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ORIGINAL RESEARCH

Scheduling activities in project network with feeding precedence relations: an earliest start forward recursion algorithm

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Abstract
In some production processes, the effort associated with a certain activity for its execution can vary over time. In this case, the amount of work per time unit devoted to each activity, so as its duration, is not univocally determined. This kind of problem can be represented by an activity project network with the so-called *feeding precedence relations*, and activity variable execution intensity. In this paper, we propose a forward recursion algorithm able to find the earliest start and finish times of each activity, in $O(m \log n)$ time, with n and m being the number of activities and the number of precedence relations, respectively. In particular, this requires the calculation of the (optimal) execution intensity profile, for each activity, that warrants the earliest start schedule and the minimum completion time of the project.

Keywords Feeding precedence relations · Minimum makespan · Earliest start schedule · Forward recursion

1 Introduction

Feeding precedence relations are a special type of precedence constraints firstly introduced by Kis et al. (2004) and Kis (2006), and used to model project management applications (like make-to-order manufacturing) in which it is not possible to calculate the exact durations of the activities of a project.

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Feeding precedences require the so called variable activity execution intensity paradigm (see, e.g., Kis (2005)), for which the amount of work done to process each activity varies over time.

Clearly, this is not the case of traditional finish-to-start precedence relations, and Generalized Precedence Relations (GPRs, see, e.g., Bartusch et al. (1988); Elmaghraby and Kamburowski (1992)), where the amount of work done is assumed to be constant over time due to the complete information on the activity durations.

Feeding precedence relations dealt with in this paper are inspired by previous works of Alfieri et al. (2011), and Bianco and Caramia (2011, 2012). They are of four types:

- Start-to-%Completed ($S\%C(g_{ij})$) precedence between two activities (i, j). This precedence imposes that the processed fraction of successor activity j of i can be greater than given portion g_{ij}, with $0 \leq g_{ij} < 1$, only if the execution of i has already started.
- Finish-to-%Completed ($F\%C(g_{ij})$) precedence between two activities (i, j). This precedence imposes that the processed fraction of successor activity j of i can be greater than given portion g_{ij}, with $0 \leq g_{ij} < 1$, only if the execution of i has already completed.
- %Completed-to-Start ($\%C(q_{ij})S$) precedence between two activities (i, j). This precedence imposes that successor activity j of i can be started only if i has been processed for at least a fractional amount q_{ij}, with $0 < q_{ij} \leq 1$.
- %Completed-to-Finish ($\%C(q_{ij})F$) precedence between two activities (i, j). This precedence imposes that successor activity j of i can be completed only if i has been processed for at least a fractional amount q_{ij}, with $0 < q_{ij} \leq 1$.

These precedence relations generalize the Start-to-Start (*SS*), Finish-to-Start (*FS*) and Finish-to-Finish (*FF*) precedence relations (of minimum type). In fact, *SS* is equivalent to $S\%C(0)$, *FS* is equivalent to $\%C(1)S$ and $F\%C(0)$, and *FF* is equivalent to $\%C(1)F$.

Furthermore and for completeness, we also directly consider Start-to-Finish (SF_{ij}) precedence relation. This precedence imposes that successor activity j of i can be completed only if the execution of i has already started. In fact, unfortunately, this constraint cannot be represented by any of the above feeding precedences.

Valls et al. (2009) and, successively, Quintanilla et al. (2012) introduce a more general type of precedence relations, useful for representing technological constraints expressed in terms of (percentage) work content, called *work* GPRs to distinguish them from the classical (*time*) GPRs. In particular, Quintanilla et al. show that feeding precedence relations are strictly included in work GPRs of minimum type. Indeed, when activity preemption is not allowed, as considered in our paper, we show that work GPRs can be mapped into feeding precedence relations, and, hence, the results provided in our paper are also valid for work GPRs of minimum type, as well as for the classical (time) GPRs.

Let $V = \{1, \ldots, n\}$ be the set of n (real) activities to be carried out without preemption and let $A = S\%C \cup \%CS \cup F\%C \cup \%CF \cup SF$ be the set of precedence relations between (ordered) pairs of activities, where $S\%C$, $\%CS$, $F\%C$, $\%CF$, and SF are the subsets of precedences of each specific type.

The set of activities and the set of precedence relations can be represented by an activity-on-node acyclic project network $N = (V, A)$, given that precedence relations respect the transitive property. An example is shown in Fig. 1: for example, the feeding precedence relation related to arc $(1, 2)$ states that the processed percentage of activity 2 can be greater than 60% only if activity 1 has already started. More in general, numbers in parentheses are the fractions of work execution associated with each constraint.

In this paper, given a project network with feeding precedences (without resource constraints) we aim at finding the execution intensity profiles π_i (i.e., the fractions of work done

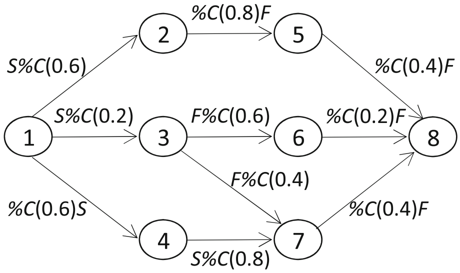


Fig. 1 Example of a project network with feeding precedence relations

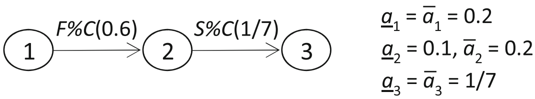


Fig. 2 A problem instance where executing activities at the maximum intensity (or at the minimum intensity) does not lead to the minimum project length

in the time slots of the planning horizon) of each activity $i \in V$ that warrant the minimum completion time (or duration) of the project, i.e., the minimum project makespan $C^*_{\max}$.

In our problem definition, we have two assigned parameters, say $\underline{a}_i$ and $\bar{a}_i$, with $0 < \underline{a}_i \leq \bar{a}_i \leq 1$, associated with each activity $i \in V$, which represent its minimum and maximum execution intensities, respectively, in each time slot in which the activity is executed (except for the last execution time slot if the remaining fraction of work to be done is less than $\underline{a}_i$). The minimum execution intensity $\underline{a}_i > 0$ forces activity i to be executed without preemption, while the maximum execution intensity $\underline{a}_i \leq 1$ fixes the maximum fraction of work of i that can be executed in each time slot, assuming that, in general, the work of an activity cannot be done in a single time slot due to limited availability of resources (not explicitly taken into account in this paper). Given the execution intensity profile π_i of activity i, the number of execution time slots of i represents its duration $d_i(\pi_i)$.

We note that finding the optimal solution to our problem is not straightforward, since adopting the solution where all the activities are executed at their maximum intensity (or at their minimum intensity) does not lead, in general, to a minimum project duration. An example of this occurrence happens for the instance depicted in Fig. 2, where the execution intensity of activity 2 ranges in $[0.1, 0.2]$, while activities 1 and 3 have a fixed execution intensity equal to 0.2 and $1/7$, respectively. In fact, we show (see Sect. 3) that the minimum project makespan is equal to 7, while forcing activity 2 to be executed at its maximum (minimum) intensity, i.e., assuming $\underline{a}_2 = \bar{a}_2 = 0.2$ ($\underline{a}_2 = \bar{a}_2 = 0.1$), gives a project duration equal to 8 (10).

Notwithstanding the aim of minimizing the project makespan, in this paper, we show that the solution guaranteeing the activities $i \in V$ to start at their earliest start time (ES_i) and to finish at their earliest finish time (EF_i) returns also the minimum project length. Therefore, we also aim to determine the execution intensity profiles π_i, $\forall i \in V$, allowing i to start at ES_i and to finish at EF_i.

In particular, in doing so, we pose the primary goal of finding a specific project network representation associated with feeding precedence relations and generalizing the network standardization introduced by Bartusch et al. (1988) for GPRs. Next, with this generalized standard representation, we generalize the forward recursion to (i) compute the activity earliest start and earliest finish times and (ii) identify the critical (longest) paths. The proposed forward recursion algorithm runs in $O(m \log n)$ time, with n and m being the number of activities and the number of precedence relations, respectively.

To the best of our knowledge, this task has not been carried out for project networks with feeding precedence relations. In fact, the literature presents several results on project networks with finish-to-start precedence relations with zero time lags, and with the more general case of GPRs (see, e.g., Bartusch et al. 1988; Bianco et al. 2022; Elmaghraby and Kamburowski 1992; Kelley 1963), but no attempt has been made to define a network model for feeding precedences with which a temporal analysis can be conducted to detect (i) the earliest and latest start schedule of the activities and (ii) the critical path(s) of the project.

The remainder of the paper is organized as follows. In Sect. 2, we present the project network standardization. In Sect. 3, we prove how to find the activity execution intensity profiles that allow to construct the earliest start schedule. We give the description of the forward recursion algorithm, along with the discussion of a few examples. In Sect. 4, we compare feeding precedences with work GPRs. Finally, in Sect. 5, we draw conclusions. Furthermore, supplementary material is given in a *supplement document*; in particular, a MIP mathematical formulation of our problem is provided, which we used to verify the optimality of the solution obtained by the proposed forward recursion algorithm on the analyzed examples.

2 Project network standardization

In addition to the n real activities, we consider two dummy activities 0 and $n+1$, representing the project beginning and completion, respectively. Therefore, we also add feeding precedences $S\%C(0)_{0i}$ (equivalent to precedence SS_{0i}) between activity 0 and activity i, and feeding precedences $\%C(1)S_{i,n+1}$ (equivalent to precedence $FS_{i,n+1}$) between activity i and activity $n+1$, for each $i \in V$.

Let $N' = (V', A')$ be the related augmented activity-on-nodes project network with feeding (and *SF*) precedence relations, where $V' = V \cup \{0, n+1\}$ is the set of nodes formed by the set V of real activities and dummy activities $0, n+1$ corresponding to the source and sink nodes of the project network, respectively, and A' is the whole set of precedence relations among all the activities in V', i.e., $A' = A \cup \{S\%C(0)_{0i} : i \in V\} \cup \{\%C(1)S_{i,n+1} : i \in V\}$.

Generalizing the standardization of project network with GPRs (with minimum time lags) given by Bartusch et al. (1988), we standardize project network N', with feeding precedences, into a (standardized) project network N'_S with only Start-to-Start precedences with minimum time lags. In particular, we substitute each (feeding) precedence relation $(i, j) \in A'$ with the related $SS^{\min}_{ij}(\ell_{ij})$ precedence relation, where ℓ_{ij} represents the minimum time lag that has to be observed between the start times S_i and S_j of activities i and j, respectively, that is: $S_i + \ell_{ij} \leq S_j$. The standardized network N'_S has the same set of nodes V' and the same set of arcs A' of the original project network N', with time lag ℓ_{ij} representing the length of arc (i, j), and it is therefore acyclic as the original project network. It is well known that the length of the longest path in N'_S from node 0 to node i represents the earliest start time of i, and, hence, the length of the longest path from node 0 to node $n+1$ is the minimum project length. However, differently from the project network with GPRs where time lags ℓ_{ij} can easily be derived from the given fixed activity durations, in our case (i.e., with feeding precedences and activity variable execution intensities) their values depend on the specific original feeding constraint among the ordered task couple (i, j) and on the execution intensity profiles π_i and π_j of activities i and j, that, therefore, deeply affect the earliest start schedule and the minimum project duration (length).

Let $k_j^-(g_{ij}, \pi_j)$ be the number of time slots (calculated as nearest integer $\lfloor\cdot\rfloor$ rounded down) needed to complete the fraction g_{ij} of work of activity j from its starting time, given its intensity execution profile π_j; let $k_i^+(q_{ij}, \pi_i)$ be the number of time slots (calculated as nearest integer $\lceil\cdot\rceil$ rounded up) needed to complete the fraction q_{ij} of work of activity i from its starting time, given its intensity execution profile π_i. Moreover, let $d_i(\pi_i)$ be the duration of i, i.e., the number of time slots (calculated as nearest integer rounded up) needed to complete the whole work of activity i from its starting time, given its intensity execution profile π_i; clearly we have: $d_i(\pi_i) = k_i^+(1, \pi_i)$. Denoted with S_i and F_i the start and finish times of activity i, respectively, it results:

- $S\%C(g_{ij})_{ij}$: according to this precedence we have $S_i \leq S_j + k_j^-(g_{ij}, \pi_j)$, since it is required that at the time when the amount of work done for activity j is (strictly) greater than g_{ij} activity i has already been started; therefore,

$$S\%C(g_{ij})_{ij} \equiv SS_{ij}^{\min}(\ell_{ij}), \text{ with } \ell_{ij} = -k_j^-(g_{ij}, \pi_j);$$

- $F\%C(g_{ij})_{ij}$: according to this precedence we have $F_i \leq S_j + k_j^-(g_{ij}, \pi_j)$, since it is required that at the time when the amount of work done for activity j is (strictly) greater than g_{ij} activity i has already been finished; therefore,

$$F\%C(g_{ij})_{ij} \equiv SS_{ij}^{\min}(\ell_{ij}), \text{ with } \ell_{ij} = d_i(\pi_i) - k_j^-(g_{ij}, \pi_j);$$

- $\%C(q_{ij})S_{ij}$: according to this precedence we have $S_i + k_i^+(q_{ij}, \pi_i) \leq S_j$, since it is required that at the time when activity j is started the amount of work done for activity i has to be at least equal to q_{ij}; therefore,

$$\%C(q_{ij})S_{ij} \equiv SS_{ij}^{\min}(\ell_{ij}), \text{ with } \ell_{ij} = k_i^+(q_{ij}, \pi_i);$$

- $\%C(q_{ij})F_{ij}$: according to this precedence we have $S_i + k_i^+(q_{ij}, \pi_i) \leq F_j$, since it is required that at the time when activity j is finished the amount of work done for activity i has to be at least equal to q_{ij}; therefore,

$$\%C(q_{ij})F_{ij} \equiv SS_{ij}^{\min}(\ell_{ij}), \text{ with } \ell_{ij} = k_i^+(q_{ij}, \pi_i) - d_j(\pi_j).$$

Figure 3 shows the effects of feeding precedence relations $S\%C(g_{ij})_{ij}$ and $F\%C(g_{ij})_{ij}$, and the related time constraints depending also on profile π_j of activity j. In particular, according to the precedence relation $S\%C(g_{ij})_{ij}$ ($F\%C(g_{ij})_{ij}$), at most the white portion of profile π_j, where the amount g_{ij} of work of activity j is done, can be completed before the start (finish) of activity i.

Similarly, Fig. 4 shows the effects of feeding precedence relations $\%C(q_{ij})S_{ij}$ and $\%C(q_{ij})F_{ij}$, and the related time constraints depending also on profile π_i of activity i. In particular, according to the precedence relation $\%C(q_{ij})S_{ij}$ ($\%C(q_{ij})F_{ij}$), activity j cannot start (finish) before the completion of the white portion of profile π_i, where amount q_{ij} of work of activity i is done.

