Infinitely many solutions for elliptic equations with non-symmetric nonlinearities

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August 24, 2022

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Abstract. - We deal with the existence of infinitely many solutions for a class of elliptic problems with non-symmetric nonlinearities. Our result, which is motivated by a well known conjecture formulated by A. Bahri and P.L. Lions, suggests a new approach to tackle these problems. The proof is based on a method which does not require to use techniques of deformation from the symmetry and may be applied to more general non-symmetric problems.

MSC: 35J20, 58E05.

Keywords: nonlinear elliptic equations, multiplicity of solutions, critical point theory, non-symmetric problems.

1 Introduction

Let us consider the problem

$$-\Delta u = |u|^{p-1}u + w \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, with $n \geq 1$, $w \in L^2(\Omega)$, $p > 1$ and $p < \frac{n+2}{n-2}$ when $n \geq 3$.
If $w \not\equiv 0$ in Ω, the corresponding energy functional $E : H^1_0(\Omega) \to \mathbb{R}$, defined by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx - \int_\Omega w\, u\, dx \tag{1.2}$$

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is not even, so the equivariant Lusternik-Schnirelmann theory for $\mathbb{Z}_2$-symmetric sets cannot be applied to find infinitely many solutions as in the case $w \equiv 0$ (see for instance [1, 3, 18, 19, 27, 28, 32, 34] and also [9, 17] for a more general framework).

In the case $w \not\equiv 0$ in Ω, a natural question (which goes back to the beginning of the eighties) is wether the infinite number of solutions still persists under perturbation.

A detailed analysis was originally carried on in [2, 3, 5–8, 25, 29, 30, 33, 35, 39] by Ambrosetti, Bahri, Berestycki, Ekeland, Ghoussoub, Krasnoselskii, Lions, Marino, Prodi, Rabinowitz, Struwe and Tanaka by introducing new perturbation methods. In particular, this question was raised to the attention by Rabinowitz also in his monograph on minimax methods (see [34, Remark 10.58]).

In [2] Bahri proved that, if $n \geq 3$ and $1 < p < \frac{n}{n-2}$, then there exists an open dense set of w in $L^2(\Omega)$ such that problem (1.1) admits infinitely many solutions. In [6] Bahri and Lions proved that, if $n \geq 3$ and $1 < p < \frac{n}{n-2}$, then problem (1.1) admits infinitely many solutions for every $w \in L^2(\Omega)$.

These results suggest the following conjecture, proposed by Bahri and Lions in [8]: the multiplicity result obtained in [8] holds also under the more general assumption $1 < p < \frac{n+2}{n-2}$.

More recently, a new approach to tackle the break of symmetry in elliptic problems has been developed by Bolle, Chambers, Ghoussoub and Tehrani (see [10, 11, 15], which include also applications to more general nonlinear problems). However that approach did not allow to solve the Bahri-Lions conjecture.

In the present paper we describe a new possible method to approach this problem. By minimizing the energy functional E in suitable subsets of $H_0^1(\Omega)$, we obtain infinitely many functions that present an arbitrarily large number of nodal regions having a prescribed structure (a check structure). Their energy tends to infinity as the number of nodal regions tends to infinity. Moreover, these functions satisfy equation (1.1) in each nodal region when the number of nodal regions is large enough (see Proposition 2.4) and they are solutions of problem (1.1) when, in addition, they satisfy the assumptions of Proposition 2.5.

The idea is to trying to piece together solutions of Dirichlet problems in subdomains of Ω chosen in a suitable way. This idea has been first used by Struwe in earlier papers (see [35–37] and references therein). In the present paper we consider as nodal regions subdomains of Ω that are suitable deformations of cubes. When the sizes of these cubes are all small enough, the nodal functions with check structure that we obtain seem to present suitable stability properties so that they persist when the problem (1.1) is perturbed by the term w. The deformations of the nodal regions we use to construct solutions of problem (1.1). are obtained in the present paper by considering a class of Lipshitz maps. It is interesting to observe that such a class also appeared in some recent works of Rabinowitz and Byeon (see [13, 14] and the references therein) concerning a rather different problem: construct solutions having certain prescribed patterns for an Allen-Cahn model equation. Also in that papers, as in the present one,

Lipschitz condition is combined with the structure of $\mathbb{Z}^n$ and the covering of $\mathbb{R}^n$ by cubes with vertices in $\mathbb{Z}^n$.
In order to verify the assumptions of Proposition 2.5, we need a technical condition (condition (2.53)). In Lemma 2.9 we show that this condition is satisfied, for example, in the case $n = 1$ (the proof may be also adapted to deal with radial solutions in domains Ω having radial symmetry).
Indeed, in dimension $n = 1$, a more general result was obtained by Ehrmann in [23] (see also [24, 26] for related results). Here it is proved that the ordinary differential equation

$$-u''(x) = f(u(x)) + w(x) \qquad \text{for } x \in (0,1),\ u(0) = u(1) = 0 \tag{1.3}$$

has infinitely many distinct solutions when f is a function with superlinear growth satisfying quite general assumptions. However, the method here used relies on a shooting argument, typical of ordinary differential equations, combined with counting the oscillations of the solutions in the interval $(0, 1)$. Therefore, this method, which gives the existence of solutions having a sufficiently large number of zeroes in dimension $n = 1$, cannot be extended to higher dimensions.
On the contrary, in the present paper we use a method which is more similar to the one introduced by Nehari in [31], that can be in a natural way extended to the case $n > 1$. In fact, for example, Nehari's work was followed up by Coffman who studied an analogous problem for partial differential equations (see [18, 19]). Independently, this problem was also studied by Hempel (see [27, 28]).
More recently, the method introduced by Nehari has been also used by Conti, Terracini and Verzini to study optimal partition problems in n-dimensional domains and related problems: in particular, existence of minimal partitions and extremality conditions, behaviour of competing species systems with large interactions, existence of changing sign solutions for superlinear elliptic equations, etc. (see [20–22, 40]).
Notice that Nehari's work deal with an odd differential operator, so the corresponding energy functional is even. Moreover, Nehari proves that for every positive integer k there exists a solution having exactly k zeroes. On the contrary, in the present paper (as Ehrmann in [23]) we find only solutions with a large number of zeroes; moreover, we prove that, for all w in $L^2(\Omega)$, the zeroes tend to be uniformly distributed in all of the domain Ω as their number tends to infinity (see Lemmas 2.9 and 3.2) The reason is that, when $w \not\equiv 0$, the Nehari type argument we use in the proof works only when the sizes of all the nodal regions are small enough, so their number is sufficiently large.
In order to show that our existence result is sharp, we prove also that the term w in problem (1.1) can be chosen in such a way that the problem does not have solutions with a small number of nodal regions. More precisely, in the case $n = 1$ we show that for all positive integer h there exists w_h in $L^2(\Omega)$ such that every solution of problem (1.1) with $w = w_h$ has at least h zeroes (see Corollary 3.6). Indeed, we show that for all $n \geq 1$ and for every eigenfunction e_k of the Laplace operator $-\Delta$ in $H^1_0(\Omega)$ there

exists $\bar{w}_k$ in $L^2(\Omega)$ such that every solution u of problem (1.1) with $w = \bar{w}_k$ must have the sign related to the sign of e_k in the sense that every nodal region of e_k has a subset where u and e_k have the same sign (see Proposition 3.5).
In the case $n > 1$, condition (2.53) seems to be more difficult to be verified because the class of all Lipschitz deformations of the nodal regions might result too large, as we explain in Remark 3.1. Therefore, in this case a useful idea might be to restrict the class of the admissible deformations so that we can verify a condition analogous to (2.53) and apply our method to construct nodal solutions having check structure. For example, as we describe in Remark 3.1, we can fix a suitable Lipschitz map $T_0 : \overline{\Omega} \to \overline{\Omega}$ and consider nodal regions deformed by Lipschitz maps suitably close to T_0. It is clear that, in order to apply our method, we need now to verify a condition analogous to (2.53) (that is condition (3.7)) which holds or fails depending on the choice of T_0 and of the neighborhood of deformations close to T_0. In a similar way, for example, we prove that if Ω is a cube of $\mathbb{R}^n$ with $n > 1$, $p > 1$, $p < \frac{n+2}{n-2}$ if $n > 2$, for all w in $L^2(\Omega)$ there exist infinitely many solutions $u_k(x)$ of problem (1.1) such that the nodal regions of the function $u_k\left(\frac{x}{k}\right)$, after translations, tend to the cube as $k \to \infty$ (the proof will be reported in a paper in preparation). We believe that this result may be extended to every interval or pluri-interval of $\mathbb{R}^n$ with $n > 1$ and then to every bounded domain Ω by a suitable choice of the deformation T_0, related to the geometrical properties of the domain Ω.
Let us point out that our method does not require techniques of deformation from the symmetry and may be applied to more general problems: for example, when the nonlinear term $|u|^{p-1}u$ is replaced by $c_+(u^+)^p - c_-(u^-)^p$ with c_+ and c_- two positive constants (see Lemma 3.2), in case of different, nonhomogeneous boundary conditions and even in case of nonlinear elliptic equations involving critical Sobolev exponents.

Acknowledgement. The authors are very much grateful to Professor P.H. Rabinowitz for several helpful comments, suggestions and informations on this work and on related subjects.

2 Existence of infinitely many nodal solutions

In order to find infinitely many solutions with an arbitrarily large number of nodal regions, we proceed as follows.
Let us set

$$\begin{aligned} C_0 &= \{x \in \mathbb{R}^n \ : \ 0 < x_i < 1 \ \text{ for } i = 1, \dots, n\}, \\ C_z &= z + C_0, \quad \sigma(z) = (-1)^{\sum_{i=1}^n z_i} \qquad \forall z \in \mathbb{Z}^n, \end{aligned} \tag{2.1}$$

$$Z_k = \left\{ z \in \mathbb{Z}^n \ : \ \frac{1}{k} C_z \subset \Omega \right\}, \qquad P_k = \bigcup_{z \in Z_k} \frac{1}{k} \overline{C}_z, \qquad \forall k \in \mathbb{N}. \tag{2.2}$$

Notice that there exists k_Ω in $\mathbb{N}$ such that $Z_k \neq \emptyset$ $\forall k \geq k_\Omega$.

For all subsets P, Q of $\mathbb{R}^n$ and for all $L \geq 1$, let us denote by $\mathcal{C}_L(P, Q)$ the set of all the functions $T : P \to Q$ such that

$$\frac{1}{L}\,|x-y| \leq |T(x)-T(y)| \leq L\,|x-y| \qquad \forall x, y \in P. \tag{2.3}$$

For all $k \geq k_\Omega$, $z \in Z_k$, $L \geq 1$, $T \in \mathcal{C}_L(P_k, \overline{\Omega})$ let us set

$$E^T_{k,z} = \inf \left\{ E(u) \ : \ u \in H^1_0\left(T\left(\frac{1}{k}\,C_z\right)\right),\ \int_{T\left(\frac{1}{k}\,C_z\right)} |u|^{p+1} dx = 1 \right\}. \tag{2.4}$$

Since $p < \frac{n+2}{n-2}$ when $n \geq 3$, one can easily verify that the infimum in (2.4) is achieved. Moreover, for all $L \geq 1$ and $k \geq k_\Omega$, also the infimum

$$\inf\{E^T_{k,z} \ : \ z \in Z_k,\ T \in C_L(P_k, \overline{\Omega})\} \tag{2.5}$$

is achieved (as one can prove by standard arguments using Ascoli-Arzelà Theorem) and the following lemma holds.

