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Worm Domains are not Gromov Hyperbolic

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Abstract
We show that Worm domains are not Gromov hyperbolic with respect to the Kobayashi distance.

Keywords Worm domain · Gromov hyperbolicity · Kobayashi metric · Scaling

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1 Introduction

A central problem in contemporary several complex variables is to determine when a complete Kobayashi hyperbolic domain $\Omega \subset\subset \mathbb{C}^n$ is Gromov hyperbolic when endowed with its Kobayashi distance. Assume in what follows that Ω is smoothly bounded.

Some families of relevant domains are Gromov hyperbolic: Balogh–Bonk [2] proved it for strongly pseudoconvex domains, and Zimmer [19] showed it for convex domains of D'Angelo finite type. The third-named author showed it [13] for pseudoconvex domains of finite type in $\mathbb{C}^2$. On the other hand, Gaussier–Seshadri [15] proved that for smoothly bounded *convex* domains $\Omega \subset\subset \mathbb{C}^n$ an analytic disk in the boundary is an obstruction to Gromov hyperbolicity. This result was later strengthened by Zimmer [19], who showed that the same is true if Ω is a smoothly bounded $\mathbb{C}$-*convex* domain. The following important question remains open.

Question *Is an analytic disk in the boundary an obstruction to Gromov hyperbolicity for a smoothly bounded complete Kobayashi hyperbolic domain* $\Omega \subset\subset \mathbb{C}^n$*?*

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In this paper, we study the Gromov hyperbolicity of the Worm domains introduced by Diederich–Fornæss [11], which have a holomorphic annulus in the boundary and are highly non-$\mathbb{C}$-convex. Worm domains play a central role in several complex variables as they provide counterexamples to several important questions. See, e.g., [17] for a review of the properties of Worm domains. We actually consider a more general class of *Worms* (see Definition 10), with an open Riemann surface in the boundary, and prove the following result:

Theorem 1 *Worms are not Gromov hyperbolic w.r.t. the Kobayashi distance.*

The proof is based on Barrett's scaling (cf. [4, Sect. 4]). We rescale the Worm W obtaining in the limit a holomorphic fiber bundle, which we call a *pre-Worm*, with base an open hyperbolic Riemann surface and with fiber the right half-plane. We show that such a pre-Worm cannot be Gromov hyperbolic. Since the Kobayashi distance is continuous with respect to this scaling, this yields the result.

2 Gromov Hyperbolicity—Basic Definitions

In this section, we will review some basic definitions and properties of Gromov hyperbolic spaces. The book [8] is one of the standard references.

Definition 2 Let (X, d) be a metric space. For every $x, y, o \in X$ the *Gromov product* is

$$(x|y)_o := \frac{1}{2}[d(x, o) + d(y, o) - d(x, y)].$$

The metric space (X, d) is δ-*hyperbolic* if for all $x, y, z, o \in X$

$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta.$$

Finally, a metric space is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Definition 3 Let (X, d) be a metric space, $I \subset \mathbb{R}$ be an interval and $A \geq 1$ and $B \geq 0$. A function $\sigma : I \to X$ is

(1) a *geodesic* if for each $s, t \in I$

$$d(\sigma(s), \sigma(t)) = |t - s|;$$

(2) a (A, B)-*quasigeodesic* if for each $s, t \in I$

$$A^{-1}|t - s| - B \leq d(\sigma(s), \sigma(t)) \leq A|t - s| + B.$$

A (A, B)-*quasigeodesic triangle* is a choice of three points in X and three (A, B)-quasigeodesic segments connecting these points, called its *sides*. If $M \geq 0$, a (A, B)-*quasigeodesic triangle* is M-*slim* if every side is contained in the M-neighborhood of the other two sides.

Finally, recall that a metric space (X, d) is *proper* if closed balls are compact, and *geodesic* if any two points can be connected by a geodesic. A fundamental property of geodesic Gromov hyperbolic spaces is that quasigeodesics are uniformly close to geodesics, a fact which implies the following characterization of Gromov hyperbolicity.

Proposition 4 [8, Corollary 1.8] *A proper geodesic metric space (X, d) is δ-hyperbolic if and only if for all $A \geq 1$ and $B \geq 0$, there exists $M \geq 0$ such that every (A, B)-quasigeodesic triangle is M-slim.*

3 Worms and Pre-Worms

Let X be an open Riemann surface, and let $\theta : X \to \mathbb{R}$ be a smooth "angle" function. Consider the domain in $X \times \mathbb{C}$ defined as follows:

$$Z(X, \theta) := \{(z, w) \in X \times \mathbb{C} \colon \Re(we^{-i\theta(z)}) > 0\},$$

which is readily seen to be a smooth fiber bundle with base X and fiber a half-plane.

Proposition 5 *If the function θ is harmonic, then $Z(X, \theta)$ is a holomorphic fiber bundle.*

Proof Let v be (minus) a local harmonic conjugate of θ, so that $F(z) = v(z) + i\theta(z)$ is a holomorphic function on an open set $U \subset X$. Then $Z(X, \theta)$ is locally defined over U by $\Re(we^{-F(z)}) = \Re(we^{-v(z)-i\theta(z)}) > 0$, and $(z, w) \mapsto (z, e^{-F(z)}w)$ is the desired local trivialization. □

Definition 6 (*pre-Worms*) If the function θ is harmonic, we call the holomorphic fiber bundle $Z(X, \theta)$ a *pre-Worm*.

Remark 7 Pre-Worms are sectorial domains in the sense of [5] (see in particular Example 2.2).