Since in addition we also consider precedence SF_{ij}, meaning that we have $S_i \leq F_j$, we recall for completeness of representation also the well known standardization for this type of precedence:

$$SF_{ij} \equiv SS_{ij}^{\min}(\ell_{ij}), \text{ with } \ell_{ij} = -d_j(\pi_j).$$

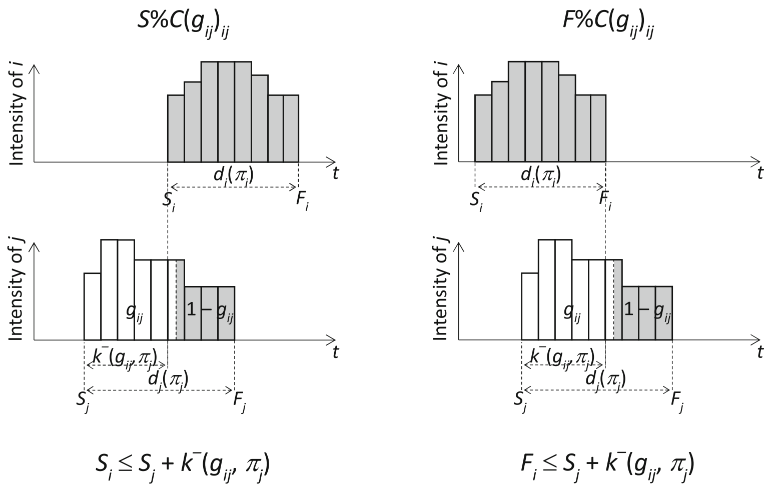


Fig. 3 $S\%C(g_{ij})_{ij}$ and $F\%C(g_{ij})_{ij}$ feeding precedence constraints

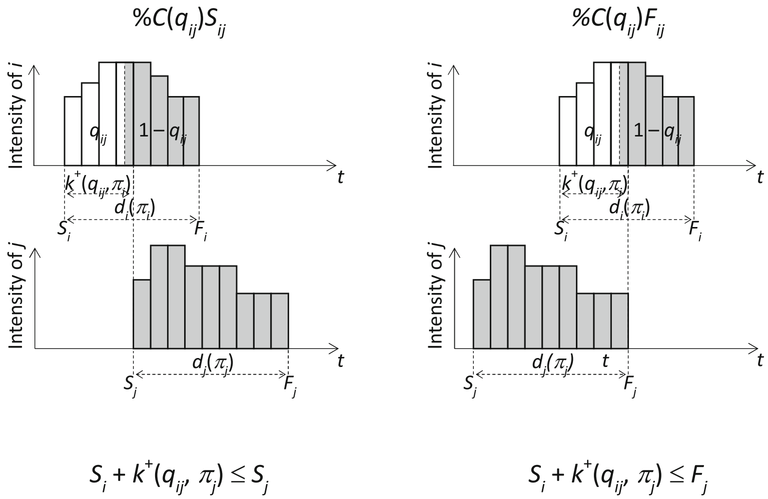


Fig. 4 $\%C(q_{ij})S_{ij}$ and $\%C(q_{ij})F_{ij}$ feeding precedence constraints

Therefore, in general, for each arc $(i, j) \in A'$ of the standardized project network N'_S, we can consider the length $\ell_{ij} = \ell^+_{ij}(\pi_i) - \ell^-_{ij}(\pi_j)$, where both $\ell^+_{ij}(\pi_i)$ and $\ell^-_{ij}(\pi_j)$ are non negative integers:

- $\ell^+_{ij}(\pi_i)$ is the contribution depending on the execution profile π_i of the preceding activity i, being $\ell^+_{ij}(\pi_i) = 0$ if the precedence is SX_{ij} (where $X \in \{\%C, F\}$), and greater than 0, otherwise: in particular, $\ell^+_{ij}(\pi_i) = k^+_i(q_{ij}, \pi_i)$ if the precedence relation is $\%C(q_{ij})X_{ij}$ (where $X \in \{S, F\}$), and $\ell^+_{ij}(\pi_i) = d_i(\pi_i)$ if the precedence is $F\%C(g_{ij})_{ij}$;
- $\ell^-_{ij}(\pi_j)$ is the contribution depending on the execution profile π_j of the succeeding activity j, being $\ell^-_{ij}(\pi_j) = 0$ if the precedence is $X\%C(0)_{ij}$, with (where $X \in \{S, F\}$) or $\%C(q_{ij})S_{ij}$, and greater or equal than 0 otherwise: in particular, $\ell^-_{ij}(\pi_j) = k^-_j(g_{ij}, \pi_j)$

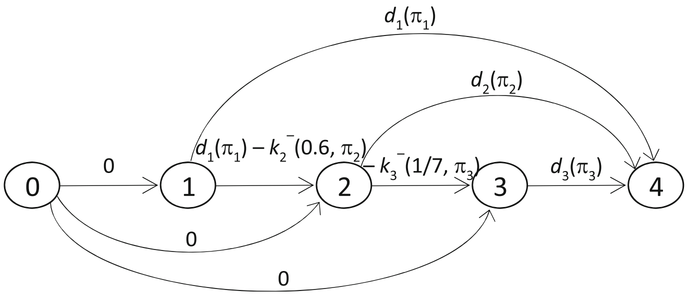


Fig. 5 The standardized network of the project network of Fig. 2

if the precedence relation is $X\%C(g_{ij})_{ij}$ (where $X \in \{S, F\}$), and $\ell_{ij}^-(\pi_j) = d_j(\pi_j)$ if the precedence is XF (where $X \in \{S, \%C\}$).

Clearly, the role of the additional set $\{S\%C(0)_{0i} : i \in V\}$ of feeding precedence relations, between dummy initial activity 0 and each real activity $i \in V$, is to assure that each activity i cannot start before the start time of dummy activity 0 assumed to be equal to 0, i.e., $S_0 = 0$. In fact feeding precedence relation $S\%C(0)_{0i}$ implies that the length of arc $(0, i)$ of the standardized project network N'_S is $\ell_{0i} = 0$, meaning that $S_i \geq S_0 + \ell_{0i} = 0$, and, then, the earliest start time of activity i must be $ES_i \geq 0$.

Analogously, the role of the additional set $\{\%C(1)S_{i,n+1} : i \in V\}$ of feeding precedence relations, between each real activity $i \in V$ and dummy final activity $n+1$, is to assure that the project makespan $C_{\max}$ cannot be less than the finish time of any real activity. In fact feeding precedence relation $\%C(1)S_{i,n+1}$ implies that the length of arc $(i, n+1)$ is $\ell_{i,n+1} = d_i(\pi_i)$, meaning that $C_{\max} = S_{n+1} \geq S_i + d_i(\pi_i)$, for each $i \in V$.

Figure 5 shows the standardized network of the project network of Fig. 2, along with the arc lengths as a function of the execution intensity profile π_i of each activity $i \in V$.

3 Earliest start schedule forward recursion algorithm

It is well known that the earliest starting time of activity i is equal to the length of the longest path from node 0 to node i in the standardized network N'_S, and, hence, that the minimum project makespan $C^*_{\max}$ is equal to the length of the longest path from node 0 to node $n+1$. If the arc lengths were given, or could be simply computed as in the case of GPRs and fixed activity durations, the above quantities could be computed by the well known forward recursion algorithm that in $O(|A'|)$ time determines the longest paths from node 0 to the other nodes of N'_S.

On the contrary, in the case of activities with variable execution intensity and with feeding precedence relations, determining the activity earliest start schedule is not equally simple, because arc lengths are not given. In fact, in our case, we do not know in advance the values of the activity durations or, more in general, the amount of time (number of time slots) needed to complete a given fraction of the work of a given activity. In fact, this amount of time depends on the activity execution intensity profile, which is not specified in advance because activities have in general variable execution intensities. Therefore, the forward recursion algorithm for finding the earliest start schedule should be properly adapted and, hence, generalized, so as to determine also the optimal profile π_i^* of each activity $i \in V$.

According to the definitions of maximum and minimum execution intensity we have that $0 < \underline{a}_i \leq \bar{a}_i \leq 1$, for any real activity $i \in V$.

If $\underline{a}_i = \bar{a}_i$, there is a unique feasible execution profile π_i for activity i in which i is executed at (let us say) maximum intensity $\bar{a}_i$ for $\lfloor \frac{1}{\bar{a}_i} \rfloor$ consecutive time slots for a total amount of work $\lfloor \frac{1}{\bar{a}_i} \rfloor \bar{a}_i$, followed by an additional time slot if the remaining fraction $1 - \lfloor \frac{1}{\bar{a}_i} \rfloor \bar{a}_i$ of work is greater than 0. Since this feasible profile is unique it is clearly also optimal (and denoted as π_i^*) with respect to the earliest start schedule, and hence the minimization of the project length.

On the contrary, in the non trivial case where $\underline{a}_i < \bar{a}_i$, we have an unlimited number of feasible execution intensity profiles for activity i, and in general executing i at its maximum intensity $\bar{a}_i$ does not assure the minimization of the project makespan, and not even if i is executed at its minimum intensity $\underline{a}_i$, as already highlighted in Sect. 1.

Example 1 Referring to the standardized network N_S' shown in Fig. 5 of the project network of Fig. 2, we have $\ell_{12} = d_1(\pi_1) - k_2^-(0.6, \pi_2)$, where $d_1(\pi_1) = 5$, since $\underline{a}_1 = \bar{a}_1 = 0.2$. Moreover, we have $\ell_{23} = -k_3^-(\frac{1}{7}, \pi_3) = -1$, and $d_3(\pi_3) = 7$, since $\underline{a}_3 = \bar{a}_3 = \frac{1}{7}$.

It is clear that starting activity 2 at the earliest time, i.e., at time 0, requires the length of path (0, 1, 2) being not greater than 0, that is, $k_2^-(0.6, \pi_2) \geq d_1(\pi_1) = 5$. Therefore profile π_2 cannot be always at maximum intensity $\bar{a}_2 = 0.2$, otherwise $k_2^-(0.6, \pi_2)$ would be equal to 3, meaning that activity 2 could not start earlier than time 2 and, in addition, the longest path from node 0 to node 4 would be (0, 1, 2, 3, 4) of length $0 + 2 - 1 + 7 = 8$. Therefore, project makespan $C_{\max}$ would be equal to 8.

On the contrary, if we execute activity 2 at its minimum intensity $\underline{a}_2 = 0.1$, $k_2^-(0.6, \pi_2)$ would be equal to 6. With this choice, activity 2 can start at time 0, but its duration $d_2(\pi_2)$ will be equal to 10. This will imply that the longest path from node 0 to node 4 will be (0, 2, 4) of length $0 + 10 = 10$, and, hence, $C_{\max} = 10$.

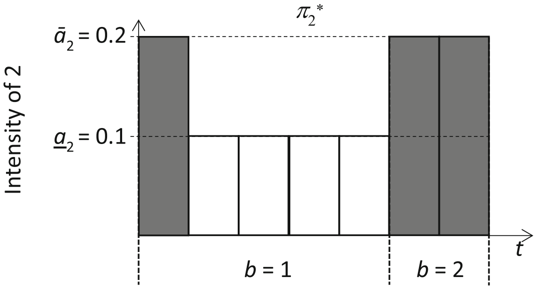
A better choice would be executing activity 2 initially at the *maximum* intensity $\bar{a}_2$ for the largest number of time slots $\alpha_2 \geq 0$ and for a total amount $z_2 = \alpha_2\, \bar{a}_2 \leq g_{12} = 0.6$ of work of the activity, followed by $\beta_2 \geq 0$ time slots, where the amount $y_2 = \beta_2\, \underline{a}_2 = g_{12} - z_2 \geq 0$ of work is executed at the *minimum* intensity $\underline{a}_2$, such that $\alpha_2 + \beta_2 = k_2^-(0.6, \pi_2) \geq 5$. This choice will in particular imply that the latter inequality is satisfied at the equality, i.e., $\alpha_2 + \beta_2$ get the minimum possible value equal to 5, while the amount of work done during these 5 time slots is equal to $g_{12} = 0.6$, because $z_2 + y_2 = \alpha_2\, \bar{a}_2 + \beta_2\, \underline{a}_2 = g_{12}$.

Note that, in this way, it is guaranteed that activity 2 can start at its earliest start time, whichever is the structure of the remaining section of the activity execution intensity profile. In particular, we can execute the remaining fraction of work $1 - (z_2 + y_2)$ of the activity at the maximum possible intensity.

This choice would be indeed overall *optimal*, because it would also guarantee that $\ell_{24}^+(\pi_2) = k_2^+(1, \pi_2) = d_2(\pi_2)$ is minimized, assuring that the activity can end at its earliest finish time.

This optimal profile π_2^* of activity 2 can be regarded as composed by 2 blocks, as shown in Fig. 6. In the first one, whose structure depends on the feeding constraint $F\%C(g_{12})_{12}$, with $g_{12} = 0.6$, activity 2 is executed at the *maximum* intensity $\bar{a}_2 = 0.2$ during the first $\alpha_2^* = 1$ time slots, for the total fraction $z_2^* = \alpha_2^*\, \bar{a}_2 = 0.2$ of work, followed by $\beta_2^* = 4$ time slots, where the total fraction $y_2^* = \beta_2^*\, \underline{a}_2 = 0.4$ of work, exactly equal to $g_{12} - z_2^*$, is executed at the *minimum* intensity $\underline{a}_2 = 0.1$. That is, $\alpha_2^*\, \bar{a}_2 + \beta_2^*\, \underline{a}_2 = g_{12}$. Finally, the remaining amount of work $1 - g_{12} = 0.4$ is done in the second block at the *maximum* intensity occupying 2 final time slots.

Fig. 6 The optimal execution intensity profile π_2^* of activity 2 of the project network of Fig. 2



With this choice, $k_2^-(0.6, \pi_2^*) = 5$, and, hence, $\ell_{12} = 0$; in addition, $\ell_{24} = d_2(\pi_2^*) = 7$ (being the minimum possible value, while assuring that the activity could start at its earliest start time), and the project makespan, equal to the length of longest path (0, 1, 2, 4), has the minimum value equal to $C^*_{\max} = 7$. Note that also paths (0, 2, 4) and (0, 3, 4) are both of length 7 and, hence, they are other longest paths. The optimality of the solution is confirmed by the optimal solution returned by a commercial solver for the MIP formulation given in Appendix D of the *supplemental document*.

Example 2 In order to generalize the previous example, let us assume that the minimum intensity of activity 2 is slightly smaller: for example, let us assume that $\underline{a}_2 = 0.09$.

In this case, the first block of the optimal profile of activity 2 cannot be simply subdivided in two sub-blocks, where in the first one the activity is executed at the *maximum* intensity and in the last second at the *minimum* intensity. This is because, differently from the previous case, for $\underline{a}_2 = 0.09$, if $\alpha_2 = 1$ and $\beta_2 = 4$, then the total amount of work of activity 2 done in these $\alpha_2 + \beta_2 = 5$ time slots will be $\alpha_2\, \bar{a}_2 + \beta_2\, \underline{a}_2 = 0.56 < g_{12} = 0.6$. With this choice for the first 5 time slots of profile π_2, we still have $k^-(0.6, \pi_2) = \alpha_2 + \beta_2 = 5$, and hence activity 2 can still start at time 0. However, the remaining work to be done after this block of 5 time slots would be $1 - 0.56 = 0.44$, that if executed at the maximum intensity, would require 3 additional time slots. Therefore, the activity duration would be equal to 8, and, hence, $C_{\max} = 8$.

We would be able to reduce the duration of activity 2, if we could anticipate a sufficiently amount of the remaining work that would be done after the first 5 time slots, while assuring that $k_2^-(g_{12}, \pi_2) = 5$. In our case, this means anticipating the amount 0.04 of work before the end of the first block, in order to execute exactly the fraction $g_{12} = 0.6$ of work during the first 5 time slots of the activity execution profile.

This suggests to correct the structure of the first block of 5 time slots, assuming that the activity could be executed at *intermediate* intensity w_2, with $\underline{a}_2 \leq w_2 < \bar{a}_2$, in the first $\gamma_2 \geq 1$ time slots after the first sub-block. Therefore, we assume that this block is composed by *three* sub-blocks of α_2, γ_2, and β_2 time slots, where the activity is executed at *maximum*, *intermediate*, and *minimum* intensity, respectively. The role of the second sub-block at intermediate intensity is to allow the total amount of work done in the block, i.e., $\alpha_2\, \bar{a}_2 + \gamma_2\, w_2 + \beta_2\, \underline{a}_2$, to reach exactly the maximum possible value (equal to $g_{12} = 0.6$), so as $\alpha_2 + \gamma_2 + \beta_2 = k_2^-(g_{12}, \pi_2) \geq 5$, and while maximizing $\alpha_2\, \bar{a}_2 + \gamma_2\, w_2$.

It is not hard to verify that the first block of the optimal execution intensity profile π_2^* of activity 2 will be composed by three sub-blocks, with the following structure: the first sub-block contains $\alpha_2^* = 1$ time slot, where the amount $z_2^* = 0.2$ of work of activity 2 is done at *maximum* intensity $\bar{a}_2 = 0.2$; the second sub-block contains $\gamma_2^* = 1$ time slot, where the amount $w_2^* = 0.13$ of work is done (at *intermediate* intensity $w_2^* = 0.13$); finally, the

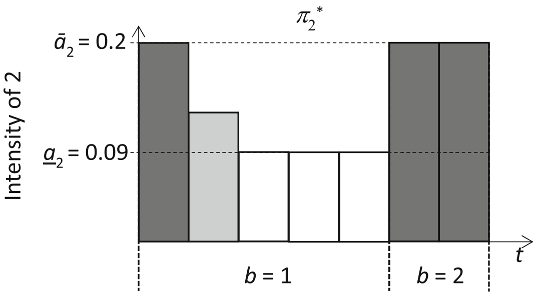


Fig. 7 The optimal execution intensity profile π_2^* of activity 2 of the project network of Fig. 2, assuming $\underline{a}_2 = 0.09$

third sub-block contains $\beta_2^* = 3$ time slots, where the amount $y_2^* = g_{12} - (z_2^* + w_2^*) = 0.27$ of the activity work is done at *minimum* intensity $\underline{a}_2 = 0.09$.

