Construction of Wedge-Local Nets of Observables through Longo-Witten Endomorphisms. II

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Dedicated to Roberto Longo on the occasion of his 60th birthday

Abstract: In the first part, we have constructed several families of interacting wedge-local nets of von Neumann algebras. In particular, we discovered a family of models based on the endomorphisms of the U(1)-current algebra $\mathcal{A}^{(0)}$ of Longo-Witten.

In this second part, we further investigate endomorphisms and interacting models. The key ingredient is the free massless fermionic net, which contains the U(1)-current net as the fixed point subnet with respect to the U(1) gauge action. Through the restriction to the subnet, we construct a new family of Longo-Witten endomorphisms on $\mathcal{A}^{(0)}$ and accordingly interacting wedge-local nets in two-dimensional spacetime. The U(1)-current net admits the structure of particle numbers and the S-matrices of the models constructed here do mix the spaces with different particle numbers of the bosonic Fock space.

Contents

1. Introduction

As already explained in Part I [Tan11a], construction of interacting models of Quantum Field Theory in (physical) four spacetime dimensions has been a long-standing open problem, and recently the algebraic approach made progress [Lec08,GL07,GL08, BS08,BLS11,Lec11] and two dimensional cases work particularly well: these works constructed models of QFT with weaker localization property, and in one case such models turned out to be strictly local and fully interacting [Lec08]. One should recall, however, that the models in [Lec08] allow a complete interpretation in terms of particles (asymptotic completeness) and the particle number is preserved under the scattering operator. On the other hand, it is known that in four dimensions an interacting model inevitably involves particle production [Aks65]. In the present paper, we construct a further new family of interacting wedge-local two-dimensional massless models and find that their S-matrices mix the spaces with different particle numbers.

In fact, the requirement to involve particle production non-perturbatively is already not simple. On the one hand, an asymptotically complete model must behave like the free theory and hence must be compatible with the Fock space structure at asymptotic time. On the other hand, a particle production process properly means a violation of the Fock structure at physical time. To overcome this difficulty, one would have to "deform" the free theory in a somewhat involved way (cf. [Lec11]) or should rely on a nice trick. Here we take the second way. Standard examples and techniques from Conformal Field Theory provide such a trick.

Conformal Field Theory has been well studied particularly on the circle, which can be seen as a chiral part of 1+1 dimensional theory. There are many important examples of such models, or nets in operator-algebraic terms, and both field-theoretic and operator-algebraic techniques allow one to analyze their interrelationships. Our trick can be briefly summarized as follows: we consider the free complex fermionic field ψ on the circle; the field ψ admits a gauge group action by U(1), and the fixed point with respect to this action is known to be isomorphic to the algebra of the U(1)-current J. Both fields are free fields acting naturally on the Fock space (fermionic and bosonic, respectively) but the correspondence between the spaces is quite involved. The passage to 1+1 dimensional models is simply the tensor product of two such chiral parts. Now, we can easily "deform" the two-dimensional Dirac field (built up from the chiral parts $\psi \otimes \mathbb{1}$ and $\mathbb{1} \otimes \psi$) in such a way that it commutes with the product action of the gauge group U(1) × U(1). Hence the deformation restricts to the algebra of the conserved current $J^\mu = (J^0, J^1) = (J \otimes \mathbb{1} + \mathbb{1} \otimes J, \mathbb{1} \otimes J - J \otimes \mathbb{1})$, and this deformation is sufficiently complicated so that the resulting S-matrix does not preserve the bosonic Fock structure, thanks to the involved fermion-boson correspondence.

In Part I, we have constructed a family of two-dimensional massless models based on the free current J^μ or more precisely its net $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$ of von Neumann algebras of observables. The main ingredient was endomorphisms of the algebra $\mathcal{A}^{(0)}(\mathbb{R}_+)$ of observables localized in the positive half-line $\mathbb{R}_+$ commuting with the translations. A family of such endomorphisms has been studied first by Longo and Witten [LW11] in order to construct Quantum Field Theory with boundary. We used those endomorphisms

to construct two-dimensional models without boundary. In the present article, we study the fermi net $\text{Fer}_{\mathbb{C}}$ generated by the free complex fermionic field ψ and its Longo-Witten endomorphisms. We construct endomorphisms of $\text{Fer}_{\mathbb{C}}$ which commute with the gauge action of U(1), hence restrict to the fixed point subnet of $\mathcal{A}^{(0)}$. It turns out that the restricted endomorphisms cannot be implemented by second quantization operators, hence are different from the ones considered in [LW11]. We again knit them up to construct S-matrices and wedge-local nets.

Then the fixed point with respect to the action of U(1) $\times$ U(1) is considered. We find that its asymptotic behaviour is the same as the free (bosonic) current J^{μ} and the S-matrix does not preserve the space of $1 + 1$ particles (1 left and 1 right moving particle) in the sense of the Fock space structure. We stress that the Fock space particle number has no intrinsic meaning as particles, because we are in a massless case where just a scattering between two waves is considered. One has to pass to the massive case to talk about particle production. We will discuss in more detail the implication of this phenomenon at the end of Sect. 5.

This paper is organized as follows. In Sect. 2 we recall the standard notions of algebraic QFT and the scattering theory of two-dimensional massless models [Buc75,DT11]. Some simple observations are given about subtheories and inner symmetries. Main examples of nets, the free complex fermionic net $\text{Fer}_{\mathbb{C}}$ and the U(1)-current net $\mathcal{A}^{(0)}$, are introduced in Sect. 3. Although it is well-known [KR87,Kac98,Reh98] that the fixed point subnet $\text{Fer}_{\mathbb{C}}$ with respect to U(1) is $\mathcal{A}^{(0)}$ at the field-theoretic level, we prove it in the framework of algebraic approach. Section 4 is devoted to the construction of new Longo-Witten endomorphisms on $\mathcal{A}^{(0)}$. They are used in Sect. 5 to construct new interacting wedge-local nets. Outlook and open problems are summarized in Sect. 6.

2. Preliminaries

2.1. Fermi nets on S^1. Here we give a summary of one-dimensional nets, since they will be our building blocks of the construction of two-dimensional interacting models. In the first part, we considered *local* nets of von Neumann algebras on S^1. Since we need to exploit the free fermionic field in this second part, a generalized concept of nets is recalled.

We follow the definition in [CKL08] and denote by $\text{Möb}^{(2)}(\cong \text{SL}(2, \mathbb{R}) \cong \text{SU}(1, 1))$ the double cover of the Möbius group $\text{PSL}(2, \mathbb{R})$. We denote by $\mathcal{I}$ **the set of proper intervals** $I \subset S^1$, where proper means that I is open and connected and neither a dense nor empty set. A (Möbius covariant) **fermi net** is an assignment of von Neumann algebras $\mathcal{F}_0(I)$ on $\mathcal{H}_{\mathcal{F}_0}$ to intervals $I \in \mathcal{I}$ on S^1 satisfying the following conditions:

(1) *Isotony.* If $I_1 \subset I_2$, then $\mathcal{F}_0(I_1) \subset \mathcal{F}_0(I_2)$.
(2) *Möbius covariance.* There exists a strongly continuous unitary representation U_0 of the group $\text{Möb}^{(2)}$ such that for any interval $I \in \mathcal{I}$ it holds that

$$U_0(g)\mathcal{F}_0(I)U_0(g)^* = \mathcal{F}_0(gI), \quad \text{for } g \in \text{Möb}^{(2)},$$

where the action of $\text{Möb}^{(2)} \cong \text{SU}(1, 1)$ on S^1 is defined through linear fractional transformation.
(3) *Positivity of energy.* The generator of the one-parameter subgroup of the lift of rotations in Möb in the representation U_0 is positive.
(4) *Existence of the vacuum.* There is a unique (up to a phase) unit vector Ω_0 in $\mathcal{H}_{\mathcal{F}_0}$ which is invariant under the action of U_0, and cyclic for $\bigvee_{I \in S^1} \mathcal{F}_0(I)$.

(5) $\mathbb{Z}_2$-*grading*. There is a unitary operator Γ_0 with $\Gamma_0^2 = \mathbb{1}$ such that $\Gamma_0\Omega_0 = \Omega_0$ and $\mathrm{Ad}\,\Gamma_0(\mathcal{F}_0(I)) = \mathcal{F}_0(I)$.

(6) *Graded locality*. If $I_1 \cap I_2 = \emptyset$, then $[\mathcal{F}_0(I_1), \mathrm{Ad}\,Z_0(\mathcal{F}_0(I_2))] = 0$, where $Z_0 := \frac{\mathbb{1}-\mathrm{i}\Gamma_0}{1-\mathrm{i}}$.

If the grading operator is trivial: $Z_0 = \mathbb{1}$, then the net $\mathcal{F}_0$ is said to be **local**.

Among the consequences of these conditions are (see [CKL08]):

(7) *Reeh-Schlieder property*. The vector Ω_0 is cyclic and separating for each $\mathcal{F}_0(I)$.

(8) *Additivity*. If $I = \bigcup_i I_i$, then $\mathcal{F}_0(I) = \bigvee_i \mathcal{F}_0(I_i)$.

(9) *Twisted Haag duality on* S^1. For an interval $I \in \mathcal{I}$, it holds that $\mathcal{F}_0(I)' = \mathrm{Ad}\,Z_0(\mathcal{F}_0(I'))$, where I' is the interior of the complement of I in S^1.

(10) *Bisognano-Wichmann property*. The modular group $\Delta_{\mathcal{F}_0(\mathbb{R}_+)}^{\mathrm{it}}$ of $\mathcal{F}_0(\mathbb{R}_+)$ with respect to Ω_0 is equal to $U_0(\delta(-2\pi t))$, where S^1 is identified as the one-point compactification of $\mathbb{R}$ as below and δ is the one-parameter group of dilations.

(11) *Irreducibility*. It holds that $\bigvee_{I \in \mathcal{I}} \mathcal{F}_0(I) = B(\mathcal{H}_{\mathcal{F}_0})$.

Each algebra $\mathcal{F}_0(I)$ is referred to as a local algebra (even for a fermi net). Note that if the grading operator Γ_0 is trivial, then the definition of fermi net coincides with the one of *local* Möbius-covariant net. We identify the circle S^1 and the compactified real line $\mathbb{R} \cup \{\infty\}$ through the Cayley transform

$$t = -\mathrm{i}\frac{z-1}{z+1} \Longleftrightarrow z = -\frac{t-\mathrm{i}}{t+\mathrm{i}}, \quad t \in \mathbb{R}, \ \ z \in S^1 \subset \mathbb{C}$$

and refer to the algebra $\mathcal{F}_0(I)$ for an interval $I \subset \mathbb{R}$. The representation U_0 of $\mathrm{M\ddot{o}b}^{(2)} \cong \mathrm{SL}(2, \mathbb{R})$ restricts indeed to a projective unitary representation of $\mathrm{PSL}(2, \mathbb{R})$ [CKL08]. Let ρ be the (4π periodic) lift of the **rotations** in $\mathrm{PSU}(1, 1)$ (acting by $\rho(\theta)z = \mathrm{e}^{\mathrm{i}\theta}z$) to $\mathrm{M\ddot{o}b}^{(2)}$ and let us denote $R_0(\theta) = U_0(\rho(\theta)) = \mathrm{e}^{\mathrm{i}\theta L_0}$. Under the identification between S^1 and $\mathbb{R} \cup \{\infty\}$, one can talk about the **translations** and **dilations** of $\mathbb{R}$, which are included in Möb. In particular, the representation of translations (which we denote by τ) plays a crucial role. Let us denote $T_0(t) = U_0(\tau(t))$.

A **Longo-Witten endomorphism** of a fermi net $\mathcal{F}_0$ is an endomorphism of the algebra $\mathcal{F}_0(\mathbb{R}_+)$ implemented by a unitary V_0 which commutes with the translation $T_0(t)$. A family of Longo-Witten endomorphisms has been found for the U(1)-current net and the real free fermion net [LW11]. The examples will be explained later in detail.

Note that a Longo-Witten endomorphism is uniquely implemented up to scalar. Indeed, since it commutes with translation, $\mathrm{Ad}\,V_0$ is an endomorphism of $\mathcal{F}_0(\mathbb{R}_+ + t)$ for any $t \in \mathbb{R}$. If there is another unitary W_0 which satisfies $\mathrm{Ad}\,W_0(x) = \mathrm{Ad}\,V_0(x)$ for any $x \in \mathcal{F}_0(\mathbb{R}_+ + t)$, $t \in \mathbb{R}$, then by the irreducibility $W_0^*V_0$ must be scalar.

2.2. Subnets and the character argument. Let $\mathcal{F}_0$ be a fermi (or local) net on $\mathcal{H}_{\mathcal{F}_0}$. Another assignment $\mathcal{A}_0$ of von Neumann algebras $\{\mathcal{A}_0(I)\}_{I \in \mathcal{I}}$ on $\mathcal{H}_{\mathcal{F}_0}$ is called a **subnet** of $\mathcal{F}_0$ if it satisfies isotony, Möbius covariance with respect to the same U_0 for $\mathcal{F}_0$ and it holds that $\mathcal{A}_0(I) \subset \mathcal{F}_0(I)$ for every interval $I \in \mathcal{I}$. We simply write $\mathcal{A}_0 \subset \mathcal{F}_0$. In this case, let us denote $\mathcal{H}_{\mathcal{A}_0} = \overline{\bigvee_{I \in \mathcal{I}} \mathcal{A}_0(I)\Omega_0}$. Then it is immediate to see that $\mathcal{A}_0(I)$ and U_0 restrict to $\mathcal{H}_{\mathcal{A}_0}$, and by this restriction $\mathcal{A}_0|_{\mathcal{H}_{\mathcal{A}_0}}$ becomes a fermi net with the representation of covariance $U_0|_{\mathcal{H}_{\mathcal{A}_0}}$. This restriction is also said to be a subnet of $\mathcal{F}_0$ if no confusion arises.

For a fermi net $\mathcal{F}_0$ on S^1, a **gauge automorphism** α_0 is a family of automorphisms $\{\alpha_{0,I}\}$ of local algebras which satisfies the consistency condition

$$\alpha_{0,I_2}|_{\mathcal{A}_0(I_1)} = \alpha_{0,I_1} \quad \text{for } I_1 \subset I_2.$$

If a gauge automorphism α_0 preserves the vacuum state $\langle \Omega_0, \cdot \, \Omega_0 \rangle$, it is said to be an **inner symmetry**. An inner symmetry α_0 can be unitarily implemented by the formula $V_{\alpha_0} x \Omega_0 = \alpha_0(x)\Omega_0$, where x is an element of some local algebra $\mathcal{F}_0(I)$. We say that a *compact* group G acts on the net $\mathcal{F}_0$ when there are automorphisms $\{\alpha_{0,g}\}_{g \in G}$ which satisfy the composition law when restricted to local algebras. The **fixed point subnet** with respect to this action of G is the subnet defined by $\mathcal{F}_0^G(I) := \mathcal{F}_0(I)^G$.

Let $\mathcal{F}_0$ be a fermi net and $\mathcal{A}_0$ be a subnet. Recall that, for a Möbius covariant fermi net, the Bisognano-Wichmann property is automatic. As a consequence, for each interval there is a conditional expectation $E_{0,I} : \mathcal{F}_0(I) \to \mathcal{A}_0(I)$ which preserves the vacuum state $\langle \Omega_0, \cdot \, \Omega_0 \rangle$ and is implemented by the projection $P_{\mathcal{A}_0}$ onto $\mathcal{H}_{\mathcal{A}_0}$ (see [Tak03, Thm. IX.4.2]). This projection $P_{\mathcal{A}_0}$ contains much information of $\mathcal{A}_0$.