Lemma 2.1 *For all $L \geq 1$, we have*

$$\lim_{k\to\infty} \min\{E^T_{k,z} \ : \ z \in Z_k,\ T \in \mathcal{C}_L(P_k, \overline{\Omega})\} = \infty. \tag{2.6}$$

Proof For all $L \geq 1$ and $k \geq k_\Omega$, let us choose $z_k \in Z_k$, $T_k \in \mathcal{C}_L(P_k, \overline{\Omega})$ and $\bar{u}_k \in H^1_0\left(T_k\left(\frac{1}{k} C_{z_k}\right)\right)$ such that

$$\int_{T_k\left(\frac{1}{k} C_{z_k}\right)} |\bar{u}_k|^{p+1} dx = 1 \ \text{ and } \ E(\bar{u}_k) = E^{T_k}_{k,z_k} = \min\{E^T_{k,z} \ : \ z \in Z_k,\ T \in \mathcal{C}_L(P_k, \overline{\Omega})\}. \tag{2.7}$$

We say that

$$\lim_{k\to\infty} \int_{T_k(\frac{1}{k}\,C_{z_k})} |\nabla \bar{u}_k|^2 dx = \infty. \tag{2.8}$$

In fact, arguing by contradiction, assume that

$$\liminf_{k\to\infty} \int_{T_k(\frac{1}{k}\,C_{z_k})} |\nabla \bar{u}_k|^2 dx < \infty. \tag{2.9}$$

It follows that (up to a subsequence) $(\bar{u}_k)_k$ is bounded in $H^1_0(\Omega)$ and there exists a function $\bar{u} \in H^1_0(\Omega)$ such that $\bar{u}_k \to \bar{u}$, as $k \to \infty$, weakly in $H^1_0(\Omega)$, in $L^{p+1}(\Omega)$, and almost everywhere in Ω (here $\bar{u}_k$ is extended by the value 0 in $\Omega \setminus T_k(\frac{1}{k}\,C_{z_k})$). Since $\text{meas}\left(T_k(\frac{1}{k}\,C_{z_k})\right) \to 0$ as $k \to \infty$, from the almost everywhere convergence we obtain $\bar{u} \equiv 0$ in Ω, which is a contradiction because $\bar{u}_k \to \bar{u}$ in $L^{p+1}(\Omega)$ and (2.7) holds for all $k \geq k_\Omega$. Thus (2.8) is proved.

Notice that

$$E_{k,z_k}^{T_k} = \frac{1}{2}\int_{T_k\left(\frac{1}{k}\,C_{z_k}\right)} |\nabla \bar{u}_k|^2 dx - \frac{1}{p+1} - \int_{T_k\left(\frac{1}{k}\,C_{z_k}\right)} \bar{u}_k w\, dx \qquad \forall k \geq k_\Omega \tag{2.10}$$

where

$$\begin{aligned} \left|\int_{T_k(\frac{1}{k}\,C_{z_k})} \bar{u}_k\, w\, dx\right| &\leq \left(\int_{T_k(\frac{1}{k}\,C_{z_k})} \bar{u}_k^2 dx\right)^{\frac{1}{2}} \left(\int_{T_k(\frac{1}{k}\,C_{z_k})} w^2 dx\right)^{\frac{1}{2}} \\ &\leq \left[\text{meas}\left(T_k(\tfrac{1}{k}\,C_{z_k})\right)\right]^{\frac{1}{2}-\frac{1}{p+1}} \left(\int_\Omega w^2 dx\right)^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

As a consequence, for all $k \geq k_\Omega$ we obtain

$$E_{k,z_k}^{T_k} \geq \frac{1}{2}\int_{T_k(\frac{1}{k}\,C_{z_k})} |\nabla \bar{u}_k|^2 dx - \frac{1}{p+1} - \left(\int_\Omega w^2 dx\right)^{\frac{1}{2}} \cdot \left[\text{meas}\left(T_k\left(\frac{1}{k}\,C_{z_k}\right)\right)\right]^{\frac{1}{2}-\frac{1}{p+1}}, \tag{2.12}$$

and, as $k \to \infty$,

$$\lim_{k\to\infty} E_{k,z_k}^{T_k} = \infty \tag{2.13}$$

which completes the proof.

q.e.d.

Corollary 2.2 *For all $L \geq 1$ there exists $k(L) \geq k_\Omega$ such that for all $k \geq k(L)$, $z \in Z_k$ and $T \in \mathcal{C}_L(P_k, \overline{\Omega})$ the minimum*

$$\min\left\{E(u) \ : \ u \in H_0^1\left(T\left(\frac{1}{k}\,C_z\right)\right),\ \int_{T\left(\frac{1}{k}\,C_z\right)} |u|^{p+1} dx < 1\right\} \tag{2.14}$$

is achieved by a unique minimizing function $\tilde{u}_{k,z}^T$. Moreover, we have

$$\lim_{k\to\infty} \sup\left\{\int_{T\left(\frac{1}{k}\,C_z\right)} |\nabla \tilde{u}_{k,z}^T|^2 dx \ : \ z \in Z_k,\ T \in \mathcal{C}_L(P_k, \overline{\Omega})\right\} = 0. \tag{2.15}$$

Proof As a consequence of Lemma 2.1, for all $L \geq 1$ there exists $k(L) \geq k_\Omega$ such that

$$0 < \min\left\{E(u) \ : \ u \in H_0^1\left(T\left(\frac{1}{k}\,C_z\right)\right),\ \int_{T\left(\frac{1}{k}\,C_z\right)} |u|^{p+1} dx = 1\right\}$$

$$\forall k \geq k(L),\ \forall z \in Z_k,\ \forall T \in \mathcal{C}_L(P_k, \overline{\Omega}). \tag{2.16}$$

On the other hand,

$$\inf\left\{E(u)\ :\ u\in H_0^1\left(T\left(\frac{1}{k}\,C_z\right)\right),\ \int_{T\left(\frac{1}{k}\,C_z\right)}|u|^{p+1}dx<1\right\}\le 0$$
$$\forall k\ge k_\Omega,\ \forall z\in Z_k,\ \forall T\in\mathcal{C}_L(P_k,\overline{\Omega})\tag{2.17}$$

because $E(u)=0$ for $u\equiv 0$ in $T\left(\frac{1}{k}\,C_z\right)$.
Now, let us consider a minimizing sequence for the infimum in (2.17). Since it is bounded in $L^{p+1}\left(\frac{1}{k}\,C_z\right)$, we infer from (2.17) that it is bounded also in $H_0^1\left(\frac{1}{k}\,C_z\right)$. Therefore, since $p<\frac{n+2}{n-2}$ when $n\ge 3$, one can prove by standard arguments that (up to a subsequence) it converges to a function $\tilde{u}_{k,z}^T\in H_0^1\left(T\left(\frac{1}{k}\,C_z\right)\right)$ such that $\int_{T\left(\frac{1}{k}\,C_z\right)}|\tilde{u}_{k,z}^T|^{p+1}dx<1$ and

$$E(\tilde{u}_{k,z}^T)=\min\left\{E(u)\ :\ u\in H_0^1\left(T\left(\frac{1}{k}\,C_z\right)\right),\ \int_{T\left(\frac{1}{k}\,C_z\right)}|u|^{p+1}dx<1\right\}.\tag{2.18}$$

In order to prove (2.15) we argue by contradiction and assume that

$$\limsup_{k\to\infty}\sup\left\{\int_{T\left(\frac{1}{k}\,C_z\right)}|\nabla\tilde{u}_{k,z}^T|^2dx\ :\ z\in Z_k,\ T\in\mathcal{C}_L(P_k,\overline{\Omega})\right\}>0.\tag{2.19}$$

Then, for all $k\ge k(L)$ there exist $z_k\in Z_k$ and $T_k\in\mathcal{C}_L(P_k,\overline{\Omega})$ such that (up to a subsequence)

$$\lim_{k\to\infty}\int_{T_k\left(\frac{1}{k}\,C_{z_k}\right)}|\nabla\tilde{u}_{k,z_k}^{T_k}|^2dx>0.\tag{2.20}$$

Since $E(\tilde{u}_{k,z_k}^{T_k})\le 0$ and the sequence $\tilde{u}_{k,z_k}^{T_k}$ (extended by the value zero outside $\frac{1}{k}\,C_{z_k}$) is bounded in $L^{p+1}(\Omega)$, we infer that it is bounded also in $H_0^1(\Omega)$. We say that, as a consequence, $\tilde{u}_{k,z_k}^{T_k}\to 0$ as $k\to\infty$ in $L^{p+1}(\Omega)$. In fact, since the sequence $\left(\tilde{u}_{k,z_k}^{T_k}\right)_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, it converges weakly in $H_0^1(\Omega)$, in $L^{p+1}(\Omega)$ and a.e. in Ω to a function $\tilde{u}\in H_0^1(\Omega)$. Since $\lim\limits_{k\to\infty}\operatorname{meas}\left(\frac{1}{k}\,C_{z_k}\right)=0$, we can say that $\tilde{u}\equiv 0$ in Ω. Thus, $\tilde{u}_{k,z_k}^{T_k}\to 0$ as $k\to\infty$ in $L^{p+1}(\Omega)$. Therefore, taking into account that

$$E\left(\tilde{u}_{k,z_k}^{T_k}\right)=\frac{1}{2}\int_\Omega|\nabla\tilde{u}_{k,z_k}^{T_k}|^2dx-\frac{1}{p+1}\int_\Omega|\tilde{u}_{k,z_k}^{T_k}|^{p+1}dx-\int_\Omega w\,\tilde{u}_{k,z_k}^{T_k}\,dx\le 0\tag{2.21}$$

it follows that $\tilde{u}_{k,z_k}^{T_k}\to 0$ also in $H_0^1(\Omega)$ in contradiction with (2.20).
Thus, we can conclude that (2.15) holds. Finally, notice that $\tilde{u}_{k,z}^T$ is the unique minimizing function for (2.14) because the functional E is strictly convex in a suitable neighborhood of zero. So the proof is complete.

q.e.d.

Taking into account Corollary 2.2, for all $k \geq k(L)$, $z \in Z_k$ and $T \in \mathcal{C}_L(P_k, \overline{\Omega})$ we can consider a minimizing function $\tilde{u}^T_{k,z}$ for the minimum (2.14). Moreover, since $p > 1$, for all $u \in H^1_0\left(\frac{1}{k}\, C_z\right)$ there exists the maximum

$$M(u) = \max\left\{E\left(\tilde{u}^T_{k,z} + t(u - \tilde{u}^T_{k,z})\right) \ : \ t \geq 0\right\} \tag{2.22}$$

and $M(u) \geq E^T_{k,z}$ when $u \not\equiv \tilde{u}^T_{k,z}$ in $T\left(\frac{1}{k}\, C_z\right)$.

Lemma 2.3 *For all $k \geq k(L)$, $z \in Z_k$ and $T \in \mathcal{C}_L(P_k, \overline{\Omega})$, there exists a function $u^T_{k,z}$ in $H^1_0\left(T\left(\frac{1}{k}\, C_z\right)\right)$ such that $u^T_{k,z} \not\equiv \tilde{u}^T_{k,z}$, $\sigma(z)[u^T_{k,z} - \tilde{u}^T_{k,z}] \geq 0$ in $T\left(\frac{1}{k}\, C_z\right)$ and*

$$E(u^T_{k,z}) = M(u^T_{k,z}) = \min\left\{M(u) \ : \ u \in H^1_0\left(T\left(\frac{1}{k}\, C_z\right)\right),\ u \not\equiv \tilde{u}^T_{k,z},\right.$$
$$\left. \sigma(z)[u - \tilde{u}^T_{k,z}] \geq 0 \quad in \quad T\left(\frac{1}{k}\, C_z\right)\right\}. \tag{2.23}$$

Moreover, we have $E(u^T_{k,z}) \geq E^T_{k,z}$.

