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On the Moduli Space of (CMC) 1-immersions of a Closed Surface Into Hyperbolic 3-Manifolds

Gabriella Tarantello[1] · **Stefano Trapani**[1]

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Abstract

Constant Mean Curvature (CMC) immersions of a closed surface S into hyperbolic 3-manifolds emerged by the work of Uhlenbeck in connection with irreducible representations of the fundamental group into the Mobious group. Moreover, Bryant revealed a bi-holomorphic (cousin) relation between (CMC) 1-immersions of surfaces into the hyperbolic 3-space (Bryant surfaces) and minimal immersions into the Euclidian 3-space. In this note, we survey recent results concerning the existence and uniqueness of (CMC) 1-immersions of a closed surface into hyperbolic 3-manifolds, labelled by Dolbeault co-homology classes. While, (CMC) c-immersions of a surface S (closed, orientable, with genus $g \geq 2$) into hyperbolic 3- manifolds are always available when $|c| < 1$ (and described in terms of the tangent bundle of the Teichmueller space of S) we find that (CMC) 1-immersions can be attained only as limits of (CMC) c-immersions for $|c| \to 1$. However, the passage to the limit can be prevented by possible blow-up phenomena, so that (after scaling) we end up with a (CMC) 1-immersion with (finitely many) conical singularities, consistently with the presence of smooth ends in Bryant surfaces. We see how to encompass the blow-up situation in terms of suitable orthogonality conditions, involving the image Z of the Kodaira map for genus $g = 2$, and the $(g-1)$-secant variety of Z, for genus $g = 3$. Consequently, we can ensure the passage to the limit under an appropriate generic condition (sharp for genus $g = 2$), yielding to (CMC) 1-immersions into suitable (germs) of hyperbolic 3-manifolds.

Keywords Blow-up Analysis · Minimiser of a Donaldson functional · CMC 1-immersions · Grassmannian · Hyperelliptic curves

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✉ Stefano Trapani
trapani@mat.uniroma2.it

Gabriella Tarantello
tarantel@mat.uniroma2.it

1 Università di Roma TorVergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy

1 Introduction

We consider an oriented closed surface S with genus $\mathfrak{g} \geq 2$ and denote by $\mathcal{T}_{\mathfrak{g}}(S)$ the Teichmüller space of conformal structures on S, modulo biholomorphisms in the homotopy class of the identity.

In this note we discuss existence and uniqueness results about constant mean curvature (CMC) 1-immersions of S into hyperbolic 3-manifolds in terms of elements of the tangent bundle of the Teichmüller space $\mathcal{T}_{\mathfrak{g}}(S)$. In this way we manage to describe the moduli space of all such immersions (up to a "natural" identification).

With respect to (CMC) c-immersions, namely immersions with (prescribed) value c of the mean curvature, the value $c = 1$ plays a special role. Indeed, Bryant [7] observed that (CMC) 1-immersions of surfaces into the hyperbolic space $\mathbb{H}^3$ share striking analogies with minimal immersions into the Euclidean space $\mathbb{E}^3$, by the validity of the so called "cousin" relation, see [42] and also [43, 53]. As a matter of fact, the value $c = 1$ enters also as a "critical" parameter in our analysis of (CMC) c-immersions, as discussed below (see also [46]).

We identify an element $X \in \mathcal{T}_{\mathfrak{g}}(S)$ with the corresponding Riemann surface obtained once we equip S with a conformal structure in the specified class.

Recall that an immersion of X into a Riemannian 3-manifold (with constant sectional curvature) is characterized by the (symmetric) quadratic forms: (I_g) (First Fundamental Form) and (II_g) (Second Fundamental Form) expressed in terms of the pull-back metric g on X and governed by the (six) Gauss-Codazzi equations.

To be more precise, we denote by g_X the unique hyperbolic metric on X (as given by the Uniformization Theorem). In this way the pull-back metic g must be conformally equivalent to g_X, that is:

$$g = e^u g_X. \tag{1.1}$$

for a suitable smooth function u on X (conformal factor).

Since we are dealing with a target manifold which is hyperbolic (i.e. admits sectional curvature -1) and the immersion admits constant mean curvature, we find that the Second Fundamental Form $(II)_g$ is completely identified by its $(2,0)-$part and of course by the metric g. Thus, if we denote by:

$$\alpha = (2,0)\text{-part of } (II_g) \text{ (Quadratic Differential)},$$

then the Gauss-Codazzi equations governing (CMC) c-immersions of X into a hyperbolic 3-manifold, reduce to an elliptic system of PDE's in terms of the pair (u, α), as derived in details in [52] and [20].

More precisely, by considering α as a smooth section of X valued in $K_X \otimes K_X$ (K_X the canonical bundle of X) in [52] and [20] one finds that the Gauss-Codazzi equations are expressed as follows:

$$-\Delta u = 2 - 2(1-c^2)e^u - 8\|\alpha\|^2 e^{-u} \tag{1.2}$$

$$\bar{\partial}\alpha = 0 \tag{1.3}$$

where,

$$\begin{aligned} \Delta =&\ \text{Laplace Beltrami operator with respect to the metric } g_X \\ \bar{\partial} =&\ \text{d-bar operator of the complex structure on } K_X \otimes K_X \text{, induced by } X \\ ||.|| =&\ \text{norm relative to the Hermitian product on } K_X \otimes K_X \text{, induced by } g_X. \end{aligned} \tag{1.4}$$

By (1.3), we see that α must correspond to a holomorphic quadratic differential (Hopf differential). Thus, we have: $\alpha \in C_2(X)$, where:

$$C_2(X) = \text{ space of holomorphic quadratic differentials on } X. \tag{1.5}$$

We know that the complex linear space $C_2(X)$ is finite dimensional and more precisely: $dim_{\mathbb{C}}(C_2(X)) = 3(\mathfrak{g} - 1)$.

Interestingly, as indicated by Taubes in [51], every solution (u, α) of the Gauss-Codazzi equations (1.2)-(1.3) gives rise to a "germ" of hyperbolic 3-manifolds of X, in the sense that it identifies a hyperbolic 3-manifold $(N, \bar{g})$ ($N \simeq X \times \mathbb{R}$ not necessarily complete) where X is immersed as a surface of constant mean curvature c, with First and Second Fundamental Form corresponding to the given solution pair. Furthermore, $(N, \bar{g})$ is unique up to local isometries of tubular neighbourhoods of X, see [51].

Therefore, to parametrize the moduli space of (CMC) c-immersions of S into hyperbolic 3-manifolds, modulo isometries between tubular neighbourhoods of X, it suffices to parametrize the solution-set of the Gauss-Codazzi equations (1.2)-(1.3). From (1.2)-(1.3), it may be tempting to describe such solution-set in terms of the pair : $(X, \alpha) \in \mathcal{T}_{\mathfrak{g}}(S) \times C_2(X)$, which provides a local trivialization for the cotangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$. In this way, we would attain a parametrization of the moduli space of (CMC) c-immersions by elements of the cotangent bundle $T^*(\mathcal{T}_{\mathfrak{g}}(S))$.

However, as discussed in [19] and [20], for a given $\alpha \in C_2(X)$ a solution of (1.2) may not exist, or when it exists, it may not be unique (see also [18]). So, in general, the pair (X, α) is not suitable to parameterized (CMC) c-immersions.

Instead, it has proved more successful the "dual" approach by Goncalves and Uhlenbeck in [15], where the authors propose to parametrize (CMC) c-immersions of S into hyperbolic 3-manifolds, in terms of elements of the tangent bundle of the Teichmüller space $\mathcal{T}_{\mathfrak{g}}(S)$.

To this purpose, we let $E = T_X^{1,0}$ be the holomorphic tangent bundle of X so that $E^* = K_X$ identifies the canonical bundle of X.

We recall the isomorphism: $C_2(X) \simeq (\mathcal{H}^{0,1}(X, E))^*$ with $\mathcal{H}^{0,1}(X, E)$ the Dolbeault (0,1)-cohomology group of X in (2.2) (cf. [16]).

We use the following standard notations:

$A^0(E) = \{$smooth sections valued in holomorphic tangent bundle $E = T^{1,0}(X)\}$,

$A^{0,1}(X, E)=\{(0, 1)$-form valued in $E\}$.

Hence each cohomology class $[\beta] \in \mathcal{H}^{0,1}(X, E))^*$ is identified by each Beltrami Differential $\beta \in A^{0,1}(X, E)$ as follows:

$$[\beta] = \{\beta + \bar{\partial}\eta,\ \eta \in A^0(E)\} \in \mathcal{H}^{0,1}(X, E).$$

Therefore, we have a parametrization of the tangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$ by the pairs: $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E)$.

The following holds:

Theorem 1 ([15, 21]) *For given $c \in (-1, 1)$ there is a one-to-one correspondence between the space of constant mean curvature c-immersions of S into a (germ of) hyperbolic 3-manifolds and the tangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$, parametrized by the pairs: $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E)$, $E = T_X^{1,0}$.*

More precisely, for fixed $c \in (-1, 1)$, the datum $(X, [\beta])$ identifies the unique solution $(u = u_c\, , \alpha = \alpha_c)$ of the Gauss-Codazzi equations (1.3)-(1.2) satisfying:

$$*_E^{-1}\,(e^{-u_c}\alpha_c) \in [\beta] \tag{1.6}$$

where $*_E$ is the Hodge star operator relative to the metric g_X with inverse $(*_E)^{-1}$, see (2.5) below.

To see that (1.6) is a natural constraint, let us recall that any Beltrami differential β admits the following Dolbeault decomposition:

$$\beta = \beta_0 + \bar{\partial}\eta$$

with unique β_0 harmonic (with respect to g_X) and η a smooth section of X valued on E. In particular, the $(0, 1)$-cohomology class $[\beta] \in \mathcal{H}^{0,1}(X, E)$ is identified by the unique harmonic differential $\beta_0 \in [\beta]$.

Therefore, for a solution pair: $(u\, , \alpha)$ of the Gauss Codazzi equations (1.2)-(1.3), we can easily formulate the constraint (1.6) by setting:

$$g = e^u g_X \quad \text{and } \alpha = e^u *_E (\beta_0 + \bar{\partial}\eta) \tag{1.7}$$

with $(u\, , \eta)$ satisfying:

$$\begin{cases} \Delta u + 2 - 2te^u - 8e^u\|\beta_0 + \bar{\partial}\eta\|^2 = 0 \quad \text{in} \quad X \\ \bar{\partial}(e^u *_E (\beta_0 + \bar{\partial}\eta)) = 0 \end{cases} \tag{1.8}$$

and $t = 1 - c^2$.

Interestingly, solutions of the "constraint" Gauss-Codazzi equations (1.8) correspond to critical points of the so called Donaldson functional F_t $(t = 1-c^2)$ introduced in [15] and defined in (3.2) below.

