# Cuntz algebras automorphisms: graphs and stability of permutations

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**Abstract**

We characterize the permutative automorphisms of the Cuntz algebra $\mathcal{O}_n$ (namely, stable permutations) in terms of two sequences of graphs that we associate to any permutation of a discrete hypercube $[n]^t$. As applications we show that in the limit of large $t$ (resp. $n$) almost all permutations are not stable, thus proving Conj. 12.5 in [3], characterize (and enumerate) stable quadratic 4 and 5-cycles, as well as a notable class of stable quadratic $r$-cycles, i.e. those admitting a compatible cyclic factorization by stable transpositions. Some of our results use new combinatorial concepts that may be of independent interest.

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## Contents



# 1 Introduction

$C^*$-algebras where first introduced for providing a suitable environment for a rigorous approach to quantum theories [14], and more recently have found applications in many areas of mathematics [5]. Symmetries of $C^*$-algebras, in the most classical sense, are provided by (unital and $*$-preserving) automorphisms. Despite the fact that many general results and constructions exist about automorphisms of $C^*$-algebras [16] and that they are used extensively, not very much is known about the automorphisms group of most $C^*$-algebras. Indeed, investigating the fine structure of the automorphism group and constructing automorphisms with specific properties are often challenging tasks [20, 21, 22, 13]. The Cuntz algebra $\mathcal{O}_n$ [10], namely the universal $C^*$-algebra generated by $n$ isometries with mutually ortogonal range projections summing up to one, is no exception. Although this definition looks simple, somewhat (not too) unexpectedly, the study of automorphisms of the Cuntz algebras soon appeared quite intriguing and revealed many interesting facets. Notably, following the deep insight by Cuntz, in a series of papers (see e.g. [9, 8, 6, 7]) a general theory of reduced Weyl groups for $\mathrm{Aut}(\mathcal{O}_n)$ has been studied both from a theoretical viewpoint as well as from the perspective of constructing explicit examples.

Later, in [3, 4], these reduced Weyl groups were further investigated from a combinatorial point of view. This approach has proved fruitful and has yielded, in particular, the first exact enumerative formulas for the cardinalities of certain families of elements inside these groups. Such numbers were previously known only in special cases through direct computer calculations. In this paper we deepen these combinatorial investigations and focus on the explicit construction of reduced Weyl group elements using combinatorial techniques, as foreseen by Cuntz [11, p. 195].

More precisely, it was shown in [9] that there is a bijection between the elements of the reduced Weyl group of $\mathrm{Aut}(\mathcal{O}_n)$ [6] and certain permutations of $[n]^t$ ($t \in \mathbb{N}$) which are called stable [3], defined by a complicated recursive procedure. In this work we associate to any permutation of $[n]^t$ a sequence of finite graphs, and characterize the stable permutations in terms of these graphs (Theorem 3.13). While this is not the first time that stable permutations have been characterized in terms of suitable graphs ([9, Corollary 4.12], [2]), our treatment is different, and more efficient. In particular, it can be easily implemented on a computer (see, e.g., the proof of Theorem 9.8) and allows the stability for permutations that previously would have needed massive computer calculations to be decided either by inspection or with a bare minimum of computations. As applications of our characterization we prove that in the limit of large $t$ (resp. $n$) almost all permutations in $[n]^t$ are not stable (Theorems 4.6 and 4.7) thus proving Conjecture 12.5 in [3], and obtain upper and lower bounds on the rank ([3, Def. 4.3]) of stable permutations in $[n]^t$. As further applications we explicitly characterize a notable subclass of the stable $r$-cycles in $S([n]^2)$ in terms of an associated subset of $[r]^2$, and we characterize and enumerate stable 4-cycles and stable 5-cycles in $S([n]^2)$ (Corollaries 9.3 and 9.4, Theorem 9.8, and Propositions 9.12 and 9.13). Some of these characterizations depend on combinatorial concepts that are new, and could be of independent interest, such as *trees with angles* (see Section 8) and the *connectivity set* of a permutation (see Theorem 8.7).

The paper is organized as follows. After collecting some background material in the next section, in Section 3 we discuss the main results: we characterize the stability of a permutation $u \in S([n]^t)$ in terms of two non-negative integers $N(u)$ and $N^{\#}(u)$, and provide lower and upper bounds for the rank of $u$; then we introduce the sequences of graphs $\Gamma_k(u)$ and $\Gamma_k^{\#}(u)$ and characterize the stability of $u$ in terms of these sequences. In Section 4 we show that almost all permutations are not stable. In Section 5 we provide new bounds on the rank of stable permutations in $S([n]^2)$. In Section 6 we identify certain subsets $R(u), C(u)$ (resp. $R^o(u), C^o(u)$) of $[n]$, for $u \in S([n]^t)$, as useful invariants and present some evidence for the conjecture (cf. Conjecture 6.10) that the stability of an $r$-cycle $u$ only depends on a suitable subset $S(u)$ of $[r]^2$; thereby we introduce notions of left/right equivalence of two permutations and prove that, for cycles, equality of the $S$-invariants captures both left and right equivalence. In Section 7 we characterize when two cycles of $S([n]^2)$ are compatible (generalizing Theorem 3.1 of [4]). Section 8 is devoted to the study of a notable subclass of stable cycles that we call strongly stable; they are defined by the existence of a compatible cyclic factorization by stable transpositions, and can be further characterized in several other equivalent ways (Theorem 8.7), including one in terms of the $S$-invariant. In Section 9 we apply the results in the previous one to obtain explicit characterizations of stable $r$-cycles $u \in S([n]^2)$ for $r \leq 5$ and enumerate the stable 4 and 5-cycles. The last section collects some conjectures and a list of topics for further studies.

## 2 Background

For $n \in \mathbb{N}$, $n \geq 2$, $u \in S([n]^t)$, and $v \in S([n]^r)$ (where $[n]^t$ denotes the cartesian product of $t$ copies of $[n] := \{1, \ldots, n\}$) we let the *tensor product* of $u$ and $v$ be the permutation $u \otimes v \in S([n]^{t+r})$ defined by

$$(u \otimes v)(\alpha, \beta) := (u(\alpha), v(\beta))$$

for all $\alpha \in [n]^t$ and $\beta \in [n]^s$. We denote by 1 the identity of $S_n := S([n])$. We refer the reader to [3, Sec. 2] for further information about the tensor product of permutations.

Given a permutation $u \in S([n]^t)$, define a sequence of permutations $\psi_k(u) \in S([n]^{t+k})$, $k \geq 0$ by setting $\psi_0(u) := u^{-1}$ and, for $k \in \mathbb{N}$,

$$\psi_k(u) = \prod_{i=0}^{k}(\underbrace{1 \otimes \ldots \otimes 1}_{k-i} \otimes u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{i}) \prod_{i=1}^{k}(\underbrace{1 \otimes 1 \ldots \otimes 1}_{i} \otimes u \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-i}) \ . \tag{1}$$

Then $u$ as above is said to be *stable* if there exists some integer $k \geq 1$ such that

$$\psi_{k+h}(u) = \psi_{k-1}(u) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{h+1}, \quad h \geq 0 \ , \tag{2}$$

and then $\mathrm{rk}(u)$, the *rank* of $u$, is the least such value of $k$. So, for example, if $t = 1$ then all permutations $u \in S([n])$ are stable of rank 1.

In the sequel, it will be convenient to write, for $k \geq 0$,

$$\mathcal{S}_k(u) = \prod_{i=0}^{k}(\underbrace{1 \otimes \ldots \otimes 1}_{k-i} \otimes u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{i}), \tag{3}$$

(so $(\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u^{-1})$ is the leftmost factor) and $\mathcal{S}_{-1}(u) := 1$, so that

$$\psi_k(u) = \mathcal{S}_k(u)\big(1 \otimes \mathcal{S}_{k-1}(u)^{-1}\big), \tag{4}$$

for all $k \geq 0$. Then, for all $k > h \geq 0$, it is not difficult to check that one has

$$\psi_k(u) = \Big(\underbrace{1 \otimes \ldots \otimes 1}_{h+1} \otimes \mathcal{S}_{k-h-1}(u)\Big)\Big(\psi_h(u) \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-h}\Big)\Big(\underbrace{1 \otimes \ldots \otimes 1}_{h+1} \otimes \mathcal{S}_{k-h-1}(u)\Big)^{-1} \tag{5}$$

For $u \in S([n]^t)$ we write

$$(u_1(x_1, \ldots, x_t), \ldots, u_t(x_1, \ldots, x_t)) := u(x_1, \ldots, x_t)$$

for all $(x_1, \ldots, x_t) \in [n]^t$, and let ${}^t u \in S([n]^t)$ be the transposed permutation defined by

$${}^t u(x_1, \ldots, x_t) := (u_t(x_t, \ldots, x_1), \ldots, u_1(x_t, \ldots, x_1)), \tag{6}$$

for all $(x_1, \ldots, x_t) \in [n]^t$. We then let $u^{\#} := {}^t(u^{-1}) (= ({}^t u)^{-1}) \in S([n]^t)$.

We recall the following properties of these operations, which can be proved in exactly the same way as Propositions 7.1 and 7.2 of [4].

**Proposition 2.1.** *Let $t, r \in \mathbb{N}$. Then:*

*i)* ${}^t(uv) = {}^t u\, {}^t v$ *for all* $u, v \in S([n]^r)$*;*

*ii)* ${}^t(u \otimes v) = {}^t v \otimes {}^t u$ *for all* $u \in S([n]^t)$ *and* $v \in S([n]^r)$*.*

**Proposition 2.2.** *Let $u, v \in S([n]^r)$, then:*

*i)* $(uv)^{\#} = v^{\#} u^{\#}$*;*

*ii)* $(u \otimes v)^{\#} = v^{\#} \otimes u^{\#}$*.*

It turns out that $u \in S([n]^t)$ is stable of rank $k$ if and only if $u^{\#}$ is stable of rank $k$ [4, Theorem 7.3].

**Definition 2.3.** Given $u, v \in S([n]^2)$ we say that $u$ is *compatible* with $v$ (or that $u$ is compatible with $v$ in this order, for emphasis) if

$$(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1) \tag{7}$$

in $S([n]^3)$. In this case we also say that $uv$ is a *compatible product* of $u$ and $v$ and write $u \bullet v$.

All directed graphs in this paper are without multiple edges, but can have loops.

# 3 Stability of permutations and some associated graphs

We introduce integers $N(u)$ and $N^{\#}(u)$ as follows.

**Definition 3.1.** Let $u \in S([n]^t)$, $t > 1$. Then

- $N(u)$ is the least integer such that, for all $k \geq N(u)$ and all $(a_1, \dots, a_{k+t}) \in [n]^{k+t}$, the last $t-1$ elements of $\psi_k(u)(a_1, \dots, a_{k+t})$ and of $(a_1, \dots, a_{k+t})$ coincide, i.e.

  $$\psi_k(u)(a_1, \dots, a_{k+t}) = (b_1, \dots, b_{k+1}, a_{k+2}, \dots, a_{k+t})$$

  for some $b_1, \dots, b_{k+1} \in [n]$ (which may depend on $a_1, \dots, a_{k+t}$). If there is no such integer, we set $N(u) = +\infty$.

- $N^{\#}(u)$ is the least integer such that, for all $k \geq N^{\#}(u)$ and all $(b_1, \dots, b_{k+t}) \in [n]^{k+t}$, the first $t-1$ elements of $\mathcal{S}_k(u)^{-1}(b_1, \dots, b_{k+t})$ and of $(\mathcal{S}_{k-1}(u)^{-1} \otimes 1)(b_1, \dots, b_{k+t})$ coincide. If there is no such integer, we set $N^{\#}(u) = +\infty$.

We first note the following simple but useful property.

**Lemma 3.2.** *Let $u \in S([n]^t)$, and $k \geq 0$. Then*

$$\mathcal{S}_k(u)^{\#} = \mathcal{S}_k(u^{\#}).$$

*Proof.* We have from our definition (3) that

$$\begin{aligned}
\mathcal{S}_k(u)^{\#} &= [(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1}) \cdots (u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})]^{\#} \\
&= (u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})^{\#} \cdots (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1})^{\#} \\
&= {}^t(u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k}) \cdots {}^t(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u) \\
&= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes {}^t u) \cdots ({}^t u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k}) \\
&= \mathcal{S}_k(u^{\#})
\end{aligned}$$

where we have used Propositions 2.1 and 2.2. $\square$

**Proposition 3.3.** *Let $u \in S([n]^t)$, $t > 1$. Then $N^{\#}(u) = N(u^{\#})$ in $\mathbb{N}_0 \cup \{+\infty\}$.*

*Proof.* Assume first that $N^{\#}(u)$ is finite. Let $a_1, \ldots, a_{t-1}, d_t, \ldots, d_{t+k} \in [n]$ and $k \in \mathbb{N}$ be such that $k \geq N^{\#}(u)$. Let

$$(b_1, \ldots, b_{t+k}) := (\mathcal{S}_{k-1}(u) \otimes 1)(a_1, \ldots, a_{t-1}, d_t, \ldots, d_{t+k}).$$

Then

$$(\mathcal{S}_{k-1}(u)^{-1} \otimes 1)(b_1, \ldots, b_{t+k}) = (a_1, \ldots, a_{t-1}, d_t, \ldots, d_{t+k})$$

and hence, since $k \geq N^{\#}(u)$,

$$\mathcal{S}_k(u)^{-1}(b_1, \ldots, b_{t+k}) = (a_1, \ldots, a_{t-1}, c_t, \ldots, c_{t+k})$$

for some $c_t, \ldots, c_{t+k} \in [n]$. This means that

$${}^t(\mathcal{S}_{k-1}(u)^{-1} \otimes 1)(b_{k+t}, \ldots, b_1) = (d_{t+k}, \ldots, d_t, a_{t-1}, \ldots, a_1)$$

and

$${}^t\mathcal{S}_k(u)^{-1}(b_{k+t}, \ldots, b_1) = (c_{t+k}, \ldots, c_t, a_{t-1}, \ldots, a_1)$$

and therefore, by Proposition 2.1 and Lemma 3.2, that

$$(1 \otimes \mathcal{S}_{k-1}(u^{\#}))(b_{k+t}, \ldots, b_1) = (d_{t+k}, \ldots, d_t, a_{t-1}, \ldots, a_1)$$

and

$$\mathcal{S}_k(u^{\#})(b_{k+t}, \ldots, b_1) = (c_{t+k}, \ldots, c_t, a_{t-1}, \ldots, a_1)$$

so

$$\psi_k(u^{\#})(d_{t+k}, \ldots, d_t, a_{t-1}, \ldots, a_1) = (c_{t+k}, \ldots, c_t, a_{t-1}, \ldots, a_1).$$

This shows that $N(u^{\#})$ is finite and that $N(u^{\#}) \leq N^{\#}(u)$.

Conversely, assume that $N(u^{\#})$ is finite and let $b_1, \ldots, b_{t+k} \in [n]$ and $k \in \mathbb{N}$ be such that $k \geq N(u^{\#})$. Let

$$(d_{t+k}, \ldots, d_t, a_{t-1}, \ldots, a_1) := (1 \otimes \mathcal{S}_{k-1}(u^{\#}))(b_{k+t}, \ldots, b_1)$$

and

$$(c_{t+k}, \ldots, c_t, f_{t-1}, \ldots, f_1) := \mathcal{S}_k(u^{\#})(b_{k+t}, \ldots, b_1).$$

Then

$$\psi_k(u^{\#})(d_{t+k}, \ldots, d_t, a_{t-1}, \ldots, a_1) = (c_{t+k}, \ldots, c_t, f_{t-1}, \ldots, f_1)$$

and so, since $k \geq N(u^{\#})$, $f_i = a_i$ for $i = 1, \ldots, t-1$. By Proposition 2.1 and Lemma 3.2 this means that

$${}^t(\mathcal{S}_{k-1}(u)^{-1} \otimes 1)(b_{k+t}, \ldots, b_1) = (d_{t+k}, \ldots, d_t, a_{t-1}, \ldots, a_1)$$

and

$$ {}^t\mathcal{S}_k(u)^{-1}(b_{k+t},\dots,b_1)=(c_{t+k},\dots,c_t,a_{t-1},\dots,a_1). $$

Therefore

$$ (\mathcal{S}_{k-1}(u)^{-1}\otimes 1)(b_1,\dots,b_{k+t})=(a_1,\dots,a_{t-1},d_t,\dots,d_{t+k}) $$

and

$$ \mathcal{S}_k(u)^{-1}(b_1,\dots,b_{k+t})=(a_1,\dots,a_{t-1},c_t,\dots,c_{t+k}), $$

which shows that $N^{\#}(u)$ is finite and $N^{\#}(u)\le N(u^{\#})$.

This proves that $N(u^{\#})$ is finite if and only if $N^{\#}(u)$ is finite, and in this case $N^{\#}(u)=N(u^{\#})$. The result follows. $\square$

**Proposition 3.4.** *Let $u\in S([n]^t)$, $t>1$, be a stable permutation. Then*

*(i)* $N(u)\le \mathrm{rk}(u)+t-2$*;*

*(ii)* $N^{\#}(u)\le \mathrm{rk}(u)+t-2$*.*

*Proof.* Let, for brevity $k=\mathrm{rk}(u)$. To prove (i) note that, by the definition of rank of a stable permutation, $\psi_{k+h+t-2}(u)=\psi_{k-1}(u)\otimes\underbrace{1\otimes\cdots\otimes 1}_{h+t-1}$ for all $h\ge 0$, and (i) follows.

To prove (ii) note first that, by Theorem 7.3 of [4], $u^{\#}$ is also stable of rank $k$. Therefore, by the definition of rank, $\psi_h(u^{\#})=\psi_{k-1}(u^{\#})\otimes\underbrace{1\otimes\cdots\otimes 1}_{h-k+1}$ for all $h\ge k$. Let, for brevity, $v:=\psi_{k-1}(u^{\#})^{\#}\in S([n]^{t+k-1})$. Then

$$ ({}^tv\otimes\underbrace{1\otimes\cdots\otimes 1}_{h-k+1})\,\psi_h(u^{\#})=1 $$

so, by (4),

$$ ({}^tv\otimes\underbrace{1\otimes\cdots\otimes 1}_{h-k+1})\,\mathcal{S}_h(u^{\#})=1\otimes\mathcal{S}_{h-1}(u^{\#}) $$

which, by Proposition 2.1, is equivalent to

$$ {}^t(\underbrace{1\otimes\cdots\otimes 1}_{h-k+1}\otimes v)\,\mathcal{S}_h(u^{\#})={}^t({}^t\mathcal{S}_{h-1}(u^{\#})\otimes 1) $$

and then to

$$ (\underbrace{1\otimes\cdots\otimes 1}_{h-k+1}\otimes v)\,{}^t\mathcal{S}_h(u^{\#})={}^t\mathcal{S}_{h-1}(u^{\#})\otimes 1 $$

and therefore, by Lemma 3.2, to

$$(\underbrace{1 \otimes \cdots \otimes 1}_{h-k+1} \otimes v)\, \mathcal{S}_h(u)^{-1} = \mathcal{S}_{h-1}(u)^{-1} \otimes 1.$$

Hence, if $h \geq k+t-2$, $\mathcal{S}_h(u)^{-1}$ and $\mathcal{S}_{h-1}(u)^{-1} \otimes 1$ agree on the first $t-1$ coordinates, which proves (ii). □

**Theorem 3.5.** *Let $u \in S([n]^t)$, $t > 1$, then $u$ is stable if and only if both $N(u)$ and $N^{\#}(u)$ are finite, and in this case*

$$\max\{N(u) - t + 2, N^{\#}(u) - t + 2\} \leq \operatorname{rk}(u) \leq N(u) + N^{\#}(u) + t - 1.$$

*Proof.* We write, for simplicity, $N$ and $N^{\#}$ in place of $N(u)$ and $N^{\#}(u)$. By the previous proposition, it is enough to show that if both $N$ and $N^{\#}$ are finite, then $u$ is stable with rank bounded by $N + N^{\#} + t - 1$. We will prove that, for all $h \geq N^{\#} + t - 1$,

$$\psi_{N+h}(u) = \psi_{N+h-1}(u) \otimes 1 \in S([n]^{N+t+h}).$$

By formula (5) we can write

$$\psi_{N+h}(u) = \Big(\underbrace{1 \otimes \ldots \otimes 1}_{N+1} \otimes \mathcal{S}_{h-1}(u)\Big)\Big(\psi_N(u) \otimes \underbrace{1 \otimes \ldots \otimes 1}_{h}\Big)\Big(\underbrace{1 \otimes \ldots \otimes 1}_{N+1} \otimes \mathcal{S}_{h-1}(u)\Big)^{-1}$$

and, similarly,

$$\psi_{N+h-1}(u) \otimes 1 = \Big(\underbrace{1 \otimes \ldots \otimes 1}_{N+1} \otimes \mathcal{S}_{h-2}(u) \otimes 1\Big)\Big(\psi_N(u) \otimes \underbrace{1 \otimes \ldots \otimes 1}_{h}\Big)\Big(\underbrace{1 \otimes \ldots \otimes 1}_{N+1} \otimes \mathcal{S}_{h-2}(u) \otimes 1\Big)^{-1}.$$

Let $(a_1, \ldots, a_{N+1}, b_1, \ldots, b_{t-1}, c_1, \ldots, c_h) \in [n]^{N+t+h}$. Since $h-1 \geq N^{\#}$ we have from the definition of $N^{\#}$ that there are $b'_1, \ldots, b'_{t-1}, c'_1, \ldots, c'_h, \tilde{c}_1, \ldots, \tilde{c}_h \in [n]$ such that

$$\mathcal{S}_{h-1}(u)^{-1}(b_1, \ldots, b_{t-1}, c_1, \ldots, c_h) = (b'_1, \ldots, b'_{t-1}, c'_1, \ldots, c'_h)$$

and

$$(\mathcal{S}_{h-2}(u)^{-1} \otimes 1)(b_1, \ldots, b_{t-1}, c_1, \ldots, c_h) = (b'_1, \ldots, b'_{t-1}, \tilde{c}_1, \ldots, \tilde{c}_h).$$

Similarly, by the definition of $N$ we have that there are $a'_1, \dots, a'_{N+1} \in [n]$ such that

$$\psi_N(u)(a_1, \dots, a_{N+1}, b'_1, \dots, b'_{t-1}) = (a'_1, \dots, a'_{N+1}, b'_1, \dots, b'_{t-1}).$$

Therefore

$$\begin{aligned}\psi_{N+h}(u)&(a_1, \dots, a_{N+1}, b_1, \dots, b_{t-1}, c_1, \dots, c_h)\\ &= (a'_1, \dots, a'_{N+1}, b_1, \dots, b_{t-1}, c_1, \dots, c_h)\end{aligned}$$

and analogously

$$\begin{aligned}(\psi_{N+h-1}(u) \otimes 1)&(a_1, \dots, a_{N+1}, b_1, \dots, b_{t-1}, c_1, \dots, c_h)\\ &= (a'_1, \dots, a'_{N+1}, b_1, \dots, b_{t-1}, c_1, \dots, c_h),\end{aligned}$$

as claimed. □

We introduce two sequences $\Gamma_k(u), \Gamma_k^{\#}(u), k \geq 0$ of simple directed graphs (we often write simply $\Gamma_k$ instead of $\Gamma_k(u)$ when there is no danger of confusion, and similarly for the other quantities).

