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Complementary Proof Nets for Classical Logic

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Abstract. A complementary system for a given logic is a proof system whose theorems are exactly the formulas that are not valid according to the logic in question. This article is a contribution to the complementary proof theory of classical propositional logic. In particular, we present a complementary proof-net system, CPN, that is sound and complete with respect to the set of all classically invalid (one-side) sequents. We also show that cut elimination in CPN enjoys strong normalization along with strong confluence (and, hence, uniqueness of normal forms).

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1. Introduction

A *complementary system* for a given logic $\mathcal{L}$ is a proof system whose theorems are exactly those formulas in $\mathcal{L}$'s language that are not valid according to $\mathcal{L}$. By extension, if $\mathcal{S}$ is any proof system that is sound and complete with respect to $\mathcal{L}$-validity, a complementary system for $\mathcal{S}$ is a proof system $\overline{\mathcal{S}}$ in the same language such that, for every well-formed formula A, $\overline{\mathcal{S}}$ proves A if and only if A is not provable in $\mathcal{S}$. Thus defined, complementary systems belong to the broad category of so-called *refutation calculi* [10, 27] (also known as *rejection calculi* [33]). Generally speaking, however, such calculi are not bound to soundness, since a refutation rule may be designed so as to infer something unprovable by combining provable and unprovable premises [25, 26]. For example, the original refutation calculus of Łukasiewicz [15] includes the "reverse modus ponens" rule: If $A \to B$ is provable and B is unprovable, then A is unprovable. Complementary systems are to be understood more strictly as refutation calculi in which such "mixed" patterns of inference are

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banished altogether; all inference rules are required to be soundness-preserving. In other words, they are *pure* refutation calculi [24]. The results presented in this paper are meant as a contribution to the structural proof theory for such systems. Specifically, we shall be concerned with complementary systems for classical propositional logic, hence proof systems that are sound and complete with respect to the set of classical non-tautologies (contradictions and contingencies). There are several systems of this sort in the literature, both Hilbert-style [3,37,38] and Gentzen-style [2,4,9,33,35] (see Pulcini and Varzi [21] for overviews and comparisons). Here we propose a new *proof-net* complementary system whose mathematical properties are significantly stronger.

Proof nets were originally introduced by Jean–Yves Girard with the intent to provide an alternative, more perspicuous syntax for linear logic [7,8]. Later developments have led to numerous variants, and proof-net systems are now available for a larger range of non-classical logics as well as for classical logic [11,14,16,17,22,23,31]. In general, proof nets are distinguished by the fact that they do not require the rigid sequentiality imposed by familiar sequent calculi; one can represent sequent proofs more freely modulo trivial permutations of the rules. As a consequence, proof nets usually induce nice properties that may fail in a standard sequent setting. This is true also of our complementary proof-net system. Particularly, we will show that cut elimination on complementary proof nets enjoys two major properties that are missing in a complementary sequent calculus of the sort pioneered by Tiomkin [35] and Goranko [9]: *strong normalization* (to the effect that every reduction strategy leads to a normal form) and *confluence*, indeed *strong confluence* (so that any two normalizing reductions will lead to the same form) [8,29,34].

With these results, we also aim to support three different philosophical thoughts. First of all, any proof-net system can be seen as a multi-conclusion natural-deduction system. In particular, the proof-net system that we introduce in Sect. 3 can be seen as a complementary natural-deduction system for classical logic alternative to the Fitch-style formalism proposed by Tamminga [33]. In the second place, given a proof system $\mathcal{S}$, any complementary $\overline{\mathcal{S}}$ can be thought of as delivering a semantic characterization of $\mathcal{S}$ itself. Any formula in the language can be interpreted as the set of its $\overline{\mathcal{S}}$-proofs, and it will count as an $\mathcal{S}$-theorem if and only if it is interpreted as the empty set. Among other things, this licenses perspicuous renderings of important metalogical notions. For instance, $\mathcal{S}$'s soundness effectively amounts to the conditional: *if* $\vdash_{\mathcal{S}} A$, *then* $\nvdash_{\overline{\mathcal{S}}} A$, i.e., anything that is provable by $\mathcal{S}$ cannot be refuted. Since this, in turn, amounts to denying the possibility that $\vdash_{\mathcal{S}} A$ and $\vdash_{\overline{\mathcal{S}}} A$, such a formulation clarifies the somewhat opaque relationship between soundness and consistency. Likewise, the converse conditional amounts to the completeness of $\mathcal{S}$: *if* $\nvdash_{\overline{\mathcal{S}}} A$, *then* $\vdash_{\mathcal{S}} A$, i.e., $\mathcal{S}$ *proves every formula that is non-refutable.* From this perspective, which is not available in "mixed" refutation calculi, any further improvement in the study of $\overline{\mathcal{S}}$'s syntax turns into an improvement in our semantic grasp of $\mathcal{S}$, and this can be especially rewarding when complementary proofs are graphs, i.e., structures having a clear mathematical *raison d'être* prior to their logical interpretation. The third and final thought concerns the fact that any complementary system $\overline{\mathcal{S}}$ can be seen as a mirror image of the complemented system $\mathcal{S}$ and, hence, as identifying the same logic, albeit "in the negative" [20]. In a completely decidable setting, such as classical propositional logic, one can actually establish that a formula A is theorem of a sound and complete proof system $\mathcal{S}$ just by excluding the possibility of an $\overline{\mathcal{S}}$-derivation ending in A. Accordingly, the availability of a proof-net system for complementary classical propositional logic may suggest novel ways to conceive and design proof nets also for classical propositional logic as normally understood, i.e., "in the positive".

2. Background: The Complementary Antisequent Calculus $\overline{\mathsf{GS4}}$

2.1. Basic Facts

A proof-net system is usually based on a suitable sequent calculus, typically a one-sided calculus with rules dealing exclusively with sequents of the form $\vdash \Gamma$ [36]. Here we focus on one-sided calculi formulated *à la* Tait [32], whose language includes only the binary connectives for conjunction ($\wedge$) and disjunction ($\vee$); negation ($\perp$) comes as a primitive on atomic formulas and is extended to compound formulas in the following way:

$$(A^{\perp})^{\perp} \equiv A, \quad (A \wedge B)^{\perp} \equiv A^{\perp} \vee B^{\perp}, \quad (A \vee B)^{\perp} \equiv A^{\perp} \wedge B^{\perp}.$$

More precisely, this means we take the set of atoms, $\mathcal{A}$, to be comprised of all literals p, $p^{\perp}$, q, $q^{\perp}, \ldots$, and the set of formulas, $\mathcal{F}$, is defined recursively by means of the following grammar:

$$\mathcal{F} ::= \mathcal{A} \,|\, \mathcal{F} \wedge \mathcal{F} \,|\, \mathcal{F} \vee \mathcal{F}.$$

We shall use capital Greek letters $\Gamma, \Delta, \ldots$ to range over finite multisets of formulas (i.e., sets $[A_1, \ldots, A_n]$ whose elements are allowed to repeat), whereas small Greek letters $\pi, \rho, \ldots$ will be reserved for proofs. To simplify notation, we shall write Γ, A and Γ, Δ for the multisets $\Gamma \uplus [A]$ and $\Gamma \uplus \Delta$, respectively. A multiset Γ is *consistent* if and only if there is no formula A such that $\{A, A^{\perp}\} \subseteq \Gamma$.