Indeed, Theorem 1 is established in [21] by showing precisely that, for $t > 0$ the functional F_t admits a unique critical point given by its global minimum.

Moreover, as discussed in [21] and [46], from Theorem 1 we can deduce useful algebraic consequences about the parametrization of all possible irreducible representations:

$$\rho : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$$

with $PSL(2, \mathbb{C})$ the (orientation preserving) isometry group of $\mathbb{H}^3$, see also [33, 34, 52] and [51] for more applications in this direction and related issues.

At this point it is natural to ask whether, for a given pair $(X, [\beta])$ analogous (CMC) c-immersions do exist also when $|c| \geq 1$.

We have an evident non-existence result when $[\beta] = 0$ (see Section 4 for details), while for $[\beta] \neq 0$ and $t \leq 0$, the functional F_t may be unbounded from below and it is not obvious how to detect possible critical points.

When $t = 0$, that is $|c| = 1$, by [7] we know that, (CMC) 1-immersions of surfaces into the hyperbolic space $\mathbb{H}^3$ admit smooth "ends" (see [43, 53]) which (by a conformal transformation) are captured into the setting of closed surfaces by the presence of "punctures" at finitely many points.

Those points occur naturally in [46] as "blow-up" points with a "quantized" blow-up mass (see [45]).

In fact, it was shown in [46] that (CMC) 1-immersions of a closed surface can be attained only as "limits" of the (CMC) c-immersions of Theorem 1, when $|c| \to 1^-$. However, the passage to the limit can be jeopardized by "blow-up" phenomena (as $|c| \to 1^-$), so that (after scaling) at the limit, we could end up with immersed surfaces with "conical" singularities at the blow up points.

Therefore, to obtain (CMC) 1-immersions, we must identify those cohomology classes $[\beta]$ for which "blow-up" can be rule out and "compactness" holds.

The first result in this direction was attained in [46] for genus $\mathfrak{g} = 2$, where only one blow-up point can occur. In this case, "compactness" can be formulated in terms of the Kodaira map:

$$\tau : X \longrightarrow \mathbb{P}(V^*), \ \ V = C_2(X) \tag{1.9}$$

a holomorphic map defined in section 12.1.3 of [14].

Since $\mathbb{P}(\mathcal{H}^{0,1}(X, E)) \simeq \mathbb{P}^{3\mathfrak{g}-4}$ then $\dim_{\mathbb{C}} \mathbb{P}(\mathcal{H}^{0,1}(X, E)) \geq 2$ for $\mathfrak{g} \geq 2$ and we derive that: $\tau(X) \subsetneq \mathbb{P}(\mathcal{H}^{0,1}(X, E))$.

Actually, $\mathbb{P}(\mathcal{H}^{0,1}(X, E)) \setminus \tau(X)$ defines a Zariski open subset (hence dense) in $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$.

The role of the projective space $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$ is readily explained once we use for $[\beta] \neq 0$ a simple scaling argument to see that:

the pair $(X, [\beta])$ yields to a (CMC) 1-immersion from a solution of (1.8) and (1.7) $\Longleftrightarrow$ the pair $(X, \lambda[\beta])$ yields to (CMC) 1-immersion from an obvious "scaling" of the given solution satisfying the corresponding (1.8) and (1.7), for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Thus, for $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$ we define,

$$[\beta]_{\mathbb{P}} = \{[\lambda\beta], \quad \lambda \in \mathbb{C} \setminus \{0\}\} \in \mathbb{P}(\mathcal{H}^{0,1}(X, E)) \tag{1.10}$$

the projective representative of the class $[\beta]$ in $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$. The following holds:

Theorem 2 ([46]) *If* $\mathfrak{g} = 2$, *then to every* $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(X) \times (\mathcal{H}^{0,1}(X, E) \setminus \{0\})$ $E = T_X^{1,0}$, *with projective representative* $[\beta]_{\mathbb{P}} \notin \tau(X)$, *there correspond a unique (CMC) 1-immersion of X into a (germ of) hyperbolic* 3*-manifold* $N(\simeq S \times \mathbb{R})$, *with pull back metric g and* $(2, 0)$*-part* α *of the second fundamental form* (II_g) *satisfying* (1.6).

Next, we recall that every Riemann surface of genus $\mathfrak{g} = 2$ is hyperelliptic. Hence it admits a unique bi-holomorphic hyperelliptic involution (see [16, 37])

$$j : X \to X \tag{1.11}$$

with exactly $2(\mathfrak{g} + 1)$ distinct fixed points. Moreover, for $\mathfrak{g} = 2$, those $2(\mathfrak{g} + 1) = 6$ points coincide with the Weierstrass points of X (cf. [37]).

On the basis of the invariance under the bi-holomorphism (1.11) of the (constrained) Gauss-Codazzi equations (1.8) and correspondingly the invariance of the Donaldson functional F_t in (3.2) (see Appendix 2 of [49]) and by employing the equivalent formulation of (1.8) in terms of Hitchin selfduality equation for a suitable nilpotent $SL(2, \mathbb{C})$ Higgs bundle, see [1] and [21] for details, the following sharp version of Theorem 2 was establish in [49] and in a forthcoming paper.

Theorem 3 ([49]) *Let* $\mathfrak{g} = 2$ *and* $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(X) \times (\mathcal{H}^{0,1}(X, E) \setminus \{0\})$. *There exist a unique (CMC) 1-immersion of X into a (germ) of hyperbolic* 3*-manifold* $N (\simeq X \times \mathbb{R})$ *with pull-back metric g and* $(2, 0)$*-part* α *of* (II_g) *satisfying* (1.6) $\iff$ $[\beta]_{\mathbb{P}} \notin \{\tau(q),\ \textit{with}\ q \in X : j(q) = q\}$.

The case of $\mathfrak{g} \geq 3$ is more delicate, since multiple (up to $\mathfrak{g} - 1$) blow up points may occur and the invariance property (mentioned above for $\mathfrak{g} = 2$) is no longer available. So reasonably, for higher genus we can only hope to establish the corresponding version of Theorem 2 .

To this purpose, we introduce the symmetric product $X^{(\nu)}$ of ν-copies of X modulo permutations, as the natural space to account for multiple blow-up points.

It is well known that $X^{(\nu)}$ defines a smooth complex manifold of dimension ν (see [16]), and it can be identified with the space of non zero effective divisors of degree $\nu \geq 1$ on X.

Recall that, an effective divisor on X is given by the formal sum: $D = \sum_{j=1}^{k} n_j p_j$, where the degree of $D := deg(D) = \sum_{j=1}^{k} n_j$, the support of D:= $supp\ D = \{p_1, \ldots, p_k\} \subset X$ (formed by distinct points in X) and we call $n_j \in \mathbb{N}$ the multiplicity of the point p_j.

Hence, to any such effective divisor of degree ν, we can associate the ν-ple in $X^{(\nu)}$ formed by each point in its support repeated according to its multiplicity.

Divisors naturally arise also in connections with holomorphic quadratic differentials. Indeed, we know that every non-trivial holomorphic quadratic differential $\alpha \in C_2(X)$ admits $4(\mathfrak{g} - 1)$ zeroes counted with multiplicity . So the zero set of α identifies in a natural way an effective divisor in $X^{(4(\mathfrak{g}-1))}$ which is denoted by: $\mathrm{div}(\alpha)$.

For an effective divisor $D \in X^{(\nu)}$ we let,

$$Q(D) = \{\alpha \in C_2(X) : \mathrm{div}(\alpha) \geq D\}$$

namely, every $\alpha \in Q(D)$ must vanish at each point of the support of D with greater or equal multiplicity.

In particular for $x_0 \in X$, by taking $D = x_0$ we have: $Q(x_0) = \{\alpha \in C_2(X) : \alpha(x_0) =$

$0\}$. As a consequence, by definition the Kodaira map is characterized as follows:

$$[\beta]_{\mathbb{P}} = \tau(x_0) \iff \int_X \beta \wedge \alpha = 0 \quad \forall \alpha \in Q(x_0). \tag{1.12}$$

For higher genus $\mathfrak{g} \geq 3$, in order to extend property (1.12) beyond the complex curve $\tau(X)$, we need to evoke, for $1 \leq \nu \leq \mathfrak{g} - 1$, the ν-Secant Variety of $\tau(X)$ (see [2]), which we denote by $\widetilde{\Sigma}_\nu$.

Indeed $\widetilde{\Sigma}_\nu \subset \mathbb{P}(\mathcal{H}^{0,1}(X, E))$, defines a closed irreducible analytic sub-variety of $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$, and according to Lemma 2.1 below, we have:

Proposition 1.1 *For any $\nu \in \{1, \cdots, \mathfrak{g} - 1\}$ let $\tilde{\Sigma}_\nu \subset \mathbb{P}(\mathcal{H}^{0,1}(X, E))$ be the ν-secant variety of $\tau(X)$ (cf. [2]) we have:*

$$\tilde{\Sigma}_1 = \tau(X) \subset \tilde{\Sigma}_2 \subset \cdots \tilde{\Sigma}_{\mathfrak{g}-1}, \quad dim(\tilde{\Sigma}_\nu) \leq 2\nu - 1, \tag{1.13}$$

($\tau(X)$ the image of the Kodaira map in (1.12)) and

$$[\beta]_{\mathbb{P}} \in \tilde{\Sigma}_\nu \iff \exists \, divisor \, D \in X^{(\nu)} : \int_X \beta \wedge \alpha = \int_X \beta_0 \wedge \alpha = 0 \tag{1.14}$$

$\forall \alpha \in Q(D)$, ($\beta_0 \in [\beta]$ the associated harmonic Beltrami differential with respect to g_X).

Again we see that, $\mathbb{P}(\mathcal{H}^{0,1}(X, E)) \setminus \widetilde{\Sigma}_{\mathfrak{g}-1}$ is a Zariski open set (hence dense) in $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$.

We prove:

Theorem 4 *If $\mathfrak{g} = 3$ and $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(X) \times \mathcal{H}^{0,1}(X, E)$ satisfies:*

$$[\beta] \neq 0 \text{ and } [\beta]_{\mathbb{P}} \notin \widetilde{\Sigma}_2$$

where $\widetilde{\Sigma}_2$ is the 2-secant variety of $\tau(X)$, then there exists a unique (CMC) 1-immersion of X into a (germ of) hyperbolic 3-manifold $N (\simeq X \times \mathbb{R})$ with pull-back metric g and $(2, 0)$-part α of (II_g) satisfying (1.6).

As in [46], Theorem 4 is established in [49] by means of a detailed asymptotic analysis of the (CMC) c-immersions of Theorem 1 (with $|c| < 1$) as $c \to 1^-$.

In the framework of Hitchin self-duality theory ([17]), we pursue the limiting behaviour of suitable Higgs bundles under the $\mathbb{C}^*$ action. In this context the role of secant varieties also appeared in [58].

