Applied Mathematics and Computation 531 (2026) 130208

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

Distributed estimation algorithms on undirected chained graphs with explicit characterization of consensus

Cristiano Maria Verrelli [a], Patrizio Tomei [a], Carmine Di Fiore [b]

[a] University of Rome "Tor Vergata", Electronic Engineering Department, Via del Politecnico 1, Rome, 00133, Italy
[b] University of Rome "Tor Vergata", Mathematics Department, Via della Ricerca Scientifica, Rome, 00133, Italy

ARTICLE INFO

Keywords:
Parallel algorithm
Identifiability
Parameter estimation
Chained graph
Consensus
Linear time-varying system

ABSTRACT

In this paper, an original solution to a distributed identification problem on an undirected chain graph is presented. A symmetrical neighbourhood-based decentralized parallel set of estimation algorithms (aiming at reducing the computational burden of a centralized estimation scheme) is designed and adopted. The positive definite nature of a quadratic form associated with a tridiagonal block structure is crucial to proving consensus achievement. Explicit characterization of persistency of excitation for the exponential convergence is innovatively provided, without resorting to existence assertions.

1. Introduction

Consider an undirected, unweighted connected graph without multiple edges and self-loops. It is said to be $l(v_\star)$-chained with initial vertex $v_\star$ if i) the set of its vertices can be subdivided into $l(v_\star)$ disjoint ordered non-empty subsets – with $v_\star$ belonging to the first of them – and ii) all vertices in one subset are adjacent only to vertices in the subsequent and/or preceding subset. The vertices in the first and the last subsets are thus adjacent only to the vertices in the second and the penultimate subsets, respectively. Furthermore, the chain length $l(v_\star)$ is the largest number of vertex subsets with such a property [1,2]. Now, a path subgraph (or linear subgraph or Hamiltonian path) of the above graph is a graph whose vertices can be listed in the order $v_1, \dots, v_p$ such that the edges are $\{v_i, v_{i+1}\}$, $i = 1, \dots, p-1$ [3,4]. The path subgraph with at least two vertices is thus connected and has two terminal vertices (vertices of degree 1), while all others (if any) have degree 2. The special structure of a tridiagonal block matrix arising from a multi-agent estimation problem on an undirected path subgraph with consensus is exploited in this paper[1]. The reader is referred to [5] and references therein, for as estimation with consensus is concerned; the reader is also referred to [6–11] for related, slightly different frameworks and to [12] for the characterization of spectra of tridiagonal matrices arising from differential equations commonly used for modeling flocking behavior. In particular, the considered problem consists of looking for p time-varying vectors $\hat{\Theta}^{[i]}(t)$, $i = 1, \dots, p$ – named Θ-estimates – that all converge exponentially to the unknown constant parameter vector $\Theta \in \mathbb{R}^m$ defined by the set of p linear time-varying equations:

$$y_1(t) = \phi_1^{\mathrm{T}}(t)\Theta \qquad \text{first node} \tag{1}$$

$$y_i(t) = \phi_i^{\mathrm{T}}(t)\Theta, \quad i = 2, \dots, p-1 \qquad \text{internal nodes} \tag{2}$$

$$y_p(t) = \phi_p^{\mathrm{T}}(t)\Theta, \qquad \text{last node} \tag{3}$$

E-mail addresses: verrelli@ing.uniroma2.it (C.M. Verrelli), tomei@ing.uniroma2.it (P. Tomei), difiore@mat.uniroma2.it (C. Di Fiore).

[1] Preliminary results have been presented at the 20th Edition of the workshop: *Due Giorni di Algebra Lineare Numerica e Applicazioni*, January 20–21, 2025, Pisa, Italy [series dedicated to Numerical Linear Algebra and Applications].

https://doi.org/10.1016/j.amc.2026.130208
Received 9 November 2025; Received in revised form 8 April 2026; Accepted 10 June 2026
Available online 26 June 2026
0096-3003/© 2026 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

where: $y_i(t)$ are the outputs locally measured at each node v_i; $\phi_i(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}^m$ are the local regressor vectors, $i = 1, \dots, p$, available at the same each node v_i. Here, measurable and available are meant at any running time $t \geq 0$. In our case – namely, in the case in which there are no communication edges connecting nodes of the path subgraph that are far from each other – the problem is not full-knowledge-based. Instead, it is partial-knowledge-based, with the burden of information that has to be communicated to the various measurement/estimation nodes of the path subgraph being largely reduced. This paper follows a general research line that is still capturing the interest of the research community [13–17]. Its original contribution consists in the following three points.