A pre-Worm $Z(X, \theta)$ with hyperbolic base X is complete Kobayashi hyperbolic by the following classical result.

Proposition 8 ([16, Theorem 3.2.15]) Let $\pi : E \to X$ be a holomorphic fiber bundle with fiber F. Assume that F and X are both (complete) Kobayashi hyperbolic. Then E is (complete) Kobayashi hyperbolic.

Now we proceed to the definition of the Worms. First of all, given two compact intervals $I, J \subset \mathbb{R}$ such that $I \subset J^\circ$, we denote by $\eta : \mathbb{R} \to [0, +\infty)$ any smooth function satisfying the following properties:

- on I, the function η vanishes identically;
- on $\mathbb{R} \setminus I$, the function η is real-analytic and satisfies $\eta'' > 0$ (in particular, η is strictly positive and $\eta' \neq 0$ on $\mathbb{R} \setminus I$);
- $J = \{\eta \leq 1\}$.

The precise choice of a function η satisfying the above properties is completely irrelevant for what follows.

Next, given an open Riemann surface Y equipped with a smooth angle function $\theta : Y \to \mathbb{R}$ and two compact intervals I, J as above, we define

$$W := \{(z, w) \in Y \times \mathbb{C} \colon |w - e^{i\theta(z)}|^2 < 1 - \eta(\theta(z))\}.$$

We assume the following:

- θ has no critical points where $\theta(z) \in \partial I$ or $\theta(z) \in \partial J$;
- $\theta^{-1}(J)$ is a compact subset of Y.

Proposition 9 *The domain $W \subset\subset Y \times \mathbb{C}$ has smooth boundary. Moreover, if θ is harmonic, then W is Levi-pseudoconvex.*

Proof The precompactness of the domain W is a consequence of our assumption that $\theta^{-1}(J)$ is compact. The domain W has defining function:

$$r(z, w) = w\overline{w} - we^{-i\theta(z)} - \overline{w}e^{i\theta(z)} + \eta(\theta(z)).$$

We show that $dr \neq 0$ for all $(z, w) \in \partial W$. If $\partial_{\bar{w}} r \neq 0$, this is clear. Since $\partial_{\bar{w}} r = w - e^{i\theta(z)}$ vanishes only if $w = e^{i\theta(z)}$, we may assume that this identity holds. Then necessarily $\eta(\theta(z)) = 1$, that is, $\theta(z) \in \partial J$, in which case

$$\partial_{\bar{z}} r = i\partial_{\bar{z}}\theta(z) we^{-i\theta(z)} - i\partial_{\bar{z}}\theta(z)\overline{w}e^{i\theta(z)} + \eta'(\theta(z))\partial_{\bar{z}}\theta(z) = \eta'(\theta(z))\partial_{\bar{z}}\theta(z) \neq 0$$

by our assumption about the critical points of θ. This proves that W has smooth boundary.

Since Levi-pseudoconvexity is a local property, we may restrict the z variable to an open set $U \subset Y$ where $\theta(z)$ admits a harmonic conjugate $v(z)$, as in the proof of Proposition 5. A local defining function for the boundary of W is then given by

$$e^{-v} r = |e^{-\frac{F}{2}} w|^2 - 2\Re(we^{-F}) + e^{-v}\eta \circ \theta,$$

where $F(z) = v(z) + i\theta(z)$ is holomorphic. Recalling that moduli squared (resp. real parts) of holomorphic functions are plurisubharmonic (resp. pluriharmonic), we see that $e^{-v} r$ is equal to a plurisubharmonic function plus $e^{-v}\eta \circ \theta$, which is a function of the variable z alone. If we show that the latter is subharmonic, we are done. One computes

$$\Delta(e^{-v}\eta \circ \theta) = \Delta(e^{-v})\eta \circ \theta + 2\nabla(e^{-v}) \cdot \nabla(\eta \circ \theta) + e^{-v}\Delta(\eta \circ \theta),$$

where Δ and ∇ are the ordinary real Laplacian and gradient in $\mathbb{C} \equiv \mathbb{R}^2$. In U, we have

$$\nabla(e^{-v}) \cdot \nabla(\eta \circ \theta) = -e^{-v}(\eta' \circ \theta)\nabla v \cdot \nabla\theta = 0,$$

by Cauchy–Riemann equations.

Next, notice that $e^{-v} = |e^{-\frac{F}{2}}|^2$ is subharmonic. Since η and e^{-v} are nonnegative, all we are left to do to check the nonnegativity of $\Delta(e^{-v}\eta\circ\theta)$ is to verify that $\Delta(\eta\circ\theta) \geq 0$. By direct computation, we see that

$$\Delta(\eta \circ \theta) = 4|\partial_{\bar{z}}\theta|^2\eta'' \circ \theta,$$

which is nonnegative thanks to our convexity assumption on the auxiliary function η. $\square$

Definition 10 (*Worms*) If the function θ is harmonic (and satisfies the assumptions on page 3), we call the domain $W \subset Y \times \mathbb{C}$ a *Worm*.

The reader may find a picture of a Worm in Fig. 1.

Remark 11 By Docquier–Grauert [12] every Worm is Stein.

