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Exploiting homogeneity in games with non-homogeneous revenue functions

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Abstract

We exploit the properties of homogeneous functions to characterize the symmetric pure-strategy Nash equilibria of n-player symmetric games in which each player's revenue function is not homogeneous but it can be decomposed into the sum of homogeneous functions with different degrees of homogeneity. Our results aim to provide a pathway for an easy computation of symmetric equilibria for this type of games. We discuss our results in a Cournot game, a contest game, and a public good game.

Keywords Equilibrium characterization · Symmetric games · Homogeneous functions' properties · Non-homogeneous revenue function

1 Introduction

Homogeneous functions frequently appear in economic theory models. For example, in consumption theory, the indirect utility function is often homogeneous of degree zero in prices and income and the expenditure function is usually homogeneous of degree one in prices; in general equilibrium, the excess demand function is often homogeneous of degree zero.

The properties of homogeneous functions have been exploited by Malueg and Yates (2006) to show sufficient conditions for the existence of a unique symmetric

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interior pure-strategy Nash equilibrium in rent-seeking contests. More specifically, in their setting, the players' revenue (utility net of cost) is described by a unique homogeneous function of degree zero. Ferrarese (2022) generalizes those results by allowing such a function to exhibit any degree of homogeneity. This allowed the author to extend the set of possible applications beyond the games considered by Malueg and Yates (2006). For instance, the results in Ferrarese (2022) can be applied to contests with an endogenous prize valuation or to Cournot games with a non-linear inverse demand.

This note aims to extend the characterization of symmetric pure-strategy Nash equilibria (henceforth, *symmetric equilibria*) to an even wider set of games. Specifically, we show that the properties of homogeneous functions can also be exploited to characterize the symmetric equilibria in symmetric games where the players' revenue function is not (necessarily) homogeneous but homogeneous-decomposable, in the sense that, it can be written as a sum of homogeneous functions, with (possibly) different degrees of homogeneity. A simple example may help the reader. Let us consider a linear, symmetric, homogeneous good Cournot game between two firms i and j. The inverse demand is $p(q_i, q_j) = \alpha - q_i - q_j$ $\alpha > 0$ and q_i and q_j represent the firms' quantities. Thus, firm i's revenue is $R(q_i, q_j) = (\alpha - q_i - q_j)q_i$. Notice that R is not homogeneous. However, one can decompose it as $R(q_i, q_j) = \alpha q_i - (q_i q_j - q_i^2)$, where the first term is a homogeneous function of degree one, and the second term is a homogeneous function of degree two (in both quantities). Hence, the results presented in this note apply not only to the games covered by Ferrarese (2022) and Malueg and Yates (2006) but to others, as well. In particular, we show how our results can be used to characterize the symmetric interior equilibria of symmetric games classified within several families of games such as, public good, contests, and imperfect competition games.

The extra generality of our setting comes at a cost: The set of symmetric equilibrium candidates cannot be reduced to a singleton and, in general, there is no closed-form solution. However, we can exploit homogeneity to (i) characterize the equilibrium candidates as the roots of a generalized polynomial whose real coefficients are represented by the partial derivatives of the component functions of a player's payoff evaluated at a specific point. This contributes a shortcut for economic theory researchers to identify the maximum number of interior equilibrium candidates, as they can use the mathematical results for counting the positive zeros of generalized polynomials (Jameson, 2006). Moreover, (ii) we provide sufficient conditions for each of these candidates to be a symmetric equilibrium, and (iii) we show sufficient conditions for excluding the null vector from the set of symmetric equilibria. Finally, we apply results (i)–(iii) to characterize the equilibria of Cournot games, Tullock contets, and public good games.

The note is structured as follows: in Sect. 2, we briefly present some useful properties of homogeneous functions; in Sect. 3, we present the model; Sect. 4 is devoted to the equilibrium analysis; in Sect. 5, we present applications and Sect. 6 concludes. All proofs are in the Appendix.

2 Preliminaries: a reminder about homogeneous functions

A real valued function $f : \mathbb{R}^I \to \mathbb{R}$ is homogeneous of degree α in $\mathbf{x} \equiv (x_1, x_2, \ldots, x_I)$ if $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$, $\forall t > 0$. This implies that, given a vector with I identical entries $\check{\mathbf{x}} \equiv (x, x, \ldots, x)$, $f(\check{\mathbf{x}}) = x^\alpha f(\mathbf{1})$, where $\mathbf{1}$ is an I-dimensional vector of ones. Furthermore, the two following properties will be useful:

- **P1** *Let $f : \mathbb{R}^I \to \mathbb{R}$ be differentiable and homogeneous of degree α. Then, for the n^{th} derivative, $\frac{\partial^n f}{\partial x_i^n}(\check{\mathbf{x}}) = x^{\alpha-n} \frac{\partial^n f}{\partial x_i^n}(\mathbf{1})$.*
- **P2** *The set of homogeneous functions is closed with respect to the operation of multiplication and is not (in general) closed with respect to the operation of sum. The sum of homogeneous functions is itself homogeneous if all the addends exhibit the same degree of homogeneity.*

3 The model

Let $\mathscr{I} = \{1, 2, \ldots, I\}$ be the index set of symmetric players. Each $i \in \mathscr{I}$ selects an $x_i \in [0, +\infty)$ simultaneously and non-cooperatively. Let $\mathbf{x} \equiv (x_1, x_2, \ldots, x_I)$ be an I-dimensional vector of strategies selected by all players and $\mathbf{x}_{-i} \in \mathbb{R}_+^{I-1}$ denote the vector of strategies excluding x_i.[1] Player i's payoff is given by

$$u(x_i, \mathbf{x}_{-i}) = R(x_i, \mathbf{x}_{-i}) - C(x_i, \mathbf{x}_{-i}), \text{ for all } i \in \mathscr{I}.$$