- To design a set of estimation algorithms, namely a set of suitably tailored differential equations for the time-dependent vectors $\hat{\Theta}^{[i]}(t)$, all of them converging to the unknown Θ, i.e., performing the *consensus*. Such parallel estimation laws are (redundantly) designed at each node v_i, $i = 1, \dots, p$, under the weakest Θ-identification condition, namely the *Cooperative PE Condition* [18] (PE stands for Persistency of Excitation) reported in (5) of Section 2 hereafter. In the aforementioned scenario where nodes are undirectedly connected in series, each estimation scheme at the node can symmetrically share information – namely, its own Θ-estimate[2] – with the neighbours only, one for the first and last nodes v_1 and v_p, two for the remaining internal nodes. Indeed, in accordance with the identifiability requirement given by the Cooperative PE Condition, it is not necessary for the regressors to be persistently exciting (PEg) at each node – or, equivalently, for the parameter vector Θ to be identifiable from each measured output. Instead, it is sufficient for the nodes of the path subgraph to cooperatively exchange information.
- To establish the convergence properties for such estimation algorithms by adopting a matrix point of view. In this respect, it is here shown how the positive definite nature of a quadratic form associated with a tridiagonal block matrix is crucial to innovatively prove the exponential achievement, with an explicit characterization of the PE, of the distributed parameter estimation task on the undirected chain graph, in which, as aforementioned, a symmetrical neighbourhood-based decentralized parallel architecture is originally designed and adopted.
- To reduce the computational burden of a centralized estimation scheme on the graph: since each agent processes information only from its local neighbors, lower communication complexity is achieved; such complexity scales more efficiently with the number of agents compared to approaches relying on strongly connected graphs.

The derivations here move along the direction of (and are in exact accordance with) [18] (look at Theorem 1 therein) and [19]. However,

- in contrast to [18], which – for general graph topologies – uses weaker contradiction arguments to prove that the aforementioned Cooperative PE Condition guarantees exponential consensus, here an original (linear-algebra-based) constructive proof of convergence is able to provide an explicit characterization of the exponentially achieved consensus in terms of PE;
- differently from [19], which derives convergence-rate estimates for networks of systems interconnected through persistently excited graphs (the reader is referred to Assumption 2 therein) while recasting the classical consensus paradigm into a problem of stability analysis for systems with PE, here the set of tailored differential equations for the time-dependent vectors $\hat{\Theta}^{[i]}(t)$ has to be designed at each node v_i, $i = 1, \dots, p$, in order to actually guarantee PE under the weakest Θ-identification condition.

The result of this paper, in our view, turns out to be of self-interest and certainly provides new effective tools to analyze error system dynamics coming from system identification problems and deeply connect them with numerical algorithms. It is worth noticing that PE is strictly related to uniform observability properties. They allow for the definition of strong Lyapunov functions with intrinsic robustness properties that come not only from the exponential stability nature of the equilibrium point but also (and mainly) from its uniform-in-time properties. This is crucial in practice, in the presence of parameters not exactly complying with the constant nature covered by this paper and demonstrated by recent experimental results (not reported here for the sake of brevity) that show how experiments involving quadrotor UAVs (equipped with a Pixhwak autopilot running PX4 firmware and a Raspberry Pi serving as the onboard computer) in an indoor experimental setup, satisfactorily reproduce simulations. Nevertheless, as an additional original contribution of the paper,

- the solution to the problem of asymptotically identifying time-varying parameters that are periodic with known periods is innovatively presented as well (see Section 4 hereafter): adaptive tools can be directly replaced by repetitive learning tools within the same theoretical framework, where the asymptotic consensus is successfully ensured under identification mechanisms based on the information exchange between neighbours.

General applicative scenario

The general applicative scenario is the one given by a multisensor network in which a parameter estimator is to be designed on the basis of space-distributed sensing. This happens, for instance, in the field of coordination of Unmanned Aerial Vehicles, when tasks such as target point localization, cooperative tracking, and state estimation under GPS denial are to be performed. In such tasks, communication constraints often make fully connected networks impractical. On the other hand, wireless channels performing communication between nodes might be unreliable, often suffering from impairments such as signal fading and additive noise. This leads to degraded tracking accuracy and compromised swarm coordination, so that a reduction in communication channels is desirable, especially when the number of agents increases. As a specific *illustrative example*, consider the case in which a set of p

[2] Sharing both output measurements and regressor vectors would correspond to a larger computational burden.

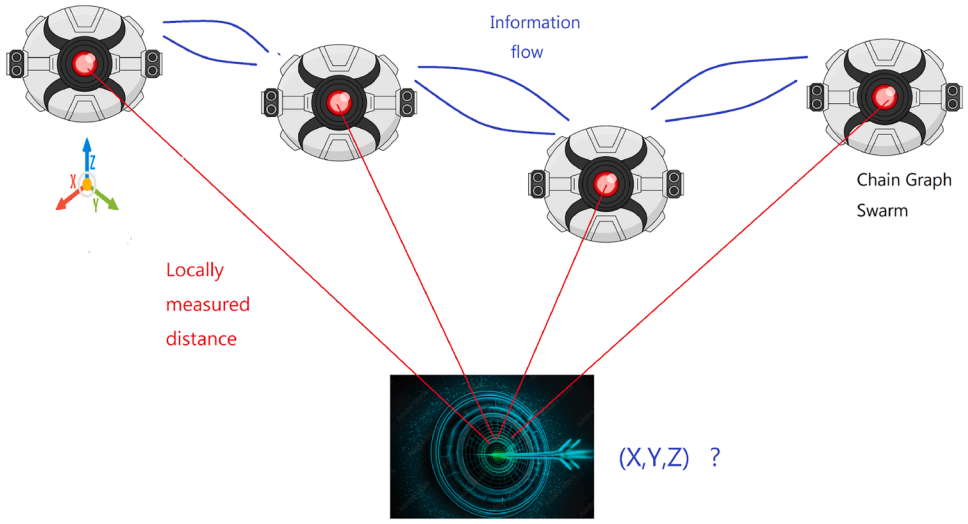


Fig. 1. *Illustrative example*: a swarm of drones estimating the position of a target in the space.