**Definition 3.6.** For a permutation $u \in S([n]^t)$, and $k \geq 0$ we define simple directed graphs $\Gamma_k(u)$ and $\Gamma_k^{\#}(u)$, as follows:

- $V(\Gamma_k) = V(\Gamma_k^{\#}) = [n]^{t-1}$;
- given two vertices $(a_1, a_2, \dots, a_{t-1}), (b_1, b_2, \dots, b_{t-1}) \in [n]^{t-1}$ there is a directed edge $(a_1, a_2, \dots, a_{t-1}) \to (b_1, b_2, \dots, b_{t-1})$ in $\Gamma_k$ if

  $$\psi_k(u)(c_1, \dots, c_{k+1}, a_1, a_2, \dots, a_{t-1}) = (d_1, \dots, d_{k+1}, b_1, b_2, \dots, b_{t-1})$$

  for some $(c_1, \dots, c_{k+1}), (d_1, \dots, d_{k+1}) \in [n]^{k+1}$;
- given two vertices $(a_1, a_2, \dots, a_{t-1}), (b_1, b_2, \dots, b_{t-1}) \in [n]^{t-1}$ there is a directed edge $(a_1, a_2, \dots, a_{t-1}) \to (b_1, b_2, \dots, b_{t-1})$ in $\Gamma_k^{\#}$ if there is $x \in [n]^{t+k}$ such that

  $$\mathcal{S}_k(u)^{-1}(x) = (a_1, a_2, \dots, a_{t-1}, c_1, \dots, c_{k+1})$$

  and

  $$(\mathcal{S}_{k-1}(u)^{-1} \otimes 1)(x) = (b_1, b_2, \dots, b_{t-1}, d_1, \dots, d_{k+1})$$

  for some $(c_1, \dots, c_{k+1}), (d_1, \dots, d_{k+1}) \in [n]^{k+1}$. Equivalently if

  $$\begin{aligned}(\mathcal{S}_{k-1}(u)^{-1} \otimes 1)\mathcal{S}_k(u)&(a_1, a_2, \dots, a_{t-1}, c_1, \dots, c_{k+1})\\ &= (b_1, b_2, \dots, b_{t-1}, d_1, \dots, d_{k+1})\end{aligned}$$

  for some $(c_1, \dots, c_{k+1}), (d_1, \dots, d_{k+1}) \in [n]^{k+1}$.

**Remark 3.7.** All graphs have the same vertex set. Note that if a permutation $u$ is stable, then $\Gamma_a(u)$ and $\Gamma_b^\#(u)$ consist only of loops if $a \geq N(u)$ and $b \geq N^\#(u)$. Actually, $N(u)$ is the least integer such that $\Gamma_k(u)$ consists only of loops for all $k \geq N(u)$, and similarly for $N^\#(u)$. We may sometimes say, for brevity, that a graph is "empty" to mean that it has no edges between different vertices, and denote this by "$\emptyset$".

**Proposition 3.8.** *Given $u \in S([n]^t)$ and $k \geq 0$. If the graph $\Gamma_k(u)$ (resp., $\Gamma_k^\#(u)$) has a path from $(a_1, a_2, \ldots, a_{t-1})$ to $(b_1, b_2, \ldots, b_{t-1})$, then $\Gamma_k(u)$ (resp., $\Gamma_k^\#(u)$) has also a path from $(b_1, b_2, \ldots, b_{t-1})$ to $(a_1, a_2, \ldots, a_{t-1})$. I.e., strongly connected components and connected components are the same notions for $\Gamma_k(u)$ (resp., $\Gamma_k^\#(u)$).*

*Proof.* It is enough to check that if there is an edge $(a_1, a_2, \ldots, a_{t-1}) \to (b_1, b_2, \ldots, b_{t-1})$ ($(a_1, a_2, \ldots, a_{t-1})$,$(b_1, b_2, \ldots, b_{t-1})$ distinct) in $\Gamma_k$ then there is a path from $(b_1, b_2, \ldots, b_{t-1})$ to $(a_1, a_2, \ldots, a_{t-1})$. Indeed, if there are $(c_1, \ldots, c_{k+1}), (d_1, \ldots, d_{k+1}) \in [n]^{k+1}$ such that $\psi_k(u)(c_1, \ldots, c_{k+1}, a_1, a_2, \ldots, a_{t-1}) = (d_1, \ldots, d_{k+1}, b_1, b_2, \ldots, b_{t-1})$ then, since $\psi_k(u) \neq \mathrm{id}$ is a permutation of a finite set, there is some $h \geq 2$ such that $\psi_k(u)^h = \mathrm{id}$. But then

$$\begin{aligned}(c_1, \ldots, c_{k+1}, a_1, a_2, \ldots, a_{t-1}) &= \psi_k(u)^{h-1}\psi_k(u)(c_1, \ldots, c_{k+1}, a_1, a_2, \ldots, a_{t-1}) \\ &= \psi_k(u)^{h-1}(d_1, \ldots, d_{k+1}, b_1, b_2, \ldots, b_{t-1})\end{aligned}$$

provides such a path. The argument for $\Gamma_k^\#$ is similar, after replacing $\psi_k(u)$ with $(\mathcal{S}_{k-1}(u)^{-1} \otimes 1)\mathcal{S}_k(u)$. $\square$

We define two "actions" of a permutation on oriented graphs that are useful for our purposes (Strictly speaking, these are not actions!).

**Definition 3.9.** Given a directed graph $G$ on vertex set $[n]^{t-1}$ and $u \in S([n]^t)$, by $\mathcal{R}_u(G)$ and $\mathcal{L}_u(G)$ we mean the directed graphs with the same set of vertices $V(\mathcal{R}_u(G)) = V(\mathcal{L}_u(G)) = [n]^{t-1}$ and edges given by the following rules:

- there is a directed edge $(x_1, \ldots, x_{t-1}) \to (x'_1, \ldots, x'_{t-1})$ in $\mathcal{R}_u(G)$ if there are $z \in [n]$ and a directed edge $(y_1, \ldots, y_{t-1}) \to (y'_1, \ldots, y'_{t-1})$ in $G$ such that $(w, x_1, \ldots, x_{t-1}) = u(y_1, \ldots, y_{t-1}, z)$ and $(w', x'_1, \ldots, x'_{t-1}) = u(y'_1, \ldots, y'_{t-1}, z)$, for some $w, w' \in [n]$;

- there is a directed edge $(x_1, \ldots, x_{t-1}) \to (x'_1, \ldots, x'_{t-1})$ in $\mathcal{L}_u(G)$ if there are $z \in [n]$ and a directed edge $(y_1, \ldots, y_{t-1}) \to (y'_1, \ldots, y'_{t-1})$ in $G$ such that $(x_1, \ldots, x_{t-1}, w) = u(z, y_1, \ldots, y_{t-1})$ and $(x'_1, \ldots, x'_{t-1}, w') = u(z, y'_1, \ldots, y'_{t-1})$ for some $w, w' \in [n]$.

**Theorem 3.10.** *Given $u \in S([n]^t)$, then*

$$\Gamma_{k+1}(u) = \mathcal{R}_{u^{-1}}\Gamma_k(u) \quad \textit{and} \quad \Gamma^{\#}_{k+1}(u) = \mathcal{L}_u\Gamma^{\#}_k(u) \quad \textit{for any } k \geq 0.$$

*Proof.* Let $(a_1, a_2, \ldots, a_{t-1}), (b_1, b_2, \ldots, b_{t-1}) \in [n]^{t-1}$. Then $(a_1, a_2, \ldots, a_{t-1}) \to (b_1, b_2, \ldots, b_{t-1})$ in $\Gamma_{k+1}(u)$ if and only if there are $c_1, \ldots, c_{k+1}, d_1, \ldots, d_{k+1}, w, v \in [n]$ such that

$$\Big(\underbrace{1 \otimes \cdots \otimes 1}_{k+1} \otimes u^{-1}\Big)\Big(\psi_k(u) \otimes 1\Big)\Big(\underbrace{1 \otimes \cdots \otimes 1}_{k+1} \otimes u\Big)(c_1, \ldots, c_{k+1}, w, a_1, \ldots, a_{t-1})$$
$$= (d_1, \ldots, d_{k+1}, v, b_1, \ldots, b_{t-1}) \ ,$$

where we have used (5). This happens if and only if there are $c_1, \ldots, c_{k+1}, d_1, \ldots, d_{k+1}, a'_1, \ldots, a'_{t-1}, b'_1, \ldots, b'_{t-1}, w, v, z \in [n]$ such that

$$u(w, a_1, \ldots, a_{t-1}) = (a'_1, \ldots, a'_{t-1}, z)$$

$$\psi_k(u)(c_1, \ldots, c_{k+1}, a'_1, \ldots, a'_{t-1}) = (d_1, \ldots, d_{k+1}, b'_1, \ldots, b'_{t-1})$$

and

$$u^{-1}(b'_1, \ldots, b'_{t-1}, z) = (v, b_1, \ldots, b_{t-1}).$$

But this means that there are $a'_1, \ldots, a'_{t-1}, b'_1, \ldots, b'_{t-1}, w, v, z \in [n]$ such that there is a directed edge $(a'_1, \ldots, a'_{t-1}) \to (b'_1, \ldots, b'_{t-1})$ in $\Gamma_k(u)$,

$$u^{-1}(a'_1, \ldots, a'_{t-1}, z) = (w, a_1, \ldots, a_{t-1})$$

and

$$u^{-1}(b'_1, \ldots, b'_{t-1}, z) = (v, b_1, \ldots, b_{t-1})$$

and this exactly means that there is a directed edge

$$(a_1, a_2, \ldots, a_{t-1}) \to (b_1, b_2, \ldots, b_{t-1})$$

in $\mathcal{R}_{u^{-1}}(\Gamma_k(u))$.

The other statement can be proved by a similar argument, using the fact that

$$(\mathcal{S}_k(u)^{-1} \otimes 1)\mathcal{S}_{k+1}(u) = (u \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k+1})(1 \otimes \mathcal{S}_{k-1}(u)^{-1} \otimes 1)(1 \otimes \mathcal{S}_k(u))(u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k+1}).$$

$\square$

For a directed graph $G = ([n]^t, A)$ we denote by $G^r = ([n]^t, A^r)$ the *edge reverse* graph where $A^r := \{(b, a) : (a, b) \in A\}$ and by $G^\tau = ([n]^t, A^\tau)$ the *vertex reverse* graph where $(a_1, \ldots, a_t) \to (b_1, \ldots b_t)$ in $G^\tau$ if and only if $(a_t, \ldots, a_1) \to (b_t, \ldots b_1)$ in $G$. Note that these two operations commute.

**Proposition 3.11.** *Let $G$ be a directed graph on vertex set $[n]^{t-1}$ and $u \in S([n]^t)$. Then*

$$\mathcal{L}_u(G^\tau) = (\mathcal{R}_{\,{}^t u}(G))^\tau$$

*and*

$$\mathcal{L}_u(G^r) = (\mathcal{L}_u(G))^r.$$

*Proof.* Let $(a_1, \ldots, a_{t-1}), (b_1, \ldots, b_{t-1}) \in [n]^{t-1}$. Then $(a_1, \ldots, a_{t-1}) \to (b_1, \ldots, b_{t-1})$ in $\mathcal{L}_u(G^\tau)$ if and only if there are $w, v, z \in [n]$ and $(a'_1, \ldots, a'_{t-1}), (b'_1, \ldots, b'_{t-1}) \in [n]^{t-1}$ such that $(a'_1, \ldots, a'_{t-1}) \to (b'_1, \ldots, b'_{t-1})$ in $G^\tau$,

$$u(z, a'_1, \ldots, a'_{t-1}) = (a_1, \ldots, a_{t-1}, w), \tag{8}$$

and

$$u(z, b'_1, \ldots, b'_{t-1}) = (b_1, \ldots, b_{t-1}, v). \tag{9}$$

Similarly, $(a_{t-1}, \ldots, a_1) \to (b_{t-1}, \ldots, b_1)$ in $\mathcal{R}_{{}^t u}(G)$ if and only if there are $w, v, z \in [n]$ and $(a'_{t-1}, \ldots, a'_1), (b'_{t-1}, \ldots, b'_1) \in [n]^{t-1}$ such that $(a'_{t-1}, \ldots, a'_1) \to (b'_{t-1}, \ldots, b'_1)$ in $G$,

$${}^t u(a'_{t-1}, \ldots, a'_1, z) = (w, a_{t-1}, \ldots, a_1),$$

and

$${}^t u(b'_{t-1}, \ldots, b'_1, z) = (v, b_{t-1}, \ldots, b_1).$$

The first equality follows.

The second equality follows immediately from the definitions. Indeed, there is an edge $(a_1, \ldots, a_{t-1}) \to (b_1, \ldots, b_{t-1})$ in $\mathcal{L}_u(G^r)$ if and only if there is an edge $(b_1, \ldots, b_{t-1}) \to (a_1, \ldots, a_{t-1})$ in $\mathcal{L}_u(G)$. □

**Corollary 3.12.** *Let $u \in S([n]^t)$. Then*

$$\Gamma_k^\#(u)^r = \Gamma_k(u^\#)^\tau$$

*for all $k \geq 0$.*

*Proof.* We proceed by induction on $k \geq 0$. Let $(a_1, \ldots, a_{t-1}), (b_1, \ldots b_{t-1}) \in [n]^{t-1}$. From our definitions we have that there is a directed edge $(b_{t-1}, \ldots, b_1) \to (a_{t-1}, \ldots, a_1)$ in $\Gamma_0^\#(u)$ if and only if there are $c, d \in [n]$ such that $u^{-1}(b_{t-1}, \ldots, b_1, d) =$

$(a_{t-1}, \ldots, a_1, c)$, and there is a directed edge $(a_1, \ldots, a_{t-1}) \to (b_1, \ldots, b_{t-1})$ in $\Gamma_0(u^\#)$ if and only if there are $c, d \in [n]$ such that ${}^t u(c, a_1, \ldots, a_{t-1}) = (d, b_1, \ldots, b_{t-1})$. Therefore $\Gamma_0^\#(u)^r = \Gamma_0(u^\#)^\tau$.

Suppose now $k \geq 1$. Then we have that, by Theorem 3.10 and Proposition 3.11

$$\Gamma_k^\#(u)^r = \mathcal{L}_u(\Gamma_{k-1}^\#(u))^r = \mathcal{L}_u(\Gamma_{k-1}^\#(u)^r)$$

while

$$\Gamma_k(u^\#)^\tau = \mathcal{R}_{\,{}^t u}(\Gamma_{k-1}(u^\#))^\tau = \mathcal{L}_u(\Gamma_{k-1}(u^\#)^\tau)$$

so the result follows by induction. □

We can now prove what is probably the main result of this work.

**Theorem 3.13.** *Given a permutation $u \in S([n]^t)$, then $u$ is stable if and only if there is $M \in \mathbb{N}$ such that $\mathcal{R}_{u^{-1}}^M(\Gamma_0)$ and $\mathcal{L}_u^M(\Gamma_0^\#)$ consist only of loops.*

*Proof.* Suppose first that $u$ is stable. Then by Theorem 3.5 we have that both $N(u)$ and $N^\#(u)$ are finite. Therefore, by our definitions (see also Remark 3.7) both $\Gamma_M(u)$ and $\Gamma_M^\#(u)$ consist only of loops if $M \geq \max\{N(u), N^\#(u)\}$, so, by Theorem 3.10, $(\mathcal{R}_{u^{-1}})^M(\Gamma_0)$ and $(\mathcal{L}_u)^M(\Gamma_0^\#)$ both consist only of loops, as claimed.

Conversely, assume that there is $M \in \mathbb{N}$ such that $\mathcal{R}_{u^{-1}}^M(\Gamma_0)$ and $\mathcal{L}_u^M(\Gamma_0^\#)$ consist only of loops. It is then easy to check from our definitions that $\mathcal{R}_{u^{-1}}^k(\Gamma_0)$ and $\mathcal{L}_u^k(\Gamma_0^\#)$ consist only of loops for all $k \geq M$. This, by Theorem 3.10, implies that $\Gamma_k(u)$ and $\Gamma_k^\#(u)$ consist only of loops for all $k \geq M$. By the definitions of the graphs $\Gamma_k(u)$ and $\Gamma_k^\#(u)$ this means that both $N(u)$ and $N^\#(u)$ are finite so, by Theorem 3.5, $u$ is stable. □

This is a very good criterion for checking stability. We will discuss bounds for $M$ in Section 5.

**Example 3.14.** Let $u := ((1,3),(1,2),(3,4)) \in S([4]^2)$ (note that, by Theorem 5.12 in [4], $u$ is stable of rank 2). Then the vertex set of all the graphs $\Gamma_k, \Gamma_k^\#, k \geq 0$ is $[4]$.

The graph $\Gamma_0$ has directed edges $E_0 = \{(1,1),(2,2),(3,3),(4,4),(4,2),(2,3),(3,4)\}$. Consider the edge $(2,3)$ in $\Gamma_0$. Then the graph $\Gamma_1$ has edges $(\pi_2(u^{-1}(2,z)), \pi_2(u^{-1}(3,z)))$ for all $z \in [4]$ (where $\pi_2$ is the projection on the second coordinate). Given that $u^{-1}(2,1) = (2,1)$, $u^{-1}(3,1) = (3,1)$, $u^{-1}(2,2) = (2,2)$, $u^{-1}(3,2) = (3,2)$, $u^{-1}(2,3) = (2,3)$, $u^{-1}(3,3) = (3,3)$, $u^{-1}(2,4) = (2,4)$, and $u^{-1}(3,4) = (1,2)$, we obtain that $\Gamma_1$ has edges $(1,1),(2,2),(3,3)$ and $(4,2)$. Similarly, for the edge $(4,2)$ we obtain edges, $(1,1),(2,2),(3,3)$ and $(4,4)$, and

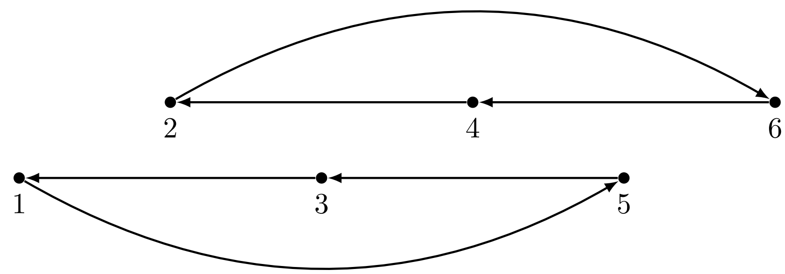

Figure 1: The directed graph $\Gamma_0^{\#}(u)$

for the edge $(3,4)$ we obtain $(1,1),(2,2),(3,3)$ and $(2,4)$. Note that any loop $(z,z)$ only gives rise to other loops, so we don't need to compute them explicitly. Thus $\Gamma_1$ has edge set $E_1 = \{(1,1),(2,2),(3,3),(4,4),\ (2,4),(4,2)\}$. Repeating this process on the non-loop edges of $\Gamma_1$ (note that we have already performed most of the computations) we obtain that $\Gamma_2$ consists only of loops, and hence that $\Gamma_k = \Gamma_2 = \emptyset$ for all $k \geq 2$. In particular, $N(u) = 2$.

The graph $\Gamma_0^{\#}$ has directed edges $\{(1,1),(2,2),(3,3),(4,4),(1,3),(3,1)\}$. Considering the edge $(1,3)$ we have that $\Gamma_1^{\#}$ has directed edges $(\pi_1(u(z,1)),\pi_1(u(z,3)))$ for all $z \in [4]$. Since $u(1,1) = (1,1)$, $u(1,3) = (1,2)$, $u(2,1) = (2,1)$, $u(2,3) = (2,3)$, $u(3,1) = (3,1)$, $u(3,3) = (3,3)$, $u(4,1) = (4,1)$, and $u(4,3) = (4,3)$ we obtain that $\Gamma_1^{\#}$ has edges $(1,1),(2,2),(3,3)$ and $(4,4)$. Performing the same computation for the edge $(3,1)$ we only obtain loops. Hence $\Gamma_1^{\#}$ consists only of loops and therefore $\Gamma_k^{\#} = \Gamma_1^{\#} = \emptyset$ for all $k \geq 1$. In particular $N^{\#}(u) = 1$.

The next example shows that in Theorem 3.13 both sequences of graphs need to be considered.

**Example 3.15.** Let us consider the product of three 3-cycles

$$u := \big((1,1),(3,1),(5,1)\big)\big((2,3),(4,3),(6,3)\big)\big((1,5),(3,5),(5,5)\big) \in S([6]^2)\ .$$

Then one can check that $\Gamma_0 = \emptyset$ (so that $\Gamma_k = \emptyset$, for all $k \geq 0$), while $\Gamma_0^{\#}$ is shown in Figure 1. Also, $\Gamma_k^{\#}$ is the union of $\Gamma_0^{\#}$ and its reverse edge graph, for all $k \geq 1$. In particular, $u$ is not stable.

The next example shows that the graphs $\Gamma_k(u)$ do not necessarily stabilize as $k \to +\infty$.

**Example 3.16.** Let $u := \big((1,2),(1,1),(3,5)\big)\big((5,2),(4,1)\big) \in S([5]^2)$. Then one can check that $\Gamma_0(u)$ has directed edges $\{(1,2),(2,1),(2,5),(5,1)\}$, while $\Gamma_{2k}(u)$ has directed edges $\{(1,2),(2,1),(2,5),(5,1),(5,2)\}$ and $\Gamma_{2k-1}(u)$ has directed edges $\{(1,2),(2,1),(2,5),(1,5),(5,2)\}$ for all $k \geq 1$.

We want to single out one important property, which we use in Sections 4 and 5. Recall that the transitive closure $T(G)$ of a directed graph $G$ is the graph which contains an edge $a \to b$ if and only if $G$ has a directed path from $a$ to $b$.

**Lemma 3.17.** *Let $u \in S([n]^t)$, and two directed graphs $G_1, G_2$ on vertex set $[n]^{t-1}$ be such that $T(G_1) = T(G_2)$. Then $T(\mathcal{R}_{u^{-1}}G_1) = T(\mathcal{R}_{u^{-1}}G_2)$ and $T(\mathcal{L}_u G_1) = T(\mathcal{L}_u G_2)$.*

*Proof.* It is clearly enough to prove the statement in the case where $T(G_1) = G_2$. So let $G$ be a directed graph on vertex set $[n]^{t-1}$. We will show that if $(a_1,\ldots,a_{t-1}) \to (b_1,\ldots,b_{t-1}) \to (c_1,\ldots,c_{t-1})$ is a directed path in $G$ then $T(\mathcal{R}_{u^{-1}}G) = T(\mathcal{R}_{u^{-1}}(G \cup \{((a_1,\ldots,a_{t-1}) \to (c_1,\ldots,c_{t-1}))\}))$ and $T(\mathcal{L}_u G_1) = T((G \cup \{((a_1,\ldots,a_{t-1}) \to (c_1,\ldots,c_{t-1}))\}))$ and the result will follow. We will check only $\mathcal{R}_{u^{-1}}$, the second statement is similar.

