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Closed-Form SWCC Functions for Granular Mixtures

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Abstract. Four closed-form water-retention functions (the Gardner, Fredlund – Xing, van Genuchten 4 and 5) were validated using the experimental, ideal water retention curve data of 7 granular fractions (fine gravel, sand and silt) and their 21 mixtures with fractal distribution (being similar to the Fuller curves), composed from 2 to 7 fractions. The measurement was approximate since some sand and kaolinite boxes designated for some fixed suction loads were used being proper for more plastic soils. The homothetic, fractal grading curves with N = 2, 3, ...7 fractions, resp.; resulted in homothetic SWCC-s. A data mining procedure was used to assess the precise shape. Then the discrimination study showed that the 5-parameter model of van Genuchten was the best and the Fredlund-Xing modelwas the secdond best. The results showed some interdependence of the parameters. Discussing the results by considering more complicated grading curve samples, the double Weibull function was found to be better for SWCC-s of gap-graded grading curves.

1 Introduction

The aim of the ongoing research is to study the relation between the soil water characteristic curve (SWCC) and the grading curve of granular matter. In the first stage of this researcg, 28 special granular mixtures with fractal distribution were tested and a model discrimination was made to select the best closed-form parametric function, after data mining.

In the experimental part of the research ([1] to [4]), the soil water characteristic curve of 7 sand fractions and 21 sand mixtures (with fractal distribution, defined by normalised entropy coordinate A=2/3) was determined with a fixed measuring system which gave approximate result only. After a data mining, the ideal homothetic, fractal grading curves resulted in homothetic retention curves, which contained a near linear part, with slope depending on the number of constituting fractions.

The processed data were used to discriminate five closed-form models [5] to [9], the model of Gardner [5], the non-simplified and simplified model of Van Genuchten [6] and, the two model-versions of Fredlund – Xing [7 to 9].

The well-known non-linear minimization methods find a local minimum of the merit function and, the global minimum is searched with a trial and error procedure which is time consuming for parameter numbers greater than 2 ([10]). To save time and to simplify the fitting of the various models, an automatic non-linear minimization method was elaborated, valid for nice (eg., convex) merit functions. It was used for the fast determination of the approximate solution of 140 inverse problems with reliability testing ([11 to 13].

Table 1. Formatting sections, subsections and subsubsections.

J	Grain size d (mm)	Soil mechanical name
1	0.03 – 0.06	silt
2	0.06 – 0.125	fine sand
3	0.125 – 0.25	fine sand
4	0.25 – 0.50	medium sand
5	0.50 – 1.0	coarse sand
6	1.0 – 2.0	coarse sand
7	2.0 – 4.0	fine gravel

2 Materials and methods

2.1 Creating the database

2.1.1 Fractions and mixtures, measurements

The fractions are numbered by integers (Table 1). The mixtures are numbered by two integers NJ, N being the number of the fractions, J being coarsest fraction, respectively. In the first stage of research, 7 fractions and 21 continuous mixtures (Table 2) were tested.

The samples were prepared in the loosest possible dry state then were saturated with distilled water from above. Two sample heights were used (5 cm and, 2.5 cm) in two repetitions. The soil water characteristic curve was determined according to the method used in the Research Institute for Soil Science ([14 to 15]). The method entailed fixed loads. In the low suction range of u_a-u_w ≤ 50 kPa sand boxes were used and, the suction

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was applied by water pressure decrease. Hanging water column or vacuum was applied (Table 3). In the higher suction range (u_a-u_w > 50 kPa) pressure membrane extractor was used. In later stages some tests were repeated, suction load steps were increased and different eg., gap-graded samples were also tested.

Table 2. Notation for mixtures (*NJ, N* is equal to 2 to 7)

Largest fraction *J*	Mixture notation *NJ*	Fractions in the mixture	Fraction number *N*
2	22	1-2	2
3	23	2-3	2
4	24	3-4	2
5	25	4-5	2
6	26	5-6	2
7	27	6-7	2
3	33	1-2-3	3
4	34	2-3-4	3
5	35	3-4-5	3
6	36	4-5-6	3
7	37	5-6-7	3
4	44	1-2-3-4	4
5	45	2-3-4-5	4
6	46	3-4-5-6	4
7	47	4-5-6-7	4
5	55	1-2-3-4-5	5
6	56	2-3-4-5-6	5
7	57	3-4-5-6-7	5
6	66	1-2-3-4-5-6	6
7	67	2-3-4-5-6-7	6
7	77	1-2-3-4-5-6-7	7