agents – the swarm of p drones with (not all zero) positions $(X_i(t), Y_i(t), Z_i(t))$ moving in the space depicted by Fig. 1 – face a local identification problem in which they have to consistently estimate a constant parameter vector (the vector of the positions (X_o, Y_o, Z_o) of the target object in Fig. 1) only by engaging symmetrical communication with their neighbours [20]. Each drone knows its own position in the space and measures its (Euclidean) distance $D_i(t) = \sqrt{(X_i(t) - X_o)^2 + (Y_i(t) - Y_o)^2 + (Z_i(t) - Z_o)^2}$ from the target object. The measured output is given by $y_i(t) = D_i^2(t) - X_i^2(t) - Y_i^2(t) - Z_i^2(t)$, which can be put in the form (1), (2), (3):

$$y_i(t) = \overbrace{[-2X_i(t), -2Y_i(t), -2Z_i(t), 1]}^{\phi_i^{\mathrm{T}}(t)} \overbrace{\begin{bmatrix} X_o \\ Y_o \\ Z_o \\ X_o^2 + Y_o^2 + Z_o^2 \end{bmatrix}}^{\Theta}, \quad i = 1, \dots, p.$$

Thus, this paper provides a feasible framework that enables each Unmanned Aerial Vehicle to achieve consensus with reduced communication load, in environments with limited bandwidth and energy, for collaborative search-and-rescue missions as well as cooperative target interception. An alternative situation also arises when mobile sensors measure the distribution of an unknown quantity over a field and, due to the size of the region, it is computationally inefficient or just infeasible to visit every point in the space in order to collect data. The reader is referred to [21–24] for related results.

Notation. In the remainder of this paper, $\mathbb{R}^+$, $\mathbb{R}_0^+$ will denote the sets of positive and non-negative real numbers, respectively, whereas $\mathbb{R}^{n\times m}$ will denote the set of real matrices with n rows and m columns; moreover, $\mathbb{I}$, $\mathbb{O}$ will represent the identity and the zero matrices of suitable dimensions; finally, for symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, $A \geqq B$ will mean that $A - B$ is positive semi-definite.

2. Assumptions & full-graph-based information (centralized estimation)

First, in order to guarantee that a solution to the problem addressed in the previous section exists, we introduce two standard assumptions, one of which has already been mentioned in the previous section.

A1. The components of the regressor vectors are assumed to be continuous and uniformly bounded over $[0, +\infty)$ as functions of time.

A2. (*Cooperative PE Condition* [18]) The corresponding regressor matrix $\Phi^{\mathrm{T}}(\cdot) \in \mathbb{R}^{p\times m}$

$$\Phi^{\mathrm{T}}(\cdot) = \begin{bmatrix} \phi_1^{\mathrm{T}}(\cdot) \\ \dots \\ \phi_p^{\mathrm{T}}(\cdot) \end{bmatrix} \tag{4}$$

is assumed to be PEg, i.e., there exist (known) positive reals c_p and T_p such that the following condition [$\mathbb{I} \in \mathbb{R}^{m\times m}$] holds:

$$\int_t^{t+T_p} \Phi(\tau)\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau \geqq c_p\mathbb{I}, \quad \forall\, t \geq 0. \tag{5}$$

Assumptions A1-A2 ensure, on the basis of [25] and references therein, that the unknown parameter vector Θ is identifiable from the entire set of available measurements (i.e., all the outputs and all the regressor vector components), even when $p < m$ and $\Phi^{\mathrm{T}}(\cdot)$ is not a full-rank matrix. Indeed, under Assumptions A1-A2, a solution to the estimation problem certainly exists, at least when all the

amount of information coming from the nodes of the considered graph is available for every node of the graph. It is provided by the set of differential equations[3]

$$\dot{\hat{\Theta}}(t) = \Phi(t)\Big(y(t) - \Phi^{\mathrm{T}}(t)\hat{\Theta}(t)\Big) \tag{6}$$

starting from the initial condition $\hat{\Theta}(0) = \hat{\Theta}_0$, where $y(t)$ is the column vector whose components are the scalar outputs $y_1(t)$, ..., $y_p(t)$. Define the Θ-estimation error $\tilde{\Theta} = \Theta - \hat{\Theta}$. It satisfies, on the basis of (1), (2), (3), (4) and (6):

$$\dot{\tilde{\Theta}}(t) = -\Phi(t)\Phi^{\mathrm{T}}(t)\tilde{\Theta}(t), \tag{7}$$

with initial condition $\tilde{\Theta}(0) = \Theta - \hat{\Theta}_0$. The reader can easily recognize that Eq. (7) complies with the structure required by Theorem 1 of [25]. According to it, its solution is guaranteed to tend to zero exponentially for any initial condition $\tilde{\Theta}(0)$. Indeed, this is in accordance with the behaviour in time, along (7), of the (strict) quadratic Lyapunov function

$$\mathcal{W}(\tilde{\Theta}) = \frac{1}{2}\tilde{\Theta}^{\mathrm{T}}\big[2Q_*(\cdot) + \big(1 + \mathrm{sgn}(m-1)\gamma_L\big)\big]\tilde{\Theta},$$

and its time derivative along the trajectories of the estimation error system

$$\dot{\mathcal{W}}(\tilde{\Theta}) \leq -\frac{\theta_c}{2}x^{\mathrm{T}}Q_*x - \left(\gamma_L - \frac{2(\beta\|\Phi(\cdot)\|)^2}{\theta_c c_p \mathrm{e}^{-\theta_c T_p}}\right)\|\Phi^{\mathrm{T}}(\cdot)x\|^2$$

