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On the concept of criticality on GPRs project network with variable activity durations

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ABSTRACT

Temporal analysis of project networks has been widely studied in the literature; basically, it consists of determining the starting and finishing times of activities respecting a set of precedence constraints among them. The main output of the temporal analysis is twofold: on the one hand, it provides information on the minimum completion time of the project and, on the other hand, it determines which activity may be considered critical. Defining and determining activity criticalities on its own is a problem that has attracted the attention of many researchers over the last decades. In this paper, in an attempt to further pursue these studies, we focus on project scheduling with generalized precedence relationships where durations are not fixed in advance, but are variable within given ranges and have to be determined to minimize the makespan of the project. Analyzing activity criticalities for the same problem where activity durations are fixed has been tackled within the literature; what happens when durations are assumed variables, to the best of our knowledge, has not been investigated. We show that, in this scenario, the current knowledge on activity criticalities is no longer valid and we give new definitions of criticality together with the rules for its identification. An extensive experimental campaign on benchmark instances is presented to show that our findings are meaningful for quantitative project management.

1. Introduction

Temporal analysis of project networks has been one of the most studied topics in the project management and scheduling area. From the early studies of Kelley (1963) and Malcolm, Roseboom, Clark, and Fazar (1959) on the Critical Path Method in the Sixties, many researchers addressed the problem of modeling project activities and constraints to minimize the project completion time (makespan). Special attention has been deserved on the relation between the minimum makespan of the project and the longest path of the project network in the quest of defining the so-called "critical" activities, i.e., those activities responsible for the increase of the project completion time when delayed from their earliest start (or finish) time and/or a variation of their durations happen.

Indeed, when activity durations are fixed and given, the temporal analysis consists of executing a forward and a backward recursion to calculate the earliest and latest start times of the activities as well as their earliest and latest finish times. In particular, for each activity, the difference between the latest and earliest start times equals the difference between the latest and earliest finish times. The common value of these two differences is the activity *total float*, and an activity is critical if and only if its total float is equal to zero.

Identifying critical activities is a crucial task in project management. Indeed, they represent the bottleneck of the project in terms of its performance and require special attention in the resource assignment task. In the literature, several authors have addressed the concept of criticality of an activity, providing different definitions based on the type of temporal relationships (classical Finish-to-Start precedences, generalized precedence relationships, feeding precedences) of the project (see, e.g., Bianco, Caramia, & Giordani, 2022; Bianco, Caramia, Giordani, & Salvatore, 2023; Chen & Huang, 2007; Elmaghraby & Kamburowski, 1992; Floyd, Barker, Rocco, & Whitman, 2017; Jaber, Marle, Vidal, & Didiez, 2018; Quintanilla, Perez, Lino, & Valls, 2012 and Valls & Lino, 2001). In particular, in Bianco et al. (2022), the authors study project scheduling with generalized precedence relations proposing a new method to analyze criticalities and flexibilities. The latter paper extends previous works Elmaghraby and Kamburowski (1992), Quintanilla et al. (2012), and Valls and Lino (2001) under the same problem

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setting. In the aforementioned works, differently from the contribution given in this paper, durations are given and fixed.

In Bianco et al. (2023), an analysis of how to find the critical path in a project network with feeding precedence relations is presented using a specific forward recursion algorithm. In Bowers (1995), the author presents a revised method of calculating resource-constrained float and its application in project management. Precedence relations are classical Finish-to-Start and durations are fixed. Several measures of an activity's importance in a network are described and compared in an application to an aircraft development. A quantitative comparison of these measures is presented through simulation.

In Nguyen and Chua (2014), the authors present a systematic method to classify and identify the criticality of schedule constraints for schedule management from the constraint perspective. In terms of criticality, it is shown how schedule constraints can be grouped into four types: project-critical, activity-critical, sequence-critical, and non-critical. The method proposed is illustrated in a case example based on the construction of the main entrance of a nursing house. Also in this case precedence relationships are not generalized and activity durations are fixed. In Tavares, Ferreira, and Coelho (2004), starting from the notion of critical activity developed for deterministic project networks, the authors show the inadequacy of the concept of critical activity for stochastic project networks and define a new surrogate indicator of criticality using a regression model applied to a large set of generated project networks. Differently from our study, in the latter paper, the environment is stochastic which is also what happens in Floyd et al. (2017).

Criticalities have also been discussed in several papers debating the effectiveness of Critical chain scheduling/Buffer management, that is, the direct application of the theory of constraints to project management. Indeed, as reported in Herroelen, Leus, and Demeulemeester (2002), there is a controversy over the merits and pitfalls of this methodology.

Notwithstanding, a shared definition brings together these contributions, that is, an activity is critical if and only if it lies on a longest path (critical path) of the project network.

In this paper, we further study activity criticalities in project scheduling with Generalized Precedence Relationships (GPRs), when activity durations are not fixed but vary in given ranges and are to be determined to minimize the project makespan. We note that durations are deterministic (decision) variables; therefore, our problem does not lie within stochastic project scheduling. To interpret and explain the rationale behind the role of variable activity durations in our problem, we may see the latter, from a modeling viewpoint, as a way to represent a continuous multimodal scheduling scenario where each activity is executable with infinite combinations of resources each one producing a different duration, from a minimum to a maximum value. For example, this happens when considering activity variable execution intensities (speeds) (Kis, 2005). In this case, the amount of work per time unit devoted to each activity and, consequently, also its duration are not fixed but are decision variables to be determined. Specifically, feasible durations range between minimum and maximum values, related to the lowest and highest intensities of activity execution, respectively. The presented problem setting has been considered in the literature in a few other papers and, in particular, to minimize the total adjustment cost in Bianco, Caramia, and Giordani (2017) and with the more general feeding precedences under the minimum project length (makespan) objective in Bianco et al. (2023).

The setting of the studied problem is as follows. Given are n non-preemptable activities; each activity i has a variable duration $d_i \in [d_i^{\min}, d_i^{\max}]$, which constitutes a deterministic variable to be determined to minimize the project makespan. Generalized Precedence Relationships (GPRs), including Start-to-Start (*SS*), Start-to-Finish (*SF*), Finish-to-Start (*FS*), and Finish-to-Finish (*FF*) relationships, are assigned between pairs of activities. Only minimum time lags are taken into account, and it is assumed that the resulting project network is acyclic.

As we will show, in this scenario the traditional concept of critical activity is no longer valid in the presence of GPRs, differently from the case with only Finish-to-Start relations among activities. Therefore, a more general characterization of criticality is required. Specifically, we pose the following Research Questions (RQs):

1. Are the activity (total) floats, calculated respectively on the start and finish times, still equal when activity durations are variable?
2. If the floats are different, is it however necessary that they should be both equal to zero to identify a critical activity?
3. Since activity durations are variable and, therefore, the optimal duration of an activity evaluated with the forward and backward recursions could be different, are these durations equal when the activity is critical?
4. Is an activity belonging to both the longest path obtained with the forward recursion and the longest path obtained with the backward recursions critical?

To the best of our knowledge, the above RQs have not been addressed in the literature and, in our opinion, deserve an answer which will be given in the next sections. It is also important to note that though Elmaghraby and Kamburowski (1992) introduced project activities with variable durations in the Time-Cost Trade-Off studied problem, their criticality analysis was conducted only assuming fixed durations.

The organization of the manuscript is as follows. In Sections 2 and 3, we define new forward and backward recursion algorithms and networks, respectively. In Section 4, we present theoretical results, answering to the RQs we posed and generalizing the concept of critical activity. Section 5 presents an extensive experimental campaign on benchmark instances to show that our findings are meaningful also for quantitative project management. Finally, in Section 6, we draw some conclusions and present future work.

2. Forward and backward recursions

Since the project network is assumed to be acyclic, in the following we consider the activities $i = 1, \dots, n$ topologically indexed. Moreover, for ease of presentation of the results, but without loss of generalities, we assume zero time lags.

Given any feasible duration $d_i \in [d_i^{\min}, d_i^{\max}]$, for each activity i, let us denote with $N(d)$ the GPRs project network related to a given vector d of (feasible) activity durations.