Let the cost function $C : \mathbb{R}_+^I \to \mathbb{R}_+$ be a homogeneous function of degree $s \in \mathbb{R}_{++}$ with $C(0, \mathbf{x}_{-i}) = 0 \; \forall \; \mathbf{x}_{-i}$, $\frac{\partial C(x_i, \mathbf{x}_{-i})}{\partial x_i} > 0$, and $\frac{\partial^2 C(x_i, \mathbf{x}_{-i})}{\partial x_i^2} \geq 0$. The revenue function $R(\mathbf{x})$ is homogeneous-decomposable, that is, it can be obtained as a sum of homogeneous functions of possibly different degrees of homogeneity. Hence, $R(\mathbf{x}) = \sum_{h \in \mathscr{H}} f_h(\mathbf{x})$, where $f_h(\mathbf{x})$ is a homogeneous function of degree $\alpha_h \in \mathbb{R}$ for all $h \in \mathscr{H}$, and $\mathscr{H} = \{1, 2, \ldots, \overline{h}\}$ is the index set of these functions holding that for any pair $(k, l) \in \mathscr{H}^2$, $k < l$ if and only if $\alpha_k < \alpha_l$. Thus, α_1 and $\alpha_{\overline{h}}$ are the lowest and highest degrees of homogeneity of the functions composing $R(\mathbf{x})$, respectively. Furthermore, $f_h(\mathbf{0}) \geq 0$ for all $h \in \mathscr{H}$, and $C(\cdot)$ and $f_h(\cdot) \; \forall \; h \in \mathscr{H}$ are continuous and differentiable over $\mathbb{R}_{++}^I$ and possibly over $\mathbb{R}_+^I$.[2] Hence, this setting allows for the payoff function to have a point of discontinuity at the origin.[3] Let $\mathscr{H}_d \subseteq \mathscr{H}$ be the index set of functions with a discontinuity at the origin. Additionally, let $\mathscr{H}_c \subseteq \mathscr{H}$ and $\mathscr{H}_{c_0} \subseteq \mathscr{H}$ be two subsets of the index set of continuous functions over $\mathbb{R}_+^I$ such that $f_h(\mathbf{0}) > 0$ when $h \in \mathscr{H}_c$ and $f_h(\mathbf{0}) = 0$ when $h \in \mathscr{H}_{c_0}$.

Let $\mathscr{G}$ denote the family of games with the above features. Notice that the set of games covered by Ferrarese (2022) is a subset of $\mathscr{G}$, as it considers cases in which $\mathscr{H}$

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[1] $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$.

[2] $\mathbf{0}$ is an I-dimensional vector of zeros.

[3] A contest success function typically satisfies this feature.

has a unique element h, with $\alpha_h \in \mathbb{R}$. The set of games analyzed by Malueg and Yates (2006) is even more restricted as $\alpha_h = 0$.

A symmetric equilibrium of $g \in \mathscr{G}$ is a strategy profile $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_I^*) \in \mathbb{R}_+^I$ with $x_i^* = x_j^*, \forall i \neq j$, such that:

$$\sum_{h \in \mathscr{H}} \left(f_h(\mathbf{x}^*) - f_h(x_i, \mathbf{x}_{-i}^*) \right) - \left(C(\mathbf{x}^*) - C(x_i, \mathbf{x}_{-i}^*) \right) \geq 0, \forall x_i \neq x_i^* \text{ and } \forall i \in \mathscr{I}.$$

4 Equilibrium analysis

We now exploit the properties of homogeneous functions to (i) characterize the interior symmetric equilibrium candidates, (ii) show sufficient conditions for each candidate to be a symmetric equilibrium, and (iii) determine sufficient conditions that exclude the null vector as a symmetric equilibrium. The first result is the following:

Proposition 1 *If an interior symmetric equilibrium of $g \in \mathscr{G}$ exists, then the equilibrium strategy is a zero of the generalized polynomial:*

$$P(x) = \sum_{h \in \mathscr{H}} \frac{\partial f_h}{\partial x_i}(\mathbf{1})\, x^{\alpha_h - 1} - \frac{\partial C}{\partial x_i}(\mathbf{1})\, x^{s-1},$$

with $\deg(P) = \max\{\alpha_{\bar{h}} - 1, s - 1\}$.

Proposition 1 contributes a reinterpretation of the necessary first-order condition for an interior maximum as the roots of a generalized polynomial, whose coefficients, because of homogeneity of the cost function and the component functions of the players' revenue, are their partial derivatives computed at the unit vector. This provides a computational advantage as, in general, $\frac{\partial u}{\partial x_i}(x_i, \mathbf{x}_{-i})\Big|_{\mathbf{x}=\mathbf{x}\Box}$ could be a rather complicated function.

When $\#\mathscr{H} = 1$, a common feature in Malueg and Yates (2006) and Ferrarese (2022), the value of the unique symmetric equilibrium candidate can be explicitly obtained as a zero of $P(x)$.[4] For those $g \in \mathscr{G}$ for which $\#\mathscr{H} > 1$, in general, this is no longer the case. However, we can take advantage of the results regarding the number of positive zeros of polynomials to offer a shortcut to identify the maximum number of interior symmetric equilibrium candidates. A known property regarding the number of positive roots of a polynomial is the Descartes' rule of sign, stating that this number is no larger than the number of sign changes of its coefficients and that the disparity between these two numbers is always even. As in our case, the exponents of $P(x)$ can be non-integers, we can make use of the following result due to Jameson (2006), which can be rewritten as:

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[4] In this case, the unique symmetric candidate is $\bar{x} = \left(\frac{\partial C}{\partial x_i}(\mathbf{1}) / \frac{\partial R}{\partial x_i}(\mathbf{1}) \right)^{\alpha - s}$.

Theorem 1 *Let $\hat{P}(x)$ be an ordered version of $P(x)$ in which the exponents of x ($s-1$ and $\alpha_h - 1$ for all $h \in \mathscr{H}$) appear in descendant order.*[5] *Let $z(\hat{P}(x))$ be the number of zeros of $\hat{P}(x)$. Then $z(\hat{P}(x))$ is no greater than the number of sign changes in $\hat{P}(x)$.*

Proof See Jameson (2006). ☐

Hence, to assess the maximum number of interior symmetric equilibrium candidates, one just needs to compute the partial derivatives, evaluated at a vector of ones, of the cost function and the functions composing $R(\mathbf{x})$.

At this point, let $\bar{x}$ be a positive solution of $P(x)$.

Proposition 2 *The interior symmetric equilibrium candidate $\bar{\mathbf{x}} \equiv (\bar{x}, \bar{x}, \ldots, \bar{x})$ is an equilibrium of $g \in \mathscr{G}$ if the following two requirements are satisfied:*

I) $C(\mathbf{1})^{-1} \sum_{h \in \mathscr{H}_d \cup \mathscr{H}_{c_0}} \left(\frac{f_h(\mathbf{1}) - f_h(0, \mathbf{1}_{-i})}{\bar{x}^{s-\alpha_h}} \right) > 1;$

II) $\sum_{h \in \mathscr{H}_d \cup \mathscr{H}_{c_0}} \bar{x}^{\alpha_h - s} \frac{\partial^2 f_h}{\partial x_i^2} \left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i} \right) - \frac{\partial^2 C}{\partial x_i^2} \left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i} \right) > 0, \ \forall x_i < \tilde{x},$ for some $\tilde{x} \in [0, \bar{x})$, and negative otherwise.

