Journal of Philosophical Logic (2026) 55:199–215
https://doi.org/10.1007/s10992-026-09832-y

A Hypersequent Calculus for Classical Contingencies

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Received: 29 November 2024 / Accepted: 20 January 2026 / Published online: 21 February 2026

Abstract
We present a hypersequent calculus that is sound and complete with respect to the truth-functionally contingent formulas of classical logic. We investigate its structural properties and provide a Gentzen-style cut-elimination procedure. The most notable feature of the calculus is that it jointly satisfies the subformula property and the property of *deductive purity*, to the effect that only contingent hypersequents occur in formal proofs. Moreover, since the negation of a contingent formula is also contingent, the calculus turns out to be *paraconsistent*, and since the conjunction of a formula with its own negation is not contingent, the paraconsistency is of the *non-adjunctive* kind.

Keywords Classical logic · Antisequent · Hypersequent · Contingency · Paraconsistency

1 Introduction

This work is part of a broader strand of studies concerning the analysis of (decidable) logical systems through the proof-theoretic characterization of their invalid formulas, or of semantically significant subsets thereof [10, 15, 34]. In particular, with regard to classical propositional logic, there is an extensive literature on proof systems for contradictions and, more generally, for non-tautologies—so-called *refutation* or *rejection* calculi [29, 30]. By providing strictly syntactic means for ascertaining non-validity, independently of countermodel analysis, such calculi have been found particularly valuable in the context of automated reasoning [8, 39]. Starting with [18, §4], the problem of characterizing the set of truth-functional contingencies—i.e., those formulas that are neither tautologies nor contradictions—has also been considered. This problem is especially interesting from a philosophical perspective, since the logic of scientific discovery typically proceeds from contingent truths to contingent truths, yet

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it has received comparatively less attention. This is partly due to the fact that the class of such formulas is not closed under substitution and, furthermore, exhibits a peculiar form of non-adjunctive paraconsistency (since contingencies may be mutually inconsistent), two properties that complicate the design of elegant and well-behaved proof-theoretic characterizations (see e.g. [20, 42]).

An elegant Gentzen-style system for classical contingencies has been proposed by Michael Tiomkin [40]. This system is aptly described by the author as a "contingency prover". However, it includes *hybrid* rules, that is, rules that involve tautologically valid premises next to contingent ones. In the literature, this kind of "deductive impurity" is generally unwelcome, for it implies that handling contingencies requires prior knowledge of what constitutes a tautology [14, 32, 44]. Tiomkin himself acknowledges this limitation:

> This calculus is very natural, because it possesses both cut elimination and the subformula property. However, it also involves derivations in a classical sequent calculus. Thus, it would be nice to have a pure self-contained contingency calculus. [40, p. 535]

Recently, this challenge has been taken up by Liang et al. [17]. These authors propose a Hilbert-style system for classical contingencies that does not rely on tautological notions or rules in order to attain completeness and is, therefore, "self-contained" in the relevant sense. Unfortunately, the system suffers from the typical disadvantages of Hilbert-style axiomatic approaches. Most notably, it lacks the subformula property, which makes proof search extremely difficult to implement.

In this paper, we present a novel proof system that overcomes these limitations by combining deductive purity with desirable proof-theoretic features, including the subformula property. This is achieved through a hypersequent adaptation of Kleene's sequent calculus G4 [16]. The resulting system, HCC (Hypersequent Contingency Calculus), proves exactly the formulas that are truth-functionally contingent in classical propositional logic and requires no recourse to derivations in the classical sequent calculus, thereby providing a full solution to Tiomkin's problem.

The focus on classical propositional logic is directly motivated by the above considerations. Of course, similar projects could be pursued with regard to other, non-classical logics that enjoy a decidable notion of validity. For example, refutation and rejection calculi have already been developed for propositional intuitionistic logic [11, 33] and for various types of modal logics [13, 35], many-valued logics [5], relevant logics [7], and non-monotonic logics [6], *inter alia*. Each of these calculi could, in principle, be supplemented with a corresponding contingency calculus, ideally self-contained in Tiomkin's sense. Moreover, while no complete calculus of this kind can be constructed when the relevant notion of (in)validity fails to be fully decidable, as in classical first-order logic [19, §3], one could still investigate the extent to which a propositional contingency calculus can be extended by adding suitable quantificational rules. All these projects are of intrinsic interest and would shed further light on the prospects of analyzing the many facets of logical invalidity in strict proof-theoretic terms; however, they lie beyond the specific aims of the present paper. Likewise, although the proof-theoretic study of logical contingency could be further motivated and enriched by considerations of algorithmic complexity (the problem 'Is A contingent?' being NP), we leave such developments to future work.

2 Tiomkin's Hybrid Sequent Calculus

2.1 Preliminary Notions

We shall initially focus on a propositional language whose primitive connectives include classical negation ($\neg$), disjunction ($\vee$), and conjunction ($\wedge$). We denote by AT the set of all atomic formulas, which includes the propositional variables $p, q, r, \ldots$ along with the usual two constants $\bot$ (false) and $\top$ (true). Additionally, we use upper-case Latin letters for arbitrary formulas whereas upper-case Greek letters represent finite *multisets* of formulas. As customary, we write Γ, Δ to mean the multiset union $\Gamma \uplus \Delta$. For $\Gamma = A_1, \ldots, A_n$, we set $\bigvee \Gamma = A_1 \vee \cdots \vee A_n$ and $\bigwedge \Gamma = A_1 \wedge \cdots \wedge A_n$. To cover the limiting cases where Γ is the empty set, we stipulate that $\bigvee \varnothing = \bot$ and $\bigwedge \varnothing = \top$.