Finally, the remaining work $1 - g_{12} = 0.4$ is executed at the *maximum* possible intensity in the second final block containing 2 time slots, where in the second last time slot activity 2 is executed at its maximum intensity $\bar{a}_2 = 0.2$ and in the last one with intensity $0.15 < \bar{a}_2$ (see Fig. 7).

With this choice, $k_2^-(g_{12}, \pi_2^*) = 5$, and, hence, $\ell_{12} = 0$; in addition, $\ell_{24} = d_2(\pi_2^*) = 7$ (being the minimum possible value, while assuring the activity could start at its earliest start time), and the project makespan, equal to the length of path (0, 1, 2, 4), has the minimum value equal to $C_{\max}^* = 7$. Also paths (0, 2, 4) and (0, 3, 4) are of length 7 and, hence, they are other longest paths. The optimality of the solution is confirmed by the optimal solution returned by a commercial solver for the MIP formulation given in Appendix D of the *supplemental document*.

We note that,

Remark 1 The number of time slots of the second sub-block of the first block of the optimal profile π_2^* is exactly equal to 1 (i.e., $\gamma_2^* = 1$).

Finally, note that also the first block of the optimal profile valid for Example 1 can be regarded as formed by three sub-blocks, where the first sub-block at the maximum intensity remains of 1 time slot, the second sub-block of 1 time slot has intermediate intensity $w_2 = 0.1$ (i.e, equal to the minimum intensity $\underline{a}_2$ in this case), and the third sub-block at the minimum intensity has length 3 time slots.

Indeed, this is always the case, in general (see proof of Proposition 1). That is, for each activity i, there is an optimal profile π_i^* subdivided in blocks, where any non-last block of π_i^* is formed by three sub-blocks. In next subsection, we formalize the structure of such a kind of execution intensity profile for a generic non-preemptive real activity $i \in V$, with variable execution intensity, of a given project network $N = (V, A)$ with feeding precedence relations.

3.1 The *max-inter-min* execution intensity profile

Let us introduce the following execution intensity profile for activity $i \in V$, where during the period in which i is executed its execution intensity profile is subdivided into r_i consecutive non-empty blocks, each one composed by three sub-blocks (sections).

The three sub-blocks of block $b \in \{1, \ldots, r_i - 1\}$ have the following sizes: the first one is composed by $\alpha_i^b \geq 0$ time slots in which activity i is executed at the *maximum* intensity

$\bar{a}_i$, followed by the second one formed exactly by $\gamma_i^b = 1$ time slot in which activity i is executed at *intermediate* intensity w_i^b, with $\underline{a}_i \le w_i^b < \bar{a}_i$, and followed by the third sub-block composed by $\beta_i^b \ge 0$ time slots in which activity i is executed at the *minimum* intensity $\underline{a}_i$. In particular, let $z_i^b = \alpha_i^b\, \bar{a}_i$ and $y_i^b = \beta_i^b\, \underline{a}_i$ be the (total) fraction of work of i executed at the maximum and the minimum intensity, respectively in the first and third sub-blocks.

It has to be noted that α_i^b and β_i^b can be equal to 0: in these two cases the first and the third sub-blocks are empty, i.e., $z_i^b = 0$, and $y_i^b = 0$, respectively. On the contrary, assuming that the second sub-block is always not empty and formed by exactly one time slot (i.e., $\gamma_i^b = 1$) is not a restriction. In fact,

Proposition 1 *The number of time slots of the second sub-block of block $b < r_i$ can be assumed exactly equal to* 1*, i.e., $\gamma_i^b = 1$.*

Proof If block b is not the last one, then activity i is not completed at the maximum intensity within this block, otherwise the block can be considered as part of the first sub-block of the next block and, hence, block b would be totally empty. Therefore, block b contains at least one time slot, where the amount of work done is less than $\bar{a}_i$, and none where the activity execution intensity is less than $\underline{a}_i$. Since the amounts of work done in each time slot of block b are arranged in non increasing order in order to keep the values of $\ell_{ij}^+(\pi_i)$ as small as possible, in the last time slots of block b the activity is not done at the maximum intensity, and the number of these time slots is greater than 0.

Therefore, without loss of generality, let us assume that in block b we have firstly $\tilde{\alpha}_i^b \ge 0$ time slots where activity i is executed at maximum intensity $\bar{a}_i$, followed by $\tilde{\gamma}_i^b + \tilde{\beta}_i^b \ge 1$ time slots, where the activity is executed with non increasing intensity order: $\tilde{\gamma}_i^b$ is the number of (consecutive) time slots where activity i is executed with intensity greater than $\underline{a}_i$ and less than $\bar{a}_i$, and $\tilde{\beta}_i^b$ is the number of (last) time slots of block b where activity i is executed exactly at minimum intensity $\underline{a}_i$.

If $\tilde{\gamma}_i^b > 1$, we could rearrange the work done in these $\tilde{\gamma}_i^b$ time slots by anticipating at most the amount of work done in the first ones. Accordingly, we would not change the total number of time slots of block b, that therefore would not increase the values of $\ell_{ij}^+(\pi_i)$. Therefore, this profile of block b would not be worse (and possibly would by better) than the given one. Note that, with this rearrangement, in the first $\tilde{\gamma}_i^{\prime b} \ge 0$ time slots of the given $\tilde{\gamma}_i^b > 1$ time slots the activity would by done at maximum intensity, followed by $0 \le \tilde{\gamma}_i^{\prime\prime b} \le 1$ time slots where the activity is done with intensity less than $\bar{a}_i$ and greater than $\underline{a}_i$, and by other $\tilde{\gamma}_i^{\prime\prime\prime b} \ge 0$ time slots where it is done at minimum intensity. Hence, block b would be composed by $\tilde{\alpha}_i^b + \tilde{\gamma}_i^{\prime b} \ge 0$ time slots at maximum intensity $\bar{a}_i$, followed by $0 \le \tilde{\gamma}_i^{\prime\prime b} \le 1$ time slots where the activity is done at intensity less than $\bar{a}_i$ and greater than $\underline{a}_i$, and followed by $\tilde{\beta}_i^b + \tilde{\gamma}_i^{\prime\prime\prime b} \ge 0$ time slots at minimum intensity $\underline{a}_i$.

Therefore, without loss of generality, we can assume that $0 \le \tilde{\gamma}_i^b \le 1$. Accordingly, we can consider the first sub-block of block b formed by $\alpha_i^b = \tilde{\alpha}_i^b \ge 0$ time slots at maximum intensity, and the second block formed anyway by a single time slot (i.e., $\gamma_i^b = 1$) at *intermediate* intensity w_i^b, with $\underline{a}_i \le w_i^b < \bar{a}_i$. The third sub-block at minimum intensity has $\beta_i^b = \tilde{\beta}_i^b \ge 0$ time slots, if $\tilde{\gamma}_i^b = 1$, and $\beta_i^b = \tilde{\beta}_i^b - 1 \ge 0$, otherwise. □

Finally, in the last block, i.e., block r_i, the remaining amount of work $\mu_i^{r_i} = 1 - \sum_{b=1}^{r_i-1}(z_i^b + w_i^b + y_i^b)$ of activity i is done at the maximum possible intensity. That is, in the first $\alpha_i^{r1} = \lfloor \frac{\mu_i^{r_i}}{\bar{a}_i} \rfloor \ge 0$ time slots of the block, activity i is executed at its maximum intensity $\bar{a}_i$ for a

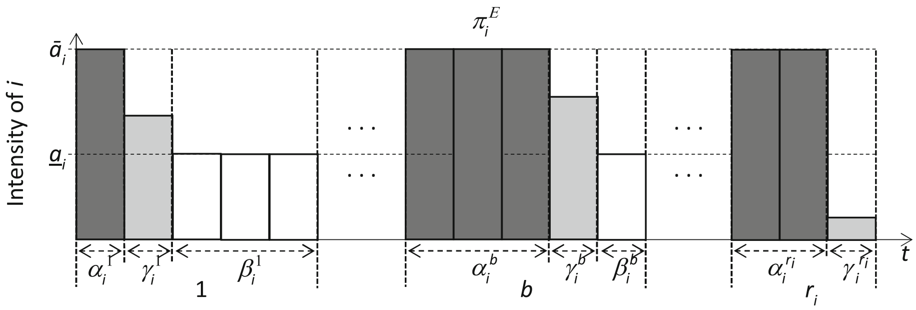


Fig. 8 A *max-inter-min* execution intensity profile π_i^E for an activity i

total amount $z_i^{r_i} = \alpha_i^{r_i}\,\bar{a}_i$ of work. At most one single time slot follows, where the remaining amount $w_i^{r_i} = \mu_i^{r_i} - z_i^{r_i} < \bar{a}_i$ of work is done, if $w_i^{r_i} > 0$. For the sake of uniformity, also last block r_i can be seen as structured in three sub-blocks, where activity i is executed at maximum, intermediate, and minimum intensities, respectively. However, in this case, not only the first block could be empty (i.e., when $\alpha_i^{r_i} = 0$), but also the second sub-block (i.e., $\gamma_i^{r_i} = 0$ if $w_i^{r_i} = 0$, and $\gamma_i^{r_i} = 1$ otherwise), while the third sub-blocks is always empty (i.e., $\beta_i^{r_i} = 0$), because $y_i^{r_i} = \mu_i^{r_i} - (z_i^{r_i} + y_i^{r_i}) = 0$. However, $\alpha_i^{r_i} + \gamma_i^{r_i} + \beta_i^{r_i} \geq 1$, because block r_i cannot be empty, as well as the others.

Definition 1 We refer to such an execution intensity profile as *max-inter-min* profile and we denote it with π_i^E because, as shown next, there exists an execution profile of this type that allows to execute activity i at its *earliest* start and finish times, and hence also to minimize the project makespan if the same is done for all the activities.

Figure 8 shows an example of *max-inter-min* execution intensity profile π_i^E for an activity i. The profile is sub-divided in r_i blocks. Looking at block $b < r_i$ in the figure, the section with dark grey bars is the first sub-block of $\alpha_i^b \geq 0$ time slots, whose total area is $z_i^b = \alpha_i^b\,\bar{a}_i$, corresponding to the amount of work done in these α_i^b time slots at the maximum intensity $\bar{a}_i$. The single light gray bar corresponds to the second sub-block of $\gamma_i^b = 1$ time slot and its area w_i^b, with $\underline{a}_i \leq w_i^b < \bar{a}_i$ is the amount of work of activity i done in this unique time slot. Finally, the white section of block b identifies the third sub-block of β_i^b time slots, where activity i is done at minimum intensity. Block r_i is an exception since its third sub-block is always empty ($\beta_i^{r_i} = 0$), and the remaining amount of work done in the last time slot can be less than the minimum intensity.

If the intensity of activity i is fixed, i.e, $\underline{a}_i = \bar{a}_i$, the activity has a unique feasible (and hence optimal) execution profile π_i^*, in which it is executed at (let us say maximum) speed $\bar{a}_i$ for $\tilde{\alpha}_i = \lfloor \frac{1}{\bar{a}_i} \rfloor > 0$ consecutive time slots for a total amount $\tilde{z}_i = \tilde{\alpha}_i\,\bar{a}_i$ of work, followed by an additional time slot where the remaining fraction of work $\tilde{w}_i = 1 - \tilde{z}_i < \underline{a}_i = \bar{a}_i$ is done, if $\tilde{w}_i > 0$. Clearly and without loss of generality, we can also regard this profile as a *max-inter-min* profile π_i^E containing a single block, i.e., $r_i = 1$, where the first sub-block contains $\alpha_i^1 = \tilde{\alpha}_i > 0$ time slots in which the amount $z_i^1 = \tilde{z}_i$ of work of activity i is done at its maximum intensity, the second sub-block contains a single time slot (i.e., $\gamma_i^1 = 1$) if $w_i^1 = \tilde{w}_i > 0$, otherwise is empty (i.e., $\gamma_i^1 = 0$), and the third sub-block is empty (i.e., $\beta_i^1 = 0$, and $y_i^1 = 0$).

In the next subsection, we show that there is a specific *max-inter-min* execution profile π_i^E for each activity i being optimal with respect to the earliest start (and finish) schedule and guaranteeing the minimization of project length, given an optimal execution profile for each preceding activity.

In particular, we will show how to calculate the structure of π_i^E for the non trivial case with $0 < \underline{a}_i < \bar{a}_i \leq 1$, and prove that the number r_i of blocks of π_i^E is $1 \leq r_i \leq \left|\Gamma^-_{X\%C}(i)\right| + \min\left[1, \left|\Gamma^-_{XF}(i)\right|\right] + 1$, where $\left|\Gamma^-_{X\%C}(i)\right|$ is the cardinality of the subset of its incoming feeding predecences $(h, i) \in X\%C$ (where $X \in \{S, F\}$), and $\left|\Gamma^-_{XF}(i)\right|$ is the cardinality of the subset of its incoming feeding predecences $(h, i) \in XF$ (where $X \in \{S, \%C\}$). For example, for the project network of Fig. 1, $\left|\Gamma^-_{X\%C}(5)\right| = 0$ and $\left|\Gamma^-_{XF}(5)\right| = 1$, i.e., $r_5 \leq 2$; $\left|\Gamma^-_{X\%C}(7)\right| = 2$ and $\left|\Gamma^-_{XF}(7)\right| = 0$, i.e., $r_7 \leq 3$.

3.2 Determining the optimal earliest start and finish execution profile

If activity $i \in V$ has fixed intensity, that is $\underline{a}_i = \bar{a}_i$, we already observed that i has a unique (optimal) execution intensity profile, i.e., a profile π_i^E with a single block ($r_i = 1$) and the following sizes for its three sub-blocks:

- The first sub-block contains $\alpha_i^1 = \lfloor \frac{1}{\bar{a}_i} \rfloor > 0$ time slots in which a (total) amount $z_i^1 = \alpha_i^1 \, \bar{a}_i > 0$ of work of i is done at the maximum speed $\bar{a}_i$;
- The second sub-block contains a single time slot (i.e., $\gamma_i^1 = 1$), containing the residual work $w_i^1 = 1 - z_i^1 < \bar{a}_i$ of i if $w_i^1 > 0$; otherwise, it is empty ($\gamma_i^1 = 0$);
- The third sub-block is empty ($y_i^1 = 0$ and, hence, $\beta_i^1 = 0$).

Now, let us assume that activity $i \in V$ has variable execution intensity ($0 < \underline{a}_i < \bar{a}_i \leq 1$). Since i is at least preceded by dummy activity 0 (that conventionally has $ES_i = 0$) with precedence $S\%C(0)_{0i}$, in the standardized network N'_S arc $(0, i) \in A'$ has length $\ell_{0i} = 0$ and $ES_i \geq 0$.

If there is no real activity h (directly) preceding activity i, we have $ES_i = ES_0 + \ell_{0i} = 0$ and since $\ell_{0i} = 0$ does not depend on the execution intensity profile π_i of i, we can execute the latter activity at its maximum possible intensity to complete it at its earliest finish time, and to allow succeeding activities to start as soon as possible. This (optimal) execution of i corresponds again to profile π_i^E with a single block ($r_i = 1$) with sizes of its three sub-blocks defined described when $\underline{a}_i = \bar{a}_i$.

Now, let us suppose that activity i has variable execution intensity, and at least one real predecessor activity. Moreover, let us assume that for each (real) activity $h \in V$ (directly) preceding i (i.e., such that $(h, i) \in A$), $ES_h \geq 0$ is known and given, as well as the optimal execution intensity profile π_h^* that allows h to start at its ES_h and guarantees, for each outgoing arc $(h, k) \in \Gamma^+(h)$ from h, that $\ell^+_{hk}(\pi_h^*)$ assumes the minimum value, among the profiles allowing h to start at its earliest start time.

In particular, we will prove (see Theorem 10) that, with such a profile π_h^*, the length of the longest path from 0 to h in the standardized network N'_S plus the positive contribution ℓ^+_{hk} of the length of arc (h, k) is minimized and equal to $ES_h + \ell^+_{hk}(\pi_h^*)$. This implies also that, for any succeeding activity k, there is an execution profile that, together with the profiles of the preceding activities, allows activity k to start at its earliest start time. Moreover, it also assures that activity h is finished at its earliest finish time $EF_h = ES_h + \ell^+_{h,n+1}(\pi_h^*) = ES_h + d_h(\pi_h^*)$.