Consider the case where $\mathcal{A}_0 = \mathcal{F}_0^G$ is the fixed point subnet with respect to an action α_0 of a compact group G by inner symmetry. Then we have a unitary representation V_{α_0} of G on $\mathcal{H}_{\mathcal{F}_0}$. If we write the set of invariant vectors with respect to V_{α_0} by $\mathcal{H}_{\mathcal{F}_0}^G$, it holds that $\mathcal{H}_{\mathcal{F}_0}^G = \mathcal{H}_{\mathcal{A}_0}$. Indeed, the inclusion $\mathcal{H}_{\mathcal{A}_0} \subset \mathcal{H}_{\mathcal{F}_0}^G$ is obvious. On the other hand, for $x \in \mathcal{F}_0(I)$, we have

$$\left(\int_G \alpha_0(x) \, \mathrm{d}g \right) \Omega_0 = \int_G \left(V_{\alpha_0}(g) x \Omega_0 \right) \mathrm{d}g,$$

which implies that any vector in $\mathcal{H}_{\mathcal{F}_0}^G$ can be approximated from $\mathcal{H}_{\mathcal{A}_0}$ by the Reeh-Schlieder property.

For later use, we put here a simple observation.

Proposition 2.1. *In the situation above, if a Longo-Witten endomorphism is implemented by W_0 and W_0 commutes with V_{α_0}, then $\mathrm{Ad}\, W_0$ restricts to a Longo-Witten endomorphism of the fixed point subnet $\mathcal{A}_0$.*

Proof. The unitary W_0 commutes with the projection $P_{\mathcal{A}_0}$, hence also with the conditional expectation E_0 onto $\mathcal{A}_0$. □

Let $\mathcal{F}_0$ be fermi (or local) net on $\mathcal{H}_{\mathcal{F}_0}$. The Hilbert space $\mathcal{H}_{\mathcal{F}_0}$ is graded by the action of the rotation subgroup $R_0(\theta) = \mathrm{e}^{\mathrm{i}\theta L_0}$:

$$\mathcal{H}_{\mathcal{F}_0} = \mathbb{C}\Omega_0 \oplus \bigoplus_{r \in \frac{1}{2}\mathbb{N}} \mathcal{H}_r = \bigoplus_{r \in \frac{1}{2}\mathbb{N}_0} \mathcal{H}_r$$

with $\mathcal{H}_r = \{\xi \in \mathcal{H}_{\mathcal{F}_0} : R_0(\theta)\xi = \mathrm{e}^{\mathrm{i}r\theta}\xi\}$ and the sum only going over $\mathbb{N}_0$ for a local net. The **conformal character** of the net $\mathcal{F}_0$ is given as a formal power series of $t = \mathrm{e}^{-\beta}$:

$$\mathrm{tr}_{\mathcal{H}_{\mathcal{F}_0}}(\mathrm{e}^{-\beta L_0}) = \sum_{r \in \frac{1}{2}\mathbb{N}_0}^{\infty} \dim \mathcal{H}_r \cdot t^r.$$

Let us assume that there is an action of $G = \mathrm{U}(1)$ by inner symmetry. We denote by $V_0(\theta)$ the implementing unitary. Then V_0 and U_0 commute and $\mathcal{H}_{F_0}$ is graded also by the gauge action $V_0(\theta) = \mathrm{e}^{\mathrm{i}\theta Q_0}$:

$$\mathcal{H}_{\mathcal{F}_0} = \mathbb{C}\Omega_0 \oplus \bigoplus_{r \in \frac{1}{2}\mathbb{N}, q \in \mathbb{Z}} \mathcal{H}_{r,q} = \bigoplus_{q \in \mathbb{Z}} \mathcal{H}_{\cdot, q}, \quad \text{with} \quad \mathcal{H}_{\cdot, q} := \bigoplus_{r \in \frac{1}{2}\mathbb{N}_0} \mathcal{H}_{r,q}$$

and the character is given as a formal power series in $t = \mathrm{e}^{-\beta}$ and $z = \mathrm{e}^{-E}$:

$$\mathrm{tr}_{\mathcal{H}_{\mathcal{F}_0}} (\mathrm{e}^{-\beta L_0 - E Q_0}) = \sum_{r \in \frac{1}{2} \mathbb{N}_0, q \in \mathbb{Z}} \dim \mathcal{H}_{r,q} \cdot t^r z^q.$$

Recall that it holds that $\mathcal{H}_{\mathcal{F}_0}^G = \mathcal{H}_{\mathcal{A}_0}$. The operator Q_0 acts by 0 on $\mathcal{H}_{\mathcal{F}_0}^G$, hence we can obtain the conformal character of $\mathcal{A}_0$ just by taking the coefficient of z^0 in $\mathrm{tr}_{\mathcal{H}_{\mathcal{F}_0}} (\mathrm{e}^{-\beta L_0 - E Q_0})$.

Later in this paper we need to compare the size of two subnets. Let $\mathcal{A}_0 \subset \mathcal{B}_0 \subset \mathcal{F}_0$ be an inclusion of three fermi nets. If the conformal characters of $\mathcal{A}_0$ and $\mathcal{B}_0$ coincide, then this means that the subspaces $\mathcal{H}_{\mathcal{A}_0}$ and $\mathcal{H}_{\mathcal{B}_0}$ coincide, since we have already an inclusion $\mathcal{H}_{\mathcal{A}_0} \subset \mathcal{H}_{\mathcal{B}_0}$ and the coefficients of the conformal character are the dimensions of eigenspaces of L_0. This in turn implies that two subnets $\mathcal{A}_0$ and $\mathcal{B}_0$ are the same since the conditional expectations which are implemented by $P_{\mathcal{A}_0}$, $P_{\mathcal{B}_0}$ are the same. We will see such an argument in an example.

2.3. Scattering theory of waves in $\mathbb{R}^2$ (revisited).

Here we just collect some basic notions regarding scattering theory of two-dimensional massless models. As recalled in Part I [Tan11a], this theory has been established by Buchholz [Buc75] and extended to the wedge-local case [DT11]. A **Borchers triple** on a Hilbert space $\mathcal{H}$ is a triple $(\mathcal{M}, T, \Omega)$ of a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$, a unitary representation T of $\mathbb{R}^2$ on $\mathcal{H}$ and a vector $\Omega \in \mathcal{H}$ such that

- Ad $T(t_0, t_1)(\mathcal{M}) \subset \mathcal{M}$ for $(t_0, t_1) \in W_R = \{(x_0, x_1) \in \mathbb{R}^2 : x_1 > |x_0|\}$, the standard right wedge.
- The joint spectrum sp T is contained in the closed forward lightcone $\overline{V}_+ = \{(p_0, p_1) \in \mathbb{R}_2 : p_0 \geq |p_1|\}$.
- Ω is a unique (up to scalar) invariant vector under T, and cyclic and separating for $\mathcal{M}$.

We recall that one interprets $\mathcal{M}$ as the algebra assigned to the wedge W_R. Let $\mathcal{W}$ be the set of wedges, i.e. the set of all $W = g W_R$, where g is a Poincaré transformation, then we define the **wedge-local net** $\mathcal{W} \ni W \mapsto \mathcal{M}(W)$ associated with the Borchers triple $(\mathcal{M}, T, \Omega)$ by $\mathcal{M}(W_R + a) = T(a) \mathcal{M} T(a)^*$ and $\mathcal{M}(W_R' + a) = T(a) \mathcal{M}' T(a)^*$. With the help of the modular objects one can define a representation of the Poincaré group extending the one of translations T [Bor92]. For details we refer to the first part.

Take a Borchers triple $(\mathcal{M}, T, \Omega)$ and $x \in B(\mathcal{H})$. We write $x(a) = \mathrm{Ad}\, T(a)(x)$ for $a \in \mathbb{R}^2$ and consider observables sent to lightlike directions with parameter $\mathcal{T}$:

$$x_\pm(h_\mathcal{T}) := \int h_\mathcal{T}(t) x(t, \pm t) \, \mathrm{d}t,$$

where $h_\mathcal{T}(t) = |\mathcal{T}|^{-\varepsilon} h(|\mathcal{T}|^{-\varepsilon}(t - \mathcal{T}))$, $0 < \varepsilon < 1$ is a constant, $\mathcal{T} \in \mathbb{R}$ and h is a nonnegative symmetric smooth function on $\mathbb{R}$ such that $\int h(t) \, \mathrm{d}t = 1$. Then for $x \in \mathcal{M}$, the limits $\Phi_+^{\mathrm{out}}(x) := \text{s-}\lim_{\mathcal{T} \to +\infty} x_+(h_\mathcal{T})$ and $\Phi_-^{\mathrm{in}}(x) := \text{s-}\lim_{\mathcal{T} \to -\infty} x_-(h_\mathcal{T})$ exist. Furthermore we set $\Phi_+^{\mathrm{in}}(y') := J_\mathcal{M} \Phi_+^{\mathrm{out}}(J_\mathcal{M} y' J_\mathcal{M}) J_\mathcal{M}$, $\Phi_-^{\mathrm{out}}(y') := J_\mathcal{M} \Phi_-^{\mathrm{in}}(J_\mathcal{M} y' J_\mathcal{M}) J_\mathcal{M}$ for $y' \in \mathcal{M}'$, where $J_\mathcal{M}$ is the modular conjugation of $\mathcal{M}$ with respect to Ω. The properties of these **asymptotic fields** are summarized in [DT11, Tan11a]. For example, it holds that $\Phi_+^{\mathrm{in}}(y') = \text{s-}\lim_{\mathcal{T} \to -\infty} y'_+(h_\mathcal{T})$ and $\Phi_-^{\mathrm{out}}(y') = \text{s-}\lim_{\mathcal{T} \to +\infty} y'_-(h_\mathcal{T})$.

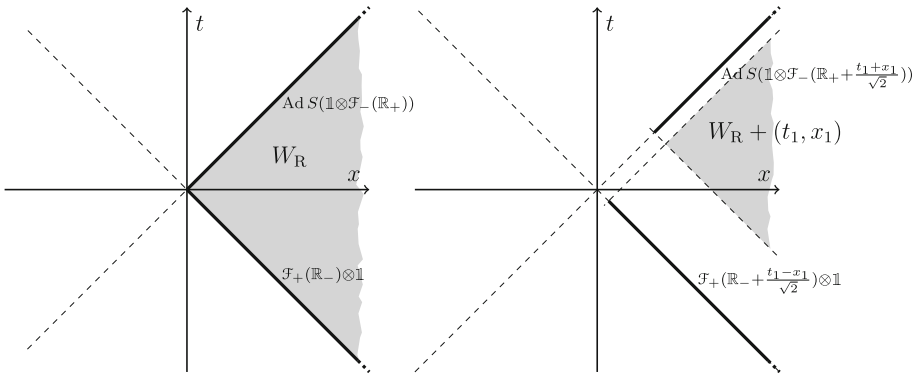Construction of Wedge-Local Nets of Observables. II

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Fig. 1. On the definition of the wedge-local net

Let $\mathcal{H}_+$ (respectively $\mathcal{H}_-$) be the space of the single excitations with positive momentum, (respectively with negative momentum), i.e. $\mathcal{H}_+ = \{\xi \in \mathcal{H} : T(t,t)\xi = \xi$ for $t \in \mathbb{R}\}$ (respectively $\mathcal{H}_- = \{\xi \in \mathcal{H} : T(t,-t)\xi = \xi$ for $t \in \mathbb{R}\}$). For $\xi_+ \in \mathcal{H}_+$, $\xi_- \in \mathcal{H}_-$, there are sequences of local operators $\{x_n\}, \{y_n\} \subset \mathcal{M}$ and $\{x'_n\}, \{y'_n\} \subset \mathcal{M}'$ such that s-$\lim_{n\to\infty} P_+ x_n \Omega = $ s-$\lim_{n\to\infty} P_+ x'_n \Omega = \xi_+$ and s-$\lim_{n\to\infty} P_- y_n \Omega = $ s-$\lim_{n\to\infty} P_+ y'_n \Omega = \xi_-$. We define collision states as in [DT11]:

$$\xi_+ \overset{\text{in}}{\times} \xi_- = \text{s-}\lim_{n\to\infty} \Phi^{\text{in}}_+(x'_n)\Phi^{\text{in}}_-(y_n)\Omega, \quad \xi_+ \overset{\text{out}}{\times} \xi_- = \text{s-}\lim_{n\to\infty} \Phi^{\text{out}}_+(x_n)\Phi^{\text{out}}_-(y'_n)\Omega.$$

We denote by $\mathcal{H}^{\text{in}}$ (respectively $\mathcal{H}^{\text{out}}$) the subspace generated by $\xi_+ \overset{\text{in}}{\times} \xi_-$ (respectively $\xi_+ \overset{\text{out}}{\times} \xi_-$). The isometry

$$S : \mathcal{H}^{\text{out}} \ni \xi_+ \overset{\text{out}}{\times} \xi_- \longmapsto \xi_+ \overset{\text{in}}{\times} \xi_- \in \mathcal{H}^{\text{in}}$$

is called the **scattering operator** or the **S-matrix** of the Borchers triple $(\mathcal{M}, T, \Omega)$. We say that the Borchers triple $(\mathcal{M}, T, \Omega)$ is **interacting** if S is not equal to the identity operator on $\mathcal{H}^{\text{out}}$ and **asymptotically complete (with respect to waves)** if it holds that $\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}} = \mathcal{H}$.

We have studied the general structure of asymptotically complete local and wedge-local nets (using Borchers triple) in [Tan11a, Sect. 3]. The point was that for a given (strictly local) $(\mathcal{M}, T, \Omega)$ we can construct the chiral net, and the original object $\mathcal{M}$ can be recovered from the chiral net and a single operator S. Here we rephrase this observation from the point of view of constructing examples based on chiral components. See also the general structure of asymptotically complete *strictly local* nets [Tan11a, Sect. 3]

Proposition 2.2. *Let $\mathcal{F}_\pm$ be two fermi nets on S^1 defined on $\mathcal{H}_\pm$ and assume that there is a unitary operator S on $\mathcal{H}_+ \otimes \mathcal{H}_-$ commuting with $T_+ \otimes T_-$, leaving $\mathcal{H}_+ \otimes \Omega_-$ and $\Omega_+ \otimes \mathcal{H}_-$ pointwise invariant, such that $x \otimes \mathbb{1}$ commutes with $\text{Ad}\,S(x' \otimes \mathbb{1})$, where $x \in \mathcal{F}_+(\mathbb{R}_-)$ and $x' \in \text{Ad}\,Z_+(\mathcal{F}_+(\mathbb{R}_+))$, and $\text{Ad}\,S(\mathbb{1} \otimes y)$ commutes with $\mathbb{1} \otimes y'$, where $y \in \mathcal{F}_-(\mathbb{R}_+)$ and $y' \in \text{Ad}\,Z_-(\mathcal{F}_-(\mathbb{R}_-))$. Then the triple (see Fig. 1)*

- $\mathcal{M}_S := \{x \otimes \mathbb{1}, \text{Ad}\,S(\mathbb{1} \otimes y) : x \in \mathcal{F}_+(\mathbb{R}_-), y \in \mathcal{F}_-(\mathbb{R}_+)\}''$,

- $T(t, x) := T_+(\frac{t-x}{\sqrt{2}}) \otimes T_-(\frac{t+x}{\sqrt{2}})$,
- $\Omega := \Omega_+ \otimes \Omega_-$

is an asymptotically complete Borchers triple with the S-matrix S.

Proof. As in Part I [Tan11a], the conditions on T and Ω are automatic because they are just tensor products of objects for fermi nets. Similarly, the condition that $\mathrm{Ad}\,T(a)\mathcal{M}_S \subset \mathcal{M}_S$ for $a \in W_R$ is easily seen from the assumption that T commutes with S and the covariance of fermi nets.