In what follows, we shall be dealing with standard Gentzen sequents of the form $\Gamma \vdash \Delta$ as well as with *antisequents* $\Gamma \dashv \Delta$. Antisequents come from the literature on rejection and refutation calculi and have been introduced specifically to handle invalid sequents within formal derivations [9, 13, 38]. In other words, asserting an antisequent $\Gamma \dashv \Delta$ is tantamount to asserting the *in*validity of the corresponding sequent $\Gamma \vdash \Delta$, hence, semantically, the existence of a Boolean valuation v falsifying the conditional $\bigwedge \Gamma \to \bigvee \Delta$. Accordingly, two subcases can be distinguished. If *every* Boolean valuation falsifies $\bigwedge \Gamma \to \bigvee \Delta$, i.e., if this conditional is a contradiction, the antisequent $\Gamma \dashv \Delta$ is itself classified as *contradictory*. Otherwise, when there also exists a Boolean valuation that verifies $\bigwedge \Gamma \to \bigvee \Delta$, the source antisequent $\Gamma \dashv \Delta$ qualifies as *contingent*.

2.2 The Hybrid Calculus CC

Figure 1 shows the basic rules of Tiomkin's Contingency Calculus, CC, restricted to a propositional language with our primitive connectives. The calculus proves to be sound and complete with respect to the set of all classical contingencies: for every formula A, there is a CC-derivation ending in the antisequent $\dashv A$ if and only if A is truth-functionally contingent in classical logic [40, Prop. 1 and Prop. 2]. CC-derivations involve only contingent antisequents, never contradictory ones. However, the structural rules of cut and extension rely also on classically valid sequents—namely, $\Gamma \vdash \Delta, A$ and $\Gamma, A \vdash \Delta$—and thus count as impure.

Example 1 Figure 2 shows a CC-derivation ending in $\dashv (q \wedge (\neg q \vee p)) \vee q$. It is easy to check that the formula $(q \wedge (\neg q \vee p)) \vee q$ is indeed contingent.

AXIOM

$$\frac{}{\Gamma \dashv \Delta}\,\overline{ax} \qquad \text{where } \Gamma \cup \Delta \subset \mathsf{AT},\ \Gamma \cap \Delta = \varnothing, \text{ and } \Gamma \cup \Delta \neq \varnothing$$

NEGATION

$$\frac{\Gamma \dashv \Delta, A}{\Gamma, \neg A \dashv \Delta}\ \neg\dashv \qquad\qquad \frac{\Gamma, A \dashv \Delta}{\Gamma \dashv \Delta, \neg A}\ \dashv\neg$$

CONJUNCTION

$$\frac{\Gamma, A, B \dashv \Delta}{\Gamma, A \wedge B \dashv \Delta}\ \wedge\dashv$$

$$\frac{\Gamma, A, B \dashv \Delta \qquad \Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B}\ \dashv\wedge_{\mathfrak{L}}$$

$$\frac{\Gamma, A, B \dashv \Delta \qquad \Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \wedge B}\ \dashv\wedge_{\mathfrak{R}}$$

DISJUNCTION

$$\frac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}\ \dashv\vee$$

$$\frac{\Gamma \dashv \Delta, A, B \qquad \Gamma, A \dashv \Delta}{\Gamma, A \vee B \dashv \Delta}\ \vee_{\mathfrak{L}}\dashv$$

$$\frac{\Gamma \dashv \Delta, A, B \qquad \Gamma, B \dashv \Delta}{\Gamma, A \vee B \dashv \Delta}\ \vee_{\mathfrak{R}}\dashv$$

CUT

$$\frac{\Gamma, A \dashv \Delta \qquad \Gamma \vdash \Delta, A}{\Gamma \dashv \Delta}\ cut_{\mathfrak{L}} \qquad\qquad \frac{\Gamma \dashv \Delta, A \qquad \Gamma, A \vdash \Delta}{\Gamma \dashv \Delta}\ cut_{\mathfrak{R}}$$

EXTENSION

$$\frac{\Gamma \dashv \Delta \qquad \Gamma \vdash \Delta, A}{\Gamma, A \dashv \Delta}\ ext_{\mathfrak{L}} \qquad\qquad \frac{\Gamma \dashv \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \dashv \Delta, A}\ ext_{\mathfrak{R}}$$

Fig. 1 The basic rules of Tiomkin's calculus CC

$$\dfrac{\dfrac{\dfrac{}{\dashv q}\,\overline{ax} \qquad \dfrac{\dfrac{}{q, \neg q \vee p \vdash q}\,ax.}{q \wedge (\neg q \vee p) \vdash q}\,\wedge}{\dashv q \wedge (\neg q \vee p), q}\,ext_{\mathfrak{R}}}{\dashv (q \wedge (\neg q \vee p)) \vee q}\,\vee$$

Fig. 2 A derivation in CC

2.3 Cut Eliminability and the Subformula Property

The "deductive impurity" stemming from the presence of hybrid rules is not the only unwelcome feature of CC. There are also structural limitations. Consider cut. As noted in [9], the multiplicative (context-mixing) formulation of the cut-rule is not admissible in refutational settings, since there are simple cases where a valid conclusion can be derived by cutting two invalid premises. In the even more specific case of contingencies, the additive (context-sharing) version of the cut-rule is not admissible either. Here is an easy counterexample:

$$\frac{\dashv p \wedge \overline{p}, q \qquad \dashv p \wedge \overline{p}, \overline{q}}{\dashv p \wedge \overline{p}}$$

It is for these reasons that CC handles cut by means of the two (hybrid) rules $cut_{\mathfrak{L}}$ and $cut_{\mathfrak{R}}$. These two rules are then shown to be redundant [40, Cor. 3]. Nonetheless, while Tiomkin uses the term 'cut elimination'—suggesting that any proof with cuts can be *transformed* into a proof without—what he actually establishes is a form of *cut eliminability*: whatever can be proved in CC using the cut rules *can* be proved without them. This result is obtained semantically as a corollary of the soundness and completeness of CC, from which the subformula property is established as a further corollary. From a strictly proof-theoretic perspective, however, this is not entirely satisfactory. Indeed, it is easy to see that the CC calculus does *not* admit the implementation of a fully syntactic, Gentzen-style normalization procedure, specifically one based on the progressive reduction of the linguistic complexity of the cut formula through the execution of parallel reductions.