It is well known (see, e.g., Bartusch, Möhring, & Radermacher, 1988) that a GPRs project network can be equivalently represented by the so-called standardized network where only one type of GPRs precedence relations is considered (e.g., *SS* or *FF*). In particular, the *SS*-standardized network represents precedence relations referred to the activity starting times. Therefore, we consider the additional dummy activities 0 (initial) and $n+1$ (final), and additional precedences $(0, i)$ of type *SS* and $(i, n+1)$ of type *FS*, for $i = 1, \dots, n$, to force activity i, if necessary, to start (finish) not earlier (later) than the project start (completion). Let $t_{ij}(d_i, d_j)$ be the length of arc (i, j) equal to the minimum difference between the starting times of activities j and i, according to precedence relation (i, j): $t_{ij}(d_i, d_j) = t^+_{ij}(d_i) - t^-_{ij}(d_j)$, with $t^+_{ij}(d_i)$ being equal to d_i if precedence relation (i, j) is of type *FS* or *FF*, and equal to 0 otherwise, and $t^-_{ij}(d_j)$ being equal to d_j if precedence relation (i, j) is of type *SF* or *FF*, and equal to 0 otherwise. Let us denote with $\ell^{N(d)}_{h \to k}$ the length of the longest path from node h to node k, with $h < k$, in the *SS*-standardized network of project network $N(d)$, if such a path exists. Similarly, the *FF*-standardized network represents precedence relations w.r.t. activity finish times. Analogously, we consider the additional dummy activities 0 (initial) and $n+1$ (final), and additional precedences $(0, i)$ of type *SF* and $(i, n+1)$ of type *FF*, for $i = 1, \dots, n$, to force activity i, if necessary, to start (finish) not

Fig. 1. Project network, with minimum and maximum activity durations, of Example 1.

earlier (later) than the project start (completion). In this representation, the length $\hat{t}_{ij}(d_i, d_j)$ of arc (i, j) is the minimum difference between the finish times of activities j and i, according to precedence relation (i, j): $\hat{t}_{ij}(d_i, d_j) = \hat{t}^+_{ij}(d_j) - \hat{t}^-_{ij}(d_i)$, with $\hat{t}^+_{ij}(d_j)$ being equal to d_j if precedence relation (i, j) is of type *SS* or *FS*, and equal to 0 otherwise, and $\hat{t}^-_{ij}(d_i)$ being equal to d_i if precedence relation (i, j) is of type *SS* or *SF*, and equal to 0 otherwise. Let us denote with $\hat{\ell}^{N(d)}_{h\to k}$ the length of the longest path from node h to node k, with $h < k$, in the *FF*-standardized network of project network $N(d)$, if such a path exists. While in general $t_{ij}(d_i, d_j)$ may be different from $\hat{t}_{ij}(d_i, d_j)$, it is easy to show that the lengths of the paths from node 0 to node $n+1$ do not depend on the type of network standardization. Therefore $\ell^{N(d)}_{0\to i} + \ell^{N(d)}_{i\to n+1} = \hat{\ell}^{N(d)}_{0\to i} + \hat{\ell}^{N(d)}_{i\to n+1}$, for any given (real) activity i, and $\ell^{N(d)}_{0\to n+1} = \hat{\ell}^{N(d)}_{0\to n+1} = C^d_{\max}$, where $C^d_{\max}$ is the minimum project length with the given activity duration $d_i \in [d_i^{\min}, d_i^{\max}]$, for each activity i. In the following, unless otherwise stated, we refer to the *SS*-standardized network representation.

Note that in a GPRs network, setting $d_i = d_i^{\min}$, for each activity i, does not necessarily conduct to the minimum project length (makespan) $C^*_{\max}$, even with minimum time lags only, because increasing an activity duration could decrease the project length (see, e.g., Bianco et al., 2022; Demeulemeester & Herroelen, 2002 and Elmaghraby & Kamburowski, 1992).

An optimal duration of activity i, denoted with d_i^{FW}, is the minimum value in the range $[d_i^{\min}, d_i^{\max}]$ that allows i to start at its earliest start time ES_i, assuming $ES_0 = 0$. We can compute d_i^{FW} by the following forward (FW) recursion. Since increasing d_i will not increase length $t_{hi}(d_h, d_i)$ of arc (h, i), for the calculation of ES_i we can initially consider $d_i = d_i^{\max}$ and calculate $ES_i = \max_{(h,i)\in\Gamma^-(i)} \{ES_h + t_{hi}\}$, where $\Gamma^-(i)$ is the set of incoming arcs (h, i) of i and $t_{hi} = t_{hi}(d_h^{FW}, d_i^{\max})$. Conversely, decreasing d_i will not increase $t_{ij}(d_i, d_j)$, for any outgoing arcs (i, j) of i. Therefore, we determine d_i^{FW} as the minimum feasible value of d_i that still allows i to start at ES_i: this can be done by looking only at the set $\Gamma^-(i)$ of incoming arcs of i. The earliest finish time of i is $EF_i = ES_i + d_i^{FW}$, and the minimum project makespan $C^*_{\max} = ES_{n+1} = \max_i\{EF_i\}$.

Analogously, another optimal duration of i, denoted with d_i^{BW}, is the maximum value in the range $[d_i^{\min}, d_i^{\max}]$ that allows i to start at its latest time LS_i, assuming $LS_{n+1} = ES_{n+1} = C^*_{\max}$. We compute d_i^{BW} by the following backward (BW) recursion. Since decreasing d_i will not increase length $t_{ij}(d_i, d_j)$ of arc (i, j), we can initially assume $d_i = d_i^{\min}$ and calculate LS_i, i.e., $LS_i = \min_{(i,j)\in\Gamma^+(i)}\{LS_j - t_{ij}\}$, where $\Gamma^+(i)$ is the set of outgoing arcs (i, j) of i and $t_{ij} = t_{ij}(d_i^{\min}, d_j^{BW})$. Conversely, increasing d_i will not increase $t_{hi}(d_h, d_i)$, for any incoming arcs (h, i) of i. Therefore, we determine d_i^{BW} as the maximum feasible value of d_i that still allows i to start at LS_i: this can be done by looking only at the set $\Gamma^+(i)$ of outgoing arcs of i. The latest finish time of i is $LF_i = LS_i + d_i^{BW}$.

Therefore, in linear time w.r.t. the cardinality of the network precedence relations, we can compute ES_i (LS_i) and the related optimal durations d_i^{FW} (d_i^{BW}) of activities $i = 1, \dots, n$.

After having introduced the above preliminary concepts, let us start to analyze the Research Questions (RQs) of Section 1 by considering the following example.

Example 1. Let us consider the project network with $n = 4$ activities and minimum and maximum activity durations depicted in Fig. 1.

Table 1
FW and *BW* durations, and earliest/latest start/finish times of Example 1.

Activity i	1	2	3	4
d_i^{FW}, d_i^{BW}	2, 2	2, 4	6, 6	4, 4
ES_i, LS_i	0, 2	0, 0	0, 0	0, 2
EF_i, LF_i	2, 4	2, 4	6, 6	4, 6

Table 1 lists, for each activity i, the activity forward and backward optimal durations (d_i^{FW}, d_i^{BW}), and its earliest/latest start/finish times (ES_i, LS_i, EF_i, LF_i), calculated by applying the forward and backward recursions. The minimum project makespan $C^*_{\max}$ is equal to 6 and $d_i^{FW} = d_i^{BW}$, for all activities i but 2; in particular, $d_2^{FW} < d_2^{BW}$.

Looking at activity 2, we note that $LS_2 - ES_2 = 0$ and $LF_2 - EF_2 = 2$. Therefore, differently from the case with known and fixed activity durations where the values of these two differences are always equal and we simply refer to them as activity float, when activity durations are variable, in the sense that their values can be chosen within given ranges, these two floats could be different. Hence, we need to differentiate between them: let us call (total) *start float* (F_i^S) of activity i the difference between its latest and earliest start times, i.e., $F_i^S = LS_i - ES_i$, and (total) *finish float* (F_i^F) of i the difference between its latest and earliest finish times, i.e., $F_i^F = LF_i - EF_i$.

Since, according to the traditional concept of criticality, an activity is considered as *critical* if it has a non-positive float, activity 3 would be therefore classified as critical. However, we would have difficulty to characterize the criticality of activity 2, since one of its floats (the one evaluated referred to the finish time) is positive. Therefore, the above example shows the need to redefine and generalize the concept of critical activity and its relation with activity floats.

3. Forward and backward networks

When activity durations are given and fixed, it is known that *start* and *finish* activity floats are equal. Moreover, we have a project network with fixed activity durations, and then it is possible to characterize activity floats and criticalities directly on this network. On the contrary, looking at Example 1, we note that the forward and backward durations, as well as the *start* and *finish* floats, of activity 2 are different.