Note that

$$\begin{aligned}&\mathcal{R}_{u^{-1}}(G \cup \{((a_1,\ldots,a_{t-1}) \to (c_1,\ldots,c_{t-1}))\})\\ &\qquad = \mathcal{R}_{u^{-1}}G \cup \mathcal{R}_{u^{-1}}\{((a_1,\ldots,a_{t-1}) \to (c_1,\ldots,c_{t-1}))\}.\end{aligned}$$

The second set is the set of all edges $(a'_1,\ldots,a'_{t-1}) \to (c'_1,\ldots,c'_{t-1})$ such that there are $z, w, v \in [n]$ such that $u^{-1}(a_1,\ldots,a_{t-1},z) = (w,a'_1,\ldots,a'_{t-1})$ and $u^{-1}(c_1,\ldots,c_{t-1},z) = (v,c'_1,\ldots,c'_{t-1})$. Hence,

$$(a'_1,\ldots,a'_{t-1}) \to (b'_1,\ldots,b'_{t-1}) \to (c'_1,\ldots,c'_{t-1})$$

is a directed path in $\mathcal{R}_{u^{-1}}G$ where $(b'_0,b'_1,\ldots,b'_{t-1}) := u^{-1}(b_1,\ldots,b_{t-1},z)$. Therefore, $T(\mathcal{R}_{u^{-1}}G) \supseteq T(\mathcal{R}_{u^{-1}}(G \cup \{((a_1,\ldots,a_{t-1}) \to (c_1,\ldots,c_{t-1}))\})$. The opposite inclusion is clear. $\square$

**Lemma 3.18.** *Let $u \in S([n]^t)$ and $G$ be a directed graph on vertex set $[n]^{t-1}$. If $T(G) = T(\mathcal{R}_{u^{-1}}G)$ (resp., $T(G) = T(\mathcal{L}_u G)$) then $T(\mathcal{R}^m_{u^{-1}}G) = T(G)$ (resp., $T(\mathcal{L}^m_u G) = T(G)$) for any $m \in \mathbb{N}$.*

*Proof.* This follows by repeated application of Lemma 3.17. $\square$

**Corollary 3.19.** *Given $u \in S([n]^t)$. If $T(\Gamma_k(u)) = T(\Gamma_{k+1}(u)) \neq \emptyset$ or $T(\Gamma^{\#}_k(u)) = T(\Gamma^{\#}_{k+1}(u)) \neq \emptyset$ for some $k \geq 0$, then $u$ is not stable.*

*Proof.* If $T(\Gamma_k(u)) = T(\Gamma_{k+1}(u))$ then, by Theorem 3.10 and Lemma 3.18 $T(\Gamma_k(u)) = T(\Gamma_m(u))$ for all $m \geq k$ so $T(\Gamma_m(u)) \neq \emptyset$ for all $m \geq k$ so $\Gamma_m(u) \neq \emptyset$ for all $m \geq k$ and hence, by Theorem 3.13, $u$ is not stable. The other statement is exactly analogous. $\square$

## 4 Almost all permutations are not stable

In this section, using the results in the previous one, we prove Conjecture 12.5 in [3]. Namely that, if we choose a permutation in $S([n]^t)$, $t \geq 2$, uniformly at random, the probability that this is stable goes to zero as either $n$ or $t$ go to infinity.

For $n, t \in \mathbb{N}$ let $X_0(n,t)$ (resp., $X_1(n,t)$) be the number of permutations $u \in S([n]^t)$ such that $\Gamma_0(u)$ is disconnected (resp., $\Gamma_0(u)$ is connected but $\Gamma_1(u)$ is disconnected). Recall that by Proposition 3.8, for any $k \in \mathbb{N}$, $\Gamma_k(u)$ is connected if and only if it is strongly connected.

**Proposition 4.1.** *Let $n, t \in \mathbb{N}$, $n, t > 1$, then*

$$\frac{X_0(n,t)}{(n^t)!} \leq \sum_{k=1}^{\lfloor \frac{n^{t-1}}{2} \rfloor} \frac{1}{\binom{n^t - n^{t-1}}{(n-1)k}}.$$

*Proof.* Let $u \in S([n]^t)$ be such that $\Gamma_0(u)$ is disconnected. Then there are $A, B \subseteq [n]^{t-1}$ such that $A \cap B = \emptyset$, $A \cup B = [n]^{t-1}$, $A, B \neq \emptyset$ and there are no directed edges between $A$ and $B$ in $\Gamma_0(u)$. Therefore, by definition of $\Gamma_0(u)$, $u([n] \times A) = [n] \times A$, and similarly for $[n] \times B$. Since there are $(n \cdot |A|)!$ permutations of $[n] \times A$ and $(n \cdot |B|)!$ permutations of $[n] \times B$, we have that

$$X_0(n,t) \leq \sum_{k=1}^{\lfloor \frac{n^{t-1}}{2} \rfloor} \binom{n^{t-1}}{k} (nk)!(n(n^{t-1} - k))!\,.$$

Hence,

$$\begin{aligned} \frac{X_0(n,t)}{(n^t)!} &\leq \sum_{k=1}^{\lfloor \frac{n^{t-1}}{2} \rfloor} \binom{n^{t-1}}{k} \frac{(nk)!(n(n^{t-1} - k))!}{(n^t)!} = \sum_{k=1}^{\lfloor \frac{n^{t-1}}{2} \rfloor} \binom{n^{t-1}}{k} \frac{1}{\binom{n^t}{nk}} \\ &\leq \sum_{k=1}^{\lfloor \frac{n^{t-1}}{2} \rfloor} \binom{n^{t-1}}{k} \frac{1}{\binom{n^t - n^{t-1}}{(n-1)k} \binom{n^{t-1}}{k}} = \sum_{k=1}^{\lfloor \frac{n^{t-1}}{2} \rfloor} \frac{1}{\binom{n^t - n^{t-1}}{(n-1)k}}. \end{aligned}$$

$\square$

**Proposition 4.2.** *Let $n, t \in \mathbb{N}$, $n, t > 1$, then*

$$\frac{X_1(n,t)}{(n^t)!} \leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{\binom{n^t - n^{t-1} - n}{(n^{t-1} - n^{t-2} - 1)k}} .$$

*Proof.* Let $u \in S([n]^t)$ be such that $\Gamma_0(u)$ is connected but $\Gamma_1(u)$ is disconnected. Since $\Gamma_1(u)$ is not connected there are $A, B \subseteq [n]^{t-1}$ such that $A \cap B = \emptyset$, $A \cup B = [n]^{t-1}$, $A, B \neq \emptyset$ and there are no directed edges between $A$ and $B$ in $\Gamma_1(u)$. For $z \in [n]$ we let

$$A(z) := \{x \in [n]^{t-1} : u^{-1}(x,z) \in [n] \times A\}$$

and

$$B(z) := \{x \in [n]^{t-1} : u^{-1}(x,z) \in [n] \times B\}.$$

Then either $A(z) = \emptyset$ or $B(z) = \emptyset$. Indeed, if $A(z), B(z)$ are not empty, then, as $\Gamma_0(u)$ is connected, there is an edge from $x \in A(z)$ to $y \in B(z)$ in $\Gamma_0(u)$ so, by Theorem 3.10, there would be a directed edge from $A$ to $B$ in $\Gamma_1(u)$.

Let $A' := \{z \in [n] : B(z) = \emptyset\}$. If $z \in A'$ then $A(z) = [n]^{t-1}$ so $u^{-1}([n]^{t-1} \times \{z\}) \subseteq [n] \times A$. Therefore $u^{-1}([n]^{t-1} \times A') \subseteq [n] \times A$. Similarly $u^{-1}([n]^{t-1} \times B') \subseteq [n] \times B$ where $B' := \{z \in [n] : A(z) = \emptyset\} = [n] \setminus A'$. So $u^{-1}([n]^{t-1} \times A') = [n] \times A$ and similarly for $B'$ and $B$. Hence $n^{t-2}|A'| = |A|$, so the following bound follows

$$X_1(n,t) \leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n^{t-1}}{n^{t-2}k} (n^{t-1}k)!\, (n^{t-1}(n-k))!$$

(if $|A'| \leq |B'|$, the first binomial coefficient is an upper bound for the number of possibilities for $A'$, the second binomial coefficient is an upper bound for the number of possibilities for $A$, and factorials are upper bounds for the number of bijections $[n]^{t-1} \times A' \to [n] \times A$ and $[n]^{t-1} \times B' \to [n] \times B$, resp.). Hence,

$$\begin{aligned}
\frac{X_1(n,t)}{(n^t)!} &\leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n^{t-1}}{n^{t-2}k} \frac{(n^{t-1}k)!\, (n^{t-1}(n-k))!}{(n^t)!} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n^{t-1}}{n^{t-2}k} \frac{1}{\binom{n^t}{n^{t-1}k}} \\
&\leq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n^{t-1}}{n^{t-2}k} \frac{1}{\binom{n^t - n^{t-1} - n}{n^{t-1}k - n^{t-2}k - k} \binom{n}{k} \binom{n^{t-1}}{n^{t-2}k}} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{\binom{n^t - n^{t-1} - n}{(n^{t-1} - n^{t-2} - 1)k}}.
\end{aligned}$$

□

We need the following lemma.

**Lemma 4.3** (Folklore)**.** *We have*

$$\lim_{m\to\infty}\left(\frac{1}{\binom{m}{1}}+\frac{1}{\binom{m}{2}}+\frac{1}{\binom{m}{3}}+\ldots+\frac{1}{\binom{m}{m-1}}\right)=0.$$

*Proof.* This immediately follows from the below inequality

$$\frac{1}{\binom{m}{1}}+\frac{1}{\binom{m}{2}}+\frac{1}{\binom{m}{3}}+\ldots+\frac{1}{\binom{m}{m-1}}\leq\frac{2}{\binom{m}{1}}+\frac{m-3}{\binom{m}{2}}\leq\frac{4}{m}.$$

□

We get two simple corollaries from Propositions 4.1,4.2, and Lemma 4.3.

**Corollary 4.4.** *Given $1 < t \in \mathbb{N}$, then*

$$\lim_{n\to\infty}\frac{X_0(n,t)}{(n^t)!}=\lim_{n\to\infty}\frac{X_1(n,t)}{(n^t)!}=0.$$

**Corollary 4.5.** *Given $1 < n \in \mathbb{N}$, then*

$$\lim_{t\to\infty}\frac{X_0(n,t)}{(n^t)!}=\lim_{t\to\infty}\frac{X_1(n,t)}{(n^t)!}=0.$$

We are now ready to prove Conjecture 12.5 of [3], the proofs of the two statements are exactly the same.

**Theorem 4.6.** *Given $1 < t \in \mathbb{N}$. Then almost all permutations from $S([n]^t)$, $n \in \mathbb{N}$ are not stable, i.e.,*

$$\lim_{n\to\infty}\frac{|\{u\in S([n]^t):\ u \textit{ is stable}\}|}{(n^t)!}=0.$$

**Theorem 4.7.** *Given $1 < n \in \mathbb{N}$. Then almost all permutations from $S([n]^t)$, $t \in \mathbb{N}$ are not stable, i.e.,*

$$\lim_{t\to\infty}\frac{|\{u\in S([n]^t):\ u \textit{ is stable}\}|}{(n^t)!}=0.$$

*Proof of Theorems 4.6 and 4.7.* By Corollary 3.19 we get that $|\{u \in S([n]^t) : u \text{ is stable}\}| \leq X_0(n,t) + X_1(n,t)$. Hence

$$\lim_{n\to\infty}\frac{|\{u\in S([n]^t):\ u\text{ is stable}\}|}{(n^t)!}\leq\lim_{n\to\infty}\frac{X_0(n,t)}{(n^t)!}+\lim_{n\to\infty}\frac{X_1(n,t)}{(n^t)!}=0$$

and

$$\lim_{t\to\infty}\frac{|\{u\in S([n]^t):\ u\text{ is stable}\}|}{(n^t)!}\leq\lim_{t\to\infty}\frac{X_0(n,t)}{(n^t)!}+\lim_{t\to\infty}\frac{X_1(n,t)}{(n^t)!}=0,$$

where the last equalities follow from Corollaries 4.4 and 4.5. □

The previous results show that the number of stable permutations of $[n]^t$ is a little $o$ of the total number of permutations for either $n$ or $t$ going to infinity. We now show, however, that the number of stable permutations is not so small in absolute terms.

Recall first that if $u \in S([n]^{t-1})$ then $(u \otimes 1)(1 \otimes u^{-1}) \in S([n]^t)$ is a stable permutation (these permutations correspond to the inner automorphisms of $\mathcal{O}_n$). Hence the number of stable permutations in $S([n]^t)$ is at least $n^{t-1}!$. We now show that this bound can be improved.

Let $n, t \in \mathbb{N}$, $n \geq 3$, $t \geq 2$, and $v \in S([n-1]^{t-1})$. We define a permutation $\widetilde{v} \in S([n]^t)$ by letting

$$\widetilde{v}(a_1, \ldots, a_t) := \begin{cases} (n, v(a_2, \ldots, a_t)), & \text{if } (a_1, \ldots, a_t) \in \{n\} \times [n-1]^{t-1}, \\ (a_1, \ldots, a_t), & \text{otherwise.} \end{cases}$$

Note that $(\widetilde{v})^{-1} = \widetilde{(v^{-1})}$.

**Proposition 4.8.** *Let $n, t \in \mathbb{N}$, $n \geq 3$, $t \geq 2$, and $v \in S([n-1]^{t-1})$. Then*

$$\psi_k(\widetilde{v}) = \widetilde{v}^{-1} \otimes 1 \otimes \cdots \otimes 1$$

*for all $k \geq 0$. In particular, $\widetilde{v}$ is stable of rank 1.*

*Proof.* Let $(a_1, \ldots, a_{t+k}) \in [n]^{t+k}$ and let

$$\{i_1, \ldots, i_r\}_< := \{i \in [k+1] : (a_i, \ldots, a_{i+t-1}) \in \{n\} \times [n-1]^{t-1}\}.$$

Note that $i_j - i_{j-1} \geq t$ for all $j = 2, \ldots, r$. If $i_1 > 1$ then by (1) and what was just observed

$$\psi_k(\widetilde{v})(a_1, \ldots, a_{k+t}) = (a_1, \ldots, a_{t+k}) = (\widetilde{v}^{-1} \otimes 1 \otimes \cdots \otimes 1)(a_1, \ldots, a_{t+k}).$$

If $i_1 = 1$ then we obtain similarly that

$$\psi_k(\widetilde{v})(a_1, \ldots, a_{k+t}) = (\widetilde{v}^{-1} \otimes 1 \otimes \cdots \otimes 1)(a_1, \ldots, a_{t+k}).$$

□

The previous proposition shows that there are at least $(n-1)^{t-1}!$ stable permutations of rank 1. One might wonder whether the corresponding permutative automorphisms of $\mathcal{O}_n$ are outer. We now show that this is the case and that, in fact, all these automorphisms are inequivalent under the action of inner automorphisms.

**Proposition 4.9.** *Let $n, t \in \mathbb{N}$, $n \geq 3$, $t \geq 2$, and $v \neq w \in S([n-1]^{t-1})$. Then there is no $u \in S([n]^{t-1})$ such that*

$$\widetilde{v} = (u \otimes 1)\widetilde{w}(1 \otimes u^{-1}).$$

*In particular, for non-identical $v \in S([n-1]^{t-1})$, $\widetilde{v}$ is an outer permutative automorphism.*

*Proof.* Let's proof by contrapositive. Assume that there is such $u$. We proof by induction that $u$ does not change the last $k \in [0, t-1]$ coordinates. The base case $k = 0$ is trivial.

Assume that $u$ does not change the last $(k-1)$. By induction hypothesis the last $k$ coordinates of $(u \otimes 1)\widetilde{w}(1 \otimes u^{-1})(1, c_1, c_2, \ldots, c_{t-1})$ coincide with the last $k$ coordinates of

$$\widetilde{w}(1 \otimes u^{-1})(1, c_1, c_2, \ldots, c_{t-1}) = \widetilde{w}(1, u^{-1}(c_1, \ldots, c_{t-1})) = (1, u^{-1}(c_1, \ldots, c_{t-1})).$$

In another hand $(u \otimes 1)\widetilde{w}(1 \otimes u^{-1})(1, c_1, c_2, \ldots, c_{t-1}) = \widetilde{v}(1, c_1, c_2, \ldots, c_{t-1}) = (1, c_1, c_2, \ldots, c_{t-1})$. Hence, $u$ does not change the last $k$ coordinates.

Therefore $u$ is the identical permutation, which contradicts to $v \neq w$. $\square$

Proposition 4.9 shows that there are at least $(n-1)^{t-1}!$ distinct classes of inner-equivalent automorphisms of $\mathcal{O}_n$, and that there are at least $(n-1)^{t-1}!\, n^{t-1}!$ stable permutations in $S([n]^t)$.

## 5 Upper bound for rank

In this section, using the graphs introduced in Section 3, we obtain an explicit upper bound for the ranks of stable permutations of $S([n]^2)$ which is at worst linear in $n$.

We need to consider two more graphs

$$\Gamma(u) = \Gamma_0(u) \cup \Gamma_1(u) \cup \Gamma_2(u) \cup \Gamma_3(u) \cup \ldots;$$

$$\Gamma^{\#}(u) = \Gamma_0^{\#}(u) \cup \Gamma_1^{\#}(u) \cup \Gamma_2^{\#}(u) \cup \Gamma_3^{\#}(u) \cup \ldots$$

on the vertex set $[n]^{t-1}$.

**Lemma 5.1.** *Given $u \in S([n]^t)$, $t \in \mathbb{N}$. Then*

$$\Gamma(u) \supseteq \mathcal{R}_{u^{-1}}\Gamma(u) \supseteq \mathcal{R}^2_{u^{-1}}\Gamma(u) \supseteq \mathcal{R}^3_{u^{-1}}\Gamma(u) \supseteq \ldots$$

*and*

$$\Gamma^{\#}(u) \supseteq \mathcal{L}_u\Gamma^{\#}(u) \supseteq \mathcal{L}^2_u\Gamma^{\#}(u) \supseteq \mathcal{L}^3_u\Gamma^{\#}(u) \supseteq \ldots.$$

*Proof.* This follows immediately from Theorem 3.10 and the fact that for any graphs $G_1, G_2$ on the vertex set $[n]^{t-1}$, $\mathcal{R}_{u^{-1}}(G_1 \cup G_2) = \mathcal{R}_{u^{-1}}(G_1) \cup \mathcal{R}_{u^{-1}}(G_2)$. $\square$

Let $\mathrm{supp}(G)$ and $c(G)$ be the number of non-isolated vertices and the number of connected components, respectively, of the graph $G$. Note that, since the number of isolated vertices of $G$ is at most $c(G)$, we have that $|V| \leq \mathrm{supp}(G) + c(G)$, with strict inequality if $G$ is non-empty.

We have the following bound

**Theorem 5.2.** *Given $u \in S([n]^t)$, $t \in \mathbb{N}$, then*

- $N(u) \leq \max\{\mathrm{supp}(\Gamma(u)) - 1, 0\}$ *or* $N(u) = +\infty$*;*
- $N^{\#}(u) \leq \max\{\mathrm{supp}(\Gamma^{\#}(u)) - 1, 0\}$ *or* $N^{\#}(u) = +\infty$*.*

*Proof.* We prove the result only for $N(u)$, the second statement being similar (cf. Proposition 3.3 and Corollary 3.12).

Assume that $N(u)$ is finite, then (cf. Remark 3.7) for any $m \geq N(u)$, $\Gamma_m(u) = \emptyset$ so $\mathcal{R}^m_{u^{-1}}\Gamma(u) = \emptyset$. If $\mathrm{supp}(\Gamma(u)) = 0$ then $N(u) = 0$ and the result follows.

So assume that $\mathrm{supp}(\Gamma(u)) > 0$. Note that we have $\Gamma(u) \supseteq \mathcal{R}_{u^{-1}}\Gamma(u) \supseteq \mathcal{R}^2_{u^{-1}}\Gamma(u) \supseteq \mathcal{R}^3_{u^{-1}}\Gamma(u) \supseteq \ldots$ by Lemma 5.1. Hence there is $k \in \mathbb{N}_0$ such that $c(\mathcal{R}^{k+1}_{u^{-1}}\Gamma(u)) = c(\mathcal{R}^k_{u^{-1}}\Gamma(u))$, and furthermore $T(\mathcal{R}^{k+1}_{u^{-1}}\Gamma(u)) = T(\mathcal{R}^k_{u^{-1}}\Gamma(u))$. Hence, by Lemma 3.18, we get $T(\mathcal{R}^m_{u^{-1}}\Gamma(u)) = T(\mathcal{R}^k_{u^{-1}}\Gamma(u))$ for any $m \geq k$. Therefore we have $\mathcal{R}^k_{u^{-1}}\Gamma(u) = \emptyset$ (otherwise $T(\mathcal{R}^{N(u)+k}_{u^{-1}}\Gamma(u)) = T(\mathcal{R}^k_{u^{-1}}\Gamma(u)) \neq \emptyset$). So $k \geq N(u)$ (for if $\mathcal{R}^i_{u^{-1}}\Gamma_0(u) = \emptyset$ for some $i \in \mathbb{N}_0$ then, by definition of $\mathcal{R}_{u^{-1}}$, $\mathcal{R}^j_{u^{-1}}\Gamma_0(u) = \emptyset$ for all $j \geq i$ so $N(u) \leq i$). So we get

$$\begin{aligned} n^{t-1} - \mathrm{supp}(\Gamma(u)) + 1 \leq c(\Gamma(u)) < c(\mathcal{R}_{u^{-1}}\Gamma(u)) < \\ \ldots < c(\mathcal{R}^{N(u)-1}_{u^{-1}}\Gamma(u)) < c(\mathcal{R}^{N(u)}_{u^{-1}}\Gamma(u)) = n^{t-1}. \end{aligned}$$

Hence, $N(u) \leq n^{t-1} - (n^{t-1} - \mathrm{supp}(\Gamma(u)) + 1) = \mathrm{supp}(\Gamma(u)) - 1$. $\square$

Recall (see [3, Def. 11.4]) that for $u \in S([n]^2)$ we let $C(u) := \{j \in [n] : \exists i \in [n]$ such that $u(i,j) \neq (i,j)\}$ and define $R(u)$ similarly.

**Corollary 5.3.** *Given a non-identical permutation $u \in S([n]^2)$, then*

- $N(u) < |C(u)|$ *or* $N(u) = +\infty$*;*
- $N^{\#}(u) < |R(u)|$ *or* $N^{\#}(u) = +\infty$*.*

*In particular, if a non-identical permutation* $u \in S([n]^2)$ *is stable, then* $\mathrm{rk}(u) \leq |C(u)| + |R(u)| - 1$.