For a more refined analysis, we split the boundary of W into four regions:

- the *spine* of the Worm

$$S := \{(z, w) \in \partial W : \theta(z) \in I,\ \ w = 0\},$$

- the *body* of the Worm

$$B := \{(z, w) \in \partial W : \theta(z) \in I,\ \ \partial_z\theta(z) \neq 0,\ \ w \neq 0\},$$

- the *exceptional* set

$$E := \{(z, w) \in \partial W :\ \ \partial_z\theta(z) = 0,\ \ w \neq 0\},$$

- the *caps*

$$C := \{(z, w) \in \partial W : \theta(z) \in J \setminus I,\ \ \partial_z\theta(z) \neq 0\}.$$

Remark 12 Identify the slice $\{w = 0\} \subset Y \times \mathbb{C}$ with the Riemann surface Y. Inside Y the spine S is the closure of the domain

$$X_{\text{in}} := \theta^{-1}(I^\circ) \subset\subset Y.$$

Since the angle function θ has no critical point $z \in \theta^{-1}(\partial I)$, the domain X_{in} is smoothly bounded. X_{in} is a Riemann surface contained in the boundary of the Worm W; hence, every point of the spine S is of D'Angelo infinite type.

In what follows an important role is also played by the smoothly bounded domain

$$X_{\text{out}} := \theta^{-1}(J^\circ) \subset\subset Y.$$

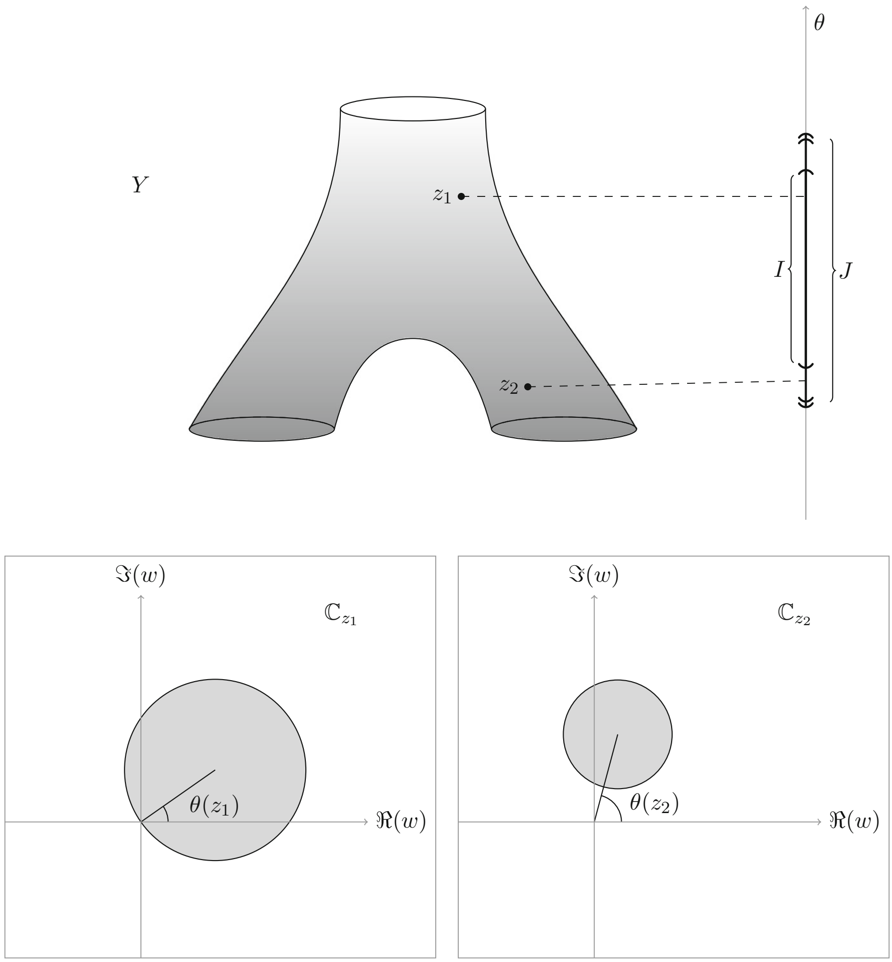

Fig. 1 A Worm, whose underlying Riemann surface Y (depicted above) has genus zero and three boundary components. In this picture, the harmonic angle function θ is represented as a height function for visual clarity. In the two boxes below, one finds a generic w-slice of the worm over a point $z_1 \in \theta^{-1}(I)$ (on the left) and $z_2 \in \theta^{-1}(J \setminus I)$ (on the right). Notice that, because of the indicated choice of I and J, the surfaces X_{in} and X_{out} have the same topology (albeit in general different conformal structures)

Remark 13 The classical Worm domains introduced by Diederich–Fornæss [11] correspond to the case where $Y = \mathbb{C}^*$, $\theta(z) = \log |z|^2$. In this case, X_{in} is a holomorphic annulus contained in the boundary of W, of which conformal class depends on the choice of the interval I.

A“genus zero”generalization of the Diederich–Fornæss Worms is obtained choosing $Y = \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ and $\theta(z) = \sum_{j=1}^{k} \lambda_j \log |z - a_j|^2$ (where $\lambda_j > 0$). If $I = [-a, b]$ with a and b large enough, the spine S has $k + 1$ boundary components.

Proposition 14 *The caps C and the body B consist of strongly pseudoconvex points, the exceptional set E consists of finite-type points, and the spine S consists of infinite-type points.*

Proof We already remarked that S consists of infinite-type points.