Table 3. The load steps at low suctions

Load step (kPa)	Material for the semi-permeable membrane	The method of the suction load application
0.1	Sand	Gravitational
0.25	sand	Gravitational
1	fine sand	Gravitational
3.15	fine sand	Gravitational
10.0	Kaolinit	Gravitational
20.0	Kaolinit	Gravitational
50.1	Kaolinit	Vacuum

2.1.2 Data mining

The load steps were fixed for each sand box ([15, 16]). The applied load steps were not proper for determining the precise shape of the retention curves. Therefore, a data mining process was made by a simple model based on the following observation. The raw data showed that the homothetic grading curves gave homothetic retention curves, with a near linear part, with slope depending on the number of constituting fractions.

A simple, tri-linear retention curve model was adopted for fraction *i*, with two parameters at the breaking points, at the air entry suction $s_{i,1}$ and at the residual suction $s_{i,2}$. The parameters were generally interdependent.

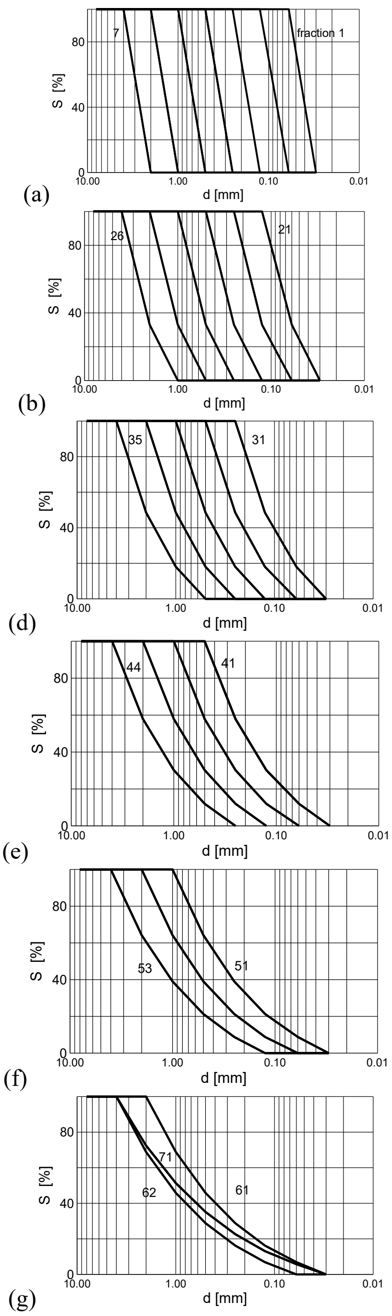

Fig. 1. (a) to (g) Theoretical grading curves. Fractions. 2 – 3 – 4 – 5 – 6 and 7 – component mixtures.

The air entry suction of fraction *i*+1 was equal to the residual suction of fraction *i* as follows:

$$s_{i+1,2} = s_{i,1} \tag{1}$$

except that a gap was between fractions 3 and 4:

$$s_{4,2} > s_{3,1} \tag{2}$$

and the last parameter was 100 kPa.

The retention curve of the mixture (w_{NJ}) was the weighted sum of the retention curve of the fractions (w_k where *k* = *J*-*N*+1..*J*) with equal weights 1/*N*:

$$w_{NJ}(s) = \sum_{J-N+1}^{J} \frac{1}{N} w_k(s) \tag{3}$$

(1)

2.2 Models and model fitting

2.2.1 Models

The tested models, by using both notations, the original form and the parameters used in the software, resp., were as follows. The Gardner (1958) equation:

$$w_G = w_r + \frac{w_s - w_r}{1+\left(\frac{s}{a}\right)^n}; w_G = w_r + \frac{w_s - w_r}{1+Cs^A} \quad (2)$$

where w is volumetric water content, s is suction and, the parameters are : w_r, w_s, a, n and w_r, w_s, C, A. The van Genuchten equation (1980) and the simplified form:

$$w_{v4} = w_r + \frac{w_s - w_r}{\left(1+\left(\frac{s}{a}\right)^n\right)^m}; w_{v4} = w_r + \frac{w_s - w_r}{\left(1+Cs^A\right)^B} \quad (3)$$

$$w_{v5} = w_r + \frac{w_s - w_r}{\left(1+\left(\frac{s}{a}\right)^n\right)^{\left(1-\frac{1}{n}\right)}}; w_{v5} =_r + \frac{w_s - w_r}{\left(1+Cs^A\right)^{\left(1-\frac{1}{A}\right)}}$$

(4)

where w_r, w_s, a, n, m and w_r, w_s, C, A, B are parameters.