What remains is the cyclicity and separating property of Ω for $\mathcal{M}_S$. Cyclicity is immediate because we have

$$\mathcal{M}_S \Omega \supset \{x \otimes \mathbb{1} \cdot S(\mathbb{1} \otimes y)S^* \cdot \Omega : x \in \mathcal{F}_+(\mathbb{R}_-), y \in \mathcal{F}_-(\mathbb{R}_+)\}$$
$$= \{x \otimes y \cdot \Omega : x \in \mathcal{F}_+(\mathbb{R}_-), y \in \mathcal{F}_-(\mathbb{R}_+)\}$$

by the assumed property of S, and the latter set is total in $\mathcal{H}_+ \otimes \mathcal{H}_-$ by the Reeh-Schlieder property for fermi nets. As for the separating property, we define:

$$\mathcal{M}_S^1 := \{\mathrm{Ad}\,S(x' \otimes \mathbb{1}), \mathbb{1} \otimes y' : x' \in \mathrm{Ad}\,Z_+(\mathcal{F}_+(\mathbb{R}_+)), y' \in \mathrm{Ad}\,Z_-(\mathcal{F}_-(\mathbb{R}_-))\}''.$$

By an analogous proof, one sees that Ω is cyclic for $\mathcal{M}_S^1$. Furthermore, $\mathcal{M}_S$ and $\mathcal{M}_S^1$ commute by assumption. Hence Ω is separating for $\mathcal{M}_S$. In other words, $(\mathcal{M}_S, T, \Omega)$ is a Borchers-triple.

It is immediate that $\Phi_+^{\mathrm{out}}(x \otimes \mathbb{1}) = x \otimes \mathbb{1}$ and $\Phi_-^{\mathrm{in}}(\mathrm{Ad}\,S(\mathbb{1} \otimes y)) = \mathrm{Ad}\,S(\mathbb{1} \otimes y)$ (the latter follows since S commutes with T). Similarly, we have $\Phi_+^{\mathrm{in}}(\mathrm{Ad}\,S(x' \otimes \mathbb{1})) = \mathrm{Ad}\,S(x' \otimes \mathbb{1})$ and $\Phi_-^{\mathrm{out}}(\mathbb{1} \otimes y') = \mathbb{1} \otimes y'$. From this, one concludes that the Borchers triple $(\mathcal{M}_S, T, \Omega)$ is asymptotically complete and its S-matrix is S. $\square$

We remark that we see $(\mathcal{M}_S, T, \Omega)$ as a fermi (i.e. twisted local) net defined by $\mathcal{M}(W_R') = \mathrm{Ad}\,Z_+ \otimes Z_-(\mathcal{M})$ and that the scattering theory of waves [Buc75] is considered to be an analogue of the Haag-Ruelle scattering theory and it is not intended to be applied to fermionic nets. But we will not pay much attention to this restriction, since our result is a construction of wedge-local nets with a free massless bosonic net as the asymptotic net, and fermionic nets appear as auxiliary objects.

2.4. Restriction of wedge-local nets. We consider a Borchers triple $(\mathcal{M}, T, \Omega)$. It is in some cases interesting to consider a subalgebra $\mathcal{N}$ of $\mathcal{M}$. Let us denote $\mathcal{H}_\mathcal{N} := \overline{\mathcal{N}\Omega}$.

Proposition 2.3. *If the subspace $\mathcal{H}_\mathcal{N}$ is invariant under T and $\mathrm{Ad}\,T(a)(\mathcal{N}) \subset \mathcal{N}$ for $a \in W_R$. Then $(\mathcal{N}|_{\mathcal{H}_\mathcal{N}}, T|_{\mathcal{H}_\mathcal{N}}, \Omega)$ is a Borchers triple on $\mathcal{H}_\mathcal{N}$.*

Proof. The components $\mathcal{N}$, T and Ω naturally restrict to $\mathcal{H}_\mathcal{N}$. The conditions on T and Ω are trivial, even restricted to $\mathcal{H}_\mathcal{N}$. The cyclicity of Ω is immediate from the definition of $\mathcal{H}_\mathcal{N}$. Since Ω is already separating for $\mathcal{M}$, so also it is for $\mathcal{N}$. Endomorphic action of T on $\mathcal{N}$ is in the hypothesis. $\square$

We call a triple $(\mathcal{N}, T, \Omega)$ a **(Borchers) subtriple** of $(\mathcal{M}, T, \Omega)$ if $\mathcal{N}$ is a subalgebra of $\mathcal{M}$, $\mathcal{H}_\mathcal{N}$ is invariant under $T(a)$, $\mathrm{Ad}\,T(a)(\mathcal{N}) \subset \mathcal{N}$ for $a \in W_R$, and $\mathcal{N}$ is invariant under $\mathrm{Ad}\,\Delta_\mathcal{M}^{it}$, where $\Delta_\mathcal{M}$ is the modular operator of $\mathcal{M}$ with respect to Ω.

Recall that a Borchers triple $(\mathcal{M}, T, \Omega)$ gives rise to a strictly local net if Ω is cyclic for $\mathcal{M} \cap \mathrm{Ad}\,T(a)(\mathcal{M})'$ for any $a \in W_R$. We call such a triple therefore **strictly local**. The following proposition shows that the concept of Borchers subtriple corresponds to the one of a local subnet.

Proposition 2.4. *If a Borchers triple* $(\mathcal{M}, T, \Omega)$ *is strictly local, any subtriple* $(\mathcal{N}, T, \Omega)$ *is again strictly local when restricted on* $\overline{\mathcal{N}\Omega}$.

Proof. Since $\mathcal{N}$ is invariant under the modular automorphism $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}$, there is a conditional expectation E from $\mathcal{M}$ onto $\mathcal{N}$ which preserves the state $\langle\Omega, \cdot\,\Omega\rangle$ and is implemented by the projection $P_{\mathcal{N}}$ (see [Tak03, Thm. IX.4.2] for the original reference and [Tan11b, App. A] for an application to nets).

We have to show that Ω is cyclic for the relative commutant $\mathcal{N} \cap \mathrm{Ad}\,T(a)(\mathcal{N})'$ on the subspace $\mathcal{H}_{\mathcal{N}}$. Let us denote $\mathcal{M}_{0,a} := \mathcal{M} \cap \mathrm{Ad}\,T(a)(\mathcal{M})'$. We claim that $E(\mathcal{M}_{0,a})$ is contained in $\mathcal{N} \cap \mathrm{Ad}\,T(a)(\mathcal{N})'$. Indeed, by the definition of E, the image of E is contained in $\mathcal{N}$. Furthermore, if $x \in \mathcal{M}_{0,a}$, $y \in \mathrm{Ad}\,T(a)(\mathcal{N}) \subset \mathrm{Ad}\,T(a)(\mathcal{M})$, then

$$E(x)y = E(xy) = E(yx) = yE(x),$$

hence they commute and the image $E(\mathcal{M}_{0,a})$ lies in the relative commutant. Now we have

$$\overline{(\mathcal{N} \cap \mathrm{Ad}\,T(a)(\mathcal{N})')\,\Omega} \supset \overline{E(\mathcal{M}_{0,a})\Omega} \supset \overline{P_{\mathcal{N}}\mathcal{M}_{0,a}\Omega} = \mathcal{H}_{\mathcal{N}}$$

by the assumed strict locality of $(\mathcal{M}, T, \Omega)$. $\quad\square$

Let $(\mathcal{B}, U, \Omega)$ be an *asymptotically complete* local Poincaré covariant net on $\mathbb{R}^2$ fulfilling the Bisognano-Wichmann property (see [Tan11a] for related definitions). We recall that one can define the (out-) asymptotic algebras $\mathcal{B}_+ \otimes \mathcal{B}_-$ and the scattering operator S which is a unitary operator, and that it is possible to recover the original net by the formula

$$\mathcal{B}(W_{\mathrm{R}}) = \{x \otimes \mathbb{1}, \mathrm{Ad}\,S(\mathbb{1} \otimes y) : x \in \mathcal{B}_+(\mathbb{R}_-), y \in \mathcal{B}_-(\mathbb{R}_+)\}''.$$

Note that $(\mathcal{B}(W_{\mathrm{R}}), U|_{\mathbb{R}^2}, \Omega)$ is an asymptotically complete, strictly local Borchers triple. Here we exhibit a simple way to construct subtriples. Let $\mathcal{A}_+, \mathcal{A}_-$ be (Möbius covariant) subnets of $\mathcal{B}_+, \mathcal{B}_-$, respectively. If we define

$$\mathcal{N} = \{x \otimes \mathbb{1}, \mathrm{Ad}\,S(\mathbb{1} \otimes y) : x \in \mathcal{A}_+(\mathbb{R}_-), y \in \mathcal{A}_-(\mathbb{R}_+)\}'',$$

then $(\mathcal{N}, U|_{\mathbb{R}^2}, \Omega)$ is a Borchers subtriple of $(\mathcal{B}(W_{\mathrm{R}}), U|_{\mathbb{R}^2}, \Omega)$. Indeed, conditions regarding $\mathcal{N}, U|_{\mathbb{R}^2}, \Omega$ are immediate. As for the invariance of $\mathcal{N}$ under $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}$, it suffices to note that S and $\Delta_{\mathcal{M}}^{it}$ commute ([Tan11a, Lem. 2.4], cf.[Buc75]) and that $\mathcal{A}_+(\mathbb{R}_-)$ and $\mathcal{A}_-(\mathbb{R}_+)$ are preserved by $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}$ because of Bisognano-Wichmann property.

The trouble is, however, that such Borchers triples constructed as above are not necessarily asymptotically complete in general. Indeed, the out-asymptotic states span the subspace $\overline{\mathcal{A}_+(\mathbb{R}_-)\Omega} \otimes \overline{\mathcal{A}_-(\mathbb{R}_+)\Omega}$. It is easy to see that this coincides with the full space $\overline{\mathcal{N}\Omega}$ if and only if it is invariant under S.

Since a clear-cut scattering theory is so far available only for asymptotically complete nets, it is worthwhile to give a general condition to assure that subnets are asymptotically complete. For simplicity, we consider the following situation: let $\mathcal{A}_0$ be a fermi net on $\mathcal{H}_0$ with an action of a compact group G by inner symmetry implemented by V_g. Suppose that there is a unitary operator S on $\mathcal{H}_0 \otimes \mathcal{H}_0$ such that $(\mathcal{M}_S, T, \Omega)$ is a Borchers triple where

- $\mathcal{M}_S := \{x \otimes \mathbb{1}, \mathrm{Ad}\, S(\mathbb{1} \otimes y) : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}''$,
- $T(t, x) := T_0(\frac{t-x}{\sqrt{2}}) \otimes T_0(\frac{t+x}{\sqrt{2}})$,
- $\Omega := \Omega_0 \otimes \Omega_0$,

as in Proposition 2.2.

Proposition 2.5. *If S commutes with $V_g \otimes V_{g'}$, $g, g' \in G$, then the triple*

- $\mathcal{N}_S := \{x \otimes \mathbb{1}, \mathrm{Ad}\, S(\mathbb{1} \otimes y) : x \in \mathcal{A}_0^G(\mathbb{R}_-), y \in \mathcal{A}_0^G(\mathbb{R}_+)\}''$ *(restricted to $\overline{\mathcal{N}_S \Omega}$),*
- $T(t, x) := T_0(\frac{t-x}{\sqrt{2}}) \otimes T_0(\frac{t+x}{\sqrt{2}})$, *(restricted to $\overline{\mathcal{N}_S \Omega}$),*
- $\Omega := \Omega_0 \otimes \Omega_0$

is an asymptotically complete Borchers triple with asymptotic algebra $\mathcal{A}_0^G \otimes \mathcal{A}_0^G$ and scattering operator $S|_{\overline{\mathcal{N}_s \Omega}}$.

Proof. As remarked above, $(\mathcal{N}_S, T, \Omega)$ is a Borchers triple on $\mathcal{H}_{\mathcal{N}_S}$, hence the only thing to be proven is asymptotic completeness. We show that the subspace $\overline{\mathcal{A}_0^G(\mathbb{R}_-) \otimes \mathcal{A}_0^G(\mathbb{R}_+)\Omega}$ is invariant under S.

We claim that $\overline{\mathcal{A}_0^G(\mathbb{R}_-)\Omega_0}$ coincides with the subspace $\mathcal{H}_0^G$ of invariant vectors under $\{V_g\}_{g \in G}$. Indeed, for any $x \in \mathcal{A}_0$, the averaging $\int_g V_g x \Omega_0 \, \mathrm{d}g = \left(\int_g \alpha_g(x) \, \mathrm{d}g\right) \Omega_0$ gives a projection onto $\mathcal{H}_0^G$. By the Reeh-Schlieder property, any vector in $\mathcal{H}_0^G$ can be approximated by vectors in $\mathcal{A}_0^G(\mathbb{R}_-)\Omega_0$. The converse inclusion is obvious.

Now it is easy to see that $\overline{\mathcal{A}_0^G(\mathbb{R}_-) \otimes \mathcal{A}_0^G(\mathbb{R}_+)\Omega} = \mathcal{H}_0^G \otimes \mathcal{H}_0^G$. This is the space of invariant vectors under the action $\{V_g \otimes V_{g'} : g, g' \in G\}$. Since S commutes with $V_g \otimes V_{g'}$ by assumption, this subspace is preserved under S. Then, as remarked before, $\overline{\mathcal{N}_S \Omega}$ coincides with $\overline{\mathcal{A}_0^G(\mathbb{R}_-) \otimes \mathcal{A}_0^G(\mathbb{R}_+)\Omega}$ and we obtain the asymptotic completeness. The statement on S-matrix is immediate from the definition and by Proposition 2.2. $\square$

3. Examples of Fermi Nets

3.1. U(1)-current net $\mathcal{A}^{(0)}$. Let U_1 be the irreducible unitary positive-energy representation of Möb with lowest weight 1 on a Hilbert space denoted by $\mathcal{H}_{\mathcal{A}^{(0)}}^1$, which can be identified with the one-particle space of the U(1)-current. This has the following concrete realization: consider $C^\infty(S^1, \mathbb{R})$, where we write the periodic function $f \in C^\infty(S^1, \mathbb{R})$ as a Fourier series

$$f(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ik\theta}, \quad \hat{f}_k = \int_0^{2\pi} e^{-ik\theta} f(\theta) \frac{\mathrm{d}\theta}{2\pi} = \overline{\hat{f}_{-k}}.$$

We introduce a semi-norm

$$\|f\|^2 = \sum_{k=1}^\infty k \cdot |\hat{f}_k|^2$$

and a complex structure, i.e. an isometry $\mathcal{J}$ w.r.t. $\|\cdot\|$ satisfying $\mathcal{J}^2 = -1$, by $\mathcal{J} : \hat{f}_k \mapsto -i\,\mathrm{sign}(k)\hat{f}_k$, and finally we get the Hilbert space $\mathcal{H}_{\mathcal{A}^{(0)}}^1 = \overline{C^\infty(S^1, \mathbb{R})/\mathbb{R}}^{\|\cdot\|}$

by completion with respect to the norm $\|\cdot\|$, where $\mathbb{R}$ is identified with the constant functions. By abuse of notation we denote also the image of $f \in C^\infty(S^1, \mathbb{R})$ in $\mathcal{H}^1_{\mathcal{A}^{(0)}}$ by f. The scalar product (linear in the second component) and the sesquilinear form $\omega(\cdot, \cdot) \equiv \Im\langle \cdot, \cdot \rangle$ are given by

$$\langle f, g \rangle = \sum_{k=1}^{\infty} k \hat{f}_k \hat{g}_{-k},$$

$$\omega(f, g) = \frac{-i}{2} \sum_{k \in \mathbb{Z}} k \hat{f}_k \hat{g}_{-k} = \frac{1}{2} \int_0^{2\pi} f(\theta) g'(\theta) \frac{d\theta}{2\pi} = \frac{1}{4\pi} \int f \, dg,$$

respectively. The unitary action U_1 of Möb on $\mathcal{H}^1_{\mathcal{A}^{(0)}}$ is induced by the action on $C^\infty(S^1, \mathbb{R})$ $(U_1(g)f)(\theta) := f(g^{-1}(\theta))$.