Consider, for instance, the derivation in Fig. 3, where one occurrence of the cut formula is introduced by applying the $\dashv\wedge_{\mathfrak{L}}$-rule while its dual in the other premise comes from the $\wedge\dashv$-rule. Since the psequent $\Gamma, p, q \vdash \Delta$ and the antisequent $\Gamma, p, q \dashv \Delta$ are mutually exclusive, there seems to be no way to properly produce the parallel reduction under consideration. More generally, we see this blockage of standard cut-elimination procedures as stemming essentially from the fact that some rules in the calculus transfer formulas from one side of the (anti)sequent symbol to the other side without introducing new negations. As a result, CC satisfies the subformula property only "weakly", in the sense that the property is not preserved when CC is reformulated as its one-sided variant *à la* Tait [37].

$$\dfrac{\dfrac{\Gamma, p, q \dashv \Delta \qquad \Gamma \dashv \Delta, p}{\Gamma \dashv \Delta, p \wedge q}\dashv\wedge_{\mathfrak{L}} \qquad \dfrac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta}\wedge\vdash}{\Gamma \dashv \Delta}cut_{\mathfrak{R}} \qquad \rightsquigarrow \quad ?$$

Fig. 3 A non-reducible cut application

3 Proving Contingency Through Rejection and Acception

3.1 The Symmetric Calculi $\overline{\mathsf{GS4}}$ and $\overline{\text{ꟼƧƆ}}$

As mentioned in the Introduction, the literature on the subject offers a wide variety of rejection calculi for classical propositional logic, i.e., proof systems that are sound and complete with respect to the set of classical non-tautologies [15, 29]. In this section, we briefly consider a way of proving contingencies that relies on such calculi, specifically on the antisequent system $\overline{\mathsf{GS4}}$ (from [21]). This is a one-sided rejection calculus for $\mathsf{GS4}$, the one-sided version of Kleene's $\mathsf{G4}$ [16, 41], and is inspired by the two-sided antisequent calculi due to Tiomkin himself [39] and Goranko [13].

Systems based on one-sided (anti)sequents require a modification of the language. In particular, in languages *à la* Tait [37], the negation connective is treated as primitive on the propositional variables and extends to compound formulas via De Morgan dualities. More formally:

$$\mathsf{AT} = \{p, q, r \ldots\} \cup \{\overline{p}, \overline{q}, \overline{r}, \ldots\} \cup \{\bot\}$$

$$\overline{\overline{A}} \equiv A \quad \overline{A \wedge B} \equiv \overline{A} \vee \overline{B} \quad \overline{A \vee B} \equiv \overline{A} \wedge \overline{B}$$

Based on these definitions, the rules of $\overline{\mathsf{GS4}}$ are shown in the left column of Fig. 4. The right column presents the rules of the symmetric system $\overline{\text{ꟼƧƆ}}$. Whereas $\overline{\mathsf{GS4}}$ is a rejection calculus for proving classical non-tautologies, $\overline{\text{ꟼƧƆ}}$ may be termed an *acception* calculus [29], since its provable antisequents correspond to the classical non-contradictions. We call it 'symmetric' and use a mirror-reversed label because the rules of $\overline{\text{ꟼƧƆ}}$ themselves may be seen as the "mirror image" of those defining $\overline{\mathsf{GS4}}$: each system can be obtained from the other by switching to the opposite side of the antisequent symbol $\dashv$ and uniformly interchanging conjunction ($\wedge$) and disjunction

$\overline{\mathsf{GS4}}$	$\overline{\text{ꟼƧƆ}}$
$\dfrac{}{\dashv \Delta}\ \overline{ax}$	$\overline{ax}\ \dfrac{}{\Delta \dashv}$
where $\Delta \subseteq \mathsf{AT}$ and $A \in \Delta \Rightarrow \overline{A} \notin \Delta$	
$\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, A \vee B}\ \vee$	$\wedge\ \dfrac{\Gamma, A, B \dashv}{\Gamma, A \wedge B \dashv}$
$\dfrac{\dashv \Gamma, A}{\dashv \Gamma, A \wedge B}\ \wedge_{\mathfrak{R}}$	$\vee_{\mathfrak{R}}\ \dfrac{\Gamma, A \dashv}{\Gamma, A \vee B \dashv}$
$\dfrac{\dashv \Gamma, B}{\dashv \Gamma, A \wedge B}\ \wedge_{\mathfrak{L}}$	$\vee_{\mathfrak{L}}\ \dfrac{\Gamma, B \dashv}{\Gamma, A \vee B \dashv}$

Fig. 4 The symmetric rules of $\overline{\mathsf{GS4}}$ and $\overline{\text{ꟼƧƆ}}$

($\vee$) in each of the logical rules. In both calculi, the $\overline{ax}$-rule is restricted so as to introduce only *consistent* clauses, i.e., multisets of atoms none of which is the dual of another. As a limiting case, the empty antisequent $\dashv$ is considered a legitimate instance of the $\overline{ax}$-rule in both $\overline{\mathsf{GS4}}$ and $\overline{\text{ᔨƧӘ}}$.