*Proof.* Note that $[n] \setminus C(u)$ are isolated vertices of $\Gamma(u)$. Hence, $\mathrm{supp}(\Gamma(u)) \leq |C(u)|$. Similarly, $\mathrm{supp}(\Gamma^{\#}(u)) \leq |R(u)|$. The conclusion follows at once from Theorem 3.5. □

The above result implies that, if $u \in S([n]^2)$ is stable, then $\mathrm{rk}(u) \leq 2n - 1$. This bound improves, for permutative automorphisms, the one of $n^2$ which can be obtained from a similar estimate in [9, Cor. 3.3] for general unitaries $u \in \mathcal{U}(\mathcal{F}_2)$.

**Remark 5.4.** $C(u)$ and non-isolated vertices of $\Gamma(u)$ are not always the same sets. For example, if $u = ((1,1),(2,1))((1,2),(2,2)) \in S([2]^2)$, then $C(u) = \{1,2\}$, but $\mathrm{supp}(\Gamma(u)) = 0$.

We have the following generalization of Lemma 5.1 for $t = 2$.

**Proposition 5.5.** *Let* $u \in S([n]^2)$*. Then*

$$T(\Gamma(u)) = T(\Gamma_0(u)) \supseteq T(\Gamma_1(u)) \supseteq T(\Gamma_2(u)) \supseteq \cdots$$

*and*

$$T(\Gamma^{\#}(u)) = T(\Gamma_0^{\#}(u)) \supseteq T(\Gamma_1^{\#}(u)) \supseteq T(\Gamma_2^{\#}(u)) \supseteq \cdots .$$

*Proof.* We first show that $T(\Gamma_1(u)) \subseteq T(\Gamma_0(u))$. Let $x \to y$ be a directed edge in $\Gamma_1(u)$. By Theorem 3.10 and our definitions this means that there is a directed edge $x' \to y'$ in $\Gamma_0(u)$ and $z \in [n]$ such that $u^{-1}(x', z) = (w, x)$ and $u^{-1}(y', z) = (v, y)$ for some $v, w \in [n]$. Therefore there are directed edges $z \to x$ and $z \to y$ in $\Gamma_0(u)$. Hence $z, x$ and $y$ belong to the same connected component of $\Gamma_0(u)$ and so, by Proposition 3.8, $x \to y$ is a directed edge in $T(\Gamma_0(u))$.

Suppose now that $T(\Gamma_{k-1}) \supseteq T(\Gamma_k)$ for some $k \geq 1$. Then $T(\mathcal{R}_{u^{-1}}(T(\Gamma_{k-1}))) \supseteq T(\mathcal{R}_{u^{-1}}(T(\Gamma_k)))$. But, by Lemma 3.17 (applied to $\Gamma_k$ and $T(\Gamma_k)$) we have $T(\mathcal{R}_{u^{-1}}(T(\Gamma_k))) = T(\mathcal{R}_{u^{-1}}(\Gamma_k)) = T(\Gamma_{k+1})$ and similarly for $T(\mathcal{R}_{u^{-1}}(T(\Gamma_{k-1})))$, so $T(\Gamma_k) \supseteq T(\Gamma_{k+1})$.

Since $T(\Gamma_0(u)) \supseteq T(\Gamma_1(u)) \supseteq T(\Gamma_2(u)) \supseteq \cdots$ and $\Gamma(u) = \Gamma_0(u) \cup \Gamma_1(u) \cup \Gamma_2(u) \cup \Gamma_3(u) \cup \ldots$, we have $T(\Gamma(u)) = T(\Gamma_0(u))$.

The second chain of set inclusions follows from the first one and Corollary 3.12. □

The previous result implies that, if $u \in S([n]^2)$, then $u$ is stable if and only if $\Gamma_n(u) = \Gamma_n^{\#}(u) = \emptyset$. Indeed, if there is an $h \in [0, n-1]$ such that $T(\Gamma_h(u)) = T(\Gamma_{h+1}(u)) \neq \emptyset$ then, by Lemma 3.17 and Theorem 3.10, $\Gamma_k(u) =$

$\Gamma_h(u)$ for all $k \geq h$ so $u$ is not stable. Similarly for $\Gamma_k^{\#}(u)$. Conversely, if $T(\Gamma_0(u)) \supset T(\Gamma_1(u)) \supset T(\Gamma_2(u)) \supset \cdots \supset T(\Gamma_n(u))$ then this means that $\Gamma_{k+1}(u)$ has at least one more connected component compared to $\Gamma_k(u)$ for all $k \in [0, n-1]$, so $\Gamma_n(u) = \emptyset$. Similarly for $\Gamma_n^{\#}(u)$.

## 6 What makes a cycle stable?

In this section we present evidence in favor of the conjecture that stability of an $r$-cycle $u$ in $S([n]^2)$ depends only on a certain subset $S(u)$ of $[r]^2$ (see equation (16)). More precisely, for $u \in S([n]^t)$ we introduce invariants $R(u)$, $C(u)$, $R^o(u)$, $C^o(u)$ as suitable subsets of $[n]$, and then use them to define some equivalence relations on permutations (see Definition 6.1). We then show that if two cycles $u, v \in S([n]^2)$ are such that $S(u) = S(v)$ then they are (left and right) equivalent. We also give evidence for the stronger conjecture that this equivalence relation is stability invariant.

For a permutation $u \in S([n]^t)$, we define $\mathrm{supp}(u) \subseteq [n]^t$ as the set of elements $x$ such that $u(x) \neq x$. Let $R(u) \subseteq [n]$ (resp. $C(u) \subseteq [n]$) be the elements that appear somewhere in the first (resp., last) $t-1$ places in $\mathrm{supp}(u)$, i.e.,

$$R(u) := \{a \in [n] \mid \exists x \in \mathrm{supp}(u) \text{ and } i \in [t-1] \text{ s.t. } x_i = a\},$$

$$C(u) := \{a \in [n] \mid \exists x \in \mathrm{supp}(u) \text{ and } i \in [t-1] \text{ s.t. } x_{i+1} = a\}.$$

Note that, for $t = 2$, these sets coincide with the ones defined in [3, Def. 11.4], and denoted there by the same notation. Let $C^\circ(u) \subseteq [n]$ (resp., $R^\circ(u) \subseteq [n]$) be the set of elements in $[n]$ that appear in the last (resp., first) position, and $u$ can change it, i.e.,

$$\begin{aligned} C^\circ(u) &:= \{a \in [n] \mid \exists x \in \mathrm{supp}(u) \text{ s.t. } x_t = a \text{ and } (u(x))_t \neq a\} \\ &= \{a \in [n] \mid \exists x \in [n]^t \text{ s.t. } x_t = a \text{ and } (u(x))_t \neq a\}, \end{aligned}$$

$$\begin{aligned} R^\circ(u) &:= \{a \in [n] \mid \exists x \in \mathrm{supp}(u) \text{ s.t. } x_1 = a \text{ and } (u(x))_1 \neq a\} \\ &= \{a \in [n] \mid \exists x \in [n]^t \text{ s.t. } x_1 = a \text{ and } (u(x))_1 \neq a\}. \end{aligned}$$

In particular, $C^\circ(u) \subseteq C(u)$ and $R^\circ(u) \subseteq R(u)$. Observe that if $u \neq 1$ then $C(u) \neq \emptyset$ and $R(u) \neq \emptyset$, but this is not true for $C^\circ(u)$ and $R^\circ(u)$ in general. For example, if $v \in S([n]^{t-2})$ then $C^\circ(1 \otimes v \otimes 1) = R^\circ(1 \otimes v \otimes 1) = \emptyset$, while $C(1 \otimes v \otimes 1) = R(1 \otimes v \otimes 1) = [n]$ if $v \neq 1$.

Note that if $u \in S([n]^t)$ then $\mathrm{supp}(u^{-1}) = \mathrm{supp}(u)$ and hence $\mathrm{supp}(u^{\#}) = w_0^{(t)}(\mathrm{supp}(u))$ where $w_0^{(t)}(x_1, \ldots, x_t) := (x_t, \ldots, x_1)$ for all $(x_1, \ldots, x_t) \in [n]^t$.

Therefore $R(u^\#) = C(u^{-1}) = C(u)$ and so $C(u^\#) = R(u)$. Furthermore, note that $R^\circ(u^\#) = C^\circ(u^{-1}) = C^\circ(u)$ and so also $C^\circ(u^\#) = R^\circ(u)$.

Note that if $u \in S([n]^t)$ and $\Gamma_k(u)$ consists only of loops for some $k \in \mathbb{N}$ then $C^\circ(\psi_k(u)) = \emptyset$. Furthermore, if $C^\circ(\psi_k(u)) = \emptyset$ for some $k \in \mathbb{N}$ and $t = 2$ then $\Gamma_k(u)$ consists only of loops. Also, note that $C^\circ(\psi_k(u)) \subseteq C^\circ(\psi_{k-1}(u)) \subseteq \cdots \subseteq C^\circ(\psi_1(u)) \subseteq C^\circ(u^{-1})$ for any $k \in \mathbb{N}$.

**Definition 6.1.** For $u, v \in S([n]^t)$ we say that $u$ and $v$ are left-equivalent if there are $\sigma \in S_n$ and $\pi \in S([n]^t)$ such that

- $u = \pi^{-1} v \pi$;
- $\sigma(R^\circ(u) \cap C(u)) = R^\circ(v) \cap C(v)$;
- For any $x \in [n]^t$ and $i \in [t]$ such that $x_i \in R^\circ(u) \cap C(u)$, $(\pi(x))_i = \sigma(x_i)$.

Similarly define right-equivalence by replacing $R^\circ(u) \cap C(u)$ with $C^\circ(u) \cap R(u)$, and the same for $v$.

**Proposition 6.2.** *Left and right equivalence are equivalence relations.*

*Proof.* This is easy to check. The only property that requires a little bit of care is symmetry which follows from the fact that if $u$ is left equivalent to $v$ and $y \in [n]^t$ and $i \in [n]$ are such that $y_i \in R^\circ(v) \cap C(v)$ then $(\pi^{-1}(y))_i \in \sigma^{-1}(R^\circ(v) \cap C(v)) = R^\circ(u) \cap C(u)$, so $y_i = \sigma((\pi^{-1}(y))_i)$. $\square$

**Proposition 6.3.** *Let $u, v \in S([n]^t)$. Then $u$ is left (resp., right) equivalent to $v$ if and only if $u^\#$ is right (resp., left) equivalent to $v^\#$.*

*Proof.* It is enough to show that if $u$ is left-equivalent to $v$ then $u^\#$ is right-equivalent to $v^\#$, the other implication being analogous. So suppose that $(\sigma, \pi) \in S_n \times S([n]^t)$ is a pair satisfying the properties as in the definition of left-equivalence for $u$ and $v$. Then it is routine to check that the pair $(\sigma, {}^t\pi) \in S_n \times S([n]^t)$ satisfies the properties as in the definition of right-equivalence for $u^\#$ and $v^\#$. $\square$

The sets $R^\circ(u)$ and $C^\circ(u)$ allow for a more precise understanding of what makes a permutation stable. The next result is a refinement, and generalization, of Proposition 5.15 in [3]. Note that, if $t = 2$, then $[n] \setminus R^\circ(u)$ (resp., $[n] \setminus C^\circ(u)$) is the set of first (resp., second) coordinates that are not changed by $u$ (so, the indices of the rows (resp., columns) that are left invariant by $u$).

**Proposition 6.4.** *Let $u, v \in S([n]^2)$ be such that $R(u) \cap C^\circ(v) = \emptyset$ and $R^\circ(u) \cap C(v) = \emptyset$. Then $u$ is compatible with $v$, that is $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$.*

*Proof.* Let $x, y, z \in [n]$. We will show that

$$(v \otimes 1)(1 \otimes u)(x, y, z) = (1 \otimes u)(v \otimes 1)(x, y, z). \tag{10}$$

If $v(x, y) = (x, y)$ and $u(y, z) = (y, z)$ then (10) is clear. So assume that either $v(x, y) \neq (x, y)$ or $u(y, z) \neq (y, z)$. Assume first that $v(x, y) \neq (x, y)$. Then $y \in C(v)$ so by our hypotheses $y \notin R^\circ(u)$ which means that

$$u_1(y, w) = y \tag{11}$$

for all $w \in [n]$. If $v_2(x, y) = y$ then, by (11), we have that

$$(v \otimes 1)(1 \otimes u)(x, y, z) = (v \otimes 1)(x, y, u_2(y, z)) = (v_1(x, y), y, u_2(y, z))$$

and

$$(1 \otimes u)(v \otimes 1)(x, y, z) = (1 \otimes u)(v_1(x, y), y, z) = (v_1(x, y), y, u_2(y, z)),$$

and (10) follows.

If $v_2(x, y) \neq y$ then $y \in C^\circ(v)$ and hence also $v_2(x, y) \in C^\circ(v)$. By our hypotheses, this implies that $y \notin R(u)$ and $v_2(x, y) \notin R(u)$ and hence that $u(y, w) = (y, w)$ and $u(v_2(x, y), w) = (v_2(x, y), w)$ for all $w \in [n]$. Therefore

$$(1 \otimes u)(v \otimes 1)(x, y, z) = (1 \otimes u)(v_1(x, y), v_2(x, y), z) = (v_1(x, y), v_2(x, y), z)$$

and

$$(v \otimes 1)(1 \otimes u)(x, y, z) = (v \otimes 1)(x, y, z) = (v_1(x, y), v_2(x, y), z)$$

and (10) again follows.

If $v(x, y) = (x, y)$ and $u(y, z) \neq (y, z)$ the reasoning is similar, and easier, and we therefore omit it. $\square$

It is easy to see that if $u \in S([n]^2)$ is such that $C^\circ(u) \cap R(u) = \emptyset$ then the graphs $\Gamma_k(u)$ are empty (i.e., consist only of loops) for all $k \geq 1$. The next result examines the case that $|C^\circ(u) \cap R(u)| = 1$.

**Proposition 6.5.** *Let $u \in S([n]^2)$ be such that $C^\circ(u) \cap R(u) = \{z_0\}$ for some $z_0 \in [n]$. Then $\Gamma_k(u) \neq \emptyset$ for all $k \in \mathbb{N}$ if and only if either*

$$u^{-1}(z_0, z)_2 = z_0 \tag{12}$$

*or*

$$u^{-1}(z_0, z_0)_2 = z \tag{13}$$

*for some $z \in [n] \setminus \{z_0\}$.*

*Proof.* Assume first that (12) holds for some $z \in [n] \setminus \{z_0\}$. Then, by our definitions, $z \to z_0$ in $\Gamma_0(u)$. Therefore, since $z \neq z_0$, $z \in C^\circ(u)$ and hence $z \notin R(u)$. Hence $u^{-1}(z, z) = (z, z)$ while $u^{-1}(z_0, z)_2 = z_0$ by (12), so $z \to z_0$ in $\Gamma_1(u)$, and therefore $z \to z_0$ in $\Gamma_k(u)$ for all $k \in \mathbb{N}$.

Assume now that (13) holds for some $z \in [n] \setminus \{z_0\}$. Then $z_0 \to z$ in $\Gamma_0(u)$, so, as above, $z \in C^\circ(u) \setminus R(u)$. Hence $u^{-1}(z, z_0) = (z, z_0)$ and, by (13), $u^{-1}(z_0, z_0)_2 = z$, so $z \to z_0$ in $\Gamma_1(u)$. By exactly this same reasoning, $z_0 \to z$ in $\Gamma_2(u)$, etc..., so again $\Gamma_k(u) \neq \emptyset$ for all $k \in \mathbb{N}$.

Conversely, assume that $\Gamma_k(u) \neq \emptyset$ for all $k \in \mathbb{N}$. Then there are $x_1, \ldots, x_r$, $y_1, \ldots, y_r, w_1, \ldots, w_r \in [n]$ and $h \in \mathbb{N}$ such that

$$x_i \to y_i$$

in $\Gamma_{h+i}(u)$, $x_i \neq y_i$,

$$u^{-1}(x_i, w_i)_2 = x_{i+1}, \tag{14}$$

and

$$u^{-1}(y_i, w_i)_2 = y_{i+1}, \tag{15}$$

for all $i \in [r]$ (where indices are modulo $r$). Since, $x_i \neq y_i$ we have from our definitions that $x_i, y_i \in C^\circ(\psi_{h+i}(u))$ and so $x_i, y_i \in C^\circ(u)$. Furthermore, since either $w_i \neq x_{i+1}$ or $w_i \neq y_{i+1}$ we have that $w_i \in C^\circ(u)$. Hence $x_1, \ldots, x_r$, $y_1, \ldots, y_r, w_1, \ldots, w_r \in C^\circ(u)$. This, in turn, implies that $z_0 \in \{x_i, y_i\}$ for all $i \in [r]$ (for if $x_i \neq z_0$ and $y_i \neq z_0$ then $x_i, y_i \notin R(u)$ so $u^{-1}(x_i, w_i) = (x_i, w_i)$ and $u^{-1}(y_i, w_i) = (y_i, w_i)$ which, by (14) and (15), implies that $x_{i+1} = w_i = y_{i+1}$, which is a contradiction).

Assume now that $z_0 = x_1 = x_2$. Then $y_1, y_2 \neq z_0$ so $y_1, y_2 \notin R(u)$. Hence, by (15), $y_2 = u^{-1}(y_1, w_1)_2 = w_1$ so $w_1 \neq z_0$ and therefore (12) follows from (14). Similarly if $z_0 = y_1 = y_2$. So assume that $x_1 = z_0 = y_2$, and $y_1, x_2 \neq z_0$. Then $u^{-1}(z_0, w_1)_2 = x_2$, $u^{-1}(y_1, w_1)_2 = z_0$ and $y_1 \in C^\circ(u) \setminus R(u)$. Hence $u^{-1}(y_1, w_1) = (y_1, w_1)$ so $w_1 = z_0$ and therefore $u^{-1}(z_0, z_0)_2 = x_2$ which proves (13). Similarly if $y_1 = z_0 = x_2$, and $x_1, y_2 \neq z_0$. $\square$

**Remark 6.6.** Note that conditions (12) and (13) imply that $\{z_0\}$ is not a connected component of $\Gamma_0(u)$.

**Remark 6.7.** We note that if $u, v \in S([n]^2)$ are right equivalent and $|C^\circ(u) \cap R(u)| = 1$ then $u$ satisfies condition (12) if and only if $v$ does, and similarly for condition (13). Indeed, let $C^\circ(u) \cap R(u) = \{z_0\}$, and $\pi \in S([n]^2)$, $\sigma \in S_n$ be as in the definition of right equivalence. Then

$$u = \pi^{-1} v \pi$$

$$\sigma(\{z_0\}) = \{w_0\}$$

where $w_0 := \sigma(z_0)$, and for all $x, y \in [n]$ we have that

$$\pi(z_0, y)_1 = \sigma(z_0)$$

and

$$\pi(x, z_0)_2 = \sigma(z_0).$$

Then the above conditions imply that there are $\tau, \rho \in S_n$ such that

$$\pi(z_0, y) = (w_0, \tau(y)) \text{ and } \pi(x, z_0) = (\rho(x), w_0).$$

In particular $\pi(z_0, z_0) = (w_0, \tau(z_0)) = (\rho(z_0), w_0) = (w_0, w_0)$. Since (12) holds for $u$ we have that $u^{-1}(z_0, z) = (U, z_0)$ for some $z \in [n] \setminus \{z_0\}$ and $U \in [n]$. Then we have that

$$v^{-1}(w_0, \tau(z)) = \pi u^{-1} \pi^{-1}(w_0, \tau(z)) = \pi u^{-1}(z_0, z) = \pi(U, z_0) = (\rho(U), w_0)$$

and $\tau(z) \neq \tau(z_0) = w_0$, so (12) holds for $v$. Similarly, if (13) holds for $u$ then $u^{-1}(z_0, z_0) = (U, z)$ for some $z \in [n] \setminus \{z_0\}$ and $U \in [n]$ and we have that

$$v^{-1}(w_0, w_0) = \pi u^{-1} \pi^{-1}(w_0, w_0) = \pi u^{-1}(z_0, z_0) = \pi(U, z)$$

so $v^{-1}(w_0, w_0)_2 = \pi(U, z)_2 \neq w_0$ since $\pi$ maps column $z_0$ into column $w_0$. So (13) holds for $v$.

One can hope that by similar methods it will be possible to deal with any cardinality $\kappa = |C^\circ(u) \cap R(u)|$.

We have started the investigation of the case $\kappa = 2$, so when $u \in S([n]^2)$ is such that $C^\circ(u) \cap R(u) = \{z_0, z_1\}$ with $z_0, z_1 \in [n]$ and $z_0 \neq z_1$. In this case, one can check that each of the following conditions implies that $\Gamma_k(u) \neq \emptyset$ for all $k$:

i) $u^{-1}(z_i, z)_2 = z_i$ for some $z \notin R(u), i \in \{0, 1\}$;

ii) $u^{-1}(z_0, z)_2 = z_1,\ u^{-1}(z_1, z')_2 = z_0$ for some $z, z' \notin R(u)$;

iii) $u^{-1}(z_i, z_i)_2 = z$ for some $z \notin R(u), i \in \{0, 1\}$;

iv) $u^{-1}(z_0, z_1)_2 = z,\ u^{-1}(z_1, z_0)_2 = z'$ for some $z, z' \notin R(u)$ ;

v) $\{u^{-1}(a, b)_2, u^{-1}(b, b)_2\} = \{a, b\}$ for some $a, b \in [n], a \neq b$
(i.e., either $u^{-1}(a, b)_2 = a, u^{-1}(b, b)_2 = b$ or $u^{-1}(a, b)_2 = b, u^{-1}(b, b)_2 = a$);

vi) $u^{-1}(z_0, w)_2 = z_1,\ u^{-1}(y_1, w)_2 = y_2,\ u^{-1}(z_1, z_0)_2 = z_0,\ u^{-1}(y_2, z_0)_2 = y_1$, for some $y_1, y_2 \in [n]$, $y_1 \neq z_0, y_2 \neq z_1$ and $w \in C^{\circ}(u)$;

vii) $u^{-1}(z_i, w)_2 = z_{i+1},\ u^{-1}(z_{i+1}, z_i)_2 = z$ for some $z \notin R(u), i \in \{0, 1\}, w \neq z_{i+1}$;

viii) $u^{-1}(z_i, w_1)_2 = z_j,\ u^{-1}(z_{i+1}, w_1)_2 = y_2,\ u^{-1}(z_j, z_{i+1})_2 = z_i$ , for some $w_1 \in [n]$, $y_2 \in [n] \setminus R(u)$, $i, j \in \{0, 1\}$;

ix) $u^{-1}(z_i, w_1)_2 = z_{i+1},\ u^{-1}(z_{i+1}, w_1)_2 = y_2,\ u^{-1}(z_{i+1}, z_i)_2 \neq z_i$ for some $w_1, y_2 \in [n] \setminus R(u)$, and $i \in \{0, 1\}$ such that $(z_i, z_{i+1})$ is a directed edge in either $\Gamma_0(u)$ or $\Gamma_1(u)$.