For $I \in \mathcal{J}$ we denote by $H(I)$ the closure of the subspace of real functions with support in I. This space is standard (i.e. $H(I) + iH(I)$ is dense in $\mathcal{H}^1_{\mathcal{A}^{(0)}}$ and $H(I) \cap iH(I) = \{0\}$) and the family $\{H(I)\}_{I \in \mathcal{J}}$ is a local Möbius covariant net of standard subspaces [Lon08, LW11].

We explain briefly the bosonic second quantization procedure in general. Let $\mathcal{H}^1$ be a separable Hilbert space, the **one-particle space**, and $\omega(\cdot, \cdot) = \Im\langle \cdot, \cdot \rangle$ the sesquilinear form. There are unitaries $W(f)$ for $f \in \mathcal{H}^1$ fulfilling

$$W(f)W(g) = e^{-i\omega(f,g)} W(f + g) = e^{-2i\omega(f,g)} W(g)W(f)$$

and acting naturally on the bosonic Fock space $e^{\mathcal{H}^1}$ over $\mathcal{H}^1$. This space is given by $e^{\mathcal{H}^1} = \oplus_{n=0}^{\infty} P_n(\mathcal{H}^1)^{\otimes n}$, where P_n is the projection $P_n(\xi_1 \otimes \cdots \otimes \xi_n) = 1/n! \sum_\sigma \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$ and the sum goes over all permutations. The set of coherent vectors $e^h := \oplus_{n=0}^{\infty} h^{\otimes n}/\sqrt{n!}$ with $h \in \mathcal{H}^1$ is total in $e^{\mathcal{H}^1}$ and it holds $\langle e^f, e^h \rangle = e^{\langle f, h \rangle}$. The vacuum is given by $\Omega = e^0$ and the action of $W(f)$ is given by $W(f)e^0 = e^{-\frac{1}{2}\|f\|^2} e^f$, in other words the vacuum representation is characterized by $\phi(W(f)) = e^{-\frac{1}{2}\|f\|^2}$, where $\phi(\cdot) = \langle \Omega, \cdot \Omega \rangle$. For a real subspace $H \subset \mathcal{H}^1$, we define the von Neumann algebra

$$R(H) = \{W(f) : f \in H\}'' \subset B(e^{\mathcal{H}^1}).$$

Let U be a unitary on the one-particle space $\mathcal{H}^1$, then $e^U := \oplus_{n=0}^{\infty} U^{\otimes n}$ acts on coherent states by $e^U e^h = e^{Uh}$ and is therefore a unitary on $e^{\mathcal{H}^1}$, the **second quantization unitary**.

We obtain the U(1)-**current net** $\mathcal{A}^{(0)}$ on $\mathcal{H}_{\mathcal{A}^{(0)}} := e^{\mathcal{H}^1_{\mathcal{A}^{(0)}}}$ with $\Omega_0 = e^0$ by defining $\mathcal{A}^{(0)}(I) := R(H(I))$ which is covariant with respect to $U(g) := e^{U_1(g)}$. For $f \in C^\infty(S^1, \mathbb{R})$ we consider a self-adjoint operator $J(f)$ given by the generator of the unitary one-parameter group $W(t \cdot f) = e^{it \cdot J(f)}$ with $t \in \mathbb{R}$. This defines the usual current (field operator) smeared with the real test function f, which fulfills $J(f)\Omega_0 = f \in \mathcal{H}^1_{\mathcal{A}^{(0)}}$ and

$$[J(f), J(g)] = 2i\omega(f, g) = \sum_k k \hat{f}_k \hat{g}_{-k} = \frac{i}{2\pi} \int f \, dg.$$

It can be extended to complex test functions via $J(f + ig) = J(f) + iJ(g)$, and one obtains the usual operator valued (z-picture) distribution $J(z)$ with the relations

$$J(f) = \sum_{n \in \mathbb{Z}} \hat{f}_n J_n = \oint_{S^1} f(z) J(z) \frac{dz}{2\pi i}, \qquad J(z) = \sum_n J_n z^{-n-1},$$

$$[J_m, J_n] = m \delta_{m+n,0},$$

where the modes $J_n = J(e_n)$ with $e_n(\theta) = e^{in\theta}$ satisfy $J_n \Omega_0 = 0$ for $n \geq 0$.

The space $\mathcal{H}_{\mathcal{A}^{(0)}}$ is spanned by vectors of the form $\xi = J_{-n_1} \cdots J_{-n_k} \Omega_0$ with $0 < n_1 \leq \cdots \leq n_k$ with "energy" $N = \sum_m n_m$, i.e. $R(\theta)\xi = e^{iN\theta}\xi$. Therefore it is graded with respect to the rotations

$$\mathcal{H}_{\mathcal{A}^{(0)}} = \mathbb{C}\Omega_0 \oplus \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{\mathcal{A}^{(0)},n}, \qquad \mathcal{H}_{\mathcal{A}^{(0)},n} = \bigoplus_{k=1}^{n} \bigoplus_{\substack{0 < n_1 \leq \cdots \leq n_k \\ n_1 + \cdots + n_k = n}} \mathbb{C} J_{-n_1} \cdots J_{-n_k} \Omega_0,$$

and $\dim \mathcal{H}_{\mathcal{A}^{(0)},n}$ is the number of partitions of n elements, whose generating function $p(t)$ is the inverse of Euler's function $\phi(t) = \prod_{k=1}^{\infty}(1 - t^k)$, and therefore the conformal character of the U(1)-current net is given by ($t = e^{-\beta}$):

$$\mathrm{tr}_{\mathcal{H}_{\mathcal{A}^{(0)}}} (e^{-\beta L_0}) = \sum_{n=0}^{\infty} \dim \mathcal{H}_{\mathcal{A}^{(0)},n} \cdot t^n = \prod_{n \in \mathbb{N}} (1 - t^n)^{-1}$$

(a conformal character is defined as a formal power series, but it is often convergent for $|t| < 1$ and here we used the formula $(1 - z)^{-1} = 1 + z + z^2 \cdots$). It will be convenient to use the real parametrization $x \in \mathbb{R} \cong S^1 \setminus \{-1\}$ of the cut circle and use the conventions

$$f(s) = \int_{\mathbb{R}} e^{-isp} \hat{f}(p) \, dp.$$

By writing $f(s) = f_0(\theta(s))$, for $f_0 \in C^\infty(S^1, \mathbb{R})$, where $\theta(s) = 2\arctan(s)$, the space $\mathcal{H}^1_{\mathcal{A}^{(0)}}$ above can be identified with the space $L^2(\mathbb{R}_+, p \, dp)$ in which the space $\mathcal{S}(\mathbb{R}, \mathbb{R})$ embeds by restriction of the Fourier transformation to $\mathbb{R}_+$. In other words $\mathcal{H}^1_{\mathcal{A}^{(0)}}$ can be seen as the closure of the space $\mathcal{S}(\mathbb{R}, \mathbb{R})$ with complex structure $\mathfrak{J}\hat{f}(p) = i\,\mathrm{sign}(p)\hat{f}(p)$ and the scalar product and sesquilinear form given by:

$$\langle f, g \rangle = \int_{\mathbb{R}_+} \hat{f}(-p)\hat{g}(p)p \, dp,$$

$$\omega(f, g) = \frac{-i}{2} \int_{\mathbb{R}} \hat{f}(-p)\hat{g}(p)p \, dp = \frac{1}{4\pi} \int_{\mathbb{R}} f(x)g'(x) \, dx.$$

Using the above identification we denote for $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ by $J(f)$ the smeared current with $J(f)\Omega_0 = f \in \mathcal{H}^1_{\mathcal{A}^{(0)}}$. In this parametrization commutation relations read:

$$[J(f), J(g)] = \frac{i}{2\pi} \int_{\mathbb{R}} f(x)g'(x) \, dx = \int_{\mathbb{R}} \hat{f}(-p)\hat{g}(p)p \, dp.$$

3.2. The free complex fermion net $\mathrm{Fer}_{\mathbb{C}}$. We construct the net of the free complex fermion on the circle, which can be seen as the chiral part of the net of the free massless Dirac (or complex) fermion on two dimensional Minkowski space. The notations of this section are basically in accordance with [Was98], but we use a different convention of positive-energy, which leads to the conjugated complex structure. For giving a simple description of the one-particle space, we consider first the Hilbert space $L^2(S^1)$ and the Hardy space $H^2(S^1)$, namely

$$H^2(S^1) := \left\{ f \, : \, \text{analytic on the unit disk } D, \ \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty \right\}.$$

Any function in $H^2(S^1)$ has a L^2-boundary value and can be considered as an element of $L^2(S^1)$. In this sense, $H^2(S^1)$ is a subspace of $L^2(S^1)$. Furthermore, it holds that

$$H^2(S^1) = \{ f \in L^2(S^1) : \hat{f}_n = 0 \text{ for } n < 0 \},$$

where $\hat{f}_n$ is the nth Fourier component of f. We denote the orthogonal projection onto $H^2(S^1)$ by P.

The group

$$\mathrm{SU}(1,1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1 \right\}$$

acts on the circle S^1 by $g \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$ and there is a unitary action of $\mathrm{SU}(1,1)$ on $L^2(S^1)$ by

$$(U(g)f)(z) := (V_g f)(z) = \frac{1}{-\bar{\beta} z + \alpha} f(g^{-1} \cdot z).$$

One sees that the projection P commutes with V_g, since $V_g f$ is still an analytic function for $|\alpha| > |\beta|$.

Then one defines a new Hilbert space $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} = \overline{P L^2(S^1)} \oplus (\mathbb{1} - P)L^2(S^1)$: namely, $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}}$ is identical with $L^2(S^1)$ as a real linear space and the multiplication by i is given by $-i(2P - \mathbb{1})$, or in other words, by $-i$ on $P L^2(S^1)$ and i on $(\mathbb{1} - P)L^2(S^1)$. Because P and $U(g)$ commute, the action of $\mathrm{SU}(1,1)$ remains unitary on $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}}$.

Then for $I \in \mathcal{J}$ one takes real Hilbert subspaces $K(I) := L^2(I)$ of $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}}$. These subspaces turn out to be standard [Was98, Thm. (p. 497)]. If I_1 and I_2 are disjoint intervals, $K(I_1)$ are real orthogonal to $K(I_2)$, in other words $K(I_1) \subset K(I_2)^\perp$, where $K^\perp = \{ \xi \in \mathcal{H} : \Re\langle \xi, K \rangle = 0 \}$. It turns out that $I \mapsto K(I)$ is a twisted-local Möbius covariant net of standard subspaces.

We briefly explain the fermionic second quantization in general. Let $\mathcal{H}^1$ be a complex Hilbert space and $\mathcal{H} = \Lambda(\mathcal{H}^1)$ be the antisymmetric (fermionic) Fock space obtained by completing the exterior algebra with the inner product. For $A \in B(\mathcal{H}^1)$ with $\|A\| \leq 1$ we define $\Lambda(A)$ to be $A^{\otimes k}$ on $\mathcal{H}^k := \Lambda^k(\mathcal{H}^1) \subset (\mathcal{H}^1)^{\otimes k}$. The space is $\mathbb{Z}_2$ graded by $\Gamma := \Lambda(-\mathbb{1})$. We define $Z = \frac{\mathbb{1} - i\Gamma}{1 - i}$ and note that $Z^2 = \Gamma$. For $f \in \mathcal{H}^1$ let $a(f)$ be the bounded operator obtained by continuing the exterior multiplication $f \wedge \cdot$. The operators fulfill the complex Clifford relations $a(f)^* a(g) + a(g)a(f)^* = \langle f, g \rangle$ and $\{a(f), a(g)\} = \{a(f)^*, a(g)^*\} = 0$ for all $f, g \in \mathcal{H}^1$. For a standard subspace $K \subset \mathcal{H}^1$ we define the von Neumann algebra

$$C(K) = \{c(f) : f \in K\}'' \subset B(\Lambda \mathcal{H}^1),$$

where $c(f) = a(f) + a(f)^*$, which fulfills the real Clifford relations $c(f)c(g) + c(g)c(f) = 2\Re\langle f, g \rangle$. By $\Omega = 1 \in \Lambda^0$ we denote the vacuum which is cyclic and separating for $C(K)$ for every standard subspace $K \subset \mathcal{H}^1$. Further **Haag-Araki duality** holds, i.e. $C(K^\perp)$ equals $C(K)^\perp := ZC(K)'Z^*$, the **twisted commutant** of $C(K)$. For a unitary U on $\mathcal{H}^1$, $\Lambda(U)c(f)\Lambda(U^*) = c(Uf)$ holds, which implies that C is covariant with respect to the unitaries $U(\mathcal{H}^1)$, i.e. $\Lambda(U)C(K)\Lambda(U)^* = C(UK)$.

We note that in the case like the complex fermion the one-particle space is obtained from a Hilbert space $\mathcal{H}^1$ (the space of test functions) and a projection P by $\mathcal{H}^1_P = P\mathcal{H}^1 \oplus \overline{P^\perp \mathcal{H}^1}$, and one gets a new representation of the complex Clifford algebra on $\Lambda(\mathcal{H}^1_P)$ by $a_P(f) = a(Pf) + a(\overline{P^\perp f})^*$, where $a(f)$ is the creation operator. For a standard subspace $K \subset \mathcal{H}^1_P$ which is invariant under the multiplication of $i_{\mathcal{H}^1}$ in $\mathcal{H}^1$, the von Neumann algebra $C(K)$ on $\Lambda(\mathcal{H}^1_P)$ coincides with the von Neumann algebra $\{a_P(f), a_P(f)^* : f \in K\}''$. Indeed, the one inclusion follows from $c(f) = a_P(f) + a_P(f)^*$ and the other follows from Araki-Haag duality and $\{a_P(f), c(g)\} = \langle g, f \rangle_{\mathcal{H}^1} = \Re\langle g, f \rangle_{\mathcal{H}^1} - i\Re\langle g, i_{\mathcal{H}^1} f \rangle_{\mathcal{H}^1_P} = 0$ for $f \in K$ and $g \in K^\perp$. We further note that the space $\Lambda(\mathcal{H}^1_P)$ is as a real Hilbert space the same as $\Lambda(\mathcal{H}^1)$ and can be identified canonically with $\Lambda(P\mathcal{H}^1) \otimes \Lambda(\overline{P^\perp \mathcal{H}^1})$.