Requirement *I)* ensures that player i prefers $\bar{x}$ to 0 when all the remaining players are selecting $\bar{x}$ as well. This condition is needed to exclude the possibility of $\bar{x}$ being just a local maximum of $u\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right)$, as 0 is at the corner of the players' strategy set. Requirement *II)* implies that $u\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right)$ could be initially convex and then concave and that the equilibrium candidate $\bar{x}$ lies on its strictly concave region. These two conditions ensure that $\bar{x} = \underset{x_i \in \mathbb{R}_+}{\operatorname{argmax}}\, u\left(\frac{x_i}{\bar{x}}, \mathbf{1}_{-i}\right)$, i.e., $\bar{x}$ is player i's best response to $\bar{x}_{-i}$.[6]

We can also exploit homogeneity to provide insights on whether a game $g \in \mathscr{G}$ does not admit the null vector as a symmetric equilibrium. This might be a relevant analysis in some games (like public good games) where the null vector is a potential equilibrium. In this regard, we have the following result:

Proposition 3 *The null vector is not a symmetric equilibrium of $g \in \mathscr{G}$ if one of the following mutually exclusive sets of conditions holds*

a) $\alpha_1 < 0$ and $f_1(1, \mathbf{0}_{-i}) > 0$;
b) $\alpha_1 = 0$, and $f_1(1, \mathbf{0}_{-i}) - f_1(\mathbf{0}) > 0$;
c) $s > \alpha_1 > 0, f_h(\mathbf{0}) = 0 \, \forall h \in \mathscr{H}_d$, and $f_1(1, \mathbf{0}_{-i}) > 0$;
d) $s = \alpha_1 > 0, f_h(\mathbf{0}) = 0 \, \forall h \in \mathscr{H}_d$, and $C(1, \mathbf{0}_{-i})^{-1} f_1(1, \mathbf{0}_{-i}) > 1$.

Conditions *a)-d)* are necessary and sufficient for having an $\hat{x} > 0$ such that $u(x_i, \mathbf{0}_{-i}) > u(\mathbf{0})$ for all $x_i \in (0, \hat{x})$, or equivalently for having a profitable deviation for player i from the null vector by selecting an infinitesimal x_i. These conditions are

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[5] Notice that $s-1$ may coincide with $\alpha_h - 1$ for at most one $h \in \mathscr{H}$.

[6] When these conditions are not satisfied, then each $\bar{x}$ can be either a global or a local maximum/minimum of $u(x_i, \bar{\mathbf{x}}_{-i})$. Hence, checking that $\bar{x}$ is a global maximum requires further analysis.

simple as they are just based on the inspection of the cost function and f_1, the component function of $R(x_i, \mathbf{x}_{-i})$ exhibiting the lowest degree of homogeneity. In this regard, it is worth noting that the above sufficient conditions can only hold when α_1 is not higher than s. Our applications show that these sufficient conditions are relevant as they are satisfied for several classes of games under quite general parametric restrictions.

We can link previous conditions to the existing literature. Point b) nests Malueg and Yates (2006). In their paper, the unique f_h composing the revenue function is the product between the Tullock contest success function

$$p(x_i, \mathbf{x}_{-i}) = \begin{cases} x_i / \sum_{j \in \mathscr{I}} x_j & \text{if } \mathbf{x} \neq \mathbf{0} \\ 1/I & \text{if } \mathbf{x} = \mathbf{0}, \end{cases} \tag{1}$$

and the exogenous value of the prize V. This f_h satisfies the two conditions of point b), as the Tullock function is homogeneous of degree zero and $p(1, \mathbf{0}_{-i})V - p(\mathbf{0})V = V - V/I > 0$. However, condition b) also applies to other games with a non-homogeneous revenue function, as illustrated by our second application. Point c) applies to the abatement game described in our third application.

5 Applications

We now present three applications in more detail: a Cournot game with a demand function exhibiting a constant curvature, a Tullock contest with externalities, and a public good (bad) game.

Application 1: *A Cournot duopoly with constant demand curvature*

Two firms $i = \{1, 2\}$ simultaneously select their output level $x_i \in \mathbb{R}_+$ of a homogeneous good. Firms face the inverse demand $p(X)$, where $X = x_1 + x_2$ is the aggregate supply. We assume that $\frac{\partial p}{\partial X} < 0$ for all X such that $p(X) > 0$. Moreover, let $\delta \equiv -\frac{X \frac{\partial^2 p}{\partial X^2}}{\frac{\partial p}{\partial X}}$ be the curvature of the inverse demand (or its degree of convexity), which is assumed to be constant. As pointed out in López and Vives (2019), the set of inverse demands with the above features can be represented as:

$$p(X) = a - bX^{1+\delta}, \ \delta \neq -1 \tag{2}$$

where $a > 0$ is a measure of market extension and $b > 0$ (resp. < 0) if $\delta \geq -1$ (resp. < -1).[7] Firm i's revenue is:

$$R(x_i, x_j) = x_i \left(a - b(x_i + x_j)^{1+\delta} \right)$$

which is not homogeneous. However, one can rewrite this revenue as $R(x_i, x_j) = f(x_i) + g(x_i, x_j)$, where $f(x_i) = ax_i$ and $g(x_i, x_j) = -bx_i(x_i + x_j)^{1+\delta}$ are

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[7] The inverse demand is concave (resp. convex) if $\delta > 0$ (resp. < 0) and it is log-concave (resp. log-convex) if $\delta > -1$ (resp. < -1). López and Vives (2019) consider the case $\delta = -1$ as well, in which $p(X) = a - b \log X$. We exclude this case since the resulting revenue function is not homogeneous-decomposable.

both homogeneous and where $\alpha_f = 1$ and $\alpha_g = 2 + \delta$.[8] We assume that firm i's cost function is $C(x_i) = cx_i^s$ with $c > 0$ and $s \geq 1$. Hence, firm i's payoff is:

$$\pi(x_i, x_j) = x_i\left(a - b\left(x_i + x_j\right)^{1+\delta}\right) - cx_i^s.$$