Proof Let us consider a minimizing sequence $(u_i)_{i\in\mathbb{N}}$ for the minimum (2.23). Whitout any loss of generality, we can assume that

$$\int_{T\left(\frac{1}{k}\, C_z\right)} |u_i - \tilde{u}^T_{k,z}|^{p+1} dx = 1 \qquad \forall i \in \mathbb{N}. \tag{2.24}$$

It follows that this sequence is bounded in $H^1_0\left(T\left(\frac{1}{k}\, C_z\right)\right)$. Therefore, since $p < \frac{n+2}{n-2}$ when $n \geq 3$, up to a subsequence it converges weakly in H^1_0, in L^{p+1} and a.e. to a function $\hat{u} \in H^1_0\left(T\left(\frac{1}{k}\, C_z\right)\right)$.
Notice that the L^{p+1} convergence and (2.24) imply

$$\int_{T\left(\frac{1}{k}\, C_z\right)} |\hat{u} - \tilde{u}^T_{k,z}|^{p+1} dx = 1, \tag{2.25}$$

so $\hat{u} \not\equiv \tilde{u}^T_{k,z}$. Moreover, the a.e. convergence implies $\sigma(z)[\hat{u} - \tilde{u}^T_{k,z}] \geq 0$ in $T\left(\frac{1}{k}\, C_z\right)$. We say that, indeed, the convergence is strong in $H^1_0\left(T\left(\frac{1}{k}\, C_z\right)\right)$. In fact, arguing by contradiction, assume that (up to a subsequence)

$$\int_{T\left(\frac{1}{k}\, C_z\right)} |\nabla \hat{u}|^2 dx < \lim_{i\to\infty} \int_{T\left(\frac{1}{k}\, C_z\right)} |\nabla u_i|^2 dx. \tag{2.26}$$

As a consequence, we obtain $M(\hat{u}) < \lim\limits_{i\to\infty} M(u_i)$ which is a contradiction because $\hat{u} \not\equiv \tilde{u}^T_{k,z}$ and $(u_i)_{i\in\mathbb{N}}$ is a minimizing sequence for (2.23) so that $M(\hat{u}) \geq \lim\limits_{i\to\infty} M(u_i)$.

Therefore, we can conclude that $u_i \to \hat{u}$ in $H^1_0\left(T\left(\frac{1}{k}\,C_z\right)\right)$ and $\lim_{i\to\infty} M(u_i) = M(\hat{u})$. Since $p > 1$, there exists $\hat{t} > 0$ such that

$$E(\tilde{u}^T_{k,z} + \hat{t}(\hat{u} - \tilde{u}^T_{k,z})) = M(\hat{u}). \tag{2.27}$$

Thus, all the assertions of Lemma 2.3 hold with $u^T_{k,z} = \tilde{u}^T_{k,z} + \hat{t}(\hat{u} - \tilde{u}^T_{k,z})$.

q.e.d.

Proposition 2.4 *There exists $k_1(L) \geq k(L)$ such that for all $k \geq k_1(L)$, $z \in Z_k$ and $T \in \mathcal{C}_L(P_k, \overline{\Omega})$ the function $u^T_{k,z}$ is a solution of the Dirichlet problem*

$$-\Delta u = |u|^{p-1}u + w \quad \text{in } T\left(\frac{1}{k}\,C_z\right), \qquad u = 0 \quad \text{on } \partial T\left(\frac{1}{k}\,C_z\right). \tag{2.28}$$

Proof It is clear that the function $\tilde{u}^T_{k,z}$ (local minimum of the functional E) is a solution of the Dirichlet problem (2.28). In order to prove that, for k large enough, also $u^T_{k,z}$ is solution of the same problem, let us consider the function $G : T\left(\frac{1}{k}\,C_z\right) \times \mathbb{R} \to \mathbb{R}$ such that $G(x,\cdot) \in \mathcal{C}^2(\mathbb{R})$ $\forall x \in T\left(\frac{1}{k}\,C_z\right)$ and

$$\begin{aligned} G(x,t) &= \tfrac{|t|^{p+1}}{p+1} + w(x)\,t \quad && \text{if } \sigma(z)[t - \tilde{u}^T_{k,z}(x)] \geq 0, \\ \tfrac{\partial^2 G}{\partial t^2}\,(x,t) &= \tfrac{\partial^2 G}{\partial t^2}\,(x, \tilde{u}^T_{k,z}(x)) \quad && \text{if } \sigma(z)[t - \tilde{u}^T_{k,z}(x)] \leq 0. \end{aligned} \tag{2.29}$$

Moreover, let us set $g(x,t) = \frac{dG}{dt}(x,t)$.
Then, consider the functional $E_{k,z,T} : H^1_0\left(T\left(\frac{1}{k}\,C_z\right)\right) \to \mathbb{R}$ defined by

$$E_{k,z,T}(u) = \frac{1}{2}\int_{T\left(\frac{1}{k}\,C_z\right)} |\nabla u|^2 dx - \int_{T\left(\frac{1}{k}\,C_z\right)} G(x,u)\,dx. \tag{2.30}$$

Let us assume, for example, $\sigma(z) = 1$ (in a similar way one can argue when $\sigma(z) = -1$). One can verify by direct computation that for all $u \not\equiv \tilde{u}^T_{k,z}$ there exists a unique $t_u > 0$ such that

$$E'_{k,z,T}(\tilde{u}^T_{k,z} + t_u(u - \tilde{u}^T_{k,z}))[u - \tilde{u}^T_{k,z}] = 0 \tag{2.31}$$

if and only if $(u - \tilde{u}^T_{k,z}) \vee 0 \not\equiv 0$. In this case $E'_{k,z,T}(\tilde{u}^T_{k,z} + t(u - \tilde{u}^T_{k,z})[u - \tilde{u}^T_{k,z}]$ is positive for $t \in]0, t_u[$ and negative for $t > t_u$, so we have

$$E_{k,z,T}(\tilde{u}^T_{k,z} + t_u(u - \tilde{u}^T_{k,z})) = \max\{E_{k,z,T}(\tilde{u}^T_{k,z} + t(u - \tilde{u}^T_{k,z}))\ :\ t > 0\}. \tag{2.32}$$

Moreover, we have

$$E''_{k,z,T}(\tilde{u}^T_{k,z} + t_u(u - \tilde{u}^T_{k,z}))[u - \tilde{u}^T_{k,z}, u - \tilde{u}^T_{k,z}] < 0. \tag{2.33}$$

Taking into account that $\tilde{u}^T_{k,z}$ is solution of problem (2.28), we obtain by direct computation

$$\begin{aligned} E_{k,z,T}(\tilde{u}^T_{k,z}+t(u-\tilde{u}^T_{k,z})) &= E_{k,z,T}(\tilde{u}^T_{k,z}+t[(u-\tilde{u}^T_{k,z})\vee 0]) \\ &\quad +\frac{t^2}{2}\int_{T\left(\frac{1}{k}C_z\right)}|\nabla[(u-\tilde{u}^T_{k,z})\wedge 0]|^2dx \\ &\quad -\frac{t^2}{2}\,p\int_{T\left(\frac{1}{k}C_z\right)}|\tilde{u}^T_{k,z}|^{p-1}[(u-\tilde{u}^T_{k,z})\wedge 0]^2dx \qquad \forall t>0. \end{aligned} \tag{2.34}$$

Notice that

$$\begin{aligned} &\int_{T\left(\frac{1}{k}C_z\right)}|\tilde{u}^T_{k,z}|^{p-1}[(u-\tilde{u}^T_{k,z})\wedge 0]^2dx \\ &\qquad \le \left(\int_{T\left(\frac{1}{k}C_z\right)}|\tilde{u}^T_{k,z}|^{p+1}dx\right)^{\frac{p-1}{p+1}}\left(\int_{T\left(\frac{1}{k}C_z\right)}|(u-\tilde{u}^T_{k,z})\wedge 0|^{p+1}dx\right)^{\frac{2}{p+1}} \end{aligned} \tag{2.35}$$

and, by (2.15),

$$\lim_{k\to\infty}\sup\left\{\int_{T\left(\frac{1}{k}C_z\right)}|\tilde{u}^T_{k,z}|^{p+1}dx\ :\ z\in Z_k,\ T\in\mathcal{C}_L(P_k,\overline{\Omega})\right\}=0. \tag{2.36}$$

Moreover, we have

$$\begin{aligned} \int_{T\left(\frac{1}{k}C_z\right)}|\nabla[(u-\tilde{u}^T_{k,z})\wedge 0]|^2dx &\ge \lambda_k\left(\int_{T\left(\frac{1}{k}C_z\right)}|(u-\tilde{u}^T_{k,z})\wedge 0|^{p+1}dx\right)^{\frac{2}{p+1}} \\ &\qquad \forall k\in\mathbb{N},\ \forall z\in Z_k,\ \forall T\in\mathcal{C}_L(P_k,\overline{\Omega}) \end{aligned} \tag{2.37}$$

where, for all $k\in\mathbb{N}$,

$$\begin{aligned} \lambda_k=\inf\Bigg\{&\int_{T\left(\frac{1}{k}C_z\right)}|\nabla\psi|^2dx\ :\ z\in Z_k,\ T\in\mathcal{C}_L(P_k,\overline{\Omega}), \\ &\psi\in H^1_0\left(T\left(\frac{1}{k}C_z\right)\right),\ \int_{T\left(\frac{1}{k}C_z\right)}|\psi|^{p+1}dx=1\Bigg\}. \end{aligned} \tag{2.38}$$

Notice that $\lim\limits_{k\to\infty}\lambda_k=\infty$ otherwise for all $i\in\mathbb{N}$ there exist $k_i\in\mathbb{N}$ $z_i\in Z_{k_i}$, $T_i\in\mathcal{C}_L(P_{k_i},\overline{\Omega})$, $\psi_i\in H^1_0\left(T_i\left(\frac{1}{k_i}C_{z_i}\right)\right)$ (extended by the value zero outside $T_i\left(\frac{1}{k_i}C_{z_i}\right)$) such that $\lim\limits_{i\to\infty}k_i=\infty$, $\int_\Omega|\psi_i|^{p+1}dx=1$ $\forall i\in\mathbb{N}$ and $\lim\limits_{i\to\infty}\int_\Omega|D\psi_i|^2dx<\infty$.
As a consequence, since $p<\frac{n+2}{n-2}$ when $n\ge 3$, there exists $\bar{\psi}$ in $H^1_0(\Omega)$ such that (up to a subsequence) $\psi_i\to\bar{\psi}$ as $i\to\infty$ weakly in $H^1_0(\Omega)$, in $L^{p+1}(\Omega)$ and a.e. in Ω.