Therefore, in the following, we call such a profile π_i^* of activity i *optimal*, since it allows to start activity i at its earliest starting time ES_i, and guarantees that for each outgoing arc $(i, j) \in \Gamma^+(i)$ of node i, $\ell^+_{ij}(\pi_i^*)$ assumes the minimum value, among the execution intensity profiles of i that allow the activity to start at time ES_i.

Let us denote with π_i^{min} the execution intensity profile of activity i in which i is entirely executed at its minimum intensity $\underline{a}_i$ (with the exception of the last time slot if the residual amount of work to do in this time slot is less than $\underline{a}_i$). Executing i with profile π_i^{min} guarantees that the value of $\ell_{hi}^-(\pi_i)$ is maximized, for each incoming arc (h, i) in node i in N'_S. Therefore, executing i with profile π_i^{min}, assures that i will start at its earliest start time $ES_i = \max[0, \max_{h:(h,i)\in A}\{ES_h + \ell_{hi}^+(\pi_h^*) - \ell_{hi}^-(\pi_i^{min})\}]$.

It is simple to evaluate $\ell_{hi}^-(\pi_i^{min}) = \max_{\pi_i}\{\ell_{hi}^-(\pi_i)\}$, according to the type of the precedence relation between activities h and i: $\ell_{hi}^-(\pi_i^{min}) = k_i^-(g_{hi}, \pi_i^{min}) = \lfloor \frac{g_{hi}}{\underline{a}_i} \rfloor$, if $(h, i) \in X\%C$ (with $X \in \{S, F\}$); $\ell_{hi}^-(\pi_i^{min}) = d_i(\pi_i^{min}) = \lceil \frac{1}{\underline{a}_i} \rceil$, if $(h, i) \in XF$ (with $X \in \{S, \%C\}$); and $\ell_{hi}^-(\pi_i^{min}) = 0$, if $(h, i) \in \%CS$. Therefore, we can suppose that the value of ES_i is known in advance.

We remark that despite the above result, profile π_i^{min} is not necessarily optimal, because it does not assure the minimization of the project makespan (as mentioned in Sect. 1). In the following, we assume that i is executed according to a profile of type π_i^E, and we will prove that there is an optimal profile of this type.

Let us introduce the following notations. Given block $b \in \{1, \ldots, r_i\}$ of π_i^E, let $\chi_i^b = \sum_{p=1}^{b-1}(z_i^p + w_i^p + y_i^p) < 1$ be the amount of work of i already processed in the first $b-1$ blocks of profile π_i^E, and let $\mu_i^b = 1 - \chi_i^b > 0$ be the residual amount of work to be done in the successive blocks.

Moreover, for each incoming arc (h, i) in node i of N'_S, we have that $\ell_{hi}^-(\pi_i^E) = \hat{\ell}_{hi}^{b-}(\pi_i^E) + \tilde{\ell}_{hi}^{b-}(\pi_i^E) + \bar{\ell}_{hi}^{b-}(\pi_i^E)$, where $\hat{\ell}_{hi}^{b-}(\pi_i^E)$ is the fraction (number of time slots) of $\ell_{hi}^-(\pi_i^E)$ due to the first $b-1$ blocks of π_i^E, $\tilde{\ell}_{hi}^{b-}(\pi_i^E)$ is the fraction due to block b, and $\bar{\ell}_{hi}^{b-}(\pi_i^E)$ is the fraction due to the other $r_i - b$ blocks following b in profile π_i^E.

Similarly, for each outgoing arc (i, j) from i, we have $\ell_{ij}^+(\pi_i^E) = \hat{\ell}_{ij}^{b+}(\pi_i^E) + \tilde{\ell}_{ij}^{b+}(\pi_i^E) + \bar{\ell}_{ij}^{b+}(\pi_i^E)$, where $\hat{\ell}_{ij}^{b+}(\pi_i^E)$ is the fraction (number of time slots) of $\ell_{ij}^+(\pi_i^E)$ due to the first $b-1$ blocks of π_i^E, $\tilde{\ell}_{ij}^{b+}(\pi_i^E)$ is the fraction due to block b, and $\bar{\ell}_{ij}^{b+}(\pi_i^E)$ is the fraction due to the other $r_i - b$ blocks following b in profile π_i^E.

In addition, considering the longest path from 0 to i traversing arc (h, i), whose length is $ES_h + \ell_{hi}$, let $\delta_{hi}^b = ES_h + \ell_{hi}^+(\pi_h^*) - \hat{\ell}_{hi}^{b-}(\pi_i^E)$ be the fraction of $ES_h + \ell_{hi}$ that does not depend on block b and on the remaining blocks of profile π_i^E.

Finally, for each incoming precedence (h, i) of i of type $X\%C(g_{hi})$ (where $X \in \{S, F\}$), let $\tilde{g}_{hi}^b = \max[0, g_{hi} - \chi_i^b]$ be the additional fraction of work of i that at most can be done, after the first $b-1$ blocks of profile π_i^E, before the start (end) of activity h assuming $X = S$ $(X = F)$.

3.2.1 The case with a single directly preceding real activity

The above notations have been introduced for the description of the more general case with multiple real activities directly preceding activity i. However, for ease of presentation, let us start by assuming that there is only one (real) activity $h \in V$ directly preceding i, with precedence $(h, i) \in A$.

Let us consider the first block of profile π_i^E, that is, $b = 1$. Therefore, if precedence (h, i) is of type $X\%C(g_{ij})$, then $g_{hi}^b = g_{hi}$, because $b = 1$.

Clearly, we assume that $\mu_i^b > 0$. We have the following two cases:

(i): $(h, i) \in \%CS$ or $(h, i) \in X\%C$ (where $X \in \{S, F\}$);

(ii): $(h,i) \in XF$ (where $X \in \{S, \%C\}$).

Let us consider case (i). We have to examine two sub-cases:

(ia): $(h,i) \in \%CS$, or $(h,i) \in X\%C$ with $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor \leq ES_i$;

(ib): (h,i) belongs to $X\%C$ with $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor > ES_i$.

If (ia) occurs, we have the following result.

Proposition 2 *If there is only one (real) activity $h \in V$ (immediately) preceding activity i such that $(h,i) \in \%CS$, or $(h,i) \in X\%C$ with $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor \leq ES_i$, then an optimal execution intensity profile π^*_i for i is a* max-inter-min *profile π^E_i, having block $r_i = b = 1$ with the following structure π^{E,r_i}_i:*

- *The first sub-block contains $\alpha^{r_i}_i = \lfloor \frac{\mu^{r_i}_i}{\bar{a}_i} \rfloor \geq 0$ time slots in which a (total) amount $z^{r_i}_i = \alpha^{r_i}_i \, \bar{a}_i \geq 0$ of work of i is done at the maximum intensity $\bar{a}_i$;*
- *The second sub-block contains a single time slot (i.e., $\gamma^{r_i}_i = 1$), where the residual work $w^{r_i}_i = \mu^{r_i}_i - z^{r_i}_i < \bar{a}_i$ of i is done, if $w^{r_i}_i > 0$, otherwise it is empty (i.e., $\gamma^{r_i}_i = 0$);*
- *The third sub-block is empty: $y^{r_i}_i = 0$ (and, hence, $\beta^{r_i}_i = 0$).*

Proof Let us suppose that i is executed with profile π^E_i, and let us consider block $b = 1$. If $(h,i) \in \%CS$, then $\ell^-_{hi}(\pi_i) = 0$, for any execution intensity profile π_i of i. Therefore, we have $\delta^b_{hi} \leq ES_i$, because the optimal profile π^*_h of preceding activity h assures that $ES_h + \ell^+_{hi}(\pi^*_h) - \ell^-_{hi}(\pi^{min}_i) \leq ES_i$, and, in our case, $\ell^-_{hi}(\pi^{min}_i) = 0$.

If $(h,i) \in X\%C$ with $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor \leq ES_i$, then, for any profile of the remaining section of profile π^E_i after the first $b-1$ blocks (that in our case are not present because we assume $b = 1$), we have that $ES_h + \ell_{hi} = \delta^b_{hi} - (\tilde{\ell}^{b-}_{hi}(\pi^E_i) + \bar{\ell}^{b-}_{hi}(\pi^E_i)) \leq \delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor \leq ES_i$, because $\tilde{\ell}^{b-}_{hi}(\pi_i) + \bar{\ell}^{b-}_{hi}(\pi^E_i) \geq \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor$, since $\lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor$ would be the value of the left hand side of the last inequality, if the remaining amount μ^b_i of work to do after the first $b-1$ block (in our case $\mu^b_i = 1$, because $b = 1$) were done at the maximum possible intensity.

Therefore, both if $(h,i) \in \%CS$ and if $(h,i) \in X\%C$ with $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor \leq ES_i$, we can execute the whole remaining amount μ^b_i of work at the maximum possible intensity, ensuring that $ES_h + \ell_{hi} \leq ES_i$. Therefore, block b can be the last one (i.e., $r_i = b = 1$) and with structure π^{E,r_i}_i.

Moreover, with this choice, it is also assured that the lengths $\ell^+_{ij}(\pi^E_i) = \hat{\ell}^{r_i+}_{ij}(\pi^E_i) + \tilde{\ell}^{r_i+}_{ij}(\pi^E_i)$ of all the outgoing arcs $(i,j) \in \Gamma^+(i)$ from i are minimized (where, in particular, $\hat{\ell}^{r_i+}_{ij}(\pi^E_i) = 0$, since $r_i = 1$).

In conclusion, this proves that profile π^E_i, having block $r_i = b = 1$ of structure π^{E,r_i}_i, is the optimal profile for activity i, under the assumptions of the proposition. □

Now, let us suppose that case (ib) occurs. Without loss of generality, we can assume that $\tilde{g}^b_{hi} \geq \underline{a}_i$.

Since $(h,i) \in X\%C$ with $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor > ES_i$, differently from case (ia), we cannot execute the remaining work μ^b_i of activity i at the maximum possible intensity, otherwise the activity cannot start at its earliest start time ES_i. In fact, we would have $ES_h + \ell_{hi} = \delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor > ES_i$.

This means that block b cannot be of type π_i^{E,r_i}, and, hence, it cannot be the last block of profile π_i^E, i.e., $b < r_i$.

Therefore, without loss of generality, we assume that the length (number of time slots) of block b is equal to $(\alpha_i^b + 1 + \beta_i^b)$, with non-negative integers α_i^b and β_i^b possibly equal to 0, being respectively the number of time slots of the first and third sub-blocks of block b, and with the second sub-block having exactly $\gamma_i^b = 1$ time slot where the amount of work w_i^b is done, with $\underline{a}_i \leq w_i^b < \bar{a}_i$, according to Proposition 1.

In addition, since the aim is also to execute the maximum amount of the work of activity i in block b, so as to execute the work of i as early as possible, we assume that the amount of work done in block b is equal to $\tilde{g}^b_{hi} \geq \underline{a}_i$, which is the maximum possible amount if we want that activity i can start at time ES_i.

With the above choice, $\tilde{\ell}^{b-}_{hi}(\pi_i^E) = (\alpha_i^b + 1 + \beta_i^b)$, and $\bar{\ell}^{b-}_{hi}(\pi_i^E) = 0$. Therefore, the size of block b must ensure that $ES_h + \ell_{hi} = \delta^b_{hi} - (\alpha_i^b + 1 + \beta_i^b) \leq ES_i$.

Finally, the amount of work $\tilde{g}^b_{hi}$ should be done by anticipating at most the amount of work done in block b, in order to minimize the value of $\tilde{\ell}^{b+}_{ij}(\pi_i^E)$, for each outgoing arc $(i, j) \in \Gamma^+(i)$ of node i. This can be achieved by choosing the minimum value for integer $\beta_i^b \geq 0$.

Therefore, let us consider a *max-inter-min* profile π_i^E where the three sub-blocks of block b have the following sizes:

- $\gamma_i^b = 1$ is the length of the second sub-block,
- $\alpha_i^b \geq 0$ and $\beta_i^b \geq 0$ are integers representing the number of time slots of the first and third sub-blocks, respectively, such that:

$$\delta^b_{hi} - (\alpha_i^b + 1 + \beta_i^b) \leq ES_i, \tag{1}$$

$$\alpha_i^b \, \bar{a}_i + w_i^b + \beta_i^b \, \underline{a}_i = \tilde{g}^b_{hi}, \tag{2}$$

$$\underline{a}_i \leq w_i^b < \bar{a}_i, \tag{3}$$

  and β_i^b is minimized.

The amounts of work done in the three sub-blocks are $z_i^b = \alpha_i^b \, \bar{a}_i$, w_i^b, and $y_i^b = \beta_i^b \, \underline{a}_i$, respectively.

Let us denote with $\pi_i^{E,b}$ the structure of the b-th block, with $b < r_i$, of the *max-inter-min* profile π_i^E, with the above settings, according to case (ib). It results that:

Proposition 3 *If there is only one (real) activity $h \in V$ (immediately) preceding activity i such that $(h, i) \in X\%C$, with $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor > ES_i$, then an optimal execution intensity profile π_i^* for i is a* max-inter-min *profile π_i^E, having block $b = 1 < r_i$ with structure $\pi_i^{E,b}$.*

Proof Let us suppose that activity i is executed with profile π_i^E, and let us consider block $b = 1$. We have already shown that the remaining work μ_i^b of activity i cannot be executed at the maximum possible intensity, otherwise, the activity cannot start at its earliest start time. Therefore, let us assume that the structure of block b of profile π_i^E is of type $\pi_i^{E,b}$.

Constraint (3) forces the second sub-block of block b to be non-empty and composed by a single time slot (i.e., $\gamma_i^b = 1$), according to Proposition 1.

From the definition of δ^b_{hi}, we have that $ES_h + \ell_{hi} = \delta^b_{hi} - (\tilde{\ell}^{b-}_{hi}(\pi_i^E) + \bar{\ell}^{b-}_{hi}(\pi_i^E))$. Therefore, by Inequality (1) and by Eq. (2), it follows that $ES_h + \ell_{hi} \leq ES_i$. In fact, by Eq.

(2) it results $\tilde{\ell}_{hi}^{b-}(\pi_i^E) = (\alpha_i^b + 1 + \beta_i^b)$. Moreover, since $\tilde{g}_{hi}^{b+1} = 0$, it results $\bar{\ell}_{hi}^{b-}(\pi_i^E) = 0$, independently of the structure of the successive blocks of π_i^E. Hence, with block b of type $\pi_i^{E,b}$, it is guaranteed that activity i can start at time ES_i.

Moreover, choosing the minimum integer value for $\beta_i^b \geq 0$, such that Constraints (1)–(3) are fulfilled (with $\alpha_i^b \geq 0$ and integer) allows us to maximize the fraction of work done on block b at the maximum intensity while guaranteeing that activity i can start at time ES_i. Therefore, the structure $\pi_i^{E,b}$ also warrants that, for each outgoing arc $(i, j) \in \Gamma^+(i)$ from node i, the fraction $\tilde{\ell}_{ij}^{b+}(\pi_i^E)$ of $\ell_{ij}^+(\pi_i^E)$ is minimized.

This implies that the value $\ell_{ij}^+(\pi_i^E) = \hat{\ell}_{ij}^{b+}(\pi_i^E) + \tilde{\ell}_{ij}^{b+}(\pi_i^E) + \bar{\ell}_{ij}^{b+}(\pi_i^E)$ is minimized, since $\hat{\ell}_{ij}^{b+}(\pi_i^E) = 0$, if $b = 1$, and $\bar{\ell}_{ij}^{b+}(\pi_i^E)$ assumes the minimum value, since in block $b+1$ it will be possible to execute the remaining amount $\mu_i^{b+1} > 0$ of work at maximum intensity, since $\tilde{g}_{hi}^{b+1}$ will be equal to 0.

In conclusion, this proves that profile π_i^E, having block $b = 1 < r_i$ of structure $\pi_i^{E,b}$, is the optimal profile for activity i, under the assumptions of the proposition.