Therefore, when activity durations are not given and fixed but are variable and have to be chosen within given ranges, if one aims to characterize its criticality and/or its floats from the network, it may not appear clear which network (along with activity durations) has to be taken into account. Since with the forward and backward recursions, we can calculate the forward and backward activity durations, let us start to consider the associated networks. Let us call *forward network* (*FW*-network) the project network $N(d^{FW})$ with activity durations d_i^{FW}, $i = 1, \dots, n$, with $d_i^{\min} \le d_i^{FW} \le d_i^{\max}$, resulting from the forward recursion. Analogously, let us call *backward network* (*BW*-network) the project network $N(d^{BW})$ with activity durations d_i^{BW}, $i = 1, \dots, n$, with $d_i^{\min} \le d_i^{BW} \le d_i^{\max}$, resulting from the backward recursion. Since the forward and backward activity durations could be different, the related *FW*-network and *BW*-network could be different too, w.r.t. the activity durations, as well as the related *SS*-standardized networks, where the lengths of an arc (i, j), therefore, could be different in the two networks. Of course, the same happens to the related *FF*-standardized networks.

Let $\ell^{FW}_{0\to i}$ and $\ell^{FW}_{i\to n+1}$ ($\ell^{BW}_{0\to i}$ and $\ell^{BW}_{i\to n+1}$) be the lengths of the longest paths from 0 to i and from i to $n+1$, respectively, of the *SS*-standardized

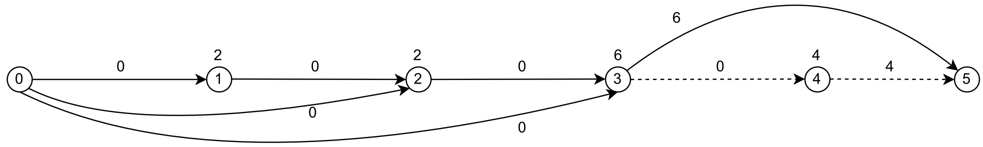

Fig. 2. The SS-standardized network of the FW-network of Example 1.

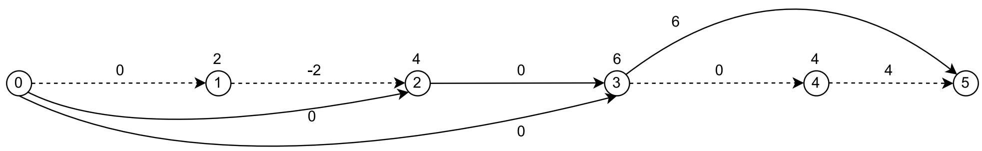

Fig. 3. The SS-standardized network of the BW-network of Example 1.

network of the FW-network (BW-network). It clearly results $ES_i = \ell^{FW}_{0\to i} \le \ell^{BW}_{0\to i}$, and $C^*_{\max} - LS_i = \ell^{BW}_{i\to n+1} \le \ell^{FW}_{i\to n+1}$, where $C^*_{\max} = \ell^{FW}_{0\to n+1} = \ell^{BW}_{0\to n+1}$. In general, for any activity i, it results $\ell^{FW}_{0\to i} + \ell^{FW}_{i\to n+1} \le C^*_{\max}$, and $\ell^{BW}_{0\to i} + \ell^{BW}_{i\to n+1} \le C^*_{\max}$, since $\ell^{FW}_{0\to i} + \ell^{FW}_{i\to n+1}$ and $\ell^{BW}_{0\to i} + \ell^{BW}_{i\to n+1}$ are the lengths of the longest path from 0 to $n+1$ traversing node i on the SS-standardized networks of the FW-network and of the BW-network, respectively.

Let $\delta^{FW}_i = C^*_{\max} - (\ell^{FW}_{0\to i} + \ell^{FW}_{i\to n+1}) \ge 0$ be the *FW-gap* of activity i, i.e., the difference between $C^*_{\max}$ and the length of the longest path from 0 to $n+1$ traversing node i in the FW-network, and let $\delta^{BW}_i = C^*_{\max} - (\ell^{BW}_{0\to i} + \ell^{BW}_{i\to n+1}) \ge 0$ be the *BW-gap* of activity i (w.r.t. the BW-network).

According to the traditional definition of critical activity, we would say that activity i is critical if and only if $\delta^{FW}_i = 0$ and $\delta^{BW}_i = 0$, i.e., if and only if it belongs to a critical (longest) path on both the FW- and BW- networks (let us denote such activity as $FW\&BW$-critical activity).

Figs. 2 and 3 represent the SS-standardized networks of the FW-network and of the BW-network of Example 1, respectively; the values associated with nodes and arcs represent activity durations and arc lengths, respectively. In particular, in the figures only the arcs belonging to a longest path from node 0 to node $n+1$ traversing an activity node are shown: among them, dashed arcs do not belong to a critical path (i.e., a longest path from 0 to $n+1$ with length $C^*_{\max} = 6$).

First of all, from Figs. 2 and 3 we note that the FW-network and the BW-network could be different, at least for the arc lengths. Moreover, it is difficult to relate the criticality of an activity with its belonging to a critical path in these networks. Let us consider activities 2 and 3 that result to be both $FW\&BW$-critical. From the earliest/latest start/finish times listed in Table 1, we note that while activity 3 has both the start and finish floats equal to zero, as one could expect, this is not the case for activity 2 that has only the start float equal to zero. In addition, activity 1 belongs to a critical path of the FW-network, but not to a critical path of the BW-network, despite both its start and finish floats are greater than zero (note that it could also happen the opposite, i.e., an activity could belong to a critical path of the BW-network but not to a critical path of the FW-network). Finally, we note that activity 4 also has both positive floats but, on the contrary, it does not belong to any critical path of the two networks.

The above analysis for Example 1 does not show all the possible issues. In fact, from the definitions of FW-gap and BW-gap of activity i, it follows that $\delta^{FW}_i \le LS_i - ES_i$, because $\ell^{FW}_{i\to n+1} \ge \ell^{BW}_{i\to n+1}$, and, analogously, $\delta^{BW}_i \le LS_i - ES_i$, because $\ell^{BW}_{0\to i} \ge \ell^{FW}_{0\to i}$. Hence, we have $\max\{\delta^{FW}_i, \delta^{BW}_i\} \le LS_i - ES_i$. Analogously, using the FF-standardized network representation, it can be shown that $\delta^{FW}_i \le LF_i - EF_i$ and $\delta^{BW}_i \le LF_i - EF_i$. Therefore, we have

$$\max\{\delta^{FW}_i, \delta^{BW}_i\} \le \min\{LS_i - ES_i, LF_i - EF_i\}, \tag{1}$$

and it could happen that a $FW\&BW$-critical activity i (i.e., with $\max\{\delta^{FW}_i, \delta^{BW}_i\} = 0$) has both start and finish floats greater than 0. Hence, an activity could have both positive start and finish floats, despite belonging to a critical path in both the FW- and BW- networks. We will discuss this occurrence in the following additional example.

Table 2
FW and BW durations, and earliest/latest start/finish times of Example 2.

Activity i	1	2	3	4	5	6	7	8
d^{FW}_i, d^{BW}_i	8, 8	5, 5	10, 9	1, 1	4, 5	4, 4	9, 9	7, 7
ES_i, LS_i	0, 0	8, 8	3, 4	3, 4	0, 0	0, 0	4, 4	4, 6
EF_i, LF_i	8, 8	13, 13	13, 13	4, 5	4, 5	4, 4	13, 13	11, 13

Example 2. Let us consider the project network, with minimum and maximum activity durations, shown in Fig. 4.

Table 2 lists, for each activity i, the activity forward and backward optimal durations (d^{FW}_i, d^{BW}_i) and its earliest/latest start/finish times (ES_i, LS_i, EF_i, LF_i), calculated by applying the forward and backward recursions. The minimum project makespan $C^*_{\max} = 13$, and $d^{FW}_i = d^{BW}_i$, for all activities i but 3 and 5; in particular, $d^{FW}_3 > d^{BW}_3$, while $d^{FW}_5 < d^{BW}_5$. As for activity floats, all the activities but 3, 4, and 5, have both start and finish floats equal to zero. However, activity 3 has a positive start float, contrarily to its finish float, while the opposite occurs to activity 5, and finally, activity 4 has both positive start and finish floats.

Figs. 5 and 6 represent the SS-standardized networks of the FW-network and of the BW-network of Example 2, respectively. In both networks, critical paths have length $C^*_{\max} = 13$; dashed arcs do not belong to a critical path. Therefore, all the activities, except activity 8, belong to a critical path and, hence, they are $FW\&BW$-critical.