The simple and the modified models of Fredlund – Xing (1994):

$$w_F = w_r + \frac{w_s - w_r}{\left\{\ln\left[e+\left(\frac{s}{a}\right)^n\right]\right\}^m}; w_{Fm} = w_r + \frac{w_s - w_r}{\left\{\ln\left[e+\left(\frac{s}{C}\right)^A\right]\right\}^B} \quad (5)$$

$$w_{FFm} = \left[1 - \frac{\ln\left(1+\frac{s}{s_r}\right)}{\ln\left(1+\frac{1000000}{s_r}\right)}\right] w_{Fm} \quad (6)$$

where w_s, s_r, a, n, m and w_r, w_s, C, A, B are parameters, the completed model described by Eq 8 gives zero at 1,000,000 kPa suction.

2.2.2 Model fitting, reliability tests, model discrimination

The dependence of the models on the parameters was partly linear (w_r, w_s), partly non-linear (a, n, m). The model fitting method was based on the assumption that the merit function was convex**.**

Geometrical exploration of the merit function was made. The number of the function value evaluations increased exponentially with the non-linear parameter number. The linearly dependent parameters were eliminated by sub-minimization.

The deepest section of the merit function concerning every non-linearly dependent parameter was constructed. This was called as minimal section of the merit function concerning parameter p_i, defined as the lowest section of the merit function with respect to p_i (Fig. 2).

The minimum was selected on these sections which were also used for reliability testing. The real-life LS merit function with the measured data was normalised by the largest measured data to compare the results of the various samples at the minimum of F:

$$F(\underline{p}) = \frac{\left\| w_{me}(s_i) - w(s_i, \underline{p}) \right\|}{\max_i \left(w_{me}(s_i)\right)} \quad (7)$$

where $F(\underline{p})$ is merit function, the 2-norm of the $\boldsymbol{h}(\boldsymbol{p})$ error vector, where w and w_{me} were computed and measured water content, $\underline{p}$ was parameter vector consisting of model parameters, s_i are the suction load steps. For model discrimination, the value of the real-life merit function at the minimum F ($\underline{p_{min}}$) was used.

The reliability tests were as follows. Using $\underline{p_{min}}$ to simulate measured data, the "follower", noise-free merit function was constructed and the minimal sections were determined for it, with the same algorithm.

The minimal sections of the follower and real-life merit function are nearly the same everywhere except in the vicinity of the global minimum where the real-life merit function is "filled up" with noise (Fig 2), defining a parameter error domain. The non-linear (generalized) standard deviation is a measure of the difference of the two merit function sections. It can approximately determined for the non-linearly dependent parameters.

The linearized standard deviation of parameter c_v (p_i) was determined analytically ([13]), and the non-linear standard deviation geometrically, as shown in Fig. 2.

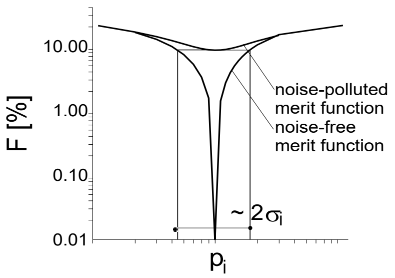

Fig. 2. Geometrical concept of the standard deviation of parameter p_i

3 Results

According to the results (Figs 3 to 6, Tables 4 to 6), the homothetic grading curves gave homothetic retention curves, which a near linear part, with slope depending on the number of fractions. The threshold suction data for fractions 1 to 7 were between about 0.1 and 100 kPa.

Table 4. Approximate threshold suction data for fractions 1-7

Fraction	Air entry suction [kPa]	Residual suction [kPa]
7	0.1	0.4
6	0.25	0.7
5	0.7	2
4	1.8	3
3	9	14
2	14	38
1	38	100

Table 5. Typical covariance matrix, van Genuchten 5 model (parameters A-B and A - C are not completely independent)

	C(=a)	*A(=n)*	*B(=m)*
C(=a)	1.00	-0.89	-0.06
A(=n)	-0.89	1,00	-0.98
B(=m)	-0.06	-0.98	1.00