Moreover, since $\lim_{i\to\infty} \operatorname{meas} T_i\left(\frac{1}{k_i}\,C_{z_i}\right) = 0$, the a.e. convergence implies that $\bar{\psi} \equiv 0$ in Ω, which is in contradiction with the convergence in $L^{p+1}(\Omega)$ because $\int_\Omega |\psi|^{p+1}dx = 1$ $\forall i \in \mathbb{N}$.
Thus, we can conclude that $\lim_{k\to\infty} \lambda_k = \infty$. It follows that, for k large enough,

$$E_{k,z,T}(\tilde{u}^T_{k,z} + t[(u-\tilde{u}^T_{k,z}) \vee 0]) \le E_{k,z,T}(\tilde{u}^T_{k,z} + t(u-\tilde{u}^T_{k,z})) \qquad \forall t > 0,\ \forall u \in H^1_0\left(\frac{1}{k}\,C_z\right). \tag{2.39}$$

As a consequence, if we denote by Γ the set defined by

$$\Gamma = \left\{u \in H^1_0\left(T\left(\frac{1}{k}\,C_z\right)\right)\ :\ u \not\equiv \tilde{u}^T_{k,z},\ E'_{k,z,T}(u)[u-\tilde{u}^T_{k,z}] = 0\right\}, \tag{2.40}$$

we have $u^T_{k,z} \in \Gamma$ and

$$E(u^T_{k,z}) = E_{k,z,T}(u^T_{k,z}) = \min_\Gamma E_{k,z,T}. \tag{2.41}$$

Therefore, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$E'_{k,z,T}(u^T_{k,z})[\varphi] = \mu\left\{E''_{k,z,T}(u^T_{k,z})[u^T_{k,z} - \tilde{u}^T_{k,z}, \varphi] + E'_{k,z,T}(u^T_{k,z})[\varphi]\right\} \quad \forall \varphi \in H^1_0\left(\frac{1}{k}\,C_z\right). \tag{2.42}$$

In particular, if we choose $\varphi = u^T_{k,z} - \tilde{u}^T_{k,z}$, we obtain $\mu = 0$ because $E'_{k,z,T}(u^T_{k,z})[u^T_{k,z} - \tilde{u}^T_{k,z}] = 0$ while $E''_{k,z,T}(u^T_{k,z})[u^T_{k,z} - \tilde{u}^T_{k,z}, u^T_{k,z} - \tilde{u}^T_{k,z}] \neq 0$. Thus, $u^T_{k,z}$ is a weak solution of the Dirichlet problem

$$-\Delta u^T_{k,z} = g(x, u^T_{k,z}) \quad \text{in } T\left(\frac{1}{k}\,C_z\right), \qquad u = 0 \quad \text{on } \partial T\left(\frac{1}{k}\,C_z\right). \tag{2.43}$$

On the other hand, since $u^T_{k,z} - \tilde{u}^T_{k,z} \ge 0$ in $T\left(\frac{1}{k}\,C_z\right)$, we have

$$g(x, u^T_{k,z}(x)) = |u^T_{k,z}(x)|^{p-1}u^T_{k,z}(x) + w(x) \qquad \forall x \in T\left(\frac{1}{k}\,C_z\right), \tag{2.44}$$

so $u^T_{k,z}$ is a solution of problem (2.28).

q.e.d.

When the function $u^T_k = \sum_{z\in Z_k} u^T_{k,z}$ satisfies a suitable stationarity property, then it is solution of problem (1.1) (here the function $u^T_{k,z}$ is extended by the value zero outside $T\left(\frac{1}{k}\,C_z\right)$). In fact, the following proposition holds.

Proposition 2.5 *Assume that $k \geq k_1(L)$ and $T \in \mathcal{C}_L(P_k, \overline{\Omega})$. Moreover, assume that the function $u_k^T = \sum_{z \in Z_k} u_{k,z}^T$ satisfies the following condition: $E'(u_k^T)[v \cdot Du_k^T] = 0$ for all vector field $v \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$ such that $v \cdot \nu = 0$ on $\partial\Omega$ (here ν denotes the outward normal vector on $\partial\Omega$). Then, u_k^T is a solution of the Dirichlet problem (1.1).*

<u>Proof</u> We have to prove that $E'(u_k^T)[\varphi] = 0\ \forall \varphi \in H_0^1(\Omega)$. Taking into account Proposition 2.4, since $u_{k,z}^T$ satisfies the Dirichlet problem (2.28) for all $z \in Z_k$, we have

$$
\begin{aligned}
E'(u_k^T)[\varphi] &= \int_\Omega [\nabla u_k^T \cdot \nabla\varphi - |u_k^T|^{p-1} u_k^T \varphi - w\varphi] dx \\
&= \sum_{z \in Z_k} \int_{T\left(\frac{1}{k} C_z\right)} [\nabla u_k^T \cdot \nabla\varphi - |u_k^T|^{p-1} u_k^T \varphi - w\varphi] dx \qquad (2.45) \\
&= \sum_{z \in Z_k} \int_{\partial T\left(\frac{1}{k} C_z\right)} \varphi\, (\nabla u_k^T \cdot \nu_{k,z}) d\sigma, \qquad (2.46)
\end{aligned}
$$

where $\nu_{k,z}$ denotes the outward normal on $\partial T\left(\frac{1}{k} C_z\right)$. Thus, in order to obtain $E'(u_k^T)$ $[\varphi] = 0$, we have to prove that if $z_1, z_2 \in Z_k$ and $|z_1 - z_2| = 1$ (that is $T\left(\frac{1}{k} C_{z_1}\right)$ and $T\left(\frac{1}{k} C_{z_2}\right)$ are adjacent subdomains of Ω) then

$$
\nabla u_{k,z_1}^T(x) = \nabla u_{k,z_2}^T(x) \qquad \forall x \in \partial T\left(\frac{1}{k} C_{z_1}\right) \cap T\left(\frac{1}{k} C_{z_2}\right). \qquad (2.47)
$$

Taking into account that $u_{k,z}^T$ satisfies problem (2.28) for all $z \in Z_k$, for all vector field $v \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$ such that $v \cdot \nu = 0$ on $\partial\Omega$ we obtain

$$
\begin{aligned}
&E'(u_k^T)[v \cdot \nabla u_k^T] = \\
&\quad = \int_\Omega [\nabla u_k^T \cdot \nabla(v \cdot \nabla u_k^T) - |u_k^T|^{p-1} u_k^T (v \cdot \nabla u_k^T) - w(v \cdot \nabla u_k^T)] dx \\
&\quad = \sum_{z \in Z_k} \int_{T\left(\frac{1}{k} C_z\right)} [\nabla u_k^T \cdot \nabla(v \cdot \nabla u_k^T) - |u_k^T|^{p-1} u_k^T (v \cdot \nabla u_k^T) - w(v \cdot \nabla u_k^T)] dx \\
&\quad = \sum_{z \in Z_k} \int_{\partial T\left(\frac{1}{k} C_z\right)} (\nabla u_k^T \cdot \nu_{k,z})^2 (v \cdot \nu_{k,z}) d\sigma. \qquad (2.48)
\end{aligned}
$$

Since $E'(u_k^T)[v \cdot \nabla u_k^T] = 0\ \forall v \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$ such that $v \cdot \nu = 0$ on $\partial\Omega$, (2.47) follows easily. Thus, we can conclude that u_k^T is a solution of problem (1.1).

q.e.d.

In order to obtain a function u_k^T which is stationary in the sense of Proposition 2.5, we can, for example, minimize $E(u_k^T)$ with respect to T for k large enough.

First notice that, since Ω is a smooth bounded domain, there exist $k'_\Omega \geq k_\Omega$ and $L'_\Omega \geq 1$ such that, for all $k \geq k'_\Omega$ and $L \geq L'_\Omega$, we have

$$\{T \in \mathcal{C}_L(P_k, \overline{\Omega}) \ : \ T(P_k) = \overline{\Omega}\} \neq \emptyset. \tag{2.49}$$

Moreover, using Ascoli-Arzelà Theorem, one can show the following lemma.

Lemma 2.6 *If (2.49) holds, there exists* $\widetilde{T}_k^L \in \mathcal{C}_L(P_k, \overline{\Omega})$ *such that* $\widetilde{T}_k^L(P_k) = \overline{\Omega}$ *and*

$$\sum_{z \in Z_k} E(u_{k,z}^{\widetilde{T}^L}) = \min \left\{ \sum_{z \in Z_k} E(u_{k,z}^T) \ : \ T \in \mathcal{C}_L(P_k, \overline{\Omega}),\ T(P_k) = \overline{\Omega} \right\}. \tag{2.50}$$

For all $L \geq 1$ and $T \in \mathcal{C}_L(P_k, \overline{\Omega})$, let us set

$$\mathcal{L}(T) = \inf \left\{ \mathcal{L} \ : \ \mathcal{L} \geq 1,\ \frac{1}{\mathcal{L}} |x - y| \leq |T(x) - T(y)| \leq \mathcal{L} |x - y| \quad \forall x, y \in P_k \right\}. \tag{2.51}$$

Using again Ascoli-Arzelà Theorem, we infer that, for all $L \geq L'_\Omega$ and $k \geq k'_\Omega$, there exists $T_k^L \in \mathcal{C}_L(P_k, \overline{\Omega})$ such that $T_k^L \left(\frac{1}{k} C_z \right) = \widetilde{T}_k^L \left(\frac{1}{k} C_z \right)$ $\forall z \in Z_k$ and

$$\mathcal{L}(T_k^L) = \min \left\{ \mathcal{L}(T) \ : \ T \in \mathcal{C}_L(P_k, \overline{\Omega}),\ T\left(\frac{1}{k} C_z \right) = \widetilde{T}_k^L \left(\frac{1}{k} C_z \right) \ \forall z \in Z_k \right\}. \tag{2.52}$$

Notice that T_k^L depends only on the geometrical properties of the subdomains $\widetilde{T}_k^L \left(\frac{1}{k} C_z \right)$ with $z \in Z_k$. A large $\mathcal{L}(T_k^L)$ means that there are large differences in the sizes and in the shape of these subdomains.
We can now state the following multiplicity result.

Theorem 2.7 *Let* $n \geq 1$, $p > 1$ *and* $p < \frac{n+2}{n-2}$ *when* $n \geq 3$. *Moreover, assume that there exists* $\bar{L} \geq L'_\Omega$ *such that*

$$\limsup_{k \to \infty} \mathcal{L}(T_k^{\bar{L}}) < \bar{L}. \tag{2.53}$$

Then, problem (1.1) admits infinitely many solutions (see also Remark 3.1 concerning condition (2.53)).

Theorem 2.7 is a direct consequence of the following proposition.

Proposition 2.8 *If the assumptions of Theorem 2.7 are satisfied, for all* $w \in L^2(\Omega)$ *there exists* $\bar{k} \geq k_\Omega$ *such that for all* $k \geq \bar{k}$ *there exists* $T_k^{\bar{L}} \in \mathcal{C}_{\bar{L}}(P_k, \overline{\Omega})$ *satisfying the following property:* $T_k^{\bar{L}}(P_k) = \overline{\Omega}$ *and the function* $u_k = \sum_{z \in Z_k} u_{k,z}^{T_k^{\bar{L}}}$ *is a solution of problem (1.1). Moreover, the number of nodal regions of* u_k *tends to infinity as* $k \to \infty$ *and*

$$\lim_{k \to \infty} \min \left\{ E\left(u_{k,z}^{T_k^{\bar{L}}} \right) \ : \ z \in Z_k \right\} = \infty. \tag{2.54}$$

Proof As a consequence of condition (2.53), there exist $\bar{k} \in \mathbb{N}$ and $\bar{\varepsilon} > 0$ such that

$$D \circ T_k^{\bar{L}} \in \mathcal{C}_{\bar{L}}(P_k, \overline{\Omega}) \quad \forall k \geq \bar{k},\ \forall D \in \mathcal{C}_{1+\bar{\varepsilon}}(\overline{\Omega}, \overline{\Omega}). \tag{2.55}$$

Moreover, from Proposition 2.4 we infer that, if we choose $\bar{k}$ large enough, for all $k \geq \bar{k}$ and $z \in Z_k$ the function $u_{k,z}^{T_k^{\bar{L}}}$ is a solution of the Dirichlet problem

$$-\Delta u = |u|^{p-1}u + w \quad \text{ in } T_k^{\bar{L}}\left(\frac{1}{k}\, C_z\right), \qquad u = 0 \quad \text{ on } \partial T_k^{\bar{L}}\left(\frac{1}{k}\, C_z\right). \tag{2.56}$$