In the proof of Proposition 9, we saw that the boundary of a worm has a local defining function admitting the representation

$$\tilde{r} = |e^{-\frac{F}{2}} w|^2 + e^{-v}\eta \circ \theta + \mathrm{ph},$$

where ph denotes a pluriharmonic function, and that

$$\Delta_z(e^{-v}\eta \circ \theta) \geq e^{-v}|\partial_{\bar{z}}\theta|^2 \eta'' \circ \theta.$$

Since the latter quantity is positive on the caps (thanks to the strict convexity assumption on η), we conclude that the Worm is strictly pseudoconvex at every point of C where ∂_w is *not tangent* to the boundary, that is $\partial_w r \neq 0$ (or, equivalently, the vector $(0, 1)$ is not in the complex tangent to ∂W). If instead $\partial_w r = 0$, then we have $\theta \in \partial J$ (cf. the beginning of the proof of Proposition 9), and we may exploit the strong plurisubharmonicity of $|e^{-\frac{F}{2}} w|^2$:

$$\partial_w \partial_{\bar{w}} |e^{-\frac{F(z)}{2}} w|^2 = |e^{-\frac{F(z)}{2}}|^2 > 0.$$

Thus, every point of C is strongly pseudoconvex.

We now study points (z, w) in the body B, where $\eta \circ \theta \equiv 0$. Calculating the Levi form $\mathcal{L}_{(z,w)}\tilde{r}$ we obtain, for $(a, b) \in \mathbb{C}^2$,

$$\mathcal{L}_{(z,w)}\tilde{r}(a, b) = \begin{pmatrix}\overline{a} & \overline{b}\end{pmatrix} |e^{-\frac{F(z)}{2}}|^2 \begin{pmatrix} |w|^2 \frac{|F'(z)|^2}{4} & -\overline{w}\,\frac{\overline{F'(z)}}{2} \\ -w\,\frac{F'(z)}{2} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence, $L_{(z,w)}\tilde{r}(a, b)$ vanishes if and only if $(a, b) \in \mathbb{C}^2$ is a multiple of $(2, wF'(z))$. This readily shows that the Worm is strongly pseudoconvex at every boundary point of the body where $(2, wF'(z))$ *is not in the complex tangent* to the boundary. But a simple computation shows that the vector $(2, wF'(z))$ is never complex tangent to the boundary since

$$(2\partial_z + wF'(z)\partial_w)\tilde{r} = wF'(z)e^{-F(z)} \neq 0.$$

This shows that every point of the body B is strongly pseudoconvex.

We are left with the proof that every point of the exceptional set E is of finite type. By the Cauchy–Riemann equations, the critical points of θ are the same as the critical points of the (locally defined) holomorphic function F, and hence, they are isolated. Thus, E is a finite union of circles and circles with one point deleted (the point with $w = 0$, in case the circle crosses the spine). Moreover, since θ has no critical points on ∂I, the boundary of the Worm is real-analytic in an open neighborhood of E. Thus, to verify that every point of E is of finite type, we need to check that no positive dimensional complex analytic variety lies in such a neighborhood (see, e.g., [3]). This is easy, because any point of such a variety would be of infinite type and, since we already checked that B and C consist of strongly pseudoconvex points, this would force the variety to be contained in E, which is impossible by dimension considerations (or by the open-mapping theorem). □

Remark 15 The Worms are examples of domains with nontrivial, yet nicely behaved, Levi core. See [9, 10], where this notion has been introduced by the second-named author and S. Mongodi. As a consequence of Proposition 14, the Levi core of a Worm is the $T^{1,0}$ bundle of its spine. A straightforward computation using [9, Proposition 4.1, part vi)] shows that the de Rham cohomology class on the spine S (or, equivalently, X_{in}) induced by the D'Angelo class of the Worm is represented by $i(\overline{\partial}-\partial)\theta$, which is exact if and only if the angle function θ is globally on X_{in} the real part of a holomorphic function (that is, the pre-Worm $Z(X_{\text{in}}, \theta|_{X_{\text{in}}})$ is trivial as a fiber bundle). This is in turn equivalent to the condition that the Diederich–Fornæss index of the Worm is 1. We refer to [9, Sect. 4] for a review of the basic theory of D'Angelo classes and to [1, 9] for the implications on the Diederich–Fornæss index.

We end this section proving that Worms are complete Kobayashi hyperbolic. For this, we observe that a Worm W is naturally associated with two pre-Worms.

Definition 16 Set

$$W_{\text{in}} := Z(X_{\text{in}}, \theta|_{X_{\text{in}}}), \quad W_{\text{out}} := Z(X_{\text{out}}, \theta|_{X_{\text{out}}}),$$

where we are using the notation of Remark 12. Notice that $W_{\text{in}} \subset W_{\text{out}}$ and $W \subset W_{\text{out}}$.

In the remaining of the paper, if M is a complex manifold, we denote by k_M its Kobayashi pseudodistance and by K_M its Kobayashi–Royden pseudometric. The following lemma is proved in [14, Lemma 2.1.3].

Lemma 17 *Let $D \subset \mathbb{C}^d$ be a domain and k_D its Kobayashi distance. If $z_n \to \xi \in \partial D$ and ξ admits a local holomorphic peak function, then for every neighborhood U of ξ, we get*

$$\lim_{n\to+\infty} k_D(z_n, D \cap U^c) = +\infty.$$

Proposition 18 *Worms are complete Kobayashi hyperbolic.*

Proof Assume by contradiction that there exists a nonconvergent Cauchy sequence $\{x_n\}_n$ in W. Passing to a subsequence, we can assume that $x_n \to \xi \in \partial W$. We write $x_n = (z_n, w_n)$ and $\xi = (z_0, w_0)$.

If $w_0 \neq 0$, then ξ is a pseudoconvex finite-type point by Proposition 14. By [6] (see also [18, Sect. 4]), ξ admits a local holomorphic peak function, and hence, it cannot be a Cauchy sequence by Lemma 17.