It remains to be seen whether these conditions, maybe with the addition of a few similar ones, characterize the fact that $\Gamma_k(u) \neq \emptyset$ for all $k$.

We note the following simple consequence of Corollary 5.8 of [3] and Lemma 5.4 of [4].

**Proposition 6.8.** *Let $u = ((a, b_1), \ldots, (a, b_r)) \in S([n]^2)$ be an $r$-cycle. Then $u$ is stable if and only if $a \notin \{b_1, \ldots, b_r\}$.*

Let $(a_1, b_1), \ldots, (a_r, b_r) \in [n]^2$, distinct, and $u := ((a_1, b_1), \ldots, (a_r, b_r)) \in S([n]^2)$. Define

$$S(u) := \{(i, j) \in [r]^2 : a_i = b_j\}. \tag{16}$$

Strictly speaking, this definition of $S(u)$ depends on the choice of an element in the cycle. So, to be precise, we should associate to any $r$-cycle an $r$-tuple of subsets of $[r]^2$. However, any two of these subsets differ by an element of the diagonal of $[r]^2$, so for notational convenience we consider only one of them. In particular, in all our statements, we use this simplification and write, for example, $S(u) \subseteq \{(1, 2), (2, 4)\}$ to mean that there is a choice of element in the cycle that makes this inclusion true. Equivalently, we might also write $S(u) \subseteq \{(1, 2), (2, 4)\} + (i, i)$, for some $i \in [r]$ (numbers modulo $r$). More in detail, the notation $S(u) = \{(1, 2), (2, 4)\}$ means that there is a $j \in [r]$ such that $a_j = b_{j+1}$ and $a_{j+1} = b_{j+3}$ (indices modulo $r$).

Keeping in mind what was just stated, we note that

$$S(u^{\#}) = \{(r + 1 - j, r + 1 - i) : (i, j) \in S(u)\}. \tag{17}$$

The next result provides some evidence that the stability of cycles $u \in S([n]^2)$ might depend only on $S(u)$. Note that, for quadratic cycles (i.e., for $t = 2$), by [4, Theorem 3.1], $u$ is stable of rank 1 if and only if $S(u) = \emptyset$, which in turn happens if and only if $R(u) \cap C(u) = \emptyset$. This last condition (since $u$ is a cycle) is equivalent to $R^{\circ}(u) \cap C(u) = \emptyset$ and $R(u) \cap C^{\circ}(u) = \emptyset$.

**Theorem 6.9.** *Let $u, v \in S([n]^2)$ be $r$-cycles such that $S(u) = S(v)$. Then $u$ and $v$ are left and right equivalent.*

*Proof.* Say $u = ((a_1, b_1), \ldots, (a_r, b_r))$ and $v = ((c_1, d_1), \ldots, (c_r, d_r))$. Note that $R(u) \cap C(u) = \{a_i : i \in [r]$ and there is $j \in [r]$ such that $(i, j) \in S(u)\}$ and similarly for $v$. Since $S(u) = S(v)$ we have that if $a_i, a_j \in R(u) \cap C(u)$ then $c_i, c_j \in R(v) \cap C(v)$ and $a_i = a_j$ if and only if $c_i = c_j$. Therefore there is $\sigma \in S_n$ such that $\sigma(c_i) = a_i$ for any $(i, j) \in S(v)$. Consider the permutation $\widetilde{u} = ((a'_1, b'_1), \ldots, (a'_r, b'_r)) = (\sigma^{-1} \otimes \sigma^{-1})u(\sigma \otimes \sigma)$, clearly $\widetilde{u}$ and $u$ are left and right equivalent. It remains to show that $\widetilde{u}$ and $v$ are left and right equivalent.

Note that by construction of $\sigma$ if $c_i = d_j$ then $c_i = a'_i = b'_j = d_j$. Similarly if $a'_i = b'_j$. In particular, $R(\widetilde{u}) \cap C(\widetilde{u}) = R(v) \cap C(v)$, $R^\circ(\widetilde{u}) \cap C(\widetilde{u}) = R^\circ(v) \cap C(v)$, and $R(\widetilde{u}) \cap C^\circ(\widetilde{u}) = R(v) \cap C^\circ(v)$. Consider any permutation $\pi \in S([n]^2)$ such that

- $\pi(a'_i, b'_i) = (c_i, d_i)$ for $i \in [r]$;
- $\pi(x)_1 = x_1$ for $x \in (R(\widetilde{u}) \cap C(\widetilde{u})) \times [n]$;
- $\pi(y)_2 = y_2$ for $y \in [n] \times (R(\widetilde{u}) \cap C(\widetilde{u}))$.

Such a permutation $\pi \in S([n]^2)$ exists since if $(a'_i, b'_i) \in (R(\widetilde{u}) \cap C(\widetilde{u})) \times [n]$ then there is $j \in [r]$ such that $a'_i = b'_j$ and so, by what was observed above, $a'_i = c_i$, and similarly if $(a'_i, b'_i) \in [n] \times (R(\widetilde{u}) \cap C(\widetilde{u}))$ then $b'_i = d_i$. Hence $\widetilde{u}$ and $v$ are left and right equivalent (the pair of permutations are $\pi \in S([n]^2)$ and $1 \in S_n$ for both left and right equivalences). $\square$

The next statement is the main motivation for the approach presented in this section.

**Conjecture 6.10.** *Given two right and left equivalent permutations $u, v \in S([n]^2)$. Then $u$ is stable if and only if $v$ is stable.*

Especially, one would like to see that if there exists some $k_0 \geq 0$ such that $\Gamma_k(u)$ (resp. $\Gamma^{\#}_k(u)$) is empty for all $k \geq k_0$ then there should exist some $h_0 \geq 0$ such that $\Gamma_h(v)$ (resp. $\Gamma^{\#}_h(v)$) is empty for all $h \geq h_0$. While this looks very difficult to prove in general, it becomes quite immediate for $k_0 = 0$.

To see this, note first that if $u \in S([n]^2)$ and $k \in \mathbb{N}_0$ have the property that all directed edges $x \to y$ in $\Gamma_k(u)$ that are not loops are such that $x, y \in C^\circ(u) \setminus R(u)$ (note that, since $x \neq y$, it is clear that $x, y \in C^\circ(\psi_k(u))$ and so, by the remarks preceding Definition 6.1, $x, y \in C^\circ(u^{-1}) = C^\circ(u)$) then, for any $z \in [n]$, $u^{-1}(x, z) = (x, z)$ and $u^{-1}(y, z) = (y, z)$ so, by Theorem 3.10, $\Gamma_{k+1}(u)$ consists only of loops.

Assume now that $\Gamma_0(u)$ has only loops, then, by the definition of $\Gamma_0(u)$, $C^\circ(u) = \emptyset$. This, since $u$ is right equivalent to $v$, implies that $C^\circ(v) \cap R(v) = \emptyset$. Therefore, if $x \to y$ in $\Gamma_0(v)$ is not a loop, by what was just observed, we have that $x, y \in C^\circ(v)$ and so $x, y \in C^\circ(v) \setminus R(v)$. Hence, by the above remark, $\Gamma_1(v)$ has only loops.

Similarly, assume that $\Gamma_0^\#(u)$ has only loops, then, by Corollary 3.12, $\Gamma_0(u^\#)$, has only loops, i.e. $C^\circ(u^\#) = \emptyset$. By Proposition 6.3, $u^\#$ and $v^\#$ are left and right equivalent, so that $C^\circ(v^\#) \cap R(v^\#) = \emptyset$ and again, as above, $\Gamma_1(v^\#)$ has only loops, i.e. $\Gamma_1^\#(v)$ has only loops.

Also, a notable special case of Conjecture 6.10 holds if $u, v$ are $r$-cycles and $r \leq 5$, see Corollary 9.9. The proof requires the concept of strong stability that will be introduced and studied in Sections 8 and 9.

Finally, it is worth to stress that the analogue of Conjecture 6.10 most likely fails when $u, v \in S([n]^t)$, $t \geq 3$. It would be quite instructive to examine in detail the following situation. Pick a permutation $u = 1 \otimes u_0 \otimes 1 \in S([n]^3)$, where $u_0$ is a nontrivial element in $S_n$. Then $u$ is stable of rank one and, moreover, $R^\circ(u) = \emptyset = C^\circ(u)$. Also, consider another permutation $\pi \in S([n]^3)$ such that $\pi(a, b, c) = (a, \sigma_{a,c}(b), c)$, where $\sigma_{a,c} \in S_n$ for all $a, c \in [n]$. Now, the permutation $v := \pi u \pi^{-1} \in S([n]^3)$ also satisfies $R^\circ(v) = \emptyset = C^\circ(v)$ and is thus left and right equivalent to $u$. In particular, it would follow from the previous conjecture, somewhat surprisingly, that $v$ is stable as well.

# 7 Quadratic cycles and compatibility

Given the importance of compatibility in the study of stable permutations (see [3, Theorem 5.2]), in this section we study the compatibility of cycles with other permutations. In particular we obtain an explicit characterization of pairs of compatible cycles. This analysis is used in the next section for our study of strongly stable cycles.

Recall from Definition 2.3 that we denote by $\bullet$ a compatible product. So, if $u, v, w \in S([n]^2)$ we write $w = u \bullet v$ to mean that $w = uv$ and $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$ holds.

Compatibility with 2-cycles (transpositions) is discussed in [3, Propositions 5.6 and 5.7]. We now examine compatibility with general cycles.

**Proposition 7.1.** *Let $u = ((a_1, b_1), \ldots, (a_r, b_r)) \in S([n]^2)$ be an $r$-cycle and $v \in S([n]^2)$. Then $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$ if and only if there exist $\sigma \in S_n$ and $s : [n] \to [r]$ such that*

$$\big(v(x, a_j), b_j\big) = \big(\sigma(x), a_{s(x)-1+j}, b_{s(x)-1+j}\big) \tag{18}$$

*for all* $j = 1, \dots, r$ *and all* $x \in [n]$ *(indices modulo* $r$*).*

*Proof.* We have that

$$1 \otimes u = \prod_{x=1}^{n} ((x, a_1, b_1), \dots, (x, a_r, b_r))$$

and so

$$(v \otimes 1)(1 \otimes u)(v^{-1} \otimes 1) = \prod_{x=1}^{n} ((v(x, a_1), b_1), \dots, (v(x, a_r), b_r)).$$

Therefore $1 \otimes u = (v \otimes 1)(1 \otimes u)(v^{-1} \otimes 1)$ if and only if there is $\sigma \in S_n$ such that

$$((\sigma(x), a_1, b_1), \dots, (\sigma(x), a_r, b_r)) = ((v(x, a_1), b_1), \dots, (v(x, a_r), b_r))$$

(as $r$-cycles) for all $x \in [n]$. The result follows. $\square$

**Remark 7.2.** Let $q = \gcd(r, s(1) - 1, \dots, s(n) - 1)$ be the greatest common divisor. Then, since $q$ is a linear combination of $r, s(1)-1, \dots, s(n)-1$, $b_{j+q} = b_j$ for any $j \in [r]$. Therefore, if $q < r$, $a_{j+q} \neq a_j$ for any $j \in [r]$, $|\{b_1, \dots, b_r\}| \leq q$, and hence $|\{a_1, \dots, a_r\}| \geq r/q$.

If there exists $x \in [n]$ such that $s(x) \neq 1$, then ($q < r$ and) $\{a_1, \dots, a_r\} \subseteq C(v)$ (indeed, if $a_j \notin C(v)$ for some $j \in [r]$ then $v(x, a_j) = (x, a_j)$ for all $x \in [n]$ so by (18) we have that $(x, a_j, b_j) = (\sigma(x), a_{s(x)-1+j}, b_{s(x)-1+j})$ for all $x$ which implies that $s(x) = 1$ for all $x \in [n]$). Suppose now that ($s(x) \neq 1$ for some $x \in [n]$ and) $v = ((c_1, d_1), \dots, (c_\ell, d_\ell)) \in S([n]^2)$ is a cycle. Then, since $\{a_1, \dots, a_r\} \subseteq C(v)$, there is $i \in [\ell]$ such that $d_i = a_1$. Therefore, we conclude from (18) that $(v(c_i, a_1), b_1) = (\sigma(c_i), a_{s(c_i)}, b_{s(c_i)})$ which implies that $\sigma(c_i) = c_{i+1}$ and $d_{i+1} = a_{s(c_i)} \in R(u)$. Continuing in this way we therefore conclude that $C(v) \subseteq R(u)$ and hence that $R(u) = C(v)$. With some more work, see also the proof of Lemma 7.4 below, one can also show that $pq = r$, where $p := |\{a_1, \dots, a_r\}|$.

**Proposition 7.3.** *Let* $u = ((a_1, b_1), \dots, (a_r, b_r)) \in S([n]^2)$ *be an* $r$*-cycle and* $v \in S([n]^2)$*. Then* $(u \otimes 1)(1 \otimes v) = (1 \otimes v)(u \otimes 1)$ *if and only if there exist* $\sigma \in S_n$ *and* $s : [n] \to [r]$ *such that*

$$\big(a_j, v(b_j, x)\big) = \big(a_{s(x)-1+j}, b_{s(x)-1+j}, \sigma(x)\big) \tag{19}$$

*for all* $j = 1, \dots, r$ *and all* $x \in [n]$ *(indices modulo* $r$*).*

*Proof.* This follows from the definition of $u^{\#}$, Proposition 7.2 of [4], and Proposition 7.1. $\square$

**Lemma 7.4.** *Let $u := ((a_1, b_1), \dots, (a_r, b_r))$ and $v := ((c_1, d_1), \dots, (c_\ell, d_\ell))$ be an $r$-cycle and an $\ell$-cycle, such that $u$ is compatible with $v$ and $R(u) \cap C(v) \neq \emptyset$. Then $R(u) = C(v), r = |R(u)||C(u)|$, and $\ell = |R(v)||C(v)|$. Furthermore, $u$ acts on the second coordinate as a cycle of length $|C(u)|$ (i.e., $b_1, \dots, b_{|C(u)|}$ are all distinct and $b_{i+|C(u)|} = b_i$ for all $i$, where indices are modulo $r$) and $v$ acts on the first coordinate as a cycle of length $|R(v)|$.*

*Proof.* By Proposition 7.1 we have that $u$ is compatible with $v$ if and only if there are $\sigma \in S_n$ and $s : [n] \to [r]$ such that

$$\big(v(x, a_j), b_j\big) = \big(\sigma(x), a_{s(x)-1+j}, b_{s(x)-1+j}\big) \tag{20}$$

for all $j \in [r]$ and all $x \in [n]$ (where indices are modulo $r$).

Since $R(u) \cap C(v) \neq \emptyset$, then, arguing similarly as in Remark 7.2, one can show that $C(v) \subseteq R(u)$ and then that $R(u) \subseteq C(v)$.

If there are $i, j$ such that $v(c_i, a_j) = (c_i, a_j)$, then $(v(c_i, a_j), b_j) = (c_i, a_j, b_j)$. Therefore, $\sigma(c_i) = c_i$ and $s(c_i) = 1$, i.e., $v(c_i, a_k) = (c_i, a_k)$ for any $k$. We get that $d_i \notin R(u)$, however $C(v) = R(u)$. Therefore $v(c_i, a_j) \neq (c_i, a_j)$ for all $i, j$. The number of such pairs is $|R(u)||R(v)| = |C(v)||R(v)|$. We immediately have two inequalities for $\ell$, then $\ell = |C(v)||R(v)|$. Furthermore, we see that $(c_{i+1}, d_{i+1}) = v(c_i, d_i) = (\sigma(c_i), \cdot)$, hence, $v$ acts on the first coordinate as a cycle of length $|R(v)|$. Similarly, $r = |C(u)||R(u)|$ and $u$ acts on the second coordinate as a cycle of length $|C(u)|$. $\square$

The previous lemma easily implies the following characterization of compatible pairs of cycles $u, v \in S([n]^2)$ such that $R(u) \cap C(v) \neq \emptyset$. By [3, Proposition 5.15] this completely characterizes the pairs of cycles in $S([n]^2)$ that are compatible since any permutations $u, v \in S([n]^2)$ such that $R(u) \cap C(v) = \emptyset$ are compatible in this order.

**Theorem 7.5.** *Given an $r$-cycle $u$ and an $\ell$-cycle $v$ such that $R(u) \cap C(v) \neq \emptyset$. Then $u$ and $v$ are compatible in this order if and only if there are $t|\gcd(r, \ell)$, three sequences $(x_1, x_2, \dots, x_t)$, $(y_1, y_2, \dots, y_{\frac{r}{t}})$, $(z_1, z_2, \dots, z_{\frac{\ell}{t}})$ of distinct elements in $[n]$ and two sequences of numbers $(p_1, p_2, \dots, p_{\frac{\ell}{t}}) \in [0, r-1]^{\frac{\ell}{t}}$, $(q_1, q_2, \dots, q_{\frac{r}{t}}) \in [0, \ell-1]^{\frac{r}{t}}$ such that:*

1. *$R(u) = C(v) = \{x_1, x_2, \dots, x_t\}$, $C(u) = \{y_1, y_2, \dots, y_{\frac{r}{t}}\}$, and $R(v) = \{z_1, z_2, \dots, z_{\frac{\ell}{t}}\}$;*
2. *$u(x_i, y_j) = (x_{i+q_j}, y_{j+1})$ for all $i \in [t]$ and $j \in [\frac{r}{t}]$;*
3. *$v(z_i, x_j) = (z_{i+1}, x_{j+p_i})$ for all $i \in [\frac{\ell}{t}]$ and $j \in [t]$.*

*(where the indexes of $x, y, z$ are taken mod $t$, mod $\frac{r}{t}$ and mod $\frac{\ell}{t}$, respectively).*

**Remark 7.6.** Note that conditions 2. and 3. in Theorem 7.5 define cycles if and only if

$$\gcd(p_1 + p_2 + \ldots + p_{\frac{\ell}{t}}, t) = \gcd(q_1 + q_2 + \ldots + q_{\frac{r}{t}}, t) = 1 \ .$$

Indeed, if $k := \gcd(p_1 + p_2 + \ldots + p_{\frac{\ell}{t}}, t)$ then by 3. we have that $v^{\frac{\ell}{k}}(z_i, x_j) = (z_{i+\frac{\ell}{k}}, x_{j+\frac{t}{k}(p_1+\ldots+p_{\frac{\ell}{t}})})$ for all $i \in [\frac{\ell}{t}]$ and $j \in [t]$. But $\frac{\ell}{k} \equiv 0 \pmod{\frac{\ell}{t}}$ and $\frac{t}{k}(p_1 + \ldots + p_{\frac{\ell}{t}}) \equiv 0 \pmod{t}$, so $v^{\frac{\ell}{k}}$ is the identity which implies that $k = 1$. Similarly for $u$.

**Lemma 7.7.** *Let $u$ be an $r$-cycle such that $r = |R(u)||C(u)|$ and $u$ acts on the first coordinate as a cycle of length $|R(u)|$ (on the second coordinate as a cycle of length $|C(u)|$). If $R(u) \cap C(u) \neq \emptyset$ and $R(u) \neq C(u)$, then $u$ is not a stable permutation.*

*Proof.* Let $u = ((a_1, b_1), \ldots, (a_r, b_r)) \in S([n]^2)$. Write, for brevity, $\Gamma_k$ rather than $\Gamma_k(u)$, and similarly for $\Gamma_k^{\#}$. Consider first the case $R(u) \not\subseteq C(u)$. Let $i$ be an index such that $a_i \in C(u)$ and $a_{i+1} \notin C(u)$. Then $u(a_i, a_{i+1}) = (a_i, a_{i+1})$. Also, since $u$ acts as a cycle on the first coordinate, $u(x, y) \neq (x, y)$ for all $(x, y) \in R(u) \times C(u)$, and $a_i \in C(u)$, $u(a_i, a_i) = (a_{i+1}, \cdot)$. This implies that $a_{i+1} \to a_i$ in $\Gamma_0^{\#}$ and that, if $a_i \to a_{i+1}$ (resp. $a_{i+1} \to a_i$) in $G$, then $a_{i+1} \to a_i$ (resp. $a_i \to a_{i+1}$) in $\mathcal{L}_u(G)$. Hence, by Theorems 3.10 and 3.13, $u$ is not a stable permutation.

Suppose now that $R(u) \subsetneq C(u)$. Since $u$ is a cycle, the transitive closure of $\Gamma_0$, $T(\Gamma_0)$, is a complete graph on $C(u)$. We claim that $T(\Gamma_k) = T(\Gamma_0)$ for all $k \geq 0$. We prove this claim by induction on $k \geq 0$. Indeed, let $x \in C(u) \setminus R(u)$ and $i \in [r]$. Since $a_i \to x$ in $T(\Gamma_k)$, then $b_{i-1} \to b_i$ in $\mathcal{R}_{u^{-1}}(T(\Gamma_k))$ but, by Lemma 3.17, $T(\mathcal{R}_{u^{-1}}(T(\Gamma_k))) = T(\mathcal{R}_{u^{-1}}(\Gamma_k))$, so by Theorem 3.10, $b_{i-1} \to b_i$ in $T(\Gamma_{k+1})$. Hence $T(\Gamma_{k+1})$ is a complete graph on $C(u)$. But $|C(u)| > 1$ so $\Gamma_k \neq \emptyset$ for all $k \geq 0$ hence $u$ is not stable.

The statement in parenthesis follows by applying what we have just proved to $u^{\#}$ and using the facts that $R(u) = C(u^{\#})$ and $R(u^{\#}) = C(u)$ (see the remarks made before Definition 6.1) and Theorem 7.3 of [4]. ☐

Note that, by [3, Proposition 5.15], if $u, v \in S([n]^2)$ are such that $R(u) \cap C(v) = \emptyset$ then $u$ is compatible with $v$. The converse is false even if $u$ and $v$ are stable cycles. For example, if $u = ((1, 2), (1, 3), (1, 4))$ and $v = ((2, 1), (3, 1), (4, 1))$ then $u$ and $v$ are stable of rank 1 and $u$ is compatible with $v$ (see also [4, Theorem 3.1] and Theorem 7.5). However, the following statement holds.

**Corollary 7.8.** *Let $u$ and $v$ be a stable $r$-cycle and a stable $\ell$-cycle such that* $\gcd(r, \ell) = 1$. *Then $uv$ is a compatible product if and only if either*

1. $R(u) \cap C(v) = \emptyset$;
2. $R(u) = C(v) = \{a\}$ *for some* $a \in [n]$;

*Furthermore, $u \bullet v$ is a stable permutation in this case.*

*Proof.* Let $u := ((a_1, b_1), \ldots, (a_r, b_r))$ and $v := ((c_1, d_1), \ldots, (c_\ell, d_\ell))$. We have already observed that if 1. holds then $u$ is compatible with $v$. Assume that 2. holds. Then $R(u) \cap C(v) \neq \emptyset$ and $a_1 = \cdots = a_r = d_1 = \cdots = d_\ell$ so $b_1, \ldots, b_r$ are all distinct and $c_1, \ldots, c_\ell$ are all distinct. Therefore, taking $t = 1$, $p_1 = \cdots = p_\ell = 1$, and $q_1 = \cdots = q_r = 1$ we see that all the conditions in Theorem 7.5 are satisfied so $u$ is compatible with $v$.