In terms of the unique solution pair: (u_c, α_c) of (1.2)-(1.3) satisfying (1.6), we can use the Liouville-type character of the Gauss equation (1.2) (cf. [30]) to find that (along a sequence) the function $\xi_c := -u_c + \log(||\alpha_c||^2_{L^2})$ can blow-up only around finitely many points (blow-up points) see [6], and as shown in [3, 4, 31] and [45], each blow-up point carries a "quantized" blow-up mass given by an integral multiple of 8π, see also [26, 27, 47] for related results.

Therefore, also to a blow-up situation we can associate an effective "blow-up" divisor, which we denote by D_0, whose support is formed by the set of (distinct) blow-up points with multiplicities given by the corresponding (integral) blow-up masses, see (3.5) and definition 3.1. By the Gauss-Bonnet Theorem we have that a "blow-up" divisor $D_0 \in X^{(\nu)}$ with $1 \leq \nu \leq \mathfrak{g} - 1$.

With respect to the available blow-up analysis for Liouville-type equations, there are new difficulties to overcome when different zeroes of α_c, converge at the blow-up point, as $|c| \to 1^-$. In this case, we call the blow up point of "collapsing" type, and in general we can no longer ensure that "blow-up" occurs with the "concentration" property, see [25, 32, 44, 45]. In turn (as expected) we are lead (after scaling) to a "limiting" metric with conical singularities exactly at the blow-up points with "conical" angle an integral multiple of 4π.

The ultimate goal of the blow-up analysis would be to show that, if blow-up occurs with blow up divisor $D_0 \in X^{(\nu)}$ $(1 \leq \nu \leq \mathfrak{g} - 1)$ then the following should hold:

$$\int_X \beta \wedge \alpha = 0 \quad \forall \alpha \in Q(D_0), \quad D_0 = \text{ blow up divisor.} \tag{1.15}$$

Unfortunately, property (1.15) has been established in [49] under the assumption that each point in the support of the "blow-up" divisor D_0 admits multiplicity at most 2, see Theorem 9. In this way, one can handle the case of genus $\mathfrak{g} = 3$ and obtain Theorem 4.

This improves already the blow-up analysis developed in [45] when blow-up occurs with the least blow-up mass 8π, namely when all points in the support of the "blow-up" divisor D_0 admit multiplicity 1. This information was used in [46] to reveal (for the first time) the crucial role of the "orthogonality" condition (1.15) towards the "compactness" issue and applies directly to the case of genus $\mathfrak{g} = 2$ and implies Theorem 2.

We point out that, even when the multiplicity of the blow-up points is at most two (as required in Theorem 9), the blow-up analysis becomes immediately more involved, with new analytical difficulties to be resolved beyond [45, 46].

At this point it is natural to ask whether also the viceversa of Theorem 4 holds. For the moment, in in a forthcoming paper we will show that indeed for $\mathfrak{g} = 3$ and for "almost" all $[\beta] \in \widetilde{\Sigma}_2$ blow-up must occur. Again this information is derived on the basis of the equivalent formulation of problem (1.8) in terms of Higgs Bundles, which in this case fails to satisfy the so called "stability" condition required by Hitchin selfduality theory [17] in order to ensure existence and uniqueness .

The case of higher genus has been handled in [50] under a different point of view, where the local blow-up analysis around a blow up point is combined with a "global" approximation procedure. We refer to [50] and the following sections for more details.

2 Preliminaries

Let $X \in \mathcal{T}_{\mathfrak{g}}(S)$ be a given Riemann surface with (unique) hyperbolic metric g_X and induced scalar product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$ and volume element dA.

Throughout this paper, we let:

$$E = T_X^{1,0} \text{ the holomorphic tangent bundle of } X \tag{2.1}$$

with dual:

$$E^* = (T_X^{1,0})^* = K_X \text{ the canonical bundle of } X.$$

The holomorphic line bundles E and E^* will be equipped with the complex structure induced by X, and with an hermitian product induced by a metric g (conformal to the metric g_X) defined in X. Thus, on sections and forms valued on E (or E^*), we have a well defined d-bar operator $\bar{\partial}$, and a fiber-wise hermitian product $\langle \cdot, \cdot \rangle_g$ and norm $\| \cdot \|_g$.

In the sequel, unless confusion arises, we shall drop the subscript g_X in the hermitian product and norm induced by g_X.

Recall that,

$$A^0(E) = \{\text{smooth sections of } X \text{ valued on } E\},$$
$$A^{0,1}(X, E) = \{(0, 1)\text{-forms valued on } E\},$$

and the elements in $A^{0,1}(X, E) = A^{0,1}(X, \mathbb{C}) \otimes E$ are also known as Beltrami differentials. By considering :

$$\bar{\partial} = \bar{\partial}_E : A^0(E) \longrightarrow A^{0,1}(X, E).$$

we can define the (0, 1)-Dolbeault cohomology group as follows:

$$\mathcal{H}^{0,1}(X, E) = A^{0,1}(X, E)/\bar{\partial}(A^0(E)). \tag{2.2}$$

So that, for any Beltrami differential $\beta \in A^{0,1}(X, E)$, there correspond the cohomology class:

$$[\beta] = \{\beta + \bar{\partial}\eta, \ \forall \, \eta \in A^0(E)\} \in \mathcal{H}^{0,1}(X, E).$$

Similarly, we let:

$$A^{1,0}(X, E^*) = \{(1, 0)\text{-forms valued on } E^*\} = A^{1,0}(X, \mathbb{C}) \otimes E^*.$$

Thus, we can consider the wedge product $\wedge$: $A^{0,1}(X, E) \times A^{1,0}(X, E^*)$, so for $\beta \in A^{0,1}(X, E)$ and $\alpha \in A^{1,0}(X, E^*)$, we have: $\beta \wedge \alpha \in A^{1,1}(X, \mathbb{C})$ satisfying the well-known properties of the wedge product, see [16] Chapter 2 Section 2. Consequently, we obtain the bilinear form:

$$A^{1,0}(X, E^*) \times A^{0,1}(X, E) \longrightarrow \mathbb{C} \, : \, (\alpha, \beta) \longrightarrow \int_X \beta \wedge \alpha, \tag{2.3}$$

which, by Serre duality (see [54]), is non-degenerate and induces the isomorphism:

$$A^{1,0}(X, E^*) \simeq (A^{0,1}(X, E))^*. \tag{2.4}$$

Also, we obtain the Hodge star operator:

$$*_E : A^{0,1}(X, E) \longrightarrow A^{1,0}(X, E^*), \tag{2.5}$$

where, for given $\beta \in A^{0,1}(X, E)$, the form $*_E\beta \in A^{1,0}(X, E^*)$ is uniquely identified by the condition:

$$\xi \wedge *_E\beta = \langle \xi, \beta \rangle \, dA, \quad \forall \, \xi \in A^{0,1}(X, E).$$

The Hodge operator $*_E$ depends on the metric g_X and it defines an isometry with inverse $*_E^{-1}$. It expresses the (metric dependent) isomophism between $A^{0,1}(X, E)$ and $A^{1,0}(X, E^*)$.

In general, for a given holomorphic line bundle L, we denote by $H^0(X, L)$ the space of global holomorphic sections of L. Clearly, $H^0(X, L)$ is a complex linear space, and since X is compact, it admits finite dimension (see [37] Proposition 3.16). Moreover, every $\alpha \in H^0(X, L) \setminus \{0\}$ admits the same number of zeroes counted with multiplicity (see [37] Lemma 1.5) and thus we can define the degree of L, denoted by $\deg L$, as given by the number of zeroes counted with multiplicity of a non trivial section in $H^0(X, L)$.

We have:

$$C_2(X) = H^0(X, \otimes^2(K_X)) = \{\alpha \in A^0(\otimes^2(K_X)) \; : \; \bar{\partial}\alpha = 0\}$$

$$= \{\alpha \in A^{1,0}(X, E^*) \; : \; \bar{\partial}\alpha = 0\}$$

and by applying the Riemann-Roch Theorem to $L = \otimes^2(K_X)$, we find:

$$\dim_{\mathbb{C}} C_2(X) = 3(\mathfrak{g} - 1), \tag{2.6}$$

(see [37] and [40]). Moreover, we know that, $deg(K_X) = 2(\mathfrak{g} - 1)$ (see [37] Chapter V Prop. 1.14), so we may conclude that,

$$\deg \otimes^2 K_X = 4(\mathfrak{g} - 1), \tag{2.7}$$

and consequently,

$$\text{any } \; \alpha \in C_2(X) \setminus \{0\} \text{ admits } 4(\mathfrak{g} - 1) \text{ zeroes counted with multiplicity.} \tag{2.8}$$

In local holomorphic z-coordinates around a given x_0 (centred at the origin), the metric g_X takes the expression:

$$g_X = e^{2u_X}\frac{i}{2}dz \wedge d\bar{z}, \quad u_X \text{ smooth: } u_X(0) = |\nabla u_X(0)| = 0, \tag{2.9}$$

and $\alpha \in C_2(X)$ is given by:

$$\alpha = h(dz)^2 \ \ (\text{and } \|\alpha\| = 2|h|e^{-2u_X}) \text{ with } h \text{ holomorphic.} \tag{2.10}$$

In this way, it is clear what we mean by a zero of α and corresponding multiplicity, as indeed those notions are independent of the chosen holomorphic coordinates. By Stokes theorem, for $\alpha \in C_2(X)$ we have: $\int_X \bar{\partial}\eta \wedge \alpha = 0, \ \ \forall\ \eta \in A^0(E)$, and therefore the bilinear form (2.3) is well defined and non degenerate when restricted on the space: $C_2(X) \times \mathcal{H}^{0,1}(X, E)$, and it induces the isomorphism:

$$C_2(X) \simeq (\mathcal{H}^{0,1}(X, E))^*. \tag{2.11}$$

Thus, for harmonic $\beta_0 \in [\beta] \in \mathcal{H}^{0,1}(X, E)$ we have: $*_E\beta_0 \in C_2(X)$, and (in analogy to (2.5)) we have the isomorphism:

$$\mathcal{H}^{0,1}(X, E) \longrightarrow C_2(X) : [\beta] \longrightarrow *_E\beta_0.$$

In other words, for $\alpha \in C_2(X) \subset A^{1,0}(X, E^*)$ there exist a unique harmonic Beltrami differential β_0: $*_E\beta_0 = \alpha$ or equivalently $*_E^{-1}\alpha = \beta_0$.

