reflecting a more general and exploratory procedure.

The UCLA folio, while more general and less formalized, relies on the same constructive logic and may be seen as a reusable geometric tool. Guidobaldo's adaptation of it to the specific constraints of the three circles problem illustrates how abstract constructions could be translated into effective problem-solving strategies.

8 Guidobaldo, Galileo, and the international mathematical community

Guidobaldo dal Monte's engagement with the three circles problem took shape within a broader network of intellectual exchange, both local and international. As we said, early studies on the topic likely began during his collaboration with Federico Commandino in the 1570s and are documented in the *Meditatiunculae*. These early investigations were later revisited and developed further in 1588, when Galileo Galilei explicitly asked Guidobaldo for assistance.[62] In a letter dated 16 September 1588, Guidobaldo recalled having previously found a solution and promised to send it, despite the disarray of his papers: a few weeks later, on 7 October, he confirmed that he had recovered the solution and sent it to Galileo.

This international network of mathematicians included figures such as Michel Coignet, with whom Galileo corresponded about various geometric problems,[63] and Adriaan van Roomen, who dedicated his *Problema Apolloniacum* (1596) to the Augustinian archbishop Angelo Rocca,[64] and sought the judgment of experts such as Clavio, Magini, and Guidobaldo dal Monte. Van Roomen's work on the three circles problem, particularly his use of conic sections, demonstrates the common interest in this topic among mathematicians of the time. Although both Guidobaldo and Van Roomen employed conic sections to solve the problem, their approaches differed in several key respects.

Van Roomen's *Problema Apolloniacum* provides a comprehensive treatment of the three circles problem, examining all possible configurations of the three given

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[61] "Secetur deinde hc in x, ita ut hx sit aequalis bi [excessus quo ch superat hb], unde erit xc aequalis hb, deinde a puncto x describatur hyperbole xnq, ita ut xh sit axis, et quartae parti figurae sit aequale utrumque rectangulorum xch et xbh.", *Par. Lat.* 10246, 37.

[62] See Sect. 2.

[63] Coignet to Galileo, 31 March 1588 (Galilei 1891-1909, X, 31–33).

[64] "Perillustri ac reverendissimo Domino F. Angelo Rocca a Camerino, Eremitae Augustiniano, sacrae theologiae doctori eximio et sacrarii Apostolici Antistiti, domino suo plurimum colendo." (Van Roomen 1596, 3).

circles.[65] He concluded his demonstration with an observation on the construction of conics, emphasizing the variety of methods proposed by different authors and the need to consult specialized works.[66] For example, he quotes the theorems III.51 and III.52 of Apollonius' *Conics* to draw hyperbolas and ellipses. He also mentions that he was aware of another demonstration of Apollonius' problem, attributed to an assistant of Viète, but that he had not included it due to lack of time.[67]

The exchange between Guidobaldo and Galileo, as well as the wider network of mathematicians involved in the three circles problem, illustrates the dynamic and interconnected nature of mathematical inquiry during the Renaissance. The shared interest in conic sections, as evidenced by Galileo's work on the center of gravity of solids and Van Roomen's use of conic sections to solve the three circles problem, underlines the importance of these curves in the development of geometry and physics.

9 Galileo and Guidobaldo dal Monte: a shared interest in conic sections

Although Galileo Galilei left no systematic treatise on the construction of conic sections, his deep-rooted interest in their properties and terminology is evident throughout his writings. His early studies on the centers of gravity of solids of revolution—later published as an appendix to the *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*—testify to his mastery of figures such as paraboloids, ellipsoids, and hyperboloids,[68] whose geometry proved essential to his later work in mechanics and physics.

Already in the *De motu antiquiora*, Galileo showed a strong grasp of conic section theory. In arguing against Aristotle's notion of an infinite increase in speed, he drew an analogy between natural motion and the asymptote of a hyperbola,[69] extending the notion of asymptotes beyond conic sections to a more general class of curves. As

65 Adriaan van Roomen challenged the mathematical community with his *Ideae mathematicae* (1593) which proposed a complex problem requiring the solution of a 45th degree equation. François Viète found a solution of the problem and published it: *Ad problema quod omnibus mathematicis totius orbis construendum proposuit Adrianus Romanus Francisci Vietae responsum* (1595); in the last page of this book (f. 14v) Viète added another challenge by presenting the famous Apollonius' problem further stimulating the debate among mathematicians: "Porro ad exercendum …studiosorum ingenia Problema …subiicio. Datis tribus circulis, quartum circulum eos contingentem describere". Van Roomen, for his part, devoted himself to the problem of Apollonius, proposing an innovative solution in the *Problema Apolloniacum* which, in Viète's opinion, did not respect the canons of classical geometry, since it relied on the intersection of second-degree curves such as hyperbolas. Dissatisfied, Viète published his solution as *Apollonius Gallus* (1600).