By symmetry, we focus on the problem faced by firm 1 only. Given that

$$\begin{aligned} \frac{\partial f}{\partial x_1}(1) &= a; \\ \frac{\partial g}{\partial x_1}(\mathbf{1}) &= -2^{\delta} b(3+\delta); \\ \frac{\partial C}{\partial x_1}(1) &= cs, \end{aligned}$$

We can easily apply Proposition 1 to conclude that any interior symmetric equilibrium must be a root of the polynomial

$$P(x) = a - 2^{\delta} b(3+\delta)x^{1+\delta} - csx^{s-1}, \tag{3}$$

with $\deg P(x) = \max\{1+\delta, s-1\}$.[9] Applying Theorem 1, the maximum number of equilibrium candidates depends on both *i)* the order between the exponents of the polynomial and *ii)* the sign of the coefficients in $P(x)$, which in turn depend on δ and s. The school case of a linear inverse demand $(\delta = 0)$ and a linear cost function $(s = 1)$ leads to $P(x) = (a - c) - 3bx$, which has at most one sign change. By the Descartes' rule of sign, this implies that there is a unique interior equilibrium candidate under the usual restriction $a > c$.[10] However, the above model is way richer, since the number of sign changes in $\hat{P}(x)$ depends on the specific values of δ and s. Next, we will show that our machinery can be applied to less straightforward cases.

Proposition 2 allows us to check whether a particular equilibrium candidate is indeed an equilibrium. For the sake of concreteness, let $\delta = -4$ and $s = 2$, namely a sufficiently log-convex inverse demand plus a quadratic cost function. In addition, let $(a, b, c) = (10, -2, 1)$. In this case, the ordered version of (3) is:

$$\hat{P}(x) = -\frac{1}{8x^3} + 10 - 2x$$

which has two sign changes, so that from the *generalized* Descartes' rule, there can be atmost two equilibrium candidates. It turns out that such candidates are $\bar{\mathbf{x}}_1 \equiv (\bar{x}_1, \bar{x}_1)$ and $\bar{\mathbf{x}}_2 \equiv (\bar{x}_2, \bar{x}_2)$ with $\bar{x}_1 \approx 0.2358$ and $\bar{x}_2 \approx 4.9995$. Since

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[8] Note that the ordering of the component functions in terms of their degrees of homogeneity depends on δ. For this reason, we depart from the notation used in the main text and label such functions as f and g.

[9] The case $\delta < -1$ and $s = 1$ is discussed in Corchón and Torregrosa (2020).

[10] The reader can easily check that the candidate is actually the unique interior symmetric equilibrium.

$$\begin{aligned} f(1) =& a; \\ g(1,1) =& -2^{1+\delta}b; \\ f(0) =& g(0,1) = 0; \\ C(1) =& c, \end{aligned}$$

it is easy to check that, given our parametric restrictions, condition I) in Proposition 2 is satisfied for both $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$, and that, since

$$\begin{aligned} \frac{\partial^2 f}{\partial x_1^2}(x_1) =& 0; \\ \frac{\partial^2 g}{\partial x_1^2}(x_1,1) =& \frac{12(x_1-1)}{(x_1+1)^5}; \\ \frac{\partial^2 C}{\partial x_1^2}(x_1) =& 2, \end{aligned}$$

condition II) in Proposition 2 needs to be checked for

$$\frac{1}{\bar{x}_k^4}\frac{12\left(\frac{x_1}{\bar{x}_k}-1\right)}{\left(\frac{x_1}{\bar{x}_k}+1\right)^5}-2,\ k=\{1,2\}.$$

For $k=2$, the previous expression is always negative, so that $\bar{\mathbf{x}}_2$ is an equilibrium of the Cournot game. For $k=1$, the previous expression is negative for sufficiently low values of x_1, then positive, and finally negative. Hence, condition II) does not hold, so confirming whether $\bar{\mathbf{x}}_1$ is an equilibrium or not requires further analysis.[11]

Finally, we can go back to our general model (with any $\delta \neq -1$, $a>0$, $b\neq 0$, $c>0$, and $s\geq 1$) to check whether there is an infinitesimal profitable deviation from the null vector. Applying Proposition 3, we need to consider the following cases:

i) $\delta \leq -2$. In this case, $b<0$. Since $g(1,0)=-b>0$ and $g(0,0)=0$, the sets of conditions *a)* and *b)* of Proposition 3 hold. Hence, the null vector is not a symmetric equilibrium.

ii) $s>\min\{\alpha_f,\alpha_g\}>0$. This may happen in three subcases:

  iia) $\delta\in(-2,-1)$. In this case, since $\mathscr{H}_d=\emptyset$, it is required that $g(1,0)>0$;
  iib) $\delta=-1$. This case is excluded from our analysis;
  iic) $\delta>-1$. In this case, it is required that $f(1)=a>0$.

  Hence, the set of conditions *c)* of Proposition 3 holds.

iii) $s=\min\{2+\delta,1\}>0$. This may happen in three subcases:

  iiia) $s=2+\delta=1$. This case is excluded from our analysis;

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[11] In that case, it can be checked that $\bar{\mathbf{x}}_1 \neq \underset{x_i\in\mathbb{R}_+}{\operatorname{argmax}}\,\pi\left(\frac{x_i}{\bar{\mathbf{x}}_1},\mathbf{1}_{-i}\right)$, so $\bar{\mathbf{x}}_1$ is not an equilibrium.

- iiib) $s = 1 < 2 + \delta$. In this case, it is required that $\mathscr{H}_d = \emptyset, f(1) = a > 0$, and $C(1) = c$;
- iiic) $s = 2 + \delta < 1$. As $s \geq 1$, this case is excluded from our analysis.

Hence, the set of conditions *d)* of Proposition 3 holds when $a > c$.

This illustrates how our results provide a useful set of sufficient conditions to easily check whether the null vector can be excluded from the equilibrium set.