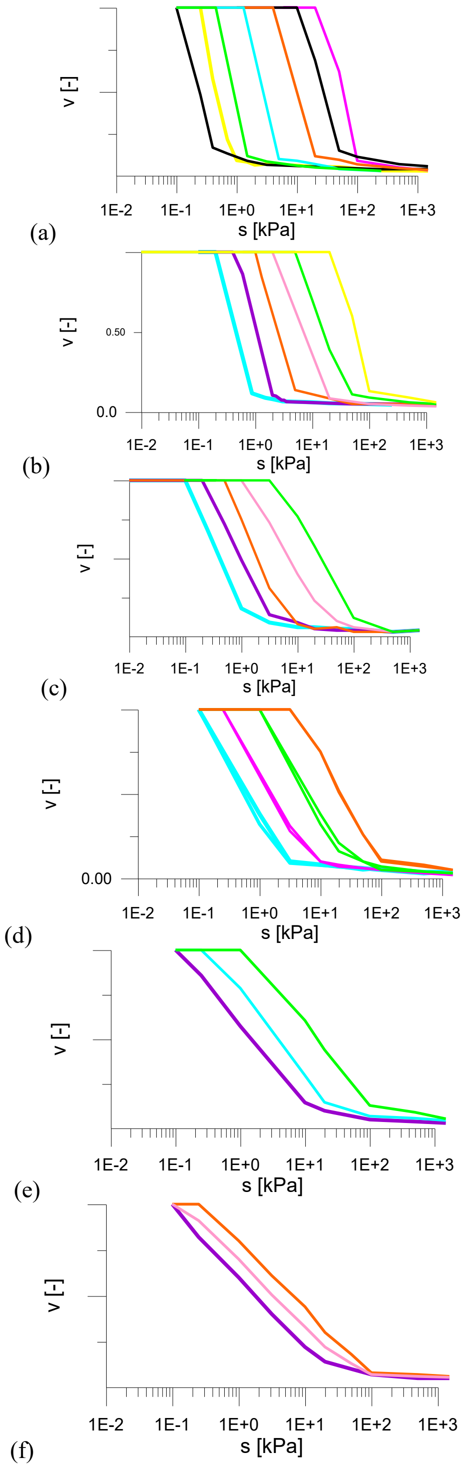

Fig. 3. Ideal retention curves, results of data mining. (a) Fraction curves. (b) 2- mixture. (c) 3- mixtures. (d) 4-mixtures. (e) 5- mixtures. (f) 6 and 7- mixtures.

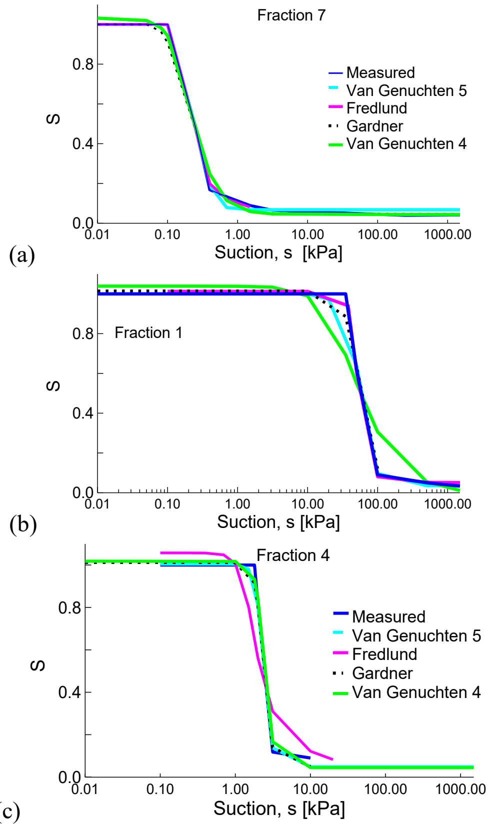

Fig. 4. Measured and the fitted data for fractions 7, 1, 4

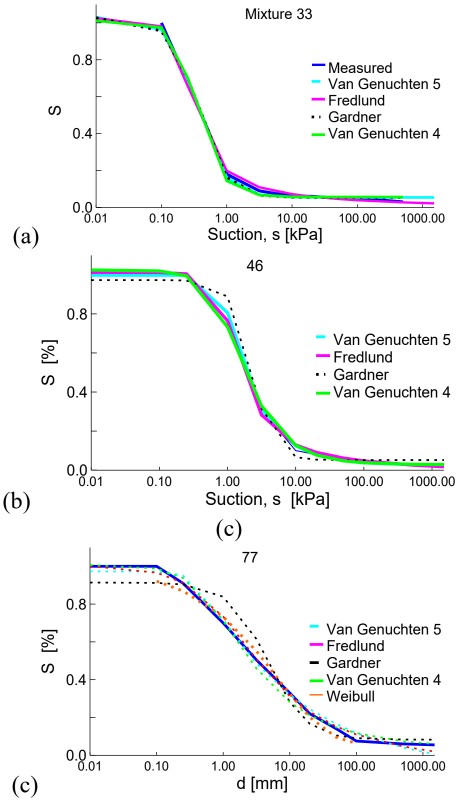

Fig. 5. (a) to (c): Measured and the fitted data for mixtures 33, 46, 77, resp.