66 "Conicas porro sectiones ducendi ratio varia a variis traditur, ideo ratio earundem describendarum ex authoribus qui de Conicis sectionibus egerunt, petenda est. Nobis namque sufficit problema solvisse. Interim tamen haec de conicis sectionibus theoremata subdere visum est, quae ingenioso lectori viam praebere poterunt ad ducendas eas sectiones." (Van Roomen 1596, 18).

67 "Interim hoc Apolloniacum problema animi mei declarandi ergo construxi. Demonstrationem auctarii tui adiecissem, si per temporis brevitatem licuisset." (Van Roomen 1596, 18).

68 In a letter dated 16 July 1588, Galileo promised to send Guidobaldo a demonstration of the center of gravity of an obtuse-angled conoid (Galileo to Guidobaldo, 16 July 1588, Galilei (1891-1909), X, 36).

69 "Verum quod, et si semper celeritas intenderetur et infinitum esset spatium motus, non tamen sequeretur, motum tandem ad infinitam celeritatem devenire et mobile ad infinitam gravitatem, non difficile erit eis intelligere, qui in mathematicis versati fuerint. Simile enim hoc est ei quod illis fere omnibus impossibile

an example, he compared the hyperbola to the conchoid of Nicomedes, described by Eutocius in his commentaries on Archimedes' *On the Sphere and Cylinder*.[70]

> That such lines exist, everyone knows who has encountered either the asymptotes of the hyperbolas in the *Conics* of Apollonius of Perga, or the first conchoid line of Nicomedes, in Eutocius of Ascalon's *Commentaries* on the second book of the inimitable Archimedes' *On the Sphere and Cylinder*. These are two lines (and many others could also be imagined) which, extended to infinity, continually draw nearer yet can never meet; their distance steadily decreases, but is never entirely eliminated.[71]

Whereas Greek mathematicians defined curves through specific geometric or mechanical constructions, Galileo grasped the more general principle underlying the asymptote, even likening that of the conchoid to the fixed ruler used in its construction—an analogy absent in Eutocius. This conceptual approach to conic curves highlights his attention to their mathematical properties. A further instance of Galileo's engagement with conic terminology appeared in the 1615 *Risposta alle opposizioni del S. Lodovico delle Colombe e del S. Vincenzio di Grazia*, largely written by Galileo and signed by his student Benedetto Castelli,[72] In defending the classical names of the conic sections—parabola, hyperbola, and ellipse—he mocked his opponents' ignorance and praised the mathematical rigor of the ancients[73]:

> But even if one wished to stir up a dispute over names, Signor Grazia should not have debased the profession he claims as a philosopher, and ought to have

Footnote 69 continued
videtur, qui demonstrationis non sono capaces: quod, scilicet, inveniri possint duo lineae, quae, in infinitum protractae, semper appropinquentur, nunquam tamen concurrant; ita ut distantia, quae inter eas est, semper in infinitum minuatur, nunquam tamen absumatur." (Galilei 1891-1909, I, 330).

70 Tartaglia also knew this curve and translated Eutocius' description into the vernacular in the fifth part of his *General trattato* adding explanatory remarks and slightly modifying the figures. See Tartaglia (1560), 47v–50r.

71 "Verum tales lineas dari, omnes norunt qui aut in asymptotos hyperboles in *Conicis* Apollonii Pergaei, aut in primam lineam conchoidem Nicomedis, apud Eutocium Ascalonitam in *Commentariis* super librum secundum inimitabilis Archimedis *De sphaera et cylindro*, inciderint: sunt enim hae duae lineae (et multae etiam aliae excogitari possent), quae, in infinitum protractae, semper magis accedunt, verum ut aliquando concurrant impossibile est; minuitur ergo semper eorum distantia, nunquam tamen absumitur." (Galilei 1891-1909, I, 330–331).