Also, the dual space $(C_2(X))^*$ can be identified with the space of harmonic Beltrami differentials (with respect to g_X), and hence with the space $\mathcal{H}^{0,1}(X, E)$. Indeed, to every harmonic $\beta_0 \in [\beta] \in \mathcal{H}^{0,1}(X, E)$, there correspond an (unique) element in $(C_2(X))^*$ defined as follows:

$$C_2(X) \longrightarrow \mathbb{C} : \alpha \longrightarrow \int_X \beta_0 \wedge \alpha = \int_X (\beta_0 + \bar{\partial}\eta) \wedge \alpha. \tag{2.12}$$

At this point, by recalling that the Teichmüller space $\mathcal{T}_{\mathfrak{g}}(S)$ has the structure of a differential cell of real dimension $6(\mathfrak{g} - 1)$, in view of (2.6), we have a well-known parametrization of $T^*(\mathcal{T}_{\mathfrak{g}}(S))$, the cotangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$, given by the pairs:

$$(X, \alpha) \in \mathcal{T}_{\mathfrak{g}}(X) \times C_2(X),$$

see e.g. [22] for details. Consequently, the tangent bundle $T(\mathcal{T}_{\mathfrak{g}}(S))$ of $\mathcal{T}_{\mathfrak{g}}(S))$ is parametrized by the pairs:

$$(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E).$$

For $p \geq 1$, we consider the L^p-space of sections and forms valued on E :

$$L^p(X,E) = \{\eta : X \longrightarrow E \ : \ \|\eta\|_{L^p} := (\int_X \|\eta\|^p dA)^{\frac{1}{p}} < +\infty\},$$
$$L^p(A^{0,1}(X,E)) = \{\beta \in A^{0,1}(X,E) \ : \ \|\beta\|_{L^p} := (\int_X \|\beta\|^p dA)^{\frac{1}{p}} < +\infty\},$$

which define Banach spaces equipped with the given norm: $\|\cdot\|_{L^p}$.

Also for $p \geq 1$, we have the Sobolev space:

$$W^{1,p}(X,E) = \{\eta \in L^p(X,E) \ : \ \bar{\partial}\eta \in L^p(A^{0,1}(X,E))\}, \tag{2.13}$$

defining a Banach space equipped with the norm:

$$\|\eta\|_{W^{1,p}} = \|\eta\|_{L^p} + \|\bar{\partial}\eta\|_{L^p}, \ \forall \ \eta \in W^{1,p}(X,E).$$

For the holomorphic line bundle $E = T_X^{1,0}$ in (2.1), we recall that the norm $\eta \in W^{1,p}(X,E)$ is equivalent to the L^p-norm of $\bar{\partial}\eta$ in view of the following Poincaré inequality:

$$\|\eta\|_{L^p} \leq C_p \|\bar{\partial}\eta\|_{L^p}, \ \forall \ \eta \in W^{1,p}(X,E).$$

for suitable $C_p > 0$, see [21].

In the finite dimensional space $C_2(X)$, where we have the equivalence of all norms, it is usual (by recalling the Weil-Patterson form [22]) to consider the following L^2-norm:

$$\|\alpha\|_{L^2} := (\int_X \langle \alpha, \alpha \rangle dA)^{\frac{1}{2}} \ \text{ for } \ \alpha \in C_2(X). \tag{2.14}$$

With respect to a basis:

$$\{s_1, \ldots, s_N\} \subset C_2(X) \ \text{ with } \ N = 3(\mathfrak{g}-1) : \ \int_X \langle s_j, s_k \rangle dA = \delta_{j,k} \tag{2.15}$$

($\delta_{j,k}$ the Kronecker symbols) we may write:

$$\alpha = \sum_{j=1}^{\nu} b_j s_j, \ b_j \in \mathbb{C}, \ \text{ and } \ \beta_0 = *_E^{-1}\alpha = \sum_{j=1}^{\nu} b_j *_E^{-1} s_j$$

and obtain:

$$\|\alpha\|_{L^2}^2 = \|\beta_0\|_{L^2}^2 = \sum_{j=1}^{\nu} |b_j|^2.$$

Clearly, any closed and bounded subsets of $C_2(X)$ (with respect to the L^2-norm) is compact. Thus, for example, if $\alpha_n \in C_2(X)$ satisfies $\|\alpha_n\|_{L^2} = 1$ then it admits a convergent subsequence $\alpha_{n_k} \longrightarrow \alpha_0 \in C_2(X)$ with $\|\alpha_0\|_{L^2} = 1$.

We conclude this section of preliminaries by discussing $\nu-$secant varieties and by providing the proof of Proposition 1.1

To his purpose, for any $\nu \in \{1, \cdots, \mathfrak{g}-1\}$ we let $\tilde{\Sigma}_\nu \subset \mathbb{P}(\mathcal{H}^{0,1}(X, E))$ defined as follows:

$$[\beta]_{\mathbb{P}} \in \tilde{\Sigma}_\nu \iff \exists \text{ divisor } D \in X^{(\nu)} : \int_X \beta \wedge \alpha = \int_X \beta_0 \wedge \alpha = 0 \quad (2.16)$$

$\forall \alpha \in Q(D)$, ($\beta_0 \in [\beta]$ the associated harmonic Beltrami differential).

It is shown in [49] that $\tilde{\Sigma}_\nu$ defines a closed analytic irreducible sub-variety of $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$ satisfying:

$$\tilde{\Sigma}_1 = \tau(X) \subset \tilde{\Sigma}_2 \subset \cdots \tilde{\Sigma}_{\mathfrak{g}-1}, \quad \dim(\tilde{\Sigma}_\nu) \leq 2\nu - 1, \quad (2.17)$$

($\tau(X)$ the image of the Kodaira map).

Thus, to establish Proposition 1.1, we need to prove that $\tilde{\Sigma}_\nu$ coincides with the ν-secant variety of $\tau(X)$. To this purpose, let $Z := \tau(X) \subseteq \mathbb{P}(V^*)$, and by recalling that we have set: $V = C_2(X)$, we consider the canonical projection:

$$\pi : V^* \setminus \{0\} \to \mathbb{P}(V^*) \text{ so that } \pi(\beta) = [\beta]_{\mathbb{P}}, \ \forall \beta \in V^* \setminus \{0\}.$$

For given $\nu = 1, \cdots, \mathfrak{g} - 1$, and distinct points $p_j \in X$, $j = 1, \cdots, \nu$, we let $\beta_j \in V^* \setminus \{0\}$, satisfying: $\pi(\beta_j) = [\beta_j]_{\mathbb{P}} := \tau(p_j) \in Z$, for $1 \leq j \leq \nu$. By Riemann-Roch theorem, it is easy to check that $\{\beta_1, \ldots, \beta_\nu\} \in V^*$ are linearly independent. Hence, if we consider the corresponding simple divisor $D := \sum_j p_j \in X^{(\nu)}$ then $W_D := \pi(span(\beta_1, \ldots \beta_\nu) \setminus \{0\})$ is a projective subspace of $\mathbb{P}(V^*)$ depending only of D, with dim(W_D)=$\nu - 1$.

By definition, the ν-secant variety $Y_\nu(Z)$ of Z is the closure, in the Zariski topology, of the union of all such projective subspaces W_D, for all simple divisor $D \in X^{(\nu)}$, see [2] Chapter VI section 1 for details. Clearly: $Y_1(Z) = Z$.

Lemma 2.1 *If $1 \leq \nu \leq g - 1$ the sub-variety $\tilde{\Sigma}_\nu$ in (2.16) coincides with the ν-secant variety $Y_\nu(Z)$ of $Z = \tau(X)$.*

Proof Let $D \in X^{(\nu)}$ be an effective divisor in X of degree ν, with $1 \leq \nu \leq \mathfrak{g} - 1$. Let:

$$Q(D)^\perp = \{\beta \in V^* : \int_X \beta \wedge \alpha = 0 \ \ \forall \alpha \in Q(D)\}.$$

It follows again by the Riemann-Roch theorem, that $dim(Q(D)^\perp) = \nu$, and moreover $\tilde{\Sigma}_\nu$ is the closed sub-variety (in Zariski topology) formed by the union of all the projective subspaces $\pi(Q(D)^\perp)$, for any $D \in X^{(\nu)}$. Notice in particular that, $dim(\pi(Q(D)^\perp)) = \nu - 1$

On the other hand, if $D = \sum_{j=1,\ldots,\nu} p_j \in X^{(\nu)}$ then

$$Q(D) \subseteq Q(p_j) \text{ and so } Q(p_j)^\perp \subseteq Q(D)^\perp \ \forall \, j \in \{1, \ldots, \nu\}.$$

Therefore, $\tau(p_j) \in \pi(Q(D)^\perp)$, $\forall \, j \in \{1, \ldots, \nu\}$, and consequently: $W_D = \pi(span(\beta_1, \ldots, \beta_\nu)) \subseteq \pi(Q(D)^\perp)$. Since dim($\pi(span(\beta_1, \ldots, \beta_\nu)$)= $\nu - 1 =$

$\dim(\pi(Q(D)^{\perp})$, we conclude that:

$$\pi(span(\beta_1, \ldots, \beta_\nu)) = \pi(Q(D)^{\perp}).$$

Thus, we obtain that, $Y_\nu(\tau(X)) \subseteq \tilde{\Sigma}_\nu$.

If $\mathfrak{g} = 3$ then it is easy to show that equality holds. Indeed, it is well known that, $dim Y_\nu(\tau(X)) = 3$ whereas $dim \tilde{\Sigma}_\nu \leq 3$, and consequently: $Y_\nu(\tau(X)) = \tilde{\Sigma}_\nu$, as claimed.

To attain the same conclusion for genus $\mathfrak{g} \geq 4$, we recall that any divisor $D \in X^{(\nu)}$ is the limit (in the topology of $X^{(\nu)}$) of a sequence of simple divisors: $D_k = \sum_{j=1,\ldots,\nu} p_{j,k} \in X^{(\nu)}$ (namely, $p_{j,k}$ all distinct). Consequently, we find that, $Q(D_k) \to Q(D)$, as $k \to \infty$, with respect to the topology of the Grassmannian $G_{N-\nu}(V)$. This implies that (dually) we have: $Q(D_k)^{\perp} \to Q(D)^{\perp}$, as $k \to \infty$ (with respect to the topology of $G_\nu(V^*)$). At this point, by using a (finite) orthonormal basis for $Q(D_k)^{\perp}$, we see that, for any $\beta \in Q(D)^{\perp}$ there exists $\beta_k \in Q(D_k)^{\perp}$ such that $\beta_k \to \beta$, as $k \to \infty$. Hence, $\pi(\beta) \in Y_\nu(\tau(X))$ that is, $\pi(Q(D)^{\perp}) \subset Y_\nu(\tau(X))$, a therefore: $\tilde{\Sigma}_\nu = Y_\nu(\tau(X))$, as claimed. □

3 The Donaldson Functional

We fix the pair: $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E)$, with $\beta_0 \in [\beta]$ the harmonic representative of the class $[\beta]$.