Assume next that $w_0 = 0$, so that in particular $\xi \in \partial W_{\text{out}}$. Since $W \subset W_{\text{out}}$, it follows that $\{x_n\}_n$ is also a Cauchy sequence w.r.t. $k_{W_{\text{out}}}$, which converges to $\xi \in \partial W_{\text{out}}$. This contradicts the completeness of the pre-Worm W_{out} (Proposition 8 below). □

4 Holomorphic Fiber Bundles are not Gromov Hyperbolic

We recall a classical result from the theory of Kobayashi hyperbolic complex manifolds. If M is a complex manifold, we denote by $B_M(p,r)$ the k_M-ball of center p and radius r.

Proposition 19 ([16, Proposition 3.1.19]) Let M be a Kobayashi hyperbolic complex manifold. Let $p \in M$ and $R, \epsilon > 0$. Then there exists a constant $C \geq 1$ depending only on ϵ such that

$$k_{B_M(p,3R+\epsilon)}(x,y) \leq Ck_M(x,y), \quad \forall x,y \in B_M(p,R),$$

and thus, the metrics k_M and $k_{B_M(p,3R+\epsilon)}$ are biLipschitz equivalent on $B_M(p,R)$.

The fact that C depends only on ϵ is not stated explicitly in [16, Proposition 3.1.19], but it is clear from the (first paragraph of the) proof. We will actually use this result in the following simplified form.

Corollary 20 *Let M be a Kobayashi hyperbolic complex manifold. Then there exists an absolute constant $C \geq 1$ such that*

$$k_{B_M(p,4R)}(x,y) \leq Ck_M(x,y), \quad \forall x,y \in B_M(p,R),$$

for all $R \geq 1$ and all $p \in M$.

We introduce the following definition.

Definition 21 Let $\pi : E \to X$ be a holomorphic fiber bundle and $z \in X$. Then define

$$r(z) := \sup\{r > 0 : \ \text{the bundle trivializes over } B_X(z,r)\}$$

Notice that $r(z) > 0$ for every $z \in X$ if X is Kobayashi hyperbolic.

We can now prove the main result of this section. Recall [16, Theorem 3.1.9] that if X and Y are two complex manifolds then

$$k_{X\times Y}((z_1,w_1),(z_2,w_2)) = \max\{k_X(z_1,z_2), k_Y(w_1,w_2)\}, \quad z_1,z_2 \in X, w_1,w_2 \in Y. \tag{1}$$

Theorem 22 *Let X, F be non-compact complete Kobayashi hyperbolic complex manifolds. Let $\pi : E \to X$ be a holomorphic fiber bundle with fiber F and such that $\sup_{z\in X} r(z) = +\infty$. Then (E, k_E) is not Gromov hyperbolic.*

Proof We will construct a sequence $\{T_n\}_n$ of quasigeodesic triangles in E violating the definition of Gromov hyperbolicity. Let $\{z_n\}_n$ in X be such that $r_n := r(z_n) \to +\infty$. We define

$$\Omega_n := \pi^{-1}(B_X(z_n, r_n/2)),$$

and let

$$\Psi_n \colon B_X(z_n, r_n/2) \times F \to \Omega_n$$

be a holomorphic trivialization. Let $q \in F$ be any point of F. Let $C \geq 1$ be the universal constant given by Corollary 20. Set $t_n := \frac{r_n}{16C}$.

We construct the triangles in the following way. Since X and F are non-compact, for all $n > 0$, we can find a geodesic of X denoted $\gamma_n : [0, t_n] \to X$ with $\gamma_n(0) = z_n$, and a geodesic of F denoted $\sigma_n : [0, t_n] \to F$ with $\sigma_n(0) = q$. Notice that $\gamma_n([0, t_n]) \subset B_X(z_n, r_n/8)$, so by Corollary 20 the curve γ_n is a $(C, 0)$-quasigeodesic w.r.t. the Kobayashi distance of $B_X(z_n, r_n/2)$ (we may assume that $r_n \geq 8$ for every n).

By (1) the curves $a_n(t) = (z_n, \sigma_n(t))$ and $b_n(t) = (\gamma_n(t), q)$ are respectively a geodesic and $(C, 0)$-quasigeodesic of $B_X(z_n, r_n/2) \times F$. Moreover, a simple computation shows that the curve $c_n : [0, 2t_n] \to B_X(z_n, r_n/2) \times F$ defined by

$$c_n(t) = \begin{cases} (\gamma_n(t), \sigma_n(t_n)) & \text{if } t \in [0, t_n] \\ (\gamma_n(t_n), \sigma_n(2t_n - t)) & \text{if } t \in [t_n, 2t_n]. \end{cases}$$

is a $(2C, 0)$-quasigeodesic of $B_X(z_n, r_n/2) \times F$. Indeed, c_n is a geodesic w.r.t. the distance $k_X + k_F$ that is $2C$-BiLipschitz to $k_{B_X(z_n, r_n/2) \times F}$ in $B_X(z_n, r_n/8) \times F$. Hence, the triangle T_n with sides a_n, b_n and c_n is a $(2C, 0)$-quasigeodesic triangle in $B_X(z_n, r_n/2) \times F$. Notice T_n is not t_n-slim because

$$k_{B_X(z_n, r_n/2) \times F}(c_n(t_n), a_n([0, t_n]) \cup b_n([0, t_n])) = t_n.$$

Now since Ψ_n is a biholomorphism between $B_X(z_n, r_n/2) \times F$ and Ω_n, the triangle $\hat{T}_n$ in Ω_n that is image of T_n via Ψ_n is again a $(2C, 0)$ quasigeodesic triangle w.r.t. k_{Ω_n}, and it is not t_n-slim.