□

According to Eq. (2), the whole amount of $\tilde{g}_{ij}^b$ is covered in block b. Hence, when we will consider next block $b+1$ we will have $\tilde{g}_{ij}^{b+1} = 0$, meaning that, for block $b+1$, we will be in case (ia) for precedence $(h, i) \in X\%C$, since we will have $\delta_{hi}^{b+1} = \delta_{hi}^b - (\alpha_i^b + 1 + \beta_i^b) \leq ES_i$, according to Inequality (1). Therefore, $r_i = b+1$. Note that such a block always exists because $\tilde{g}_{hi}^{b+1} < \mu_i^{b+1}$, since $g_{hi} < 1$.

For case (ib), let us show how to find the optimal numbers α_i^{*b}, γ_i^{*b}, and β_i^{*b}, of time slots of the three sub-blocks of block b, respectively, that minimize β_i^q, while fulfilling Constraints (1)–(3), with integers $\alpha_i^b \geq 0$ and $\beta_i^b \geq 0$.

Theorem 4 *For case (ib), block $b < r_i$ of* max-inter-min *execution profile π_i^E of activity i, whose structure is $\pi_i^{E,b}$, has three sub-blocks with the following optimal profiles:*

- *First sub-block contains $\alpha_i^{*b} = \left\lfloor \frac{\tilde{g}_{hi}^b - (\delta_{hi}^b - ES_i)\underline{a}_i}{\bar{a}_i - \underline{a}_i} \right\rfloor \geq 0$ time slots, where a total amount $z_i^{*b} = \alpha_i^{*b}\,\bar{a}_i \geq 0$ of work of i is done;*
- *Second sub-block contains $\gamma_i^{*b} = 1$ time slot, where a total amount of work w_i^{*b} of work of i is done, with $\underline{a}_i \leq w_i^{*b} = \tilde{g}_{hi}^b - (\delta_{hi}^b - ES_i)\underline{a}_i - \alpha_i^{*b}(\bar{a}_i - \underline{a}_i) + \underline{a}_i < \bar{a}_i$;*
- *Third sub-block contains $\beta_i^{*b} = \delta_{hi}^b - ES_i - \alpha_i^{*b} - 1 \geq 0$ time slots, where a total amount $y_i^{*b} = \beta_i^{*b}\,\underline{a}_i \geq 0$ of work of i is done.*

Proof From Eq. (2), we have

$$\beta_i^b\,\underline{a}_i = \tilde{g}_{hi}^b - \alpha_i^b\,\bar{a}_i - w_i^b. \tag{4}$$

Multiplying both sides of Inequality (1) by $\underline{a}_i$ and rearranging the resulting inequality, we have

$$\alpha_i^b\,\underline{a}_i + \underline{a}_i + \beta_i^b\,\underline{a}_i \geq (\delta_{hi}^b - ES_i)\underline{a}_i. \tag{5}$$

Therefore, substituting Eq. (4) into Inequality (5) and rearranging the obtained inequality, we have

$$\alpha_i^b(\bar{a}_i - \underline{a}_i) + w_i^b \leq \underline{a}_i + \tilde{g}_{hi}^b - (\delta_{hi}^b - ES_i)\underline{a}_i. \tag{6}$$

It is clear that minimizing β_i^b, while satisfying that Constraints (1)–(3), corresponds to maximizing the left hand side of Inequality (6), with α_i^b integer and with $0 \le \alpha_i^b \le \lfloor \frac{\tilde{g}_{hi}^b}{\bar{a}_i} \rfloor$, while fulfilling Constraint (3).

From Inequalities (3) and (6), it results

$$\alpha_i^b(\bar{a}_i - \underline{a}_i) \le \tilde{g}_{hi}^b - (\delta_{hi}^b - ES_i)\underline{a}_i. \tag{7}$$

We note that the above inequality is well posed, because from Inequalities (1)–(3) it follows that its right hand side is non negative. Moreover, we note that $(\bar{a}_i - \underline{a}_i) > 0$. Let, therefore,

$$\alpha_i'^b = \frac{\tilde{g}_{hi}^b - (\delta_{hi}^b - ES_i)\underline{a}_i}{\bar{a}_i - \underline{a}_i} \ge 0. \tag{8}$$

According to Eq. (8), $\alpha_i'^b$ is the maximum value that can be chosen for α_i^b, if we remove the integer restriction on it, while assuming $w_i^b = \underline{a}_i$.

Let us show that the value $\alpha_i^{*b} = \lfloor \alpha_i'^b \rfloor$ is the searched maximum integer value for (integer) $\alpha_i^b \ge 0$, that, together with the optimal value w_i^{*b} of w_i^b that can be derived from the former, maximize the left hand side of Inequality (6), while fulfilling Constraint (3).

If $\alpha_i'^b$ is an integer and, hence, $\alpha_i^{*b} = \alpha_i'^b$, this value is clearly the maximum searched value for (integer) $\alpha_i^b \ge 0$. Moreover, from Inequalities (6) and (3) it results $w_i^{*b} = \underline{a}_i$, that together with α_i^{*b} maximize the left hand side of Inequalities (6), being satisfied at the equality.

If, oppositely, $\alpha_i^{*b} < \alpha_i'^b$, we assign the value w_i^{*b} to variable w_i^b so as to maximize its value, that is, forcing Constraint (6) binding. We can show that in this way the value w_i^{*b} is feasible with respect to Constraint (3). In fact, assigning the value to variable w_i^b that satisfy Inequality (6) at the equality, that is, letting

$$w_i^{*b} = \tilde{g}_{hi}^b - (\delta_{hi}^b - ES_i)\underline{a}_i - \alpha_i^{*b}(\bar{a}_i - \underline{a}_i) + \underline{a}_i \ge 0, \tag{9}$$

it is easy to verify that $w_i^{*b} \ge \underline{a}_i$, given Inequality (7). Moreover, since $\alpha_i^{*b} + 1 > \alpha_i'^b$, from Eq. (8) we have

$$(\bar{a}_i - \underline{a}_i) > \tilde{g}_{hi}^b - (\delta_{hi}^b - ES_i)\underline{a}_i - \alpha_i^{*b}(\bar{a}_i - \underline{a}_i), \tag{10}$$

which, by Eq. (9), implies $w_i^{*b} < \bar{a}_i$.

It is clear, therefore, that considering the above (optimal) values α_i^{*b} and w_i^{*b} for variables α_i^b and w_i^b, respectively, makes Constraint (1) binding. Then, it follows that the optimal value of β_i^b is $\beta_i^{*b} = \delta_{hi}^b - ES_i - \alpha_i^{*b} - 1$. It is clear that value β_i^{*b} is integer and it easy to verify that it is non-negative.

Summarizing, for case (ib), the amounts of work of activity i done in the three sub-blocks of block b are $z_i^{*b} = \alpha_i^{*b}\,\bar{a}_i$, w_i^{*b} calculated according to Eq. (9), and $y_i^{*b} = \beta_i^{*b}\,\underline{a}_i$. □

Let us consider case (ii): $(h, i) \in XF$ (where $X \in \{S, \%C\}$). Also in this case we have to consider two alternative sub-cases:

(iia): $\delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil \le ES_i$;

(iib): $\delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil > ES_i$.

Let us suppose that case (iia) occurs. For this case, analogously to case (ia), next we prove that the optimal profile is executing activity i at the maximum possible execution intensity, that is, with a *max-inter-min* profile π_i^E with a single block, i.e., $r_i = b = 1$, with the same structure introduced for case (ia) and denoted as π_i^{E,r_i} (see Proposition 2).

Proposition 5 *If there is only one (real) activity $h \in V$ (immediately) preceding activity i such that $(h, i) \in XF$, with $\delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil \leq ES_i$, then an optimal execution intensity profile π_i^* for i is a* max-inter-min *profile π_i^E, having block $r_i = b = 1$ with structure π_i^{E,r_i}.*

Proof Let us suppose that activity i is executed with profile π_i^E, and let us consider block $b = 1$.

If $(h, i) \in XF$ with $\delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil \leq ES_i$, we are free to execute the whole remaining amount μ_i^b of work at the maximum possible intensity, ensuring that $ES_h + \ell_{hi} = \delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil \leq ES_i$. Therefore block b can be the last one, i.e., $r_i = b = 1$, and with structure π_i^{E,r_i}.

Moreover, with this choice it is also assured that the lengths $\ell_{ij}^+(\pi_i^E) = \hat{\ell}_{ij}^{r_i+}(\pi_i^E) + \tilde{\ell}_{ij}^{r_i+}(\pi_i^E)$ of all the outgoing arcs $(i, j) \in \Gamma^+(i)$ of node i are minimized (where in particular $\hat{\ell}_{ij}^{r_i+}(\pi_i^E) = 0$, since $r_i = 1$).

In conclusion, this proves that profile π_i^E, having block $r_i = b = 1$ of structure π_i^{E,r_i}, is the optimal profile for activity i, under the assumptions of the proposition.

□

Now, let us suppose that case (iib) occurs. In this case, $\delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil > ES_i$, and, hence, differently from case (iia), we cannot execute the remaining amount of work μ_i^b of activity i at the maximum possible intensity, otherwise the activity cannot start at its earliest start time ES_i. In fact, we would have $ES_h + \ell_{hi} = \delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil > ES_i$.

This means that block b cannot be of type π_i^{E,r_i}, and, hence, it cannot be the last block of profile π_i^E, i.e., $b < r_i$.

Therefore, without loss of generality, we assume that the length (number of time slots) of block b is equal to $(\alpha_i^b + 1 + \beta_i^b)$, with non-negative integers α_i^b and β_i^b possibly equal to 0, being respectively the number of time slots of the first and third sub-blocks of block b, and with the second sub-block having exactly $\gamma_i^b = 1$ time slot where the amount of work w_i^b is done, with $\underline{a}_i \leq w_i^b < \bar{a}_i$, according to Proposition 1.

In addition, since the aim is also to execute the maximum amount of the work of activity i in block b, so as to execute the work of i as early as possible, we assume that the amount of work done in block b is equal $\mu_i^b - \epsilon > 0$, with $\epsilon > 0$ being a sufficiently small remaining amount of work of i that can be done in exactly one time slot after the end of block b, because $b < r_i$.

With the above choice, $\tilde{\ell}_{hi}^{b-}(\pi_i^E) = (\alpha_i^b + 1 + \beta_i^b)$, and $\bar{\ell}_{hi}^{b-}(\pi_i^E) = 1$. Therefore, the size of block b must ensure that $ES_h + \ell_{hi} = \delta_{hi}^b - (\alpha_i^b + 1 + \beta_i^b) - 1 \leq ES_i$, if we want that activity i can start at its earliest start time.

Finally, the amount of work $\mu_i^b - \epsilon$ should be done by anticipating at most the amount of work done in block b, in order to minimize the value of $\tilde{\ell}_{ij}^{b+}(\pi_i^E)$, for each outgoing arc $(i, j) \in \Gamma^+(i)$ of node i. This can be achieved by choosing the minimum value for integer $\beta_i^b \geq 0$.

Therefore, in case (iib), let us consider a *max-inter-min* profile π_i^E where the three sub-blocks of block b have the following sizes:

- $\gamma_i^b = 1$ is the length of the second sub-block;

- $\alpha_i^b \geq 0$ and $\beta_i^b \geq 0$ are integers representing the number of time slots of the first and third sub-blocks, respectively, such that:

$$\alpha_i^b + 1 + \beta_i^b \geq \delta_{hi}^b - ES_i - 1, \tag{11}$$

$$\alpha_i^b \, \bar{a}_i + w_i^b + \beta_i^b \, \underline{a}_i = \mu_i^b - \epsilon, \tag{12}$$

$$\underline{a}_i \leq w_i^b < \bar{a}_i, \tag{13}$$

with ϵ being a sufficiently small positive value, and β_i^b is minimized.

The amounts of work done in the three sub-blocks are $z_i^b = \alpha_i^b \, \bar{a}_i$, w_i^b, and $y_i^b = \beta_i^b \, \underline{a}_i$, respectively.

Let us denote with $\pi_i^{E,b'}$ the structure of the b-th block, with $b < r_i$, of the *max-inter-min* profile π_i^E, with the above settings, according to case (iib). It results that:

Proposition 6 *If there is only one (real) activity $h \in V$ (immediately) preceding activity i such that $(h, i) \in XF$, with $\delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil > ES_i$, then an optimal execution intensity profile π_i^* for i is a* max-inter-min *profile π_i^E, having block $b = 1 < r_i$ with structure $\pi_i^{E,b'}$.*

Proof Let us suppose that activity i is executed with profile π_i^E, and let us consider block $b = 1$. We have already shown that the remaining work μ_i^b of activity i cannot be executed at the maximum possible intensity, otherwise the activity cannot start at its earliest start time. Therefore, let us assume that the structure of block b of profile π_i^E is of type $\pi_i^{E,b'}$.

Constraint (13) forces the second sub-block of block b to be non-empty and composed by a single time slot (i.e., $\gamma_i^b = 1$), according to Proposition 1.

From the definition of δ_{hi}^b, we have that $ES_h + \ell_{hi} = \delta_{hi}^b - (\tilde{\ell}_{hi}^{b-}(\pi_i^E) + \bar{\ell}_{hi}^{b-}(\pi_i^E))$. Therefore, from Inequality (11) and Eq. (12), it follows that $ES_h + \ell_{hi} \leq ES_i$. In fact, from Eq.s (12) it results $\tilde{\ell}_{hi}^{b-}(\pi_i^E) = (\alpha_i^b + 1 + \beta_i^b)$. Moreover, since $\mu_i^{b+1} = \epsilon > 0$ and sufficiently small, it results $\bar{\ell}_{hi}^{b-}(\pi_i^E) = 1$, because the remaining amount of work μ_i^{b+1} can be done within 1 time slot. Hence, with block b of type $\pi_i^{E,b'}$, it is assured that activity i can start at time ES_i.

Moreover, choosing the minimum integer value for $\beta_i^b \geq 0$, such that Constraints (11)–(13) are fulfilled (and with $\alpha_i^b \geq 0$ and integer) allows to maximize the fraction of work done on block b at the maximum intensity, while assuring that activity i can start at time ES_i. Therefore, the structure $\pi_i^{E,b'}$ also assures that, for each outgoing arc $(i, j) \in \Gamma^+(i)$ of node i, the fraction $\tilde{\ell}_{ij}^{b+}(\pi_i^E)$ of $\ell_{ij}^+(\pi_i^E)$ is minimized, assuming the remaining work to be done in block $b + 1$ being equal to $\mu_i^{b+1} = \epsilon > 0$, with $\epsilon > 0$ and sufficiently small.

This implies that the value $\ell_{ij}^+(\pi_i^E) = \hat{\ell}_{ij}^{b+}(\pi_i^E) + \tilde{\ell}_{ij}^{b+}(\pi_i^E) + \bar{\ell}_{ij}^{b+}(\pi_i^E)$ is minimized, since $\hat{\ell}_{ij}^{b+}(\pi_i^E) = 0$, if $b = 1$, and $\bar{\ell}_{ij}^{b+}(\pi_i^E) \leq 1$ assumes the minimum value, with $\mu_i^{b+1} = \epsilon > 0$ sufficiently small.

In conclusion, this proves that profile π_i^E, having block $b = 1 < r_i$ of structure $\pi_i^{E,b}$, is the optimal profile for activity i, under the assumptions of the proposition.

□

At the end of block b, the amount $\mu_i^{b+1} = \epsilon > 0$ of work of activity i will remain to do. Therefore, for block $b + 1$, case (iia) will occur for precedence $(h, i) \in XF$, because $\delta_{hi}^{b+1} - \left\lceil \frac{\mu_i^{b+1}}{\bar{a}_i} \right\rceil = \delta_{hi}^{b+1} - 1 \leq ES_i$, since $\epsilon > 0$ and sufficiently small, and $\delta_{hi}^{b+1} = \delta_{hi}^b - (\alpha_i^b + 1 + \beta_i^b) \leq ES_i - 1$, according to Inequality (11). Therefore, $r_i = b + 1$.

For case (iib), following the same proof of Theorem 4, we can show that the optimal numbers α_i^{*b}, γ_i^{*b}, and β_i^{*b}, of time slots of the three sub-blocks of block b, respectively, that minimize β_i^q, while fulfilling Constraints (11)–(13), with integers $\alpha_i^b \geq 0$ and $\beta_i^b \geq 0$, are as follows.