We are looking for a solution pair (u, α) of (1.2), (1.3) such that: the pull-back metric g on X and the $(2, 0)$-part α of the second fundamental form (II_g) satisfy:

$$g = e^u g_X \quad \text{and } \alpha = e^u *_E (\beta_0 + \bar{\partial}\eta), \text{ with suitable } \eta \in A^0(E),$$

Correspondingly, the pair (u, η) satisfies the following “constraint" Gauss-Codazzi equations:

$$\begin{cases} \Delta u + 2 - 2te^u - 8e^u \|\beta_0 + \overline{\partial}\eta\|^2 = 0 \quad \text{in } X, \\ \overline{\partial}(e^u *_E (\beta_0 + \overline{\partial}\eta)) = 0, \end{cases} \tag{3.1}$$

with $t = 1 - c^2$, (recall (1.4))

Please note that the Beltrami differential $\beta_0 + \bar{\partial}\eta \in [\beta]$ in the second equation in (3.1) is harmonic with respect to the metric $g = e^u g_X$.

With this point of view, the system (3.1) can be formulated in terms of Hitchin self-duality equations [17] for a suitable nilpotent $SL(2, \mathbb{C})$ Higgs bundle, we refer to [1] and [21] for details. Therefore, on the ground of Hitchin's sefduality theory, the existence and uniqueness for (3.1) is equivalent to the "stability" of the given Higgs bundle (cfr [17]). In various context (see e.g. [1, 17, 28, 33] and [34]) the "stability" property has been successfully verified, but still it appears difficult to be directly checked in our context.

On the other hand, it is easy to check that solutions of (3.1) correspond to critical points of the following Donaldson functional (in the terminology of [15])

$$F_t(u,\eta) = \int_X \left(\frac{|\nabla u|^2}{4} - u + te^u + 4e^u \|\beta_0 + \overline{\partial}\eta\|^2 \right) dA, \tag{3.2}$$

$t \in \mathbb{R}$, with “natural" (convex) domain:

$$\Lambda = \left\{ (u,\eta) \in H^1(X) \times W^{1,2}(X,E) \ : \ \int_X e^u \|\beta_0 + \bar{\partial}\eta\|^2 \, dA < \infty \right\},$$

where $H^1(X)$ is the usual Sobolev spaces of function of X and $W^{1,2}(X,E)$ is the Sobolev space of sections of E (see (2.13)).

We refer to [46] for a discussion about the Gateaux differentiability of F_t along "smooth" directions and the corresponding notion of "weak" critical point and relative regularity.

For $t > 0$ the functional F_t is clearly bounded from below in Λ, and it was shown in [21] that it admits a unique (smooth) critical point (u_t, η_t) corresponding to its global minimum in Λ. However, for $t \leq 0$ the functional F_t may no longer admit critical points, as indeed non-existence does occur for system (3.1). For example, we check easily that this is the case when we take $[\beta] = 0$ (i.e. $\beta_0 = 0$).

Even for the case $t = 0$, it is a delicate task to identify the cohomology classes $[\beta]$ for which the functional:

$$F_0(u,\eta) = \int_X \left(\frac{1}{4}|\nabla u|^2 - u + 4e^u \|\beta_0 + \overline{\partial}\eta\|^2 \right) dA,$$

admits a critical point in Λ.

In this respect, in [46] was pointed out that a critical points for F_0 can be attained only as "limit" of (u_t, η_t) for $t \to 0^+$. Indeed, the following holds:

Proposition 3.1 (Theorem 8 [46]) *If (u_0, η_0) is a solution for the system (3.1) with $t = 0$, then*

(i) $(u_t, \eta_t) \to (u_0, \eta_0)$ uniformly in $C^\infty(X)$, as $t \to 0^+$;
(ii) F_0 is bounded from below in Λ and attains its global minimum at (u_0, η_0) which defines its only critical point.
Hence, (u_0, η_0) is the only solution of (3.1) with $t = 0$.

In fact, for $[\beta] = 0$ (i.e. $\beta_0 = 0$), we know that (3.1) admits no solutions at $t = 0$, and indeed we find: $u_t = \ln \frac{1}{t} \to +\infty$, $\eta_t = 0$ and $F_t(u_t, \eta_t) \to -\infty$, as $t \to 0^+$.

.

Therefore, to identify possible critical points for F_0, we must investigate when the pair (u_t, η_t) survives the passage to the limit, as $t \to 0^+$.

For $t > 0$, set:

$$\beta_t = \beta_0 + \overline{\partial}\eta_t \in A^{0,1}(X,E) \ \text{ and } \ \alpha_t = e^{u_t} *_E \beta_t \in C_2(X) \setminus \{0\},$$

and define:

$$s_t \in \mathbb{R} \ : \ e^{s_t} = \|\alpha_t\|^2_{L^2} \ \text{ and } \ \hat{\alpha}_t = \frac{\alpha_t}{\|\alpha_t\|_{L^2}} = e^{-\frac{s_t}{2}}\alpha_t,$$

where $\|\alpha_t\|_{L^2}$ is the L^2-norm of $\alpha_t \in C_2(X)$ (see (2.14)).

Clearly, we have: $\text{div}(\widehat{\alpha}_t) = \text{div}(\alpha_t) := D_t$. So D_t is the zero divisor of α_t, with support $Z_t := supp\ D_t$ given by the finite set of distinct zeroes of α_t whose multiplicities adds up to $4(\mathfrak{g}-1)$ (see (2.8)).

In order to attain an accurate asymptotic description about the behaviour of (u_t, η_t), as $t \to 0^+$, we shall account for possible blow-up phenomena (cf. [6]) for the function:

$$\xi_t := -u_t + s_t, \tag{3.3}$$

satisfying the Liouville-type equation:

$$-\Delta\xi_t = 8\|\widehat{\alpha}_t\|^2 e^{\xi_t} - f_t \text{ in } X, \tag{3.4}$$

with $f_t = 2(1 - te^{u_t})$ and $0 \le f_t \le 2$ in X.

We can combine the blow-up information in Theorem 3 of [45] with Theorem 3.1 above, to conclude the following facts about ξ_t as $t \to 0^+$:

FACT 1. If ξ_t does not blow up (i.e $\limsup_{t\to 0^+} max_X\ \xi_t < +\infty$) then $(u_t, \eta_t) \to (u_0, \eta_0)$ uniformly in $C^\infty(X)$, as $t \to 0^+$; and (u_0, η_0) is the unique critical point of F_0 in Λ corresponding to its global minimum point.

FACT 2. If ξ_t does blow up (i.e. $\limsup_{t\to 0^+} max_X\ \xi_t = +\infty$,) then $\liminf_{t\to 0^+} max_X\ \xi_t = +\infty$, and along any sequence $t_k \to 0^+$, we have that, $\xi_k := \xi_{t_k}$ admits a finite set $\mathcal{S}$ of blow up points, (depending possibly on the sequence t_k) such that, for any blow-up point $x \in \mathcal{S}$, the following holds:

$$\lim_{k\to+\infty} max_{B(x;r)}\ \xi_k = +\infty, \quad \text{for all small } \ r > 0;$$

Furthermore, the blow-up mass at x is defined by:

$$\sigma(x) := \lim_{r\to 0^+}\left(\lim_{k\to+\infty} 8\int_{B(x;r)} \|\widehat{\alpha}_{t_k}\|^2 e^{\xi_k} dA\right) \tag{3.5}$$

and it satisfies the quantization property:

$$\sigma(x) \in 8\pi\mathbb{N};$$

see [45, 46] and [48] for details.

We introduce the following notion of "blow-up" divisor:

Definition 3.1 If ξ_k blows-up with (non-empty) blow up set $\mathcal{S}$ then we define the blow-up divisor of ξ_k the formal sum:

$$D_0 = \sum_{x\in\mathcal{S}} m_x x \in X^{(\nu)} \ \text{ with } \ m_x = \frac{1}{8\pi}\sigma(x) \in \mathbb{N},$$

so that, supp $D_0 = \mathcal{S}$ and $deg(D_0) = \sum_{x\in\mathcal{S}} m_x = \nu$.

After integration of (3.4) we have:

$$8\int_X \|\widehat{\alpha}_{t_k}\|^2 e^{\xi_k} dA = 8\int_X \epsilon^{u_{t_k}} \|\beta_0 + \overline{\partial}\eta_{t_k}\|^2 dA = 8\pi(\mathfrak{g}-1) - 2t_k \int_X \epsilon^{u_{t_k}} dA$$

and therefore we find:

$$\nu = deg(D_0) = \sum_{x\in\mathcal{S}} m_x \in \{1, ..., \mathfrak{g}-1\} \tag{3.6}$$

that is, any blow-up divisor belongs to $X^{(\nu)}$ with $\nu \in \{1, ..., \mathfrak{g}-1\}$.

One of the main contribution of [46] concerns the case of blow-up with minimal blow-up mass 8π, namely when each (blow-up) point admits multiplicity one.

This is always the case for genus $\mathfrak{g} = 2$, since by (3.6), we know that necessarily the corresponding blow-up divisor is formed by a single point of multiplicity one.

By combining Theorem 9 and Theorem 10 in [46], we have:

Theorem 5 ([46]) *Suppose that ξ_k in (3.3) admits a (non-empty) blow-up set $\mathcal{S}$ with the corresponding blow up divisor D_0 satisfying: $D_0 = \sum_{q\in\mathcal{S}} q$, then (1.15) holds.*

In particular if the genus $\mathfrak{g} = 2$ then $\mathcal{S} = \{q\}$, $D_0 = q$ and $[\beta]_{\mathbb{P}} = \tau(q) \in \tau(X)$, (recall (1.10)).

In terms of the Donaldson functional, for genus $\mathfrak{g} = 2$ the following holds:

Theorem 6 ([46]) *If $\mathfrak{g} = 2$, then for every $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$ the functional F_0 is bounded from below in Λ. Moreover, if $[\beta]_{\mathbb{P}} \notin \tau(X)$ then F_0 attains its infimum in Λ at a (smooth) minimum point which is its only critical point.*

See Theorem 6 in [46] for details.

As it turns out, the case of genus $\mathfrak{g} = 2$ is very special.

Firstly, it was observed in [49] that the Donaldson functional is equivariant under bi-holomorphisms of X (see Appendix 2 of [49]). For $\mathfrak{g} = 2$, we recall the unique bi-holomorphic hyperelliptic involution (see [16, 37])

$$j : X \to X \tag{3.7}$$

with exactly $2(\mathfrak{g}+1) = 6$ (distinct) fixed points (the Weierstrass points). Actually, in this case we find that the Donaldson functional is <u>invariant</u> under (3.7) and the following improvement of Theorem 6 holds:

Theorem 7 ([49]) *Let $\mathfrak{g} = 2$ and $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$.*

If $\lim_{t\to 0^+} max_X \xi_t = +\infty$ (blow-up), then there exist a <u>unique</u> (blow-up) point $q \in X : [\beta]_{\mathbb{P}} = \tau(q)$, with $j(q) = q$ and $\lim_{t\to 0^+} max_K \xi_t = -\infty$, for any compact $K \subset X \setminus \{q\}$.