Now the map $\pi : E \to X$ is non-expanding, so

$$B_E(\Psi(z_n, q), r_n/2) \subset \Omega_n.$$

The triangle $\hat{T}_n$ is contained in $B_{\Omega_n}(\Psi(z_n, q), r_n/8)$, and hence, it is contained in $B_E(\Psi(z_n, q), r_n/8)$. By another application of Corollary 20, the distances k_E and k_{Ω_n} are C-BiLipschitz in $B_E(\Psi(z_n, q), r_n/8)$, so $\hat{T}_n$ are a $(2C^2, 0)$-quasigeodesic triangle not $(C^{-1}t_n)$-slim w.r.t. the distance k_E. It follows that E is not Gromov hyperbolic. □

We conclude this section highlighting an interesting class of holomorphic fiber bundles satisfying the condition $\sup_X r = +\infty$.

Proposition 23 *Let Y be a complex manifold and let $\pi : E \to Y$ be a holomorphic fiber bundle. Let $X \subset Y$ be a domain. Assume that there exists a point $\xi \in \partial X$ which admits a local holomorphic peak function. Then the restricted holomorphic bundle $E|_X$ has the property* $\sup_X r = +\infty$.

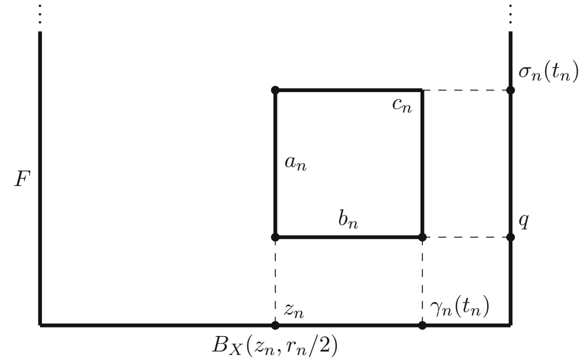

Fig. 2 The triangle T_n in $B_X(z_n, r_n/2) \times F$

Proof Let U be an open neighborhood of ξ in Y such that $\pi : E \to Y$ trivializes over U. Let $\{z_n\}_n$ be a sequence in X converging to ξ. By Lemma 17, we have that

$$\lim_{n\to+\infty} k_X(z_n, X \cap U^c) = +\infty.$$

Hence, for each $R > 0$, we have $B_X(z_n, R) \subset X \cap U$ for n large enough, which implies that $r(z_n) \to +\infty$. □

Corollary 24 *The pre-Worms W_{in} and W_{out} are not Gromov hyperbolic w.r.t. its Kobayashi distance.*

Proof The domains

$$X_{\text{in}} \subset\subset X_{\text{out}} \subset\subset Y$$

are smoothly bounded (see Remark 12), and thus, every point in their boundary admits a local holomorphic peak function. Hence by the previous proposition the pre-Worms W_{in} and W_{out} satisfy $\sup_X r = +\infty$ and Theorem 22 yields the result. □

5 Barrett's Scaling and Proof of Main Theorem

In what follows, we denote by TM the holomorphic tangent bundle of a complex manifold M and by $\pi : TM \to M$, the canonical projection. We denote by $\mathbb{D} \subset \mathbb{C}$ the unit disk. Recall the following classical definition.

Definition 25 Let M be a complex manifold and let $X \subset M$ be a domain. Then X has *simple boundary* in M if for all $\phi : \mathbb{D} \to M$ holomorphic mappings with $\phi(\mathbb{D}) \subset \overline{X}$ and $\phi(\mathbb{D}) \cap \partial X \neq \varnothing$ one has $\phi(\mathbb{D}) \subseteq \partial X$.

The proof of Theorem 1 is based on the following result, showing the stability of the Kobayashi distance and of the Kobayashi–Royden metric under a particular type of convergence of domains $D_n \to D_\infty$.

Proposition 26 *Let M be a taut complex manifold and let $\{D_n\}_n$ be a sequence of domains of M. Let $D_\infty \subset M$ be a complete Kobayashi hyperbolic domain with simple boundary. Assume that*

(i) *if $\{x_n\}_n$ is a sequence converging to $x_\infty \in M$ and $x_n \in D_n$ for all $n \in \mathbb{N}$, then $x_\infty \in \overline{D}_\infty$;*
(ii) *for every compact $H \subset D_\infty$, there exists N such that $H \subset D_n$ for $n \geq N$.*

Then as $n \to +\infty$ we have $K_{D_n} \to K_{D_\infty}$ uniformly on compact subsets of TD_∞, and $k_{D_n} \to k_{D_\infty}$ uniformly on compact subsets of $D_\infty \times D_\infty$.

See, e.g., [16, Chap. 5] for the notion of tautness. The idea of the proof of Proposition 26 is similar to [7, Theorem 4.3]. The proof is based on two lemmas, valid under the assumptions of the proposition.

Lemma 27 *For every $H \subset D_\infty$ compact and $\epsilon > 0$, there exists N such that for all $n \geq N$ and for all $v \in \pi^{-1}(H)$, we have*

$$K_{D_n}(v) \leq (1+\epsilon) K_{D_\infty}(v).$$

Proof Set $r := (1+\epsilon)^{-1} \in (0,1)$. Define $\widehat{H} \subset D_\infty$ as

$$\widehat{H} := \{\phi(\zeta) | \, \phi : \mathbb{D} \to D_\infty \text{ holomorphic}, \ \phi(0) \in H, \ |\zeta| \leq r\}.$$

The set $\widehat{H}$ is compact. Indeed, let $\{z_n\}_n$ be a sequence in $\widehat{H}$, i.e., there exist $\phi_n : \mathbb{D} \to D_\infty$ such that $\phi_n(0) \in H$, and $|\zeta_n| \leq r$ such that $z_n = \phi_n(\zeta_n)$. Since $\phi_n(0) \in H$ for all $n \in \mathbb{N}$ and D_∞ is taut (by [16, Theorem 5.1.3]), we can assume that ϕ_n converges uniformly on compact sets to a holomorphic map $\hat{\phi} : \mathbb{D} \to D_\infty$ and that ζ_n converges to $\hat{\zeta}$ with $|\hat{\zeta}| \leq r$. But then $z_n \to \hat{\phi}(\hat{\zeta}) \in \widehat{H}$. This proves that $\widehat{H}$ is compact.