Theorem 7 *For case (iib), block $b < r_i$ of* max-inter-min *execution profile π_i^E of activity i, whose structure is $\pi_i^{E,b'}$, has three sub-blocks with the following optimal profiles:*

- *First sub-block contains $\alpha_i^{*b} = \left\lfloor \frac{\mu_i^b - \epsilon - (\delta_{hi}^b - ES_i - 1)\underline{a}_i}{\bar{a}_i - \underline{a}_i} \right\rfloor \geq 0$ time slots, where a total amount $z_i^{*b} = \alpha_i^{*b}\, \bar{a}_i \geq 0$ of work of i is done;*
- *Second sub-block contains $\gamma_i^{*b} = 1$ time slot, where a total amount w_i^{*b} of work of i is done, with $\underline{a}_i \leq w_i^{*b} = \mu_i^b - \epsilon - (\delta_{hi}^b - ES_i - 1)\underline{a}_i - \alpha_i^{*b}(\bar{a}_i - \underline{a}_i) + \underline{a}_i < \bar{a}_i$;*
- *Third sub-block contains $\beta_i^{*b} = \delta_{hi}^b - ES_i - \alpha_i^{*b} - 2 \geq 0$ time slots, where a total amount $y_i^{*b} = \beta_i^{*b}\, \underline{a}_i \geq 0$ of work of i is done,*

where $\epsilon > 0$ is a given sufficiently small real value.

We close this subsection showing the calculation of the optimal execution intensity profiles for the activities of Example 2, where case (ib) occurs. Another small example where case (iib) happens, and the details for the calculation of a sufficiently small value for $\epsilon > 0$ are given in Appendix A of the *supplemental document*.

Let us reconsider Example 2, corresponding to the project network of Fig. 2 and its standardized network N'_S shown in Fig. 5, and assuming that $\underline{a}_2 = 0.09$.

Activities 1 and 3 have a unique execution intensity profile, being therefore also optimal (namely, profiles π_1^*, π_3^*, resp.), since for both these two activities the minimum and the maximum intensities have the same value. In particular, for activities $i = 1, 3$, profile π_i^* can be regarded as of type π_i^E with one block (i.e., $r_i = 1$), where the first sub-block contains $\alpha_i^1 = \lfloor \frac{1}{\bar{a}_i} \rfloor \geq 0$ time slots (i.e., $\alpha_1^1 = \lfloor \frac{1}{\bar{a}_1} \rfloor = 5$ and $\alpha_3^1 = \lfloor \frac{1}{\bar{a}_3} \rfloor = 7$), in which the amount $z_i^1 = \alpha_i^1\, \bar{a}_i \geq 0$ of work of i is done at (maximum) intensity $\bar{a}_i$. Since, for both the two activities, the whole amount of work is done in the first sub-block (i.e., $z_i^1 = \alpha_i^1\, \bar{a}_i = 1$), the second and third sub-blocks are empty.

According to the determined optimal profile $\pi_1^* = \pi_1^E$, the activity duration is $d_1(\pi_1^*) = 5$. Since activity 1 has no real precedence activity, we have $ES_1 = 0$ and, hence, $EF_1 = ES_1 + d_1(\pi_1^*) = 5$. Moreover, $\ell_{12}^+(\pi_1^*) = d_1(\pi_1^*) = 5$, since arc $(1, 2)$ models feeding precedence $F\%C(0.6)_{12}$, and $\ell_{14}^+(\pi_1^*) = k^+(1.0, \pi_1^*) = d_1(\pi_1^*) = 5$, since arc $(1, 4)$ models feeding precedence $\%C(1.0)S_{14}$.

As for activity 2, first of all we calculate $ES_2 = \max[0, ES_1 + (\ell_{12}^+(\pi_1^*) - \ell_{12}^-(\pi_2^{min}))] = \max[0, ES_1 + (d_1(\pi_1^*) - k^-(0.6, \pi_2^{min}))] = \max[0, 0 + (5 - \lfloor \frac{0.6}{0.09} \rfloor)] = 0$.

Since the minimum and maximum execution intensities of activity 2 are different and the latter has activity 1 as a real preceding activity, the determination of π_2^* is not trivial and depends on incoming feeding precedence $F\%C(g_{12})_{12}$ of activity 2 with respect to activity 1, with $g_{12} = 0.6$.

Let us find the optimal profile π_2^*, calculated as profile π_2^E, guaranteeing that the activity can start at time $ES_2 = 0$ and minimizing the positive contribution $\ell_{2j}^+(\pi_2)$ of the length ℓ_{2j} of the outgoing arcs $(2, j) \in \Gamma^+(2)$ of node 2 in the standardized network N'_S.

Let us calculate the structure of block $b = 1$ of π_2^E. Clearly, the fraction of work of activity 2 already done before the start of block 1 is $\chi_2^1 = 0$, and, hence, $\mu_2^1 = 1 - \chi_2^1 = 1$ is the fraction of work not yet done.

We calculate $\delta_{12}^{1} = ES_1 + \ell_{12}^{+}(\pi_1^*) - \hat{\ell}_{12}^{1-}(\pi_2^E) = 5$, where $ES_1 = 0$, $\ell_{12}^{+}(\pi_1^*) = 5$, and $\hat{\ell}_{12}^{1-}(\pi_2^E) = 0$ because $b = 1$.

Since feeding precedence $(1, 2)$ is of type $X\%C$ with $\tilde{g}_{12}^{1} = \max[0, g_{12} - \chi_2^1] = 0.6$, and $\delta_{12}^{1} - \lfloor \frac{\tilde{g}_{12}^{1}}{\bar{a}_2} \rfloor = 2 > ES_2 = 0$, case (ib) occurs. According to Theorem 4, we calculate the structure of block $b = 1$:

- $\alpha_2^1 = \left\lfloor \frac{\tilde{g}_{12}^{1} - (\delta_{12}^{1} - ES_2)\underline{a}_2}{\bar{a}_2 - \underline{a}_2} \right\rfloor = \left\lfloor \frac{0.6-(5-0)0.09}{0.2-0.09} \right\rfloor = 1$, and $z_2^1 = \alpha_2^1\, \bar{a}_2 = 0.2$;
- $\gamma_2^1 = 1$, and $w_2^1 = \tilde{g}_{12}^{1} - (\delta_{12}^{1} - ES_2)\underline{a}_2 - \alpha_2^1(\bar{a}_2 - \underline{a}_2) + \underline{a}_2 = 0.6 - (5 - 0)0.09 - 1(0.2 - 0.09) + 0.09 = 0.13$;
- $\beta_2^1 = \delta_{12}^{1} - ES_2 - \alpha_2^1 - 1 = 5 - 0 - 1 - 1 = 3$, and $y_2^1 = \beta_2^1\, \underline{a}_2 = 0.27$.

In conclusion, at the end of block 1 of profile π_2^E, the fraction of work of activity 2 already completed is $\chi_2^2 = (z_2^1 + w_2^1 + y_2^1) = 0.6$, and $\mu_2^2 = 1 - \chi_2^2 = 0.4$ is the fraction of work still to be done in next block $b + 1 = 2$, since $\mu_2^2 > 0$.

Let us, therefore, continue the determination of profile π_2^E, by calculating the structure of block $b = 2$ of π_2^E.

We calculate $\delta_{12}^{2} = ES_1 + \ell_{12}^{+}(\pi_1^*) - \hat{\ell}_{12}^{2-}(\pi_2^E) = 0$, where $ES_1 = 0$, $\ell_{12}^{+}(\pi_1^*) = 5$, and $\hat{\ell}_{12}^{2-}(\pi_2^E) = 5$ is the fraction of $\ell_{12}^{-}(\pi_2^E)$ due to the previous blocks of π_2^E.

The whole amount of $\tilde{g}_{12}^{1}$ has been covered in block 1, meaning that for block 2 we have $\tilde{g}_{12}^{2} = \max[0, g_{12} - \chi_2^2] = 0$.

Since feeding precedence $(1, 2)$ is of type $X\%C$ and $\delta_{12}^{2} - \lfloor \frac{\tilde{g}_{12}^{2}}{\bar{a}_2} \rfloor = 0 \leq ES_2 = 0$, case (ia) occurs. Therefore, according to Proposition 2, block 2 is the last block of profile π_2^E, i.e., $r_2 = 2$, and has structure $\pi_2^{E, r_2=2}$, that is:

- $\alpha_2^2 = \left\lfloor \frac{\mu_2^2}{\bar{a}_2} \right\rfloor = \left\lfloor \frac{0.4}{0.2} \right\rfloor = 2$, and $z_2^2 = \alpha_2^2\, \bar{a}_2 = 0.4$;
- $\gamma_2^2 = 0$, since $w_2^2 = \mu_2^2 - z_2^2 = 0.4 - 0.4 = 0$;
- $\beta_2^2 = 0$, since $y_2^2 = 0$.

In conclusion, at the end of block 2 of profile π_2^E, the whole work of activity 2 is completed.

According to the determined optimal profile $\pi_2^* = \pi_2^E$, the activity duration is $d_2(\pi_2^*) = 7$. Therefore, $EF_2 = ES_2 + d_2(\pi_2^*) = 7$. Moreover, $\ell_{12}^{-}(\pi_2^*) = k^{-}(0.6, \pi_2^*) = 5$, since arc $(1, 2)$ models feeding precedence $F\%C(0.6)_{12}$. Finally, for the outgoing arcs $(2, j) \in \Gamma^{+}(2)$ of node 2 in the standardized network N_S', we have: $\ell_{23}^{+}(\pi_2^*) = 0$, since arc $(2, 3)$ models feeding precedence $S\%C(1/7)_{23}$; finally, $\ell_{24}^{+}(\pi_2^*) = k^{+}(1, \pi_2^*) = d_2(\pi_2^*) = 7$, since arc $(2, 4)$ models feeding precedence $\%C(1)S_{24}$.

Finally, for activity 3, first of all we calculate $ES_3 = \max[0, ES_2 + (\ell_{23}^{+}(\pi_2^*) - \ell_{23}^{-}(\pi_3^{min}))] = \max[0, ES_2 + (0 - k^{-}(1/7, \pi_3^{min}))] = \max[0, 0 + (0 - \lfloor \frac{1/7}{1/7} \rfloor)] = 0$.

According to the determined optimal profile $\pi_3^* = \pi_3^E$, the activity duration is $d_3(\pi_3^*) = 7$. Therefore, $EF_3 = ES_3 + d_3(\pi_3^*) = 7$. Moreover, $\ell_{23}^{-}(\pi_3^*) = k^{-}(1/7, \pi_3^*) = 1$, since arc $(2, 3)$ models feeding precedence $S\%C(1/7)_{23}$, and $\ell_{34}^{+}(\pi_3^*) = d_3(\pi_3^*) = 7$, since arc $(3, 4)$ models feeding precedence $\%C(1)S_{34}$.

In conclusion, $\ell_{01} = \ell_{02} = \ell_{03} = 0$, $\ell_{12} = \ell_{12}^{+}(\pi_1^*) - \ell_{12}^{-}(\pi_2^*) = d_1(\pi_1^*) - k^{-}(0.6, \pi_2^*) = 5 - 5 = 0$, $\ell_{14} = \ell_{14}^{+}(\pi_1^*) = d_1(\pi_1^*) = 5$, $\ell_{23} = -\ell_{23}^{-}(\pi_3^*) = -k_3^{-}(1/7, \pi_3^*) = -1$, $\ell_{24} = \ell_{24}^{+}(\pi_2^*) = d_2(\pi_2^*) = 7$, and $\ell_{34} = \ell_{34}^{+}(\pi_3^*) = d_3(\pi_3^*) = 7$. Therefore, path $(0, 1, 2, 4)$ and path $(0, 3, 4)$ are the longest paths from node 0 to node 4 in the standardized network of Fig. 5, with length equal to $C_{\max}^* = 7$. The optimality of the solution is confirmed by the optimal solution returned by a commercial solver for the MIP formulation given in Appendix D of the *supplemental document*.

3.2.2 The case with multiple directly preceding real activities

All the above propositions and theorems can be generalized to the case where real activity $i \in V$ has multiple directly preceding real activities, assuming that, when we are calculating the structure of block b of execution intensity profile π_i^E of i, what follows next was iteratively already applied for the calculation of the structure of each one of the previous blocks $p = 1, \ldots, b-1$.

Let us denote with $\pi_i^{E(b-1,min)}$, the execution intensity profile of activity i, in which the activity is executed at the minimum intensity after the first $b-1$ blocks of profile π_i^E.

We assume that the first $b-1$ blocks of π_i^E were optimally determined, assuring, in particular, that profile $\pi_i^{E(b-1,min)}$ allows activity i to start at its earliest start time, that is, $\delta_{hi}^b - (\ell_{hi}^-(\pi_i^{E(b-1,min)}) - \hat{\ell}_{hi}^{b-}(\pi_i^{E(b-1,min)})) \leq ES_i$, for each arc $(h,i) \in \Gamma^-(i)$.

We note that this is certainly true if $b = 1$. In this case, it results $\delta_{hi}^1 - (\ell_{hi}^-(\pi_i^{E(0,min)}) - \hat{\ell}_{hi}^{1-}(\pi_i^{E(0,min)})) \leq ES_i$, because $\hat{\ell}_{hi}^{1-}(\pi_i^{E(0,min)}) = 0$, $\pi_i^{E(0,min)} \equiv \pi_i^{min}$, and ES_i was determined in advance by initially assuming profile π_i^{min} for activity i. Theorem 8 assures that this happens also for $2 \leq b < r_i$, and legitimates the above assumption.

Considering block b of π_i^E, let $\tilde{\Gamma}_{X\%C}^{b-}(i) = \{(h,i) \in \Gamma_{X\%C}(i) : \delta_{hi}^b - \lfloor \frac{\tilde{g}_{hi}^b}{\bar{a}_i} \rfloor \geq ES_i\}$ and $\tilde{\Gamma}_{XF}^{b-}(i) = \{(h,i) \in \Gamma_{XF}^-(i) : \delta_{hi}^b - \lceil \frac{\mu_i^b}{\bar{a}_i} \rceil > ES_i\}$ be the subsets of incoming direct feeding precedences (h,i) of activity i (incoming arcs (h,i) of node i in the standardized network N'_S), for which case (ib) and case (iib) occur, respectively. Clearly, for all that precedences, activity h is real.

According to the above propositions and theorems, we calculate the fractions of work that should be done in the three sub-blocks of block b, for each feeding precedence $(h,i) \in \tilde{\Gamma}_{X\%C}^{b-}(i) \cup \tilde{\Gamma}_{XF}^{b-}(i)$, as if activity h were the unique real activity directly preceding activity i. Since, in general, these triples of values differ among the direct incoming feeding precedences (h,i) of activity i, we denote them as triples $(z_{hi}^b, w_{hi}^b, y_{hi}^b)$. Contrarily, for all the other incoming direct feeding precedences $(h,i) \in \Gamma^-(i) \backslash (\tilde{\Gamma}_{X\%C}^{b-}(i) \cup \tilde{\Gamma}_{XF}^{b-}(i))$ (i.e., for which cases (ia) or (iia) occur), we know that $z_{hi}^b = \lfloor \frac{\mu_i^b}{\bar{a}_i} \rfloor$, $w_{hi}^b = \mu_i^b - \lfloor \frac{\mu_i^b}{\bar{a}_i} \rfloor$, and $y_{hi}^b = 0$.

Therefore, if $\tilde{\Gamma}_{X\%C}^{b-}(i) \cup \tilde{\Gamma}_{XF}^{b-}(i) = \emptyset$ (i.e, when all the direct incoming feeding precedences (h,i) of activity i belong to case (ia) or case (iia)), we can simply let $z_i^b = \lfloor \frac{\mu_i^b}{\bar{a}_i} \rfloor$, $w_i^b = \mu_i^b - \lfloor \frac{\mu_i^b}{\bar{a}_i} \rfloor$, and $y_i^b = 0$; then, $\alpha_i^b = z_i^b / \bar{a}_i$ (since z_i^b is a multiple of $\bar{a}_i$), γ_i^b equal to 1 if $w_i^b > 0$ and 0 otherwise (since $w_i^b < \bar{a}_i$), and $\beta_i^b = 0$. Clearly, this choice is optimal for block b and guarantees that activity i can start at time ES_i, according to Propositions 2 and 5. Moreover, since $\chi_i^b = 1$, block b is the last block of π_i^E, that is, $r_i = b$. Note, in fact, that the structure of this last block is π_i^{E,r_i}.

Let us consider now the case in which block $b < r_i$, because there exists at least one feeding precedence (h,i) of activity i belonging to case (ib) or case (iib), that is, $\left|\tilde{\Gamma}_{X\%C}^{b-}(i)\right| + \left|\tilde{\Gamma}_{XF}^{b-}(i)\right| \geq 1$.

