ESTIMATES FOR THE AVERAGE SCALAR CURVATURE OF THE WEIL-PETERSSON METRIC ON THE MODULI SPACE $\overline{\mathcal{M}}_g$

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Abstract. We give a precise estimate for the average scalar curvature of the Weil-Petersson metric on the moduli space $\overline{\mathcal{M}}_g$ as $g \to \infty$ up to the order $1/g^2$.

1. Introduction and Statement of the Result

The curvature of the Weil-Petersson metric recently attracted further interest. In this note we will give the precise estimate for the average of the scalar curvature S_{WP} of the Weil-Petersson metric on the moduli space $\overline{\mathcal{M}}_g$ as g tends to infinity. The result is the value

$$\frac{1}{(g-1)} \frac{\int_{\overline{\mathcal{M}}_g} (-S_{WP})\, dV_{WP}}{\int_{\overline{\mathcal{M}}_g} dV_{WP}} = \frac{13}{4\pi} + \frac{\pi}{12}\frac{1}{g} + \left(\frac{1}{4\pi} + \frac{\pi}{12}\right)\frac{1}{g^2} + O\left(\frac{1}{g^3}\right).$$

The proof of the asymptotics will be based upon methods of Algebraic Geometry. Wolpert showed in [W1, W2] that Mumford's canonical class κ_1 (the tautological class obtained from the universal curve) from [Mu1, Mu2] (cf. [A-C1, A-C2]) is the cohomology of the Weil-Petersson form extended to the compactification up to the factor $2\pi^2$ together with the fact that its restriction to the boundary equals to the Weil-Petersson cohomomology of the boundary (interpreted as related to moduli of punctured surfaces of lower genus).

The finiteness of the Weil-Petersson volume itself is a consequence of Masur's estimates [Ma], whereas the curvature was computed by Wolpert [W3] and Fischer-Tromba [Tro]. These results implied strong negativity properties, in particular the strict negativity of the scalar curvature. It is known that the scalar curvature tends to $-\infty$ towards the boundary. Precise estimates of the curvature of the Weil-Petersson metric towards the boundary are contained in [S] and [T] with a later developments by Liu-Sun-Yau [L-S-Y1, L-S-Y2].

Estimates of the Weil-Petersson volume had been given by Mirzakhani [Mi], Mirzakhani-Zograf [M-Z], Penner [Pe], Grushevsky [Gr], Zograf [Zo1, Zo2] and previously in [S-T]. The algebraic aspect is contained in the push-pull formulas by Arbarello and Cornalba [A-C1, A-C2].

The Weil-Petersson volume of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of Riemann surfaces of genus g with n punctures is denoted by

$$V_{g,n} = \int_{\mathcal{M}_{g,n}} \kappa_1^{3g-3+n}.$$

Finally the relationship of intersection numbers and volumes as related to two dimensional gravity ought to pointed out. Pertinent references are [Dij, F-P, Ge, Ko, M-Z, Wi].

We showed the following estimates.

Theorem 1 ([S-T]). (i) *Let $g > 1$. Then*

$$V_{g,0} \geq \frac{1}{28}V_{g-1,2} + \frac{1}{672}V_{g-1,1} + \frac{1}{14}\sum_{j=2}^{[g/2]} V_{j,1}V_{g-j,1} - \frac{1}{28}(V_{\frac{g}{2},1})^2, \tag{1}$$

with $V_{\frac{g}{2},1} = 0$, if g is odd.

(ii) *There exist constants $0 < c < C$, independent of n such that*

$$c^g(2g)! \leq \frac{V_{g,n}}{(3g-3+n)!} \leq C^g(2g)! \tag{2}$$

for all fixed $n \geq 0$ and large g.

Concerning (2), a lower estimate for $n = 1$ is due to Penner [Pe, Theorem 6.2.2], and the upper estimates for $n \geq 1$ were first shown by Grushevsky in [Gr, Sec. 7].

Bounds for the curvature of the Weil-Petersson metric were proven by Wu and Wolf in [W-W] and Wu in [Wu1, Wu2]. A recent result is the following:

Theorem 2 (Bridgeman-Wu, [B-W]). *Denote by S_{WP} the scalar curvature of the Weil-Petersson form ω_{WP}, and by dV_{WP} its volume element. There exist constants $0 < c < C$ such that*

$$c \cdot g \leq \frac{\int_{\overline{\mathcal{M}}_g} (-S_{WP})\, dV_{WP}}{\int_{\overline{\mathcal{M}}_g} dV_{WP}} \leq C \cdot g \tag{3}$$

for all g.

We will show that the Bridgeman-Wu estimate follow in an algebraic way using (1).

The actual asymptotics of the Weil-Petersson volume (containing (2)) was computed by Mirzakhani and Zograf. (We use the above normalization for the Weil-Petersson volume.)