The proof-net system we shall be concerned with is based on the one-sided sequent calculus $\overline{\mathsf{GS4}}$ summarized in Fig. 1 below [19]. $\overline{\mathsf{GS4}}$ is a complementary system for $\mathsf{GS4}$, the one-sided version of Kleene's $\mathsf{G4}$ [13], and is inspired by the (cut-free) two-sided complementary calculi of Tiomkin [35] and Goranko [9] mentioned above. Its peculiarity lies in its dealing with *antisequents*, which is to say sequents that are classically invalid. Such sequents are indicated by reversing the turnstile symbol. Thus, generally speaking, whereas a standard sequent $\Delta \vdash \Gamma$ expresses the fact that $\bigwedge \Delta$ (the conjunction of every formula in Δ) entails $\bigvee \Gamma$ (the disjunction of every formula in Γ) and is classically valid if and only if every valuation that verifies $\bigwedge \Delta$ verifies $\bigvee \Gamma$, an antisequent $\Delta \dashv \Gamma$ expresses the failure of the entailment, i.e., the fact that there is at least one valuation that verifies $\bigwedge \Delta$ but not $\bigvee \Gamma$. (Thus $\Delta \dashv \Gamma$ is not tantamount to its mirror-image standard sequent $\Gamma \vdash \Delta$, since we have, e.g., $p \dashv q$ but not $q \vdash p$.) In particular, a right-side antisequent $\dashv \Gamma$ expresses the fact that

AXIOM

$$\dfrac{}{\dashv \Gamma}\ \overline{ax} \quad \text{where } \Gamma \text{ is a finite (possibly empty) consistent multiset of atoms}$$

LOGICAL RULES

$$\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, A \vee B}\ \overline{\vee} \qquad \dfrac{\dashv \Gamma, A}{\dashv \Gamma, A \wedge B}\ \overline{\wedge}_{\mathcal{R}} \qquad \dfrac{\dashv \Gamma, B}{\dashv \Gamma, A \wedge B}\ \overline{\wedge}_{\mathcal{L}}$$

CUT RULE

$$\dfrac{\dashv \Gamma, A}{\dashv \Gamma}\ cut$$

FIGURE 1. The sequent calculus $\overline{\mathsf{GS4}}$

there is at least one valuation under which every formula in Γ is false, hence $\bigvee \Gamma$ is not a tautology. (Again, this is not tantamount to $\Gamma \vdash$, which holds only when $\bigwedge \Gamma$ is a contradiction.)

The rest of this section and the next will be devoted to illustrating the working of $\overline{\mathsf{GS4}}$ and to reviewing its most relevant computational properties.

Example 2.1. The following is a $\overline{\mathsf{GS4}}$-derivation for the classical contingency $(p \wedge p^{\perp}) \vee (p \vee q)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\dashv p, p, q}\ \overline{ax}}{\dashv p \wedge p^{\perp}, p, q}\ \overline{\wedge}_{\mathcal{R}}}{\dashv p \wedge p^{\perp}, p \vee q}\ \overline{\vee}}{\dashv (p \wedge p^{\perp}) \vee (p \vee q)}\ \overline{\vee}$$

Theorem 2.2 (Soundness and completeness). *$\overline{\mathsf{GS4}}$ proves an antisequent $\dashv \Gamma$ if and only if the sequent $\vdash \Gamma$ is classically invalid.*

Proof. Soundness: By induction on the length of the proof π ending in $\dashv \Gamma$, i.e., the number of rules occurring in π. By definition, there always exists a valuation falsifying all the atomic propositions in any instance of $\overline{ax}$. Concerning the inductive step, the case of the Cut-rule is straightforward, since any valuation that falsifies each formula in Γ, A will *a fortiori* falsify each formula in the sub-multiset Γ. Consider, then, the $\overline{\wedge}_{\mathcal{R}}$-rule. Assume there is a valuation v falsifying all the formulas occurring in the multiset Γ, A. Clearly, if $v(A) = 0$, then $v(A \wedge B) = 0$ for every formula B. Therefore, v falsifies all the formulas in $\Gamma, A \wedge B$ as well. The other cases can be treated similarly.

Completeness: It suffices to reason in terms of proof search, by induction on the number of occurrences of (binary) connectives in Γ. The backwards construction of a proof π of $\dashv \Gamma$ must be implemented by considering one of the valuations under which every formula in Γ is false.

The base case is simple. In fact, for any multiset Γ of atoms, if there is a valuation v falsifying each element of Γ, then Γ contains no pair of dual literals p, $p^{\perp}$, so $\dashv \Gamma$ can be derived directly as an instance of $\overline{ax}$.

As for the inductive step, the key part concerns formulas obtained by the conjunction rules. Consider, for instance, the $\overline{\wedge}_{\mathcal{R}}$-rule and assume the invalidity of the sequent $\vdash \Gamma, A \wedge B$. This means that there is a valuation v under which all the formulas in $\Gamma, A \wedge B$ are false. In particular, we must have $v(A \wedge B) = 0$, and so $v(A) = 0$ or $v(B) = 0$. Let's say that $v(A) = 0$. Then v will also falsify all the formulas in Γ, A. This means that the antisequent $\dashv \Gamma, A \wedge B$ can be further analyzed by applying the $\overline{\wedge}_{\mathcal{R}}$-rule upwardly to obtain $\dashv \Gamma, A \wedge B$ from $\dashv \Gamma, A$. The other cases can be handled similarly, essentially as in Proposition 1 by Tiomkin [35]. □

The significance of these soundness and completeness results will be further discussed and explained in Sect. 2.3. Moreover, it is worth noticing that the proof-search algorithm employed to establish the completeness of $\overline{\mathsf{GS4}}$ does not (need to) resort to any application of the Cut rule. Accordingly, the same algorithm can also be read as providing a semantic proof of cut-eliminability. In the next section we shall however look at a fully syntactical, Gentzen-style cut-elimination algorithm.

2.2. Cut Elimination and Other Properties

Obviously, $\dashv$ and $\vdash$ do not behave alike. Consider, for instance, the negative counterpart of the Cut rule in its standard multiplicative formulation.

$$\frac{\Gamma \dashv \Delta, A \qquad \Gamma', A \dashv \Delta'}{\Gamma, \Gamma' \dashv \Delta, \Delta'}$$

As noted by Carnielli and Pulcini [4], as well as by Tiomkin [35], this rule is not sound in complementary classical logic. We have, for instance, $p \dashv q$ and $q \dashv p$ even though $p \vdash p$, showing that a classically valid sequent may sometimes be obtained by cutting two invalid antisequents.

On the other hand, Carnielli and Pulcini [4] argue that the following inverse Weakening rules, which clearly violate the subformula property, may be treated as (unary) complementary Cut rules for all intents and purposes:

$$\frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta} \qquad \frac{\Gamma, A \dashv \Delta}{\Gamma \dashv \Delta}$$

On this basis, they outline a corresponding proof of Cut elimination for the two-sided Tiomkin–Goranko calculus. Here we show that Cut elimination holds of $\overline{\mathsf{GS4}}$ as well, with Cut understood literally as the one-sided rule listed in Fig. 1.

To this end, the complete enumeration of Cut reductions is listed in Fig. 2 below. We use the notation $\pi \to \pi'$ to signify that proof π' is obtained from proof π after *exactly one* reduction step. When a specific reduction is applied, we generally write the corresponding number above the arrow. In addition, we shall use the star-arrow notation $\pi \xrightarrow{*} \rho$ to indicate that ρ is a *normal form* of π, meaning that the following two conditions are simultaneously satisfied: (i) there is a finite chain of reductions transforming π into ρ, and (ii) no application of the Cut rule occurs in ρ. The procedure yielding a normal form ρ of a proof π by means of a finite series of reductions is called *normalization* [8,29,34].