Theorem 1 *An antisequent $\dashv \Gamma$ is provable in $\overline{\mathsf{GS4}}$ if and only if the formula $\bigvee \Gamma$ is not a tautology.*

Theorem 2 *An antisequent $\Gamma \dashv$ is provable in $\overline{\text{ᔨƧӘ}}$ if and only if the formula $\bigwedge \Gamma$ is not a contradiction.*

Proof We only provide a sketch of the proof; a detailed argument can be found in [29, Thm. 3.2.3]. Focusing on Theorem 1, let us consider the two directions of the biconditional separately.

Left-to-right (soundness): This direction of the proof is perfectly standard and can be carried out by a routine induction on the length of the derivation ending in $\dashv \Gamma$.

Right-to-left (completeness): For this direction it suffices to reason in terms of proof search, by induction on the number of occurrences of connectives in Γ. The backwards construction of a derivation of $\dashv \Gamma$ proceeds by considering a Boolean valuation under which $\bigvee \Gamma$ is false. The key observation, then, is that such a valuation provides enough information to guide the construction of the entire derivation.

Given the symmetric format of $\overline{\text{ᔨƧӘ}}$, the proof of Theorem 2 can be obtained in an analogous fashion. □

3.2 A Combined Strategy

Taken together, Theorems 1 and 2 deliver a simple hybrid method for determining whether a given formula A is truth-functionally contingent. On the one hand, a proof of $\dashv A$ in $\overline{\mathsf{GS4}}$ entails that A is not a taoutology; on the other, proving $A \dashv$ in the symmetric system $\overline{\text{ᔨƧӘ}}$ will ensure that A is not a contradiction.

Example 2 In Fig. 5, we provide two derivations of the formula $(q \wedge (\overline{q} \vee p)) \vee q$. The one on the left qualifies as a proof in $\overline{\mathsf{GS4}}$ while the other is a proof in $\overline{\text{ᔨƧӘ}}$. Given Theorems 1 and 2, these two derivations establish that $(q \wedge (\overline{q} \vee p)) \vee q$ is neither a tautology nor a contradiction. Hence we conclude that the formula is contingent.

This combined "divide and conquer" strategy is, of course, reminiscent of the familiar procedure for determining contingencies using the method of analytic tableaux [36]. One begins by ruling out that A is a tautology by constructing the tree associated with $\neg A$, and then the possibility of A being a contradiction is excluded by constructing a further tree for A itself. Such indirect methods are perfectly sensible and easy to implement. Nevertheless, again, they are not fully satisfactory from a proof-theoretic

$$\dfrac{\dfrac{\dfrac{}{\dashv q, q}\,\overline{ax}}{\dashv q \wedge (\overline{q} \vee p), q}\,\wedge_{\mathfrak{R}}}{\dashv (q \wedge (\overline{q} \vee p)) \vee q}\,\vee \qquad\qquad \vee_{\mathfrak{L}}\,\dfrac{\overline{ax}\,\dfrac{}{q \dashv}}{(q \wedge (\overline{q} \vee p)) \vee q \dashv}$$

Fig. 5 A $\overline{\mathsf{GS4}}$-proof and its $\overline{\text{ᔨƧӘ}}$-counterpart standpoint. What we want is a non-hybrid, direct method—a pure calculus for contingencies.

4 A Pure Hypersequent Calculus for Contingencies

4.1 From (anti)sequents to Hypersequents

The solution we propose involves shifting from the landscape of (anti)sequent calculi to a hypersequent calculus.

Hypersequents come into play quite naturally once we pay attention on the following key consideration. In classical logic, any sequent $\vdash \Gamma$ can be decomposed into a set of clauses $\vdash \Gamma_1, \ldots, \vdash \Gamma_n$ by maximally applying, in a bottom-up fashion, the following two rules from Kleene's GS4.

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\vee \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\wedge$$

Since both of these rules are *invertible* (the provability of the conclusion always entails the provability of the premises), it is easy to see that the disjunction corresponding to the given sequent, $\bigvee \Gamma$, is logically equivalent to the conjunction $\bigvee \Gamma_1 \wedge \cdots \wedge \bigvee \Gamma_n$ [3]. More generally, $\bigvee \Gamma$ is always equivalent to the conjunction of the disjunctions corresponding to the displayed top sequents at *any* stage of the decomposition process, even when these top sequents have not yet been reduced to clauses. This important property is crucial for our purposes, as it suggests that the preservation of contingency is essentially a matter of selecting the right parameter to consider. Specifically, if we examine a sequent $\vdash \Gamma$ that is contingent, it becomes clear that contingency can be preserved, at each decomposition step, so long as the leaves of the deductive tree are considered as a single "package". The notion of (conjunctive) hypersequents that we will elaborate in the following is intended to formally represent the structure of these packages.

4.2 Conjunctive Hypersequents

To transition from (one-sided) sequents to (one-sided) hypersequents, we need to update our notation [2, 4]. First, when there is no need to distinguish between sequents and antisequents, we will henceforth drop the prefixes '$\vdash$' and '$\dashv$' and directly write one-sided (anti)sequents as sets of formulas $\Gamma, \Delta, \ldots$ The empty sequent is denoted by the symbol '$\varnothing$'.