We turn to the concrete case where $\mathcal{H}^1 = L^2(S^1)$ and define the net $\mathrm{Fer}_\mathbb{C}(I) := C(K(I)) = \{a_P(f), a_P(f)^* : f \in L^2(I)\}''$ (where here $a_P(f) := a(\overline{Pf}) + a(P^\perp f)^*$) on $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} = \Lambda(\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}}) \cong \Lambda(\overline{PL^2(S^1)}) \otimes \Lambda(P^\perp L^2(S^1))$ which is isotonic by definition and fulfills twisted duality, namely by Haag-Araki duality $\mathrm{Fer}_\mathbb{C}(I') = C(K(I)^\perp) = C(K(I))^\perp = \mathrm{Fer}_\mathbb{C}(I)^\perp$. In addition, the net $\mathrm{Fer}_\mathbb{C}$ is Möbius covariant. Indeed, we can take the representation $\Lambda U(\cdot)$ by promoting the one-particle representation U to the second quantization operator. It is easy to see that the covariance of this net $\mathrm{Fer}_\mathbb{C}$ follows from the covariance of the net of standard spaces K. The representation ΛU has positive energy as does the representation U, and leaves invariant the vacuum vector Ω_0 of the Fock space. Summing up, the net $\mathrm{Fer}_\mathbb{C}$ is a fermi net (cf. [Was98]). This net is referred to as the **free complex fermi net** on S^1. The scalar multiplication by a constant phase $e^{-i\vartheta}$ in the original structure of the one-particle space is still a unitary operator in the new structure. Its promotion by the second quantization $V(\theta)$ implements an action of U(1) on $\mathrm{Fer}_\mathbb{C}$ by inner symmetry. This will be referred to as the U(1)-gauge action.

For $r \in \frac{1}{2} + \mathbb{Z}$, let $\psi_r = a_P(e_{-r-\frac{1}{2}})$ and $\bar{\psi}_r = a_P(e_{r-\frac{1}{2}})^*$, where $e_r \in L^2(S^1)$ with $e_r(\theta) = e^{i\theta r}$. The $\psi_r, \bar{\psi}_r$ are the modes of the free complex fermion, namely

$$\{\psi_n, \psi_m\} = \{\bar{\psi}_m, \bar{\psi}_n\} = 0$$
$$\{\bar{\psi}_n, \psi_m\} = \delta_{m+n,0},$$
$$\psi_n^* = \bar{\psi}_{-n},$$

and it holds that $\psi_r \Omega_0 = \bar{\psi}_r \Omega_0 = 0$ for $r \in \frac{1}{2} + \mathbb{N}_0$. Each of ψ_r or $\bar{\psi}_r$ has norm 1 following from the commutation relation. We can introduce the usual fields ($f, g \in L^2(S^1)$) and operator valued distributions in the z-picture:

$$\Psi(f) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} \hat{f}_r \Psi_r = \oint_{S^1} f(z) z^{-\frac{1}{2}} \Psi(z) \frac{dz}{2\pi i}, \quad \Psi(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} \Psi_r z^{-r-\frac{1}{2}},$$

$$\bar{\psi}(f) = \psi(\bar{f})^* = a_P(e_{-\frac{1}{2}}f)^*, \quad \{\bar{\psi}(f), \psi(g)\} = \oint f(z)g(z)\frac{dz}{2\pi i z},$$

where Ψ is either ψ or $\bar{\psi}$. The fields ψ, $\bar{\psi}$ are covariant, e.g. $U(g)\psi(f)U(g)^* = \psi(f_g)$ with $f_g(z) = \frac{1}{|\alpha - \bar{\beta}z|}f(g^{-1}z)$ for $g = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \in SU(1,1)$.

We note that vectors of the form

$$\xi = \psi_{-r_1}\cdots\psi_{-r_k}\bar{\psi}_{-s_1}\cdots\bar{\psi}_{-s_\ell}\Omega_0$$

with $0 < r_1 < \cdots < r_k$ and $0 < s_1 < \cdots < s_\ell$ form a basis of $\mathcal{H}_{\text{Fer}_\mathbb{C}} = \Lambda(\mathcal{H}^1_{\text{Fer}_\mathbb{C}})$ and that such a ξ is an eigenvector for the rotations, $R(\theta)\xi \equiv e^{i\theta L_0}\xi = e^{iN\theta}\xi$ with $N = \sum_{j=1}^k r_j + \sum_{j=1}^\ell s_j$ and of the gauge action $V(\theta)\xi \equiv e^{i\theta Q}\xi = e^{i(k-\ell)\theta}\xi$. In each vector of this basis the rth energy level can either be empty, be occupied by ψ_{-r} or $\bar{\psi}_{-r}$ or be occupied by both. The contribution of this level to the character $\text{tr}_{\mathcal{H}_{\text{Fer}_\mathbb{C}}}(e^{-\beta L_0 - EQ})$ is then 1, zt^r, $z^{-1}t^r$ or t^{2r}, respectively, where $t = e^{-\beta}$ and $z = e^{-E}$. By summing over all possibilities one gets that the character of $\text{Fer}_\mathbb{C}$ is given by (cf. [Kac98, Reh98]):

$$\text{tr}_{\mathcal{H}_{\text{Fer}_\mathbb{C}}}(e^{-\beta L_0 - EQ}) = \text{tr}_{\mathcal{H}_{\text{Fer}_\mathbb{C}}}(t^{L_0}z^Q) = \prod_{r\in\mathbb{N}_0+\frac{1}{2}}(1 + zt^r + z^{-1}t^r + t^{2r})$$

$$= \prod_{r\in\mathbb{N}_0+\frac{1}{2}}(1 + zt^r)(1 + z^{-1}t^r)$$

$$= p(t)\sum_{q\in\mathbb{Z}}z^q t^{\frac{q^2}{2}},$$

where the last equality follows directly from the Jacobi triple product formula (see [Apo76, Thm. 14.6])

$$\prod_{r\in\mathbb{N}}(1 + zw^{2r-1})(1 + z^{-1}w^{2r-1})(1 - w^{2r}) = \sum_{q\in\mathbb{Z}}z^q w^q$$

by setting $2r - 1 = 2n$ and $t = w^2$. In particular, for the local net $\text{Fer}_\mathbb{C}^{U(1)}$ the character is given by $\text{tr}_{\mathcal{H}_{\text{Fer}_\mathbb{C}}^{U(1)}}(e^{-\beta L_0}) = p(t)$, since it is the fixed point with respect to the $U(1)$-gauge action and the conformal character is the coefficient of z^0.

3.3. $U(1)$-*current net as a subnet of* $\text{Fer}_\mathbb{C}$. In this section we use the well-known fact that the Wick product $:\bar{\psi}\psi:$ of the complex fermion ψ equals the $U(1)$-current and give an analogue of the boson-fermion correspondence (see e.g. [Kac98, 5.2]) in the operator algebraic setting. Let us denote by $\mathcal{D}_0$ the subspace of $\Lambda(\mathcal{H}_P^1)$ of vectors with finite energy:

$$\mathcal{D}_0 := \text{span}\left\{\psi_{-r_1}\cdots\psi_{-r_k}\bar{\psi}_{-s_1}\cdots\bar{\psi}_{-s_l}\Omega_0 : k, l \in \mathbb{N}_0, r_i, s_j \in \mathbb{N} + \frac{1}{2}\right\}.$$

Then we define the unbounded operators on the domain $\mathcal{D}_0$:

$$J_n = \sum_{r+s=n} :\bar{\psi}_r \psi_s: = \sum_{r<0} \bar{\psi}_r \psi_{n-r} - \sum_{r>0} \psi_{n-r} \bar{\psi}_r$$

$$= \sum_r \left(\bar{\psi}_r \psi_{n-r} - \langle \Omega_0, \bar{\psi}_r \psi_{n-r} \Omega_0 \rangle \right)$$

with $r, s \in \frac{1}{2} + \mathbb{Z}$. Note that any vector in $\mathcal{D}_0$ is annihilated by ψ_r for sufficiently large r, thus the action of J_n on such a vector can be defined and remains in $\mathcal{D}_0$. In particular, we have $J_n \Omega_0 = 0$ for $n \in \mathbb{N}_0$.

Lemma 3.1. *On $\mathcal{D}_0$ it holds that*

1. $[J_n, \psi_k] = -\psi_{n+k}$ *and* $[J_n, \bar{\psi}_k] = \bar{\psi}_{n+k}$,
2. $[J_m, J_n] = m\delta_{m+n,0}$.

Proof. Using $[ab, c] = a\{b, c\} - \{a, c\}b$, one obtains $[\bar{\psi}_r \psi_n, \psi_k] = -\delta_{r+k,0}\psi_n$ and $[\psi_n \bar{\psi}_r, \psi_k] = \delta_{r+k,0}\psi_n$ from which directly follows $[J_n, \psi_k] = \sum_{r<0}[\bar{\psi}_r \psi_{n-r}, \psi_k] - \sum_{r>0}[\psi_{n-m}\bar{\psi}_r, \psi_k] = -\psi_{n+k}$. Analogously one shows $[J_n, \bar{\psi}_k] = \bar{\psi}_{n+k}$.

From the Jacobi identity, it follows immediately that $[J_n, J_m]$ commutes with all ψ_k and $\bar{\psi}_k$ and hence $[J_n, J_m]$ is a multiple of the identity, therefore $[J_n, J_m] = \langle \Omega_0, [J_n, J_m]\Omega_0 \rangle \mathbb{1}$. It is

$$[J_n, J_p] = \sum_{r<0}[J_n, \bar{\psi}_r \psi_{p-r}] - \sum_{r>0}[J_n, \psi_{p-r}\bar{\psi}_r]$$

$$= -\sum_{r<0}\left(\bar{\psi}_r \psi_{p-r+n} - \bar{\psi}_{r+n}\psi_{p-r} \right) - \sum_{r>0}\left(\psi_{p-r}\bar{\psi}_{r+n} - \psi_{p-r+n}\bar{\psi}_r \right),$$

and in the case $p \neq -n$, we get $\langle \Omega_0, [J_n, J_p]\Omega_0 \rangle = 0$, and otherwise

$$\langle \Omega_0, [J_n, J_{-n}]\Omega_0 \rangle$$

$$= \begin{cases} \sum_{r<0}\langle \Omega_0, \bar{\psi}_{r+n}\psi_{-r-n}\Omega_0 \rangle = \sum_{r=\frac{1}{2}}^{n-\frac{1}{2}}\langle \Omega_0, \{\bar{\psi}_r, \psi_{-r}\}\Omega_0 \rangle & n > 0 \\ -\sum_{r>0}\langle \Omega_0, \psi_{-r-n}\bar{\psi}_{r+n}\Omega_0 \rangle = -\sum_{r=\frac{1}{2}}^{-n-\frac{1}{2}}\langle \Omega_0, \{\psi_r, \bar{\psi}_{-r}\}\Omega_0 \rangle & n < 0 \end{cases}$$

$$= n,$$

which completes the proof. $\square$

Let L_0 be the generator of the rotation: $R(\theta) = e^{i\theta L_0}$. From its action (see the end of Sect. 3.2), one verifies that $\mathcal{D}_0$ is a core for L_0.

Lemma 3.2 (Linear energy bounds). *It holds that $[L_0, J_n] = -nJ_n$ on $\mathcal{D}_0$. For a trigonometric polynomial $f = \sum_n \hat{f}_n e_n$, where the sum is finite and $\xi \in \mathcal{D}_0$, we have*

$$\|J(f)\xi\| \leq c_f \|(L_0 + 1)\xi\|$$
$$\|[L_0, J(f)]\xi\| \leq c_{\partial_\theta f} \|(L_0 + 1)\xi\|,$$

where c_f depends only on f.

Proof. For the commutation relation, it is enough to choose an energy eigenvector $\xi \in \mathcal{D}_0$, i.e. $L_0\xi = N\xi$. It is $J_nL_0\xi = NJ_n\xi$ and

$$L_0 J_n\xi = L_0\left(\sum_{r<0}\bar\psi_r\psi_{n-r}\xi - \sum_{r>0}\psi_{n-r}\bar\psi_r\xi\right) = (N-n)J_n\xi,$$

and the first statement follows.

We have seen that ψ_r and $\bar\psi_r$ have norm 1 in Sect. 3.2. First we claim that $\|J_n\xi\| \le \|(2(L_0+1)+|n|)\xi\|$. Let ξ be again an eigenvector of L_0, i.e. $L_0\xi = N\xi$. From the defining sum of J_n, one sees that only $2N + |n| + 2$ terms contribute to $J_n\xi$. Hence we have $\|J_n\xi\| \le (2N+|n|+2)\|\xi\| = \|2(L_0+1)+|n|\xi\|$. If the inequality holds for eigenvectors, then for $\{\xi_r\}$ with different eigenvalues, we have $\xi_r \perp \xi_s$ and $J_n\xi_r \perp J_n\xi_s$, and hence

$$\left\|J_n\sum_r\xi_r\right\|^2 = \sum_r\|J_n\xi_r\|^2$$
$$\le \sum_r\|(2(L_0+1)+|n|)\xi_r\|^2$$
$$= \left\|(2(L_0+1)+|n|)\sum_r\xi_r\right\|^2,$$

and the general case follows.

For a smeared field, we have

$$\|J(f)\xi\| = \left\|\sum_n\hat f_n J_n\xi\right\| \le 2\tilde c_f\|(L_0+1)\xi\| + \tilde c_{\partial_\theta f}\|\xi\| \le (2\tilde c_f + \tilde c_{\partial_\theta f})\|(L_0+1)\xi\|,$$

where $\tilde c_f = \sum_n|\hat f_n|$. By defining $c_f = 2\tilde c_f + \tilde c_{\partial_\theta f}$, we obtain the first inequality of the statement. The rest follows by noting that $[L_0, J(f)] = J(i\partial_\theta f)$. $\square$

For a smooth function $f = \sum_{n\in\mathbb{Z}}\hat f_n e_n \in C^\infty(S^1)$, its Fourier coefficients $\hat f_n$ are strongly decreasing and, in particular, it is summable: $\sum_n|\hat f_n| = \tilde c_f < \infty$. Hence we can naturally extend the definition of the smeared current to smooth functions using the above estimate by

$$J(f) = \sum_{n\in\mathbb{Z}}f_n J_n = \sum_{r,s\in\frac12+\mathbb{Z}}f_{r+s}:\psi_r\bar\psi_s:,$$

and the same inequality in Lemma 3.2 holds. The operator is closable since we have $J(f) \subset J(\bar f)^*$ and we still denote the closure by $J(f)$. We note that from the above definition it follows that $J(f)$ is obtained by a limit $\sum_n:\psi(h_n)\bar\psi(k_n):$ with suitable functions such that $\sum_n h_n(\theta)k_n(\vartheta) \to 2\pi f(\theta)\delta(\theta-\vartheta)$. This implies covariance of the "field", i.e. $U(g)J(f)U(g)^* = J(f\circ g^{-1})$.

Recall that $\|\psi_r\| = 1$, hence the smeared field is still bounded: $\|\psi(g)\| \le \tilde c_g$. We claim that, for $f, g \in C^\infty(S^1)$ and $\xi \in \mathcal{D}_0$, $\psi(g)\xi$ is in the domain of $J(f)$. Indeed, for

a trigonometric polynomial g, we have the estimate

$$\|J(f)\psi(g)\xi\| \leq c_f\|(L_0 + 1)\psi(g)\xi\|$$
$$\leq c_f(\tilde{c}_g\|\xi\| + \|[L_0, \psi(g)]\xi + \psi(g)L_0\xi\|)$$
$$\leq c_f(\tilde{c}_g(\|\xi\| + \|L_0\xi\|) + \tilde{c}_{\partial_\theta g}\|\xi\|).$$

Then if we have a sequence of trigonometric polynomial g_n converging to a smooth function $g \in C^\infty(S^1)$, the sequence $\{J(f)\psi(g_n)\xi\}$ is also converging.