Application 2: *A Tullock contest with externalities*

Two risk-neutral players $i = \{1, 2\}$ exert an irreversible effort $x_i \in \mathbb{R}_+$. Contingent upon winning or losing, each player obtains a prize $W, L \in \mathbb{R}$, with $W > L$, respectively. Player i's utility is:

$$u(x_i, x_j) = p(x_i, x_j)\left(W + \beta_1 x_j^{\gamma_1} - \theta_1 x_i^{s_1}\right) + \left(1 - p(x_i, x_j)\right)\left(L + \beta_2 x_j^{\gamma_2} - \theta_2 x_i^{s_2}\right),$$

where $p(x_i, x_j)$ is the two-player version of the contest success function (1), (β_1, β_2) are spillover parameters, and (θ_1, θ_2) are cost parameters. For simplicity, we assume that $s_1 = s_2 = s$. Moreover, $\gamma_1, \gamma_2 > 0$ and $s \geq 1$. Notice that player i's utility can be decomposed into a revenue $R(x_i, x_j)$ and a cost function $C(x_i, x_j)$, where

$$R(x_i, x_j) = p(x_i, x_j)\left(W + \beta_1 x_j^{\gamma_1}\right) + \left(1 - p(x_i, x_j)\right)(L + \beta_2 x_j^{\gamma_2}),$$

and

$$C(x_i, x_j) = x_i^s\left[p(x_i, x_j)\theta_1 + \left(1 - p(x_i, x_j)\right)\theta_2\right],$$

with $i = \{1, 2\}$ and $i \neq j$. To ensure a costly effort we let $\theta_1 > 0$ and $\theta_2 \geq 0$. Notice also that $R(x_i, x_j)$ is not homogeneous. However, one can write that $R(x_i, x_j) = f_1(x_i, x_j) + f_2(x_i, x_j) + f_3(x_i, x_j)$, where $f_1(x_i, x_j) = p(x_i, x_j)(W - L) + L$, $f_2(x_i, x_j) = p(x_i, x_j)\beta_1 x_j^{\gamma_1}$, and $f_3(x_i, x_j) = \left(1 - p(x_i, x_j)\right)\beta_2 x_j^{\gamma_2}$, which are all homogeneous.

Consistent with our notation $\alpha_1 = 0$, $\alpha_2 = \gamma_1$, $\alpha_3 = \gamma_2$, and s is the degree of homogeneity of the cost function. By symmetry, we focus on the problem of contender 1 only. Given

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1}(1, 1) &= \frac{W - L}{4};\\
\frac{\partial f_2}{\partial x_1}(1, 1) &= \frac{\beta_1}{4};\\
\frac{\partial f_3}{\partial x_1}(1, 1) &= -\frac{\beta_2}{4};\\
\frac{\partial C}{\partial x_1}(1, 1) &= \frac{s}{2}(\theta_1 + \theta_2) + \frac{1}{4}(\theta_1 - \theta_2),
\end{aligned}$$

from Proposition 1 we know that if a symmetric equilibrium exists, it is a root of the polynomial

$$P(x) = \frac{W-L}{4}x^{-1} + \frac{\beta_1}{4}x^{\gamma_1 - 1} - \frac{\beta_2}{4}x^{\gamma_2 - 1} - \left(\frac{s(\theta_1+\theta_2)}{2} + \frac{\theta_1 - \theta_2}{4}\right)x^{s-1}. \qquad (4)$$

Applying Theorem 1, the upper bound of the number of positive roots of this polynomial would depend on the order among the exponents and the sign of the parameters of $P(x)$, which, in turn, depend on the values of the parameter constellation $(W, L, \beta_1, \beta_2, \theta_1, \theta_2)$ and the degrees of homogeneity (γ_1, γ_2, s).

For the sake of concreteness, let us consider the case $(W, L, \beta_1, \beta_2, \theta_1, \theta_2) = (10, 1, 2, 2, 2, 1)$ and $(\gamma_1, \gamma_2, s) = (4, 3, 2)$. In this case, the ordered version of (4) is:

$$\hat{P}(x) = \frac{9}{4x} - \frac{13}{4}x - \frac{1}{2}x^2 + \frac{1}{2}x^3$$

which has two sign changes, so that from the *generalized* Descartes' rule, there can be atmost two equilibrium candidates. It turns out that such candidates are $\bar{\mathbf{x}}_1 \equiv (\bar{x}_1, \bar{x}_1)$ and $\bar{\mathbf{x}}_2 \equiv (\bar{x}_2, \bar{x}_2)$, where $\bar{x}_1 = 3$ and $\bar{x}_2 \approx 0.8228$. Since:

$$\begin{aligned} f_1(1,1) - f_1(0,1) &= \frac{W-L}{2}; \\ f_2(1,1) - f_2(0,1) &= \frac{\beta_1}{2}; \\ f_3(1,1) - f_3(0,1) &= -\frac{\beta_2}{2}; \\ C(1,1) &= \frac{\theta_1 + \theta_2}{2}, \end{aligned}$$

it is easy to check that condition I) in Proposition 2 is satisfied for both $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$. Since:

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x_1^2}(x_1, 1) &= \frac{-2}{(1+x_1)^3}(W-L); \\ \frac{\partial^2 f_2}{\partial x_1^2}(x_1, 1) &= \frac{-2\beta_1}{(1+x_1)^3}; \\ \frac{\partial^2 f_3}{\partial x_1^2}(x_1, 1) &= \frac{2\beta_2}{(1+x_1)^3}; \\ \frac{\partial^2 C}{\partial x_1^2}(x_1, 1) &= \frac{x_1^{s-2}}{(x_1+1)^3}\big(s^2\theta_2 - s\theta_2 - 2x_1^2\theta_1 + 2x_1^2\theta_2 + 2s^2x_1^2\theta_1 + s^2x_1^2\theta_2 + s^2x_1^3\theta_1 \\ &\quad + sx_1\theta_1 - 4sx_1\theta_2 + s^2x_1\theta_1 - 3sx_1^2\theta_2 - sx_1^3\theta_1 + 2s^2x_1\theta_2\big), \end{aligned}$$

and given our parametric restrictions, condition II) in Proposition 2 needs to be checked for

$$-\frac{2}{\left(1+\frac{x_1}{\bar{x}_k}\right)^3}\left(\frac{9}{\bar{x}_k^2}+2\bar{x}_k^2-2\bar{x}_k+2\left(\frac{x_1}{\bar{x}_k}\right)^3+6\left(\frac{x_1}{\bar{x}_k}\right)^2+6\left(\frac{x_1}{\bar{x}_k}\right)+1\right),\ k=\{1,2\}.$$

Previous expression is always negative for $k=\{1,2\}$, so that both candidates are equilibria of the Tullock contest with externalities. Another special case of the general model presented here is provided by Chowdhury and Sheremeta (2011a; b), where $\gamma_1=\gamma_2=s=1$.[12]

Finally, we can go back to our general case (with any β_1; β_2; $\theta_1>0$; $\theta_2\geq 0$; $\gamma_1>0$; $\gamma_2>0$; and $s\geq 1$) to analyze whether there exists an infinitesimal profitable deviation from the null vector. Proposition 3 allows us to immediately conclude that the null vector cannot be an equilibrium, as $\alpha_1=0$ and $f_1(1,0)-f_1(0,0)=\frac{W-L}{2}>0$. This holds for any parametric restriction as $W>L$ is a fundamental of the model.