Thus, taking into account Proposition 2.5 we have to prove that $E'(u_k)[v \cdot Du_k] = 0$ for all vector field $v \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$ such that $v \cdot \nu = 0$ on $\partial\Omega$.
Therefore, for all vector field $v \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$ such that $v \cdot \nu = 0$ on $\partial\Omega$ and for all $\tau \in \mathbb{R}$, let us consider the function $D_\tau : \overline{\Omega} \to \overline{\Omega}$ defined by the Cauchy problem

$$\frac{\partial D_\tau(x)}{\partial \tau} = v \circ D_\tau(x), \qquad D_0(x) = x \qquad \forall \tau \in \mathbb{R},\ \forall x \in \overline{\Omega}. \tag{2.57}$$

Then, we have $D_\tau(\overline{\Omega}) = \overline{\Omega}\ \forall \tau \in \mathbb{R}$ and

$$\lim_{\tau \to 0} \mathcal{L}(D_\tau \circ T_k^{\bar{L}}) = \mathcal{L}(T_k^{\bar{L}}), \tag{2.58}$$

so there exists $\bar{\tau} > 0$ such that $D_\tau \circ T_k^{\bar{L}} \in \mathcal{C}_{\bar{L}}(P_k, \overline{\Omega})\ \forall \tau \in [-\bar{\tau}, \bar{\tau}]$. It follows that

$$E(u_k) = \sum_{z \in Z_k} E(u_{k,z}^{T_k^{\bar{L}}}) \leq \sum_{z \in Z_k} E(u_{k,z}^{D_\tau \circ T_k^{\bar{L}}}) = E(u_k^{D_\tau \circ T_k^{\bar{L}}}) \quad \forall \tau \in [-\bar{\tau}, \bar{\tau}]. \tag{2.59}$$

Notice that

$$\frac{d}{d\tau}\, E(u_k \circ D_\tau^{-1})_{|\tau=0} = -E'(u_k)[v \cdot \nabla u_k]. \tag{2.60}$$

Thus, we have to prove that

$$\frac{d}{d\tau}\, E(u_k \circ D_\tau^{-1})_{|\tau=0} = 0. \tag{2.61}$$

For the proof, we argue by contradiction and assume that (2.61) does not hold. For example, we assume that

$$\frac{d}{d\tau}\, E(u_k \circ D_\tau^{-1})_{|\tau=0} < 0 \tag{2.62}$$

(otherwise we replace v by $-v$). As a consequence, there exists a sequence of positive numbers $(\tau_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \tau_i = 0$ and $E(u_k \circ D_{\tau_i}^{-1}) < E(u_k)\ \forall i \in \mathbb{N}$. From Corollary 2.2 we infer that, if we choose $\bar{k}$ large enough, for all $k \geq \bar{k}$, $z \in Z_k$ and $i \in \mathbb{N}$ there exists a unique minimizing function $\tilde{u}_{k,z}^{D_{\tau_i} \circ T_k^{\bar{L}}}$ and $\tilde{u}_{k,z}^{D_{\tau_i} \circ T_k^{\bar{L}}} \to \tilde{u}_{k,z}^{T_k^{\bar{L}}}$ as $i \to \infty$ in $H_0^1(\Omega)$ $\forall k \geq \bar{k}$, $\forall z \in Z_k$.

As in the proof of Proposition 2.4, let us consider the functions $G^i_{k,z}$ verifying

$$\begin{aligned} &G^i_{k,z}(x,t) = \tfrac{|t|^{p+1}}{p+1} + w(x)\,t && \text{if } \sigma(z)[t - \tilde u_{k,z}^{D_{\tau_i}\circ T_k^{\bar L}}(x)] \geq 0, \\ &\tfrac{\partial^2 G^i_{k,z}}{\partial t^2}(x,t) = \tfrac{\partial^2 G^i_{k,z}}{\partial t^2}(x, \tilde u_{k,z}^{D_{\tau_i}\circ T_k^{\bar L}}(x)) && \text{otherwise,} \end{aligned} \tag{2.63}$$

the functional $E^i_{k,z} : H^1_0\left(T_k^{\bar L}\left(\frac{1}{k}\,C_z\right)\right) \to \mathbb{R}$ defined by

$$E_{k,z,i}(u) = \frac{1}{2}\int_{T_k^{\bar L}\left(\frac{1}{k}\,C_z\right)} |\nabla u|^2 dx - \int_{T_k^{\bar L}\left(\frac{1}{k}\,C_z\right)} G^i_{k,z}(x,u)\,dx \tag{2.64}$$

and the manifold

$$\Gamma^i_{k,z} = \left\{ u \in H^1_0\left(T_k^{\bar L}\left(\frac{1}{k}\,C_z\right)\right) \ :\ u \not\equiv \tilde u_{k,z}^{D_{\tau_i}\circ T_k^{\bar L}},\ E'_{k,z,i}(u)[u - \tilde u_{k,z}^{D_{\tau_i}\circ T_k^{\bar L}}] = 0 \right\}. \tag{2.65}$$

We say that

$$\max\left\{ \sum_{z\in Z_k} E\left(\tilde u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1} + t_z\left(u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1} - \tilde u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1}\right)\right) : t_z \geq 0,\ \forall z \in Z_k \right\} < E(u_k) \tag{2.66}$$

for i large enough.
In fact, arguing by contradiction, assume that (up to a subsequence still denoted by $(\tau_i)_{i\in\mathbb{N}}$) the inequality (2.66) does not hold. Then, for all $i \in \mathbb{N}$ and $z \in Z_k$, there exists $t_{z,i} \geq 0$ such that

$$\sum_{z\in Z_k} E\left(\tilde u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1} + t_{z,i}\left(u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1} - \tilde u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1}\right)\right) \geq E(u_k) \qquad \forall i \in \mathbb{N}. \tag{2.67}$$

It follows that $\lim\limits_{i\to\infty} t_{z,i} = 1\ \forall z \in Z_k$ and

$$\begin{aligned} &\sum_{z\in Z_k} E\left(\tilde u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1} + t_{z,i}\left(u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1} - \tilde u_{k,z}^{T_k^{\bar L}} \circ D_{\tau_i}^{-1}\right)\right) \\ &\qquad \geq \sum_{z\in Z_k} E\left(\tilde u_{k,z}^{T_k^{\bar L}} + t_{z,i}\left(u_{k,z}^{T_k^{\bar L}} - \tilde u_{k,z}^{T_k^{\bar L}}\right)\right) \qquad \forall i \in \mathbb{N} \end{aligned} \tag{2.68}$$

which, as $i \to \infty$, implies

$$\frac{d}{d\tau} E\left(u_{k,z}^{T_k^{\bar L}} \circ D_\tau^{-1}\right)_{|\tau=0} \geq 0 \tag{2.69}$$

in contradiction with (2.62). Thus, (2.66) holds.

Notice that, if $\bar{k}$ is chosen large enough,

$$E'\left(\tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} + t\left(u_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} - \tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1}\right)\right) \cdot$$
$$\cdot \left[\tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} - \tilde{u}_{k,z}^{D_{\tau_i} \circ T_k^{\bar{L}}} + t\left(u_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} - \tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1}\right)\right], \tag{2.70}$$

for i large enough, is positive for $t = \|u_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} - \tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1}\|_{H_0^1}^{-1}$ and tends to $-\infty$ as $t \to \infty$. As a consequence, there exists $t_{k,z}^i \in \mathbb{R}$ such that

$$\tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} + t_{k,z}^i\left(u_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} - \tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1}\right) \in \Gamma_{k,z}^i. \tag{2.71}$$

Therefore, from (2.66) and (2.71) we obtain

$$\begin{aligned} E(u_k) \quad &> \max\left\{\sum_{z\in Z_k} E\left(\tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} + t_z\left(u_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1} - \tilde{u}_{k,z}^{T_k^{\bar{L}}} \circ D_{\tau_i}^{-1}\right)\right) : t_z \geq 0\ \forall z \in Z_k\right\} \\ &\geq \sum_{z\in Z_k} E\left(u_{k,z}^{D_{\tau_i} \circ T_k^{\bar{L}}}\right) = E(u_k^{D_{\tau_i} \circ T_k^{\bar{L}}}) \end{aligned} \tag{2.72}$$

for i large enough, in contradiction with (2.59).
Thus, we can conclude that $\frac{d}{d\tau} E(u_k \circ D_\tau^{-1})_{|\tau=0} = 0$ that is $E'(u_k)[v \cdot Du_k] = 0$ for all vector field $v \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$ such that $v \cdot \nu = 0$ on $\partial\Omega$, so u_k is a solution of problem (1.1).
Notice that, if $J(k)$ denotes the number of elements of Z_k, the solution u_k has at least $J(k)$ nodal regions for k large enough. Moreover, we have

$$\lim_{k\to\infty} \frac{J(k)}{k^n} = \text{meas}(\Omega), \tag{2.73}$$

so the number of nodal regions of u_k tends to infinity as $k \to \infty$.
Finally, notice that (2.54) follows directly from Lemma 2.1 and Lemma 2.3.
So the proof is complete.

q.e.d.

Let us point out that, if $n = 1$, condition (2.53) in Theorem 2.7 is satisfied. In fact, it is a consequence of the following lemma (see also Remark 3.1 concerning the case $n > 1$).

Lemma 2.9 *Assume $n = 1$, $p > 1$ and $w \in L^2(\Omega)$. Then, for all $L > 1$ there exists $\bar{k}(L) \in \mathbb{N}$ such that*

$$\{T \in \mathcal{C}_L(P_k, \overline{\Omega}) \ : \ T(P_k) = \overline{\Omega}\} \neq \emptyset \qquad \forall k \geq \bar{k}(L). \tag{2.74}$$

Moreover,

$$\lim_{k\to\infty} \mathcal{L}(T_k^L) = 1 \qquad \forall L > 1. \tag{2.75}$$

Proof Let $\Omega =]a, b[$. First notice that, if Z_k consists of $J(k)$ points $z_1, \ldots, z_{J(k)}$, then $\frac{J(k)}{k} \leq b-a$ and $\lim_{k\to\infty} \frac{J(k)}{k} = b-a$. Since $L > 1$, it follows that there exists $\bar{k}(L) \in \mathbb{N}$ such that $\frac{J(k)}{k} > \frac{b-a}{L}$ $\forall k \geq \bar{k}(L)$, which implies (2.74) as one can easily verify. Also, notice that in this case we have

$$\mathcal{L}(T_k^L) = \min \left\{ \mathcal{L} \ : \ \mathcal{L} \geq 1, \ \frac{1}{\mathcal{L}} \leq k \text{ meas} \left[T_k^L \left(\frac{1}{k} C_{z_j} \right) \right] \leq \mathcal{L} \right. \tag{2.76}$$

$$\left. \text{for } j = 1, \ldots, J(k) \right\} \qquad \forall k > \bar{k}(L).$$

Moreover, if we denote by $\bar{u}_{k,z}^{T_k^L}$ the function $u_{k,z}^{T_k^L}$ obtained when $w = 0$, we get by direct computation

$$\limsup_{k\to\infty} \frac{1}{k^{\frac{p+3}{p-1}}} \max \left\{ E(\bar{u}_{k,z}^{T_k^L}) \ : \ z \in Z_k \right\} < \infty \tag{2.77}$$

and

$$\liminf_{k\to\infty} \frac{1}{k^{\frac{p+3}{p-1}}} \min \left\{ E(\bar{u}_{k,z}^{T_k^L}) \ : \ z \in Z_k \right\} > 0. \tag{2.78}$$