72 On the hydrostatic controversy sparked by Galileo's *Discorso delle cose che stanno in su l'acqua*, see Camerota (2004), 227–238.

73 "Ma quando pur sopra i nomi si avesse a suscitar contesa, non doveva il Sig. Grazia abbassar tanto la profession che e' fa di filosofo, ma lasciar tal lite a' grammatici. Ben è: stata ventura di Archimede e d'Apollonio Pergeo, che il Sig. Grazia non si sia incontrato ne i nomi che loro imposero a tre delle sezzioni coniche, chiamando questa parabola, quella iperbole, e quell'altra ellipsi, perchè, avendo egli forse saputo che questi prima furon nomi di figure retoriche che di figure matematiche, ne arebbe loro conteso l'uso. Aggiugnasi di più, che di queste definizioni veramente il Sig. Grazia non ne ha intese nissuna, e perciò forse le ha volute rimutare, ed aggiugnendo errore sopra errore gli è parso poi che il Sig. Galileo non ritrovi nè i veri sintomi ne le buone dimostrazioni; come accaderebbe a quello che prima dannasse Euclide del chiamar cerchio quel che egli vuol nominar triangolo, e triangolo quello che egli vuol chiamar cubo, e poi dicesse che le passioni dimostrate da Euclide ne' cerchi, ne' triangoli e ne' cubi fosser tutte false, e le dimostrazioni difettose, consistendo veramente tutto 'l male nella sua gravissima ignoranza." (Galilei 1891-1909, IV, 698).

> left such quarrels to the grammarians. It was indeed fortunate for Archimedes and Apollonius of Perga that Signor Grazia did not happen upon the names they gave to three of the conic sections—calling one a parabola, another a hyperbola, and the third an ellipse—for had he known that these were originally names of rhetorical figures rather than mathematical ones, he would no doubt have disputed their use. Moreover, it must be added that Signor Grazia did not understand any of these definitions, and perhaps for that very reason sought to alter them. And by adding error upon error, he concluded that Signor Galileo failed to recognize either the true symptoma or sound demonstrations—just as would happen with someone who first condemned Euclid for calling a circle what he himself wished to call a triangle, and a triangle what he preferred to call a cube, and then went on to claim that all the properties Euclid demonstrated in circles, triangles, and cubes were false, and the demonstrations defective—when in truth the entire fault lies in his own profound ignorance.

This passage not only illustrates Galileo's polemical wit, but also confirms his precise knowledge of the classical definitions of conic sections and his commitment to preserving their terminological and conceptual integrity. Such mastery of the theoretical aspects of conics found a natural counterpart in his exchanges with Guidobaldo dal Monte, whose work often combined geometric theory with practical experimentation. Their shared interest in conic sections thus extended beyond terminology, re-emerging in collaborative studies on problems—such as the motion of projectiles—in which the geometry of curves played a central role.

Around 1592, this collaboration gave rise to pivotal developments in experimental mechanics.[74] Their joint experiments on inclined planes and projectile trajectories, documented in Guidobaldo's *Meditatiunculae* and later incorporated into Galileo's *Discorsi*, provide valuable insights into their approach to scientific investigation. These experiments, using suspended chains and inclined planes to trace parabolic curves, highlight the importance of empirical observation combined with theoretical analysis. By integrating theoretical insights with experimental evidence, Galileo and Guidobaldo made significant contributions to the study of conic sections and their applications in physics. Beyond these specific achievements, their shared mathematical interests extended to other topics, most notably the Archimedean screw (*coclea*).

Two letters from Guidobaldo to Galileo attest to this dialogue. In the first, dated 3 August 1589, Guidobaldo mentioned some preliminary notes—*poche cosette sopra la cochlea*—which he intended to copy and send because of the many corrections. In the second, dated 10 April 1590, he reported having found further results that were not yet well organized, but that he hoped to submit for Galileo's opinion.

Material on the *coclea* appears in the *Meditatiunculae* in both Latin and vernacular. The heavily corrected Latin notes (pp. 57–58) likely correspond to the *poche cosette* mentioned in 1589, while a later, cleaner version (p. 134) may represent a fair copy of those earlier drafts.