where: $Q_*(t)$, such that $c_p\mathrm{e}^{-\theta_c T_p}\mathbb{I} \leqq Q_*(t) \leqq \beta\mathbb{I}$ for any $t \geq 0$, is the solution to the matrix-differential equation:

$$\dot{Q}_*(t) = -\theta_c Q_*(t) + \Phi(t)\Phi^{\mathrm{T}}(t),$$

starting from the symmetric positive definite matrix $Q_*(0)$ (with sufficiently large minimum eigenvalue) as initial condition and involving the positive real θ_c; γ_L is a sufficiently large positive real; $\mathrm{sgn}(\phi)$ is the signum function satisfying $\mathrm{sgn}(\phi) = 0$ for $\phi \leq 0$, $\mathrm{sgn}(\phi) = 1$ for $\phi > 0$. Indeed, estimates of the convergence rate can be obtained in a straightforward manner on the basis of the parameters characterizing the PE condition (Φ_M is a uniform bound on the norm of $\Phi(\cdot)$): $\dot{\mathcal{W}}(\tilde{\Theta}) \leq -\frac{\theta_c}{2r}\mathcal{W}(\tilde{\Theta})$ with $r = 1 + \frac{1}{2c_p\mathrm{e}^{-\theta_c T_p}} + \frac{\beta^2\Phi_M^2}{\theta_c(c_p\mathrm{e}^{-\theta_c T_p})^2}$.

3. Symmetrical partial-graph-based information (distributed estimation)

The determination of the differential Eqs. (6) relies on the availability of the whole information coming from all the nodes of the considered graph in terms of output and regressor vector components. The assumption that only symmetrical partial-graph-based amount of information is available for each node increases the complexity of the design of the differential equations for the Θ-estimate. In particular, in our chain graph scenario, each node v_i, i, $i = 1, \ldots, p$, has to include a (redundant) set of $\hat{\Theta}^{[i]}$-differential equations (here referred to as local estimators of Θ at the node v_i) just based on $\phi_i(t)$ and $y_i(t)$ and the only amount of information that can be shared with the node v_i is symmetrically constituted, in this paper, by the Θ-estimate $\hat{\Theta}^{[i_\star]}$ [$i_\star = 2$ for $i = 1$, $i_\star = i-1, i+1$ for $i = 2, \ldots, p-1$, $i_\star = p-1$ for $i = p$] provided by the (one or two) closest neighbours. We start by shaping the differential equation to be satisfied by the Θ-estimate at node v_i, $i = 1, \ldots, p$, denoted by $\hat{\Theta}^{[i]}$, as[4]

$$\dot{\hat{\Theta}}^{[i]}(t) = \phi_i(t)\Big(y_i(t) - \phi_i^{\mathrm{T}}(t)\hat{\Theta}^{[i]}(t)\Big) - \eta_i(t) \tag{8}$$

starting from $\hat{\Theta}^{[i]}(0) = \hat{\Theta}_{0i}$. This just corresponds to designing the Θ-estimator by the same design process of the previous section but restricted to the node v_i. However, the main issue regards how the correction terms $\eta_i(t)$ are to be ingeniously conceived i) to preserve the global convergence properties outlined in the previous section under (5) and then ii) to guarantee that all the redundant estimate vectors converge to the (same) unknown parameter vector Θ. In particular, such a consensus has to be achieved through a sort of penalization of the mismatch between the parameter estimates [26]. Now, from (1), (2), (3) and (8), the corresponding estimation errors $\tilde{\Theta}^{[i]} = \Theta - \hat{\Theta}^{[i]}$, $i = 1, \ldots, p$, satisfy the differential equations:

$$\dot{\tilde{\Theta}}^{[i]}(t) = -\phi_i(t)\phi_i^{\mathrm{T}}(t)\tilde{\Theta}^{[i]}(t) + \eta_i(t) \tag{9}$$

starting from $\tilde{\Theta}^{[i]}(0) = \Theta - \hat{\Theta}_{0i}$.

Remark 1. By Theorem 1 of [25], for $\eta_i \equiv 0$, $i = 1, \ldots, p$, each differential equation in (9) has a solution converging to zero provided that there exist (known) positive reals c_{pi} and T_{pi} such that [$\mathbb{I} \in R^{m\times m}$]

$$\int_t^{t+T_{pi}} \phi_i(\tau)\phi_i^{\mathrm{T}}(\tau)\mathrm{d}\tau \geqq c_{pi}\mathbb{I}, \ \forall\, t \geq 0. \tag{10}$$

The verification of (10) for any $i = 1, \ldots, p$ is, however, more restrictive than (5). This might be just illustrated by the simple example $\phi_1^{\mathrm{T}} = [1, 2]$, $\phi_2^{\mathrm{T}} = [2, 2]$ ($p = 2$). In such a case, (5) is merely satisfied (Φ is constant and invertible) whereas (10) is not.

[3] For the sake of simplicity, no gain matrix (namely, different from the identity matrix) multiplies $\Phi(t)$. Such gain might be used, in practice, to shape the promptness of response and, possibly, the robustness of the scheme by modifying the relative weight of the estimation error within the estimation scheme.