Theorem 3 ([M-Z, Theorem 1.2]). *There exists a constant $C \in (0, \infty)$ such that for any given $k \geq 1$, $n \geq 0$*

$$V_{g,n} = C_{MZ} \frac{(3g-3+n)!\,(2g-3+n)!\,2^{g-3+n}}{\pi^{2g}\sqrt{g}} \left(1 + \sum_{j=1}^{k} \frac{c_n^{(j)}}{g^j} + O\left(\frac{1}{g^{k+1}}\right)\right) \tag{4}$$

as $g \to \infty$.

The theorem contains further characterizations of the polynomials $c_n^{(j)}$ – we will need the case $k \leq 1$, and $n \leq 2$. The Mirzakhani-Zograf constant C_{MZ} is conjectured to be $C_{MZ} = 1/\sqrt{\pi}$.

Let again δ be the $\mathbb{Q}$-divisor related to the boundary components of the moduli space (see also below), and denote by κ_1 the Mumford class. Recall that the class $[\omega_{WP}]$ of the Weil-Petersson form was computed in [W1, W2] as $2\pi^2\kappa_1$. Set $\eta = [-Ric(\omega_{WP})/2\pi]$.

Main Theorem. *The average total scalar curvature of the Weil-Petersson metric on the moduli space $\overline{\mathcal{M}}_g$ satisfies the estimate*

$$\frac{13}{4\pi} \leq \frac{3}{\pi}\frac{\eta \cdot \kappa_1^{3g-4}}{\kappa_1^{3g-3}} = \frac{1}{(g-1)} \frac{\int_{\overline{\mathcal{M}}_g} (-S_{WP})\, dV_{WP}}{\int_{\overline{\mathcal{M}}_g} dV_{WP}} = \frac{13}{4\pi} + \frac{1}{4\pi}E_g, \quad \textit{where} \tag{5}$$

$$\frac{1}{4\pi}E_g = \frac{\pi}{12}\frac{1}{g} + \left(\frac{1}{4\pi} + \frac{\pi}{12}\right)\frac{1}{g^2} + O\left(\frac{1}{g^3}\right) \ \textit{for } g \to \infty.$$

The precise value is

$$E_g = \frac{\kappa_1^{3g-4} \cdot \delta}{\kappa_1^{3g-3}} \ \textit{where } \delta = \sum_{j=0}^{[g/2]} \delta_j. \tag{6}$$

Note that the ampleness of κ_1 implies that E_g is positive.

2. Proof

Let D denote the compactifying divisor of $\mathcal{M}_g$ with components Δ_j for $j = 0, \ldots, [g/2]$. These give rise to $\mathbb{Q}$-divisors δ_j such that in terms of the generally used notation

$$[\Delta_1] = 2\delta_1 \text{ and } [\Delta_j] = \delta_j \text{ for } j \neq 1.$$

Then $\delta := \sum_{j=0}^{[g/2]} \delta_j$ so that

$$D = \delta + \delta_1.$$

Note that there exist branched two-sheeted coverings $\overline{\mathcal{M}}_{g-1,2} \to \Delta_0$, $\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,1} \to \Delta_1$, and $\overline{\mathcal{M}}_{g/2,1} \times \overline{\mathcal{M}}_{g/2,1} \to \Delta_{g/2}$ for even g yielding extra factors 1/2 in Lemma 1.

.

Two classical facts are needed. Mumford's direct image bundle λ satisfies [Mu1]

$$\kappa_1 = 12\lambda - \delta,$$

and by [H-M] the canonical bundle $K_{\overline{\mathcal{M}}_{g,0}}$ is equal to

$$K_{\overline{\mathcal{M}}_{g,0}} = 13\lambda - 2\delta - \delta_1.$$

In [T, Corollary 5.5] it was shown by means of a a singular Mumford good hermitian metric that η is the Chern class of the dual of the logarithmic tangent bundle $T_{\overline{\mathcal{M}}_{g,0}}(\log D)$. Therefore

$$[\eta] = K_{\overline{\mathcal{M}}_{g,0}} + [D]. \tag{7}$$

Now (5) can be shown:

$$\frac{\eta \cdot \kappa_1^{3g-4}}{\kappa_1^{3g-3}} = \frac{(K_{\overline{\mathcal{M}}_{g,0}} + \delta + \delta_1) \cdot \kappa_1^{3g-4}}{\kappa_1^{3g-3}} = \frac{(13\lambda - \delta) \cdot \kappa_1^{3g-4}}{\kappa_1^{3g-3}} = \frac{(\frac{13}{12}\kappa_1 + \frac{1}{12}\delta) \cdot \kappa_1^{3g-4}}{\kappa_1^{3g-3}}$$

The main result consists of the estimate for E_g. The special value for $V_{1,1}$ etc. are taken from [S-T].