Taking into account that

$$\left| \int_{T_k^L\left(\frac{1}{k} C_z\right)} u\, w\, dx \right| \leq \left(\int_{T_k^L\left(\frac{1}{k} C_z\right)} u^2\, dx \right)^{\frac{1}{2}} \|w\|_{L^2(\Omega)} \tag{2.79}$$

$$\leq \left[\text{meas}\, T_k^L \left(\frac{1}{k} C_z \right) \right]^{\frac{1}{2} - \frac{1}{p+1}} \|u\|_{L^{p+1}\left(T_k^L\left(\frac{1}{k} C_z\right)\right)} \|w\|_{L^2(\Omega)} \quad \forall z \in Z_k,$$

for all $w \in L^2(\Omega)$ we obtain

$$\limsup_{k\to\infty} \frac{1}{k^{\frac{5-p}{2(p-1)}}} \max\{ |E(u_{k,z}^{T_k^L}) - E(\bar{u}_{k,z}^{T_k^L})| \ : \ z \in Z_k \} < \infty \tag{2.80}$$

and, as a consequence,

$$\limsup_{k\to\infty} \frac{J(k)}{k^{\frac{p+3}{2(p-1)}}} \max\{ |E(u_{k,z}^{T_k^L}) - E(\bar{u}_{k,z}^{T_k^L})| \ : \ z \in Z_k \} < \infty. \tag{2.81}$$

It is clear that $1 \leq \mathcal{L}(T_k^L) \leq L$, so $\liminf_{k\to\infty} \mathcal{L}(T_k^L) \geq 1$. Arguing by contradiction, assume that (2.75) does not hold, that is

$$\limsup_{k\to\infty} \mathcal{L}(T_k^L) > 1. \tag{2.82}$$

Thus, there exists a sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\lim_{i\to\infty} \mathcal{L}(T_{k_i}^L) > 1. \tag{2.83}$$

Taking into account (2.76), it follows that there exists a sequence $(z'_i)_{i\in\mathbb{N}}$ such that $z'_i \in Z_{k_i}$ $\forall i \in \mathbb{N}$ and (up to a subsequence)

$$\lim_{i\to\infty} k_i \,\mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_{z'_i}\right)\right] > 1, \tag{2.84}$$

or there exists a sequence $(z''_i)_{i\in\mathbb{N}}$ such that $z''_i \in Z_{k_i}$ $\forall i \in \mathbb{N}$ and (up to a subsequence)

$$\lim_{i\to\infty} k_i \,\mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_{z''_i}\right)\right] < 1. \tag{2.85}$$

Let us consider first the case where (2.84) holds. In this case, for all $i \in \mathbb{N}$, choose $\tilde{z}'_i$ in Z_{k_i} such that

$$\mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_{\tilde{z}'_i}\right)\right] = \min\left\{\mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_z\right)\right] \ :\ z \in Z_{k_i}\right\}. \tag{2.86}$$

Then, taking into account that

$$(b-a) = \sum_{z\in Z_{k_i}} \mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_z\right)\right] \geq J(k_i)\,\mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_{\tilde{z}'_i}\right)\right] \tag{2.87}$$

and that $\lim\limits_{i\to\infty} \frac{J(k_i)}{k_i} = b-a$, we obtain

$$\limsup_{i\to\infty} k_i \cdot \mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_{\tilde{z}'_i}\right)\right] \leq 1. \tag{2.88}$$

As one can easily verify, for all $i \in \mathbb{N}$, there exists a function $T'_i \in \mathcal{C}_L(P_{k_i}, \overline{\Omega})$ such that $T'_i(P_{k_i}) = \overline{\Omega}$,

$$\mathrm{meas}\left[T'_i\left(\frac{1}{k_i}\,C_z\right)\right] = \mathrm{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_z\right)\right] \qquad \forall z \in Z_{k_i} \setminus \{z'_i, \tilde{z}'_i\}, \tag{2.89}$$

$$\mathrm{meas}\left[T'_i\left(\frac{1}{k_i}\,C_{z'_i}\right)\right] = \mathrm{meas}\left[T'_i\left(\frac{1}{k_i}\,C_{\tilde{z}'_i}\right)\right] \tag{2.90}$$

and

$$\lim_{i\to\infty} \max\{|T'_i(x) - T^L_{k_i}(x)| \ :\ x \in P_{k_i}\} = 0. \tag{2.91}$$

Taking into account that $E(\bar{u}^{T'_i}_{k_i,z}) = E(\bar{u}^{T^L_{k_i}}_{k_i,z})$ $\forall z \in Z_{k_i} \setminus \{z'_i, \tilde{z}'_i\}$, from (2.81) we infer that

$$\lim_{i\to\infty} \frac{1}{k_i^{\frac{p+3}{p-1}}} \sum_{z\in Z_{k_i}\setminus\{z'_i,\tilde{z}'_i\}} |E(u^{T'_i}_{k_i,z}) - E(u^{T^L_{k_i}}_{k_i,z})| = 0. \tag{2.92}$$

On the other hand,

$$\liminf_{i\to\infty} \frac{1}{k_i^{\frac{p+3}{p-1}}}[E(u^{T_i'}_{k_i,z_i'}) + E(u^{T_i'}_{k_i,\tilde{z}_i'})] > 0 \tag{2.93}$$

as one can verify by direct computation. Moreover, (2.84) and (2.88) imply

$$\liminf_{i\to\infty} \frac{E(u^{T^L_{k_i}}_{k_i,z_i'}) + E(u^{T^L_{k_i}}_{k_i,\tilde{z}_i'})}{E(u^{T_i'}_{k_i,z_i'}) + E(u^{T_i'}_{k_i,\tilde{z}_i'})} > 1. \tag{2.94}$$

It follows that

$$\liminf_{i\to\infty} \frac{1}{k_i^{\frac{p+3}{p-1}}} \sum_{z\in Z_{k_i}} [E(u^{T^L_{k_i}}_{k_i,z}) - E(u^{T_i'}_{k_i,z})] > 0, \tag{2.95}$$

which is a contradiction because

$$\sum_{z\in Z_{k_i}} E(u^{T^L_{k_i}}_{k_i,z}) \leq \sum_{z\in Z_{k_i}} E(u^{T_i'}_{k_i,z}) \qquad \forall i\in\mathbb{N}. \tag{2.96}$$

When the case (2.85) occurs, we argue in analogous way. In this case, for all $i\in\mathbb{N}$ we choose $\tilde{z}''$ in Z_{k_i} such that

$$\text{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_{\tilde{z}_i''}\right)\right] = \max\left\{\text{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_z\right)\right] \;:\; z\in Z_{k_i}\right\}, \tag{2.97}$$

which implies

$$\liminf_{i\to\infty} k_i\cdot\text{meas}\left[T^L_{k_i}\left(\frac{1}{k_i}\,C_{\tilde{z}_i''}\right)\right] \geq 1. \tag{2.98}$$

Moreover, we can consider a function $T_i''\in\mathcal{C}_L(P_{k_i},\overline{\Omega})$ satisfying all the properties of T_i' with z_i'' and $\tilde{z}_i''$ instead of z_i' and $\tilde{z}_i'$.
Then, we can repeat for T_i'', z_i'' and $\tilde{z}_i''$ the same arguments as before. In particular, the property

$$\liminf_{i\to\infty} \frac{E(u^{T^L_{k_i}}_{k_i,z_i''}) + E(u^{T^L_{k_i}}_{k_i,\tilde{z}_i''})}{E(u^{T_i''}_{k_i,z_i''}) + E(u^{T_i''}_{k_i,\tilde{z}_i''})} > 1 \tag{2.99}$$

(analogous to (2.94)) now follows from (2.85) and (2.98). Thus, also in this case we obtain again a contradiction with the minimality property of $\sum_{z\in Z_{k_i}} E(u^{T^L_{k_i}}_{k_i,z})$. So the proof is complete.

q.e.d.

As a direct consequence of Theorem 2.7, Proposition 2.8 and Lemma 2.9 we obtain the following corollary.

Corollary 2.10 *Assume $n=1$ and $p>1$. Then, for all $w\in L^2(\Omega)$, problem (1.1) has infinitely many solutions. More precisely, for all $L>1$ there exists $\bar{k}(L)\geq k_\Omega$ such that for all $k\geq\bar{k}(L)$ there exists $T^L_k\in\mathcal{C}_L(P_k,\overline{\Omega})$ such that $T^L_k(P_k)=\overline{\Omega}$ and the function $u_k=\sum_{z\in Z_k}u_{k,z}^{T^L_k}$ is a solution of problem (1.1).*
Moreover, the number of nodal regions of u_k tends to infinity as $k\to\infty$ and

$$\lim_{k\to\infty}\mathcal{L}(T^L_k)=1,\quad \lim_{k\to\infty}\min\{E(u_{k,z}^{T^L_k})\ :\ z\in Z_k\}=\infty\quad\forall L>1. \tag{2.100}$$

3 Final remarks

Notice that the method we used in Section 2 to find infinitely many solutions of problem (1.1) with a large number of nodal regions having a prescribed structure (a check structure) may be used also in other elliptic problems as we show in this section.
It is clear that in this method condition (2.53) plays a crucial role. In Section 2 this condition is proved only in the case $n=1$. In next remark, we discuss about the case $n>1$.

Remark 3.1 Assume that condition (2.53) does not hold. Then, there exists a sequence $(L_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\lim_{i\to\infty}L_i=\infty\qquad\text{and}\qquad\limsup_{k\to\infty}\mathcal{L}(T^{L_i}_k)=L_i\quad\forall i\in\mathbb{N}. \tag{3.1}$$

As a consequence, we can construct a sequence $(k_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty}k_i=\infty,\qquad\lim_{i\to\infty}\min\left\{E\big(u_{k_i,z}^{T^{L_i}_{k_i}}\big)\ :\ z\in Z_{k_i}\right\}=\infty,\qquad\lim_{i\to\infty}\mathcal{L}(T^{L_i}_{k_i})=\infty. \tag{3.2}$$

Notice that $\mathcal{L}(T^{L_i}_{k_i})$ is large, for example, when there are large differences in the sizes or in the shapes of the subdomains $T^{L_i}_{k_i}\left(\frac{1}{k_i}C_z\right)$ with $z\in Z_{k_i}$. For k_i large enough, too large differences seem to be incompatible with the minimality property

$$\sum_{z\in Z_{k_i}}E\big(u_{k_i,z}^{T^{L_i}_{k_i}}\big)=\min\left\{\sum_{z\in Z_{k_i}}E\big(u^T_{k_i,z}\big)\ :\ T\in\mathcal{C}_{L_i}(P_{k_i},\overline{\Omega}),\ T(P_{k_i})=\overline{\Omega}\right\}\quad\forall i\in\mathbb{N}. \tag{3.3}$$

This fact explains why condition (2.53) holds in the case $n=1$. In the case $n>1$, on the contrary, even if the subdomains $T^{L_i}_{k_i}\left(\frac{1}{k_i}C_z\right)$ with $z\in Z_{k_i}$ have all the same shape and the same size, we cannot exclude that $\mathcal{L}\left(T^{L_i}_{k_i}\right)$ is large as a consequence of the fact that the shape of these subdomains is very different from the cubes of $\mathbb{R}^n$. This explains why it is difficult to prove that condition (2.53) holds also for $n>1$.