[4] A learning-like, non-implicit node-based form on the basis of the explicit expression for η_i can be obtained by rearranging terms to resemble the Jacobi method for positive definite matrices once an α is introduced as a discretization step.

Taking into account the amount of information that can be symmetrically shared by the nodes in terms of $i_\star$, we design[5]

$$\begin{aligned}\eta_1 &= \hat{\Theta}^{[1]} - \hat{\Theta}^{[2]}\\ \eta_i &= \left(\hat{\Theta}^{[i]} - \hat{\Theta}^{[i-1]}\right) + \left(\hat{\Theta}^{[i]} - \hat{\Theta}^{[i+1]}\right), \quad i = 2, \dots, p-1\\ \eta_p &= \hat{\Theta}^{[p]} - \hat{\Theta}^{[p-1]},\end{aligned} \tag{11}$$

which can be equivalently rewritten in terms of the Θ-estimation errors $\tilde{\Theta}^{[i]}$ (just add and subtract Θ within any pair of brackets) as follows

$$\begin{aligned}\eta_1 &= -\left(\tilde{\Theta}^{[1]} - \tilde{\Theta}^{[2]}\right)\\ \eta_i &= -\left(\tilde{\Theta}^{[i]} - \tilde{\Theta}^{[i-1]}\right) - \left(\tilde{\Theta}^{[i]} - \tilde{\Theta}^{[i+1]}\right), \quad i = 2, \dots, p-1\\ \eta_p &= -\left(\tilde{\Theta}^{[p]} - \tilde{\Theta}^{[p-1]}\right).\end{aligned} \tag{12}$$

By (12), the error system (9) becomes

$$\begin{bmatrix} \dot{\tilde{\Theta}}^{[1]}(t) \\ \vdots \\ \dot{\tilde{\Theta}}^{[p]}(t) \end{bmatrix} = -(\Lambda(t) + T) \begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \vdots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix} \tag{13}$$

where

$$\Lambda(t) = \mathrm{diag}[\phi_1(t)\phi_1^{\mathrm{T}}(t), \dots, \phi_p(t)\phi_p^{\mathrm{T}}(t)] \tag{14}$$

and (blocks are compatible in dimension with $[\tilde{\Theta}^{[1]}, \dots, \tilde{\Theta}^{[p]}]^{\mathrm{T}}$)

$$T = \begin{bmatrix} \mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \dots & \dots & \dots & \mathbb{O} \\ -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \dots & \dots & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \dots & \mathbb{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & \dots & -\mathbb{I} & \mathbb{I} \end{bmatrix}. \tag{15}$$

Define $G(t) = \Lambda(t) + T$ and note that $G(t)$ is a tridiagonal block matrix in $\mathbb{R}^{pm\times pm}$ that is, for any $t \geq 0$, symmetric and positive semi-definite ($G(\cdot) \geqq 0$), because both $\Lambda(t)$ and T are symmetric and positive semi-definite ($\Lambda(\cdot) \geqq 0$, $T \geqq 0$). In particular, T is so ($T \geqq 0$) since for any $w \in \mathbb{R}^{pm}$ – partitioned as $[w_1^{\mathrm{T}}, \dots, w_p^{\mathrm{T}}]^{\mathrm{T}}$ with $w_i \in \mathbb{R}^m$, $i = 1, \dots, p$ – the following chain of equalities

$$w^{\mathrm{T}}Tw = \left(\|w_1\|^2 + 2\sum_{i=2}^{p-1}\|w_i\|^2 + \|w_p\|^2 - 2\sum_{i=2}^{p}\langle w_i, w_{i-1}\rangle\right) = \sum_{i=1}^{p-1}\|w_i - w_{i+1}\|^2 \geq 0 \tag{16}$$

holds true[6] The crucial role of the non-negative term $\sum_{i=1}^{p-1}\|w_i - w_{i+1}\|^2$ will be properly highlighted hereafter. First, by looking at the structure of (13), if $G(t)$ is PEg (i.e., if (17) below holds) and $G(t)$ can be written as $\Gamma(t)\Gamma^{\mathrm{T}}(t)$ with the elements of $\Gamma(t)$ continuous on time and uniformly bounded over $[0, +\infty)$, then Theorem 1 in [25] can be used again to our advantage as the proof of the next theorem shows. A bound on the exponential rate can be derived as well on the basis of the parameters characterizing the PE, in accordance with the strict Lyapunov function and its derivative reported in the previous section.

Theorem 1. *Consider system (13) with $\Phi(t)$ satisfying A1. Assume that there exist (known) positive reals c_{pG} and T_{pG} such that the condition:*

$$\int_t^{t+T_{pG}} G(\tau)\mathrm{d}\tau \geqq c_{pG}\mathbb{I}, \quad \forall\, t \geq 0 \tag{17}$$

holds. Then the n-dimensional extended error vector $\tilde{\Theta}^{[e]}(t) = [\tilde{\Theta}^{[1]\mathrm{T}}(t), \dots, \tilde{\Theta}^{[p]\mathrm{T}}(t)]^{\mathrm{T}}$ ($n = pm$) globally exponentially converges to zero.