Conversely, suppose that $u$ is compatible with $v$ and that $R(u) \cap C(v) \neq \emptyset$. Then by Theorem 7.5 and our hypotheses we conclude that 2. holds.

The last statement follows from [3, Theorem 5.2]. □

**Remark 7.9.** Note that $uv$ is not necessarily a cycle.

Some consequences of the analysis carried out so far are the following ones.

**Corollary 7.10.** *Let $u$ and $v$ be a stable $r$-cycle and a stable $\ell$-cycle and $(a, b) \in [n]^2$ be such that $(a, b) \neq u(a, b), v(a, b)$. If $uv$ is a compatible product then either $R(u) \cap C(v) = \emptyset$ or $R(u) = C(u) = R(v) = C(v)$.*

*Proof.* Assume that $R(u) \cap C(v) \neq \emptyset$. Then, by Lemma 7.4, $R(u) = C(v)$, $r = |R(u)||C(u)|$, $\ell = |R(v)||C(v)|$, $u$ acts on the second coordinate as a cycle of length $|C(u)|$, and $v$ acts on the first coordinate as a cycle of length $|R(v)|$. This, by Lemma 7.7, implies that either $R(u) \cap C(u) = \emptyset$ or $R(u) = C(u)$, and that either $R(v) \cap C(v) = \emptyset$ or $R(v) = C(v)$. But, by our hypothesis, $a \in R(u) \cap R(v)$ and $b \in C(u) \cap C(v)$. Since $R(u) = C(v)$, the result follows. □

**Corollary 7.11.** *Let $u, v \in S([n]^2)$ be a stable $r$-cycle and a stable $\ell$-cycle such that $u$ and $v$ are not of rank 1 and either $r = \ell$ is not a square or $r \neq \ell$. Then $uv$ is a compatible product if and only if $R(u) \cap C(v) = \emptyset$.*

*Proof.* As observed before Corollary 7.8 we already know that if $R(u) \cap C(v) = \emptyset$ then $u$ is compatible with $v$. So assume that $u$ is compatible with $v$ and that $R(u) \cap C(v) \neq \emptyset$. Then, by Lemma 7.4, $R(u) = C(v), r = |R(u)||C(u)|$, $\ell = |R(v)||C(v)|$, $u$ acts on the second coordinate as a cycle of length $|C(u)|$, and $v$ acts on the first coordinate as a cycle of length $|R(v)|$. Since $u$ and $v$ are not of rank 1 we have from [4, Theorem 3.1] that $R(u) \cap C(u) \neq \emptyset$ and $R(v) \cap C(v) \neq \emptyset$. Therefore, by Lemma 7.7, we conclude that $R(u) = C(u)$ and $R(v) = C(v)$, so $r = \ell = |R(u)|^2$ which contradicts our hypotheses. $\square$

**Corollary 7.12.** *Let $u, v \in S([n]^2)$ be a stable $r$-cycle and a stable $\ell$-cycle such that exactly one of $u$, $v$ has rank $1$, and $\gcd(r, \ell) = 1$. Then $uv$ is a compatible product if and only if $R(u) \cap C(v) = \emptyset$.*

*Proof.* The proof follows the lines of that of Corollary 7.11. Suppose that $u$ is compatible with $v$ and $R(u) \cap C(v) \neq \emptyset$. Then by Corollary 7.8 we have that $R(u) = C(v)$ and $|R(u)| = |C(v)| = 1$. Also, by Lemma 7.4, $r = |R(u)||C(u)|$, $\ell = |R(v)||C(v)|$, $u$ acts on the second coordinate as a cycle of length $|C(u)|$, and $v$ acts on the first coordinate as a cycle of length $|R(v)|$. If $v$ is not of rank 1 we have from [4, Theorem 3.1] that $R(v) \cap C(v) \neq \emptyset$ so, by Lemma 7.7, $R(v) = C(v)$ so $\ell = 1$ which is a contradiction. Similarly, if $u$ is not of rank 1 then $R(u) \cap C(u) \neq \emptyset$ so, by Lemma 7.7, $R(u) = C(u)$ so $r = 1$. $\square$

**Corollary 7.13.** *Let $(a_1, b_1), \ldots, (a_r, b_r) \in [n]^2$, be distinct pairs such that $v := ((a_1, b_1), \ldots, (a_{r-1}, b_{r-1}))$ is stable, and consider the $r$-cycle $w := ((a_1, b_1), \ldots, (a_r, b_r))$. Then $u := ((a_1, b_1), (a_r, b_r))$ is stable and $w = uv$ is a compatible product if and only if*

$$\{a_1, a_r\} \cap C(w) = \emptyset. \tag{21}$$

*Proof.* If $\{a_1, a_r\} \cap C(w) = \emptyset$ then by [3, Proposition 5.15] and [3, Theorem 8.1], the transposition $u$ is stable and $u$ is compatible with $v$.

Conversely, assume that $u$ is stable and $u$ is compatible with $v$. Then by [3, Theorem 8.1] we have that $\{a_1, a_r\} \cap \{b_1, b_r\} = \emptyset$ and thus $\{a_1, a_r\} \cap C(w) = \{a_1, a_r\} \cap C(v)$. If $\{a_1, a_r\} \cap C(v) \neq \emptyset$, then, by Theorem 7.5, $\{a_1, a_r\} = C(v)$, so $b_1 \in \{a_1, a_r\}$, which is a contradiction . $\square$

This last result should be useful for some reduction/inductive step. Related statements, in a slightly more specific context, are [4, Theorem 5.2] and [4, Theorem 6.2].

The following is a quite explicit special case of Theorem 7.5.

**Corollary 7.14.** *Let $u := ((a_1, b_1), \ldots, (a_r, b_r))$ and $v := ((c_1, d_1), \ldots, (c_r, d_r))$ be two $r$-cycles, with $r$ prime. Then $u$ is compatible with $v$ if and only if either:*

1. *$R(u) \cap C(v) = \emptyset$;*
2. *$a_1 = \cdots = a_r = d_1 = \cdots = d_r$;*

3. $b_1 = \cdots = b_r, c_1 = \cdots = c_r$ *and there are* $k \in [r]$ *and* $p \in [r-1]$ *such that* $d_i = a_{k+(i-1)p}$ *for all* $i = 1, \ldots, r$.

*Proof.* Assume first that $u$ is compatible with $v$. Suppose that $R(u) \cap C(v) \neq \emptyset$. We can apply Theorem 7.5 for some $t$. Since $t|gcd(r, r)$ and $r$ is a prime, $t$ equals either 1 or $r$.
The first case: $t = 1$. Hence, $R(u) = C(v) = \{x\}$ for some $x \in [n]$. Therefore, $a_1 = \cdots = a_r = d_1 = \cdots = d_r$ holds.
The second case: $t = r$. Hence, $C(u) = \{y\}$, $R(v) = \{z\}$ for some $y, z \in [n]$. Therefore, $b_1 = \cdots = b_r$ and $c_1 = \cdots = c_r$ hold. There is $q$ such that $u(x_i, y) = (x_{i+q}, y)$, where $\{x_1, x_2, \ldots, x_r\}$ is $R(u)$. Since $q$ and $r$ are coprime, we can assume without loss of generality that $a_1 = x_1$ and $q = 1$ in Theorem 7.5. Therefore $x_i = a_i$ for any $i$. Let $j \in [r]$ be such that $d_1 = x_j$. Then $(z, d_2) = v(z, d_1) = v(z, x_j) = (z, x_{j+p})$ so $d_2 = x_{j+p} = a_{j+p}$. Similarly $(z, d_3) = v(z, d_2) = v(z, x_{j+p}) = (z, x_{j+2p})$ so $d_3 = x_{j+2p} = a_{j+2p}$, etc., so 3. holds.

Conversely. If 1. holds then, as already observed before Corollary 7.8, $u$ is compatible with $v$. Assume that 2. holds. Then $R(u) \cap C(v) \neq \emptyset$ and $(b_1, b_2, \ldots, b_r)$ and $(c_1, c_2, \ldots, c_r)$ are distinct elements of $[n]$ such that $R(u) = C(v) = \{a_1\}$, $C(u) = \{b_1, b_2, \ldots, b_r\}$, $R(v) = \{c_1, c_2, \ldots, c_r\}$, $u(a_1, b_j) = (a_1, b_{j+1})$ for all $j \in [r]$, and $v(c_i, a_1) = (c_{i+1}, a_1)$ for all $i \in [r]$. So, by Theorem 7.5 (with $p_1 = \cdots = p_r = q_1 = \cdots = q_r = 0$) $u$ is compatible with $v$. Finally, if 3. holds then $(a_1, a_2, \ldots, a_r)$ are distinct elements in $[n]$, and $b_1, c_1 \in [n]$ and $p \in [r-1]$ are such that $R(u) = C(v) = \{a_1, a_2, \ldots, a_r\}$, $C(u) = \{b_1\}$, $R(v) = \{c_1\}$ and $u(a_i, b_1) = (a_{i+1}, b_1)$ for all $i \in [r]$. Furthermore, let $j \in [r]$ then, since $r$ is prime, there is $i \in [r]$ such that $j \equiv k + (i-1)p \pmod{r}$. Therefore $v(c_1, a_j) = v(c_1, d_i) = (c_1, d_{i+1}) = (c_1, a_{j+p})$. So, again by Theorem 7.5 (with $p_1 = p$ and $q_1 = 1$) $u$ is compatible with $v$. $\square$

Note that, for $r = 3$, condition 3. in the previous result can be stated more simply as $b_1 = b_2 = b_3, c_1 = c_2 = c_3$ and $\{a_1, a_2, a_3\} = \{d_1, d_2, d_3\}$.

# 8 Strongly stable quadratic cycles

In this section we define and study a notable class of stable cycles, which we call strongly stable, whose definition relies on the concept of a cyclic factorization. Although this concept seems to be new, it is in fact closely related to many well studied combinatorial objects including non-crossing partitions and Fuss-Catalan numbers. We give several explicit characterizations of strongly stable cycles (Theorem 8.7) including in terms of the subset $S(u) \subseteq [r]^2$ defined in (16).

Some of these characterizations depend on new combinatorial concepts, such as the connectivity set of a permutation, which may be of independent interest.

Let $t_1, \dots, t_r \in S([n]^2)$ be transpositions. By an *ordered product* of $(t_1, \dots, t_r)$, we mean a total order on the $r-1$ products in the expression $t_1 \cdots t_r$. We denote this by $t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-1}}{\cdot} t_r$ where $\{a_1, a_2, \cdots, a_{r-1}\}$ is a subset of $\mathbb{Z}$, and interpret this in the obvious way. So, for example, $t_1 \underset{2}{\cdot} t_2 \underset{3}{\cdot} t_3 \underset{1}{\cdot} t_4$ means that we multiply first $t_3$ and $t_4$ (in this order), then $t_1$ and $t_2$, and finally $(t_1 t_2)$ and $(t_3 t_4)$. Note that this notation contains more information than a bracketing. If $u = t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-1}}{\cdot} t_r$ then we also say that $t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-1}}{\cdot} t_r$ is an *ordered factorization* of $u$. If all the involved products are compatible then we write $u = t_1 \underset{a_1}{\bullet} t_2 \underset{a_2}{\bullet} \cdots \underset{a_{r-1}}{\bullet} t_r$ and call this a *compatible ordered factorization* of $u$.

Let $u \in S([n]^2)$ be an $r$-cycle. Let $t_1, \dots, t_{r-1}$ be transpositions and $t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-2}}{\cdot} t_{r-1}$ be an ordered factorization of $u$. We say that such an ordered factorization is a *cyclic factorization* of $u$ if, after performing the first $k$ of these products ($1 \leq k \leq r-2$), the resulting permutations are all cycles. For example, the factorization $(P_1, P_2, P_3, P_4, P_5) = (P_1, P_2) \underset{2}{\cdot} (P_2, P_3) \underset{3}{\cdot} (P_3, P_4) \underset{1}{\cdot} (P_4, P_5)$ is cyclic, while the factorization $(P_1, P_2, P_3, P_4, P_5) = (P_1, P_3) \underset{3}{\cdot} (P_4, P_5) \underset{2}{\cdot} (P_1, P_2) \underset{1}{\cdot} (P_3, P_5)$ is not. Note that, in a cyclic factorization of an $r$-cycle $(P_1, \dots, P_r) = t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-2}}{\cdot} t_{r-1}$ all transpositions $t_i$ necessarily only involve elements in $\{P_1, \dots, P_r\}$, as can be easily seen by induction on $r$.

For $r \geq 2$ let $C_r$ denote the number of cyclic factorizations of an $r$-cycle. So, for example, $C_2 = 1$ and $C_3 = 3$ (corresponding to $(P_1, P_2, P_3) = (P_1, P_2)(P_2, P_3) = (P_2, P_3)(P_3, P_1) = (P_3, P_1)(P_1, P_2)$). Note that, if $r \geq 3$ and $u := (P_1, \dots, P_r) = t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-2}}{\cdot} t_{r-1}$ is a cyclic factorization then the shift $P_j \mapsto P_{j+1}$ for all $j \in [r]$ (indices modulo $r$) produces a different cyclic factorization of $u$ since $t_1, \dots, t_{r-1}$ are all transpositions and $r \geq 3$. Thus $r \,|\, C_r$ if $r \geq 3$.

Although the concept of cyclic factorization seems to be new, it is in fact closely related to many combinatorial objects that have been widely studied in the literature, as we now show.

Consider $r$ points on a circle, labeled clockwise from 1 to $r$. Let $T$ be a tree, embedded in the plane, having these points as vertices, and rectilinear edges. Recall (see, e.g., [15], and the references cited there) that such a tree is *non-crossing* (or, an *nc-tree*, for short) if its edges do not cross. Note that, for any such tree $T$ and any vertex $x$ of $T$ of degree $d(x)$, there are $d(x)$ angles in the vertex $x$, so $2r-2$ such angles in total. Now assign a total order (i.e., a number in $[r-2]$) to all the angles of the tree except the exterior ones (so

$(2r-2)-r=r-2$ angles in total). We call such a decorated tree a *tree with angles*.

**Proposition 8.1.** *There is a bijection between cyclic factorizations of an $r$-cycle and trees with angles on $r$ vertices. In particular,*

$$C_{r+1}=\frac{(r-1)!}{2r+1}\binom{3r}{r} \tag{22}$$

*for all $r\in\mathbb{N}$.*

*Proof.* We associate to any cyclic factorization of the cycle $u=(P_1,\dots,P_r)$ a non-crossing tree with angles $T$, on the vertex set $\{P_1,\dots,P_r\}$, embedded in the plane so that the vertices $P_1,\dots,P_r$ appear clockwise on a circle. We make the association inductively. If $r=2$ there is only one cyclic factorization of a 2-cycle and we associate to this the only tree with angles on 2 vertices. Suppose now that we have a cyclic factorization of an $r$-cycle $u$. Consider the last (i.e., the $(r-2)$-nd) multiplication in the factorization. By definition of cyclic factorization, the permutations on the left and right of this last multiplication are cycles, they have exactly one element $P_i$ in common, and we have a cyclic factorization of each one of them. Since the product of these two cycles is $u=(P_1,\dots,P_r)$ we conclude that either the left one is $(P_{j+1},P_{j+2},\dots,P_r,P_1,P_2,\dots,P_{i-1},P_i)$ and the right one is $(P_i,P_{i+1},\dots,P_j)$ for some $j\in[i+1,r]$ (if $P_1$ is in the left cycle) or the left one is $(P_j,P_{j+1},\dots,P_i)$ and the right one is $(P_i,P_{i+1},\dots,P_r,P_1,P_2,\dots,P_{j-1})$ for some $j\in[i-1]$ (if $P_1$ is in the right cycle). Now take the trees with angles $T_l$ and $T_r$ associated to the left and right cyclic factorizations, respectively, glue them together at the vertex corresponding to $P_i$ so that the two exterior (i.e, unnumbered) angles are joined and number with $r-2$, among the two new angles just formed in vertex $P_i$, the one so that $T_l$ is on the left when entering the angle numbered with $r-2$. Note that there is a canonical way to embed such a tree with angles so that the vertices $P_1,\dots,P_r$ appear clockwise on a circle. It is easy to see that this is a bijection: to any non-crossing tree with angles we can associate a cyclic factorization. Therefore, $C_r=(r-2)!t_r$, where $t_r$ is the number of nc-trees with $r$ vertices. But it is well known (see, e.g., [15, Theorem 1.1], or [12]) that $t_{r+1}=\binom{3r}{r}/(2r+1)$ for all $r\in\mathbb{N}$, so the result follows. $\square$

We illustrate the bijection just described on a couple of examples. Consider the cyclic factorization

$$\begin{aligned}(P_4,&P_5)\underset{4}{\cdot}(P_2,P_3)\underset{3}{\cdot}(P_3,P_5)\underset{2}{\cdot}(P_1,P_6)\underset{1}{\cdot}(P_1,P_5)\\ &=(P_4,P_5)\cdot\Big((P_2,P_3)\cdot\Big((P_3,P_5)\cdot\big((P_1,P_6)\cdot(P_1,P_5)\big)\Big)\Big)\end{aligned}$$

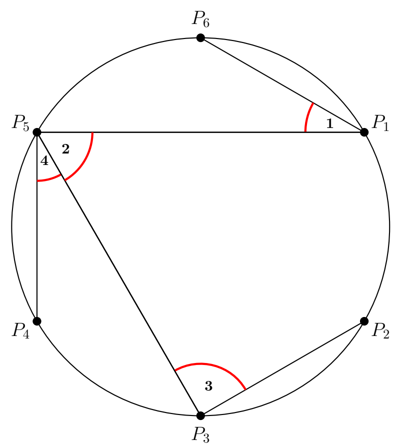


Figure 2: The tree with angles corresponding to the cyclic factorization $(P_4, P_5) \underset{4}{\cdot} (P_2, P_3) \underset{3}{\cdot} (P_3, P_5) \underset{2}{\cdot} (P_1, P_6) \underset{1}{\cdot} (P_1, P_5)$.

of the 6-cycle $(P_1, P_2, P_3, P_4, P_5, P_6)$. Then the corresponding tree with angles is shown in Figure 2.

Conversely, the cyclic factorization corresponding to the tree with angles depicted in Figure 3 is:

$$(P_1, P_2) \underset{4}{\cdot} (P_2, P_3) \underset{3}{\cdot} (P_3, P_8) \underset{6}{\cdot} (P_4, P_5) \underset{1}{\cdot} (P_3, P_5) \underset{5}{\cdot} (P_5, P_6) \underset{2}{\cdot} (P_6, P_7).$$

The sequence $\{t_r\}_{r=1,2,\ldots}$ appears often in enumerative combinatorics and has a large number of combinatorial interpretations including in terms of trees, lattice paths and noncrossing partitions (see, e.g., [17], sequence A001764, and the references cited there). In particular, $t_{r+1}$ is also the 2nd Fuss-Catalan number $\mathrm{Cat}^{(2)}(S_r)$ of the symmetric group $S_r$, so the number of 2-divisible noncrossing partitions of $S_r$ (see, e.g., [1, Section 3.5]). Finally it may be worth noting that the sequence $C_r$ satisfies the recursion $C_r = r \sum_{i=2}^{r-1} \binom{r-3}{i-2} C_i C_{r-i+1}$ for all $r \geq 3$, as can be deduced directly from the definition of these numbers, but not easily from (22).

Given a cyclic factorization $u = t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-2}}{\cdot} t_{r-1}$ of a cycle $u \in S([n]^2)$ we let $u'_k$ and $u''_k$ be the left and right cycles, respectively, that are being multiplied when performing the $k$-th product in the factorization. So, for example, for $u = (P_1, P_2, P_3, P_4, P_5) = (P_1, P_2) \underset{2}{\cdot} (P_2, P_3) \underset{3}{\cdot} (P_3, P_4) \underset{1}{\cdot} (P_4, P_5)$ we have that $u'_1 = (P_3, P_4)$, $u''_1 = (P_4, P_5)$, $u'_2 = (P_1, P_2)$, $u''_2 = (P_2, P_3)$, $u'_3 = (P_1, P_2, P_3)$,

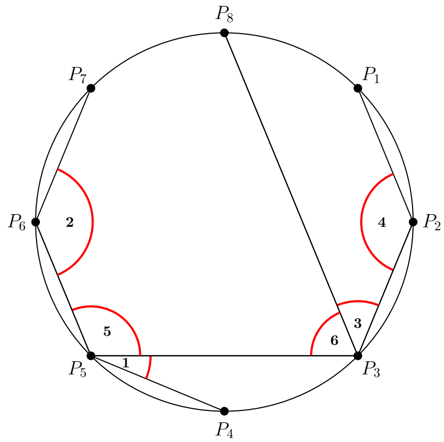


Figure 3: Another tree with angles

and $u''_3 = (P_3, P_4, P_5)$. Let $u \in S([n]^2)$ be an $r$-cycle. The following is the main definition of this section. We say that $u$ is a *strongly stable cycle* if there is a compatible cyclic factorization $u = t_1 \underset{a_1}{\bullet} t_2 \underset{a_2}{\bullet} \cdots \underset{a_{r-2}}{\bullet} t_{r-1}$ such that $t_1, \dots, t_{r-1}$ are stable transpositions. Note that, in this case, $u'_k$ and $u''_k$ are strongly stable cycles for all $k = 1, \dots, r-2$. By [3, Theorem 5.2] we have that a strongly stable cycle is stable. Conversely, by [4, Theorem 5.12], a stable 3-cycle is strongly stable. Recall the definitions of the sets $R(u)$ and $C(u)$ appearing at the beginning of Sect.6 (see also [3, Def. 11.4]).

**Lemma 8.2.** *Let $u \in S([n]^2)$ be a strongly stable cycle. Then $C(u) \nsubseteq R(u)$. In particular, $R(u) \neq C(u)$.*

*Proof.* Suppose $u$ is an $r$-cycle. We proceed by induction on $r$, the result being true by [3, Theorem 8.1] for $r = 2$. So assume that $r \geq 3$. Let $u = t_1 \underset{a_1}{\bullet} t_2 \underset{a_2}{\bullet} \cdots \underset{a_{r-2}}{\bullet} t_{r-1}$ be a compatible cyclic factorization of $u$ where $t_1, \dots, t_{r-1}$ are stable transpositions. Let, for brevity, $u' := u'_{r-2}$ and $u'' := u''_{r-2}$. Then $u = u' \bullet u''$ and, by induction, $C(u') \nsubseteq R(u')$ and $C(u'') \nsubseteq R(u'')$. Furthermore, since $u$, $u'$, and $u''$ are all cycles there is $(a, b) \in [n]^2$ such that $u'(a, b) \neq (a, b) \neq u''(a, b)$. Hence, by Corollary 7.10, $R(u') \cap C(u'') = \emptyset$. Therefore $C(u'') \nsubseteq R(u') \cup R(u'')$ so the result follows since $C(u) = C(u') \cup C(u'')$ and $R(u) = R(u') \cup R(u'')$. □

**Lemma 8.3.** *Let $u = t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-2}}{\cdot} t_{r-1}$ be a cyclic factorization by stable transpositions of an $r$-cycle $u \in S([n]^2)$. Then the factorization is compatible if* *and only if* $R(u'_k) \cap C(u''_k) = \emptyset$ *for all* $1 \leq k \leq r-2$.