Lemma 3.3. *For $\xi, \eta \in \mathcal{D}_0$, it holds that*

$$[J(f), \psi(g)]\xi = -\psi(f \cdot g)\xi$$
$$[J(f), \bar{\psi}(g)]\xi = \bar{\psi}(f \cdot g)\xi$$
$$\langle J(\bar{f})\xi, J(g)\eta \rangle = \langle J(\bar{g})\xi, J(f)\eta \rangle + 2i\omega(f, g)\langle \xi, \eta \rangle.$$

Proof. For trigonometric polynomials f, g, the statements can be proved easily from Lemma 3.1. The general case is shown by approximating first f by polynomials, then g, according to the convergence considered above (as for the third statement, obviously the order of limits does not matter). □

We need the following well-known result [DF77, Thm. 3.1]:

Theorem 3.4 (The commutator theorem). *Let H be a positive self-adjoint operator and A, B symmetric operators defined on a core $\mathcal{D}_0$ for $(H + \mathbb{1})^2$. Assume that there is a constant C such that*

$$\|A\xi\| \leq C\|(H + \mathbb{1})\xi\|, \quad \|B\xi\| \leq C\|(H + \mathbb{1})\xi\|,$$
$$\|[H, A]\xi\| \leq C\|(H + \mathbb{1})\xi\|, \quad \|[H, B]\xi\| \leq C\|(H + \mathbb{1})\xi\|,$$
$$\langle A\xi, B\eta \rangle = \langle B\xi, A\eta \rangle \quad \text{for any } \xi, \eta \in \mathcal{D}_0.$$

Then A and B are essentially self-adjoint on any core of H and any bounded functional calculus of A and B commute.

Remark 3.5. In the original literature [DF77], this theorem is proved under the assumption of certain operator inequalities. In fact, what is really used in the proof of commutativity of bounded functions is the norm estimates $\|A(H + \mathbb{1})^{-1}\| < C$, $\|[H, A](H + \mathbb{1})^{-1}\| < C$ etc. and they follow from the assumptions here. The essential self-adjointness of A and B can be proved by [RS75, Thm. X.37]. An analogous application of this theorem with norm estimates can be found in [BSM90].

By the commutator theorem, we get that $J(f)$ is self-adjoint for $f \in C^\infty(S^1, \mathbb{R})$ and that all bounded functions of $J(f)$ commute with all bounded functions of $J(g)$ for $f, g \in C^\infty(S^1, \mathbb{R})$ with disjoint support.

Let I be a proper interval and let us define the von Neumann algebra

$$\mathcal{B}(I) = \{e^{iJ(f)} : \operatorname{supp} f \subset I\}''.$$

The local net $\mathcal{B}(I)$ restricted to $\overline{\mathcal{B}(I)\Omega_0}$ can be identified with the U(1)-current net $\mathcal{A}^{(0)}$ on $\mathcal{H}_{\mathcal{A}^{(0)}}$, in particular we can identify $\overline{\mathcal{B}(I)\Omega_0} \cong \mathcal{H}_{\mathcal{A}^{(0)}}$.

Proposition 3.6. *Let I be a proper interval, then $\mathcal{B}(I) \subset \operatorname{Fer}_{\mathbb{C}}^{U(1)}(I)$.*

Proof. We see that $\mathcal{B}(I)$ commutes with $\mathrm{Fer}_{\mathbb{C}}(I') = \{c(g) : g \in L^2(I')\}''$ because, for f, g with disjoint supports, $c(g)$ commutes with $J(f)$ on a core by Lemma 3.3 and therefore any spectral projection of $c(g)$ commutes with $J(f)$, and hence with any bounded functions of $J(f)$.

Further because $J(f)$ commutes by construction with the gauge action $V(t)$ and is in particular even because $V(\pi) = \Gamma$, it follows that $\mathcal{B}(I)$ lies in the twisted commutant $\mathrm{Fer}_{\mathbb{C}}(I')^{\perp}$. By twisted Haag duality it is $\mathcal{B}(I) \subset \mathrm{Fer}_{\mathbb{C}}(I')^{\perp} = \mathrm{Fer}_{\mathbb{C}}(I)$ and therefore $\mathcal{B}(I) = \mathcal{B}(I)^{\mathrm{U}(1)} \subset \mathrm{Fer}_{\mathbb{C}}^{\mathrm{U}(1)}(I)$. $\square$

Since the covariance has been seen, we have the following.

Corollary 3.7. $\mathcal{B}$ *is a subnet of* $\mathrm{Fer}_{\mathbb{C}}^{\mathrm{U}(1)}$.

Now the following is straightforward.

Proposition 3.8. *The* U(1)-*fixed point subnet of the complex free fermion net* $\mathrm{Fer}_{\mathbb{C}}$ *is the* U(1)-*current net, i.e.* $\mathrm{Fer}_{\mathbb{C}}^{\mathrm{U}(1)} = \mathcal{B} \cong \mathcal{A}^{(0)}$.

Proof. Let us see $\mathcal{B}$ as a subnet of the fermi net $\mathrm{Fer}_{\mathbb{C}}^{\mathrm{U}(1)}$ on $\mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}^{\mathrm{U}(1)}} \equiv \mathcal{H}_{\cdot,0}$. Further $\overline{\mathcal{B}(I)\Omega}$ does not depend on I by the same proof of the Reeh-Schlieder property and is clearly a subspace of $\mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}^{\mathrm{U}(1)}} \equiv \mathcal{H}_{\cdot,0}$.

In fact they coincide, since we have confirmed that $\mathrm{tr}_{\mathcal{H}_{\mathcal{A}^{(0)}}}(e^{-\beta L_0}) = \mathrm{tr}_{\mathcal{H}_{\cdot,0}}(e^{-\beta L_0}) = p(t)$, where $e^{-\beta} = t$, namely, their conformal characters coincide (see also Sect. 2.2). $\square$

We finish this section by giving the parametrization in x-picture, where the action of the translation is more natural. With

$$f(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\left|\frac{\partial \theta(x)}{\partial x}\right|} e^{i\theta(x)/2} f_0(\theta(x))$$

we identify $L^2(\mathbb{R}) = L^2(\mathbb{R}, dx)$ with $L^2(S^1) = L^2([0, 2\pi], d\theta/(2\pi))$ and therefore the space $\mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}}^1$ is given by $PL^2(\mathbb{R}) \oplus \overline{P^{\perp}L^2(\mathbb{R})}$ with $P : \hat{f}(p) \mapsto \Theta(p)\hat{f}(p)$ and it can be identified in "momentum space" with $L^2(\mathbb{R}_+, 2\pi\,dp) \oplus L^2(\mathbb{R}_+, 2\pi\,dq)$ by

$$f(x) \longmapsto \widehat{Pf}(p) \oplus \overline{\widehat{P^{\perp}f}(-q)} \quad p, q > 0.$$

The field operators are defined for $f \in L^2(\mathbb{R})$ by $\psi(f) = a_P(f)$ and $\bar{\psi}(f) = a_P(\bar{f})^*$. For $\Psi \in \mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}}$ we write its components

$$\Psi_{m,n} \in \mathcal{H}_{m,n} := L^2(\mathbb{R}_+^{m+n}, (2\pi)^{m+n}\,dp_1 \cdots dp_m\,dq_1 \cdots q_n)_-, \tag{1}$$

where $-$ means the antisymmetrization within $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$. By this notation $(\psi(f)\Omega_0)_{1,0}(p) = \hat{f}(p)$ and $(\bar{\psi}(f)\Omega_0)_{0,1}(q) = \hat{f}(q)$. Further the bi-field $:\bar{\psi}(f)\psi(g): = \bar{\psi}(f)\psi(g) - \langle\Omega_0, \bar{\psi}(f)\psi(g)\Omega_0\rangle\mathbb{1}$ creates from the vacuum Ω_0 a fermionic 1+1 particle state $\Psi_{f,g} := :\bar{\psi}(f)\psi(g):\Omega_0$ with $(\Psi_{f,g})_{1,1}(p, q) = -\hat{f}(q)\hat{g}(p)$ and it follows for $h \in C^{\infty}(\mathbb{R}, \mathbb{R})$ that for the U(1)-current J, it holds $(J(h)\Omega_0)_{1,1}(p, q) = -\frac{1}{2\pi}\hat{h}(p + q)$ which is obtained by taking a limit $\sum_n \Psi_{f_n,g_n}$ with test functions $\sum_n f_n(x)\bar{g}_n(y) \to h(x)\delta(x - y)$. We make the important observation that the $J(f)\Omega_0$ generate the one-particle space which we can identify with $\mathcal{H}_{\mathcal{A}^{(0)}}^1$ and this is obviously a proper subspace of the fermionic 1+1-particle space $\mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}}^{1,1}$.

4. A New Family of Longo-Witten Endomorphisms on U(1)-Current Net

We use the description of $\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}} = \overline{PL^2(S^1)} + P^\perp L^2(S^1)$ which equals $L^2(S^1)$ as a real Hilbert space and is described in the beginning of Sect. 3.2. First we decompose $\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}}$ into irreducible representations of $SU(1, 1)$ in a compatible way with $K(I)$. Let us define

$$\mathcal{H}_\mathfrak{R} := \{f \in \mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}} : z^{\frac{1}{2}} f(z) \text{ is real}\},$$

$$\mathcal{H}_\mathfrak{I} := \{f \in \mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}} : z^{\frac{1}{2}} f(z) \text{ is pure imaginary}\}.$$

By their definition, it is clear that $\mathcal{H}_\mathfrak{R}$ and $\mathcal{H}_\mathfrak{I}$ are real Hilbert subspaces of $L^2(S^1)$. In fact, they are complex subspaces with respect to the new complex structure. To see this, we take another description of $\mathcal{H}_\mathfrak{R}$: in terms of Fourier components, it holds that $f \in \mathcal{H}_\mathfrak{R}$ if and only if $f_n = \overline{f_{-n-1}}$. Recall that, on $L^2(S^1)$, the new scalar multiplication by i is given by $\mathrm{i} \cdot f_n = -\mathrm{i} f_n$, $\mathrm{i} \cdot f_{-n-1} = \mathrm{i} f_{-n-1}$ for $n \geq 0$. Hence this condition is preserved under the multiplication by i and $\mathcal{H}_\mathfrak{R}$ is a complex subspace. An analogous argument holds for $\mathcal{H}_\mathfrak{I}$. Next we see that $\mathcal{H}_\mathfrak{R}$ and $\mathcal{H}_\mathfrak{I}$ are orthogonal. Note that because of the change of the complex structure, for $f(z) = \sum_n f_n z^n$ and $h(z) = \sum_n h_n z^n$ the inner product is written as follows:

$$\langle f, h \rangle = \sum_{n \geq 0} f_n \overline{h_n} + \sum_{n < 0} \overline{f_n} h_n.$$

Now $f \in \mathcal{H}_\mathfrak{R}$ implies $f_n = \overline{f_{-n-1}}$ and $h \in \mathcal{H}_\mathfrak{I}$ implies $h_n = -\overline{h_{-n-1}}$ for non-negative n, hence it is easy to see that

$$\langle f, h \rangle = \sum_{n \geq 0} f_n \overline{h_n} + \sum_{n < 0} \overline{f_n} h_n = -\sum_{n \geq 0} \overline{f_{-n-1}} h_{-n-1} + \sum_{n < 0} \overline{f_n} h_n = 0.$$

In other words, these two complex subspaces are mutually orthogonal.

Furthermore, $\mathcal{H}_\mathfrak{R}$ and $\mathcal{H}_\mathfrak{I}$ are invariant under the action of $SU(1, 1)$. We recall that the action is given by $(V_g f)(z) = \frac{1}{-\overline{\beta}z+\alpha} f\left(\frac{\overline{\alpha}z-\beta}{-\overline{\beta}z+\alpha}\right)$. Then if $z^{\frac{1}{2}} f(z)$ is real then it holds that

$$z^{\frac{1}{2}} (V_g f)(z) = \frac{1}{z^{\frac{1}{2}}(-\overline{\beta}z + \alpha)} \left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta}z + \alpha}\right)^{-\frac{1}{2}} \cdot \left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta}z + \alpha}\right)^{\frac{1}{2}} f\left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta}z + \alpha}\right)$$

$$= \frac{1}{(-\overline{\beta} + \alpha\overline{z})^{\frac{1}{2}}(\overline{\alpha}z - \beta)^{\frac{1}{2}}} \cdot \left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta}z + \alpha}\right)^{\frac{1}{2}} f\left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta}z + \alpha}\right)$$

and both factors are real. Similarly one shows that $\mathcal{H}_\mathfrak{I}$ is preserved under V_g. It is obvious that these two representations are intertwined by the multiplication by i in the old complex structure. This is still a unitary map, thus they are unitarily equivalent. One can see that each representation is indeed irreducible, and when restricted to $PSU(1, 1) = PSL(2, \mathbb{R})$, it is the projective positive energy representation with lowest weight $\frac{1}{2}$.

It is easy to see that $e_n^\mathfrak{R} := \{e_n + e_{-n+1}, n \geq 0\}$ and $e_n^\mathfrak{I} := \{\mathrm{i}(e_n + e_{-n+1}), n \geq 0\}$ form bases of $\mathcal{H}_\mathfrak{R}$ and $\mathcal{H}_\mathfrak{I}$, respectively, where $e_n(z) = z^n$ and the multiplication by i is given

in the old structure. Now we describe the gauge action in terms of this basis. By the definition, for a given complex number α with modulus 1, the action is given by the multiplication in the old structure. Hence if $\alpha = \cos\theta + i\sin\theta$, we have $U_\alpha e_n^{\mathfrak{R}} = \cos\theta e_n^{\mathfrak{R}} + \sin\theta e_n^{\mathfrak{I}}$ and $U_\alpha e_n^{\mathfrak{I}} = -\sin\theta e_n^{\mathfrak{R}} + \cos\theta e_n^{\mathfrak{I}}$. This means that U_α acts as the real rotation by θ in this basis.

Construction of endomorphisms. We construct Longo-Witten endomorphisms on the free fermion net $\mathrm{Fer}_{\mathbb{C}}$ commuting with the gauge action. The key is the following theorem. We recall that a **standard pair** $(\tilde{H}, \tilde{T})$ is a standard subspace $\tilde{H} \subset \tilde{\mathcal{H}}$ of a Hilbert space $\tilde{\mathcal{H}}$ and a positive energy representation $\tilde{T}$ of $\mathbb{R}$ on $\tilde{\mathcal{H}}$, such that $\tilde{T}(a)\tilde{H} \subset \tilde{H}$ for $a \geq 0$. If $\tilde{T}$ is maximally abelian, the standard pair is said to be irreducible and there is a (up to unitary equivalence) unique irreducible standard pair.

Theorem 4.1 ([LW11, Thm. 2.6]). *Let $(\tilde{H}, \tilde{T})$ be a standard pair with multiplicity n, i.e. it decomposes into an n-fold direct sum of irreducible standard pairs, each unitarily equivalent to the unique standard pair (H, T) and $T(t) = \mathrm{e}^{\mathrm{i}tP}$. Then a unitary $\tilde{V}$ commuting with the translation $\tilde{T}$ preserves $\tilde{H}$ if and only if $\tilde{V}$ is a $n \times n$ matrix (V_{hk}) (with respect to the decomposition of $\tilde{\mathcal{H}}$ into an n direct sum as above) such that $V_{hk} = \varphi_{hk}(P)$, where $\varphi_{hk} : \mathbb{R} \to \mathbb{C}$ are complex Borel functions such that (φ_{hk}) is a unitary matrix for almost every $p > 0$, each φ_{hk} is the boundary value of a function in $\mathbb{H}(\mathbb{S}_\infty)$ and is symmetric, i.e. $\varphi_{hk}(-p) = \overline{\varphi_{hk}(p)}$.*

Consider the one-particle space $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}}$ for $\mathrm{Fer}_{\mathbb{C}}$. The pair of the standard space $K(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ defined in Sect. 3.2 (under the identification of S^1 and $\mathbb{R} \cup \{\infty\}$) and the natural translation has multiplicity 2. If we take a matrix-valued function (φ_{hk}) as above and take the second quantization operator $\Lambda(V)$ of the (matrix-valued) operator $(V_{kh}) = (\varphi_{hk}(P))$, then it implements a Longo-Witten endomorphism of $\mathrm{Fer}_{\mathbb{C}}$ (see [LW11]).