Proof. By condition A1., the elements of $G(t)$ are continuous on time. By Theorem X.1.1 in [28] (see also page 5 therein), the principal square root $\sqrt{G(t)} = U(t)\sqrt{D(t)}U^{\mathrm{T}}(t)$ of $G(t)$ – where $U(t)$ is orthogonal and $D(t)$ is the diagonal matrix collecting, for any t, the non-negative real eigenvalues of $G(t)$ – is continuous on time since $t^{1/2}$ is a monotonic function on $[0, +\infty)$] with $f(0) = 0$. On the other hand, the entries of $\sqrt{G(t)}$ are uniformly bounded over $[0, +\infty)$ since $U(t)$ is orthogonal and the eigenvalues of $G(t)$ are uniformly bounded over $[0, +\infty)$ because the maximum eigenvalue of $G(t)$ satisfies

$$\lambda_M(G(\cdot)) = \|G(\cdot)\| \leq \|\Lambda(\cdot)\| + \|T\|, \tag{18}$$

where $\|\Lambda(\cdot)\|$ is uniformly bounded over $[0, +\infty)$ by Assumption A1. Finally, apply Theorem 1 in [25] (with m therein equaling n) to get the thesis. □

Remark 2. Two alternative proofs for Theorem 1 can be provided as well. They are of self-interest and are thus reported for the sake of exhaustiveness. 1. Pick for any positive α_G and write $G(t) = \left(G(t) + \mathrm{e}^{-\alpha_G t}\mathbb{I}\right) - \mathrm{e}^{-\alpha_G t}\mathbb{I} \doteq G_I(t) - \mathrm{e}^{-\alpha_G t}\mathbb{I}$. Then take the Choleski decomposition $G_I(t) = R_c(t)R_c^{\mathrm{T}}(t)$ for the positive definite matrix $G_I(t)$ [29], where $R_c(t)$ is continuous on time[7] and uniformly bounded

[5] See [27] for an analogous use of parallel estimations in a different scenario.

[6] The Gershgorin theorems can be used as well.

[7] Applying the Choleski decomposition [29] directly to the positive semi-definite matrix $G(t)$ would require imposing continuity constraints to the freely choosable elements of the Choleski factor.

over $[0,+\infty)$ [use, to this purpose, Lemma 12.1.6 in [30] and recall that for any symmetric positive-definite matrix $M=[m_{ij}]_{i,j=1,\dots,n}$, the inequalities $|m_{ij}|\leq\sqrt{m_{ii}}\sqrt{m_{jj}}$ hold true]. Finally, recognize that

$$\int_t^{t+T_{pG}} G_I(\tau)\mathrm{d}\tau \geqq \int_t^{t+T_{pG}} G(\tau)\mathrm{d}\tau \geqq c_{pG}\mathbb{I}, \quad \forall\, t\geq 0$$

is satisfied and apply Theorem 1 in [25] (with m therein equaling n) by imposing $R(t,x,w)=-\mathrm{e}^{-\alpha_G t}x$ within its statement. 2. Matrix $\Lambda(t)$ takes the form $\Lambda(t)=R_1(t)R_1^{\mathrm{T}}(t)$ with $R_1(t)=\mathrm{diag}[\phi_1(t),\dots,\phi_p(t)]$, whereas there exists for T an orthogonal matrix U, $U^{-1}=U^{\mathrm{T}}$, such that $T=U\sqrt{D}\sqrt{D}U^{\mathrm{T}}\doteq R_2R_2^T$ with the diagonal matrix D collecting the non-negative real eigenvalues of T. Then recognize that the proof of Theorem 1 in [25] can be adapted, in a straightforward manner, to account for a system of the form $\dot{x}=-\sum_{i=1,\dots,r}R_i(t)R_i^{\mathrm{T}}(t)x$ with the elements of $R_i(t)$ continuous on time and uniformly bounded over $[0,+\infty)$. It suffices in fact to separately deal with the addends of the decomposition – by allowing for different m therein – while handling the time derivative of the Lyapunov function involved there.

Apparently, the strongest and most restrictive condition (10) implies condition (17). To show this, recall (16) and compute

$$\begin{aligned}\mathcal{P}_{(w,t)} &\doteq w^{\mathrm{T}}\int_t^{t+\max_{i=1,\dots,p}\{T_{pi}\}} G(\tau)\mathrm{d}\tau w=\sum_{i=1}^{p} w_i^{\mathrm{T}}\int_t^{t+\max_{i=1,\dots,p}\{T_{pi}\}}\phi_i\phi_i^{\mathrm{T}}\mathrm{d}\tau w_i+\max_{i=1,\dots,p}\{T_{pi}\}w^{\mathrm{T}}Tw\\ &\geq \min_{i=1,\dots,p}\{c_{pi}\}\sum_{i=1}^{p}\|w_i\|^2=\min_{i=1,\dots,p}\{c_{pi}\}\|w\|^2.\end{aligned}\tag{19}$$

Meaningfully, we now state the following Theorem 2, which relevantly shows how the weakest and least restrictive condition (5), namely, the Cooperative PE Condition, actually implies condition (17) and thus, used in conjunction with Theorem 1, provides the proof that the solution to system (13), under Assumptions A1-A3, globally exponentially converges to zero, with an explicit characterization of such a convergence in terms of PE.