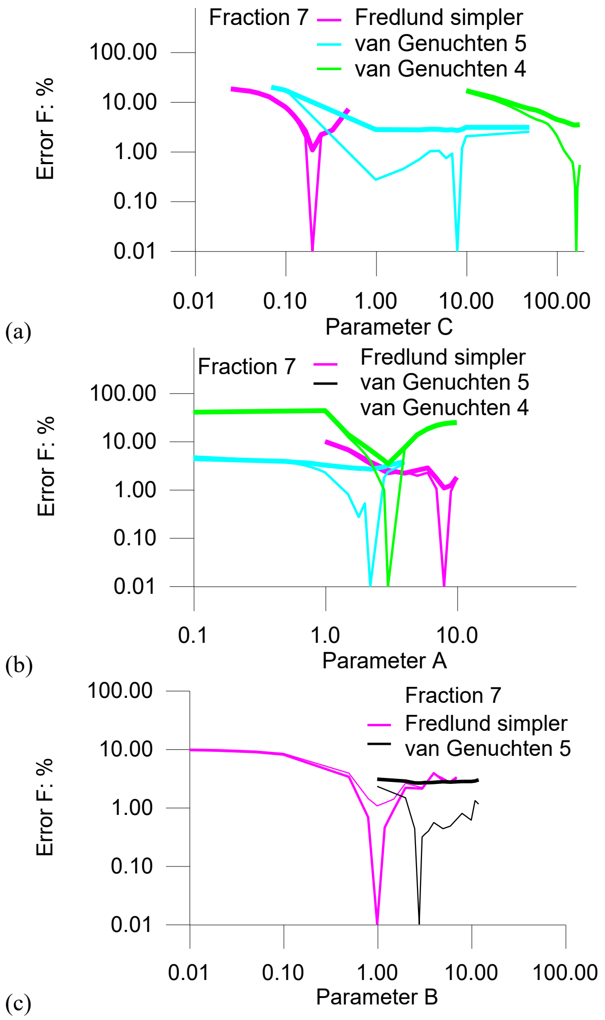

Fig. 6. The minimal (deepest) section of the follower and the real-life merit functions concerning parameters *A, B, C*, fraction 7, comparing the van Genuchten models and the Fredlund-Xing model.

Table 6. Model fitting: mean fitting error *F*, smallest in bold, mean coefficient of variation of parameters, 28 samples

Model	*F* %	c_V: w_r	c_V: w_s	c_V: a	c_V: n	c_V: m
VanGenuchten 4	3,28	1,95	0,11	0,54	0,61	
Gardner	3,62	3,12	0,13	0,55	0,27	-
Fredlund – Xing	2,70	10,49	0,08	0,13	0,62	0,34
Fredlund–Xing s	2,33	-1,47	0,08	0,10	0,48	0,25
VanGenuchten 5	**2,13**	1,86	0,07	0,25	0,14	0,20

All models gave locally one-to-one solution for every inverse problem (Fig. 6). However, the parameters were not completely independent, if we consider the covariance matrices (see the terms $|c_{ij}| > 0.8$ in Table 5).

Concerning the mean fitting error *F*, the best was the fit with the non-modified and, the worst was the fit with the modified van Genuchten model (Table 6).

The mean of coefficient of variations of the identified parameters computed with the linearized models (Table 6) was the largest for the residual water content and were generally non-negligible concerning the “non-linear” parameters *a, n, m* or *C, B, A*.

The minimal (deepest) sections of the follower and the real-life merit functions concerning the various parameters in case of fraction 7 are shown for three models, van Genuchten 4 and 5 and the simpler Fredlund-Xing model in Fig. 6.

Results indicated non-convex follower merit function for the van Genuchten 5 model, where some deep, local minima occurred. This indicated larger parameter error than the linear estimation (Fig. 6),

3 Discussion

3.1 Previous results, Weibull fit

In an earlier model discrimination study [3], beyond the four coarse fractions and their three two-component continuous mixtures, three gap-graded mixtures were also were tested. The model discrimination results were the same, except that the fitting error was larger due to the fact that gap-graded soils were also used (in Table 7).