As the gauge group acts by real rotation $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, any matrix-valued function of p which commutes with them must have the form $\begin{pmatrix} a(p) & ib(p) \\ -ib(p) & a(p) \end{pmatrix}$. If each component is symmetric, then a is symmetric and b is antisymmetric. Such a matrix-valued function can be diagonalized by the matrix $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and becomes $\begin{pmatrix} a(p)+b(p) & 0 \\ 0 & a(p)-b(p) \end{pmatrix}$. We claim that such a and b exist. Indeed, let φ be a inner function (not necessarily symmetric), namely the boundary value with modulus 1 of a bounded analytic function on the upper half-plane $\mathbb{H}$, and define $a(p) = \frac{1}{2}(\varphi(p) + \overline{\varphi(-p)})$, $b(p) = \frac{1}{2}(\varphi(p) - \overline{\varphi(-p)})$. Then it is obvious that a is symmetric and b is antisymmetric. In addition, $a(p)+b(p) = \varphi(p)$ and $a(p)-b(p) = \overline{\varphi(-p)}$, hence the diagonalized matrix is unitary for almost every p. By the theorem of Longo-Witten, the operator $\varphi(P_1) := \begin{pmatrix} a(P) & ib(P) \\ -ib(P) & a(P) \end{pmatrix}$ preserves the real Hilbert space $\tilde{H} := K(\mathbb{R}_+)$, where P_1 is the generator of the translation in $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}}$ which has multiplicity 2 and P is the generator of T of the irreducible standard pair (H, T).

It is easy to see that the above diagonalization is given exactly by the decomposition $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} = \overline{PL^2(S^1)} \oplus (\mathbb{1} - P)L^2(S^1)$.

We recall that $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}}$ can be identified with $L^2(\mathbb{R})$ as a real space. In $L^2(\mathbb{R})$ the function $\varphi(P_1)f$ is the function with Fourier transform $\varphi(p)\hat{f}(p)$ and we remark that it also follows directly from the Paley-Wiener theorem that $\varphi(P_1)$ leaves $L^2(\mathbb{R}_+) \subset L^2(\mathbb{R})$ invariant for φ inner. Further using that the space $\mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}}$ decomposes in $\mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}} = \bigoplus_{m,n\in\mathbb{N}_0}\mathcal{H}_{m,n}$ like in (1) with the gauge action given by $V(\theta)\Psi_{m,n} = \mathrm{e}^{\mathrm{i}(m-n)\theta}\Psi_{m,n}$, the action of the Longo-Witten unitary $V_\varphi = \Lambda(\varphi(P_1))$ is given by

$$(V_\varphi\Psi)_{m,n}(p_1,\ldots,p_m,q_1,\ldots,q_n)$$
$$= \varphi(p_1)\cdots\varphi(p_m)\overline{\varphi(-q_1)}\cdots\overline{\varphi(-q_n)}\Psi_{m,n}(p_1,\ldots,p_m,q_1,\ldots,q_n).$$

Lemma 4.2. *Let $\iota : \Psi \in L^2(\mathbb{R}_+,\, p\,\mathrm{d}p) \equiv \mathcal{H}^1_{\mathcal{A}^{(0)}} \hookrightarrow \mathcal{H}_{1,1} \subset \mathcal{H}_{\mathrm{Fer}_{\mathbb{C}}}$ be the embedding given by $\iota(\Psi)_{1,1}(p,q) = -\frac{1}{2\pi}\Psi(p+q)$. A second quantization Longo-Witten unitary V_φ commuting with the gauge action $V(\cdot)$ satisfies $V_\varphi\iota\mathcal{H}^1_{\mathcal{A}^{(0)}} \subset \iota\mathcal{H}^1_{\mathcal{A}^{(0)}}$ if and only if $V_\varphi = V(\theta)T(t)$ with $t \geq 0$.*

Proof. The translations commute with the gauge action and it follows immediately that they leave $\mathcal{H}^1_{\mathcal{A}^{(0)}}$ invariant. We note that $\varphi(p)\overline{\varphi(-q)}\Psi(p+q)$ belongs to $\mathcal{H}^1_{\mathcal{A}^{(0)}}$ only if it can be written as a function of $g(p+q)$. This means that $\varphi(p)\overline{\varphi(-q)} = \widetilde{\varphi}(p+q)$ for $p,q \geq 0$, where $\widetilde{\varphi}$ is another function. Then, putting $q = 0$ and $p = 0$ respectively, we see that $\varphi(p)\overline{\varphi(0)} = \widetilde{\varphi}(p)$ for $p \geq 0$ and $\varphi(0)\overline{\varphi(-q)} = \widetilde{\varphi}(q)$ for $q \geq 0$, in particular $\widetilde{\varphi}(0) = 1$. Multiplying each side of these equations, one sees that $\widetilde{\varphi}(p+q) = \widetilde{\varphi}(p)\widetilde{\varphi}(q)$ because $|\varphi(0)| = 1$. Then it follows that $\widetilde{\varphi}(p) = \mathrm{e}^{\mathrm{i}\kappa p}$ for some $\kappa \geq 0$, and $\varphi(p) = \mathrm{e}^{\mathrm{i}(\kappa p+\theta)}$ for some $\theta \in \mathbb{R}$ (in fact, the arguments here should be treated with care because the relation is given only almost everywhere, but both φ and $\check{\varphi}$ analytically continue and in the domain of analyticity it holds everywhere).

Such a φ is a Longo-Witten unitary only for $\kappa \geq 0$. The constant factor $\mathrm{e}^{\mathrm{i}\theta}$ corresponds to the factor $V(\theta)$. □

Theorem 4.3. *Let φ be an inner function as above. The endomorphism implemented by the second quantization V_φ of the operator constructed above restricts to the $\mathrm{U}(1)$-current subnet. The restriction cannot be implemented by any second quantization operator if $\varphi(p) \neq \mathrm{e}^{\mathrm{i}(\kappa p+\theta)}$.*

Proof. The operator V_φ restricts to the subnet $\mathcal{A}^{(0)}$ by the general argument in Proposition 2.1. It cannot be implemented by a second quantization operator, since any second quantization operator preserves the particle number, while V_φ does not for non-exponential φ as we saw above, and a Longo-Witten endomorphism is uniquely implemented up to scalar (see Sect. 2.1). □

Remark 4.4. By the construction in [LW11], each unitary $V = V_\varphi|_{\mathcal{H}_{\mathcal{A}^{(0)}}}$ related to an inner function φ from above gives rise to a local, time-translation covariant net of von Neumann algebras on the Minkowski half-space $M_+ = \{(t,x) \in \mathbb{R}^2 : x > 0\}$. This net is associated with the $\mathrm{U}(1)$-current net $\mathcal{A}^{(0)}$ and defined by $\mathcal{A}^{(0)}_V(O) = \mathcal{A}^{(0)}(I_1) \vee V\mathcal{A}^{(0)}(I_2)V^*$, where $O = I_1 \times I_2 = \{(t,x) \in \mathbb{R}^2 : t - x \in I_1, t + x \in I_2\}$ is a double cone with $\overline{O} \subset M_+$ corresponding uniquely to the two intervals I_1 and I_2 with disjoint closures. In the case where φ is not exponential V_φ does not come from second quantization—in contrast to the unitaries constructed by Longo and Witten in [LW11]—and therefore gives new examples.

5. Interacting Wedge-Local Net with Particle Production

5.1. Construction of scattering operators. In the previous section we saw that, in the basis $\{e_n + e_{-n}, e_n - e_{-n}\}$ the matrix operator $\begin{pmatrix} a(P) & ib(P) \\ -ib(P) & a(P) \end{pmatrix}$ implements a Longo-Witten endomorphism if a is symmetric and b is antisymmetric, and after the simultaneous diagonalization it becomes $\begin{pmatrix} \varphi(P) & 0 \\ 0 & \check{\varphi}(P) \end{pmatrix}$, where φ is an inner function and $\check{\varphi}(p) = \overline{\varphi(-p)}$ (note that if φ extends to an analytic function $\varphi(z)$ on $\mathbb{H}$, then $\check{\varphi}(z) = \overline{\varphi(-\bar{z})}$ also extends to $\mathbb{H}$, hence $\check{\varphi}$ is again an inner function). By the same argument one sees that $\begin{pmatrix} \check{\varphi}(P) & 0 \\ 0 & \varphi(P) \end{pmatrix}$ implements an endomorphism since $\check{\check{\varphi}} = \varphi$.

With respect to the basis after diagonalization, we split the Hilbert space $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} =: \mathcal{H}_+ \oplus \mathcal{H}_-$ and the generator of translation $P_1 =: P_+ \oplus P_-$. Then the tensor product space can be written as follows:

$$\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} \otimes \mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} = (\mathcal{H}_+ \otimes \mathcal{H}_+) \oplus (\mathcal{H}_+ \otimes \mathcal{H}_-) \oplus (\mathcal{H}_- \otimes \mathcal{H}_+) \oplus (\mathcal{H}_- \otimes \mathcal{H}_-).$$

According to this decomposition into a direct sum of four subspaces, we define an operator

$$M_\varphi := \varphi(P_+ \otimes P_+) \oplus \check{\varphi}(P_+ \otimes P_-) \oplus \check{\varphi}(P_- \otimes P_+) \oplus \varphi(P_- \otimes P_-).$$

Then this restricts to the subspace $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} \otimes \mathcal{H}_+ = (\mathcal{H}_+ \oplus \mathcal{H}_-) \otimes \mathcal{H}_+$ and it is $\varphi(P_+ \otimes P_+) \oplus \check{\varphi}(P_- \otimes P_+)$, or we can decompose it with respect to the spectral measure of P_+:

$$\int_{\mathbb{R}_+} \begin{pmatrix} \varphi(pP_+) & 0 \\ 0 & \check{\varphi}(pP_-) \end{pmatrix} \otimes \, dE_+(p).$$

Similarly, the restriction to $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} \otimes \mathcal{H}_-$ is written as

$$\int_{\mathbb{R}_+} \begin{pmatrix} \check{\varphi}(pP_+) & 0 \\ 0 & \varphi(pP_-) \end{pmatrix} \otimes \, dE_-(p).$$

Using the two-point set $\mathbb{Z}_2 = \{+, -\}$ we define

$$\varphi_+(p, +) := \varphi(p), \quad \varphi_+(p, -) = \check{\varphi}(p), \quad \varphi_-(p, +) = \check{\varphi}(p), \quad \varphi_-(p, -) = \varphi(p).$$

By defining the spectral measure $E_1 = E_+ \oplus E_-$ on $\mathcal{H}^1$, M_φ can be simply written as

$$M_\varphi = \int_{\mathbb{R}_+ \times \mathbb{Z}_2} \begin{pmatrix} \varphi_+(pP_+, \iota) & 0 \\ 0 & \varphi_-(pP_-, \iota) \end{pmatrix} \otimes \, dE_1(p, \iota),$$

where $\iota = \pm$.

As in [Tan11a], we construct the scattering matrix first on the unsymmetrized Fock space, then restrict it to the antisymmetric space. For an operator A on $\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}} \otimes \mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}}$, we denote by $A^{m,n}_{i,j}$ on $(\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}})^{\otimes m} \otimes (\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}})^{\otimes n}$ the operator which acts only on the ith factor in $(\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}})^{\otimes m}$ and j^{th}-th factor in $(\mathcal{H}^1_{\mathrm{Fer}_{\mathbb{C}}})^{\otimes n}$ as A. As a convention, $A^{m,n}_{i,j}$ equals the identity operator if m or n is 0. Let us denote simply $\widetilde{\varphi}(p, \iota) := \begin{pmatrix} \varphi_+(p, \iota) & 0 \\ 0 & \varphi_-(p, \iota) \end{pmatrix}$ and

$\widetilde{\varphi}(P_1, \iota) := \begin{pmatrix} \varphi_+(P_+, \iota) & 0 \\ 0 & \varphi_-(P_-, \iota) \end{pmatrix}$. From the observation above, it is straightforward to see that

$$(M_\varphi)_{i,j}^{m,n} = \int \left(\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\widetilde{\varphi}(p_j P_1, \iota_j)} \otimes \cdots \otimes \mathbb{1} \right) \otimes dE_1(p_1, \iota_1) \otimes \cdots \otimes dE_1(p_n, \iota_n)$$

(the case where m or n is 0 is treated separately). Then we define, as in [Tan11a],

$$S_\varphi^{m,n} := \prod_{i,j} (M_\varphi)_{i,j}^{m,n},$$

$$S_\varphi := \bigoplus_{m,n} S_\varphi^{m,n}.$$

Let $\mathcal{H}_{\text{Fer}_\mathbb{C}}^\Sigma$ be the unsymmetrized Fock space based on $\mathcal{H}_{\text{Fer}_\mathbb{C}}^1$. Note that S_φ is defined on $\mathcal{H}_{\text{Fer}_\mathbb{C}}^\Sigma \otimes \mathcal{H}_{\text{Fer}_\mathbb{C}}^\Sigma$, and it naturally restricts to $\mathcal{H}_{\text{Fer}_\mathbb{C}} \otimes \mathcal{H}_{\text{Fer}_\mathbb{C}}^\Sigma$, $\mathcal{H}_{\text{Fer}_\mathbb{C}}^\Sigma \otimes \mathcal{H}_{\text{Fer}_\mathbb{C}}$ and $\mathcal{H}_{\text{Fer}_\mathbb{C}} \otimes \mathcal{H}_{\text{Fer}_\mathbb{C}}$. This S_φ will be interpreted as the scattering matrix. In order to confirm this, we have to take the spectral decomposition of S_φ only with respect to the right or left component. Namely,

$$S_\varphi := \bigoplus_{m,n} \prod_{i,j} (M_\varphi)_{i,j}^{m,n}$$

$$= \bigoplus_{m,n} \prod_{i,j} \int \left(\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\widetilde{\varphi}(p_j P_1, \iota_j)} \otimes \cdots \otimes \mathbb{1} \right) \otimes dE_1(p_1, \iota_1) \otimes \cdots \otimes dE_1(p_n, \iota_n)$$

$$= \bigoplus_{m,n} \int \prod_{i,j} \left(\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\widetilde{\varphi}(p_j P_1, \iota_j)} \otimes \cdots \otimes \mathbb{1} \right) \otimes dE_1(p_1, \iota_1) \otimes \cdots \otimes dE_1(p_n, \iota_n)$$

$$= \bigoplus_n \int \bigoplus_m \prod_j \left(\widetilde{\varphi}(p_j P_1, \iota_j) \right)^{\otimes m} \otimes dE_1(p_1, \iota_1) \otimes \cdots \otimes dE_1(p_n, \iota_n)$$

$$= \bigoplus_n \int \prod_j \bigoplus_m \left(\widetilde{\varphi}(p_j P_1, \iota_j) \right)^{\otimes m} \otimes dE_1(p_1, \iota_1) \otimes \cdots \otimes dE_1(p_n, \iota_n)$$

$$= \bigoplus_n \int \prod_j \Lambda(\widetilde{\varphi}(p_j P_1, \iota_j)) \otimes dE_1(p_1, \iota_1) \otimes \cdots \otimes dE_1(p_n, \iota_n),$$

where the integral and the product commute in the third equality since the spectral measure is disjoint for different values of p's and ι's, and the sum and the product commute in the fifth equality since the operators in the integrand act on mutually disjoint spaces, namely on $(\mathcal{H}_{\text{Fer}_\mathbb{C}}^1)^{\otimes m} \otimes \mathcal{H}_{\text{Fer}_\mathbb{C}}^\Sigma$ for different m. In the final expression, all operators appearing in the integrand are the second quantization operators, thus this formula naturally restricts to the partially antisymmetrized space $\mathcal{H}_{\text{Fer}_\mathbb{C}} \otimes \mathcal{H}_{\text{Fer}_\mathbb{C}}^\Sigma$.