Theorem 2. *Under Assumption A2 [namely, condition (5)], there exist explicitly computable positive reals c_{pG} and T_{pG} such that the condition (17) holds true.*

Proof. Consider the matrix $G(t)$ and compute $w^{\mathrm{T}}\int_t^{t+T_p}G(\tau)\mathrm{d}\tau w$. First restrict the analysis to all the non-zero vectors $w_*\in\mathbb{R}^n$ that make $w_*^{\mathrm{T}}Tw_*=0$ and satisfy (w_{*i} are the m-dimensional partitions of w_* that comply with the T-structure)

$$w_{*1}=w_{*2}=\dots=w_{*p}.\tag{20}$$

Such vectors, excepting for the zero-vector, coincide with the vectors that belong to the kernel of the symmetric positive semi-definite[8] T. For them, the following equality:

$$w_*^{\mathrm{T}}\int_t^{t+T_p}\Lambda(\tau)d\tau w_*=w_{*1}^{\mathrm{T}}\int_t^{t+T_p}\sum_{i=1}^{p}\phi_i(\tau)\phi_i^{\mathrm{T}}(\tau)\mathrm{d}\tau w_{*1}$$

holds. It obeys, based on the definition of $\Phi(\cdot)$ and (5), the following:

$$w_*^{\mathrm{T}}\int_t^{t+T_p}\Lambda(\tau)w_*=w_{*1}^{\mathrm{T}}\int_t^{t+T_p}\Phi(\tau)\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau w_{*1}\geq c_p\|w_{*1}\|^2=\frac{c_p}{p}\sum_{i=1}^{p}\|w_{*i}\|^2=\frac{c_p}{p}\|w_*\|^2$$

for any $t\geq 0$. Therefore, for all the non-zero vectors $w_*\in\mathbb{R}^n$ satisfying (20), the PE condition holds with T_p and c_p/p. Now analyze the remaining non-zero vectors $w_\diamond\in\mathbb{R}^{pm}$, different from w_*, for which a positive contribution comes from the term $T_pw_\diamond^{\mathrm{T}}Tw_\diamond$ [recall (16)]. The idea is to resort to the reconstruction of the previous w_{*1}-based term that collects the contribution coming from all the ϕ_i, $i=1\dots,p$, but with a small weight (namely, μ, hereafter) in order to reduce the effect, on the quadratic form, of the difference between $w_{\diamond 1}$ and any other component of $w_\diamond$ while leaning on the non-negative term $\sum_{i=1}^{p-1}\|w_{\diamond i}-w_{\diamond i+1}\|^2$ of (16). To this purpose, compute

$$w_\diamond^{\mathrm{T}}\int_t^{t+T_p}G(\tau)\mathrm{d}\tau w_\diamond=\mathcal{T}_1+\mathcal{T}_2$$

for any aforementioned $w_\diamond$, as the sum of two addends $\mathcal{T}_1$ and $\mathcal{T}_2$, the former being given by

$$\mathcal{T}_1=w_\diamond^{\mathrm{T}}\int_t^{t+T_p}\Lambda(\tau)\mathrm{d}\tau w_\diamond=\sum_{i=1}^{p}w_{\diamond i}^{\mathrm{T}}\int_t^{t+T_p}\phi_i(\tau)\phi_i^{\mathrm{T}}(\tau)\mathrm{d}\tau w_{\diamond i},$$

the latter equalling

$$\mathcal{T}_2=w_\diamond^{\mathrm{T}}\int_t^{t+T_p}T\mathrm{d}\tau w_\diamond=T_pw_\diamond^{\mathrm{T}}Tw_\diamond=T_p\sum_{i=1}^{p-1}\|w_{\diamond i}-w_{\diamond i+1}\|^2.$$

[8] Just impose $w_*^{\mathrm{T}}Tw_*=w_*^{\mathrm{T}}WD_TW^{\mathrm{T}}w_*=0$ for D_T being the diagonal form of T and W being the orthogonal matrix performing the diagonalization of T, and recognize that the vectors characterizing the equality are the same vectors characterizing the equality $Tw_*=WD_TW^{\mathrm{T}}w_*=0$.

Now let $\mu \in (0,1)$ and note that

$$\mathcal{T}_1 \geq \mu \sum_{i=1}^{p} w_{\diamond i}^{\mathrm{T}} \int_t^{t+T_p} \phi_i(\tau)\phi_i^{\mathrm{T}}(\tau)\mathrm{d}\tau w_{\diamond i}.$$

Then define the change of coordinates: $\alpha_{\diamond 0} = w_{\diamond 1}$, $\alpha_{\diamond i} = w_{\diamond i} - w_{\diamond i+1}$, $i = 1, \ldots, p-1$, which leads to the formula: $w_{\diamond i} = \alpha_{\diamond 0} - \sum_{k=1}^{i-1} \alpha_{\diamond k}$, $i = 1, \ldots, p$. Apply it to get

$$\begin{aligned}
\mathcal{T}_1 &\geq \mu \alpha_{\diamond 0}^{\mathrm{T}} \beta_1(t)\alpha_{\diamond 0} + \mu \sum_{i=2}^{p} \overbrace{\left(\alpha_{\diamond 0} - \sum_{k=1}^{i-1}\alpha_{\diamond k}\right)^{\mathrm{T}}}^{w_{\diamond i}^{\mathrm{T}}} \beta_i(t)\left(\alpha_{\diamond 0} - \sum_{k=1}^{i-1}\alpha_{\diamond k}\right) \\
&\geq \mu \alpha_{\diamond 0}^{\mathrm{T}} \sum_{i=1}^{p} \beta_i(t)\alpha_{\diamond 0} - \mu \sum_{i=2}^{p}\sum_{k=1}^{i-1} \|\alpha_{\diamond k}\| M_i T_p \left(\|\alpha_{\diamond 0}\| + \sum_{k=1}^{i-1}\|\alpha_{\diamond k}\|\right) - \mu \sum_{i=2}^{p} \|\alpha_{\diamond 0}\| M_i T_p \sum_{k=1}^{i-1}\|\alpha_{\diamond k}\|
\end{aligned}$$