*Proof.* If the factorization is compatible then the same reasoning used to prove Lemma 8.2 shows that $R(u'_k) \cap C(u''_k) = \emptyset$ for all $1 \leq k \leq r-2$. Conversely, if $R(u'_k) \cap C(u''_k) = \emptyset$ for all $1 \leq k \leq r-2$ then, by [3, Proposition 5.15], $u'_k$ is compatible with $u''_k$, for all $1 \leq k \leq r-2$ so the factorization is compatible. □

**Theorem 8.4.** *Given two* $r$*-cycles* $u, v \in S([n]^2)$ *such that* $S(v) \subseteq S(u)$*. If* $u$ *is strongly stable then* $v$ *is strongly stable.*

*Proof.* Let $u = ((a_1, b_1), \ldots, (a_r, b_r))$ and $u = t_1 \underset{\alpha_1}{\bullet} t_2 \underset{\alpha_2}{\bullet} \cdots \underset{\alpha_{r-2}}{\bullet} t_{r-1}$ be a compatible cyclic factorization with $t_1, \ldots, t_{r-1}$ stable transpositions. Then each transposition $t_i$ only involves elements of $\{(a_1, b_1), \ldots, (a_r, b_r)\}$. Furthermore, by Lemma 8.3, the fact that this cyclic factorization is compatible depends on certain $a_i$'s being different from certain $b_j$'s, so on $S(u)$. If $v = ((c_1, d_1), \ldots, (c_r, d_r))$ then, since $S(v) \subseteq S(u)$, if $a_i \neq b_j$ for some $i, j \in [r]$ then $c_i \neq d_j$, so substituting $a_i$ with $c_i$ and $b_j$ with $d_j$ in the cyclic factorization for $u$ yields a compatible cyclic factorization of $v$ by stable transpositions. □

Note that the previous result does not hold if we replace "strongly stable" by "stable". For example, any 5-cycle $v$ such that $S(v) = \{(4,1),(1,4)\}$ is not stable but any 5-cycle $u$ such that $S(u) = \{(1,3),(1,4),(2,3),(2,4),(4,1)\}$ is stable. Indeed, let $v = ((a_1, b_1), \ldots, (a_5, b_5))$ be such that $S(v) = \{(1,4),(4,1)\}$. Then $\Gamma_0(v)$ contains the directed edge $b_1 \to b_5$. Therefore, by Theorem 3.10, $\Gamma_1(v)$ contains the directed edge $v^{-1}(b_1, b_4)_2 \to v^{-1}(b_5, b_4)_2$ namely $b_3 \to b_4$. But then $\Gamma_2(v)$ contains the directed edge $v^{-1}(b_3, b_1)_2 \to v^{-1}(b_4, b_1)_2$ namely $b_1 \to b_5$. Therefore none of the graphs $\Gamma_k(v)$ ($k \in \mathbb{N}$) is empty so, by Theorem 3.13, $v$ is not stable. Similarly, let $u = ((a_1, b_1), \ldots, (a_5, b_5))$ be such that $S(u) = \{(1,3),(1,4),(2,3),(2,4),(4,1)\}$. Then $u = ((x,y),(x,b_2),(a_3,x),(y,x),(a_5,b_5))$ for some $x, y, a_3, a_5, b_2, b_5 \in [n]$ where possibly $a_3 = a_5$ and $b_2 = b_5$. Then $\Gamma_0(u)$ has directed edges $b_5 \to x, x \to b_2, b_2 \to y, y \to b_5$. So proceeding as above we see that $\Gamma_1(u)$ has directed edges $b_5 \to y, y \to b_5, y \to b_2, b_2 \to y$, and then that $\Gamma_2(u)$ is empty. Since $S(u^{\#}) = S(u)$ we have that the computation for $\Gamma_k(u^{\#})$ is exactly the same and thus by Corollary 3.12 also $\Gamma_2^{\#}(u)$ is empty so we conclude from Theorem 3.13 that $u$ is stable.

Let $w = ((a_1, b_1), \ldots, (a_r, b_r)) \in S([n]^2)$ be an $r$-cycle. We define a directed graph $S^+(w) := ([r], E)$ by letting $E := \{(i,j) \mid a_i = b_j \text{ or } a_{i+1} = b_j\}$ (where indices are modulo $r$). So, for example, if $w = ((1,4),(6,2),(7,2),(1,3),(2,4),(2,5))$ $\in S([7]^2)$ then $S^+(w)$ is the directed graph shown in Figure 4. Note that a comment similar to that made for $S(w)$ must be made here. Namely, $S^+(w)$ is

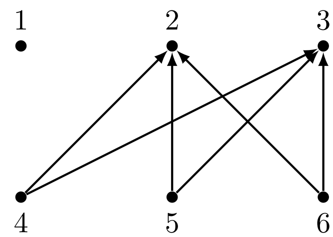

Figure 4: The directed graph $S^+(((1,4),(6,2),(7,2),(1,3),(2,4),(2,5)))$

defined for a "rooted" cycle (one in which an element is distinguished). However, all the graphs obtained from a cycle $w$ for different choices of the root differ only by a cyclic shift of the vertex set $[r]$. Since the properties that we are interested in do not change with such cyclic shifts we omit to write the dependence of $S^+(w)$ from the choice of root. We say that $S^+(w)$ is *acyclic* if it has no directed cycles (in particular, it has no loops and no antiparallel edges). Note that this property does not depend on the choice of root. For an $r$-cycle $u \in S([n]^2)$ we find it convenient to let $\ell(u) := r$.

**Theorem 8.5.** *Given cycles $u, v, w \in S([n]^2)$ such that $w = uv$, $R(u) \cap C(v) = \emptyset$, and $\ell(w) = \ell(u) + \ell(v) - 1$. Then $S^+(w)$ is acyclic if and only if both $S^+(u)$ and $S^+(v)$ are acyclic.*

*Proof.* Let $u = ((a_1, b_1), \ldots, (a_k, b_k))$, $v = ((a_k, b_k), \ldots, (a_r, b_r))$, and $w = ((a_1, b_1), \ldots, (a_r, b_r))$ $(1 < k < r)$. Note that, since $R(u) \cap C(v) = \emptyset$, there are no edges in $S^+(w)$ from $[k-1]$ to $[k, r]$ and there are no edges in $S^+(u)$ from $[k-1]$ to $k$. Therefore, $S^+(w)$ has a cycle if and only if it has a cycle on $[k-1]$ or on $[k, r]$. We consider both cases.

It is easy to see that $(i, j), i, j \in [k-1]$ is an edge in $S^+(w)$ if and only if $(i, j)$ is an edge in $S^+(u)$. Hence, $S^+(w)$ has a cycle on vertex set $[k-1]$ if and only if $S^+(u)$ has a cycle (both graphs do not have edges from $[k-1]$ to $k$).

Since $\{a_1, a_k\} \cap \{b_k, \ldots, b_r\} = \emptyset$, $(i, j), i, j \in [k, r]$ is an edge in $S^+(w)$ if and only if $(i, j)$ is an edge in $S^+(v)$. Hence, $S^+(w)$ has a cycle on vertex set $[k, r]$ if and only if $S^+(v)$ has a cycle. $\square$

**Corollary 8.6.** *Given a cycle $w \in S([n]^2)$. If $w$ is strongly stable then $S^+(w)$ is acyclic.*

*Proof.* This is easily proved by induction on $\ell(w)$. If $\ell(w) = 2$ then, by Theorem 8.1 of [3], $S^+(w)$ is empty. If $\ell(w) > 2$ then this follows immediately by induction from our definition of strong stability, Lemma 8.3 and Theorem 8.5. $\square$

For $\pi \in S_r$ we let

$$CS(\pi) := \{(i,j) \in [r]^2 \mid \max(\pi(i), \pi(i-1)) < \pi(j)\}$$

and

$$CS^{\#}(\pi) := \{(i,j) \in [r]^2 \mid \pi(i) > \max(\pi(j), \pi(j+1))\}$$

(where $\pi(0) := \pi(r)$ and $\pi(r+1) := \pi(1)$).

**Theorem 8.7.** *For an $r$-cycle $u \in S([n]^2)$ the following conditions are equivalent:*

*(1) $u$ is strongly stable;*

*(2) $u$ is strongly stable of rank $\leq r-1$;*

*(3) $S^+(u)$ is acyclic;*

*(4) $u$ is a compatible product of a stable transposition and a strongly stable $(r-1)$-cycle, in this order;*

*(5) $u$ has a compatible cyclic factorization of the form $u = t_{r-1} \underset{r-2}{\bullet} t_{r-2} \underset{r-3}{\bullet} \cdots \underset{1}{\bullet} t_1$ ($= t_{r-1} \bullet (t_{r-2} \bullet \ldots \bullet (t_3 \bullet (t_2 \bullet t_1)) \ldots)$) where $t_1, \ldots, t_{r-1}$ are stable transpositions;*

*(6) there exists $\pi \in S_r$ such that $S(u) \subseteq CS(\pi)$.*

*($1^{\#}$) $u^{\#}$ is strongly stable;*

*($2^{\#}$) $u^{\#}$ is strongly stable of rank $\leq r-1$;*

*($3^{\#}$) $S^+(u^{\#})$ is acyclic;*

*($4^{\#}$) $u$ is a compatible product of a strongly stable $(r-1)$-cycle and a stable transposition, in this order;*

*($5^{\#}$) $u$ has a compatible cyclic factorization of the form $u = t_1 \underset{1}{\bullet} t_2 \underset{2}{\bullet} \cdots \underset{r-2}{\bullet} t_{r-1}$ ($= (\ldots((t_1 \bullet t_2) \bullet t_3) \bullet \ldots \bullet t_{r-2}) \bullet t_{r-1}$) where $t_1, \ldots, t_{r-1}$ are stable transpositions;*

*($6^{\#}$) there exists $\pi \in S_r$ such that $S(u) \subseteq CS^{\#}(\pi)$.*

*Proof.* From the definition of strong stability and [3, Theorem 5.2 and Theorem 8.1] we immediately get (1) $\Leftrightarrow$ (2). By Corollary 8.6, we have (1) $\Rightarrow$ (3). Also, it is clear that (4) $\Rightarrow$ (1), and it is easy to see by induction on $r$ that (4) $\Leftrightarrow$ (5).

We now prove (3) $\Rightarrow$ (4). We proceed by induction on $r$. It is easy to check that if $r = 2$ and $S^+(u)$ is acyclic then $S^+(u) = \emptyset$ so, by [3, Theorem 8.1], $u$ is a stable transposition. Let $u = ((a_1, b_1), \ldots, (a_r, b_r))$ be an $r$-cycle such that $S^+(u)$ is acyclic, $r \geq 3$. Then we can find a sink $i_0 \in [r]$, i.e., $C(u) \cap \{a_{i_0}, a_{i_0+1}\} = \emptyset$. Hence, by [3, Proposition 5.15],

$$u = ((a_{i_0}, b_{i_0}), (a_{i_0+1}, b_{i_0+1}))\bullet((a_{i_0+1}, b_{i_0+1}), (a_{i_0+2}, b_{i_0+2}), \ldots, (a_{i_0+r-1}, b_{i_0+r-1})).$$

Therefore, by Theorem 8.5, we get that $S^+((a_{i_0}, b_{i_0}), (a_{i_0+1}, b_{i_0+1}))$ and $S^+(u')$ are acyclic, where $u' := ((a_{i_0+1}, b_{i_0+1}), (a_{i_0+2}, b_{i_0+2}), \ldots, (a_{i_0+r-1}, b_{i_0+r-1}))$. Hence, by the induction hypothesis, (4) holds for $((a_{i_0}, b_{i_0}), (a_{i_0+1}, b_{i_0+1}))$ and $u'$ so they are both strongly stable, so $u$ is a compatible product of a stable transposition and a strongly stable $(r-1)$-cycle, in this order.

This shows the equivalence of (1), (2), (3), (4), and (5).

We prove (5) $\Leftrightarrow$ (6) now. Let $u = ((a_1, b_1), \ldots, (a_r, b_r)) = (P_1, \ldots, P_r)$ be an $r$-cycle in $S([n]^2)$ and assume that there exist stable transpositions $t_1, \ldots, t_{r-1}$ such that $u = t_{r-1}(\ldots(t_3\ (t_2 t_1))\ldots)$ is a compatible cyclic factorization. Define inductively a permutation $\pi \in S_r$ associated to this factorization by

$$t_1 = (P_{\pi^{-1}(1)}, P_{\pi^{-1}(2)}),\ t_2 = (P_{\pi^{-1}(3)}, P_{\pi^{-1}(k_2)}), \ldots, t_{r-1} = (P_{\pi^{-1}(r)}, P_{\pi^{-1}(k_{r-1})}),$$

for uniquely determined $k_i \in [i]$, $i = 2, \ldots, r-1$. Of course, $P_{\pi^{-1}(1)}$ and $P_{\pi^{-1}(2)}$ are only defined up to a switch. Note also that $\pi$ depends on the rooted cycle $(P_1, \ldots, P_r)$ (i.e., on the writing of the cycle with $P_1$ as the first element). If we write the cycle in a different way (e.g., as $(P_2, \ldots, P_r, P_1)$) then we obtain a different permutation $\pi'$. However, it is easy to see that then $\pi' = \pi(r)\pi(1)\ldots\pi(r-1)$ in one-line notation. Setting $k_1 := 1$ we can write the above equations more compactly as

$$t_i = (P_{\pi^{-1}(i+1)}, P_{\pi^{-1}(k_i)}) \tag{23}$$

for all $i \in [r-1]$. Since the factorization is a compatible cyclic factorization by stable transpositions we have by [3, Theorem 8.1] that

$$\{a_{\pi^{-1}(i+1)}, a_{\pi^{-1}(k_i)}\} \cap \{b_{\pi^{-1}(i+1)}, b_{\pi^{-1}(k_i)}\} = \emptyset$$

(stability of $t_i$) for all $i \in [r-1]$ and by Lemma 8.3 that

$$\{a_{\pi^{-1}(i+1)}, a_{\pi^{-1}(k_i)}\} \cap \{b_{\pi^{-1}(1)}, \ldots, b_{\pi^{-1}(i)}\} = \emptyset$$

(compatibility of $t_i$ with $t_{i-1}(\ldots t_3(t_2t_1)\ldots)$) for all $i \in [2, r-1]$. These conditions can be rewritten as

$$\{a_{\pi^{-1}(i+1)}, a_{\pi^{-1}(k_i)}\} \cap \{b_{\pi^{-1}(1)}, b_{\pi^{-1}(2)}, \ldots, b_{\pi^{-1}(i+1)}\} = \emptyset \tag{24}$$

for all $i \in [r-1]$. We now show that (24) implies (6). Let $(i,j) \in [r]^2$. Suppose that $a_i = b_j$. Then, by (24), $\pi(j) > \pi(i)$. It remains to show that $\pi(j) > \pi(i-1)$ as well. Assume first that $i > 1$. Let, for brevity, $h := \pi(i-1)$. If $h = 1$ then $\pi(i-1) = 1 < \pi(i) < \pi(j)$. If $h > 1$ then, by (23), $t_{h-1} = (P_{i-1}, P_{\pi^{-1}(k_{h-1})})$. Since $u(P_{i-1}) = P_i$ there is some $s \in [h-1, r-1]$ such that $t_s$ contains $P_i$. If $t_s = (P_{\pi^{-1}(s+1)}, P_i)$, then $i = \pi^{-1}(k_s)$. Since $a_{\pi^{-1}(k_s)} = a_i = b_j$ then, by (24), $\pi(j) > s+1 \geq h = \pi(i-1)$. If $t_s = (P_i, P_{\pi^{-1}(k_s)})$ then $i = \pi^{-1}(s+1)$. Since $a_i = b_j$ then, by (24), $\pi(j) > \pi(i) = s+1 \geq h = \pi(i-1)$. If $i = 1$ then the reasoning is exactly the same except that "$i-1$" should be replaced by "$r$".

Conversely, assume that (6) holds. Let

$$t_i := (P_{\pi^{-1}(i+1)}, P_{\pi^{-1}(k_i)})$$

for $i \in [r-1]$, where

$$\pi^{-1}(k_i) := \min\{c \in [\pi^{-1}(i+1)+1, \pi^{-1}(i+1)+r-1] : \pi(c) < i+1\} \tag{25}$$

for $i \in [r-1]$ (where $\pi(j) := \pi(j+r)$ for all $j \in \mathbb{Z}$). Note that this implies that

$$\pi(c) \geq i+1 \tag{26}$$

for $\pi^{-1}(i+1) \leq c < \pi^{-1}(k_i)$ and $k_i \in [i]$ $(i \in [r-1])$.

We first prove that $t_{r-1}(\ldots(t_3(t_2t_1))\ldots)$ is a cyclic factorization of $(P_1, \ldots, P_r)$. More precisely, we claim that

$$t_{j-1}(\ldots(t_3(t_2t_1))\ldots) = (P_1, \ldots, \widehat{P_{a_1}}, \ldots, \widehat{P_{a_{r-j}}}, \ldots, P_r), \tag{27}$$

for all $2 \leq j \leq r$ where $\{a_1, \ldots, a_{r-j}\}_< := \{\pi^{-1}(j+1), \ldots, \pi^{-1}(r)\}$. We prove this claim by induction on $j \geq 2$, (27) being clear if $j = 2$. So let $3 \leq j \leq r-1$ and assume that (27) holds. Let $a_\ell := \pi^{-1}(j+1)$ (so $\ell \in [r-j]$). Then, by (25),

$$\begin{aligned} \pi^{-1}(k_j) &= \min\{c \in [a_\ell+1, a_\ell+r-1] \ : \ \pi(c) < j+1\} \\ &= \min_{\prec}([r] \setminus \{a_1, \ldots, a_{r-j}\}) \end{aligned}$$

where the second minimum is taken with respect to the order $a_\ell \prec a_\ell + 1 \prec \cdots \prec r \prec 1 \prec \cdots \prec a_\ell - 1$. Therefore,

$$\begin{aligned} t_j(\ldots(t_3(t_2t_1))\ldots) &= (P_{a_\ell}, P_{\pi^{-1}(k_j)})\ (P_1, \ldots, \widehat{P_{a_1}}, \ldots, \widehat{P_{a_{r-j}}}, \ldots, P_r) \\ &= (P_1, \ldots, \widehat{P_{a_1}}, \ldots, \widehat{P_{a_{\ell-1}}}, \ldots, \widehat{P_{a_{\ell+1}}}, \ldots, \widehat{P_{a_{r-j}}}, \ldots, P_r) \end{aligned}$$

which proves our claim since $\{a_1, \dots a_{\ell-1}, a_{\ell+1}, \dots, a_{r-j}\}_< = \{\pi^{-1}(j+2), \dots, \pi^{-1}(r)\}$. Note that up to this point we have not used our hypothesis (6) but only that $\pi \in S_r$.

Now, observe that (6) is equivalent to

$$a_i = b_j \Rightarrow \pi(i) < \pi(j) \quad \text{and} \quad \pi(i-1) < \pi(j) \tag{28}$$

for all $(i,j) \in [r]^2$.

Finally, we claim that (28) implies that

$$\{a_{\pi^{-1}(i+1)}, a_{\pi^{-1}(k_i)}\} \cap \{b_{\pi^{-1}(1)}, b_{\pi^{-1}(2)}, \dots, b_{\pi^{-1}(i+1)}\} = \emptyset$$

for all $i \in [r-1]$. Indeed, if $a_{\pi^{-1}(i+1)} = b_{\pi^{-1}(j)}$ for some $j \in [i+1]$ then by (28) we have that $i+1 < j$, which is a contradiction. Similarly, if $a_{\pi^{-1}(k_i)} = b_{\pi^{-1}(j)}$ for some $j \in [i+1]$ then by (26) we have that $\pi(\pi^{-1}(k_i) - 1) \geq i+1$ while, by (28), we conclude that $\pi(\pi^{-1}(k_i) - 1) < j$, also a contradiction. This, by [3, Theorem 8.1] and Lemma 8.3, shows that (27) is a compatible factorization by stable transpositions.

Using [4, Proposition7.2] and [3, Theorem 8.1], it is easy to see that (1) $\Rightarrow$ ($1^\#$), so (1) $\Leftrightarrow$ ($1^\#$). The statements ($2^\# - 6^\#$) correspond, using [4, Proposition7.2] and (17), to $(2-6)$ for $u^\#$, hence they are also equivalent. □

We illustrate the previous theorem and proof with some examples.

**Example 8.8.** Let $r = 9$ and consider $\pi = 392175846$. Then one can easily compute $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$, $k_5 = 3$, $k_6 = 5$, $k_7 = 4$ and $k_8 = 2$. Therefore, $t_1 = (P_3, P_4)$, $t_2 = (P_1, P_3)$, $t_3 = (P_8, P_1)$, $t_4 = (P_6, P_8)$, $t_5 = (P_9, P_1)$, $t_6 = (P_5, P_6)$, $t_7 = (P_7, P_8)$ and $t_8 = (P_2, P_3)$, and we obtain

$$\begin{aligned}
t_8(t_7(t_6(t_5(t_4(t_3(t_2 t_1)))))) &= t_8(t_7(t_6(t_5(t_4(t_3(P_3, P_4, P_1)))))) \\
&= t_8(t_7(t_6(t_5(t_4(P_1, P_3, P_4, P_8))))) \\
&= t_8(t_7(t_6(t_5(P_1, P_3, P_4, P_6, P_8)))) \\
&= t_8(t_7(t_6(P_1, P_3, P_4, P_6, P_8, P_9))) \\
&= t_8(t_7(P_1, P_3, P_4, P_5, P_6, P_8, P_9)) \\
&= t_8(P_1, P_3, P_4, P_5, P_6, P_7, P_8, P_9) \\
&= (P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9)\ .
\end{aligned}$$

**Example 8.9.** Let $r = 6$ and consider the factorization

$$(P_1, \dots, P_6) = (P_4, P_5)((P_2, P_3)((P_3, P_5)((P_6, P_1)(P_1, P_5)))).$$

It is easy to check that this is a cyclic factorization and one obtains either $\pi^{-1} = 156324$ or $\pi^{-1} = 516324$.

**Example 8.10.** If $w = ((1,4),(6,2),(7,2),(1,3),(2,4),(2,5)) \in S([7]^2)$ then (see Figure 4) $S^+(w)$ is acyclic so, by Theorem 8.7, $w$ is strongly stable. In particular, $w$ is stable.