Next, we shall use the letters $\mathcal{G}, \mathcal{H}, \ldots$ to represent hypersequents. A *hypersequent* is defined as a finite list of one-sided (anti)sequents separated by the bar symbol, written as $\Gamma_1 \mid \cdots \mid \Gamma_n$. Hypersequents are considered unordered; for example, $\mathcal{G} \mid \mathcal{H} \mid \mathcal{I}$ is treated as the same hypersequent as $\mathcal{G} \mid \mathcal{I} \mid \mathcal{H}$. We write $\Gamma \in \mathcal{G}$ to mean that the sequent Γ is displayed in $\mathcal{G}$, i.e., there is a hypersequent $\mathcal{H}$ such that $\mathcal{G} \equiv \mathcal{H} \mid \Gamma$. If $\Gamma \in \mathcal{G}$, then Γ is called a *component* of $\mathcal{G}$. By a *hyperclause* we mean any hypersequent whose components are all *clauses*, i.e., mulisets of atoms [4, 12]. Depending on context, any sequent Γ may also be regarded as a hypersequent with Γ as its sole component. In particular, $\varnothing$ may denote both the empty clause and the empty hyperclause.

Lastly, for any hypersequent $\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n$, we define $\mathsf{For}(\mathcal{G}) \equiv \bigvee \Gamma_1 \wedge \cdots \wedge \bigvee \Gamma_n$. If $\mathcal{G} \equiv \varnothing$, then $\mathsf{For}(\mathcal{G}) = \bot$. Thus, the separating bar in a hypersequent is here understood, as in [22, 23], conjunctively, unlike the customary interpretation where the bar expresses some sort of alternative among the components [4, 24, 25]. We say that a hypersequent $\mathcal{G}$ is *contingent* just in case the formula $\mathsf{For}(\mathcal{G})$ is truth-functionally contingent. Otherwise $\mathcal{G}$ is *tautological* or *contradictory*, depending on whether $\mathsf{For}(\mathcal{G})$ itself is a tautology or a contradiction.

4.3 Testing Hyperclauses for Contingency

Before introducing our hypersequent calculus for contingency, it will be convenient to establish a couple of preliminary facts about contingent hyperclauses. This detour will help clarify why the axiomatic clauses of the calculus come with specific side conditions.

The first fact registers an important characterizing property of non-tautological hyperclauses:

Lemma 3 *A hyperclause $\mathcal{H}$ is non-tautological if and only if $\mathcal{H}$ displays at least one consistent clause.*

Proof Left-to-right: By contraposition. If $\mathcal{H}$ displays no consistent clause, all clauses of $\mathcal{H}$ are tautological. Consequently, $\mathsf{For}(\mathcal{H})$ must also be a tautology.

Right-to-left: Let $\mathcal{H} \equiv \mathcal{H}' \mid \Gamma$ be a hyperclause with Γ consistent. Since Γ does not include pairs of dual atoms, there exists a valuation v that falsifies every formula therein. This ensures that $v(\mathsf{For}(\mathcal{H})) = \mathsf{F}$, hence $\mathcal{H}$ is non-tautological. □

We now now turn to non-contradictory hyperclauses and prove a corresponding characterizing property:

Lemma 4 *A hyperclause $\mathcal{H} \equiv \Gamma_1 \mid \cdots \mid \Gamma_n$ is non-contradictory if and only if it is possible to form a consistent clause $A_1, \ldots, A_n$ by selecting exactly one formula from each component of $\mathcal{H}$, i.e., so that $A_i \in \Gamma_i$ for every i ($1 \leqslant i \leqslant n$).*

Proof Left-to-right: By contraposition. Consider the formula in disjunctive normal form $\bigwedge \Delta_1 \vee \cdots \vee \bigwedge \Delta_k$ obtained from $\mathsf{For}(\mathcal{H})$ by maximally distributing $\wedge$ over $\vee$. Since $\mathsf{For}(\mathcal{H})$ is in conjunctive normal form, the following hold for combinatorial reasons: (i) each Δ_i in $\bigwedge \Delta_1 \vee \cdots \vee \bigwedge \Delta_k$ is formed by selecting exactly one element from each component $\Gamma_1, \ldots, \Gamma_n$, and ($ii$) the clauses in the list $\Delta_1, \ldots, \Delta_k$ exhaust all the possible combinations of literals that can be formed according to (i). Now, if $\mathcal{H}$ did not possess the property claimed by the theorem, $\bigwedge \Delta_1 \vee \cdots \vee \bigwedge \Delta_k$ would be contradictory. This in turn would make $\mathsf{For}(\mathcal{H})$ contradictory, and therefore $\mathcal{H}$ as well.

Right-to-left: If a consistent clause $A_1, \ldots, A_n$ can be "extracted" from $\mathcal{H}$ in the way described by the theorem, then there exists a valuation v that verifies each literal A_i in the clause. This ensures that $v(\mathsf{For}(\mathcal{H})) = \mathsf{T}$, hence $\mathcal{H}$ is not contradictory. □

Example 3 Consider the hyperclause $\mathcal{G} \equiv p, r \mid q, r \mid \overline{q} \mid \overline{r}$ and the related formula

$$\mathsf{For}(\mathcal{G}) = (p \vee r) \wedge (q \vee r) \wedge \overline{q} \wedge \overline{r}.$$

By applying distributivity of conjunction over disjunction, $\mathsf{For}(\mathcal{G})$ can be turned into the following conjunctive normal form:

$$(p \wedge q \wedge \overline{q} \wedge \overline{r}) \vee (r \wedge q \wedge \overline{q} \wedge \overline{r}) \vee (p \wedge r \wedge \overline{q} \wedge \overline{r}) \vee (r \wedge r \wedge \overline{q} \wedge \overline{r})$$

This formula is clearly a contradiction and indeed none of the following four clauses is consistent, in agreement with Lemma 4.

$$\{p, q, \overline{q}, \overline{r}\} \qquad \{r, q, \overline{q}, \overline{r}\} \qquad \{p, r, \overline{q}, \overline{r}\} \qquad \{r, \overline{q}, \overline{r}\}$$