IMMEDIATE REDUCTION

$$
① \quad \dfrac{\dfrac{}{\dashv \Gamma, p}\,\overline{ax}}{\dashv \Gamma}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{}{\dashv \Gamma}\,\overline{ax}
$$

SERIAL REDUCTIONS

$$
② \quad \dfrac{\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, A \vee B}\,\overline{\vee}}{\dashv \Gamma}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, A}\,\textit{cut}}{\dashv \Gamma}\,\textit{cut} \quad \text{or} \quad \dfrac{\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, B}\,\textit{cut}}{\dashv \Gamma}\,\textit{cut}
$$

$$
③ \quad \dfrac{\dfrac{\dashv \Gamma, A}{\dashv \Gamma, A \wedge B}\,\overline{\wedge}_{\mathcal{R}}}{\dashv \Gamma}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{\dashv \Gamma, A}{\dashv \Gamma}\,\textit{cut}
$$

$$
④ \quad \dfrac{\dfrac{\dashv \Gamma, B}{\dashv \Gamma, A \wedge B}\,\overline{\wedge}_{\mathcal{L}}}{\dashv \Gamma}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{\dashv \Gamma, B}{\dashv \Gamma}\,\textit{cut}
$$

COMMUTATIONS

$$
⑤ \quad \dfrac{\dfrac{\dashv \Gamma, A, B, C}{\dashv \Gamma, A, B \vee C}\,\overline{\vee}}{\dashv \Gamma, B \vee C}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{\dfrac{\dashv \Gamma, A, B, C}{\dashv \Gamma, B, C}\,\textit{cut}}{\dashv \Gamma, B \vee C}\,\overline{\vee}
$$

$$
⑥ \quad \dfrac{\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, A, B \wedge C}\,\overline{\wedge}_{\mathcal{R}}}{\dashv \Gamma, B \wedge C}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, B}\,\textit{cut}}{\dashv \Gamma, B \wedge C}\,\overline{\wedge}_{\mathcal{R}}
$$

$$
⑦ \quad \dfrac{\dfrac{\dashv \Gamma, A, C}{\dashv \Gamma, A, B \wedge C}\,\overline{\wedge}_{\mathcal{L}}}{\dashv \Gamma, B \wedge C}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{\dfrac{\dashv \Gamma, A, C}{\dashv \Gamma, C}\,\textit{cut}}{\dashv \Gamma, B \wedge C}\,\overline{\wedge}_{\mathcal{L}}
$$

$$
⑧ \quad \dfrac{\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, A}\,\textit{cut}}{\dashv \Gamma}\,\textit{cut} \qquad \longrightarrow \qquad \dfrac{\dfrac{\dashv \Gamma, A, B}{\dashv \Gamma, B}\,\textit{cut}}{\dashv \Gamma}\,\textit{cut}
$$

FIGURE 2. Cut reductions for $\overline{\mathsf{GS4}}$-proofs

Definition 2.3 (*Size of formulas, contexts, and proofs*). The *size* $\|A\|$ of a formula A is given by the number of occurrences of binary connectives in A. A multiset $\Gamma = [A_1, \ldots, A_n]$ has size $\|\Gamma\| = \|A_1\| + \cdots + \|A_n\|$. For any proof π, the size of π is defined as $\|\pi\| = \|\Gamma_1\| + \cdots + \|\Gamma_n\|$, where $[\dashv \Gamma_1, \ldots, \dashv \Gamma_n]$ is the multiset of the antisequents displayed in π.

Theorem 2.4 (Cut elimination). *If an antisequent $\dashv \Gamma$ is provable in $\overline{\mathsf{GS4}}$, then it is provable also without using the Cut rule.*

Proof. Consider the set of Cut reductions listed in Fig. 2. The Cut-elimination algorithm needed to establish the theorem will proceed by successive reductions of the uppermost Cut application occurring in the normalizing proof of $\dashv \Gamma$. This means that the last reduction, ⑧, is not allowed. Now, it is easy to see that any chain of reductions produced by following the strategy just explained will terminate. It suffices to observe that each Cut reduction (except for ⑧) decreases the size of the normalizing proof π, i.e., if $\pi \to \pi'$, then $\|\pi'\| < \|\pi\|$. □

Example 2.5. Figure 3 illustrates an application of the Cut-elimination algorithm to a proof ending in the empty antisequent. Its normal form just consists of one axiom-application introducing the empty antisequent.

The proof of Theorem 2.4 exploits a well-defined reduction strategy. One may wonder whether this is just a matter of convenience, i.e., whether any other strategy would have led unproblematically to the normal form. In other words, $\overline{\mathsf{GS4}}$ normalizes, but does it *strongly* normalize? The answer to this question is in the negative. As a counterexample, consider the reduction strategy illustrated in Fig. 4, which consists in invariably reducing the lowermost Cut application. Clearly, the process never terminates, since the second reduction always returns the initial proof.

A related question concerns the uniqueness of normal forms, i.e., whether $\pi \xrightarrow{*} \rho_1$ and $\pi \xrightarrow{*} \rho_2$ always entail the identity $\rho_1 = \rho_2$. As noted by Carnielli

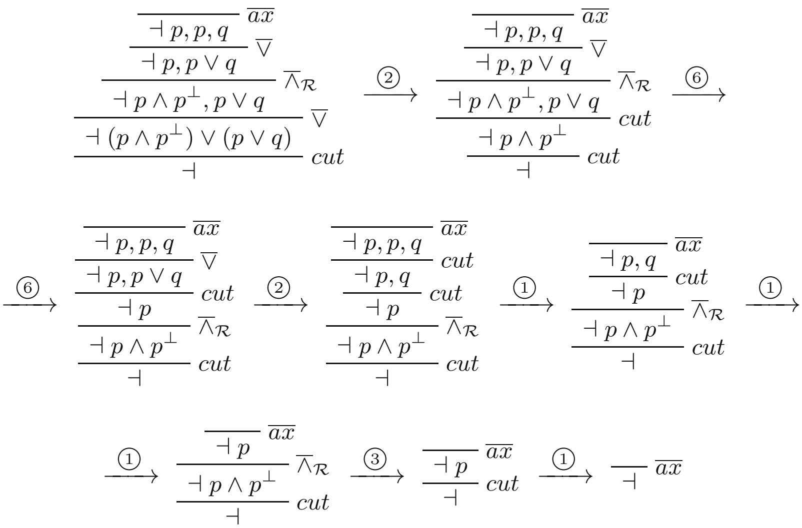

FIGURE 3. An application of the Cut-elimination algorithm

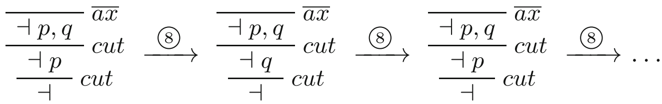

FIGURE 4. A counterexample to strong normalization in $\overline{\mathsf{GS4}}$

and Pulcini [4], one might consider a version of the Tiomkin–Goranko calculus in which logical contexts are construed as *sequences* of formulas, and such a calculus would enjoy the uniqueness property. However, in the present setting it seems more appropriate to construe the contexts of $\overline{\mathsf{GS4}}$ as ordinary multisets, since the Exchange rule

$$\frac{\dashv \Gamma, A, B}{\dashv \Gamma, B, A}$$

automatically becomes "transparent" when viewed from a graph-theoretic perspective. And if contexts are multisets, then again it is easy to see that our question has a negative answer; normal forms in $\overline{\mathsf{GS4}}$ need not be unique.

Consider, for instance, the case in Fig. 5 below. In the initial proof, on the left, the final cut transforms the multiset $[p \wedge q,\ p \wedge q]$ into $[p \wedge q]$. Multisets are not order-sensitive, and so there is no information concerning which occurrence of $p \wedge q$ has been erased in this Cut-application step. Thus, depending on how a choice is made (the occurrence introduced by $\overline{\wedge}_{\mathcal{R}}$ or the one introduced by $\overline{\wedge}_{\mathcal{L}}$), two diverging chains of reductions are produced, resulting in non-identical normal forms.