Let $(h^b, i) \in \Gamma^-(i)$ be the incoming direct precedence of i, such that $(z_{h^b i}^b + w_{h^b i}^b) \leq (z_{hi}^b + w_{hi}^b)$ for any $(h,i) \in \Gamma^-(i)$, and such that $y_{h^b i}^b \leq y_{h'i}^b$ for any arc $(h',i) \in \Gamma^-(i)$ with $z_{h'i}^b + w_{h'i}^b = z_{h^b i}^b + w_{h^b i}^b$.

Clearly, $(h^b, i) \in \tilde{\Gamma}_{X\%C}^{b-}(i) \cup \tilde{\Gamma}_{XF}^{b-}(i)$, and, hence, $\underline{a}_i \leq w_{h^b i}^b < \bar{a}_i$, and then $\gamma_{h^b i}^b = 1$, according to Proposition 1. In fact, if this were not the case, we would have $z_{h^b i}^b + w_{h^b i}^b = \mu_i^b$, which is the maximum possible value, meaning that $\tilde{\Gamma}^{b-}_{X\%C}(i) \cup \tilde{\Gamma}^{b-}_{XF}(i) = \emptyset$, and, hence, b would be equal to r_i, contrarily to the initial hypothesis.

Let the actual amounts of work of activity i done in the three sub-blocks of b be equal to the related amounts calculated for precedence (h^b, i). That is, $z^b_i = z^b_{h^b i}$, $w^b_i = w^b_{h^b i}$, and $y^b_i = y^b_{h^b i}$. Therefore, the number of time slots of the first sub-block is $\alpha^b_i = z^b_i / \bar{a}_i$, since z^b_i is a multiple of $\bar{a}_i$, the second sub-block contains exactly 1 time slot (i.e., $\gamma^b_i = \gamma^b_{h^b i} = 1$), and, finally, the number of time slots of the third sub-block of block b is $\beta^b_i = y^b_i / \underline{a}_i$, since y^b_i is a multiple of $\underline{a}_i$.

With the above choice for the structure of block $b < r_i$, it is guaranteed that activity i can start at time ES_i, at least by executing at the minimum intensity the remaining work $1 - \chi^b_i > 0$ still to be done after block b, that is, by executing activity i with profile $\pi_i^{E(b,min)}$. In fact, we can prove that:

Theorem 8 *Executing activity i with profile $\pi_i^{E(b,min)}$, with the structure of block $b < r_i$ assumed equal to that calculated for direct incoming feeding precedence (h^b, i), assures that i can start at time ES_i.*

Proof For each incoming arc (h, i) of node i, with $h \neq 0$, let us denote with $\pi_i^{E(b,(h,i))}$ the (optimal) *max-inter-min* execution intensity profile of activity i with the structure of block b computed as if (real) activity h were the unique direct predecessor of i. Accordingly, block b will be the last block of $\pi_i^{E(b,(h,i))}$ if precedence $(h, i) \in \Gamma^-(i) \backslash (\tilde{\Gamma}^{b-}_{X\%C}(i) \cup \tilde{\Gamma}^{b-}_{XF}(i))$ (i.e., for which cases (ia) or (iia) occur), otherwise (i.e, when for precedence (h, i) cases (ib) or (iib) occur) block b will be the second last block of $\pi_i^{E(b,(h,i))}$ and in the last block $b + 1$ the remaining amount $\mu^{b+1}_i > 0$ of work of i will be done at the maximum possible intensity.

Accordingly, profiles $\pi_i^{E(b,(h,i))}$ and $\pi_i^{E(b,min)}$ are the same up to the end of the last time slot of block $b-1$. Therefore, δ^b_{hi} has the same value for both these two profiles. Comparing the remaining sections of these two profiles, we have that in each time slot from the end of block $b-1$ until the last time slot of profile $\pi_i^{E(b,(h,i))}$, the execution intensity of i in profile $\pi_i^{E(b,min)}$ is not greater than that in profile $\pi_i^{E(b,(h,i))}$, because $z^b_i + w^b_i \leq z^b_{hi} + w^b_{hi}$, and from the end of block b of $\pi_i^{E(b,min)}$ onward, activity i is executed at the minimum intensity according to this profile. Therefore, $\tilde{\ell}^{b-}_{hi}(\pi_i^{E(b,min)}) + \bar{\ell}^{b-}_{hi}(\pi_i^{E(b,min)}) \geq \tilde{\ell}^{b-}_{hi}(\pi_i^{E(b,(h,i))}) + \bar{\ell}^{b-}_{hi}(\pi_i^{E(b,(h,i))})$.

Since the structure of block b of profile $\pi_i^{E(b,(h,i))}$ guarantees that $\delta^b_{hi} - (\tilde{\ell}^{b-}_{hi}(\pi_i^{E(b,(h,i))}) + \bar{\ell}^{b-}_{hi}(\pi_i^{E(b,(h,i))})) \leq ES_i$, we have that $\delta^b_{hi} - (\tilde{\ell}^{b-}_{hi}(\pi_i^{E(b,min)}) + \bar{\ell}^{b-}_{hi}(\pi_i^{E(b,min)})) \leq ES_i$. Since this is true for each precedence $(h, i) \in \Gamma^-(i)$, it follows that executing activity i with profile $\pi_i^{E(b,min)}$ would guarantee that i can start at time ES_i. □

If feeding precedence $(h^b, i) \in \tilde{\Gamma}^{b-}_{X\%C}(i)$ (i.e., for which case (ib) occurs), then after executing the fraction $\tilde{g}^b_{h^b i}$ of work of activity i in block b of profile π^E_i, we will clearly have $\tilde{g}^{b+1}_{h^b i} = 0$. Hence, $\bar{\ell}^{b-}_{h^b,i}(\pi^E_i) = 0$, whichever the structures of remaining blocks $b+1, \ldots, r_i$ of π^E_i are.

If feeding precedence $(h^b, i) \in \tilde{\Gamma}^{b-}_{XF}(i)$ (i.e., for which case (iib) occurs), then after the execution of block b of π^E_i, it will remain a (sufficiently small) residual amount $\mu^{b+1}_i = \epsilon > 0$ of work to do after block b, meaning that $b + 1 = r_i$, that is, block $b + 1$ is the last one of profile π^E_i.

Finally, if $\left|\tilde{\Gamma}^{b-}_{X\%C}(i)\right| + \left|\tilde{\Gamma}^{b-}_{XF}(i)\right| = 0$ (i.e., for all incoming direct precedence (h, i) of activity i, cases (ia) or (iia) occur) block b is the last one of profile π^E_i, i.e., $b = r_i$.

This proves the following result.

Theorem 9 *The number r_i of blocks of profile π_i^E is not greater than* $\left|\Gamma^-_{X\%C}(i)\right| + \min\left[1, \left|\Gamma^-_{XF}(i)\right|\right] + 1$.

Applying iteratively the result of Theorem 8, for each block $b = 1, \ldots, r_i - 1$ of profile π_i^E, and considering last block r_i with structure π_i^{E,r_i} guarantee that i can start at its earliest start time ES_i, if i is executed according to profile π_i^E.

In addition, profile π_i^E also guarantees that, for all outgoing arcs $(i, j) \in \Gamma^+(i)$ of node i in the standardized network N'_S, $\ell^+_{ij}(\pi_i^E)$ has the minimum value, among the values of $\ell^+_{ij}(\pi_i)$ obtained for any execution profiles π_i of i that allow the activity to start at time ES_i. Next, we show that this implies that profile π_i^E also allows activity i to finish at its earliest finish time EF_i, and makes it possible to start the successive activities of i at their earliest start time. In this regard, profile π_i^E can be therefore considered as an optimal profile π_i^* for activity i.

Theorem 10 *Let π_i^* be an execution intensity profile of activity i, such that the activity can start at time ES_i, and, for each outgoing arc $(i, j) \in \Gamma^+(i)$ of node i in N'_S, it holds that $\ell^+_{ij}(\pi_i^*) \le \ell^+_{ij}(\pi_i^0)$, for any profile π_i^0 allowing activity i to start at time ES_i. For each arc $(i, j) \in \Gamma^+(i)$, it follows that:*

$$ES_i + \ell^+_{ij}(\pi_i^*) \le ES_i + \lambda + \ell^+_{ij}(\pi_i^\lambda),$$

for any feasible profile π_i^λ of activity i, for which the activity cannot start before time $ES_i + \lambda$, with $\lambda \ge 0$ and integer.

Proof We note that execution intensity profile π_i^E respects the hypothesis of the theorem. Therefore, we assume $\pi_i^* = \pi_i^E$.

Profile π_i^E dominates any other profile π_i^0, since $ES_i + \ell^+_{ij}(\pi_i^E) \le ES_i + \ell^+_{ij}(\pi_i^0)$, for any outgoing arc $(i, j) \in \Gamma^+(i)$ of node i. Let us show that profile π_i^E also dominates any other profiles π_i^λ, for which the activity cannot start before time $ES_i + \lambda$, with $\lambda > 0$ and integer, i.e., that there is no profile π_i^λ, such that $ES_i + \ell^+_{ij}(\pi_i^E) > ES_i + \lambda + \ell^+_{ij}(\pi_i^\lambda)$, for any outgoing arc $(i, j) \in \Gamma^+(i)$ of node i. In particular, let us show that the above inequality is not true with respect to the (best) profile $\pi_i^{*\lambda}$, for which $\ell^+_{ij}(\pi_i^{*\lambda})$ has the minimum value among profile π_i^λ.

Indeed, profile $\pi_i^{*\lambda}$ can be obtained from π_i^E, reducing by λ the number of time slots of block $b = 1$, while maintaining the amount of work done in this block and minimizing the amount of work done at the minimum execution intensity. This could be achieved by anticipating the work done in the last $\lambda \le \beta_i^b$ time slots of the third sub-block of block b, as early as possible within the block.

Since $\beta_i^b \ge \lambda > 0$, block b is not the last block of profile π_i^E (i.e., $b < r_i$). Therefore, block b is composed by the first sub-block of $\alpha_i^b \ge 0$ time slots where activity i is executed at the maximum intensity $\bar{a}_i$, followed by the second sub-block of $\gamma_i^b = 1$ time slot where the activity is executed with intensity w_i^b, with $\underline{a}_i \le w_i^b < \bar{a}_i$, and completed with the third sub-block of $\beta_i^b > 1$ time slots where the activity is executed at the minimum intensity $\underline{a}_i$.

In addition, to anticipate the fraction of work $\lambda\, \underline{a}_i$ it is required that $(\bar{a}_i - w_i^b) + (\beta_i^b - \lambda)(\bar{a}_i - \underline{a}_i) \ge \lambda\, \underline{a}_i$, where $(\bar{a}_i - w_i^b)$ is the amount of additional work that could be done in the single time slot of the second sub-block, and analogously $(\beta_i^b - \lambda)(\bar{a}_i - \underline{a}_i)$ is the amount of additional work that could be done in the first $\beta_i^b - \lambda$ time slots of the third sub-block. Therefore, the reduction of the length of block b by λ time slots can be done only if $\lambda \leq \lfloor \frac{\bar{a}_i - w_i^b + (\bar{a}_i - \underline{a}_i)\beta_i^b}{\bar{a}_i} \rfloor$.

Assuming that it were possible to reduce by λ the length of block $b = 1$, for block $b = 2$ this cannot be done. In fact, if $r_i = 2$ the block length is not reducible a priori. Otherwise, since the value of δ_{hi}^2 will be increased by λ, for any incoming precedence (h, i) of activity i, i.e., by the same amount of time we have assumed to delay the starting time of i, then the (optimal) size of this block $b = 2$ will remain unchanged. Clearly, the same reasoning also applies to the other blocks, if any.

In any case, if the reduction by λ of the length of block $b = 1$ could be done, this could not imply a reduction greater than λ for the value of ℓ_{ij}^+, for any $(i, j) \in \Gamma^+(i)$. In fact, for any precedence $(i, j) \in S\%C$, $\ell_{ij}^+(\pi_i) = 0$ for any feasible profile π_i. In the other cases, i.e., $(i, j) \in \%C(q_{ij})X \cup F\%C$, the value of $\ell_{ij}^+(\pi_i) = k^+(q_{ij}, \pi_i)$ depends also by q_{ij} (with $q_{ij} = 1$, if $(i, j) \in F\%C$, since $k^+(q_{ij} = 1, \pi_i) = d_i(\pi_i)$). Therefore, denoting with z_i^1 and w_i^1 the amount of work done in the first and second sub-block of block 1 of profile π_i^E, respectively, for $q_{ij} \leq z_i^1 + w_i^1$, we have $\ell_{ij}^+(\pi_i^{*\lambda}) = \ell_{ij}^+(\pi_i^E)$, otherwise, for $q_{ij} > z_i^1 + w_i^1$, it can be shown that $\ell_{ij}^+(\pi_i^{*\lambda}) \geq \ell_{ij}^+(\pi_i^E) - \lambda$.

In conclusion, profile π_i^E dominates all other profiles π_i^λ, for any $\lambda \geq 0$.

□

Clearly, profile π_i^* of Theorem 10 is optimal, because it allows activity i to start at time ES_i, and, in addition, assures that $ES_i + \ell_{ij}^+(\pi_i^*)$ has the minimum possible value. In fact, the latter guarantees that also succeeding activities j of i might start at their earliest start time. Since this is in particular valid with respect to the succeeding dummy activity $n + 1$, it follows that, $ES_i + \ell_{i,n+1}^+(\pi_i^*) = ES_i + d_i(\pi_i^*)$ is minimum, and, hence, equal to EF_i.

Since profile π_i^E fulfills the hypothesis of Theorem 10, this proves that

Theorem 11 *The* max-inter-min *execution profile π_i^E with the structures of its blocks $b = 1, \ldots, r_i$ defined above allows to start activity i at its earliest start time ES_i and to finish it at its earliest finish time EF_i, assuming that the execution profiles of its (immediate) predecessors guarantee the same for the preceding activities. Therefore such an execution profile π_i^E for activity i is optimal for the earliest start (and finish) schedule.*

As a corollary, we have that

Corollary 1 *Executing each activity $i \in V$ with the (optimal)* max-inter-min *execution profile π_i^E, and starting the activities at the earliest times, assures the minimization of the project makespan.*

3.3 The forward recursion algorithm

Assuming the activities being indexed according to a topological order of the nodes of the acyclic standardized network N'_S, applying, in an iterative fashion for each node $i \in V$ of N'_S, the calculation of the optimal *max-inter-min* execution profile π_i^E, according to Sect. 3.2, allows to determine the optimal activity execution intensity profiles, the activity earliest start and finish times, and the minimum project makespan. This procedure generalizes the forward recursion algorithm of the critical path method and its extension for the project network with GPRs, with minimum time lags.

It is worth to notice that the optimal activity execution profiles and the related earliest start (and finish) schedule can be computed in polynomial time. In fact, for each block b of profile π_i^E of activity i, the calculation of the sizes z_i^b, w_i^b, y_i^b of the three sub-blocks of b requires the calculation of z_{hi}^b, w_{hi}^b, y_{hi}^b, for each incoming feeding precedence (arc) $(h, i) \in \Gamma^-(i)$ of i, in order to find the precedence (h^b, i) for which $y_{h^b i}^b$ is equal to the minimum value of y_{hi}^b, among those calculated for the incoming precedences (h, i) of i for which the value $z_{hi}^b + w_{hi}^b$ is minimum. Since the number of blocks $r_i \leq \left|\Gamma^-(i)\right| + 1$, the total number of times in which the above three calculations should be done is not greater than $\sum_{i \in V} \left|\Gamma^-(i)\right|(\left|\Gamma^-(i)\right| + 1)$, that is, $O(\left|A'\right|^2)$ times. Since the values of z_{hi}^b, w_{hi}^b, y_{hi}^b, related to precedence (h, i), can be computed in constant time, it follows that our forward recursion runs in $O(\left|A\right|^2)$ time.

The pseudocode of the proposed *Forward recursion* algorithm is listed in Algorithm 1.

If we were able to determine in constant time the incoming feeding precedence $(h^b, i) \in \tilde{\Gamma}_{X\%C}^{b-}(i) \cup \tilde{\Gamma}_{XF}^{b-}(i)$ of activity i, that fixes the sizes of the three sub-blocks of block $b < r_i$, we would be able to find in $O(\left|\Gamma^-(i)\right|)$ time the (optimal) profile π_i^E and the value of ES_i of activity i, along with the length ℓ_{hi} of each incoming arc $(h, i) \in \Gamma^-(i)$, and length $\ell_{i,n+1} = d_i(\pi_i^E)$.