Please note that for $\mathfrak{g} = 2$ the Kodaira map is 2 to 1, and it is only because we know that the blow-up point is one of the Weierstrass point that we can conclude that it is unique, i.e. independent of the chosen sequence.

Actually, by exploiting Hitchin selfduality theory (see [17]), in a forthcoming paper we will show that Theorem 7 is "sharp", as the following holds:

Theorem 8 ([49]) *Assume that $\mathfrak{g} = 2$ and let $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$.*

$$\lim_{t\to 0} max_X \xi_t < +\infty, \quad \Longleftrightarrow [\beta]_{\mathbb{P}} \notin \{\tau(q),\ q \in X \ \ with \ \ j(q) = q\}$$

($\Longleftrightarrow$ F_0 attains its infimum in Λ, and the (smooth) minimum point is the only critical point of F_0.)

Concerning the more delicate case of higher genus, a rather sharp result is established in [49] in case of genus $\mathfrak{g} = 3$, which relies on the following extension of Theorem 5:

Theorem 9 *Suppose that ξ_k in (3.3) admits (non-empty) blow-up set $\mathcal{S}$ with corresponding blow up divisor: $D_0 = \sum_{x\in\mathcal{S}} m_x x$. If*

$$m_x \in \{1, 2\} \ \ \forall x \in \mathcal{S}, \tag{3.8}$$

then (1.15) holds.

It is reasonable to expect that in Theorem 9 the assumption (3.8) should be dropped.

In this respect, a main contribution has been obtained in [50] where the blow up divisor D_0 in (1.15) is replaced by a "lager" (but still appropriate) divisor, as follows:

Theorem 10 ([50]) *Suppose that ξ_k in (3.3) admits (non-empty) blow-up set $\mathcal{S}$ with corresponding blow up divisor: $D_0 = \sum_{x\in\mathcal{S}} m_x x$.*

For every $x \in \mathcal{S}$ there exists:

$$N_x \in \mathbb{N} \cup \{0\} : 0 \le N_x \le 2(m_x - 1), \tag{3.9}$$

so that, for the divisor $D := \sum_{x\in\mathcal{S}} (N_x + 1)x \ge D_0$ the following holds:

$$\int_X \beta \wedge \alpha = 0, \quad \forall \alpha \in Q(D). \tag{3.10}$$

Furthermore, in [50] we have shown that the "orthogonality" condition (3.10) identifies precisely a (possibly reducible) complex analytic sub-variety $\Sigma_{\mathfrak{g}} \subset \mathbb{P}(\mathcal{H}^{0,1}(X, E))$. More precisely, the following holds:

$$\Sigma_{\mathfrak{g}} \subset \mathbb{P}(\mathcal{H}^{0,1}(X, E)) : \quad [\beta]_{\mathbb{P}} \in \Sigma_{\mathfrak{g}} \iff \beta \ \text{ satisfies (3.10)},$$

$$\dim(\Sigma_{\mathfrak{g}}) \le 2\mathfrak{g} - 3 \quad \text{and} \quad \Sigma_{\mathfrak{g}=2} = \tau(X),$$

$$\tilde{\Sigma}_{\mathfrak{g}-1} \subseteq \Sigma_{\mathfrak{g}}, \quad \text{with} \ \ \tilde{\Sigma}_{\mathfrak{g}-1} = (\mathfrak{g} - 1)\text{-Secant Variety of } \tau(X).$$

In particular, $\mathrm{codim}(\Sigma_{\mathfrak{g}}) \geq \mathfrak{g} - 1$ and therefore $\mathbb{P}(\mathcal{H}^{0,1}(X, E)) \setminus \Sigma_{\mathfrak{g}}$ defines a non-empty Zariski open set (thus dense) in $\mathbb{P}(\mathcal{H}^{0,1}(X, E))$ where compactness holds. Hence an analogue of Theorem 4 about the existence and uniqueness of (CMC) 1-immersions can be established for any genus. We refer to [50] for details.

Please note that: $D_0 \leq D$, and it is a main open problem to see when equality holds, namely when we can replace the divisor D in Theorem 10 with D_0.
For genus $\mathfrak{g} = 2$ we have: $\mathcal{S} = \{x_0\}$ and $m_{x_0} = 1$, so we see that (3.10) reduces exactly to (1.15) in this case. Also, for genus $\mathfrak{g} = 3$, we have shown that indeed we can take $D = D_0$, that is $N_x + 1 = m_x, \ \forall x \in \mathcal{S}$. Such an improvement was possible on the basis of a very accurate blow-up analysis carried out in [49] describing the asymptotic profile of ξ_{t_k} as $k \to +\infty$.
At the moment, it seems extremely difficult (or even impossible) to extend to higher genus the description of the asymptotic blow up profile of ξ_{t_k} with the same accuracy as in [49], or along the lines of [10–13] and [55–57]. In addition, we face a new and delicate situation when blow-up occurs at a point of "collapsing" zeroes of α_{t_k} as $k \to +\infty$, namely at an accumulation point of several different zeroes of $\widehat{\alpha}_t$, as $t \to 0^+$. In this situation, the phenomenon of "blow-up without concentration" may manifest, see [25, 32] and [45] .

Instead, to established Theorem 10 in [50] we change completely point of view and relay on an appropriate approximation property of "global" nature rather than the "local" viewpoint of [46] and [49] focusing on description of ξ_{t_k}, around a blow-up point.

4 Asymptotics

We let:

$$u_t = w_t + d_t, \quad \text{with} \quad \int_X w_t dA = 0 \ \text{ and } \ d_t = \fint_X u_t dA$$

$$\beta_t = \beta_0 + \bar{\partial}\eta_t \in A^{0,1}(X, E) \ \text{ and } \ \alpha_t = e^{u_t} *_E \beta_t \in C_2(X)$$

and recall that,

$$s_t \in \mathbb{R} \ : \ e^{s_t} = \|\alpha_t\|^2_{L^2} \ \text{ and } \ \hat{\alpha}_t = e^{-s_t/2}\alpha_t.$$

It was observed in [46] that the map:

$$t \to 4 \int_X e^{u_t} \|\beta_0 + \bar{\partial}\eta_t\|^2 dA = 4\pi(\mathfrak{g} - 1) - t \int_X e^{u_t} dA$$

is decreasing in $(0, +\infty)$ (see Lemma 3.6 of [46]). So it is well defined the value:

$$\begin{aligned} \rho([\beta]) = \rho(\beta_0) &:= 4 \lim_{t\to 0^+} \textstyle\int_X e^{u_t} \|\beta_0 + \bar{\partial}\eta_t\|^2 dA \\ &= 4 \lim_{t\to 0^+} \textstyle\int_X e^{\xi_t} \|\widehat{\alpha}_t\|^2 dA \in [0, 4\pi(\mathfrak{g} - 1)] \end{aligned} \tag{4.1}$$

The following easy bounds were derived in [46]:

Lemma 4.1

$$\begin{aligned}
&\forall\, q \in [1,2) \;\; \exists\, C_q > 0 \;:\; \|w_t\|_{W^{1,q}(X)} \leq C_q,\\
&w_t \leq C \;\; in \;\; X \;\; and \;\; te^{d_t} \leq 1,\\
&\int_X e^{-u_t} dA \geq C ⨍_X \|\beta_0\|^2 dA,\\
&s_t \leq d_t + C,
\end{aligned} \tag{4.2}$$

for a suitable constant $C > 0$.

See Lemma 3.7 and Remark 3.1 of [46].

Therefore, along a (positive) sequence $t_k \longrightarrow 0^+$, we set,

$$d_k := d_{t_k}, \;\; u_k = u_{t_k}, \;\; w_k := w_{t_k},$$

and we may assume that,

$$w_k \longrightarrow w_0 \;\text{ and }\; e^{w_k} \longrightarrow e^{w_0} \;\text{ pointwise and in }\; L^p(X),$$

$$t_k e^{d_k} \longrightarrow \mu \geq 0 \;\text{ and so }\; t_k e^{u_k} \longrightarrow \mu e^{w_0} \;\text{ pointwise and in }\; L^p(X), \tag{4.3}$$

for any $p > 1$, and as $k \longrightarrow +\infty$.

In addition, for suitable $1 \leq N \leq 4(\mathfrak{g}-1)$, the zero divisor of $\widehat{\alpha}_k := \widehat{\alpha}_{t_k} \in C_2(X) \setminus \{0\}$ takes the form (for k large),

$$\text{div}(\widehat{\alpha}_k) = \sum_{j=1}^{N} n_j z_{j,k} \;\text{ and }\; \sum_{j=1}^{N} n_j = 4(\mathfrak{g}-1).$$

with $\{z_{j,k}\}$, $j \in \{1, \ldots, N\}$ the distinct zeroes of $\widehat{\alpha}_k$, and $n_j \in \mathbb{N}$ the corresponding multiplicity.

Moreover, as $k \to +\infty$, we may let,

$$\widehat{\alpha}_k \to \widehat{\alpha}_0, \;\; z_{j,k} \longrightarrow z_j, \;\; \text{with } \widehat{\alpha}_0(z_j) = 0, \;\; j \in \{1, \ldots, N\}.$$

Since the total multiplicity of each z_j adds up to the value: $4(\mathfrak{g}-1)$, we see that $\widehat{\alpha}_0$ cannot vanish anywhere else.

We let: $D^0 = div(\widehat{\alpha}_0)$, be the zero divisor of $\widehat{\alpha}_0$ and define:

$$Z^{(0)} = supp\; D^0$$

so that,

$$D^0 = \text{div}(\widehat{\alpha}_0) = \sum_{p \in Z^{(0)}} n_p p \;\text{ with }\; \sum_{p \in Z^{(0)}} n_p = 4(\mathfrak{g}-1)$$

Hence, we can identify the set Z_0 of points in $Z^{(0)}$ (possibly empty) which correspond to the limit of "collapsing" zeroes, as given by:

$$Z_0 := \{p \in Z^{(0)} : n_p > n_j \text{ with } p = z_j \text{ for some } 1 \leq j \leq N\}, \tag{4.4}$$

We define:

$$\xi_k = -(u_{t_k} - s_{t_k})$$

and let,

$$R_k = 8\|\widehat{\alpha}_k\|^2 \tag{4.5}$$

so that R_k and $|\nabla R_k|$ are uniformly bounded in X. Moreover, we have:

$$-\Delta\xi_k = R_k e^{\xi_k} - f_k \ \text{ in } \ X \qquad \text{and} \quad \int_X R_k e^{\xi_k} \le C \tag{4.6}$$

with $f_k := 2(1 - t_k e^{u_{t_k}})$ satisfying: $\|f_k\|_{L^\infty(X)} \le 2$.