This result highlights a computational "defect" that $\overline{\mathsf{GS4}}$ shares with Gentzen's original calculus for classical logic, LK [6]. In LK, pathological behaviors of this sort are typically induced by the structural rules of Weakening and Contraction [8] (which for this reason are restricted to specific formulas in the sequent calculus for linear logic). Here the blame is entirely on the Cut rule. Nonetheless it is noteworthy that the problematic Cut application in the proof of Fig. 5 *could* be read as an instance of Contraction:

$$\frac{\dashv \Delta, A, A}{\dashv \Delta, A}$$

This is not among the fundamental rules of $\overline{\mathsf{GS4}}$. It is, however, an admissible rule and, indeed, a special case of the one-side Cut rule (with $\Gamma = \Delta, A$).

2.3. $\overline{\mathsf{GS4}}$ from a Broader Perspective

Before moving to proof nets, it is worth adding a few remarks concerning the nature of the soundness and completeness results for $\overline{\mathsf{GS4}}$. Consider the sequent system $\overline{\mathsf{GS4}}$ summarized in Fig. 6, where $\mid\!\!\overset{1}{-}$ and $\mid\!\!\overset{0}{-}$ stand for $\vdash$ and $\dashv$,

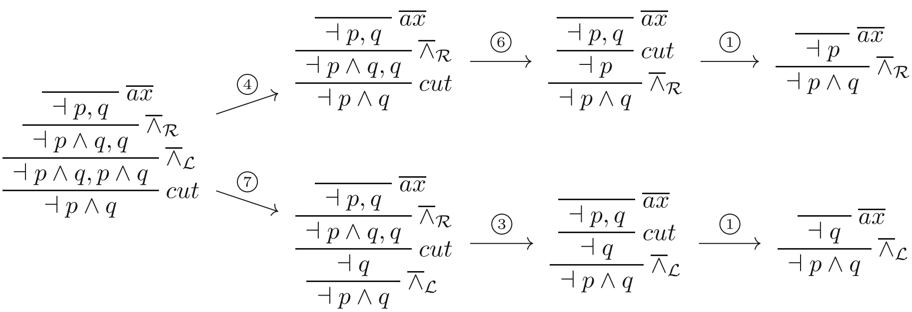

FIGURE 5. A counterexample to the uniqueness of normal forms in $\overline{\mathsf{GS4}}$

AXIOMS

$$\frac{}{\vdash^{1} \Gamma, A, A^{\perp}}\ ax \qquad\qquad \frac{}{\vdash^{0} \Gamma}\ \overline{ax}\ \text{(with } \Gamma \text{ restricted as in } \overline{\mathsf{GS4}}\text{)}$$

LOGICAL RULES

$$\frac{\vdash^{i} \Gamma, A, B}{\vdash^{i} \Gamma, A \vee B}\ \vee \qquad\qquad \frac{\vdash^{i} \Gamma, A \qquad \vdash^{j} \Gamma, B}{\vdash^{i \cdot j} \Gamma, A \wedge B}\ \wedge$$

CUT RULE

$$\frac{\vdash^{i} \Gamma, A \qquad \vdash^{j} \Gamma, A^{\perp}}{\vdash^{i \cdot j} \Gamma}\ cut$$

FIGURE 6. The sequent calculus $\overset{\text{- - -}}{\mathsf{GS4}}$

respectively, and $0 \leq i, j \leq 1$. This system comes as the one-sided version of Kleene's $\mathsf{G4}$ enriched with the complementary axiom rule of $\overline{\mathsf{GS4}}$ (subject to the same restrictions) [12,18]. It turns out that $\overset{\text{- - -}}{\mathsf{GS4}}$ is a "mixed" system for classical logic in the sense that it proves a sequent $\vdash \Gamma$ just in case the formula $\bigvee \Gamma$ is a tautology, and an antisequent $\dashv \Delta$ just in case $\bigvee \Delta$ is non-tautological [1,19]. It is easy to verify that $\overset{\text{- - -}}{\mathsf{GS4}}$-proofs, in their bottom-up reading, are isomorphic to tableaux *à la* Smullyan [1,28]. An illustrative example is given in Fig. 7 below, which displays, one after the other, a $\overset{\text{- - -}}{\mathsf{GS4}}$-proof ending in the antisequent $\dashv (p \wedge q) \vee (q^{\perp} \wedge r)$ along with its corresponding tableau, with an open branch indicating a successful refutation strategy. More generally, any $\overset{\text{- - -}}{\mathsf{GS4}}$-proof ending in an antisequent must display at least one application of the complementary axiom, i.e., in terms of tableaux, an open branch.

$$\dfrac{\dfrac{\dfrac{}{\dashv p, q^{\perp}}\ \overline{ax} \qquad \dfrac{}{\vdash q, q^{\perp}}\ ax}{\dashv p \wedge q, q^{\perp}}\ \wedge \qquad \dfrac{\dfrac{}{\dashv p, r}\ \overline{ax} \qquad \dfrac{}{\dashv q, r}\ \overline{ax}}{\dashv p \wedge q, r}\ \wedge}{\dfrac{\dashv p \wedge q, q^{\perp} \wedge r}{\dashv (p \wedge q) \vee (q^{\perp} \wedge r)}\ \vee}\ \wedge$$

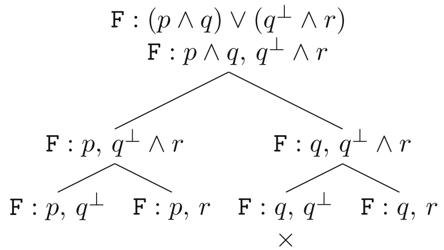

FIGURE 7. A $\overset{\text{- - -}}{\mathsf{GS4}}$-proof (top) and its tableaux-style counterpart (bottom)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\dashv p, q^{\perp}}\;\overline{ax}}{\dashv p \wedge q, q^{\perp}}\;\overline{\wedge}_{\mathcal{R}}}{\dashv p \wedge q, q^{\perp} \wedge r}\;\overline{\wedge}_{\mathcal{R}}}{\dashv (p \wedge q) \vee (q^{\perp} \wedge r)}\;\overline{\vee}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\dashv p, r}\;\overline{ax}}{\dashv p \wedge q, r}\;\overline{\wedge}_{\mathcal{R}}}{\dashv p \wedge q, q^{\perp} \wedge r}\;\overline{\wedge}_{\mathcal{L}}}{\dashv (p \wedge q) \vee (q^{\perp} \wedge r)}\;\overline{\vee}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\dashv q, r}\;\overline{ax}}{\dashv p \wedge q, r}\;\overline{\wedge}_{\mathcal{L}}}{\dashv p \wedge q, q^{\perp} \wedge r}\;\overline{\wedge}_{\mathcal{L}}}{\dashv (p \wedge q) \vee (q^{\perp} \wedge r)}\;\overline{\vee}
$$

FIGURE 8. Three "negative" slices of a $\overset{\text{- - -}}{\mathsf{GS4}}$-proof

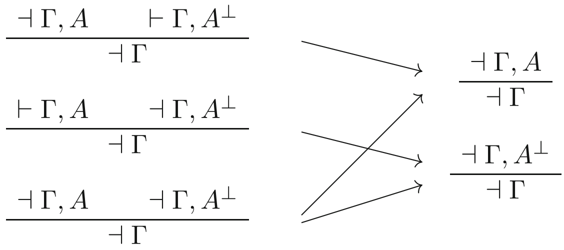

FIGURE 9. From binary mixed Cut rules to unary complementary Cut rules

Put in this way, $\overline{\mathsf{GS4}}$'s rules can be thought of as designing a proof system explicitly devoted to producing single "negative" threads in $\overset{\text{- - -}}{\mathsf{GS4}}$-proofs, i.e., paths in the whole proof tree connecting the root antisequent with a specific application of the complementary axiom. (For instance, the three $\overline{\mathsf{GS4}}$-proofs in Fig. 8 are obtained by "unthreading" the negative paths occurring in the $\overset{\text{- - -}}{\mathsf{GS4}}$-proof of Fig. 7.) From the tableaux viewpoint, a $\overline{\mathsf{GS4}}$-proof ending in the antisequent $\dashv \Delta$ guarantees the existence of a non-closing branch in each one of the possible tableaux associated with the multiset of formulas Δ.