Indeed, we show in Appendix B of the *supplemental document* that this is possible, if we initially sort the incoming precedences $(h, i) \in \Gamma^-(i)$ of i in lexicographic non-decreasing order of the values of couples $[(z_{hi}^1 + w_{hi}^1), y_{hi}^1]$, and we find the (optimal) profile π_i^E of activity i, using the information provided by that ordering. Since this sorting can be done in $O(|\Gamma^-(i)| \log|\Gamma^-(i)|)$ time, the total time required to find π_i^E will be $O(|\Gamma^-(i)| \log|\Gamma^-(i)|)$. Hence, the whole algorithm will run in $O(|A| \log|N|)$ time.

Since the set of feeding precedence relations, along with precedence *SF*, include the GPRs, our algorithm is therefore able to find the optimal activity durations d_i^* and the earliest start (and finish) schedule for the non-preemptive unconstrained project scheduling problem with GPRs, with minimum time lags, and variable activity durations $d_i^{\min} \leq d_i \leq d_i^{\max}$, where $d_i^{\min} = \lceil \frac{1}{\bar{a}_i} \rceil$ and $d_i^{\max} = \lceil \frac{1}{\underline{a}_i} \rceil$. However, in this case, the algorithm runs in $O(\left|A\right|)$ time, because it can be shown that $r_i \leq 2$, since the incoming precedence of activity i are only of type $X\%C(0)$ (since $XS \equiv X\%C(0)$) or of type XF, with $X \in \{S, \%C(1)\}$ (since $FF \equiv \%C(1)F$).

3.3.1 A complete example

Let us consider as a complete example the project network shown in Fig. 1, and assume that for all the activities $i \in V \setminus \{6\}$ the minimum and maximum execution intensities are $\underline{a}_i = 0.1$ and $\bar{a}_i = 0.2$, respectively, while for activity 6 they are $\underline{a}_6 = 0.08$ and $\bar{a}_6 = 0.2$.

Table 1 summarizes the output of the *Forward recursion* algorithm (the detail of the calculations made by the algorithm is given in Appendix C of the *supplemental document*). Column one lists the numbering of the real activities. Columns two and three list the values of ES_i and EF_i, respectively, of each activity $i \in V$. Column four lists the number r_i of blocks of the optimal *max-inter-min* profile π_i^E of each activity i, and the last groups of three columns list the values $(\alpha_i^b; z_i^b)$, $(\gamma_i^b; w_i^b)$, $(\beta_i^b; y_i^b)$, detailing the structure of each block $b = 1, \ldots, r_i$ of profile π_i^E of activity i: number of time slots and amount of work done, for each one of the three sub-blocks of block b.

Table 2shows the results of the optimal solution returned by a commercial solver for the MIP formulation given in Appendix D of the *supplemental document*. Column one lists the numbering of the real activities. Columns two and three list the values of ES_i and EF_i,

Algorithm 1 *Forward recursion*

for $i = 1$ to n **do**
  Set $ES_i := \max[0, \max_{h \in V:(h,i) \in A}\{ES_h + \ell^{+}_{hi}(\pi^E_h) - \ell^{-}_{hi}(\pi^{min}_i)\}]$
  Set $b := 1$, and $\mu^b_i := 1$
  while $\mu^b_i > 0$ **do**
    Set $\alpha^b_i := \lfloor \frac{\mu^b_i}{\bar{a}_i} \rfloor$, $z^b_i := \alpha^b_i \bar{a}_i$, and $w^b_i := \mu^b_i - z^b_i$
    if $w^b_i > 0$ **then** set $\gamma^b_i := 1$ **else** set $\gamma^b_i := 0$
    Set $\beta^b_i := 0$, and $y^b_i := 0$
    if $\bar{a}_i > \underline{a}_i$ **then**
      for each $h \in V : (h, i) \in \Gamma^{-}(i)$ **do**
        Set *last-block* := **true**
        Set $\delta^b_{hi} := ES_h + \ell^{+}_{hi}(\pi^*_h) - \hat{\ell}^{b-}_{hi}(\pi^E_i)$
        if $(h, i) \in X\%C$ **then**
          Set $\tilde{g}^b_{hi} := \max[0, g_{hi} - (1 - \mu^b_i)]$
          if $\delta^b_{hi} - \lfloor \frac{\tilde{g}^b_{hi}}{\bar{a}_i} \rfloor > ES_i$ **then**
            Compute $\alpha^b_{hi}, z^b_{hi}, w^b_{hi}, \beta^b_{hi}$, and y^b_{hi}, according to Th. 4
            Set *last-block* := **false**
          end if
        else if $(h, i) \in XF$ **and** $\delta^b_{hi} - \lceil \frac{\mu^b_i}{\bar{a}_i} \rceil > ES_i$ **then**
          Compute $\alpha^b_{hi}, z^b_{hi}, w^b_{hi}, \beta^b_{hi}$, and y^b_{hi} according to Th. 7
          Set *last-block* := **false**
        end if
        if *last-block* = **false then**
          if $(z^b_{hi} + w^b_{hi} < z^b_i + w^b_i)$ **or**
              $(z^b_{hi} + w^b_{hi} = z^b_i + w^b_i$ **and** $y^b_{hi} < y^b_i)$ **then**
            Set $\alpha^b_i := \alpha^b_{hi}$, $z^b_i := \alpha^b_i \bar{a}^b_i$
            Set $\gamma^b_i := 1$, $w^b_i := w^b_{hi}$
            Set $\beta^b_i := \beta^b_{hi}$, $y^b_i := y^b_{hi}$
          end if
        end if
      end for
    end if
    Set $\mu^{b+1}_i := \mu^b_i - (z^b_i + w^b_i + y^b_i)$
    Set $b := b + 1$
  end while
  Set $r_i := b - 1$
  Set π^E_i be the optimal profile of activity i with r_i blocks determined so far
  Set $EF_i := ES_i + d_i(\pi^E_i)$
end for
Set $C^*_{\max} := \max_{i \in V}\{EF_i\}$

respectively, of each activity $i \in V$. The minimum makespan $C^*_{\max} = \max_{i \in V}\{EF^*_i\} = 8$. The other columns show, for each activity i, the optimal execution intensity profile π^*_i, listing in each row i the total amount of work x^*_{it} done for activity i within the first t time slots of the planning horizon, with $1 \leq t \leq C^*_{\max} = 8$.

From the optimal profiles showed in Table 1, it is not hard to retrieve the optimal values x^*_{it} of the variables x_{it} of the MIP formulation, swowed in Table 2, as well as the values for the earliest start and finish time. Therefore, the optimal solution of the MIP formulation given in Appendix D of the *supplemental document* confirms the the optimal solution obtained by the proposed *Forward recursion* algorithm.

Table 1 The (optimal) solution returned by the *Forward recursion* algorithm for the project network of Fig. 1, with $\underline{a}_i = 0.1$ and $\bar{a}_i = 0.2$, with the exception of activity $i = 6$ for which $\underline{a}_6 = 0.08$ and $\bar{a}_6 = 0.2$

i	ES_i	EF_i	r_i	$(\alpha_i^b; z_i^b)$, $(\gamma_i^b; w_i^b)$, $(\beta_i^b; y_i^b)$	
				$b = 1$	$b = 2$
1	0	5	1	(5; 1.00), (0; 0.00), (0; 0.00)	
2	0	5	1	(5; 1.00), (0; 0.00), (0; 0.00)	
3	0	5	1	(5; 1.00), (0; 0.00), (0; 0.00)	
4	3	8	1	(5; 1.00), (0; 0.00), (0; 0.00)	
5	0	5	1	(5; 1.00), (0; 0.00), (0; 0.00)	
6	0	7	2	(1; 0.20), (1; 0.16), (3; 0.24)	(2; 0.40), (0; 0.00), (0; 0.00)
7	1	8	2	(0; 0.00), (1; 0.10), (3; 0.30)	(3; 0.60), (0; 0.00), (0; 0.00)
8	0	5	1	(5; 1.00), (0; 0.00), (0; 0.00)	

Table 2 The MIP formulation optimal solution for the project network of Fig. 1, with $\underline{a}_i = 0.1$ and $\bar{a}_i = 0.2$, with the exception of activity $i = 6$ for which $\underline{a}_6 = 0.08$ and $\bar{a}_6 = 0.2$

i	ES_i	EF_i	x_{it}^*							
			$t = 1$	2	3	4	5	6	7	8
1	0	5	0.20	0.40	0.60	0.80	1.00	1.00	1.00	1.00
2	0	5	0.20	0.40	0.60	0.80	1.00	1.00	1.00	1.00
3	0	5	0.20	0.40	0.60	0.80	1.00	1.00	1.00	1.00
4	3	8	0.00	0.00	0.00	0.20	0.40	0.60	0.80	1.00
5	0	5	0.20	0.40	0.60	0.80	1.00	1.00	1.00	1.00
6	0	7	0.20	0.36	0.44	0.52	0.60	0.80	1.00	1.00
7	1	8	0.00	0.10	0.20	0.30	0.40	0.60	0.80	1.00
8	0	5	0.20	0.40	0.60	0.80	1.00	1.00	1.00	1.00

Finally, Table 3 shows the values of length ℓ_{ij} of the arcs (i, j) of the standardized network N'_S, according to the optimal activity execution intensity profiles determined by the algorithm. Therefore, (0, 3, 7, 9) and (0, 1, 4, 9) are the longest (critical) paths from source node 0 to sink node 9 in the standardized network N'_S, with length equal to $C^*_{\max} = 8$.

4 Feeding precedence relations vs work GPRs

In addition to the previous four types of feeding precedence relations, we can consider a fifth one of type $\%C(q_{ij})\%C(g_{ij})$ between two activities (i, j), meaning that the processed fraction of successor activity j of activity i can be greater than $0 \le g_{ij} < 1$ only if i has been processed for at least a fractional amount $0 < q_{ij} \le 1$.

Feeding precedence $\%C(q_{ij})\%C(g_{ij})$, between ordered activity couple (i, j), includes those of types $\%C(q_{ij})S$ (when $g_{ij} = 0$) and $F\%C(g_{ij})$ (when $q_{ij} = 1$).

We note, that it is not necessary to modify the project network N and the related standardized network N'_S to represent also this fifth feeding precedence relation. In fact, for the feeding precedence $\%C(q_{ij})\%C(g_{ij})_{ij}$ we have $S_i + k_i^+(q_{ij}, \pi_i) \le S_j + k_j^-(g_{ij}, \pi_j)$, since at the time when the amount of work done for activity j is (strictly) greater than g_{ij} the

Table 3 The values of the length ℓ_{ij} of the arcs (i, j) of the standardized network N'_S, derived from the activity optimal profiles calculated by the *Forward recursion* algorithm

ℓ_{ij} / i \ j	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	.
1	.	−3	−1	3	.	.	.	.	5
2	.	.	.	.	−1	.	.	.	5
3	.	.	.	.	.	0	1	.	5
4	.	.	.	.	.	.	−6	.	5
5	.	.	.	.	.	.	.	−3	5
6	.	.	.	.	.	.	.	−4	7
7	.	.	.	.	.	.	.	−1	7
8	.	.	.	.	.	.	.	.	5

amount of work done for activity i has to be at least equal to q_{ij}. Hence,

$$\%C(q_{ij})\%C(g_{ij})_{ij} \equiv SS^{\min}_{ij}(\ell_{ij}), \text{ with } \ell_{ij} = k^+_i(q_{ij}, \pi_i) - k^-_j(g_{ij}, \pi_j).$$

Moreover, we note that the results of Sect. 3.2, for the calculation of the optimal execution intensity profile π^*_i of activity i, depend only on the values of $\ell^-_{hi}(\pi_i)$ of the incoming arcs (h, i) of node i, while the values of $\ell^+_{hi}(\pi^*_h)$ are assumed known and given. Therefore, the results of Sect. 3.2, and, hence, the proposed *Forward recursion* algorithm, for finding the optimal execution intensity profiles of the activities and the related earliest start (and finish) schedule, continue to be valid.

Valls et al. (2009) and, successively, Quintanilla et al. (2012) introduce a more general type of precedence relations, for representing technological constraints expressed in terms of (percentage) work content, that Quintanilla et al. call *work* GPRs. Referring, without loss of generality, to the subclass of work GPRs of minimum type, like the ones considered in our work, precedence constraints of this type are the following:

- $(SS^{\min}_{ij}, p_i, p_j, w)$: the initial percentage $0 \le p_j \le 100$ of activity j can be completed only if the initial percentage $0 \le p_i \le 100$ of activity i has been completed;
- $(SF^{\min}_{ij}, p_i, p_j, w)$: the process of the final percentage $0 \le p_j \le 100$ of activity j can be started only if the initial percentage $0 \le p_i \le 100$ of activity i has been completed;
- $(FS^{\min}_{ij}, p_i, p_j, w)$: the initial percentage $0 \le p_j \le 100$ of activity j can be completed only if the process of the final percentage $0 \le p_i \le 100$ of activity i has been started;
- $(FF^{\min}_{ij}, p_i, p_j, w)$: the process of the final percentage $0 \le p_j \le 100$ of activity j can be started only if process of the final percentage $0 \le p_i \le 100$ of activity i has been started.

The above work GPRs generalizes our feeding constraints, in the general case. However, when activity preemption is not allowed, as assumed in our paper, all the work GPRs can be represented with the feeding precedences we consider (including the (*time*) GPR of type *SF*), together with the generalized feeding precedence of type $\%C\%C$, that we define.

In fact, in the non-preemptive case, any type of work GPRs can be converted into any other type of work GPRs (as well as for time GPRs). In particular, if activity preemption is not allowed, it easy to prove that:

$$(SS^{\min}_{ij}, p_i, p_j, w) \equiv (SF^{\min}_{ij}, p_i, 100 - p_j, w),$$
$$(FS^{\min}_{ij}, p_i, p_j, w) \equiv (SF^{\min}_{ij}, 100 - p_i, 100 - p_j, w),$$

$$(FF_{ij}^{\min}, p_i, p_j, w) \equiv (SF_{ij}^{\min}, 100 - p_i, p_j, w).$$

Moreover, it is simple to show that:

$$(SF_{ij}^{\min}, 0, p_j > 0, w) \equiv S\%C(g_{ij} = 1 - p_j/100)_{ij},$$
$$(SF_{ij}^{\min}, 100, p_j > 0, w) \equiv F\%C(g_{ij} = 1 - p_j/100)_{ij};$$
$$(SF_{ij}^{\min}, p_i > 0, 0, w) \equiv \%C(q_{ij} = p_i/100)F_{ij},$$
$$(SF_{ij}^{\min}, p_i > 0, 100, w) \equiv \%C(q_{ij} = p_i/100)S_{ij},$$

that:

$$(SF_{ij}^{\min}, 0, 0, w) \equiv SF_{ij},$$

and finally that:

$$(SF_{ij}^{\min}, p_i > 0, p_j > 0, w) \equiv \%C(q_{ij} = p_i/100)\%C(g_{ij} = 1 - p_j/100)_{ij}.$$

In conclusion, the results of Sect. 3, and hence the proposed *Forward recursion* algorithm for the earliest start (and finish) schedule for a project network of non-preemptive activities with feeding precedence relations, are also valid in case of work GPRs. Therefore, also for this more general types of precedence relations we are able to find the (earliest start and finish) activity execution intensity profiles and the related earliest start (and finish) schedule in in $O(|A| \log|N|)$ time, for the non-preemptive resource unconstrained project scheduling problem, and this schedule also minimizes the project makespan.

5 Conclusions

The goal of this paper was twofold. On the one hand, we aimed at finding a specific project network representation associated with feeding precedence relations. The latter representation resulted in a generalization of the network standardization used in Generalized Precedence Relationships. Next, exploiting the network so defined we generalized the forward recursion for the calculation of (i) the earliest start times and of (ii) the earliest finish times of the project activities, and (iii) the critical (longest) paths. The proposed forward recursion algorithm was shown to run in $O(m \log n)$ time, with n and m being the number of activities and the number of precedence relations, respectively. Future work will be devoted to defining the backward recursion algorithm, as well as the calculation of the latest start and finish times of the activities along with activity floats and criticality.

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Conflict of interest All the authors declare that they have no conflict of interest.

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