So, we can further assume (recall (4.3)) that:

$$\begin{aligned} &f_k \to f_0 =: 2(1-\mu e^{w_0}) \ \text{ in } \ L^p(X), \ \ p > 1; \ \text{ as } \ k \to +\infty, \\ &\int_X f_0 = 2\rho([\beta]) > 0, \ \ \text{for} \ \ [\beta] \neq 0, \end{aligned} \tag{4.7}$$

and

$$R_k \to R_0 \ \text{ in } \ C^0(X) \ \text{ as } \ k \to +\infty, \ \ \text{with} \ R_0(z) = 8\|\hat{\alpha}_0\|^2. \tag{4.8}$$

We can apply Theorem 3 of [45] and deduce the following alternatives about the asymptotic behaviour of ξ_k:

Theorem 11 (Theorem 3 [45]) *Let ξ_k satisfy (4.6), (4.7) and (4.8). Then one of the following alternatives holds (along a subsequence):*
(i) (compactness) : $\xi_k \longrightarrow \xi_0$ in $C^2(X)$ with

$$-\Delta\xi_0 = R_0 e^{\xi_0} - f_0, \quad \text{in } \ X \tag{4.9}$$

(ii) (blow-up) : There exists a finite blow-up set

$$\mathcal{S} = \{x \in X \ : \ \exists\, x_k \to x \ \text{ and } \ \xi_k(x_k) \to +\infty, \ \text{ as } \ k \to +\infty\}$$

such that, ξ_k is uniformly bounded from above on compact sets of $X \setminus \mathcal{S}$ and, as $k \to +\infty$,

a) either (blow-up with concentration) :

$\xi_k \longrightarrow -\infty$ *uniformly on compact sets of* $X \setminus \mathcal{S}$,
$R_k e^{\xi_k} \rightharpoonup \sum_{x\in\mathcal{S}} \sigma(x)\delta_x$ *weakly in the sense of measures, and* $\sigma(x) \in 8\pi\mathbb{N}$.

In particular, $\int_X f_0\, dA \in 8\pi\mathbb{N}$,

$$\sigma(x) = 8\pi \ \text{ if } \ x \notin Z^{(0)} \ \text{ and } \ \sigma(x) = 8\pi(1+n_i) \ \text{ if } \ x = z_i \in Z^{(0)} \setminus Z_0. \tag{4.10}$$

Such an alternative always holds when $\mathcal{S} \setminus Z_0 \neq \emptyset$.

b) or (blow-up without concentration) :

$$\begin{aligned} &\xi_k \to \xi_0 \ \ in \ \ C^2_{loc}(X \setminus \mathcal{S}), \\ &R_k e^{\xi_k} \rightharpoonup R_0 e^{\xi_0} + \sum_{x \in \mathcal{S}} \sigma(x)\delta_x \ \ \textit{weakly in the sense of measures,} \\ &\sigma(x) \in 8\pi\mathbb{N} \quad and \ \ \mathcal{S} \subset Z_0, \end{aligned} \tag{4.11}$$

with ξ_0 satisfying:

$$-\Delta\xi_0 = R_0 e^{\xi_0} + \sum_{x \in \mathcal{S}} \sigma(x)\delta_x - f_0 \ \ in \ \ X.$$

It is useful to emphasise that if alternative (ii)-b) holds then blow-up occurs at points of "collapsing" zeroes . Actually, in this case the (CMC) immersion of X corresponding to the data $(u_t - s_t, \hat{\alpha}_t)$ can be taken to the limit, as $t \to 0^+$ along a sequence, to give a (CMC) immersion of X into a hyperbolic cone-manifold of dimension 3 (characterized by the presence of conical singularities along lines). See ([23]) for details about hyperbolic cone-manifolds. In particular, at the limit, the induced metric on X admits conical singularities at the blow-up points with conical angles an integral multiple of 8π, (and not the usual 4π due to our normalization (1.1) of the conformal factor) see e.g. [35, 36, 38, 39]. This fact reflects the presence of "ends" characterising Bryant surfaces into $\mathbb{H}^3$. In addition, in self-dual Guage Field Theory, it expresses the energy "quantization" property of vortex configurations.

Remark 4.1 If alternative (i) holds then F_0 is bounded from below in Λ and $(u_t, \eta_t) \to (u_0, \eta_0)$ in Λ, as $t \to 0^+$, with (u_0, η_0) the global minimum and only critical point of F_0, and $\rho([\beta]) = 4\pi(\mathfrak{g} - 1)$, see [46] for details.

If alternative (ii) holds with blow-up divisor D_0 then degree $D_0 \leq \frac{\rho([\beta])}{4\pi}$ (recall (4.1)) and "blow-up with concentration" (i.e. (ii)-a) occurs if and only if degree $D_0 = \frac{\rho([\beta])}{4\pi}$.

We investigate in more details the sequence ξ_k in case of blow-up (in the sense of alternative (ii) of Theorem 11) in order to arrive at the orthogonality relation (1.15) for the given class $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$.

For this purpose, let :

$$\mathcal{S} = \{x_1, \ldots, x_n\} \text{ and } D_0 = \sum_{l=1}^{n} m_j x_j \tag{4.12}$$

the blow-up set and corresponding blow-up divisor of ξ_k. Please notice that,

$$\xi_k = -(d_k - s_k) + O(1) \ \text{ on compact sets of } \ X \setminus \mathcal{S}. \tag{4.13}$$

and therefore "blow-up with concentration" occurs if and only if $d_k - s_k \longrightarrow +\infty$.

Lemma 4.2 *For any $r > 0$ sufficiently small and $\alpha \in C_2(X)$ we have:*

$$\int_X \beta \wedge \alpha = \int_X \beta_0 \wedge \alpha = e^{\frac{-s_k}{2}} \left(\sum_{l=1}^m \int_{B(x_l;r)} e^{\xi_k} < \alpha, \hat{\alpha_k} > dA \right) + o(1) \quad \text{as } k \to +\infty. \tag{4.14}$$

See (3.78) in [46].

So our effort in the following will be to estimate each of the integral terms in (4.14).

We start our "local" analysis around a given blow-up point. For fixed $j_0 \in \{1, ..., n\}$ we set:

$$x_0 = x_{j_0} \in \mathcal{S} \quad \text{with} \quad m_0 = m_{j_0}, \tag{4.15}$$

and for small $r > 0$ we let,

$$x_k \in X : \xi_k(x_k) := \max_{B(x_0;r)} \xi_k \to +\infty \text{ and } x_k \to x_0, \text{ as } k \to +\infty.$$

In $B(x_0; r)$, we introduce local "normal" holomorphic $z-$coordinate at x_0 centred at the origin, and denote by Ω_r the image of $B(x_0; r)$ in $\mathbb{C}$.

We can write:

$$\hat{\alpha}_k = \hat{a}_k(z)(dz)^2, \; \hat{\alpha}_0 = \hat{a}_0(z)(dz)^2 \; \text{ and for } \alpha \in C_2(X) \Rightarrow \; \alpha = a(z)(dz)^2, \tag{4.16}$$

with $\hat{a}_k$ and $\hat{a}$ and a holomorphic functions in Ω_r, and

$$\hat{a}_k \to \hat{a}_0 \; \text{ uniformly in } \Omega_r \quad \text{as } k \to +\infty. \tag{4.17}$$

Moreover, we let:

$$z_k \; \text{ the local expression of } x_k \text{ in the given z-coordinates (at } x_0\text{)}, \tag{4.18}$$

so that: $z_k \to 0, \quad \text{as } k \to \infty$.

The case $m_0 = 1$ has been handled in [46] (see also Proposition 3.1 in [50]).

Theorem 12 *Let $x_0 \in \mathcal{S}$ with $m_0 = 1$. Then, for $r > 0$ sufficiently small and for every $\alpha \in C_2(X)$, according to the local expressions in (4.16) at x_0, we have: $\hat{a}_k(z_k) \neq 0$ with z_k in (4.18) and*

$$\int_{B(x_0;r)} e^{\xi_k} < \alpha \; , \widehat{\alpha}_k > dA = \frac{\pi}{|\hat{a}_k(z_k)|}\left(a(0)\frac{\overline{\hat{a}_k(z_k)}}{|\hat{a}_k(z_k)|} + o(1)\right) + o_r(1) \quad \text{as } k \to +\infty \text{ and where } o_r(1) \to 0 \text{ as } r \to 0^+, \text{ uniformly on } k. \tag{4.19}$$

By [31] and [4] we know that, for $m_0 = 1$ either $x_0 \notin Z^{(0)}$ (i.e. x_0 is not a zero of $\hat{\alpha}_0$) or $x_0 \in Z_0$ (i.e. x_0 corresponds to a zero for $\hat{\alpha}_0$ of "collapsing" type). In either case, the asymptotic expression (4.19) relies in a crucial way upon the local pointwise estimates for the blow-up profile of ξ_k around x_0 as established in [3, 28] and

Corollary 3.1 of [45] respectively. Indeed, if we identify ξ_k with its local expression in the z-coordinates, there holds:

$$\xi_k(z+z_k) = \ln\left(\frac{e^{\xi_k(z_k)}}{\left(1+\frac{1}{8}W_k(0)e^{\xi_k(z_k)}|z|^2\right)^2}\right) + O(1) \quad \text{with} \quad W_k(0) > 0, \tag{4.20}$$
$$\text{for } z \in \Omega_{r,k} := \Omega_r - z_k,$$

When $m_0 \geq 2$, then necessarily: $x_0 \in Z^{(0)}$ (see [4, 31]), namely: $\hat{\alpha}_0(x_0) = 0$, or equivalently in local coordinates $\hat{a}_0(0) = 0$.

So, for suitable $s \in \{1, \dots, N\}$ we have:

$$\hat{a}_k(z) = \prod_{j=1}^{s}(z - \hat{p}_{j,k})^{n_j}\psi_k(z) \to \hat{a}_0(z) = z^{n_0}\psi_0(z), \text{ uniformly on } \Omega_r$$
$$n_0 := \sum_{j=1}^{s} n_j, \qquad \hat{p}_{j,k} \to 0 \quad \text{as } k \to +\infty, \tag{4.21}$$

where ψ_k, ψ_0 are holomorphic functions never vanishing in $\bar{B}_\delta$, such that:

$$\psi_k \to \psi_0 \text{ uniformly in } \bar{B}_\delta \quad \text{as } k \to +\infty. \tag{4.22}$$

We notice that, when the blow-up point is a zero but not of "collapsing" type (i.e. $x_0 \in Z^{(0)} \setminus Z_0$) then $s = 1$, and by [4] we know that: $m_0 = (n_0 + 1)$, and furthermore the blow-up profile of ξ_k has been extensively analysed in [5, 55, 56] and [57].

Therefore, we expect to find a suitable version of Theorem 12 in this case.