Now we define

- $\mathcal{M}_\varphi := \{x \otimes \mathbb{1}, \text{Ad}\, S_\varphi(\mathbb{1} \otimes y) : x \in \text{Fer}_\mathbb{C}(\mathbb{R}_-), y \in \text{Fer}_\mathbb{C}(\mathbb{R}_+)\}''$,
- $T(t, x) := T_0(\frac{t-x}{\sqrt{2}}) \otimes T_0(\frac{t+x}{\sqrt{2}})$,
- $\Omega := \Omega_0 \otimes \Omega_0$.

As the net $\mathrm{Fer}_\mathbb{C}$ is fermionic by nature, the interpretation of the scattering theory of [Buc75] is not clear. Nevertheless, we can show the following by an almost same proof as in [Tan11a, Lem. 5.2, Thm. 5.3].

Lemma 5.1. *The triple* $(\mathcal{M}_\varphi, T, \Omega)$ *is a Borchers triple.*

Proof. To apply Proposition 2.2, it is immediate that S_φ commutes with translation since it is defined through the spectral measure as above. It preserves $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} \otimes \Omega_0$ and $\Omega_0 \otimes \mathcal{H}_{\mathrm{Fer}_\mathbb{C}}$ pointwise, since these subspaces correspond to the case where m or n is 0 in the above decomposition and S_φ acts as the identity operator by definition. What remains to show is the commutation property.

As we saw above, the operator S_φ can be written as

$$S_\varphi = \bigoplus_n \int \prod_j \Lambda(\widetilde{\varphi}(p_j P_1, \iota_j)) \otimes \, dE_1(p_1, \iota_1) \otimes \cdots \otimes \, dE_1(p_n, \iota_n).$$

The point is that the operators which appear in the integrand implement Longo-Witten endomorphisms as we saw above since $p_j \geq 0$ in the support of the integration.

Let $x' \in \mathrm{Fer}_\mathbb{C}(\mathbb{R}_+)$ and consider $x' \otimes \mathbb{1}$ as an operator on $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} \otimes \mathcal{H}_{\mathrm{Fer}_\mathbb{C}}^\Sigma$. We have

$$\mathrm{Ad}\, S_\varphi(x' \otimes \mathbb{1}) = \bigoplus_n \int \mathrm{Ad}\left(\prod_j \Lambda(\widetilde{\varphi}(p_j P_1, \iota_j))\right)(x') \otimes \, dE_1(p_1, \iota_1)$$
$$\otimes \cdots \otimes \, dE_1(p_n, \iota_n).$$

Although this formula is not closed on $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} \otimes \mathcal{H}_{\mathrm{Fer}_\mathbb{C}}$, the left hand side obviously restricts there. One sees that the integrand remains in $\mathrm{Fer}_\mathbb{C}(\mathbb{R}_+)$.

Recall the operator Z_0 which gives the graded locality of $\mathrm{Fer}_\mathbb{C}$. One has to remind that $Z_0 = \frac{\mathbb{1} - i\Gamma_0}{1 - i}$ where $\Gamma_0 = \Lambda(-\mathbb{1})$, hence Z_0 commutes with any second quantization operator. Then by the disintegration above (and the corresponding disintegration with respect to the left component), it is easy to see that $Z_0 \otimes \mathbb{1}$ commutes with S_φ.

Let us check the commutation property of the assumptions in Proposition 2.2. Note that $\mathrm{Ad}\, Z_0(x) \otimes \mathbb{1}$ and $\mathrm{Ad}\, Z_0(x) \in \mathrm{Fer}_\mathbb{C}(\mathbb{R}_+)'$ for $x \in \mathrm{Fer}_\mathbb{C}(\mathbb{R}_-)$. Since $Z_0 \otimes \mathbb{1}$ and S_φ commute as we saw above, to prove the first commutation relation, it is enough to show that $[\mathrm{Ad}\, Z_0(x) \otimes \mathbb{1}, \mathrm{Ad}\, S_\varphi(x' \otimes \mathbb{1})] = 0$ for $x \in \mathrm{Fer}_\mathbb{C}(\mathbb{R}_-)$ and $x' \in \mathrm{Fer}_\mathbb{C}(\mathbb{R}_+)$. As operators acting on $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} \otimes \mathcal{H}_{\mathrm{Fer}_\mathbb{C}}^\Sigma$, this is done by the above disintegration of $\mathrm{Ad}\, S_\varphi(x' \otimes \mathbb{1})$. Then both operators naturally restrict to $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} \otimes \mathcal{H}_{\mathrm{Fer}_\mathbb{C}}$, and we obtain the claim (cf. [Tan11a, Lem. 5.2, Thm. 5.3]). The second commutation relation for Proposition 2.2 can be proven analogously. $\square$

Finally we arrive at a new family of interacting Borchers triples with asymptotic algebra $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$.

Theorem 5.2. *Let us define*

- $\mathcal{N}_\varphi := \{x \otimes \mathbb{1}, \mathrm{Ad}\, S_\varphi(\mathbb{1} \otimes y) : x \in \mathrm{Fer}_\mathbb{C}^{\mathrm{U}(1)}(\mathbb{R}_-), y \in \mathrm{Fer}_\mathbb{C}^{\mathrm{U}(1)}(\mathbb{R}_+)\}''$,
- $T(t, x) := T_0(\frac{t-x}{\sqrt{2}}) \otimes T_0(\frac{t+x}{\sqrt{2}})$,
- $\Omega := \Omega_0 \otimes \Omega_0$.

Then the triple $(\mathcal{N}_\varphi, T, \Omega)$, *restricted to* $\overline{\mathcal{N}_\varphi \Omega}$, *is an asymptotically complete, interacting Borchers triple with the asymptotic algebra* $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$ *and scattering operator* $S_\varphi|_{\overline{\mathcal{N}_\varphi \Omega}}$. *It also holds that* $\overline{\mathcal{N}_\varphi \Omega} = \overline{\mathcal{A}^{(0)}(I_+) \otimes \mathcal{A}^{(0)}(I_-)\Omega}$ *for arbitrary intervals* I_+, I_-.

Proof. Substantial arguments are already done: In Lemma 5.1 we constructed Borchers triples with $\text{Fer}_{\mathbb{C}} \otimes \text{Fer}_{\mathbb{C}}$ as the asymptotic algebra. We have seen in Sect. 3.3 the U(1)-current net $\mathcal{A}^{(0)}$ is the fixed point subnet of $\text{Fer}_{\mathbb{C}}$ with respect to the action of U(1). From the construction in Sect. 5.1 and Theorem 4.3, it is easy to see that S_φ commutes with the product action of the inner symmetries. Then all the statements of the theorem follow from the general consideration of Proposition 2.5. □

5.2. Action of the S-matrix on the 1+1 particle space.

In this Section we want to analyze the action of the S-matrix of the models constructed in Sect. 5.1 on the 1+1 particle space $\mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$, i.e. one left and one right moving particle, where we use the word particle in the sense of Fock space excitations. We note that on the $n{+}0$ and $0{+}n$ particle spaces $\mathcal{H}_n \otimes \mathbb{C}\Omega_0$ and $\mathbb{C}\Omega_0 \otimes \mathcal{H}_n$, respectively, the S-matrix S acts trivially. A typical vector in $\mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$ is of the form $\Psi := J(f)\Omega_0 \otimes J(g)\Omega_0$ which we express as the function $\Psi(p, \bar{p}) = \hat{f}(p)\hat{g}(\bar{p})$. The embedding $\iota : L^2(\mathbb{R}_+, p\,dp) \otimes L^2(\mathbb{R}_+, \bar{p}\,d\bar{p}) \cong \mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}} \hookrightarrow \mathcal{H}_{1,1} \otimes \mathcal{H}_{1,1} \subset \mathcal{H}_{\text{Fer}_{\mathbb{C}}} \otimes \mathcal{H}_{\text{Fer}_{\mathbb{C}}}$ is given by $\iota(\Psi)_{1,1;1,1}(p, q, \bar{p}, \bar{q}) = \frac{1}{(2\pi)^2}\Psi(p+q, \bar{p}+\bar{q})$. We have an analogue of Lemma 4.2.

Proposition 5.3. *Let φ be some inner function. The unitary S_φ satisfies $S_\varphi(\mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}) \subset \mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$ if and only if $\varphi(p) = e^{i(kp+\theta)}$.*

Proof. The action of S_φ on $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by

$$S_\varphi \Psi(p+q, \bar{p}+\bar{q}) = \varphi(p \cdot \bar{p})\check{\varphi}(q \cdot \bar{p})\check{\varphi}(p \cdot \bar{q})\varphi(q \cdot \bar{q})\Psi(p+q, \bar{p}+\bar{q}),$$

which is again in $\mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$ if it can be written as a function $\tilde{\Psi}(p+q, \bar{p}+\bar{q})$, in particular if $\varphi(p \cdot \bar{p})\check{\varphi}(q \cdot \bar{p})\check{\varphi}(p \cdot \bar{q})\varphi(q \cdot \bar{q}) = \tilde{\varphi}(p+q, \bar{p}+\bar{q})$. Setting $\bar{p} = 1$ and $\bar{q} = 0$, we have $\varphi(p)\check{\varphi}(q) = \tilde{\varphi}(p+q, 1)$. The rest follows as Lemma 4.2. □

Remark 5.4. In the case $\varphi(p) = e^{i\kappa p}$, one gets the models obtained in [DT11] using warped convolution.

Proposition 5.5. *Let e be the projection on $\mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$, then $eS_\varphi e = \tilde{\varphi}(P \otimes P)$, where $\tilde{\varphi}$ is boundary value of an analytic function in $\mathbb{H}$ with $|\tilde{\varphi}(p)| \leq 1$ and P is the generator of translation restricted to the one-particle space (which gives rise to the irreducible standard pair).*

Proof. It can be checked that

$$(e_0 f)(p, q) = \frac{1}{p+q} \int_0^{p+q} f(p+q-x, x)\,dx$$

is the projection on $\mathcal{H}^1_{\mathcal{A}^{(0)}} \subset \mathcal{H}_{1,1}$. Then the action of eS_φ on a $f \in \mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$ can be calculated to be $\varphi'(P \otimes 1, 1 \otimes P)$ with

$$\varphi'(p, q) = \frac{1}{p \cdot q} \int_0^p \int_0^q \varphi((p-x) \cdot (q-y))\varphi(x \cdot y)\check{\varphi}((p-x) \cdot y)\check{\varphi}(x \cdot (q-y))\,dy\,dx,$$

and it is easy to check that with $\tilde{\varphi}(p) := \varphi'(p, 1)$, it holds that $\varphi'(p, q) = \tilde{\varphi}(p \cdot q)$ for all $p, q > 0$. That $|\tilde{\varphi}(p)| \leq 1$ can be checked directly or follows from the fact that S_φ is unitary. □

Remark 5.6. It is a general feature of asymptotically complete Borchers triples with asymptotic algebra $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$ that the restriction of the scattering matrix S to eSe is a functional calculus of $P \otimes P$. Indeed, both e and S commute with the translation T, but T is maximally abelian when restricted to $\mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$, hence there is a function φ_S such that $eSe = \varphi_S(P \otimes \mathbb{1}, \mathbb{1} \otimes P)$. Furthermore, both e and S commute with boosts, so does φ_S and one obtains the form $eSe = \varphi'_S(P \otimes P)$.

We note that the proof above shows that $|\tilde{\varphi}(M^2/2)|$ is the probability that an improper state in $\mathcal{H}^1_{\mathcal{A}^{(0)}} \otimes \mathcal{H}^1_{\mathcal{A}^{(0)}}$ with mass M^2 is scattered elastically in the sense of Fock space particles, where

$$\tilde{\varphi}(p) = \frac{1}{p} \int_0^p \int_0^1 \varphi((p-x)(1-y))\varphi(xy)\check{\varphi}((p-x)y)\check{\varphi}(x(1-y)) \, dy \, dx.$$

As we discussed in 3.1, the Hilbert space of the U(1)-current net, and hence the tensor product of two copies of it, admit the bosonic Fock space structure, hence we can consider the particle number. Although we admit that this concept does not have an intrinsic meaning, we claim that it is possible to interpret this as the number of massless particles.

An evidence comes from the comparison with massive cases. In [Lec08] Lechner has constructed a family of massive interacting models parametrized by so-called scattering functions, and later he reinterpreted them as deformations of the massive free field [Lec11]. If one applies the same deformation procedure to the derivative of the massless free field whose net is $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$ (with scattering functions satisfying $S_2(0) = 1$), he obtains the Borchers triples with $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$ as the asymptotic net constructed in [Tan11a].[1] Hence the models in [Tan11a] should be considered as the massless versions of the models in [Lec08]. Likewise, it can be said that the models constructed in the present paper are the deformed (in an appropriate sense) version of the massless free field.

In massive case, there is a mass gap in the spectrum of the spacetime translation and the one-particle space of the Fock space has an intrinsic meaning. In massless case, such an intrinsic interpretation is lost but there is still the Fock space structure. Thus we think that, if the two-particle space in the Fock structure is not preserved by the S-matrix, as in the case where φ is not exponential (see Proposition 5.3), then it represents massless particle production.

6. Conclusion and Outlook

In this paper we have constructed a new family of Longo-Witten endomorphisms on $\mathcal{A}^{(0)}$ through the inclusion $\mathcal{A}^{(0)} = \mathrm{Fer}_{\mathbb{C}}^{\mathrm{U}(1)} \subset \mathrm{Fer}_{\mathbb{C}}$. We combined them to construct interacting wedge-local nets with $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$ as the asymptotic algebra and showed that their S-matrices do not preserve the n-particle space of the bosonic Fock space. Particle production is a necessary feature of interacting models in higher dimensions [Aks65], thus this result opens up some hope for algebraic construction of higher dimensional interacting models.

However, there are at least two shortcomings with the present method. The first is that we proved only wedge-locality of the models. As already shown in [Tan11a], a wedge-local net can be dilation-covariant and at the same time interacting. On the other hand, a

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[1] Private communication with Gandalf Lechner and Jan Schlemmer. This will be presented elsewhere.

strictly local dilation-covariant (asymptotically complete) net is necessarily not interacting [Tan11b]. Hence, interaction of wedge-local nets could be just a false-positive and strict locality is desired. The second is the fact that the concept of particle in massless case is not intrinsically defined. Although the Fock space structure is easily understood, its interpretations should be treated with care.

These issues could be overcome by considering massive cases. As for strict locality, it has been shown that the deformation of the massive free field by a suitably regular function is again strictly local [Lec08,Lec11]. On the other hand, in massless situation, even the simplest case $\varphi(p) = -1$ (where φ is an inner symmetric function used in [Tan11a] to deform *directly* $\mathcal{A}^{(0)} \otimes \mathcal{A}^{(0)}$) is already not strictly local [Tan11a]. Hence we believe that strict locality should be addressed in massive models. Furthermore, for a massive asymptotically complete model, the notion of particle production is intrinsic. Fortunately, it is known that the construction in [Tan11a] coincides with the deformation of the massive free field as we remarked in the last section, hence a further correspondence between massive and massless cases are expected. We hope to investigate this problem in a future publication.

Of course, interacting models in higher dimensions are always one of the most important issues. Although conformal nets themselves are not interacting [BF77], some new constructions based on CFT could be possible and ideas from the present article could be useful.

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