Therefore, in the case $n > 1$, the natural idea is to restrict the class of the admissible deformations of the nodal regions.
For example, we can fix $\tilde{L} \geq L'_\Omega$, $T_0 \in \mathcal{C}_{\tilde{L}}(\overline{\Omega}, \overline{\Omega})$, $r > 0$ and consider the set of deformations

$$\Theta_k^{\tilde{L}}(T_0, r) = \{T \in \mathcal{C}_{\tilde{L}}(P_k, \overline{\Omega}) \ : \ T(P_k) = \overline{\Omega}, \ d_k(T, T_0) \leq r\} \tag{3.4}$$

where

$$d_k(T, T_0) = \sup_{P_k} |T - T_0| + \mathrm{Lip}(T - T_0) \tag{3.5}$$

with

$$\mathrm{Lip}(T - T_0) = \sup\{|x - y|^{-1}|T(x) - T_0(x) - T(y) + T_0(y)| \ : \ x, y \text{ in } P_k, \ x \neq y\}. \tag{3.6}$$

Then, arguing exactly as in Section 2 but minimizing in the subset $\Theta_k^{\tilde{L}}(T_0, r)$ (instead of the set (2.49)), we obtain a minimizing deformation $T_k^{\tilde{L},r}$ which, for k large enough, gives rise to a solution $u_k^{\tilde{L},r}$ of problem (1.1) provided the condition

$$\limsup_{k\to\infty} \mathcal{L}(T_k^{\tilde{L},r}) < \tilde{L}, \quad \limsup_{k\to\infty} d_k(T_k^{\tilde{L},r}, T_0) < r \tag{3.7}$$

(analogous to condition (2.53)) is satisfied.
It is clear that condition (3.7) holds or fails depending on the choice of $\tilde{L}, T_0$ and r that have to be chosen in a suitable way. For example, in the case $n = 1$, if we choose $T_0(x) = x \ \forall x \in \Omega$, (3.7) holds for all $\tilde{L} > 1$ and $r > 0$ as follows from Lemma 2.9.
In the case $n > 1$, condition (3.7) seems to have more chances than condition (2.53) to be satisfied. In fact, as we show in a paper in preparation, a variant of this method works for example when Ω is a cube of $\mathbb{R}^n$ with $n > 1$, $p > 1$, $p < \frac{n+2}{n-2}$ if $n > 2$ and, for all $w \in L^2(\Omega)$, allows us to find infinitely many solutions $u_k(x)$ such that the nodal regions of $u_k\left(\frac{x}{k}\right)$, after translations, tend to the cube as $k \to \infty$.
Therefore, it seems quite natural to expect that, by a suitable choice of $\tilde{L}$, T_0 and r, for every bounded domain Ω in $\mathbb{R}^n$ with $n > 1$ and for all $w \in L^2(\Omega)$ one can find infinitely many nodal solutions of problem (1.1) with $p > 1$ and $p < \frac{n+2}{n-2}$ if $n > 2$.
$\square$

Notice that this method to construct solutions with nodal regions having this check structure works for more general nonlinearities, even when they are not perturbations of symmetric nonlinearities: for example when in problem (1.1) the term $|u|^{p-1}u + w$ is replaced by $c_+(u^+)^p - c_-(u^-)^p + w$ with $c_+ > 0$ and $c_- > 0$.
In fact, this method does not require any technique of deformation from the symmetry. For example, let us show how Lemma 2.9 has to be modified in this case.

In this case the energy functional is

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{c_+}{p+1}\int_\Omega (u^+)^{p+1} dx - \frac{c_-}{p+1}\int_\Omega (u^-)^{p+1} dx - \int_\Omega wu\, dx. \tag{3.8}$$

We denote by F_0 the functional F when $w = 0$.
Now, consider the number $\hat{L} \geq 1$ defined by $\hat{L} = \frac{1}{\min\{\hat{t}, 2-\hat{t}\}}$ where $\hat{t} \in]0,2[$ is the unique number such that

$$\begin{aligned} &\min\{F_0(u) \ : \ u \in H_0^1(]0,2[),\ u \geq 0 \text{ in }]0,\hat{t}[,\ u \leq 0 \text{ in }]\hat{t},2[, \\ &\qquad\qquad u^+ \not\equiv 0,\ u^- \not\equiv 0,\ F_0'(u)[u^+] = 0,\ F_0'(u)[u^-] = 0\} \qquad (3.9) \\ &= \min\{F_0(u) \ : \ u \in H_0^1(]0,2[),\ u^+ \not\equiv 0,\ u^- \not\equiv 0,\ F_0'(u)[u^+] = 0,\ F_0'(u)[u^-] = 0\}. \end{aligned}$$

Notice that $\hat{t} = 1$ (and so $\hat{L} = 1$) if and only if $c_+ = c_-$.
Then, we have the following lemma which extends Lemma 2.9.

Lemma 3.2 *Assume $n = 1$, $c_+ > 0$, $c_- > 0$, $p > 1$. Then, for all $L > \hat{L}$ there exists $\hat{k}(L) \in \mathbb{N}$ such that $\{T \in \mathcal{C}_L(P_k, \overline{\Omega}) \ : \ T(P_k) = \overline{\Omega}\} \neq \emptyset\ \forall k \geq \hat{k}(L)$ and $\lim_{k\to\infty} \mathcal{L}(T_k^L) = \hat{L}$.*

Proof Here we describe only how the proof of Lemma 2.9 has to be modified in order to be adapted in this case.
First notice that, since $L > \hat{L}$ and $\hat{L} \geq 1$, there exists $\hat{k}(L) \in \mathbb{N}$ such that

$$\{T \in \mathcal{C}_L(P_k, \overline{\Omega}) \ : \ T(P_k) = \overline{\Omega}\} \neq \emptyset \qquad \forall k \geq \hat{k}(L) \tag{3.10}$$

as one can verify as in the proof of Lemma 2.9.
In order to prove that $\lim_{k\to\infty} \mathcal{L}(T_k^L) = \hat{L}$, we argue by contradiction and assume that there exists a sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $\lim_{i\to\infty} k_i = \infty$ and $\lim_{i\to\infty} \mathcal{L}(T_{k_i}^L) \neq \hat{L}$.
First, notice that the case

$$\lim_{i\to\infty} \mathcal{L}(T_{k_i}^L) < \hat{L} \tag{3.11}$$

cannot happen. In fact, for all $i \in \mathbb{N}$ we can choose $\hat{z}_i$ and $\hat{z}_i + 1$ in Z_{k_i} such that (up to a subsequence)

$$\lim_{i\to\infty} k_i \left[\operatorname{meas} T_{k_i}^L\left(\frac{1}{k_i}\, C_{\hat{z}_i}\right) + \operatorname{meas} T_{k_i}^L\left(\frac{1}{k_i}\, C_{\hat{z}_i+1}\right)\right] \leq 2. \tag{3.12}$$

Taking into account the minimality property

$$\sum_{z\in Z_{k_i}} F(u_{k_i,z}^{T_{k_i}^L}) = \min\left\{\sum_{z\in Z_{k_i}} F(u_{k_i,z}^T) \ : \ T \in \mathcal{C}_L(P_k, \overline{\Omega}),\ T(P_{k_i}) = \overline{\Omega}\right\} \qquad \forall i \in \mathbb{N}, \tag{3.13}$$

it follows that

$$\lim_{i\to\infty} \frac{2\min\{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\hat z_i}\right),\, \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\hat z_i+1}\right)\}}{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\hat z_i}\right) + \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\hat z_i+1}\right)} = \min\{\hat t, 2-\hat t\}. \tag{3.14}$$

As a consequence, we obtain

$$\lim_{i\to\infty} k_i \min\left\{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\hat z_i}\right),\, \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\hat z_i+1}\right)\right\} \le \min\{\hat t, 2-\hat t\} \tag{3.15}$$

which implies

$$\lim_{i\to\infty} \mathcal{L}(T^L_{k_i}) \ge \frac{1}{\min\{\hat t, 2-\hat t\}} = \hat L. \tag{3.16}$$

In order to prove that $\lim_{i\to\infty} \mathcal{L}(T^L_{k_i}) = \hat L$, arguing by contradiction, assume that $\lim_{i\to\infty} \mathcal{L}(T^L_{k_i}) > \hat L$.
As a consequence, since

$$\mathcal{L}(T^L_{k_i}) = \min\left\{\mathcal{L} \ :\ \mathcal{L} \ge 1,\ \frac{1}{\mathcal{L}} \le k_i \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_z\right) \le \mathcal{L}\ \ \forall z \in Z_{k_i}\right\}, \tag{3.17}$$

there exists a sequence $(z_i')_{i\in\mathbb{N}}$ such that $z_i' \in Z_{k_i}$ $\forall i \in \mathbb{N}$ and

$$\lim_{i\to\infty} k_i \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z_i'}\right) > \hat L, \tag{3.18}$$

or there exists a sequence $(z_i'')_{i\in\mathbb{N}}$ such that $z_i'' \in Z_{k_i}$ $\forall i \in \mathbb{N}$ and

$$\lim_{i\to\infty} k_i \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z_i''}\right) < \frac{1}{\hat L}. \tag{3.19}$$

Assume, for example, that $\hat t \le 1$ (otherwise we argue in a similar way but with $\hat t$ replaced by $2-\hat t$). Then, $\hat L = \frac{1}{\hat t}$ and, if $\hat t = 1$, Lemma 2.9 applies. Thus, it remains to consider the case $\hat t \in]0,1[$.
Consider first the case where (3.18) holds. Notice that there exists a sequence $(\zeta_i')_{i\in\mathbb{N}}$ such that $\zeta_i' \in Z_{k_i}$ and $|z_i' - \zeta_i'| = 1$ $\forall i \in \mathbb{N}$.
Then, the minimality property (3.13) implies

$$\lim_{i\to\infty} \frac{2\max\{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z_i'}\right),\, \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta_i'}\right)\}}{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z_i'}\right) + \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta_i'}\right)} = 2-\hat t \tag{3.20}$$

and, arguing as in the proof of Lemma 2.9 but with $\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z'_i}\right) + \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta'_i}\right)$ instead of $\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z'_i}\right)$ also

$$\lim_{i\to\infty} k_i \left[\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z'_i}\right) + \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta'_i}\right)\right] = 2. \tag{3.21}$$

As a consequence of (3.20) and (3.21), we obtain

$$\lim_{i\to\infty} k_i \max\left\{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z'_i}\right), \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta'_i}\right)\right\} = 2-\hat{t} \tag{3.22}$$

which is a contradiction because

$$\lim_{i\to\infty} k_i \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z'_i}\right) > \hat{L} = \frac{1}{\hat{t}} \tag{3.23}$$

with $\frac{1}{\hat{t}} > 2-\hat{t}$ for $\hat{t} \in]0,1[$.
Thus, we can conclude that the case (3.18) cannot happen.
In a similar way we argue in order to obtain a contradiction in the case (3.19). In fact, assume that (3.19) holds. Notice that there exists a sequence $(\zeta''_i)_{i\in\mathbb{N}}$ such that $\zeta''_i \in Z_{k_i}$ and $|z''_i - \zeta''_i| = 1\ \forall i \in \mathbb{N}$.
As before, the minimality property (3.13) implies that

$$\lim_{i\to\infty} \frac{2\min\{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z''_i}\right), \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta''_i}\right)\}}{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z''_i}\right) + \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta''_i}\right)} = \hat{t} \tag{3.24}$$

and

$$\lim_{i\to\infty} k_i \left[\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z''_i}\right) + \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta''_i}\right)\right] = 2. \tag{3.25}$$

As a consequence, we infer that

$$\lim_{i\to\infty} k_i \min\left\{\operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z''_i}\right), \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{\zeta''_i}\right)\right\} = \hat{t}, \tag{3.26}$$

which is in contradiction with (3.19) because

$$\lim_{i\to\infty} k_i \operatorname{meas} T^L_{k_i}\left(\frac{1}{k_i}\,C_{z''_i}\right) < \frac{1}{\hat{L}} = \hat{t}. \tag{3.27}$$

Thus, we can conclude that $\lim_{i\to\infty} \mathcal{L}(T^L_{k_i}) = \hat{L}$ so the proof is complete.

q.e.d.