More generally, we can deal with the situation where,

$$m_0 \geq 2 \quad 1 \leq n_0 \leq 2(m_0 - 1), \quad (n_0 \text{ in } (4.21)). \tag{4.23}$$

In fact, if $a^{(n)}$ denotes the n-complex derivative of the holomorphic function a, then the following asymptotic expression holds:

Theorem 13 *Let $x_0 \in \mathcal{S}$ with blow-up mass m_0 and suppose that (4.23) holds.*
Then for every $\alpha \in Q(D)$ $\exists$ $\alpha_k \in Q(D_k)$: $D_k \to D$ and $\alpha_k \to \alpha$, as $k \to \infty$, such that:

$$\int_{B(x_0;r)} e^{\xi_k} < \alpha_k\ , \widehat{\alpha}_k > dA = \pi m_0 \frac{a^{(n_0)}(0)}{n_0!}\overline{\psi_0(0)} + o(1) \ \text{ as } k \to +\infty, \tag{4.24}$$

for $r > 0$ sufficiently small.

When the genus $\mathfrak{g} = 2$ or $\mathfrak{g} = 3$, then the asymptotic expansions in Theorem 12 and Theorem 13 suffice to derive the "orthogonality" condition in Theorem 3 and Theorem 4, see [46] and [49].

For higher genus one needs to handle the case where $n_0 > 2(m_0 - 1)$, which requires a much more refined asymptotic analysis in order to derive Theorem 10. We refer the interested reader to [50] for details.

In concluding this section, we wish to justify the origin of (4.24), and for simplicity, we concern ourselves with the case: $s = 1$ where we have: $m_0 = (n_0 + 1)$.

For the analysis of this case, we need to distinguish between "simple" and "not simple " blow-up, see [5, 24, 55, 56] and [57].

In case of "simple -blow up" then the profile of ξ_k is controlled by the solutions of the following "limiting" equation:

$$\begin{cases} -\Delta\varphi = |z - p_0|^{2n_0} e^{\varphi} \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |z - p_0|^{2n} e^{\varphi} < \infty \end{cases} \tag{4.25}$$

with suitable $p_0 \in \mathbb{R}^2$.

By the classification result in [41] (see also [8] and [9]) we know that necessarily,

$$\varphi(z) = \log\left(\frac{1}{\left(1 + \frac{1}{8(n_0+1)^2}|(z - p_0)^{n_0+1} - (-p_0)^{n_0+1}|^2\right)^2}\right) \tag{4.26}$$

and

$$\int_{\mathbb{R}^2} |z - p_0|^{2n_0} e^{\varphi} = 8\pi(n_0 + 1). \tag{4.27}$$

In view of such information, it is possible to derive the following expansion (see [49]):

$$\int_{B_\delta(x_k)} e^{\xi_k} < \alpha_k, \hat{\alpha_k}, > dA = \int_{\mathbb{R}^2} \left(\frac{(z - p_0)^{n_0} \bar{z}^{n_0}}{(1 + \frac{1}{8(n_0+1)^2}|(z - p_0)^{n_0+1} - (-p_0)^{n_0+1}|^2)^2}\right)$$
$$\frac{a^{(n_0)}(0)}{n_0!}\overline{\Psi_0}(0) + o(1), \tag{4.28}$$

as $k \to +\infty$.

Thus, in order to deduce (4.24), we need to establish the following identity (of independent interest).

Lemma 4.3 *For any $n \in \mathbb{N}$ there holds:*

$$\int_{\mathbb{R}^2} \left(\frac{(z - p_0)^n \bar{z}^n}{(1 + \frac{1}{8(n+1)^2}|(z - p_0)^{n+1} - (-p_0)^{n+1}|^2)^2}\right) \frac{i}{2} dz \wedge d\bar{z} = 8\pi(n + 1) \tag{4.29}$$

Proof In order to establish (4.32) we make the change of variable $w = z - p_0$ so that ,

$$\int_{\mathbb{R}^2} \left(\frac{(z - p_0)^n \bar{z}^n}{(1 + \frac{1}{8(n+1)^2}|(z - p_0)^{n+1} - (-p_0)^{n+1}|^2)^2}\right) \frac{i}{2} dz \wedge d\bar{z} = \tag{4.30}$$
$$\int_{\mathbb{R}^2} \left(\frac{w^n (\bar{w} + \bar{p}_0)^n}{(1 + \frac{1}{8(n+1)^2}|w^{n+1} - (-p_0)^{n+1}|^2)^2}\right) \frac{i}{2} dw \wedge d\bar{w} =$$

$$\int_{\mathbb{R}^2}\left(\frac{|w|^{2n}}{(1+\frac{1}{8(n+1)^2}|w^{n+1}-(-p_0)^{n+1}|^2)^2}\right)\frac{i}{2}dw\wedge d\bar{w}+$$

$$\sum_{l=0}^{n-1}\binom{n}{l}(\bar{p_0})^{n-l}\int_{\mathbb{R}^2}\left(\frac{w^n\bar{w}^l}{(1+\frac{1}{8(n+1)^2}|w^{n+1}-(-p_0)^{n+1}|^2)^2}\right)\frac{i}{2}dw\wedge d\bar{w}.$$

On the other hand:

$$I_l:=\int_{\mathbb{R}^2}\left(\frac{w^n\bar{w}^l}{(1+\frac{1}{8(n+1)^2}|w^{n+1}-(-p_0)^{n+1}|^2)^2}\right)\frac{i}{2}dw\wedge d\bar{w}=0\ \forall l=0,\dots n-1. \tag{4.31}$$

Indeed, if we use the change of variable $w=e^{\frac{2\pi i}{n+1}}z$ we find:

$$I_l=e^{\frac{2\pi i(n-l)}{n+1}}\int_{\mathbb{R}^2}\left(\frac{z^n\bar{z}^l}{(1+\frac{1}{8(n+1)^2}|z^{n+1}-(-p_0)^{n+1}|^2)^2}\right)=e^{\frac{2\pi i}{n+1}}I_l,$$

and since: $e^{\frac{2\pi i}{n+1}}\neq 1\ \forall l=0,\dots,n$ we derive (4.31). Thus, from (4.30) we conclude:

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\frac{(z-p_0)^n\bar{z}^n}{(1+\frac{1}{8(n+1)^2}|(z-p_0)^{n+1}-(-p_0)^{n+1}|^2)^2}\right)\frac{i}{2}dz\wedge d\bar{z}\\&=\int_{\mathbb{R}^2}\left(\frac{|z|^{2n}}{(1+\frac{1}{8(n+1)^2}|(z-p_0)^{n+1}-(-p_0)^{n+1}|^2)^2}\right)\frac{i}{2}dz\wedge d\bar{z}\\&=8\pi(n+1)\end{aligned}$$

and by recalling (4.27), we derive (4.32) . □

In case of "non-simple blow up" the expression (4.28) must be modified accordingly, and the following holds:

$$\begin{aligned}&\int_{B_\delta(x_k)}e^{\xi_k}<\alpha_k,\hat{\alpha_k}>dA=8\pi\left(\sum_{k=0}^{n_0}z_0^{n_0}e^{\frac{2\pi ikn_0}{n_0+1}}\left(\bar{z_0}e^{-\frac{2\pi ik}{n_0+1}}+\bar{z_0}\right)^{n_0}\right)\\&\frac{a^{(n_0)}(0)}{n!}\overline{\Psi_0}(0)+o(1)\text{ as }k\to+\infty,\\&\text{where }z_0\in\mathbb{C}:|z_0|=1.\end{aligned} \tag{4.32}$$

In this case, we prove:

Lemma 4.4 *Let* $z_0\in\mathbb{C}:\ |z_0|=1$, *and* $n\in\mathbb{N}$, *then*

$$\sum_{k=0}^{n}z_0^ne^{\frac{2\pi ikn}{n+1}}\left(\bar{z_0}e^{\frac{-2\pi ik}{n+1}}+\bar{z_0}\right)^n=n+1$$

Proof We observe:

$$\sum_{k=0}^{n}z_0^ne^{\frac{2\pi ikn}{n+1}}\left(\bar{z_0}e^{\frac{-2\pi ik}{n+1}}+\bar{z_0}\right)^n=\sum_{k=0}^{n}\left(1+e^{\frac{2\pi ik}{n+1}}\right)^n=$$

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}e^{\frac{2\pi ikj}{n+1}}=\sum_{j=0}^{n}\binom{n}{j}\left(\sum_{k=0}^{n}e^{\frac{2\pi ikj}{n+1}}\right). \tag{4.33}$$

Claim:

$$\sum_{k=0}^{n}e^{\frac{2\pi ikj}{n+1}}=0 \quad \forall j=1,\dots,n. \tag{4.34}$$

To prove (4.34) let $j=dj_1$ and $n+1=dm_1$ where d is the greatest common divisor of j and $n+1$, and so $m_1>1$. Then

$$\sum_{k=0}^{n}e^{\frac{2\pi ikj}{n+1}}=\sum_{k=0}^{n}e^{\frac{2\pi ikj_1}{m_1}}=\sum_{k=0}^{m_1-1}e^{\frac{2\pi ikj_1}{m_1}}+\sum_{k=m_1}^{2m_1-1}e^{\frac{2\pi ikj_1}{m_1}}+\dots+\sum_{k=(d-1)m_1}^{n}e^{\frac{2\pi ikj}{m_1}}=$$

$$\left(\sum_{k=0}^{m_1-1}e^{\frac{2\pi ikj_1}{m_1}}\right)\left(\sum_{l=0}^{d-1}e^{2\pi il}\right)=d\sum_{k=0}^{m_1-1}e^{\frac{2\pi ikj_1}{m_1}}.$$

So we need to show that,

$$\sum_{k=0}^{m_1-1}e^{\frac{2\pi ikj_1}{m_1}}=0.$$

If we have: $0\le k_1\le k_2\le m_1-1$ and $e^{\frac{2\pi ij_1k_1}{m_1}}=e^{\frac{2\pi ij_1k_2}{m_1}}$ then necessarely $(k_2-k_1)j_1$ must be a multiple of m_1. But since j_1 and m_1 are co-prime, then k_2-k_1 is a multiple of m_1, and therefore $k_1=k_2$. As a consequence, $\{e^{\frac{2\pi ikj_1}{m_1}}\}_{0\le k\le m_1-1}$ is the set of all distinct roots of the polynomial $z^{m_1}-1$. At this point, we recall that for given polynomial of degree $r: P(z)=z^r+a_{r-1}z^{r-1}+\dots+a_0=(z-z_1)\dots(z-z_r)$ we have: $a_{r-1}=-\sum_{k=0}^{r-1}z_k$, (it can be easily proved by induction on r), and in this way the claim follows.

In view of (4.34), we conclude that only the term with $j=0$ survives in (4.33) and the desired identity is proved. □

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