Lemma 1.

$$\frac{\kappa_1^{3g-4} \cdot \delta_0}{\kappa_1^{3g-3}} = \frac{1}{2}\frac{V_{g-1,2}}{V_{g,0}} \tag{8}$$

$$\frac{\kappa_1^{3g-4} \cdot \delta_1}{\kappa_1^{3g-3}} = \frac{1}{2} \cdot \frac{V_{1,1} \cdot V_{g-1,1}}{V_{g,0}} = \frac{1}{48} \cdot \frac{V_{g-1,1}}{V_{g,0}} \tag{9}$$

$$\frac{\kappa_1^{3g-4} \cdot \delta_j}{\kappa_1^{3g-3}} = \frac{V_{j,1} \cdot V_{g-j,1}}{V_{g,0}} \quad \textit{for} \quad 2 \leq j \ \leq [(g-1)/2] \tag{10}$$

$$\frac{\kappa_1^{3g-4} \cdot \delta_{g/2}}{\kappa_1^{3g-3}} = \frac{1}{2}\frac{(V_{g/2,1})^2}{V_{g,0}} \quad \textit{for } g \textit{ even.} \tag{11}$$

Remark 1. *Lemma 1, and* (5) *together with* (1) *imply the Bridgeman-Wu Theorem 2.*

We will apply the Mirzakhani-Zograf estimate (4), and the following values contained in [M-Z, Remark 1.2]:

$$c_0^{(1)} = \frac{7}{12} - \frac{17}{6\pi^2} \tag{12}$$

$$c_1^{(1)} = \frac{1}{3} - \frac{5}{6\pi^2} \tag{13}$$

$$c_2^{(1)} = \frac{1}{12} + \frac{1}{6\pi^2}. \tag{14}$$

Lemma 2.

$$\frac{1}{2}\frac{V_{g-1,2}}{V_{g,0}} = \frac{\pi^2}{3g}\left(1 + \left(\frac{3}{\pi^2} + 1\right)\frac{1}{g} + O\left(\frac{1}{g^2}\right)\right) \tag{15}$$

$$\frac{1}{48}\frac{V_{g-1,1}}{V_{g,0}} = \frac{1}{144}\sqrt{\frac{g}{g-1}}\frac{\pi^2}{(g-1)(3g-4)(2g-3)}\left(1 + O\left(\frac{1}{g}\right)\right) = O\left(\frac{1}{g^3}\right) \tag{16}$$

$$\frac{V_{j,1} \cdot V_{g-j,1}}{V_{g,0}} = \frac{C_{MZ}}{2}\sqrt{\frac{g}{j(g-j)}}\frac{1}{(3g-3)\binom{3g-4}{3j-2}(2g-3)\binom{2g-4}{2j-2}}\left(1 + O\left(\frac{1}{g}\right)\right) \tag{17}$$

for $2 \leq j \leq [(g-1)/2]$

$$\frac{1}{2}\frac{(V_{g/2,1})^2}{V_{g,0}} = \frac{C_{MZ}}{2}\frac{1}{\sqrt{g}}\frac{1}{(3g-3)\binom{3g-4}{(3g-4)/2}(2g-3)\binom{2g-4}{g-2}}\left(1 + O\left(\frac{1}{g}\right)\right) \tag{18}$$

for g *even.*

The computation of (15) is the following.

$$\frac{1}{2}\frac{V_{g-1,2}}{V_{g,0}} = \frac{1}{2}\frac{(3g-4)!\,(2g-3)!\,2^{g-2}\pi^{2g}\sqrt{g}}{(3g-3)!\,(2g-3)!\,2^{g-3}\,\pi^{2g-2}\sqrt{g-1}}\cdot \widetilde{C}_g =$$
$$= \frac{\pi^2}{3g}\left(\frac{g}{g-1}\right)^{3/2}\cdot \widetilde{C}_g = \frac{\pi^2}{3g}\left(1+\frac{3}{2g}+O\left(\frac{1}{g^2}\right)\right)\cdot \widetilde{C}_g$$

where

$$\widetilde{C}_g = \left(1+c_2^{(1)}\frac{1}{g-1}+O\left(\frac{1}{g^2}\right)\right)\Big/\left(1+c_0^{(1)}\frac{1}{g}+O\left(\frac{1}{g^2}\right)\right)$$

The coefficients $c_n^{(1)}$ do not contribute to the remaining terms up to order three in $1/g$.

We need an elementary estimate:

Let $l \leq k \leq n$ be positive integers, then $(n-k)(k-l) \geq 0$, i.e. $\frac{n-k+l}{l} \geq \frac{n}{k}$. So we derive the well known inequality

$$\binom{n}{k} = \prod_{l=1}^{k}\frac{n-k+l}{l} \geq \left(\frac{n}{k}\right)^k. \tag{19}$$

The function

$$f(x) := \log\left(\frac{h}{x}\right)^x = x(\log h - \log x)$$

attains its maximum value at $x = h/e$ so that its minimum will be taken at either end of a given interval.

The binomial coefficients in (17) attain asymptotic lower estimates for $2 \leq j \leq [g/2]$ at $j = 2$ namely

$$\binom{3g-4}{3j-2} \geq \left(\frac{3g-4}{4}\right)^4 \quad \text{and} \quad \binom{2g-4}{2j-2} \geq (g-2)^2.$$

Hence, summing up all terms of the type (17) yields a term of growth order at most $O(1/g^7)$. The last term (18) has a at most a negative exponential growth with respect to g and can also be disregarded.

This proves the main theorem. □

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