For $\pi \in S_r$ there are at most $\frac{r!}{2}$ sets of the form $CS(\pi)$ and at most $\frac{r!}{2}$ of the form $CS^{\#}(\pi)$ (because $\pi$ and $(1,2)\pi$ give the same sets). Note that, if $r \geq 4$, the sets obtained (for all $\pi \in S_r$) in conditions (6) and ($6^{\#}$) are not the same (see also below) so conditions (6) and ($6^{\#}$) are not identical. However, if one considers only the maximal sets under inclusion (for $\pi \in S_r$) then the two families of sets obtained in (6) and ($6^{\#}$) do coincide, as we now show.

Let $R \subseteq [r]^2$. We say that $R$ is *acceptable* if there is a a cyclic factorization of the cycle $(P_1, \ldots, P_r) = t_1 \underset{a_1}{\cdot} t_2 \underset{a_2}{\cdot} \cdots \underset{a_{r-2}}{\cdot} t_{r-1}$ such that:

1. if at some point in the multiplication sequence $P_i$ is in the left cycle and $P_j$ is in the right one then $(i,j) \notin R$;
2. if $(P_i, P_j) = t_k$ for some $k \in [r-1]$ then $(i,j),(j,i) \notin R$.

Then reasoning exactly as in the proof of Theorem 8.7 we have the following.

**Lemma 8.11.** *Let $R \subseteq [r]^2$. Then the following are equivalent:*

1. *$R$ is acceptable;*
2. *there is $\pi \in S_r$ such that $R \subseteq CS(\pi)$;*
3. *there is $\pi \in S_r$ such that $R \subseteq CS^{\#}(\pi)$.*

*Proof.* The proof of (1) $\Leftrightarrow$ (2) is exactly the same as the proof of (5) $\Leftrightarrow$ (6) in Theorem 8.7, because we used there only properties 1. and 2. above. The proof of (1) $\Leftrightarrow$ (3) corresponds to (1) $\Leftrightarrow$ (2) under the action of $\#$. $\square$

**Corollary 8.12.** *For any $r$, the set of maximal sets in* (6) *and the set of maximal sets in* ($6^{\#}$) *coincide.*

For example, let $r = 4$. Then the 12 sets $CS(\pi)$ are

$$\{(j,j+1),(j,j+2),(j+1,j+2)\}, \tag{29}$$

$$\{(j,j+1),(j,j+2),(j-1,j+1)\}, \tag{30}$$

(where $j \in [4]$ and numbers are modulo 4), and

$$\{(1,2),(4,2)\},\{(1,3),(2,3)\},\{(2,4),(3,4)\},\{(3,1),(4,1)\}$$

while the 12 sets $CS^{\#}(\pi)$ are the ones in (29), (30), and

$$\{(1,2),(1,3)\},\{(2,3),(2,4)\},\{(3,1),(3,4)\},\{(4,1),(4,2)\}.$$

The maximal ones are, in both cases, the ones in (29) and (30).

Given its ubiquitous appearance in the previous results we feel that, given a permutation $\pi \in S_r$, the set $CS(\pi)$ could be worthy of further investigation. We propose calling $CS(\pi)$ the *connectivity set* of the permutation $\pi$. The reason for this terminology lies in the fact that

$$\begin{aligned} |CS(\pi)| &= \sum_{a=2}^{r} |\{i \in [r] : \max(\pi(i), \pi(i-1)) < a\}| \\ &= \binom{r}{2} - \sum_{a=2}^{r} |\{\text{connected components of } \pi^{-1}([a-1])\}| \end{aligned} \tag{31}$$

where the numbers from 1 to $r$ are arranged in order around a circle, and *conncected component* has the obvious meaning.

We note the following simple property of the connectivity set of a permutation that follows easily from (31). Recall that a valley of a permutation $\pi \in S_r$ is an index $i \in [r]$ such that $\pi(i-1) > \pi(i) < \pi(i+1)$, where $\pi(0) := \pi(r)$ and $\pi(r+1) := \pi(1)$.

**Proposition 8.13.** *If $\pi \in S_r$ then*

$$|CS(\pi)| \leq \binom{r-1}{2}.$$

*Furthermore, equality holds if and only if $\pi$ has exactly one valley.*

We point out an interesting consequence of Theorems 8.5 and 8.7.

**Corollary 8.14.** *Given cycles $u, v, w \in S([n]^2)$ such that $w = uv$, $R(u) \cap C(v) = \emptyset$, and $\ell(w) = \ell(u) + \ell(v) - 1$. Then $w$ is strongly stable if and only if $u$ and $v$ are both strongly stable.*

# 9 Stable quadratic $r$-cycles, $r \leq 5$

Using the results of the previous section and [3, 4], we are able to present a very clear picture for the stability of $r$-cycles in $S([n]^2)$, up to $r = 5$.

As corollaries of Theorem 8.7, we get explicit characterizations of the strongly stable 2, 3, 4, and 5-cycles. The proofs are a straightforward check.

**Corollary 9.1.** *A transposition $u \in S([n]^2)$ is strongly stable if and only if $S(u) = \emptyset$.*

*Proof.* This follows immediately from Theorem 8.7 since $CS(\pi) = \emptyset$ for all $\pi \in S_2$. $\square$

**Corollary 9.2.** *Let $u \in S([n]^2)$ be a 3-cycle. Then $u$ is strongly stable if and only if $S(u)$ is $\{(1,2)\}$, $\{(2,3)\}$, $\{(3,1)\}$, or empty.*

*Proof.* Again, this follows immediately from Theorem 8.7 by computing $CS(\pi)$ for all (even) permutations $\pi \in S_3$. $\square$

The next corollary follows in the same way using the example computed after Corollary 8.12.

**Corollary 9.3.** *Let $u \in S([n]^2)$ be a 4-cycle. Then $u$ is strongly stable if and only if $S(u)$ is contained in any one of the following sets for some $j \in [4]$:*

*(1)* $\{(j, j+1), (j, j+2), (j+1, j+2)\}$;

*(2)* $\{(j, j+1), (j, j+2), (j-1, j+1)\}$;

*where numbers are modulo* 4.

Carrying out a similar computation (preferably with the aid of a computer) for the (even) permutations of $S_5$ one obtains the following characterization.

**Corollary 9.4.** *Let $u \in S([n]^2)$ be a 5-cycle. Then $u$ is strongly stable if and only if $S(u)$ is contained in any one of the following sets for some $j \in [5]$:*

*(1)* $\{(j+2, j+3), (j+2, j+4), (j+2, j), (j+3, j+4), (j+3, j), (j+4, j)\}$;

*(2)* $\{(j+3, j+4), (j+3, j+1), (j+2, j), (j+4, j+1), (j+3, j), (j+4, j)\}$;

*(3)* $\{(j+3, j+1), (j+4, j+1), (j+2, j), (j+4, j+2), (j+3, j), (j+4, j)\}$;

*(4)* $\{(j+2, j+4), (j+3, j+4), (j+2, j), (j+3, j+1), (j+3, j), (j+4, j)\}$;

*where numbers are modulo* 5.

**Remark 9.5.** Note that these sets $S(u)$, for $u$ a strongly stable cycle, are usually proper subsets. For example, there is no 4-cycle $u \in S([n]^2)$ such that $S(u) = \{(j, j+1), (j, j+2), (j+1, j+2)\}$ (otherwise $a_{j+1} = b_{j+2} = a_j = b_{j+1}$, which implies $(j+1, j+1) \in S(u)$).

So, the 20 sets in the previous corollary are the maximal ones among the 50 sets of the form $CS(\pi)$ for all the (even) permutations $\pi \in S_5$, they are also the only ones of cardinality 6 (the other 30 all have size either 4 or 5).

**Remark 9.6.** One can check that the $r \cdot 2^{r-3}$ sets $CS(\pi)$ corresponding to the $r \cdot 2^{r-3}$ even permutations $\pi \in S_r$ that have exactly one valley are all distinct (and they are all maximal). For $r \leq 5$, there are no other maximal sets. However, this is not true in general. For $r \geq 6$, there are at least $r \cdot 2^{r-3}$ maximal sets of size $\binom{r-1}{2}$ but also maximal sets of smaller sizes.

**Remark 9.7.** If an $r$-cycle $u = (P_1, \ldots, P_r) \in S([n]^2)$, where $P_i := (a_i, b_i)$ for all $i \in [r]$, is such that $S(u) \subseteq CS(\pi)$ for some $\pi \in S_r$ that has exactly one valley then $u$ can be written as a compatible product of two strongly stable cycles of lengths $a$ and $r-a+1$, for *any* $2 \leq a \leq r-1$. In fact, since both $u$ and $\pi$ can be rotated cyclically (see also the comments preceding equation (23) above) we may assume that $\pi$ has exactly one valley and $\{\pi(r-a+1), \ldots, \pi(r)\} = [a]$. This implies that $CS(\pi) \cap ([r-a+1] \times [r-a+1, r]) = \emptyset$. Hence, since $S(u) \subseteq CS(\pi)$, $a_i \neq b_j$ if $i \in [r-a+1]$ and $j \in [r-a+1, r]$ and this, by [3, Proposition 5.15], implies that $(P_1, \ldots, P_r) = (P_1, \ldots, P_{r-a+1})(P_{r-a+1}, \ldots, P_r)$ is a compatible product. Furthermore, the two cycles on the right of this equation are strongly stable by Corollary 8.14. One can check that this one-valley condition holds, by Corollaries 9.2, 9.3, and 9.4, for any strongly stable $r$-cycle if $3 \leq r \leq 5$ (cf. also Proposition 9.10 below).

Corollaries 9.1 and 9.2 show, by [3, Theorem 8.1] and [4, Theorem 5.12], that transpositions and 3-cycles are stable if and only if they are strongly stable. This is also true for 4-cycles, but not for 5-cycles.

**Theorem 9.8.** *For an $r$-cycle $u \in S([n]^2)$, $r \leq 5$, the following conditions are equivalent:*

*(1) $u$ is stable;*

*(2) $u$ is stable of rank $\leq r-1$;*

*(3) $u$ is a compatible product of $r-1$ stable transpositions, in some order;*

*(4) $u$ is a compatible product of $r-1$ stable transpositions $t_i$ ($i \in [r-1]$) of the form $u = t_{r-1} \underset{r-2}{\bullet} t_{r-2} \underset{r-3}{\bullet} \cdots \underset{1}{\bullet} t_1$ $(= t_{r-1} \bullet (t_{r-2} \bullet \ldots \bullet (t_3 \bullet (t_2 \bullet t_1)) \ldots))$;*

*(5) either $u$ is strongly stable or*

$r = 5$ *and* $S(u) = \{(j, j+2), (j, j+3), (j+1, j+2), (j+1, j+3), (j+3, j)\}$

*for some $j \in [5]$, where numbers are modulo 5.*

*Proof.* We have checked (5) $\Leftrightarrow$ (1) by a program written in $C$++ using Theorem 3.13 and Corollary 9.4. The main idea is based on the fact that the property of stability of $r$-cycle $((a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r))$ depends only on the set partition of $\{a_1, b_1, a_2, b_2, a_3 \ldots, a_r, b_r\}$ induced by equalities, as follows immediately from the definition of stability.

The implications (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are obvious. The implication (3) $\Rightarrow$ (2) holds by [3, Theorem 5.2].

It remains to check (5) $\Rightarrow$ (4). By Theorem 8.7 we know that (5) $\Rightarrow$ (4) for all strongly stable cycles. Let $u$ be an exceptional case from (5). Therefore $u = ((a, c), (a, d), (b, a), (c, a), (e, f))$, where $a, b, c, d, e, f \in [n]$ are all distinct, except possibly $b = e$ and $d = f$. Then

$$((b, a), (e, f)) \bullet \Big( ((a, c), (a, d)) \bullet \big( ((b, a), (c, a)) \bullet ((a, d), (e, f)) \big) \Big) \tag{32}$$

is a compatible product of stable transpositions, as a consequence of [3, Propositions 5.6, 5.7 and Theorem 8.1]. □

The careful reader will have noticed that the factorization in (32) is not cyclic (see also Figure 5 for a special case). Indeed, $u$ is not strongly stable since it does not satisfy any of the conditions in Corollary 9.4.

As a consequence of Theorem 9.8 we can prove a special case of Conjecture 6.10 for short cycles.

**Corollary 9.9.** *Let $u, v \in S([n]^2)$ be two $r$-cycles such that $S(u) = S(v)$ and $r \leq 5$. Then $u$ is stable if and only if $v$ is stable.*

*Proof.* By Theorem 8.4, $u$ is strongly stable if and only if $v$ is strongly stable. This, in turn, by Theorem 9.8, implies that if $r \leq 5$, then $u$ is stable if and only if $v$ is stable. □

We have one more interesting property of short strongly stable cycles.

**Proposition 9.10.** *Let $w$ be an $r$-cycle, $3 \leq r \leq 5$ and $a \in [2, r-1]$. Then $w$ is strongly stable if and only if $w$ is a compatible product of a strongly stable $a$-cycle and a strongly stable $(r+1-a)$-cycle, in this order.*

*Proof.* One direction is immediate by definition (for any $r \geq 3$). Conversely, assume that $w := ((a_1, b_1), \ldots, (a_r, b_r)) \in S([n]^2)$ is strongly stable. By Theorem 8.7 we know the result for $a = 2$ or $a = r-1$. It remains the case $r = 5$ and $a = 3$. Then, by Corollary 9.4, we have that $\{a_k, a_{k+1}, a_{k+2}\} \cap \{b_{k+2}, b_{k+3}, b_{k+4}\} = \emptyset$ where $k = j$ in cases (2) and (3), and $k = j-1$ in cases (1) and (4). Hence, by [3, Proposition 5.15], $w$ is a compatible product of two 3-cycles. The conclusion now follows from Corollary 8.14. □

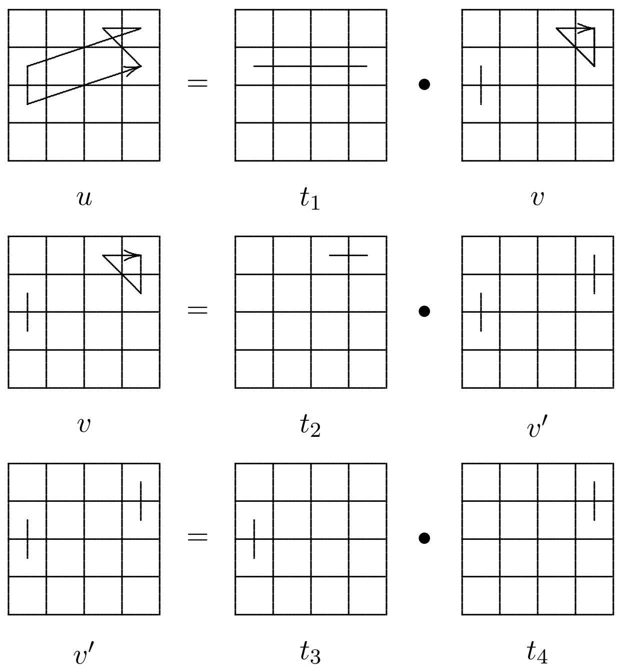


Figure 5: An example of a stable, but not strongly stable, 5-cycle $u \in S([4]^2)$ written as a compatible product of 4 stable transpositions, $u = t_1 \bullet (t_2 \bullet (t_3 \bullet t_4))$.

**Remark 9.11.** The previous proposition does not hold for large cycles. For example, consider a $6k$-cycle $u = (P_1, \dots, P_{6k})$ such that $S(u) = \{(k, 4k), (3k, 6k), (5k, 2k)\}$. It is easy to see that, for $k \geq 2$, $S^+(u)$ is acyclic, so $u$ is strongly stable by Theorem 8.7. Suppose now that $u = v \bullet w$ where $v = (P_{i-3k+1}, \dots, P_i)$, $w = (P_i, \dots, P_{i+3k})$, $v$ and $w$ are strongly stable, and the product is compatible ($i \in [6k]$, and the indices are modulo $6k$). Then, by Corollary 7.8, $R(v) \cap C(w) = \emptyset$, which implies that $P_k$, $P_{3k}$, and $P_{5k}$ must be in the right cycle $w$, which contradicts the fact that $w$ is a $(3k+1)$-cycle.

It is known that the number of stable transpositions in $S([n]^2)$ is $\frac{1}{2}(n)_2(n-1)(n-2)$ and the number of stable 3-cycles is $\frac{1}{3}(n)_3(n^3 - 3n^2 - 2n + 9)$ see [3, Corollary 8.3] and [4, Corollary 5.13]. Theorem 9.8 gives a method to count the number of stable cycles of length 4 and 5.

**Proposition 9.12.** *The number of stable 4-cycles in* $S([n]^2)$ *is*

$$\frac{1}{4}(n)_4\,(n^4 - 2n^3 - 13n^2 + 40n - 18).$$

*Proof.* By [4, Proposition 4.5], the number of stable 4-cycles of rank one is

$$\begin{aligned}
&3!\sum_{i=1}^{4}\binom{n}{i}\sum_{(a_1,\dots,a_i)\in\mathcal{C}_i(4)}\prod_{j=1}^{i}\binom{n-i}{a_j} \\
&= 3!\Big\{\binom{n}{1}\binom{n-1}{4} + 2\binom{n}{2}\binom{n-2}{1}\binom{n-2}{3} + \binom{n}{2}\binom{n-2}{2}^2 \\
&\quad + 3\binom{n}{3}\binom{n-3}{1}^2\binom{n-3}{2} + \binom{n}{4}\binom{n-4}{1}^4\Big\} \\
&= n(n-1)(n-2)(n-3) \times [(n-4)/4 + (n-2)(n-4) + 3(n-2)(n-3)/4 \\
&\quad + 3(n-3)^2(n-4)/2 + (n-4)^4/4],
\end{aligned}$$

where $\mathcal{C}_i(4)$ denotes the set of all compositions of 4 in $i$ parts. Also, by [4, Theorems 3.1 and 6.2, and Proposition 6.3], the number of stable 4-cycles of rank at least 2, which are compatible products of a stable transposition and a stable 3-cycle, in this order, is $2(n)_4(n^3 - 7n^2 + 17n - 13)$. By Theorems 8.7 and 9.8 (see also Proposition 9.10), the number of stable 4-cycles is the sum of these two figures. $\square$

So, for example, the number of stable 4-cycles in $S([n]^2)$, for $4 \leq n \leq 7$, is $372, 6960, 55620$, and $281400$, respectively.

It is easy to see that the number of exceptional 5-cycles in $S([n]^2)$ described in Theorem 9.8 is given by

$$(n)_6 + 2(n)_5 + (n)_4 = (n)_4 \left[n^2 - 7n + 13\right] .$$

Using Corollary 9.4 one can count the number of strongly stable 5-cycles in $S([n]^2)$, and thus obtain the number of stable 5-cycles in $S([n]^2)$. This computation, however, involves a considerably higher number of sets, and hence is quite long. We therefore proceed in a different way with the help of a computer.

**Proposition 9.13.** *The number of stable* 5*-cycles in* $S([n]^2)$ *is*

$$29(n)_4 + 318(n)_5 + \left(595 + \frac{2}{5}\right)(n)_6 + 354(n)_7 + 80(n)_8 + 7(n)_9 + \frac{1}{5}(n)_{10}.$$

*Proof.* The idea is to split the stable cycles $((a_1, b_1), \ldots, (a_5, b_5)) \in S([n]^2)$ according to the cardinality of $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ and then reduce the computation to a finite number of cases. For any $k \in [10]$, the number of stable 5-cycles in $S([n]^2)$ such that $|\{a_1, \ldots, a_5, b_1, \ldots, b_5\}| = k$ is given by $C_k\binom{n}{k}$ where $C_k$ is the number of stable 5-cycles in $S([n]^2)$ such that $\{a_1, \ldots, a_5, b_1, \ldots, b_5\} = [k]$. By Theorem 7.8 of [3] $C_k$ equals the number of stable 5-cycles in $S([k]^2)$ such that $\{a_1, \ldots, a_5, b_1, \ldots, b_5\} = [k]$, i.e., $C_k$ does not depend on $n$ (this also follows from Theorem 3.5). So the total number of stable 5-cycles in $S([n]^2)$ is

$$C_1\binom{n}{1} + \ldots + C_9\binom{n}{9} + C_{10}\binom{n}{10} = \frac{C_1}{1!}(n)_1 + \ldots + \frac{C_9}{9!}(n)_9 + \frac{C_{10}}{10!}(n)_{10}.$$

The result then follows by a brute force $C$++ calculation using Theorem 3.13. □

**Remark 9.14.** Note that, for all $k \neq 6, 10$, $C_k$ is divisible by $k!$. On the other hand, $\frac{2}{5}(n)_6$ corresponds to permutations of the following two types

$$((a, b_1), (a, b_2), (a, b_3), (a, b_4), (a, b_5)) \in S([n]^2), \text{ where } |\{a, b_1, b_2, b_3, b_4, b_5\}| = 6$$

and

$$((a_1, b), (a_2, b), (a_3, b), (a_4, b), (a_5, b)) \in S([n]^2), \text{ where } |\{a_1, a_2, a_3, a_4, a_5, b\}| = 6;$$

and $\frac{1}{5}(n)_{10}$ corresponds to permutations of the following type

$$((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5)) \in S([n]^2),$$

where $|\{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5\}| = 10$.

We close this section by recording the following simple albeit interesting observation.

**Corollary 9.15.** *For all integers $t$ and $a_1, \dots, a_k$ ($k \in \mathbb{N}$) larger than 1, the number of stable permutations $u \in S([n]^t)$ with cycle-type $(a_1, \dots, a_k, \underbrace{1, \dots, 1}_{n^t - \sum a_i})$ is a polynomial in $n$.*

The proof follows the same lines as that of Proposition 9.13, mutatis mutandis.

# 10 What's next and conjectures

In this section we discuss some conjectures arising from this work and some possible directions of further research.

We know that the rank of any stable $r$-cycle in $S([n]^2)$ is bounded from above by $2r - 1$, see Corollary 5.3. We think that the sharp bound should be twice smaller.

**Conjecture 10.1.** *The rank of any stable $r$-cycle in $S([n]^2)$ is bounded from above by $r - 1$.*

By definition a strongly stable cycle in $S([n]^2)$ has a cyclic compatible factorization in stable transpositions. While this is not true for stable cycles (see Theorem 9.8) we do feel that the following holds.

**Conjecture 10.2.** *Every stable $r$-cycle in $S([n]^2)$ is a compatible product of $r - 1$ stable transpositions in some order.*

By [3, Theorem 5.2], a proof of the second conjecture would settle the first one as well. By Theorem 9.8 these conjectures hold for $r \leq 5$.

There are a number of natural possible directions for further work on the problems addressed in this paper. Among these, we mention

- Continue the investigation of stable $r$-cycles for $r \geq 6$.
- Decide Conjecture 6.10 and/or whether the stability of a cycle $u \in S([n]^2)$ depends only on $S(u)$.
- Examine the stability of cycles in $S([n]^t)$ for $t > 2$.
- Examine the stability of permutations that are not cycles. The first cases to consider could be the product of two disjoint transpositions and the product of a transposition with a 3-cycle.

- Examine the case of stable involutions.

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