The topic was one that Galileo further developed in his own work. His treatment of the Archimedean screw appears in a dedicated chapter of the *Le mecaniche*, composed

[74] For the dating of the collaboration between the two scholars in conducting mechanical experiments, see the extensive discussion in Renn et al. (2000), 324–336.

in the early 1590s,[75] reinforcing the impression of an ongoing mathematical dialogue between the two scholars, with ideas and problems circulating between manuscripts and letters.

Such exchanges reflect not only their common interest in specific problems, but also a collaborative method grounded in correspondence and mutual critique. The examples of centers of gravity, conic sections, and the Archimedean screw vividly demonstrate how these shared mathematical pursuits, often involving the concrete construction of curves and instruments, fueled their intellectual exchange. This dynamic, which bridged theoretical reflection and practical construction, led to both innovative solutions in geometry and significant applications in mechanics and physics.[76]

10 Conclusions

The research of Guidobaldo dal Monte and Galileo Galilei on conic sections is a fundamental chapter in the history of mathematics and science in the sixteenth and seventeenth centuries. Both scientists showed a deep interest in these curves for their intrinsic geometric properties and for their applications in various fields, from mechanics to astronomy.

Folio 85r of the UCLA manuscript, attributed to Guidobaldo dal Monte, presents a series of geometric constructions of conic sections. The construction of the hyperbola, in particular, could be connected to the solution of the three circles problem, discussed in their correspondence. The presence of these notes among Guidobaldo's papers suggests a personal deepening of the subject by the scientist, who went beyond mere practical applications, such as the construction of astronomical instruments, and delved into more complex geometrical problems, such as the Apollonian problem.

The enigmatic note on the same folio "Del Galileo" testifies to the lively scientific exchange between the two scholars, which was particularly intense from the late 1580s onward. Although it is not possible to establish with certainty the exact origin of this note, it is undoubtedly true that both scientists were interested in the geometrical properties of conics and that their research influenced each other.

While Galileo's early research, as evidenced by the *De motu antiquiora* or the *Theoremata*, shows a solid mastery of the properties of conics, his primary interest was in the physical applications of these curves. This focus on physics continued into adulthood, as demonstrated by his use of the parabola to describe the motion of projectiles. Guidobaldo dal Monte, on the other hand, while recognizing the practical usefulness of conics, delved deeper into the geometric aspects of these curves.

In conclusion, the collaboration between Guidobaldo dal Monte and Galileo Galilei was a fundamental moment in the development of the study of conic sections. Their research not only deepened the understanding of the geometric properties of these curves but also paved the way for new applications in various fields of science. Folio 85r of the UCLA manuscript, with its original constructions and the enigmatic annotation

[75] See Galilei, *Le Mecaniche* in Galilei (1891-1909), vol. II, 186–187.

[76] Tassora has identified additional themes beyond mathematics—such as physical and methodological concerns—that suggest further areas of intellectual exchange between Guidobaldo and Galileo. See *Galileo e Guidobaldo: suggestioni galileiane nelle Meditatiunculae* (Tassora 2001, 173–186).

"Del Galileo", provides a valuable starting point for further investigations into the nature of this collaboration and its impact on the history of mathematics.

11 Appendix: the text of the UCLA manuscript

11.1 Two versions of the construction of the parabola

The UCLA manuscript contains two different versions of the construction of the parabola (and their diagrams). One version is drawn on a leaflet (folio 85bis recto and 85bis verso) attached to the left margin of the folio, covering the text of the hyperbola construction; the other version follows the proof of the ellipse and hyperbola in folio 85r.

The text of the two versions is essentially identical, except for the following details:

- The description in 85r begins with the title *La parabola*, which is missing in the leaflet.
- In the leaflet, the line *iuxta qua possunt* is not defined as *lato retto*, as in 85r.
- In the construction of the leaflet, it is mentioned twice to draw a *semicircle*, while in 85r it is a *circle*. This peculiarity can also be seen in the two diagrams accompanying the proof: in 85bis recto there are signs of tracing a semicircle, resulting in a semiparabola; in 85r, there are signs of tracing a full circle, resulting in a complete parabola.
- In 85r, the short proof is introduced by the expression *La dimostrazione è questa*, absent in the leaflet version.
- The leaflet contains some abbreviations which are missing in 85r (*Ap* for Apollonius, *parab* for parabola).
- The quotation from Apollonius in the leaflet, originally written at the beginning of the proof, is deleted and rewritten at the place where it is reported, without corrections, in 85r.