It should be clearer, now, why every $\overline{\mathsf{GS4}}$-rule comes in the form of a unary inference, including the Cut rule. That may seem odd, considering that Cut applications are expected to combine at least two independent proofs or processes. However, in all three possible cases, listed on the left of Fig. 9, exactly one of the two premises proves completely superfluous. This allows the deductive machinery to be optimized by reducing the three binary version of the Cut rule to the two unary rules reported on the right side of the same figure, without any loss of information. The interactional view normally associated with the Cut rule can be easily restored once $\overline{\mathsf{GS4}}$-derivations are reframed within the broader deductive context provided by $\overset{\text{- - -}}{\mathsf{GS4}}$.

3. Complementary Proof Nets

3.1. Complementary Proof Structures and Proof Nets

We are now ready to introduce our *complementary proof-net system*. We shall refer to this system with the acronym CPN.

Traditionally, a proof-net system comes in three steps, and CPN is no exception. To begin with, a set of *links* is provided in such a way that each

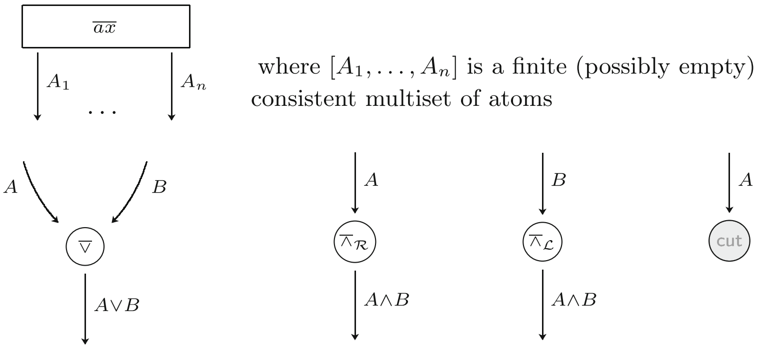

FIGURE 10. CPN links

link corresponds to an inference rule taken from a suitable one-sided sequent calculus—in our case, $\overline{\mathsf{GS4}}$ (Fig. 10). Each link involves a node and one or more directed edges, represented by arrows. Nodes are labeled with rule names and edges with formulas. In particular, incident and emerging edges represent the *premises* and the *conclusions*, respectively, of the rule. In the case of CPN, each of the five links shown in Fig. 10 is designed so as to represent a corresponding $\overline{\mathsf{GS4}}$-rule graph-theoretically. For instance, the $\overline{\vee}$-link has two premises and one conclusion, whereas the Cut-link has one premise and no conclusion. Cut-links will be graphically highlighted by shading them in grey.

The second step is the definition of a *proof structure*. Generally speaking, proof structures are directed graphs (see West [39, § 1.4]) constructed by composing the given set of links, subject to minor constraints intended to rule out meaningless constructions. The proof structures of CPN are specified in the following definition.

Definition 3.1 (*Proof structures*). A *proof structure* is a directed labeled graph recursively built from the basic links displayed in Fig. 10 in accordance with the following conditions: (i) each formula/edge is the conclusion of *exactly* one rule/node; (ii) each formula/edge is a premise of *at most* one rule/node; (iii) no edge can be premise and conclusion of the same rule/node.

In the following, proof structures will generally be referred to by capital Greek letters, $\Pi, \Sigma, \ldots$.

Finally, a *correctness criterion* must be provided. In the proof-net jargon, a correctness criterion is an algorithm capable of deciding whether or not a given proof structure is correct, i.e., encodes a proof in the sequent calculus of reference. A *proof net* is then defined as a correct proof structure. Typically, this third step is the hardest one, for a truly informative correctness criterion is expected to rely solely on extra-logical information involving the geometrical structure of the graph under consideration. In CPN, however, this task turns out to be relatively straightforward. The reason is that the antisequent calculus of reference, $\overline{\mathsf{GS4}}$, comprises only 0-ary and 1-ary rules, so proofs in

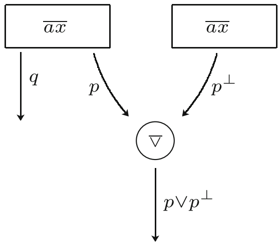

FIGURE 11. A proof structure that is not a proof net

this calculus turn out to be simple chains of antisequents. As a result, the correctness criterion for CPN can be stated quite easily as follows.

Definition 3.2 (*Proof nets*). A *proof net* is a proof structure in which exactly one $\overline{ax}$-link occurs.

Example 3.3. The simplest proof net is the graph formed by only one node, the $\overline{ax}$-link, and no edges. Its counterpart in $\overline{\mathsf{GS4}}$ will be the axiom introducing the empty antisequent. By contrast, the proof structure in Fig. 11 is not a proof net, since it displays two $\overline{ax}$-links.

We conclude this section by proving the adequacy of our correctness criterion, i.e., that a proof structure Π is a proof net if, and only if, Π corresponds to a $\overline{\mathsf{GS4}}$-proof.

Definition 3.4 (*Conclusions, terminal links*). The multiset of the *conclusions* of a proof structure Π, written $\mathcal{C}(\Pi)$, is the multiset of those formulas labeling edges in Π that are premises of no link. Links introducing the non-atomic conclusions and Cut-links of Π (if any) are said to be *terminal.*

Example 3.5. Let Π be the proof structure shown in Fig. 11. Then $\mathcal{C}(\Pi) = [\, q, p \vee p^\perp \,]$. Moreover, the $\vee$-link is Π's unique terminal link.

Theorem 3.6 (Soundness/Desequentialization). *Every $\overline{\mathsf{GS4}}$-proof π of an antisequent $\dashv \Gamma$ can be turned into a proof net Π such that $\mathcal{C}(\Pi) = \Gamma$.*

Proof. By induction on the length of π. □

Example 3.7. In Fig. 12, the $\overline{\mathsf{GS4}}$-proof of Example 2.1 is turned into its corresponding proof net.

Theorem 3.8 (Sequentialization). *Every proof net Π can be turned into a $\overline{\mathsf{GS4}}$-proof π ending in the antisequent $\dashv \mathcal{C}(\Pi)$.*

Proof. By induction on $n(\Pi)$, the total number of nodes in Π.

When $n(\Pi) = 1$, Π is just an instance of the $\overline{ax}$-link. Since no pair of dual atoms can occur in $\mathcal{C}(\Pi)$, the antisequent $\dashv \mathcal{C}(\Pi)$ is clearly an instance of the $\overline{\mathsf{GS4}}$ axiom.

For $n(\Pi) > 1$ (inductive step), we distinguish three cases, one for each rule that may be associated with the terminal nodes of Π.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\dashv p, p, q}\,\overline{ax}}{\dashv p \wedge p^{\perp}, p, q}\,\overline{\wedge}_{\mathcal{R}}}{\dashv p \wedge p^{\perp}, p \vee q}\,\overline{\vee}}{\dashv (p \wedge p^{\perp}) \vee (p \vee q)}\,\overline{\vee} \qquad \longrightarrow$$

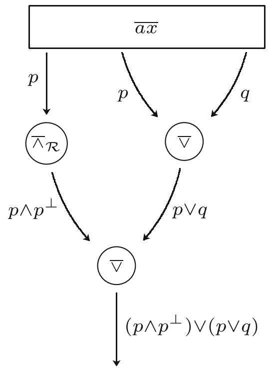

FIGURE 12. An antisequent proof and the corresponding proof net

(i) $\overline{\vee}$-*link*: Suppose $\mathcal{C}(\Pi) = \Gamma, A \vee B$ for some Γ (possibly empty). Consider the proof structure Π' in which the terminal $\overline{\vee}$-link introducing the conclusion $A \vee B$ has been removed, so that $\mathcal{C}(\Pi') = \Gamma, A, B$. Clearly, Π' is still a proof net and, moreover, $n(\Pi') < n(\Pi)$. Hence, by induction hypothesis, there exists a $\overline{\mathsf{GS4}}$-proof π' of the antisequent $\dashv \Gamma, A, B$. A $\overline{\mathsf{GS4}}$-proof π of $\dashv \Gamma, A \vee B$ can easily be obtained by extending π' by one final application of the $\overline{\vee}$-rule with A and B as premise labels and the disjunction $A \vee B$ as conclusion label.