Application 3: *Abatement games*

This application is based on Barrett (1994). Each i out of I countries emits a quantity of a pollutant $x_i\in\mathbb{R}_+$ damaging a shared natural resource. Country i's revenue is:

$$R(x_i,\mathbf{x}_{-i})=\psi\left(\sum_{j=1}^{I}x_j\right)-\sigma\left(\sum_{j=1}^{I}x_j\right)^2,$$

where ψ and σ are two positive parameters. Notice that $R(x_i,\mathbf{x}_{-i})$ is not homogeneous. However, one can write that $R=f_1(x_i,\mathbf{x}_{-i})+f_2(x_i,\mathbf{x}_{-i})$, where $f_1(x_i,\mathbf{x}_{-i})=\psi\left(\sum_{j=1}^{I}x_j\right)$ and $f_2(x_i,\mathbf{x}_{-i})=-\sigma\left(\sum_{j=1}^{I}x_j\right)^2$ are both homogeneous. Since player i's cost function is $C(x_i)=cx_i^s$ with $c>0$ and $s\geq 1$, then her payoff is:

$$u(x_i,\mathbf{x}_{-i})=\psi\left(\sum_{j=1}^{I}x_j\right)-\sigma\left(\sum_{j=1}^{I}x_j\right)^2-cx_i^s,$$

and according to our notation $\alpha_1=1$ and $\alpha_2=\alpha_{\bar{h}}=2$.[13] As:

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[12] In this case, the number of positive zeros of the above polynomial $P(x)$ is one when $\beta_1-\beta_2<3\theta_1+\theta_2$. This positive root is given by $\bar{x}=\frac{W-L}{(3\theta_1+\theta_2)-(\beta_1-\beta_2)}$, so that $(x_1,x_2)=(\bar{x},\bar{x})$ is the unique interior symmetric equilibrium candidate. Condition I) in Proposition 2 is $\frac{(3\theta_1+\theta_2)-(\beta_1-\beta_2)}{2}+\frac{\beta_1-\beta_2}{2}>\frac{\theta_1+\theta_2}{2}$, which is satisfied as $\theta_1>0$. Condition II) in Proposition 2 needs to be checked for $-\frac{2}{(1+\frac{x_1}{\bar{x}})^3}\left(\frac{W-L+\bar{x}(\beta_1-\beta_2+\theta_1-\theta_2)}{\bar{x}}\right)$, which becomes $-\frac{8\theta_1}{(1+\frac{x_1}{\bar{x}})^3}$ after substituting $\bar{x}$. This expression is negative, so that the unique interior symmetric equilibrium is $(x_1^*,x_2^*)=(\bar{x},\bar{x})$ if $\beta_1-\beta_2<3\theta_1+\theta_2$. Notice also that Chowdhury and Sheremeta (2011a; b) mistakenly compare the equilibrium payoffs with the payoff of losing which, in this case, is not equal to the payoff of deviating from the equilibrium by exerting a zero effort. This is the reason why our condition $\theta_1>0$ does not coincide with theirs, $\beta_2+\theta_1\geq 0$.

[13] The specification of the abatement game in Barrett (1994) is a special case of *quadratic games* as defined in Dokka et al. (2021), a class of n-player games in which $u(x_i,\mathbf{x}_{-i})=\psi\left(\sum_{i=1}^{I}x_i\right)+\sigma\sum_{i=1}^{I}\sum_{j=1}^{I}x_ix_j+tx_i^2$.

$$\begin{aligned}\frac{\partial f_1}{\partial x_i}(\mathbf{1}) &= \psi;\\ \frac{\partial f_2}{\partial x_i}(\mathbf{1}) &= -2\sigma I;\\ \frac{\partial C}{\partial x_i}(1) &= sc,\end{aligned}$$

using Proposition 1, any symmetric equilibrium candidate must be a root of the polynomial

$$P(x) = \psi - 2\sigma I x - csx^{s-1}.$$

If $s = 1$ or $s = 2$, $P(x)$ has only two terms. Hence, applying Theorem 1, the upper bound of the number of equilibrium candidates can be zero or one. When $s = 1$, this upper bound is one if and only if $\psi - c > 0$ and zero otherwise; when $s > 1$, $P(x)$ can have up to three terms. However, given that $\psi > 0$, and since both cs and $2\sigma I$ are positive, $\hat{P}(x)$ only has one sign change. Hence, by the Descartes' rule of sign, there is one equilibrium candidate.

As a special case, consider the original version in Barrett (1994) in which $s = 2$. In this case, the unique interior symmetric equilibrium candidate $(x_1, x_2, \ldots, x_I) = (\bar{x}, \bar{x}, \ldots, \bar{x})$, where $\bar{x}$ is the positive root of $P(x)$:

$$\bar{x} = \frac{\psi}{2(I\sigma - c)}.$$

Notice that $\bar{x} > 0$ if and only if $\sigma > c/I$. Since:

$$\begin{aligned}f_1(\mathbf{1}) &= \psi I;\\ f_2(\mathbf{1}) &= -\sigma I^2;\\ f_1(0, \mathbf{1}_{-i}) &= \psi(I-1);\\ f_2(0, \mathbf{1}_{-i}) &= -\sigma(I-1)^2;\end{aligned}$$

condition I) in Proposition 2 is:

$$2(\sigma I - c) + \sigma(1 - 2I) > c,$$

which requires $\sigma > 3c$. Notice that, since $I \geq 2$ and $c > 0$, $\max\{3c, c/I\} = 3c$. Hence, $\sigma > 3c$ is more stringent than $\sigma > c/I$. It only remains to check whether $x_i = \bar{x}$ is the best reply to $\mathbf{x}_{-i} = (\bar{x}, \bar{x}, \ldots, \bar{x})$. Since:

$$\begin{aligned}\frac{\partial^2 f_1}{\partial x_i^2}(x_i) &= 0;\\ \frac{\partial^2 f_2}{\partial x_i^2}(x_i) &= -2\sigma;\\ \frac{\partial^2 C}{\partial x_i^2}(x_i) &= 2c,\end{aligned}$$

condition II) in Proposition 2 needs to be checked for $-2(\sigma + c)$, which is negative. Hence, the unique interior symmetric equilibrium is $(x_1^*, x_2^*, \ldots, x_I^*) = (\bar{x}, \bar{x}, \ldots, \bar{x})$ if $\sigma > 3c$.