It is possible to hypothesize the succession of the two versions. Indeed, some of the corrections made in the interline of the leaflet (such as the phrase *et uno sia* or the quotation from Apollonius, which was first deleted and then moved to the end of the proof) find their final place in the second version on folio 85r, suggesting that the latter is a more accurate and complete revision of the former. It is more difficult, however, to explain the reason why Guidobaldo produced two versions. The construction of the parabola, which is certainly simpler and more intuitive than that of the other conic sections, and placed last, suggests that Guidobaldo decided to include it in addition to the other two for the sake of thematic completeness, to have the constructions of the three conic sections in a single folio. For this reason, the version originally sketched, somewhat carelessly in some points, was copied on the same folio where the ellipse and the hyperbola were present, resulting in a more accurate revision.

The text of the version of the parabola construction contained in the leaflet, together with its diagram, is published as an appendix to the transcription of the three conic sections of 85r, giving the reader the opportunity to compare the two versions directly.

11.2 Editorial criteria

The original spelling of the manuscript has been retained for technical terms such as "hiperbola" and "hyperbola" in the Italian text. Modern orthography is used for all other words; specifically, "cioè" is used throughout, not "cio'e" or "cio è". Linguistic abbreviations used in the manuscript have been written out in the transcription. The abbreviated form "11.^{a}" and "p.^{o}", present in the only specific citation, has been retained, as it is a commonly used abbreviated form for *undecima* and *primo*. The letters denoting geometric elements, which appear in lowercase in the manuscript, have been rendered in uppercase in both the transcribed text and the figures for the convenience of the reader and to conform to modern usage. The text is divided into numbered paragraphs set in smaller, boldface type.

Two versions of the parabola construction appear on folio 85r.[77] The earlier version is on a small, attached leaflet. Some words on this leaflet are illegible due to ligature. This version, nearly identical to the first version, is transcribed in the appendix.

The UCLA manuscript exhibits some erasures and imperfections; significant instances have been noted.

The hyperbola diagram is partially obscured by the attached leaflet. The obscured segment has been reconstructed using a dashed line. All other diagrams have been reproduced faithfully.

11.3 Sources and internal references

The UCLA manuscript contains two mathematical references to other works:

1. *Che si cava dal Planispherio*;
2. *per la 11.*^{a} *del p.*^{o} *di Apollonio*.

The first reference is to Guidobaldo's *Planisphere*, mentioned in connection with the ellipse but without a precise reference.[78] The second refers to proposition 11 of the first book of Apollonius' *Conics*, where the *symptoma* of the section is obtained,[79]

Sigla

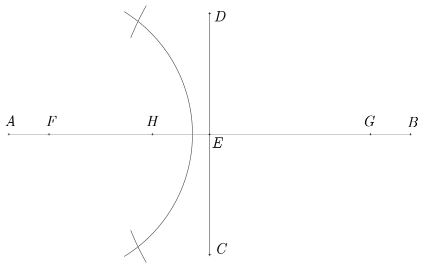
U Los Angeles, Charles E. Young Research Library, University of California-Los Angeles, MS 170/624, folio 85
canc. deleted
corr. corrected
[ ] expunction
< > integration

[77] See Sect. 11.1.

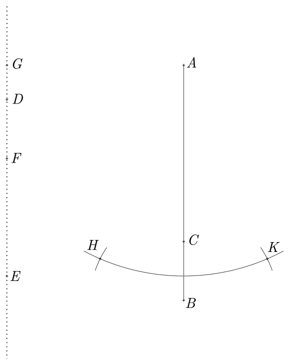
[78] See dal Monte (1579), 103–104 (103 misprinted as p. 107 in the original). This quotation is discussed in detail in Sect. 4.

[79] See Apollonius *Conics* I.11, ed. Heiberg (Apollonius 1974, 36–39).