However, looking at Table 2, it results that $ES_4 = 3$, $EF_4 = 4$, and $LS_4 = 4$, $LF_4 = 5$, and, hence, both the start float $LS_4 - ES_4$ and the finish float $LF_4 - EF_4$ of activity 4 are strictly greater than 0, despite this activity results to be $FW\&BW$-critical.

Therefore, considering the well-known concept of critical activity (recalled in Section 2) and in the light of the analysis just performed on the given examples, we cannot establish if an activity is critical just by analyzing if it belongs to a critical path on both the FW- and BW- networks, because even a $FW\&BW$-critical activity can have both the start and finish floats greater than zero (see, e.g., activity 4 of Example 2).

In conclusion, from the above analysis, two questions arise: (i) Since start and finish floats could be different, what are the conditions on the activity floats that allow one to characterize if an activity is critical? (ii) Is there a specific network related to a particular set of optimal durations (possibly distinct from the FW- and BW- networks) useful for analyzing the floats of an activity i and for determining its criticality if the activity lies on a critical path of this network?

In the next sections, we show that such a network (that we will call $best(i)$-network) exists, for each activity i, and we provide a new and more general definition of critical activity.

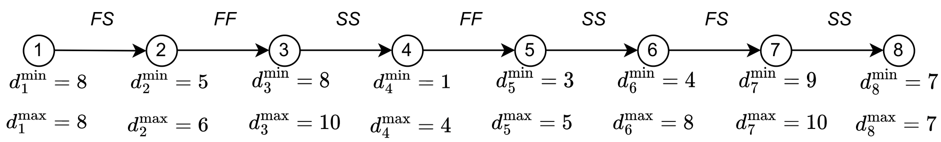

Fig. 4. Project network, with minimum and maximum activity durations, of Example 2.

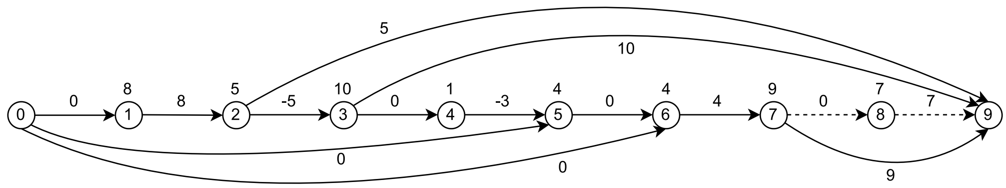

Fig. 5. The *SS*-standardized network of the *FW*-network of Example 2.

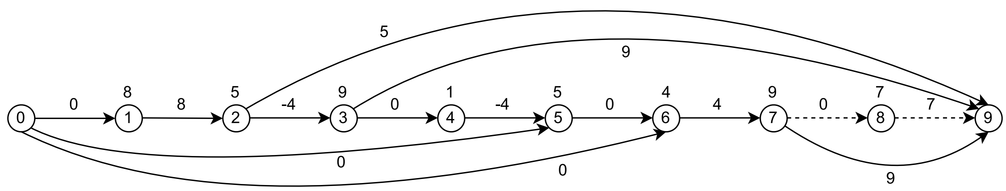

Fig. 6. The *SS*-standardized network of the *BW*-network of Example 2.

4. Theoretical results - Generalizing the concept of critical activity

Inequality (1) can be clearly generalized for any project network $N(\bar{d})$ related to given activity durations $\bar{d}_k \in [d_k^{\min}, d_k^{\max}]$, with $k = 1, \dots, n$, minimizing the project length (equal to $C^*_{\max}$). That is,

$$\delta_i^{N(\bar{d})} \leq \min\{LS_i - ES_i, LF_i - EF_i\}, \quad (2)$$

where $\delta_i^{N(\bar{d})} = C^*_{\max} - (\ell_{0\to i}^{N(\bar{d})} + \ell_{i\to n+1}^{N(\bar{d})}) \geq 0$ is the gap between $C^*_{\max}$ and the length of the longest path from 0 to $n+1$ traversing i in network $N(\bar{d})$. In fact $\delta_i^{N(\bar{d})} \leq LS_i - ES_i$, since in the *SS*-standardized network representation we have $\ell_{0\to i}^{N(\bar{d})} \geq ES_i$ and $\ell_{i\to n+1}^{N(\bar{d})} \geq C^*_{\max} - LS_i$. Analogously, considering the *FF*-standardized network representation, it can be shown that $\delta_i^{N(\bar{d})} = C^*_{\max} - (\hat{\ell}_{0\to i}^{N(\bar{d})} + \hat{\ell}_{i\to n+1}^{N(\bar{d})}) \leq LF_i - EF_i$. In fact, since $\ell_{0\to i}^{N(\bar{d})} + \ell_{i\to n+1}^{N(\bar{d})} = \hat{\ell}_{0\to i}^{N(\bar{d})} + \hat{\ell}_{i\to n+1}^{N(\bar{d})}$, we remark that the gap $\delta_i^{N(\bar{d})}$ does not depend on the adopted network standardization.

Let $d_k^{(i)} \in [d_k^{\min}, d_k^{\max}]$ be the activity durations, with $k = 1 \dots n$, minimizing the project length, and maximizing the gap between the minimum project length $C^*_{\max}$ and the longest path from 0 to $n+1$ traversing activity i. Let us denote with $\delta_i^{\max}$ the maximum value of this gap. In particular, we show that the above (best) activity durations can be derived from the forward and backward durations, as follows: $d_h^{(i)} = d_h^{FW}$, for $h = 1, \dots, i-1$, $d_j^{(i)} = d_j^{BW}$, for $j = i+1, \dots, n$, and $d_i^{(i)} = \min\{d_i^{FW}, d_i^{BW}\}$. Accordingly, let us call $best(i)$-network the project network $N(d^{(i)})$ with the above (fixed) activity durations $d_k^{(i)}, k = 1 \dots n$. Moreover, we show that $\delta_i^{\max} = \min\{LS_i - ES_i, LF_i - EF_i\}$, and, hence, the $best(i)$-network is the network for which Inequality (2) is fulfilled at the equality.

Lemma 1. *The lengths of the longest paths in the SS and in the FF standardized networks of* $best(i)$*-network from node* 0 *to node* $n+1$*, without traversing node* i*, are not greater than* $C^*_{\max}$.

Proof. Given the *SS*-standardized network of $best(i)$-network, it is clear that length $\ell_{0\to h}^{best(i)}$, with $0 < h < i < n+1$, is minimum, because $\ell_{0\to h}^{best(i)} = \ell_{0\to h}^{FW} = ES_h$. Analogously, the length $\ell_{j\to n+1}^{best(i)}$, with $0 < i < j < n+1$, is minimum, because $\ell_{j\to n+1}^{best(i)} = \ell_{j\to n+1}^{BW} = C^*_{\max} - LS_j$. Moreover, the length t_{hj} of arc (h, j), if it exists, is equal to $t_{hj}(d_h^{FW}, d_j^{BW}) = t_{hj}^{+}(d_h^{FW}) - t_{hj}^{-}(d_j^{BW}) \leq LS_j - ES_h$, because $ES_h + t_{hi}^{+}(d_h^{FW})$ and $-t_{hj}^{-}(d_j^{BW}) + (C^*_{\max} - LS_j)$ are both minima, according to the forward and backward recursions, respectively, and then their sum $ES_h + t_{hi}^{+}(d_h^{FW}) - t_{hj}^{-}(d_j^{BW}) + (C^*_{\max} - LS_j)$ being equal to the length $\ell_{0\to h}^{best(i)} + t_{hj}(d_h^{FW}, d_j^{BW}) + \ell_{j\to n+1}^{best(i)}$ of the longest path $(0, \dots, h, j, \dots, n+1)$ is minimum and, therefore, it cannot be greater than $C^*_{\max}$. Therefore, the length of the longest path in the *SS*-standardized network of $best(i)$-network from node 0 to node $n+1$ without traversing node i is equal to $\max_{(h,j)\in A: h<i<j}\{\ell_{0\to h}^{best(i)} + t_{hj} + \ell_{j\to n+1}^{best(i)}\} \leq C^*_{\max}$, where A is the arc set of the network.

The same result occurs on the FF-standardized network since the length of any path from node 0 to node $n+1$ in the SS-standardized network is equal to the length of the same path in the FF-standardized network. □

Theorem 1. *If* $d_i^{FW} \leq d_i^{BW}$*, the critical path length of* $best(i)$*-network is equal to* $C^*_{\max}$ *and* $\delta_i^{\max} = \delta_i^{best(i)} = LS_i - ES_i$.