Now, for each $v \in \pi^{-1}(H)$ let $\phi : \mathbb{D} \to D_\infty$ be such that $\phi(0) = \pi(v)$ and

$$K_{D_\infty}(v)\phi'(0) = v.$$

Using property (ii), there exists N such that for all $n \geq N$, we have $\widehat{H} \subset D_n$, which implies that if $\phi_r : \mathbb{D} \to D_\infty$ is defined by $\phi_r(z) := \phi(rz)$ then $\phi_r(\mathbb{D}) \subset \widehat{H} \subset D_n$. Finally, using the definition of the Kobayashi–Royden metric, we have

$$K_{D_n}(v) \leq r^{-1} K_{D_\infty}(v) = (1+\epsilon) K_{D_\infty}(v).$$

□

Lemma 28 *For every $H \subset D_\infty$ compact and $\epsilon > 0$, there exists N such that for all $n \geq N$ and for all $v \in \pi^{-1}(H)$, we have*

$$K_{D_\infty}(v) \leq (1+\epsilon) K_{D_n}(v). \tag{2}$$

Proof Fix an Hermitian metric on TD_∞. The result immediately follows if we prove (2) for all $v \in \pi^{-1}(H)$ such that $\|v\| = 1$. Assume by contradiction that there exist $H \subset D_\infty$ compact, $\epsilon > 0$, and $n_k \to +\infty$, $v_k \in \pi^{-1}(H)$ such that $\|v_k\| = 1$ and

$$K_{D_\infty}(v_k) > (1+\epsilon)K_{D_{n_k}}(v_k).$$

We can assume that $v_k \to v_\infty \in \pi^{-1}(H)$. Let $\phi_k : \mathbb{D} \to D_{n_k}$ be a holomorphic map such that $\phi_k(0) = \pi(v_k)$ and $\alpha_k\phi_k'(0) = v_k$, where $\alpha_k \le (1+\epsilon)^{1/2}K_{D_{n_k}}(v_k)$. In particular, $\alpha_k \le (1+\epsilon)^{-1/2}K_{D_\infty}(v_k)$ and hence, α_k is uniformly bounded in k. We may, therefore, assume that α_k converges to a limit α as $k \to +\infty$.

Since M is taut and $\phi_k(0) \in H$, we can assume that the sequence $\{\phi_k\}_k$ converges uniformly on compact sets to a holomorphic map $\phi : \mathbb{D} \to M$, which satisfies the identity $\alpha\phi'(0) = v_\infty$. Using property (i), we have $\phi(\mathbb{D}) \subset \overline{D}_\infty$. Since D_∞ has simple boundary in M it follows from $\phi(0) = \pi(v_\infty) \in D_\infty$ that $\phi(\mathbb{D}) \subset D_\infty$. Finally using the definition of the Kobayashi–Royden metric, we have

$$K_{D_\infty}(v_\infty) \le \alpha \le \lim_k (1+\epsilon)^{-1/2}K_{D_\infty}(v_k) = (1+\epsilon)^{-1/2}K_{D_\infty}(v_\infty),$$

which is a contradiction. □

Proof of Proposition 26 The uniform convergence on compact subsets of the Kobayashi–Royden metric follows from Lemmas 27 and 28. We now prove the local uniform convergence of the Kobayashi distance. In what follows, we denote by $\ell_M(\gamma)$ the Kobayashi–Royden length of a curve γ on the manifold M.

Let $H \subset D_\infty$ be a compact set, and set $R := \mathrm{diam}(H)$. Given $p, q \in H$ and $\epsilon \in (0, 1)$, let $\gamma : [0, 1] \to D_\infty$ be a piecewise C^1 curve joining p with q and satisfying $\ell_{D_\infty}(\gamma) \le k_{D_\infty}(p, q) + \epsilon$. Fix $o \in H$. Then, for all $t \in [0, 1]$,

$$\begin{aligned} k_{D_\infty}(o, \gamma(t)) &\le k_{D_\infty}(o, p) + k_{D_\infty}(p, \gamma(t)) \\ &\le R + \ell_{D_\infty}(\gamma) \le R + k_{D_\infty}(p, q) + \epsilon \le 2R + 1, \end{aligned}$$

i.e., the support of γ is contained in $\overline{B_{D_\infty}(o, 2R+1)}$ which is a compact subset of D_∞ by the completeness of D_∞. By Lemma 27, there exists N such that for all $n \ge N$ and for all $v \in \pi^{-1}(\overline{B_{D_\infty}(o, 2R+1)})$ we have $K_{D_n}(v) \le (1+\epsilon)K_{D_\infty}(v)$, which implies $\ell_{D_n}(\gamma) \le (1+\epsilon)\ell_{D_\infty}(\gamma)$. Hence,

$$\begin{aligned} k_{D_n}(p, q) &\le \ell_{D_n}(\gamma) \le (1+\epsilon)\ell_{D_\infty}(\gamma) \le (1+\epsilon)(k_{D_\infty}(p, q) + \epsilon) \\ &\le k_{D_\infty}(p, q) + O((1+R)\epsilon). \end{aligned}$$

In particular,

$$k_{D_n}(p, q) = O(1+R) \tag{3}$$

for $n \ge N$.