Critical edition of the text in the UCLA manuscript



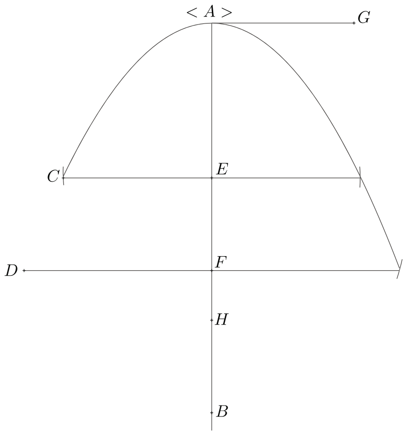
1 Siano i diametri dell'ellipse AB CD che si seghino in E. Trovati i punti F G e che la FD sia eguale alla AE, si fanno diversi punti a caso tra F G e ne sia uno H. E si pigli lo spatio AH e con il centro F si descriva una circonferenza. **2** Poi si pigli lo spatio BH e col centro G si descriva un'altra circonferenza che seghi la prima nei punti K L, li quali sono nell'ellipse. Che si cava dal Planispherio.



3 Nella linea AB segnati li punti A B et il C, dove deve esser il vertice dell'hiperbola, si facci un'altra linea nella quale si segni il punto D, et alla AC si facci eguale la DE et alla BC la DF. **4** Li punti F E reggano ogni cosa imperoché sopra il punto D se ne[80] facino molti, dei quali uno sia G. E si piglia EG e nel centro A si descrive una circonferenza, poi si piglia FG e con il centro B

[80] se ne *corr. U*.

se ne descrive un'altra che seghi la prima in H K, li quali sono nell'hyperbole.



La parabola

5 Sia il diametro della parabola che si vuol descrivere AB e sia la AG il lato retto, cioè la linea (iuxta quam possunt); si piglino nella AB quanti punti si vogliano, et uno sia E. E si facci EC perpendicolare alla AB, poi si facci EH eguale alla AG e fatto diametro AH si descriva il circolo il qual seghi EC in C. Et il punto C è nella parabola.

6 Similmente si pigli il punto F dal qual si tiri all'AB la perpendicolare FD. Poi si facci FB eguale alla AG e fatto diametro AB si descriva il circolo il qual seghi la FD in D. Et il punto D sarà medesimamente nella parabola. La dimostrazione è questa: che essendo la CE media proporzionale fra AE et EH, cioè fra AE et AG, il quadrato di CE sarà[81] eguale al rettangolo GAE.

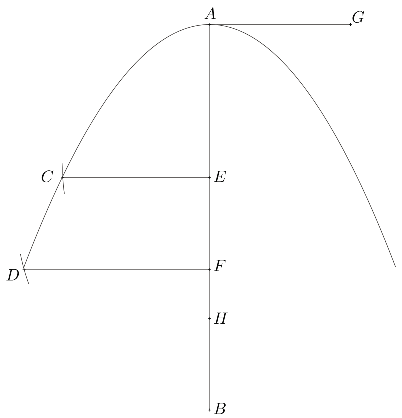
7 Il punto adunque C è nella parabola per la 11.^{a} del p.^{o} di Apollonio. E così si dimostreranno gl'altri punti esser nella parabola come è il D etc.

[verticali perché tutti passano per il vertice che è il Zenit]

del Galileo

81 CE sarà *corr. U*.

Construction of the parabola, version from the leaflet



8 Sia AB il diametro della parabola che si vuol descrivere e sia la AG la linea (iuxta quam possunt), si piglino nella AB quanti punti si vogliano, et uno sia[82] E; e si facci EC perpendicolare alla AB, poi si facci EH eguale alla AG e fatto diametro AH si descriva il semicircolo il qual seghi EC in C. Il punto C è nella parabola.

9 Similmente si pigli un altro punto F dal qual si tiri alla AB la perpendicolare FD poi si facci FB eguale alla AG e fatto diametro AB si descriva il semicircolo il qual seghi la FD in D. Il puncto D sarà medesimamente nella parabola.

10 È[83] perché essendo la CE ⟨media⟩ proportionale fra AE et EH, cioè fra AE et AG adunque il quadrato di EC[84] è eguale ⟨al⟩ rettangolo[85] GAE. Il punto adunque C è nella parabola per la 11ª del primo d'Ap⟨ollonio⟩ e così si dimostreranno gl'altri punti essere[86] nella parab⟨ola⟩ come è il D.

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[82] et uno sia *corr. da* come EF *U*.

[83] *prima di* È *canc.* La dimostrazione è per la undecima del p.º di Apollonio *U*.

[84] EC *corr. U*.

[85] *prima di* rettangolo *canc.* q.to di *U*.

[86] *prima di* essere *canc.* avere *U*.

Declarations

Conflict of interest The authors have no financial or proprietary interests in any material discussed in this article.

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