It can be noted that the two-parameter Weibull function [16, 17] is frequently used for grading curves. Its modified form with three parameters for SWCC:

$$w = 1 - \left[-\left(\frac{x}{k}\right)\right]^{\lambda} + w_r \qquad (9)$$

.

Fig. 5 shows that the measured and fitted curves did not agree for elongated mixture 77 (Gardner model) even it was “continuous”. In case of elongated or gap-graded curves Weibull fit may help as follows.

According to the previous results (Fig. 7, the gap-graded mixture), the fit is bad for van Genuchten 5 but for Weibull fit is excellent which is applied in a double form in case of gap-graded SWCC functions.

Table 7. Result of [3], mean fitting error *F*, mean coefficient of variation of the parameter, 10 gap-graded and fractal soils

Model	*F* %	c_V: w_r	c_V: w_s	c_V: a	c_V: n	c_V: m
Van Genuchten 4	6,72	1,83	0,10	0,49	0,57	
Gardner	4,19	3,01	0,13	0,53	0,27	-
Fredlund – Xing	4	>10	0,08	0,13	0,60	0,33
Van Genuchten 5	3,57	1,86	0,07	0,25	0,14	0,20

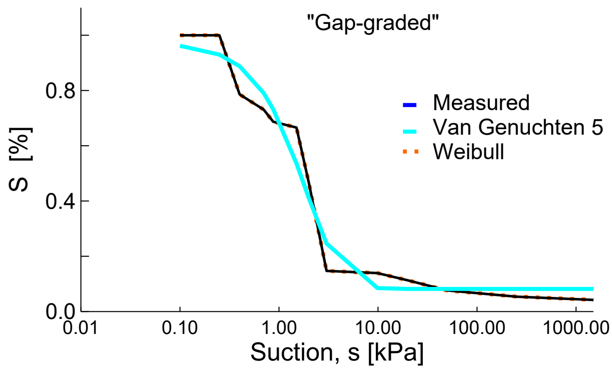

Fig. 7. Gap-graded SWCC 1-4 of [3], van Genuchten 5 and Weibull fit.

3.2 Model fitting, identified parameters

Fig. 8. summarizes the parameters *a* and *n* of the Fredlund –Xing model (identified from non-modified data) in the entropy diagram [18]. The *n* parameter values are hardly changing in a row (for N = const).

It follows from this work that only the identified parameter *a* is expected to change significantly. The two other parameters could vary less for a fixed N value, being not completely independent (Table 5), so different parameter combinations may give similarly good fits.

A great advantage of the van Genuchten 5 model is to have immediately a closed-form permeability function with the identified parameters.

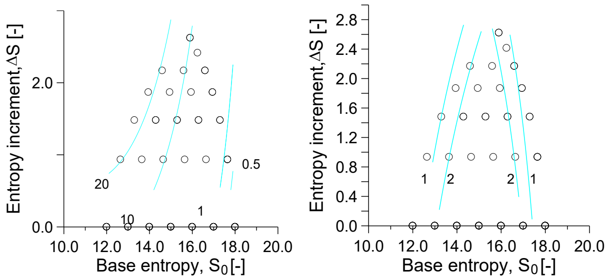

Fig. 8. Left to right: Identified parameters *a* and *n* of Fredlund -Xing model in grading entropy diagram, samples are indicated by points. ΔS entropy increment: originated from the mixing of the fractions, zero for fractions, increasing for mixtures with N; S_0 base entropy: equal to the mean log diameter of sample.

4 Conclusion

The data were produced within this research, using ideal, the homothetic, fractal grading curves for N = 1 to 7. The ideal, homothetic water retention curves of sand fractions and continuous, fractal sand mixtures were used for the model fitting. The preliminary results of a model discrimination work concerning the well-known, closed form SWCC equations (the Gardner, Fredlund – Xing, van Genuchten 4 and 5) were as follows.

The agreement between the measured and computed data was the best for van Genuchten 5 model and was the worst of the van Genuchten 4 model. The parameters were found not to be not completely independent.

It can be noted that the result was the same for less nice samples [3], including gap-graded samples, the fit was with larger error. A double Weibull model may overcome the difficulty. Further research is suggested on the application of the Weibull model.

The results of the fast model fitting method presented here [19] is approximate (available upon request). The solutions are planned to be refined starting by using the T-secant method ([20]), and the convexity assumptions are planned to be examined and discussed later on.

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