(ii) *Cut-link*: Suppose Π contains a Cut-link whose incident edge is labeled with formula A. Consider the graph Π' obtained from Π by removing the Cut-link under consideration, so that $\mathcal{C}(\Pi') = \mathcal{C}(\Pi), A$. Since Π is a proof net, Π' will be a proof net as well. Moreover, we have that $n(\Pi') < n(\Pi)$ and thus, by induction hypothesis, there exists a $\overline{\mathsf{GS4}}$-proof π' ending in $\dashv \mathcal{C}(\Pi), A$. Given π', we can obtain a $\overline{\mathsf{GS4}}$-proof π of $\dashv \mathcal{C}(\Pi)$ just by adding a final application of the Cut rule erasing one occurrence of A from the end antisequent.

(iii) $\overline{\wedge}_{\mathcal{R}}$-*and* $\overline{\wedge}_{\mathcal{L}}$-*links*: Similar to the previous cases. □

Let us observe that the correctness criterion provided in Definition 3.2 proves equivalent to a "weakened" version of the better known Acyclic-and-Connected criterion (AC) designed to characterize proof nets in multiplicative linear logic [5]. In particular, whereas AC demands acyclicity and connectedness of the switching graphs, its transposition into the complementary setting of CPN requires only to test connectedness.

Before proving this fact, we need to introduce the two notions of *switching function* and *switching graph*.

Definition 3.9 *(Switching function, switching graph).* Given a proof structure Π, let $\Pi_{\overline{\vee}} = \{\ell_1, \ell_2, \ldots, \ell_n\}$ be a complete enumeration of the $\overline{\vee}$-links occurring in Π. A *switching* for Π is a function $\mathscr{S} : \Pi_{\overline{\vee}} \mapsto \{\texttt{left}, \texttt{right}\}$. For any switching function $\mathscr{S}$, the *switching graph* $\mathscr{S}(\Pi)$ is defined as the graph obtained by transforming Π according to the following instructions for each $\ell_i \in \Pi_{\overline{\vee}}$: (i) if

$\mathscr{S}(\ell_i) = \texttt{left}$, then erase the left premise of ℓ_i; (*ii*) if $\mathscr{S}(\ell_i) = \texttt{right}$, then erase the right premise of ℓ_i.

If $\Pi_\triangledown$ has cardinality n, then there are 2^n switchings functions associated with Π and, therefore, 2^n corresponding switching graphs. Let us indicate with $\mathbf{S}_\Pi$ the set collecting the switching functions associated with Π; the $\overline{\mathsf{AC}}$ criterion can be then formulated as follows:

Definition 3.10 ($\overline{\mathsf{AC}}$ *proof structures*). A proof structure Π is said to satisfy $\overline{\mathsf{AC}}$ just in case $\mathscr{S}(\Pi)$ is a connected graph for all $\mathscr{S} \in \mathbf{S}_\Pi$.

Example 3.11. Figure 13 displays the four switching graphs associated with the proof net Π of Fig. 12. Each of them turns out to be connected, thus Π satisfies $\overline{\mathsf{AC}}$. By contrast, the proof structure in Fig. 11 does not satisfy $\overline{\mathsf{AC}}$, since the switching functions associated with it return a disconnected graph.

The following theorem establishes the fact that the two notions of *proof net* (Definition 3.2) and $\overline{\mathsf{AC}}$ *proof structure* (Definition 3.10) are extensionally equivalent, i.e., characterize the same class of proof structures.

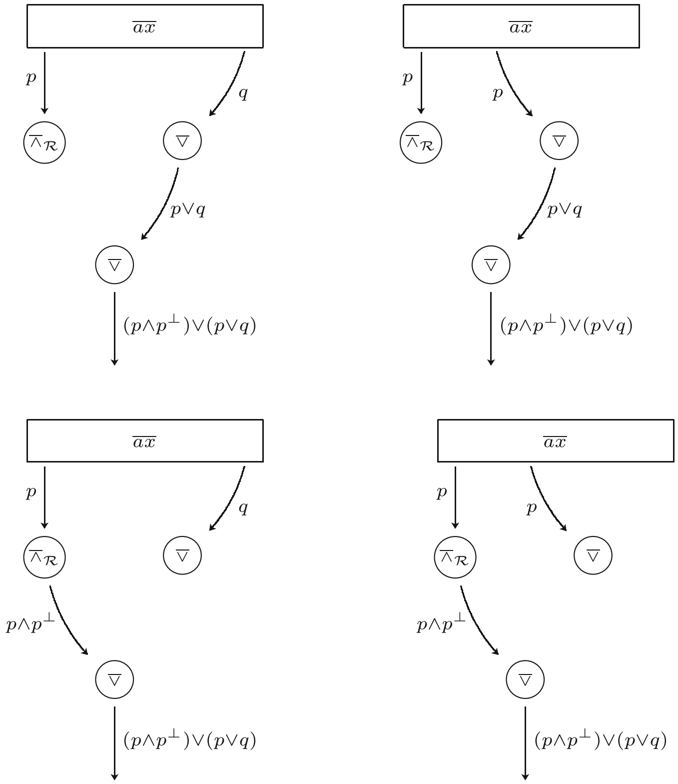

FIGURE 13. The four switching graphs associated with the proof net of Fig. 12

Theorem 3.12. *A proof structure* Π *is a proof net if and only if it is an* $\overline{\mathsf{AC}}$ *proof structure.*

Proof. By induction on $n(\Pi)$, the number of nodes of Π.

If $n(\Pi) = 1$, then Π is just an instance of the $\overline{ax}$-link. Therefore, $\mathbf{S}_\Pi = \varnothing$ and the claim of the theorem is trivially verified.

For $n(\Pi) > 1$ (inductive step), we need to consider two cases.

(i) $\mathcal{C}(\Pi) = \Gamma, A \wedge B$. We can assume without loss of generality that the terminal link introducing the conclusion $A \wedge B$ is a $\overline{\wedge}_\mathcal{R}$-link. Consider the proof structure Π' obtained from Π by removing that link, so that $\mathcal{C}(\Pi') = \Gamma, A$. By inductive hypothesis, Π' satisfies the theorem. Since $\mathbf{S}_\Pi = \mathbf{S}_{\Pi'}$, we easily get the desired conclusion.

(ii) $\mathcal{C}(\Pi) = \Gamma, A \vee B$. Consider the proof structure Π' obtained from Π by removing the terminal $\overline{\vee}$-link ℓ introducing the conclusion $A \vee B$, so that $\mathcal{C}(\Pi') = \Gamma, A, B$. By inductive hypothesis, Π' is a proof net just in case it is an $\overline{\mathsf{AC}}$ proof structure. We proceed by proving the following biconditionals:
   (1) Π is a proof net if and only if Π' is a proof net;
   (2) Π is an $\overline{\mathsf{AC}}$ proof structure if and only so is Π'.