Back to our general case (with any $s \geq 1$), we can easily apply Proposition 3 to check whether there exists an infinitesimal profitable deviation from the null vector. First, notice that $\mathscr{H}_d = \emptyset$. There are two possible cases: (i) $s > \alpha_1 = 1$ or (ii) $s = \alpha_1 = 1$, coinciding with cases c) and d) in our Proposition 3, respectively. The null vector can be excluded from the set of symmetric equilibria if, in case (i) $f_1(1, \mathbf{0}_{-i}) = \psi > 0$, and in case (ii) $f_1(1, \mathbf{0}_{-i}) - C_1(1, \mathbf{0}_{-i}) = \psi - c > 0$.

6 Conclusion

We characterized the symmetric equilibrium candidates of n-person symmetric games in which the players' revenue takes the form of a finite sum of homogeneous functions. By breaking down this revenue function into smaller pieces, we take advantage of the homogeneity properties to present the equilibrium candidates as the positives zeros of a generalized polynomial, whose coefficients are represented by the partial derivatives of the component functions of a player's payoff evaluated at the unit vector. This permits to use the mathematical results for counting the number of positive zeros of this kind of polynomials to determine the maximum number of interior symmetric equilibrium candidates. We also provided sufficient conditions for each candidate to be a symmetric equilibrium and sufficient conditions for excluding the null vector as a symmetric equilibrium. Since the set of homogeneous functions is not (generally) closed with respect to the operation of sum, our results extend the set of applications of the existing literature.

Appendix

Proof (Proposition 1) First, by symmetry, we focus on the problem of a representative player i. In a symmetric vector of strategies with positive entries $\mathbf{x}$, player i's payoff is:

$$\sum_{h \in \mathscr{H}} f_h(\mathbf{x}) - C(\mathbf{x}).$$

By homogeneity and Remark 1, we can write that $\frac{\partial f_h}{\partial x_i}(\mathbf{x}) = x^{\alpha_h - 1} \frac{\partial f_h}{\partial x_i}(\mathbf{1})$ and $\frac{\partial C}{\partial x_i}(\mathbf{x}) = x^{s-1} \frac{\partial C}{\partial x_i}(\mathbf{1})$. Thus, the necessary first-order conditions for an interior equilibrium $0 = \frac{\partial u(x_i, \mathbf{x}_{-i})}{\partial x_i}$ can be rewritten as:

$$0 = \sum_{h \in \mathscr{H}} x^{\alpha_h - 1} \frac{\partial f_h}{\partial x_i}(\mathbf{1}) - x^{s-1} \frac{\partial C}{\partial x_i}(\mathbf{1}).$$

□

Proof (Proposition 2) We first establish the following result:

Lemma 1 If a function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous, homogeneous of degree α with $f(\mathbf{0}) = C > 0 \Rightarrow f$ is the constant function $f(x) = C$.

Proof By homogeneity $f(t\mathbf{0}) = t^\alpha f(\mathbf{0})\ \forall t > 0$. Since $f(t\mathbf{0}) = f(\mathbf{0}) = C > 0$, then $\alpha = 0$. Since f is homogeneous of degree 0, then $f(t\mathbf{x}) = t^\alpha f(\mathbf{x}) = f(\mathbf{x})\ \forall \mathbf{x} \in \mathbb{R}^n_+$, namely for a given $\mathbf{x} \in \mathbb{R}^n_+$, f is constant along the line passing through $(0, t\mathbf{x})$. By continuity at the origin, then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $| f(\mathbf{x}) - C | < \varepsilon\ \forall x \in \mathbb{R}^n_+$ with $||\mathbf{x}|| < \delta$. Assume that $\exists \tilde{\mathbf{x}} \in \mathbb{R}^n_+$ such that $f(\tilde{\mathbf{x}}) \neq f(\mathbf{0}) = C$. Given $0 < \bar{\varepsilon} := | f(\tilde{\mathbf{x}}) - C |$, by the continuity of f at the origin $\exists \bar{\delta}$ such that $| f(\mathbf{x}) - C | < \bar{\epsilon}\ \forall x \in \mathbb{R}^n_+$ with $||\mathbf{x}|| < \bar{\delta}$. Let t^* be small enough such that $||t^* \mathbf{x}|| < \bar{\delta}$. Hence $| f(\tilde{\mathbf{x}}) - C | = | f(t^* \tilde{\mathbf{x}}) - C | < \bar{\epsilon} = | f(\tilde{\mathbf{x}}) - C |$, a contradiction. Thus, $f(\mathbf{x}) = C\ \forall \mathbf{x} \in \mathbb{R}^n_+$. □

We must show that $\bar{x}$ is the best reply of player i to $\bar{\mathbf{x}}_{-i}$. The first requirement is that any player i is strictly better off in $\bar{\mathbf{x}}$ rather than after deviating from $\bar{\mathbf{x}}$ by selecting $x_i = 0$ (the corner of the set of available strategies). Hence, we compare $x_i = \bar{x}$ and $x_i = 0$, given that $\mathbf{x}_{-i} = \bar{\mathbf{x}}_{-i}$. If $x_i = \bar{x}$, player i's payoff is:

$$\sum_{h \in \mathscr{H}} f_h(\bar{\mathbf{x}}) - C(\bar{\mathbf{x}}). \tag{A − 1}$$

By homogeneity, (A-1) becomes:

$$\sum_{h \in \mathscr{H}} \bar{x}^{\alpha_h} f_h(\mathbf{1}) - \bar{x}^s C(\mathbf{1}).$$

If instead player i selects $x_i = 0$ his payoff is:

$$\sum_{h \in \mathscr{H}} f_h(0, \bar{\mathbf{x}}_{-i}) - C(0, \bar{\mathbf{x}}_{-i}). \tag{A − 2}$$

By homogeneity, (A-2) becomes:

$$\sum_{h \in \mathscr{H}} \bar{x}^{\alpha_h} f_h(0, \mathbf{1}_{-i}),$$

since $C(0, \mathbf{1}_{-i}) = 0$. Thus, $\bar{x}$ is a better option than 0 if and only if:

$$\sum_{h \in \mathscr{H}} \bar{x}^{\alpha_h} f_h(\mathbf{1}) - \bar{x}^s C(\mathbf{1}) > \sum_{h \in \mathscr{H}} \bar{x}^{\alpha_h} f_h(0, \mathbf{1}_{-i}). \tag{A − 3}$$

Dividing both sides of (A-3) by $\bar{x}^s$, and according to Lemma 1, yields:

$$\frac{1}{C(\mathbf{1})}\sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{c_0}}\left(\frac{f_h(\mathbf{1})-f_h(0,\mathbf{1}_{-i})}{\bar{x}^{s-\alpha_h}}\right)>1,$$

namely condition (I).