4.4 The Hypersequent Calculus HCC

We are finally ready to introduce our Hypersequent Contingency Calculus, HCC. The rules of HCC are listed in Fig. 6. Hypersequents are prefixed with the symbol '$\Rightarrow$' to emphasize that derivations deal with provably contingent hypersequents. An HCC-*proof* is any sequence of hypersequents constructed by recursively applying the rules of HCC. Given Lemmas 3 and 4, the specific conditions constraining the introduction of hyperclauses via the $\overline{ax}$-rule ensure that all and only contingent hyperclauses can serve as the starting point (or the end point, when working bottom-up) of an HCC-proof. We require, in particular, that any axiomatic hyperclause $\mathcal{H}$ displays at least

AXIOM

$$\frac{}{\Rightarrow \Gamma_1, A_1 \mid \cdots \mid \Gamma_n, A_n} \, \overline{ax}$$

where $\Gamma_1, A_1 \mid \cdots \mid \Gamma_n, A_n$ is a hyperclause,
Γ_1, A_1 is a consistent clause,
and $A_1, \ldots, A_n$ form a consistent clause

CONJUNCTION

$$\frac{\Rightarrow \mathcal{G} \mid \Gamma, A \mid \Gamma, B}{\Rightarrow \mathcal{G} \mid \Gamma, A \wedge B} \wedge$$

DISJUNCTION

$$\frac{\Rightarrow \mathcal{G} \mid \Gamma, A, B}{\Rightarrow \mathcal{G} \mid \Gamma, A \vee B} \vee$$

CUT

$$\frac{\Rightarrow \mathcal{G} \mid \Gamma, A \mid \Gamma, \overline{A}}{\Rightarrow \mathcal{G} \mid \Gamma} \, cut$$

Fig. 6 The hypersequent calculus HCC

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow p, \overline{p} \mid q, r, \overline{p} \mid r, \overline{q}, \overline{p}}\ \overline{ax}}{\Rightarrow p, \overline{p} \mid q \vee r, \overline{p} \mid r, \overline{q}, \overline{p}}\ \vee}{\Rightarrow p \wedge (q \vee \overline{r}), \overline{p} \mid r, \overline{q}, \overline{p}}\ \wedge}{\Rightarrow p \wedge (q \vee \overline{r}), \overline{p} \mid r \vee \overline{q}, \overline{p}}\ \vee}{\Rightarrow (p \wedge (q \vee \overline{r})) \wedge (r \vee \overline{q}), \overline{p}}\ \wedge}{\Rightarrow ((p \wedge (q \vee \overline{r})) \wedge (r \vee \overline{q})) \vee \overline{p}}\ \vee$$

Fig. 7 An example of an HCC-proof

one consistent component and, in addition, that a consistent clause can be formed by selectiong exacly one formula from each of $\mathcal{H}$'s components.

Example 4 In Fig. 7, we prove the contingency of the formula $((p \wedge (q \vee \overline{r})) \wedge (r \vee \overline{q})) \vee \overline{p}$ by constructing an HCC-derivation ending in $\Rightarrow ((p \wedge (q \vee \overline{r})) \wedge (r \vee \overline{q})) \vee \overline{p}$. The hyperclause $p, \overline{p} \mid q, r, \overline{p} \mid r, \overline{q} \mid \overline{p}$ is a legitimate instance of the $\overline{ax}$-rule, as required by the calculus. Notably, $p, \overline{p}$ is the only non-consistent clause within the hyperclause. Moreover, it is easy to select one element from each component so as to form a consistent clause, for instance, $\{p, q, r\}$.

It is worth noting that soundness-violating "pure" cut applications, like the one given at the beginning of Section 2.3, cannot find a viable counterpart in terms of hypersequents, since the premise would be contradictory, too:

$$\frac{p \wedge \overline{p}, p \mid p \wedge \overline{p}, \overline{p}}{p \wedge \overline{p}}$$

More generally, it is easy to verify that the pure (additive) *cut* rule of HCC, as listed in Fig. 6, is admissible:

Lemma 5 *If the hypersequent* $\mathcal{G} \mid \Gamma, A \mid \Gamma, \overline{A}$ *is contingent, so is* $\mathcal{G} \mid \Gamma$.

Proof Let $\mathcal{G} \mid \Gamma, A \mid \Gamma, \overline{A}$ be contingent. Suppose, for *reductio*, that $\mathcal{G} \mid \Gamma$ is tautological. Then $\mathcal{G}$ is tautological, and since Γ must also be tautological, so are the components Γ, A and $\Gamma, \overline{A}$ in the premise. By definition, the entire hypersequent $\mathcal{G} \mid \Gamma, A \mid \Gamma, \overline{A}$ would then be tautological, contrary to our assumption.

On the other hand, suppose $\mathcal{G} \mid \Gamma$ is contradictory. Then so is $\mathcal{G} \mid \Gamma, A \wedge \overline{A}$, which means that $\mathsf{For}(\mathcal{G}) \wedge (\bigvee \Gamma \vee (A \wedge \overline{A}))$ is a contradiction. But then $\mathsf{For}(\mathcal{G}) \wedge (\bigvee \Gamma \vee A) \wedge (\bigvee \Gamma \vee \overline{A})$ is also a contradiction, hence the hypersequent $\mathcal{G} \mid \Gamma, A \mid \Gamma, \overline{A}$ would be contradictory, again contrary to our assumption. □

Incidentally, we speak of a single cut rule but, in fact, the HCC version of *cut* is a rule schema. The reason is that, in the formulation of the rule, the formula A is not required to be atomic, hence '$\overline{A}$' need not stand for a negation *stricto sensu*. Instead, it can represent the definitional abbreviation of a formula in which the negation applies to proper subformulas. (See again the remarks at the beginning of Section 3.1.) As a result, HCC derivations may require definitional unpacking when using *cut* applications.