Biconditional (1) is an immediate consequence of Definition 3.2. As for (2), let $\Pi'_{\overline{\vee}} = \{\ell_1, \ell_2, \dots, \ell_k\}$ and $\Pi'_{\overline{\vee}} \cup \{\ell\}$ be two complete enumerations of the $\overline{\vee}$-links occurring in Π' and Π, respectively. For every $\mathscr{S} \in \mathbf{S}_{\Pi'}$, the switching functions $\mathscr{S}^{\mathtt{R}}, \mathscr{S}^{\mathtt{L}} \in \mathbf{S}_\Pi$ are defined as follows:

$\mathscr{S}^{\mathtt{R}}(\ell) = \mathtt{right}$ and $\mathscr{S}^{\mathtt{L}}(\ell) = \mathtt{left}$;
$\mathscr{S}^{\mathtt{R}}(\ell_i) = \mathscr{S}^{\mathtt{L}}(\ell_i) = \mathscr{S}(\ell_i)$, for all $\ell_i \in \mathbf{S}_{\Pi'}$.

It suffices to observe that, for any $\mathscr{S} \in \mathbf{S}_{\Pi'}$, the switching graph $\mathscr{S}(\Pi')$ turns out to be connected precisely when the switching graphs $\mathscr{S}^{\mathtt{R}}(\Pi)$ and $\mathscr{S}^{\mathtt{L}}(\Pi)$ are both connected. □

3.2. Cut Elimination, Normalization, and Confluence

In this section we show that the complementary proof-net system CPN enjoys Cut elimination and offers significant improvements over the antisequent calculus $\overline{\mathsf{GS4}}$, both with respect to normalization and with respect to uniqueness of normal forms.

The complete list of Cut reductions for CPN is given in Fig. 14. We may see how Cut elimination concretely works in CPN by considering the example in Fig. 15, where the initial proof net on the left corresponds to the $\overline{\mathsf{GS4}}$-proof presented in Example 2.5. Starting from this graph, we implement Cut elimination until we reach a normal form, i.e., a proof net with no Cut-links, which consists of just one node, the $\overline{ax}$-link, introducing no conclusion.

The example of Fig. 15 generalizes, yielding a proof-net analogue of the Cut-elimination Theorem 2.4 (see Theorem 3.14 below). Furthermore, as with any respectable proof-net system, CPN allows us to identify $\overline{\mathsf{GS4}}$-proofs modulo trivial permutations of the rules they involve. The easiest example is given by the two derivations in Fig. 16, which share the single proof net reported underneath.

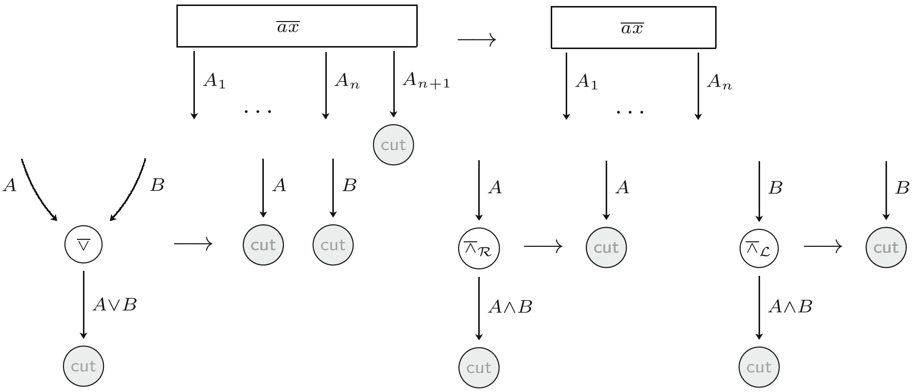

Figure 14. Cut reductions for proof nets

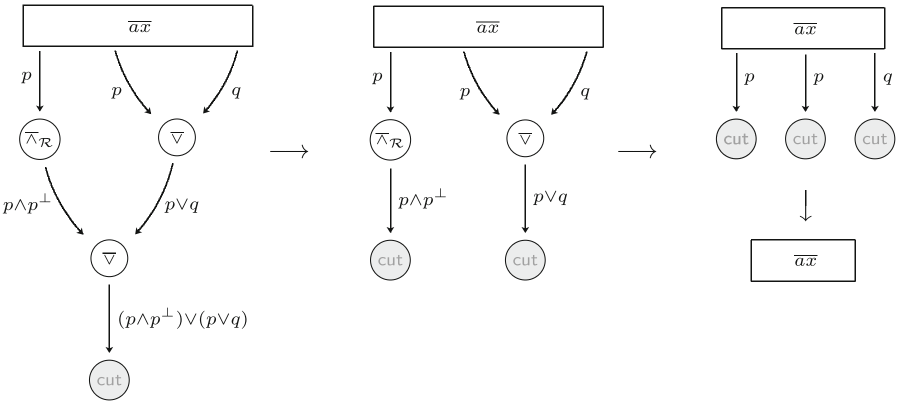

Figure 15. An example of normalization

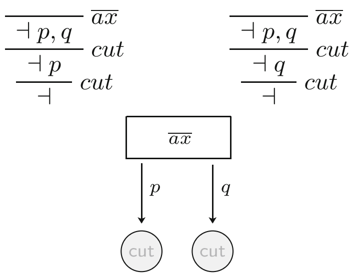

Figure 16. One proof net for two sequent proofs

This fact allows us to rule out counterexamples to strong normalization. For instance, the non-terminating chain of $\overline{\mathsf{GS4}}$-reductions considered in Sect. 2.2 (Fig. 4), which consists in an infinite alternation of the two sequent

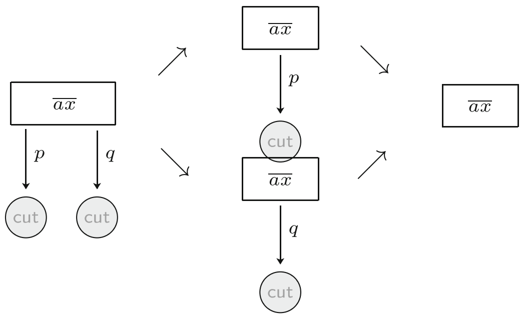

FIGURE 17. A strongly normalizing proof net

proofs just mentioned, is immediately defused in CPN. There are only two possible reduction strategies for the corresponding proof net and both normalize. Indeed, both yield the same normal form, the empty axiom-link (Fig. 17).

The following theorems establish more rigorously the technical improvements just underlined.

Definition 3.13 (*Size of proof structures*). The *size* $\|\Pi\|$ of a proof structure Π is defined as $n(\Pi) + e(\Pi)$, where $n(\Pi)$ is the number of nodes in Π and $e(\Pi)$ the number edges.

Theorem 3.14 (Cut elimination). *Every proof net Π can be reduced to a proof net Π' containing no Cut-links and such that $\mathcal{C}(\Pi) = \mathcal{C}(\Pi')$.*

Proof. Consider the set of reductions in Fig. 14. It is easy to see that: (i) proof nets are closed under those Cut reductions, and (ii) if $\Pi \to \Pi'$, then $\|\Pi'\| < \|\Pi\|$. □

Corollary 3.15 (Strong normalization). *Every reduction strategy leads to a normal form.*

Proof. Straightforward, since the proof of Theorem 3.14 does not mention any specific normalization strategy. □

Turning now to the issue of uniqueness, we can see that the sort of ambiguity that is responsible for the non-uniqueness of normal forms in $\overline{\mathsf{GS4}}$ cannot find an analogue in CPN. Consider again the two diverging reductions displayed in Fig. 5. Because of the lack of information induced by the order-insensitivity of multisets, the initial sequent proof can in principle be turned into two different proof nets, depending on whether the Cut-link is attached below the $\overline{\wedge}_{\mathcal{R}}$-link or below the $\overline{\wedge}_{\mathcal{L}}$-link (Fig. 18).

However, it is easy to see that each of these proof nets yields a unique normal form. Whereas $\overline{\mathsf{GS4}}$ allows for proofs whose normalization requires specific (even if arbitrary) choices, yielding divergent solutions, CPN treats those choices like any other, as the sort of strategic decision that is involved in the very process of setting up a proof.