For the converse, notice that by (ii) H is eventually contained in the domains D_n. Given $p, q \in H$ and $\epsilon \in (0, 1)$, let $\gamma_n : [0, 1] \to D_n$ be a piecewise C^1 curve joining

p with q and satisfying $\ell_{D_n}(\gamma_n) \leq k_{D_n}(p,q)+\epsilon$. Fix $o \in H$ and define

$$t_n := \sup\{t \in [0,1] : \gamma_n([0,t]) \subset B_{D_\infty}(o, 2R)\}.$$

We have that $k_{D_\infty}(p, \gamma_n(t_n)) \geq k_{D_\infty}(p,q)$. Indeed, this clearly holds if $t_n = 1$. If $t_n < 1$, then $k_{D_\infty}(o, \gamma_n(t_n)) = 2R$ and thus

$$k_{D_\infty}(p, \gamma_n(t_n)) \geq k_{D_\infty}(o, \gamma_n(t_n)) - k_{D_\infty}(p, o) \geq 2R - R = R \geq k_{D_\infty}(p,q).$$

Since $\overline{B_{D_\infty}(o, 2R)}$ is compact, by Lemma 28 there exists N such that for all $n \geq N$ and for all $v \in \pi^{-1}(\overline{B_{D_\infty}(o, 2R)})$, we have that $K_{D_n}(v) \geq (1+\epsilon)^{-1} K_{D_\infty}(v)$. Hence,

$$\begin{aligned} k_{D_n}(p,q)+\epsilon &\geq \ell_{D_n}(\gamma_n) \geq \ell_{D_n}(\gamma_n|_{[0,t_n]}) \geq (1+\epsilon)^{-1}\ell_{D_\infty}(\gamma_n|_{[0,t_n]}) \\ &\geq (1+\epsilon)^{-1} k_{D_\infty}(p, \gamma_n(t_n)) \geq (1+\epsilon)^{-1} k_{D_\infty}(p,q), \end{aligned}$$

that is

$$k_{D_\infty}(p,q) \leq (1+\epsilon)(k_{D_n}(p,q)+\epsilon) \leq k_{D_n}(p,q) + O((1+R)\epsilon),$$

where we used (3). □

Let W be a Worm. We call *Barrett's scaling* the one-parameter group of automorphisms of $Y \times \mathbb{C}$ given by

$$\mathrm{B}_\lambda : (z, w) \mapsto (z, \lambda w) \qquad (\lambda > 0),$$

which played a key role in [4, Sect. 4].

For all $n \geq 1$ we set $D_n := \mathrm{B}_n(W)$, $D_\infty := W_{\mathrm{in}}$, and $M := W_{\mathrm{out}}$.

Remark 29 Properties (i) and (ii) of Proposition 26 are satisfied in this case.

Lemma 30 *The domain W_{in} has simple boundary in W_{out}.*

Proof Let $\varphi\colon \mathbb{D} \to W_{\mathrm{out}}$ be a holomorphic map such that $\varphi(\mathbb{D}) \subset \overline{W}_{\mathrm{in}}$. Assume that there exists $\zeta_0 \in \mathbb{D}$ such that

$$(z_0, w_0) := \phi(\zeta_0) \in \partial W_{\mathrm{in}}.$$

Clearly $z_0 \in \partial X_{\mathrm{in}}$. If $\pi_1 : X_{\mathrm{out}} \times \mathbb{C} \to X_{\mathrm{out}}$ denotes the projection to the first variable, then $\pi_1 \circ \phi : \mathbb{D} \to X_{\mathrm{out}}$ is a holomorphic function with image contained in $\overline{X}_{\mathrm{in}}$ and such that $(\pi_1 \circ \phi)(\zeta_0) \in \partial X_{\mathrm{in}}$, hence by the open-mapping theorem $\pi_1 \circ \phi$ is constant. Thus, $\phi(\mathbb{D}) \subset \partial W_{\mathrm{in}}$. □

We are finally able to prove our main theorem.

Proof of Theorem 1 By contradiction, assume that there exists $\delta \geq 0$ such that for each $o, x, y, z \in W$ we have

$$\min\{(x|y)_o^{k_W}, (y|z)_o^{k_W}\} - (x|z)_o^{k_W} \leq \delta.$$

Now since for all $n \geq 1$, the Barrett's scaling B_n is an isometry between W and $B_n(W)$ we have, for each $o, x, y, z \in B_n(W)$,

$$\min\{(x|y)_o^{k_{B_n(W)}}, (y|z)_o^{k_{B_n(W)}}\} - (x|z)_o^{k_{B_n(W)}} \leq \delta.$$

By Proposition 26, we have, for all $o, x, y, z \in W_{\text{in}}$,

$$\begin{aligned} &\min\left\{(x|y)_o^{k_{W_{\text{in}}}}, (y|z)_o^{k_{W_{\text{in}}}}\right\} - (x|z)_o^{k_{W_{\text{in}}}} \\ &= \lim_{n\to+\infty} \min\left\{(x|y)_o^{k_{B_n(W)}}, (y|z)_o^{k_{B_n(W)}}\right\} \\ &\quad -(x|z)_o^{k_{B_n(W)}} \leq \delta. \end{aligned}$$

Thus, W_{in} is Gromov hyperbolic, which contradicts Corollary 24. □

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