Lemma 6 (Semantic invertibility) *For every instance of the* HCC *rules* $\wedge$ *and* $\vee$*, the premise is a contingent hypersequent if and only if the conclusion is also contingent.*

Proof The case of the $\vee$-rule is trivial because, in sequent components, commas and disjunction can be freely interchanged without altering the semantic properties.

With regard to the $\wedge$-rule, it is easy to see that any valuation that verifies the premise $\mathcal{G} \mid \Gamma, A \mid \Gamma, B$ must also verify the conclusion $\mathcal{G} \mid \Gamma, A \wedge B$, and vice versa. We may distinguish three subcases. First, if there is a valuation v that falsifies all formulas in some component of $\mathcal{G}$, this trivially applies to both the premise and the conclusion. Next, we clearly have that a valuation v falsifies all the formulas in either Γ, A or Γ, B if and only if v falsifies all formulas in $\Gamma, A \wedge B$. Finally, suppose v falsifies all formulas in $\Gamma, A \wedge B$. Then we must have either $v(A) = \mathtt{F}$ or $v(B) = \mathtt{F}$, and in both cases we may conclude that $v\big(\mathsf{For}(\mathcal{G} \mid \Gamma, A \mid \Gamma, B)\big) = \mathtt{F}$ (owing to the conjunctive interpretation of the bar symbol). □

4.5 Soundness and Completeness

We now show that HCC has the required adequacy properties for proving classical contingencies. To this end, we need to extend the notion of HCC-proof to the more general notion of a decomposition chain.

Definition A *decomposition chain* is any sequence of hypersequents that can be constructed by recursively applying the HCC-rules $\wedge$, $\vee$, and *cut* to *any* starting hyperclause (not necessarily contingent) without using the prefix '$\Rightarrow$'.

To distinguish actual HCC-proofs from decomposition chains, we denote the former by simple lowercase Greek letters ($\pi, \delta, \ldots$) while the latter are marked with a 'hat' ($\hat{\pi}, \hat{\delta}, \ldots$).

Lemma 7 *For any hypersequent* $\mathcal{G}$*, there always exists a cut-free decomposition chain that terminates in* $\mathcal{G}$*.*

Proof To construct the desired decomposition chain, begin with $\mathcal{G}$ and apply the $\wedge$ and $\vee$ rules recursively in an upward manner (without using the prefix '$\Rightarrow$') until a hyperclause is finally reached. Termination is guaranteed because the process is bounded: a decomposition chain cannot exceed a height of $m + (2^m \cdot n)$, where m and n represent the number of occurrences of conjunctions and disjunctions in the end hypersequent, respectively. □

Theorem 8 (Soundness and completeness) *A hypersequent* $\mathcal{G}$ *is provable in* HCC *if and only if it is contingent, i.e., if and only if the formula* $\mathsf{For}(\mathcal{G})$ *is truth-functionally contingent.*

Proof Left-to-right (soundness): Let π an HCC-proof ending in $\Rightarrow \mathcal{G}$, and let $\Rightarrow \mathcal{H}$ be its top hypersequent. By Lemmas 3 and 4, $\mathcal{H}$ is a contingent hyperclause, and Lemma 6 ensures that HCC rules preserve contingency. It follows that $\mathcal{G}$ must also be contingent.

Right-to-left (completeness): By Lemma 7, there exists a decomposition chain $\hat{\pi}$ that terminates in $\mathcal{G}$. Since $\mathcal{G}$ is contingent, Lemma 6 guarantees that the top hypersequent $\mathcal{H}$ of $\hat{\pi}$ is also contingent. Furthermore, Lemmas 3 and 4 ensure that $\mathcal{H}$ is a correct instance of the $\overline{ax}$-rule. By prefixing the derivation symbol '$\Rightarrow$' to all hypersequents in $\hat{\pi}$, we convert it into an HCC-proof π ending in $\Rightarrow \mathcal{G}$. □

Remark (Weak paraconsistency) A formula A is contingent if and only if its negation $\overline{A}$ is also contingent. Thus, a consequence of completeness is that HCC yields a *paraconsistent* logic [26]. This is a distinctive feature that HCC shares with the antisequent calculi $\overline{\text{GS4}}$ and $\overline{\text{ᔭSפ}}$ presented in Section 3 and, more generally, with any refutation or rejection calculi for classical non-tautologies [28, 29]. On the other hand, the soundness of HCC guarantees that the system does not prove any explicit contradiction of the form $A \wedge \overline{A}$. Therefore, the paraconsistency of HCC is of the *non-adjunctive* type: it permits the provability of two formulas without thereby proving their conjunction [43]. In this regard, HCC differs from those other calculi (which do include explicit contradictions among their theorems) and may be classified as *weakly paraconsistent* [1].

4.6 Cut Elimination

An immediate consequence of Theorem 8 is cut eliminability, as no use of the *cut*-rule is required in the proof of soundness and completeness. This property parallels the eliminability property established by Tiomkin for his system CC (see again Section 2.3). However, HCC advances beyond CC in an important respect: it supports a thorough Gentzen-style normalization procedure for proofs involving cuts. In other words, HCC enjoys fully fledged cut elimination.