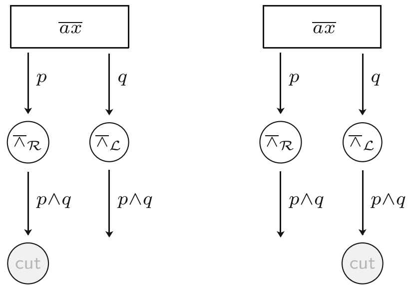

FIGURE 18. Two proof nets for the same sequent proof

It is perhaps surprising that a single sequent proof corresponds to *two* proof nets. Proof nets are typically called upon in relation to the complex problem of the identity of proofs (see e.g., [30]), the general idea being that sequent proofs that yield the same proof net could in principle be identified. This is the case, for instance, in multiplicative linear logic [5]. In the case of $\overline{\mathsf{GS4}}$-proofs, however, such a result cannot be fully accomplished, since complementary proof nets will sometimes force the disambiguation of critical cases such as the one under consideration. In such cases, the disambiguation may produce, not fewer, but more proof nets than the corresponding sequent proofs. Nonetheless this is once again due to the fact that the antisequents of $\overline{\mathsf{GS4}}$ have been defined in terms of multisets. It would suffice to shift from multisets to sequences of formulas to increase the number of sequent proofs, thereby restoring the canonical functional relationship. (This means that revising $\overline{\mathsf{GS4}}$ in terms of sequences would secure the uniqueness of normal forms, although the system would still lack strong normalization, as the counterexample in Fig. 4 would still apply, as well as strong confluence, as shown in Fig. 19 below.)

Now, to return to our question, of course the uniqueness of the normal forms for the proof nets in Fig. 18 does not by itself establish the general fact we are interested in, namely, that in CPN normal forms are always unique. Nevertheless the result holds. Indeed, we can prove the following stronger result, which generalizes the case illustrated in Fig. 17.

Theorem 3.16 (Strong confluence). *For any proof nets* Π, Π', *and* Π'', *if* $\Pi \to \Pi'$ *and* $\Pi \to \Pi''$, *then there exists a proof net* Λ *such that* $\Pi' \to \Lambda$ *and* $\Pi'' \to \Lambda$.

Proof. We exploit a standard result for rewriting systems, to the effect that termination + confluence = convergence (existence of a common reduct); see, e.g., Terese [34]. Let $Cut(1)$, $Cut(2)$, ..., $Cut(n)$ be an arbitrary enumeration of the Cut-links occurring in Π, and let $\Pi \xrightarrow{i} \Pi'$ indicate that Π' has been obtained from Π by specifically reducing $Cut(i)$. Similarly for $\Pi \xrightarrow{j} \Pi''$ etc. We can show that, for every $i, j \leq n$, if $\Pi \xrightarrow{i} \Pi'$, $\Pi \xrightarrow{j} \Pi''$, $\Pi' \xrightarrow{j} \Lambda$, and $\Pi'' \xrightarrow{i} \Lambda'$, then $\Lambda = \Lambda'$. The proof is by cases, according to the kind of reductions that are applied. Figure 20 illustrates the case in which $Cut(i)$ and $Cut(j)$ are the conclusions of a $\overline{\wedge}_{\mathcal{R}}$-link and of a $\overline{\vee}$-link, respectively. The other cases can be treated similarly. □

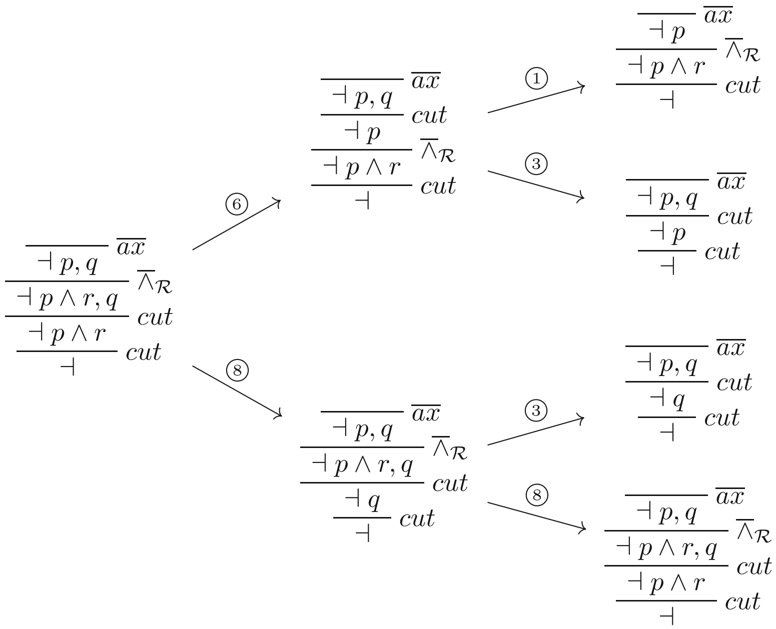

FIGURE 19. A counterexample to strong confluence in $\overline{\mathsf{GS4}}$

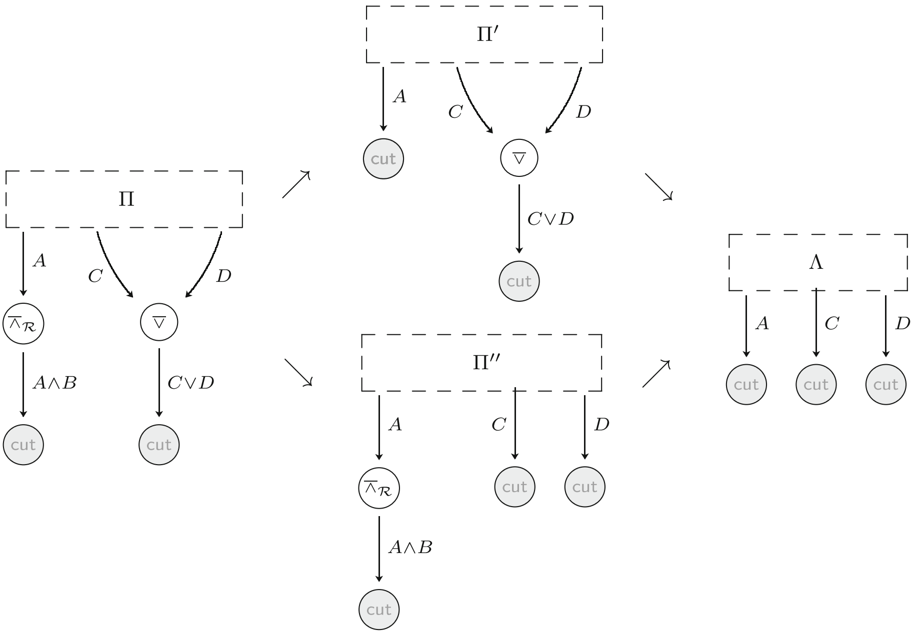

FIGURE 20. A case of strong confluence in CPN

It is now a straightforward matter to derive the uniqueness of normal forms from the confluence property [8,34].

Corollary 3.17 (Uniqueness of the normal form). *If* $\Pi \xrightarrow{*} \Lambda$ *and* $\Pi \xrightarrow{*} \Lambda'$, *then* $\Lambda = \Lambda'$.

Proof. Suppose $\Lambda \neq \Lambda'$. Since Λ and Λ' are both Cut-free proof nets, they cannot be further rewritten by means of some Cut reduction. Therefore they would constitute a counterexample to Theorem 3.16. □

This concludes our presentation. While patterned after the antisequent calculus $\overline{\mathsf{GS4}}$, the proof-net system CPN offers significant improvements to our complementary grasping of classical propositional logic. $\overline{\mathsf{GS4}}$ enjoys cut elimination; however, it does not strongly normalize and, while its failure to secure uniqueness of normal forms may be seen as a accidental feature, it is at best weakly confluent. CPN preserves cut elimination, strongly normalizes, and is strongly confluent.

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