Proof. Let us consider the *SS*-standardized network of the $best(i)$ -network. If $d_i^{FW} \leq d_i^{BW}$, then $d_i^{(i)} = d_i^{FW}$, and, hence, $\ell_{0\to i}^{best(i)} = \ell_{0\to i}^{FW} = ES_i$. Moreover, the lengths of the outgoing arcs of node i in the *SS*-standardized network of $best(i)$-network are not greater than those in the *SS*-standardized network of the BW-network, since $d_i^{FW} \leq d_i^{BW}$. Therefore, we have that $\ell_{i\to n+1}^{best(i)} \leq \ell_{i\to n+1}^{BW}$. However, since $\ell_{i\to n+1}^{BW}$ is minimum and equal to $C^*_{\max} - LS_i$, we have that $\ell_{i\to n+1}^{best(i)} = \ell_{i\to n+1}^{BW} = C^*_{\max} - LS_i$ and minimum too. Therefore, $\ell_{0\to i}^{best(i)} + \ell_{i\to n+1}^{best(i)} = ES_i + C^*_{\max} - LS_i \leq C^*_{\max}$ is the length of the longest path from 0 to $n+1$ traversing activity i. Since, by Lemma 1, also all the other paths from 0 to $n+1$ and not traversing i are not greater than $C^*_{\max}$, with the given activity durations, defining the $best(i)$-network, the project length is still equal to the minimum value $C^*_{\max}$.

In addition, $\delta_i^{best(i)} = C^*_{\max} - (\ell_{0\to i}^{best(i)} + \ell_{i\to n+1}^{best(i)}) = C^*_{\max} - (ES_i + C^*_{\max} - LS_i)$. Hence, $\delta_i^{best(i)} = LS_i - ES_i$; therefore, the $best(i)$-network is the network for which Inequality (2) is fulfilled at the equality and, hence, $\delta_i^{\max} = \delta_i^{best(i)} = LS_i - ES_i$. □

Fig. 7 shows the *SS*-standardized network of the $best(4)$-network of Example 2, i.e., of the project network with activity durations $d_h = d_h^{(4)} = d_h^{FW}$, for $h = 1, 2, 3$, $d_4 = d_4^{(4)} = d_4^{FW} = \min\{d_4^{FW}, d_4^{BW}\}$, and

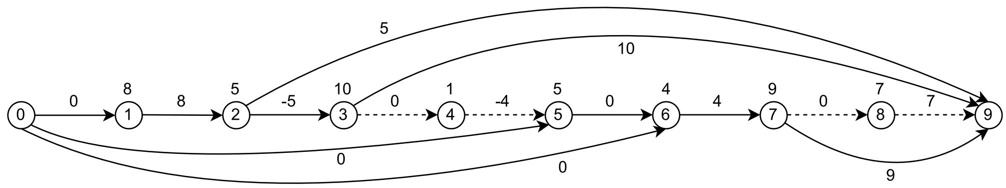

Fig. 7. The *SS*-standardized network of the *best*(4)-network of Example 2.

$d_j = d_j^{(4)} = d_j^{BW}$, for $j = 5,6,7,8$. We can easily note that activity 4 does not belong to a critical path of this network and that the length of the longest path from 0 to $n+1$ traversing activity 4 is equal to 12 (since $\ell^{best(4)}_{0\to4} = 3$ and $\ell^{best(4)}_{4\to9} = 9$), while critical paths have length equal to $C^*_{\max} = 13$.

Therefore, $\delta_4^{\max} = \delta_4^{best(4)} = LS_4 - ES_4 = 1$. The earliest/latest start values of activity 4 listed in Table 2 confirm the value of the *start* float of this activity, derived from the *best*(4)-network, according to Theorem 1. In addition, from the table, we note that the *start* and *finish* float are equal. Therefore, we also have $\delta_4^{\max} = \delta_4^{best(4)} = LF_4 - EF_4 = 1$. In fact, for this activity, both Theorems 1 and (next) 2 apply, since $d_4^{FW} = d_4^{BW}$.

Theorem 2. *If $d_i^{FW} \ge d_i^{BW}$, the critical path length of best(i)-network is equal to $C^*_{\max}$ and $\delta_i^{\max} = \delta_i^{best(i)} = LF_i - EF_i$.*

Proof. For easy of proof, let us consider the *FF*-standardized network of *best*(i)-network. If $d_i^{FW} \ge d_i^{BW}$, then $d_i^{(i)} = d_i^{BW}$, and, hence, $\hat{\ell}^{best(i)}_{i\to n+1} = \hat{\ell}^{BW}_{i\to n+1} = C^*_{\max} - LF_i$. Moreover, the lengths of the incoming arcs of node i in the *FF*-standardized network of *best*(i)-network are not greater than those in the *FF*-standardized network of the FW-network, since $d_i^{FW} \ge d_i^{BW}$. Therefore, we have that $\hat{\ell}^{best(i)}_{0\to i} \le \hat{\ell}^{FW}_{0\to i}$. However, since $\hat{\ell}^{FW}_{0\to i}$ is minimum and equal to EF_i, we have that $\hat{\ell}^{best(i)}_{0\to i} = \hat{\ell}^{FW}_{0\to i} = EF_i$ and minimum too. Therefore, $\hat{\ell}^{best(i)}_{0\to i} + \hat{\ell}^{best(i)}_{i\to n+1} = EF_i + C^*_{\max} - LF_i \le C^*_{\max}$ is the length of the longest path from 0 to $n+1$ traversing activity i. Since, by Lemma 1, also all the other paths from 0 to $n+1$ and not traversing i are not greater than $C^*_{\max}$, with the given activity durations, defining the *best*(i)-network, the project length is still equal to the minimum value $C^*_{\max}$.

Since the lengths of the paths between 0 and $n+1$ do not depend on the type of network standardization, we have $\delta_i^{best(i)} = C^*_{\max} - (\ell^{best(i)}_{0\to i} + \ell^{best(i)}_{i\to n+1}) = C^*_{\max} - (\hat{\ell}^{best(i)}_{0\to i} + \hat{\ell}^{best(i)}_{i\to n+1}) = C^*_{\max} - (EF_i + C^*_{\max} - LF_i)$. Hence, $\delta_i^{best(i)} = LF_i - EF_i$; therefore, the *best*($i$)-network is the network for which Inequality (2) is fulfilled at the equality and, hence, $\delta_i^{\max} = \delta_i^{best(i)} = LF_i - EF_i$. □

In conclusion, according to Theorems 1 and 2, Inequality (2) is valid at equality that is:

Corollary 1. *For the best(i)-network $N(d^{(i)})$ related to activity i, it results*

$$\delta_i^{\max} = \delta_i^{N(d^{(i)})} = \delta_i^{best(i)} = \min\{LS_i - ES_i, LF_i - EF_i\}.$$

Moreover,

Corollary 2. *For any activity i, it results $LS_i - ES_i = \max\{0, (d_i^{FW} - d_i^{BW})\} + \delta_i^{\max}$, and $LF_i - EF_i = \max\{0, (d_i^{BW} - d_i^{FW})\} + \delta_i^{\max}$.*

As an example, let us consider activity 3 of Example 2. Since $d_3^{FW} > d_3^{BW}$, we have that the duration of activity 3 in the *best*(3)-network is equal to $d_3^{BW} = 9$, while the durations of activities $h = 1,2$ are equal to d_h^{FW}, and the durations of activities $j = 4,\ldots,8$ are equal d_j^{BW}. Since, in this example, we have that $d_h^{FW} = d_h^{BW}$, with $h = 1,2$, in the resulting *best*(3)-network for any activity i we have $d_i^{(3)} = d_i^{BW}$, and then the *best*(3)-network is equivalent to the BW-network, whose *SS*-standardized network is shown in Fig. 6. We can easily note that activity 3 belongs to a critical path of this network and that the length of the longest path from 0 to $n+1$ traversing activity 3 is equal to $C^*_{\max} = 13$ (since $\ell^{best(3)}_{0\to3} = 4$ and $\ell^{best(3)}_{3\to9} = 9$). Therefore, $\delta_3^{\max} = \delta_3^{best(3)} = 0$, which is equal to $LF_3 - EF_3 = 0$, while $LS_3 - ES_3 = (d_3^{FW} - d_3^{BW}) + \delta_3^{\max} = 1$, since $d_3^{FW} = 10$ and $d_3^{BW} = 9$.