Lemma 9 *Any decomposition chain can be transformed into a cut-free chain that terminates in the same hypersequent.*

Proof With only minor adjustments, any standard Gentzen-style procedure tailored to Kleene's one-sided sequent system GS4 can be applied to reduce cut applications in any decomposition chain $\hat{\pi}$ (cf. [31, 37]). In particular, we can follow the algorithm presented in [27], where only parallel reductions need to be considered. The procedure illustrated here always focuses on topmost cut applications and relies only on the following observations: (i) The set of hyperclauses is closed under the *cut*-rule. Hence, once an application of the rule reaches the very top of the chain (via an instance of the axiom-rule), the *cut* can be immediately removed using the reduction indicated in Fig. 8.

(ii) In GS4, parallel reductions $\wedge/\vee$ typically require Weakening (from $\vdash \Delta$ to $\vdash \Delta, A$), a rule that has no admissible counterpart in HCC. Thus, here we focus

$$\dfrac{\dfrac{}{\mathcal{G} \mid \Gamma, A \mid \Gamma, \overline{A}}\,\overline{ax}}{\mathcal{G} \mid \Gamma}\,cut \quad \rightsquigarrow \quad \dfrac{}{\mathcal{G} \mid \Gamma}\,\overline{ax}$$

Fig. 8 A *cut*-reduction in a decomposition chain

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\hat{\pi}\\ \vdots\\ \mathcal{G}\mid\Gamma,A\mid\Gamma,B\mid\Gamma,\overline{A},\overline{B}\end{matrix}}{\mathcal{G}\mid\Gamma,A\wedge B\mid\Gamma,\overline{A},\overline{B}}\,\wedge}{\mathcal{G}\mid\Gamma,A\wedge B\mid\Gamma,\overline{A}\vee\overline{B}}\,\vee}{\mathcal{G}\mid\Gamma,A\wedge B\mid\Gamma,\overline{A\wedge B}}\,\mathit{def.}}{\mathcal{G}\mid\Gamma}\,\mathit{cut}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\begin{matrix}\hat{\pi}'\\ \vdots\\ \mathcal{G}\mid\Gamma,A,\overline{B}\mid\Gamma,B\mid\Gamma,\overline{A},\overline{B}\end{matrix}}{\mathcal{G}\mid\Gamma,\overline{B}\mid\Gamma,B}\,\mathit{cut}}{\mathcal{G}\mid\Gamma}\,\mathit{cut}
$$

Fig. 9 Another *cut*-reduction (via Lemma 7)

on decomposition chains. For example, consider the reduction in Fig. 9. Assuming $\hat{\pi}$ is cut-free, the reduction strategy can be implemented smoothly thanks to Lemma 7, which ensures the existence of a cut-free decomposition chain $\hat{\pi}'$ ending in $\mathcal{G}\mid\Gamma, A, \overline{B}\mid\Gamma, B\mid\Gamma, \overline{A}, \overline{B}$.

(iii) Commutation steps can be reshaped as parallel reductions by applying a procedure similar to the one proposed in [27]. Such a procedure essentially stresses the fact that any GS4-proof of $\vdash\Gamma, A\bullet B$, with $\bullet\in\{\wedge,\vee\}$, can be turned into a proof whose last rule is exactly the $\bullet$-rule forming the compound $A\bullet B$ (cf. [27, *Lemma 3*]). □

Theorem 10 (Cut elimination) *Any* HCC*-proof* π *can be turned into a cut-free proof* π' *that terminates in the same hypersequent.*

Proof Consider an HCC-proof π of $\Rightarrow\mathcal{G}$. As a first step, we remove all prefixes '$\Rightarrow$' from π to obtain the corresponding decomposition chain $\hat{\pi}$. Using the procedure from Lemma 9, we then transform $\hat{\pi}$ into a cut-free decomposition chain $\hat{\pi}'$, still ending in $\mathcal{G}$. Since $\mathcal{G}$ is contingent (by hypothesis) and the top hypersequent of $\hat{\pi}'$ must also be contingent (by Lemma 6), reintroducing all prefixes '$\Rightarrow$' in $\hat{\pi}'$ results in a cut-free HCC-proof for $\Rightarrow\mathcal{G}$. □

As a corollary, we immediately obtain that HCC obeys the subformula property, as desired:

Corollary 11 *Any* HCC*-provable hypersequent* $\mathcal{G}$ *admits an analytic proof, i.e., a proof that uses only subformulas of the formulas already present in* $\mathcal{G}$.

The combination of these two last results reestablishes, in a complete syntactic fashion, the fact that the HCC calculus can be thought of as a simple and completely deterministic algorithm for checking the contingency of any given formula A. Such a procedure can be easily extended so as to cover propositional formulas involving the material conditional and in which negation is treated as a primitive operator. As a matter of fact, a two-sided version of HCC can be obtained easily by resorting to the full spectrum of logical rules that define Kleene's G4. Hypersequents will then come in the form of unordered lists of standard Gentzen-style sequents. In particular, hyperclauses will be set of standard two-sided clauses testable against the side conditions that define contingent axioms simply by turning any clause $A_1,\ldots,A_n\vdash B_1,\ldots,B_m$ into its the one-sided version $\vdash\overline{A}_1,\ldots,\overline{A}_n, B_1,\ldots,B_m$.

Acknowledgements We are grateful to our referees and to the editors of the *Journal of Philosophical Logic* for their insightful and constructive comments on an earlier draft of the paper. G.P. acknowledges support from the MUR (Italian Ministry of University and Research) through the PRIN project *Proof and Understanding in Mathematics* (PUMa), protocol number 2022TYNY32_005.

Author Contributions Both authors made equivalent contributions to the paper

Funding Open access funding provided by Università degli Studi di Roma Tor Vergata within the CRUI-CARE Agreement.

Availability of data and materials Not applicable

Declarations

Ethical approval Not applicable

Competing interests Not applicable

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