We now focus on requirement (II). By homogeneity, the second derivative of $u\left(\frac{x_i}{\bar{x}},\mathbf{1}_{-i}\right)$ with respect to x_i, given $\mathbf{x}_{-i}=\bar{\mathbf{x}}_{-i}$, is:

$$\left.\frac{\partial^2 u}{\partial x_i^2}(x_i,\mathbf{x}_{-i})\right|_{\mathbf{x_{-i}}=\bar{\mathbf{x}}_{\mathbf{-i}}}=\sum_{h\in\mathscr{H}}\bar{x}^{\alpha_h-2}\frac{\partial^2 f_h}{\partial x_i^2}\left(\frac{x_i}{\bar{x}},\mathbf{1}_{-i}\right)-\bar{x}^{s-2}\frac{\partial^2 C}{\partial x_i^2}\left(\frac{x_i}{\bar{x}},\mathbf{1}_{-i}\right).$$

Using Lemma 1, condition (II) in the proposition is equivalent to player i's payoff being (possibly) initially convex and eventually concave and that the equilibrium candidate lies on its concave region.

Conditions (I) and (II) ensure that $\bar{x}$ is the best reply of player i to $\bar{\mathbf{x}}_{-i}$. □

Proof (Proposition 3) The null vector is not a symmetric equilibrium if player i has an incentive to deviate by selecting an infinitesimal x_i. This deviation is profitable when there exists an $\hat{x}>0$ such that $u(x_i,\mathbf{0}_{-i})>u(\mathbf{0})$ for all $x_i\in(0,\hat{x})$. By homogeneity, this condition can be written as:

$$\sum_{h\in\mathscr{H}}x_i^{\alpha_h}f_h(1,\mathbf{0}_{-i})-x_i^sC(1,\mathbf{0}_{-i})>\sum_{h\in\mathscr{H}}f_h(\mathbf{0}),\text{ for all}\,x_i\in(0,\hat{x}).$$

Distinguishing the different types of functions f_h, the previous condition can be rewritten as:

$$\begin{aligned}&\frac{1}{C(1,\mathbf{0}_{-i})}\left(\sum_{h\in\mathscr{H}_c}\left(\frac{f_h(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}}-\frac{f_h(\mathbf{0})}{x_i^s}\right)\right.\\&\left.+\sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{c_0}}\left(\frac{f_h(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}}-\frac{f_h(\mathbf{0})}{x_i^s}\right)\right)>1,\text{ for all}\,x_i\in(0,\hat{x}).\end{aligned}\qquad(\mathrm{A}-4)$$

Since any positive constant function is homogeneous of degree zero,

$$\sum_{h\in\mathscr{H}_c}\left(\frac{f_h(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}}-\frac{f_h(\mathbf{0})}{x_i^s}\right)=\sum_{h\in\mathscr{H}_c}\left(\frac{f_h(1,\mathbf{0}_{-i})-f_h(\mathbf{0})}{x_i^s}\right).$$

Notice also that Lemma 1 implies that $\sum_{h\in\mathscr{H}_c}\left(\frac{f_h(1,\mathbf{0}_{-i})-f_h(\mathbf{0})}{x_i^s}\right)=0$. In consequence, (A-4) simplifies to:

$$\frac{1}{C(1,\mathbf{0}_{-i})}\sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{c_0}}\left(\frac{f_h(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}}-\frac{f_h(\mathbf{0})}{x_i^s}\right)>1,\text{ for all}\,x_i\in(0,\hat{x}).\qquad(\mathrm{A}-5)$$

Notice that there will exist an $\hat{x}$ such that (A-5) is satisfied if and only if

$$\frac{1}{C(1,\mathbf{0}_{-i})}\times\lim_{x_i\to 0^+}\sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{c_0}}\left(\frac{f_h(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}}-\frac{f_h(\mathbf{0})}{x_i^{s}}\right)>1.$$

This inequality holds whenever the fraction with the denominator with the highest exponent has a positive numerator, since the limit of this fraction goes to infinity as x_i approaches 0 from the right and it dominates the remaining terms on the LHS. The cases in hand are the following:

$\alpha_1 < 0$. (A-5) is satisfied if and only if $f_1(1,\mathbf{0}_{-i}) > 0$.
$\alpha_1 = 0$. (A-5) is satisfied if and only if $f_1(1,\mathbf{0}_{-i}) - f_1(\mathbf{0}) > 0$.
$\alpha_1 > 0$. Here, we need to analyze three subcases:

First, if $\alpha_1 > s$ then

$$\lim_{x_i\to 0^+}\sum_{h\in\mathscr{H}_d\cup\mathscr{H}_{c_0}}\left(\frac{f_h(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}}-\frac{f_h(\mathbf{0})}{x_i^{s}}\right)=\lim_{x_i\to 0^+}\sum_{h\in\mathscr{H}_d}-\frac{f_h(\mathbf{0})}{x_i^{s}}<0,$$

as $f_h(\mathbf{0}) \geq 0$. Hence, (A-5) cannot hold.
Second, if $\alpha_1 = s$ then

$$\lim_{x_i\to 0^+}\frac{f_h(1,\mathbf{0}_{-i})}{x_i^{s-\alpha_h}}=\begin{cases} f_1(1,\mathbf{0}_{-i}) & , \text{ if } h=1\\ 0 & , \text{ otherwise.}\end{cases}$$

Thus, given that $f_h(\mathbf{0}) = 0$ for any $h \in \mathscr{H}_{c_0}$ and $f_h(\mathbf{0}) \geq 0$ $\forall h \in \mathscr{H}_d$, condition (A-5) holds if and only if

$$C(1,\mathbf{0}_{-i})^{-1}f_1(1,\mathbf{0}_{-i}) > 1,$$

and

$$f_h(\mathbf{0}) = 0,\ \forall h \in \mathscr{H}_d.$$

Third, if $\alpha_1 < s$ and $f_h(\mathbf{0}) > 0$ for some $h \in \mathscr{H}_d$, then condition (A-5) cannot hold. Instead, if $f_h(\mathbf{0}) = 0$ for all $h \in \mathscr{H}_d$, then condition (A-5) holds if and only if $f_1(1,\mathbf{0}_{-i}) > 0$. □

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