From the above examples, we note that in general $\delta_i^{FW} \le \delta_i^{\max} = \delta_i^{best(i)} = \min\{LS_i - ES_i, LF_i - EF_i\}$ and $\delta_i^{BW} \le \delta_i^{\max} = \delta_i^{best(i)} = \min\{LS_i - ES_i, LF_i - EF_i\}$ and these inequalities could be strict. In particular, it could happen that $\delta_i^{FW} = \delta_i^{BW} = 0$ despite $\min\{LS_i - ES_i, LF_i - EF_i\} > 0$, that is, a $FW\&BW$-critical activity could have positive floats w.r.t. start and finish times (see, e.g., activity 4 of Example 2).

Therefore, these results call for a new and more general definition of critical activity.

Definition 1. An activity i is critical if $\min\{LS_i - ES_i, LF_i - EF_i\} = 0$ and strongly critical if both the start and the finish floats are equal to 0.

From Definition 1 and Theorems 1 and 2, it follows that activity i is critical if and only if $\delta_i^{\max} = 0$ (i.e., it belongs to a critical path of the *best*(i)-network), and it is strongly critical if, in addition, it has $d_i^{FW} = d_i^{BW}$.

Going back to our Research Questions (RQs) listed in Section 1, we can therefore conclude that on GPRs project networks with variable activity durations:

1. The start and finish floats of an activity can be different.
2. It is not required that both the start and finish floats must be equal to zero for a critical activity, but it is required for at least one of them.
3. The activity durations evaluated with the forward and backward recursion are not necessarily equal, even when the activity is critical.
4. An activity that belongs to the longest path both when the activity durations are evaluated with the forward recursion, and when they are evaluated with the backward recursions, is not necessarily critical.

We close this section, providing a complete analysis of FW and BW activity durations, activity floats, path length gaps, and criticality for all the activities of Examples 1 and 2. The results are summarized in Tables 3 and 4, respectively. As for Example 1, from the *FW* and *BW* activity durations, *best*(i)-network $N(d^{(i)})$ is equal to the BW-network for activity $i = 1$ (see Fig. 3), while it is equal to the FW-network (see Fig. 2) for the remaining activities (i.e. $i = 2,3,4$). In particular, looking at the *best*(1)-network, we note that activity 1 does not belong to any critical path, and $\delta_1^{\max} = \delta_1^{best(1)} = 2$ which is equal to both its *start* and *finish* floats ($F_1^S = LS_1 - ES_1 = 2$ and $F_1^F = LF_1 - EF_1 = 2$), since $d_1^{FW} = d_1^{BW}$, and then the activity is not critical. For activity 2, we note that it belongs to a critical path of the *best*(2)-network, then $\delta_2^{\max} = \delta_2^{best(2)} = 0$. Therefore, it is critical, and in fact $F_2^S = LS_2 - ES_2 = 0$; however, $F_2^F = LF_2 - EF_2 = 2$, since $d_2^{BW} = d_2^{FW} + 2$. Activity 3 lies on a critical path of *best*(3)-network and $d_3^{FW} = d_3^{BW}$; therefore, $\delta_3^{\max} = \delta_3^{best(3)} = LS_3 - ES_3 = LF_3 - EF_3 = 0$. Finally, activity 4 does not belong to any critical path of the *best*(4)-network and $\delta_4^{\max} = \delta_4^{best(4)} = 2$. Since $d_4^{FW} = d_4^{BW}$, we have $\delta_4^{best(4)} = LS_4 - ES_4 = LF_4 - EF_4 = 2$. Based on these results, summarized in Table 3, and according to Definition 1, we have that activities 1 and 4 are not critical, since they have both

Table 3
Results for Example 1.

Activity i	1	2	3	4
d_i^{FW}, d_i^{BW}	2, 2	2, 4	6, 6	4, 4
F_i^S	2	0	0	2
F_i^F	2	2	0	2
δ_i^{FW}	0	0	0	2
δ_i^{BW}	2	0	0	2
$\delta_i^{\max}$	2	0	0	2
Critical	No	Yes	Yes	No

Table 4
Results for Example 2.

Activity i	1	2	3	4	5	6	7	8
d_i^{FW}, d_i^{BW}	8, 8	5, 5	10, 9	1, 1	4, 5	4, 4	9, 9	7, 7
F_i^S	0	0	1	1	0	0	0	2
F_i^F	0	0	0	1	1	0	0	2
δ_i^{FW}	0	0	0	0	0	0	0	2
δ_i^{BW}	0	0	0	0	0	0	0	2
$\delta_i^{\max}$	0	0	0	1	0	0	0	2
Critical	Yes	Yes	Yes	No	Yes	Yes	Yes	No

positive *start* and *finish* floats, activity 2 is critical because its *start* float is zero (even if its *finish* float is positive), while activity 3 is critical, with both its *start* and *finish* floats are equal to zero, meaning that it is strongly critical.

As for Example 2, with the results summarized in Table 4, for activity 4 we already showed that $\delta_4^{\max} = 1$ and $d_4^{FW} = d_4^{BW}$, with the consequence that $F_i^S = LS_4 - ES_4 = LF_4 - EF_4 = F_i^F = 1$. Consequently, despite this activity belongs to a critical path on both the FW- and BW-networks (i.e., $\delta_4^{FW} = \delta_4^{BW} = 0$), it is not critical since it does not belong to any critical path in the $best(4)$-network (see Fig. 7), because $\delta_4^{\max} = \delta_4^{best(4)} = 1$; we can note that for activity 4 the $best(4)$-network network is different from both the FW- and BW- networks. As for the other activities, the $best(i)$-network is equal to the BW-network, for activities $i = 1, 2, 3$, while it is equal to the FW-network for activities $i = 5, \dots, 8$. Using these networks, analog evaluations can be done on the path length gaps $\delta_i^{\max}$, ending up that all the activities are critical, but activities 4 and 8.

We conclude the analysis by showing that, for the case with only Finish-to-Start precedences, the traditional concepts of activity float and criticality are again found by applying the more general concepts provided in this work. In fact, in that case, it is easy to see that $d_i^{FW} = d_i^{BW} = d_i^{\min}$, for each activity $i = 1, \dots, n$. Therefore, the FW- and BW-networks are the same and equal to the $best(i)$-network, because also $d_i^{(i)} = d_i^{\min}$, for all activity $i = 1, \dots, n$. Hence, according to Corollary 2, we have $LS_i - ES_i = LF_i - EF_i = \delta_i^{\max} = \delta_i^{best(i)}$ for any activity i; therefore, *start* and *finish* floats F_i^S, F_i^F are equal, and we simply refer to them as activity float. Finally, an activity float is equal to zero if and only if $\delta_i^{\max} = 0$, meaning that the activity is critical if and only if it belongs to a critical path of the project network, as stated by the traditional concept of activity criticality.

5. Computational results

To further highlight the importance of addressing the Research Questions outlined in the previous sections, we conducted an extensive experimental campaign. As previously discussed, variable activity durations can be conceptualized as a multi-modal scheduling scenario. In such a scenario, each activity i is executable with infinite combinations of resources, resulting in varying durations ranging from a minimum ($d_i^{\min}$) to a maximum ($d_i^{\max}$) value, even if in our specific scenario resources are considered unconstrained. Consequently, to perform our analysis, we generated a new set of test instances, by adapting to our scenario some known benchmark instances available in the PSPLib repository (PSPLib, 0000). In particular, we consider the multi-mode resource-constrained project scheduling instance sets "Jn.mm", with $n = 10, 12, 14, 16, 18, 20, 30$ activities. Moreover, since these multi-mode instances are with at most 30 activities, we also adapted the single-mode resource-constrained project scheduling instance sets "Jn.sm", with $n = 60, 90, 120$ activities, for generating additional larger instances.

All the benchmark instances that we adapted to our scenario assume that the precedences are of type FS. To generate instances with GPRs (named "Jn.gpr"), as considered in our case, from each given benchmark instance, we generated 5 instances preserving the network structure and randomly assigning the type of precedence from the sets $\{SS, SF, FS, FF\}$.

As for the values of the minimum and maximum activity durations $d_i^{\min} \leq d_i^{\max}$, for the (smaller) instances derived from the multi-mode test cases "Jn.mm", we preserve their values which are integers ranging in $[1, 10]$. In particular, for the smaller instances, Table 5 reports the average distribution (in percentage) of the values of the range widths $\Delta_i = d_i^{\max} - d_i^{\min}$ of the activity durations, with $\Delta_i = 0, 1, \dots, 9$ (according to the possible values of $d_i^{\min}$ and $d_i^{\max}$).