Remark 3.3 The results we present in this paper concern the existence of solutions with a large number of nodal regions. In particular, when $\Omega \subset \mathbb{R}^n$ with $n = 1$, these solutions must have, as a consequence, a large number of zeroes. In next propositions we show that the term w can be chosen in such a way that the sign of the solutions is related to the nodal regions of the eigenfunctions of the Laplace operator $-\Delta$ in $H^1_0(\Omega)$. In particular, if $n = 1$ we show that for suitable terms w in $L^2(\Omega)$, problem (1.1) does not have solutions with a small number of zeroes: more precisely, we show that for all positive integer h there exists $w_h \in L^2(\Omega)$ such that every solution of problem (1.1) has at least h zeroes (it follows from Corollary 3.6).

Lemma 3.4 *Let D be a piecewise smooth, bounded open subset of $\mathbb{R}^n$, $n \geq 1$, and λ_1 be the first eigenvalue of the Laplace operator $-\Delta$ in $H^1_0(D)$.*
Let $g : D \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$\inf\{g(x,t) - \lambda_1 t \ : \ x \in D, \ t \geq 0\} > 0. \tag{3.28}$$

Let $u \in H^1(D)$ be a weak solution of the equation

$$-\Delta u = g(x,u) \qquad in\ D. \tag{3.29}$$

Then $\inf_D u < 0$. Moreover, if

$$\sup\{g(x,t) - \lambda_1 t \ : \ x \in D, \ t \leq 0\} < 0, \tag{3.30}$$

then $\sup_D u > 0$.

<u>Proof</u> Let e_1 be a positive eigenfunction corresponding to the eigenvalue λ_1, that is

$$\Delta e_1 + \lambda_1 e_1 = 0, \quad e_1 > 0 \ \text{ in } D, \quad e_1 \in H^1_0(D). \tag{3.31}$$

Arguing by contradiction, assume that (3.28) holds and $u \geq 0$ in D. Then, from (3.29) we infer that

$$-\int_D \Delta u\, e_1\, dx = \int_D g(x,u) e_1\, dx \tag{3.32}$$

which implies

$$\int_D Du\, De_1\, dx = \int_{\partial D} u\, De_1 \cdot \nu\, d\sigma - \int_D u\, \Delta e_1\, dx = \int_D g(x,u) e_1\, dx, \tag{3.33}$$

where ν denotes the outward normal on ∂D, so that

$$\int_{\partial D} u\, De_1 \cdot \nu\, d\sigma \leq 0, \tag{3.34}$$

and $g(x,t) \geq \lambda_1 t + c \ \forall x \in D$, $\forall t \in \mathbb{R}$ for a suitable constant $c > 0$. It follows that

$$\lambda_1 \int_D u e_1 \, dx \geq \int_D g(x,u) e_1 dx \geq \lambda_1 \int_D u e_1 \, dx + c \int_D e_1 \, dx, \tag{3.35}$$

which implies $c \int_D e_1 \, dx \leq 0$, that is a contradiction. Thus, the function u cannot be a.e. nonnegative in D.
In a similar way one can show that we cannot have $u \leq 0$ a.e. in D when (3.30) holds, so the proof is complete.

q.e.d.

In particular, Lemma 3.4 may be used to obtain informations on the effect of the term w on the sign changes of the solutions of problem (1.1), as we describe in the following proposition.

Proposition 3.5 *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ and $e_k \in H_0^1(\Omega)$ be an eigenfunction of the Laplace operator $-\Delta$ with eigenvalue λ_k, that is $\Delta e_k + \lambda_k e_k = 0$ in Ω. Assume that $w \in L^2(\Omega)$ satisfies*

$$e_k w \geq 0 \quad \text{in } \Omega, \quad \inf_\Omega |w| > \max\{\lambda_k t - t^p \ : \ t \geq 0\}. \tag{3.36}$$

Let $\Omega_k \subseteq \Omega$ be a nodal region of e_k, that is $e_{k|\Omega_k} \in H_0^1(\Omega_k)$, $e_k \neq 0$ everywhere in Ω_k and the sign of e_k is constant in Ω_k.
Then, there exists no function u in $H^1(\Omega)$ satisfying

$$u \, e_k \geq 0 \quad \text{and} \quad -\Delta u = |u|^{p-1} u + w \qquad \text{in } \Omega_k. \tag{3.37}$$

In particular, if w satisfies (3.36), *every solution u of problem* (1.1) *must satisfy*

$$\inf_{\Omega_k} u < 0 \quad \text{if} \quad e_k > 0 \quad \text{in} \quad \Omega_k \quad \text{and} \quad \sup_{\Omega_k} u > 0 \quad \text{if} \quad e_k < 0 \quad \text{in} \quad \Omega_k. \tag{3.38}$$

Proof Notice that λ_k is the first eigenvalue of the Laplace operator $-\Delta$ in $H_0^1(\Omega_k)$ and $|e_k|$ is a corresponding positive eigenfunction. Moreover, if we set $g(x,t) = |t|^{p-1} t + w(x)$, we infer from (3.36) that, if $w(x) > 0$,

$$g(x,t) \geq \lambda_k t + \tilde{c} \qquad \forall t \geq 0 \tag{3.39}$$

and, if $w(x) < 0$,

$$g(x,t) \leq \lambda_k t - \tilde{c} \qquad \forall t \leq 0 \tag{3.40}$$

where

$$\tilde{c} = \inf_\Omega |w| - \max\{\lambda_k t - t^p \ : \ t \geq 0\} > 0. \tag{3.41}$$

Since $ue_k \geq 0$ and $we_k \geq 0$ in Ω_k and e_k has constant sign in Ω_k, we have $u \geq 0$ and $w > 0$ in Ω_k if $e_k > 0$ in Ω_k and $u \leq 0$, $w < 0$ in Ω_k in the opposite case. Thus, our assertion follows from Lemma 3.4. In fact, for example, in the case $e_k > 0$ in Ω_k we cannot have

$$-\Delta u = |u|^{p-1}u + w \quad \text{ in } \Omega_k \tag{3.42}$$

otherwise $\inf_{\Omega_k} u < 0$, because of Lemma 3.4, while $u \geq 0$ in Ω_k.
In the opposite case, when $e_k < 0$ in Ω_k, one can argue in a similar way, so the proof is complete.

q.e.d.

Corollary 3.6 *Assume $n = 1$ and $\Omega =]a, b[$. Let us denote by $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ the eigenvalues of the Laplace operator $-\Delta$ in $H^1_0(]a, b[)$ and, for all $k \in \mathbb{N}$, consider an eigenfunction e_k with eigenvalue λ_k. Moreover, assume that $w \in L^2(]a, b[)$ satisfy condition* (3.36)*.*
Set $h = k - 1$ (h is the number of zeroes of e_k in $]a, b[$).
Then, every solution of problem (1.1) *has in $]a, b[$ at least h zeroes $\zeta_1, \zeta_2, \dots, \zeta_h$ such that*

$$\left| a + \frac{b-a}{k}\, i - \zeta_i \right| < \frac{b-a}{k} \qquad \text{for } i = 1, \dots, h. \tag{3.43}$$

Proof Notice that the points $\nu_i = a + \frac{b-a}{k}\, i$, for $i = 0, 1, \dots, k$, are the zeroes of e_k in $[a, b]$ and the intervals $I_i =]\nu_{i-1}, \nu_i[$, for $i = 1, \dots, k$, are the nodal regions of e_k.
Assume, for example, that $e_k > 0$ on I_1 (in a similar way one can argue if $e_k < 0$ in I_1). Then, from Proposition 3.5 we infer that for every solution u of problem (1.1) we have $\inf_{I_i} u < 0$ for i odd and $\sup_{I_i} > 0$ for i even.
Therefore, the function u has at least h zeroes $\zeta_1, \dots, \zeta_h$ such that $\zeta_i \in]\nu_{i-1}, \nu_{i+1}[$ for $i = 1, \dots, h$, so the proof is complete.

q.e.d.

Remark 3.7 Notice that all the assertions in Proposition 3.5 and Corollary 3.6 still hold when the nonlinear term $|u|^{p-1}u$ is replaced by $c_+(u^+)^p - c_-(u^-)^p$ where c_+ and c_- are two positive constants. In this case we have only to replace $\max\{\lambda_k t - t^p \ : \ t \geq 0\}$ by $\max\{\lambda_k t - \bar{c}t^p \ : \ t \geq 0\}$, where $\bar{c} = \min\{c_+, c_-\} > 0$.

Notice that this method to construct solutions with nodal regions having a check structure may be used for nonlinear elliptic problems with different boundary conditions, for systems and also when the nonlinear term has critical growth. For example, for all $\lambda \in \mathbb{R}$ consider the Dirichlet problem

$$-\Delta u = |u|^{\frac{4}{n-2}}u + \lambda u + w \quad \text{ in } \Omega, \qquad u = 0 \quad \text{ on } \partial\Omega \tag{3.44}$$

whose solutions are critical points of the energy functional $\mathcal{F} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{n-2}{2n}\int_\Omega |u|^{\frac{2n}{n-2}} dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega w\, u\, dx \qquad u \in H_0^1(\Omega). \tag{3.45}$$

Using this method, if the functional $\mathcal{F}$ satisfies condition (2.53), one can prove that for $n \geq 4$ and $\lambda > 0$ the functional $\mathcal{F}$ has an unbounded sequence of critical levels. More precisely, the following theorem can be proved.

Theorem 3.8 *Let $n \geq 4$, $\lambda > 0$, $w \in L^2(\Omega)$ and assume that condition (2.53) holds for the functional $\mathcal{F}$. Then, there exists $\bar{k} \geq k_\Omega$ such that for all $k \geq \bar{k}$ there exists $T_k^{\bar{L}} \in \mathcal{C}_{\bar{L}}(P_k, \overline{\Omega})$ and a solution u_k of problem (3.44) such that $T_k^{\bar{L}}(P_k) = \overline{\Omega}$ and, if for all $z \in Z_k$ we set $u_k^z(x) = u_k(x)$ when $x \in T_k^{\bar{L}}\left(\frac{1}{k}\, C_z\right)$, $u_k^z(x) = 0$ otherwise, then we have $[\sigma(z)u_k^z]^+ \not\equiv 0$,*

$$\mathcal{F}(u_k^z) \leq \frac{1}{n}\, S^{n/2} \quad \forall z \in Z_k \quad \text{ and } \quad \lim_{k\to\infty} \min\left\{\mathcal{F}(u_k^z) \ : \ z \in Z_k\right\} = \frac{1}{n}\, S^{n/2}, \tag{3.46}$$

where S (the best Sobolev constant) is defined by

$$S = \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^2 dx \ : \ u \in H^1(\mathbb{R}^n),\ \int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx = 1\right\}. \tag{3.47}$$

Let us point out that Theorem 3.8 gives a new result also when $w \equiv 0$ in Ω. In fact, in this case the functional $\mathcal{F}$ is even but well known results (see [12, 16, 38]) guarantee only the existence of a finite number of solutions (because some compactness conditions hold only at suitable levels of $\mathcal{F}$). On the contrary our method, combined with some estimates as in [12] and in [16], allows us to construct infinitely many solutions with many nodal regions and arbitrarily large energy level.

Funding. The authors have been supported by the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)" of the *Istituto Nazionale di Alta Matematica (INdAM).*
R.M. acknowledges also the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

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