For the other larger instances, derived from the single-mode test cases "Jn.sm", for which the values of $d_i^{\min}$ and $d_i^{\max}$ cannot be directly derived from, we randomly generated their values as integers in the range $[1, 10]$, assuring (on average) the same distribution for the activity duration range widths $\Delta_i = d_i^{\max} - d_i^{\min}$ evaluated on the smaller instances (last row of Table 5). Table 6 reports the distribution of the values of $\Delta_i = d_i^{\max} - d_i^{\min}$ on the larger instances.

The overall analyzed dataset consists of 27,450 generated instances. These instances are categorized into two groups: 19,650 smaller instances and 7800 larger instances, derived from multi-mode and single-mode benchmark test cases, respectively. Collectively, these instances encompass a total of 1,063,270 activities. Among them, 343,270 stem from multi-mode benchmark instances, while 720,000 activities are from single-mode benchmark instances.

After describing the generated instances, we will analyze the results obtained, following the theoretical exploration of the Research Questions (RQs) presented in the previous section. Specifically, we will focus on the following evaluations:

1. How often does a critical activity have only the start (finish) float equal to zero, and how often does it have both the two floats equal to zero?
2. How often are the durations d_i^{FW} and d_i^{BW} (computed by the forward and backward recursions, respectively) of an activity i different? Additionally, which is the frequency in which d_i^{FW} (d_i^{BW}) is equal to $d_i^{\min}$ or $d_i^{\max}$ or in between them?
3. How often does a non-critical activity belong to a critical path of networks $N(d^{FW})$ and/or $N(d^{BW})$?

Starting with the analysis of the critical activities, the second and third columns of Table 7 show the total number of critical activities for each set of instances and the related percentage compared to the total number of activities. The last three columns of Table 7 provide details on the average percentage of critical activities having only zero start float (i.e., $F_i^S = 0$ & $F_i^F > 0$), only zero finish float (i.e., $F_i^F = 0$ & $F_i^S > 0$), and both these floats equal to zero (i.e., $F_i^S = F_i^F = 0$), respectively. We note, as could be expected, that the percentage of critical activities decreases as the size of the instances increases. However, the percentage of critical activities with only one zero float increases (while the percentage of critical activities having both zero floats decreases) by increasing the size of the instances. This is because by increasing the number of activities, the frequency with which $d_i^{FW} \neq d_i^{BW}$ (and, then, by Corollary 2, $F_i^S \neq F_i^F$) increases. This trend could be justified by the heightened flexibility of the activity durations that are, in general, less constrained with the increase in the number of project activities. In addition, comparing the values of the third last and the second last columns of Table 7, we note that the average percentage

Table 5
Characteristics of small/medium size instances.

Inst. set	# inst.	# act.	avg. distribution (%)									
			$\Delta_i = 0$	$\Delta_i = 1$	$\Delta_i = 2$	$\Delta_i = 3$	$\Delta_i = 4$	$\Delta_i = 5$	$\Delta_i = 6$	$\Delta_i = 7$	$\Delta_i = 8$	$\Delta_i = 9$
J10.gpr	2680	26,800	1.03	5.35	10.32	12.39	14.40	15.21	15.26	11.96	8.45	5.63
J12.gpr	2735	32,820	0.98	5.10	10.31	12.42	13.89	15.05	13.56	13.33	9.98	5.38
J14.gpr	2755	38,570	0.99	5.38	9.66	12.65	13.77	15.63	13.84	12.69	9.92	5.47
J16.gpr	2750	44,000	0.94	5.77	9.40	12.56	14.23	14.58	14.41	13.35	9.56	5.20
J18.gpr	2760	49,680	0.97	5.52	9.59	12.64	14.59	15.06	14.81	12.31	9.47	5.04
J20.gpr	2770	55,400	1.05	5.56	10.05	12.97	14.31	14.90	14.03	12.64	9.33	5.16
J30.gpr	3200	96,000	0.90	5.11	9.54	12.13	14.60	15.66	14.68	12.43	9.74	5.22
tot./avg.	19,650	343,270	0.98	5.40	9.84	12.54	14.26	15.15	14.37	12.67	9.49	5.30

Note: # inst. = total number of instances, # act. = total number of activities, $\Delta_i = d_i^{\max} - d_i^{\min}$.

Table 6
Characteristics of large size instances.

Inst. set	# inst.	# act.	avg. distribution (%)									
			$\Delta_i = 0$	$\Delta_i = 1$	$\Delta_i = 2$	$\Delta_i = 3$	$\Delta_i = 4$	$\Delta_i = 5$	$\Delta_i = 6$	$\Delta_i = 7$	$\Delta_i = 8$	$\Delta_i = 9$
J60.gpr	2400	144,000	0.95	5.43	9.97	12.41	14.25	15.18	14.31	12.62	9.62	5.26
J90.gpr	2400	216,000	0.99	5.39	9.90	12.36	14.27	15.19	14.40	12.67	9.50	5.33
J120.gpr	3000	360,000	0.97	5.44	9.83	12.62	14.26	15.20	14.31	12.62	9.43	5.31
tot./avg.	7800	720,000	0.97	5.42	9.90	12.47	14.26	15.19	14.34	12.64	9.51	5.30

Note: Note: # inst. = total number of instances, # act. = total number of activities, $\Delta_i = d_i^{\max} - d_i^{\min}$.

Table 7
Critical activities and floats statistics.

Inst. set	# c.a.	% c.a.	$\%(F_i^S = 0\ \&\ F_i^F > 0)$	$\%(F_i^F = 0\ \&\ F_i^S > 0)$	$\%(F_i^S = F_i^F = 0)$
J10.gpr	8438	31.49	10.41	10.44	79.15
J12.gpr	10,237	31.19	10.50	10.56	78.94
J14.gpr	11,335	29.39	11.87	11.54	76.59
J16.gpr	12,179	27.68	12.48	12.48	75.04
J18.gpr	12,969	26.11	12.56	12.61	74.82
J20.gpr	13,727	24.78	12.92	13.23	73.85
J30.gpr	19,271	20.07	14.79	14.13	71.08
J60.gpr	18,820	13.07	15.67	15.67	68.66
J90.gpr	22,057	10.21	16.54	16.38	67.08
J120.gpr	30,177	8.38	17.32	16.30	66.38

Note: #, % c.a. = total number and percentage of critical activities, $F_i^S = LS_i - ES_i$, $F_i^F = LF_i - EF_i$.

of critical activities having only zero start float is almost equal to that of those having only zero finish float, for each the instance set. In particular, the (average) percentage of critical activities having only the start (finish) float equal to zero is in the range $[10.41\%; 17.32\%]$ ($[10.44\%; 16.30\%]$), and, hence, it is not negligible in all the test cases.

The results just described are consistent with those reported in Table 8, which shows the comparison between the values of activity durations d_i^{FW} and d_i^{BW}, computed by the forward and backward recursions, respectively, for all the activities i having $d_i^{\min} < d_i^{\max}$. Again, the percentage of activities with different FW and BW durations increases with increasing the number of activities, for the reason we have already mentioned. In particular, the occurrences $d_i^{FW} < d_i^{BW}$ range from 12.08% to 23.31% while the occurrences $d_i^{FW} > d_i^{BW}$ range from 12.01% to 21.41%. Accordingly, the cases with $d_i^{FW} = d_i^{BW}$ decrease from 74.89% to 54.31%, and, in all the cases, the fraction of activities having different FW and BW durations is not negligible.

Table 9 illustrates the average distribution of FW and BW activity durations for the activities having $d_i^{\min} < d_i^{\max}$. The last two columns of the table show the relative positioning of d_i^{FW} and d_i^{BW} within the activity duration range $[d_i^{\min}, d_i^{\max}]$. For this purpose, we introduce and evaluate the following two indicators $\alpha_i^{FW} = \frac{d_i^{FW} - d_i^{\min}}{d_i^{\max} - d_i^{\min}}$ and $\alpha_i^{BW} = \frac{d_i^{BW} - d_i^{\min}}{d_i^{\max} - d_i^{\min}}$, ranging between 0 and 1: the closer α_i^{FW} (α_i^{BW}) is to 0, the closer duration d_i^{FW} (d_i^{BW}) is to $d_i^{\min}$; vice-versa, the closer they are to 1, the closer that durations are to $d_i^{\max}$. From the results listed in the table, it appears that both indicators α_i^{FW} and α_i^{BW} on average are very close to 0: durations d_i^{FW} and d_i^{BW} are very close to $d_i^{\min}$, with d_i^{FW} being slightly closer to $d_i^{\min}$ than d_i^{BW}. In addition, on average, we record a small increase of both α_i^{FW} and α_i^{BW} by increasing the number of the project activities, meaning that accordingly both d_i^{FW} and d_i^{BW} tend on average to align less closely to $d_i^{\min}$. Again, this trend could be justified by the heightened flexibility of the activity durations on projects with larger sets of activities.

Moreover, looking at columns 2, 3, 4 of the table, it appears that durations d_i^{FW} are often equal to $d_i^{\min}$ (with decreasing average percentages going from 87.04% to 73.38% by increasing the project sizes), sometimes are in between $d_i^{\min}$ and $d_i^{\max}$ (with average percentages ranging from 9.87% to 13.70%), and only a few times equal to $d_i^{\max}$ (with average percentages ranging from 3.10% to 12.92%). The same happens for durations d_i^{BW} (see columns 5, 6, and 7 of the table), even if on average they are slightly larger than d_i^{FW}, as it appears also from the last two columns of the table. These results can be justified because, besides the aim of minimizing the project makespan, on the one hand, we also wish to finish the activities as earliest as possible (obtained with the forward recursion), and, on the other hand, we wish to start the activities as latest as possible (obtained with the backward recursion), and of course, in general, all this is obtained by selecting the activity execution modality with sufficiently small duration.

Finally, Table 10 lists the statistics on non-critical activities (i.e., activities having both start and finish floats greater than 0). Columns 2 and 3 report the total number of non-critical activities and their percentage compared to the number of all activities, for each instance set: we may note that the number of non-critical activities increases by increasing the number of project activities from 68.51% to 91.62%. The last 5 columns of the table report the statistics on the subset of non-critical activities that belongs to a critical path in the FW and/or BW

Table 8
Comparison between FW and BW activities durations (d_i^{FW}, d_i^{BW}) when $d_i^{\min} < d_i^{\max}$.

Inst. set	$\#(d_i^{FW} < d_i^{BW})$	$\%(d_i^{FW} < d_i^{BW})$	$\#(d_i^{FW} = d_i^{BW})$	$\%(d_i^{FW} = d_i^{BW})$	$\#(d_i^{FW} > d_i^{BW})$	$\%(d_i^{FW} > d_i^{BW})$
J10.gpr	3237	12.08	20 070	74.89	3218	12.01
J12.gpr	4905	14.95	22 838	69.59	4757	14.49
J14.gpr	6092	15.79	26 152	67.80	5946	15.42
J16.gpr	7237	16.45	29 117	66.18	7231	16.43
J18.gpr	8373	16.85	32 657	65.73	8170	16.45
J20.gpr	9687	17.49	35 880	64.77	9253	16.70
J30.gpr	18 582	19.36	58 801	61.25	17 752	18.49
J60.gpr	30 703	21.32	83 047	57.67	28 881	20.06
J90.gpr	48 832	22.61	119 919	55.52	45 103	20.88
J120.gpr	83 925	23.31	195 529	54.31	77 060	21.41

Note: $\#, \%(\cdot)$ = number and percentage of activities for which condition $(\cdot)$ is true.

Table 9
Avg. distribution of FW and BW activities durations (d_i^{FW}, d_i^{BW}) when $d_i^{\min} < d_i^{\max}$.

Inst. set	avg. occurrences (%)						avg.	
	$d_i^{FW} = d_i^{\min}$	$d_i^{\min} < d_i^{FW} < d_i^{\min}$	$d_i^{FW} = d_i^{\max}$	$d_i^{BW} = d_i^{\min}$	$d_i^{\min} < d_i^{BW} < d_i^{\max}$	$d_i^{BW} = d_i^{\max}$	α_i^{FW}	α_i^{BW}
J10.gpr	87.04	9.87	3.10	87.00	9.68	3.32	0.07	0.07
J12.gpr	84.04	11.68	4.28	83.72	11.82	4.46	0.09	0.10
J14.gpr	82.77	12.23	5.00	82.52	12.19	5.28	0.10	0.11
J16.gpr	81.65	12.50	5.85	81.61	12.45	5.94	0.11	0.11
J18.gpr	81.40	12.44	6.16	81.03	12.78	6.19	0.12	0.12
J20.gpr	80.92	12.62	6.46	80.20	12.74	7.06	0.12	0.13
J30.gpr	78.51	13.53	7.96	77.65	13.72	8.63	0.14	0.15
J60.gpr	75.89	13.48	10.63	74.69	13.89	11.41	0.17	0.18
J90.gpr	74.38	13.69	11.93	72.82	14.03	13.15	0.18	0.20
J120.gpr	73.38	13.70	12.92	71.71	13.82	14.48	0.19	0.21

Note: $\alpha_i^{FW} = \frac{d_i^{FW} - d_i^{\min}}{d_i^{\max} - d_i^{\min}}$, $\alpha_i^{BW} = \frac{d_i^{BW} - d_i^{\min}}{d_i^{\max} - d_i^{\min}}$.

Table 10
Non-critical activities statistics.

Inst. set	# n.-c.a.	% n.-c.a.	*Hidden* non-critical activities				
			# *h.*	% *h.*	$\in N_c(d^{FW})$ $\notin N_c(d^{BW})$	$\in N_c(d^{BW})$ $\notin N_c(d^{FW})$	$\in N_c(d^{FW})$ $\in N_c(d^{BW})$
J10.gpr	18,362	68.51	686	3.74	50.73%	49.13%	0.15%
J12.gpr	22,583	68.81	1312	5.81	49.77%	49.47%	0.76%
J14.gpr	27,235	70.61	1561	5.73	49.58%	49.39%	1.02%
J16.gpr	31,821	72.32	1812	5.69	52.48%	46.47%	1.05%
J18.gpr	36,711	73.89	1913	5.21	48.41%	50.65%	0.94%
J20.gpr	41,673	75.22	2072	4.97	52.36%	46.04%	1.59%
J30.gpr	76,729	79.93	3466	4.52	50.43%	47.75%	1.82%
J60.gpr	125,180	86.93	4055	3.24	50.38%	47.57%	2.05%
J90.gpr	193,943	89.79	5217	2.69	49.17%	48.07%	2.76%
J120.gpr	329,823	91.62	7204	2.18	48.89%	49.50%	1.61%

Note: #, % n.-c.a. = total number and percentage of non-critical activities, #, % *h.* = total number and percentage (respect to # n.-c.a) of *hidden* non-critical activities, $N_c(\cdot)$ = critical subnetwork of $N(\cdot)$.

networks, i.e., which belong to critical subnetwork $N_c(d^{FW})$ of network $N(d^{FW})$ and/or to critical subnetwork $N_c(d^{BW})$ of network $N(d^{BW})$. Since these non-critical activities would (on the contrary) appear as critical to at least one of these two networks, we refer to them as *hidden* non-critical activities. By column 5 of the table, the *hidden* non-critical activities exist and are between 2.18% and 3.74% of the non-critical activities, i.e. and their number is not negligible because on average they turn out to be $3,31\%$ of the total project activities. The last three columns of the table show that among the *hidden* non-critical activities about, half are on a critical path of network $N(d^{FW})$ and about the other half are on a critical path of network $N(d^{BW})$, while only very few (0.05% of the total activities) belong to a critical path on both the two networks (e.g., as activity 4 in Example 2).

In conclusion, from the experimental analysis, the cases in which activity start and finish floats are different are on average 36.23%; in particular, this occurs in 26.84% of the critical activities. In addition, 3.31% of the activities are non-critical even if they belong to a critical path of the FW or BW networks. This shows that our findings are not simply theoretical but also important from the application point of view.

6. Conclusions

We have shown that in the presence of variable activity durations, that is when the duration of each activity can be chosen within a range of values (e.g., in the context of multi-modal activity execution), the traditional conditions to identify criticalities on a GPRs project network appear to be no longer valid. In this paper, we proposed a new (and more general) definition of critical activity and gave the corresponding rules for its identification. An extensive experimental analysis has been carried out, demonstrating that our findings are important from a theoretical point of view and meaningful for quantitative project management. We believe that this may open new perspectives on related project scheduling problems; in fact, the circumstance under which a critical activity may have a variable duration and one of its floats different from zero introduces flexibility in the resource assignment. Therefore, future work will be devoted to analyzing how to exploit the results contained in this paper when resource constraints are considered explicitly.

CRediT authorship contribution statement

L. Bianco: Conceptualization, Methodology, Writing – original draft. **M. Caramia:** Writing – original draft, Conceptualization, Methodology. **S. Giordani:** Software, Methodology, Conceptualization, Writing – original draft. **A. Salvatore:** Conceptualization, Methodology, Software, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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