and

$$\mathcal{T}_2 = T_p \sum_{i=1}^{p-1} \|\alpha_{\diamond i}\|^2,$$

where $\beta_i(t) = \int_t^{t+T_p} \phi_i(\tau)\phi_i^{\mathrm{T}}(\tau)\mathrm{d}\tau$, $M_i \geq \left\|\phi_i(t)\phi_i^{\mathrm{T}}(t)\right\|$ and thus $\|\beta_i(t)\| \leq T_p M_i$, with M_i independent of t. Therefore, by (5)

$$\mathcal{T}_1 + \mathcal{T}_2 \geq \mu c_p \|\alpha_{\diamond 0}\|^2 + T_p \sum_{i=1}^{p-1} \|\alpha_{\diamond i}\|^2 - \mu\sigma$$

where $\sigma = \sigma_1 + \sigma_2$ with

$$\begin{aligned}
\sigma_1 &= \sum_{i=2}^{p}\sum_{k=1}^{i-1} \|\alpha_{\diamond k}\| M_i T_p \left(\|\alpha_{\diamond 0}\| + \sum_{k=1}^{i-1}\|\alpha_{\diamond k}\|\right) \\
\sigma_2 &= \sum_{i=2}^{p} \|\alpha_{\diamond 0}\| M_i T_p \sum_{k=1}^{i-1}\|\alpha_{\diamond k}\|.
\end{aligned}$$

Note that

$$\sigma_1 = \sigma_2 + \sum_{i=2}^{p} M_i T_p \left(\|\alpha_{\diamond 1}\| + \ldots + \|\alpha_{\diamond i-1}\|\right)^2 \leq \sigma_2 + \sum_{i=2}^{p} M_i T_p (i-1)\left(\sum_{k=1}^{i-1}\|\alpha_{\diamond k}\|^2\right).$$

Thus

$$\begin{aligned}
\sigma &\leq 2\sigma_2 + \sum_{i=2}^{p} M_i T_p (i-1)\left(\sum_{k=1}^{i-1}\|\alpha_{\diamond k}\|^2\right) \\
&\leq 2MT_p\|\alpha_{\diamond 0}\|\Big[(p-1)\|\alpha_{\diamond 1}\| + (p-2)\|\alpha_{\diamond 2}\| + \ldots + 1\|\alpha_{\diamond p-1}\|\Big] \\
&\quad + MT_p\Big[\|\alpha_{\diamond 1}\|^2(1+2+\ldots+(p-1)) + \|\alpha_{\diamond 2}\|^2(2+3+\ldots+(p-1)) + \|\alpha_{\diamond p-1}\|^2(p-1)\Big] \\
&= 2MT_p\|\alpha_{\diamond 0}\|\Big[(p-1)\|\alpha_{\diamond 1}\| + (p-2)\|\alpha_{\diamond 2}\| + \ldots + 1\|\alpha_{\diamond p-1}\|\Big] + MT_p \sum_{j=p-1}^{1} \|\alpha_{\diamond p-j}\|^2 \frac{j(2p-j-1)}{2},
\end{aligned}$$

where $M = \max_{i=2,\ldots,p}\{M_i\}$. So we have that

$$\mathcal{T}_1 + \mathcal{T}_2 \geq w_\alpha^{\mathrm{T}} \mathcal{Y} w_\alpha, \tag{21}$$

where $w_\alpha = [\|\alpha_{\diamond 0}\|, \ldots, \|\alpha_{\diamond p-1}\|]^{\mathrm{T}}$ and

$$\mathcal{Y} = \begin{bmatrix}
\mu c_p & -\mu T_p b_{p-1} & \ldots & \ldots & -\mu T_p b_1 \\
-\mu T_p b_{p-1} & T_p(1-\mu d_{p-1}) & 0 & \ldots & 0 \\
\vdots & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
-\mu T_p b_1 & 0 & \ldots & 0 & T_p(1-\mu d_1)
\end{bmatrix},$$

with $b_i = Mi$ and $d_i = Mi(2p-i-1)/2$, $i = 1, \ldots, p-1$. Meanwhile, notice that, for any $i = 1, \ldots, p$, $0 < b_i \leq M(p-1)$, $0 < d_i \leq Mp(p-1)/2$. Denote by $\mathbf{0} \in \mathbb{R}^{p-1}$ the column vector with all zero entries and by b the vector $[b_{p-1}, \ldots, b_1]^{\mathrm{T}}$. Then consider the Schur representation of $\mathcal{Y}$:

$$\mathcal{Y} = L\mathcal{Y}_S L^{\mathrm{T}}$$

with ($\mathbb{I} \in \mathbb{R}^{(p-1)\times(p-1)}$)

$$L = \begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ -(T_p/c_p)b & \mathbb{I} \end{bmatrix}, \quad \mathcal{Y}_s = \begin{bmatrix} \mu c_p & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & T_p S \end{bmatrix}$$

and ($\mathbb{I} \in \mathbb{R}^{(p-1)\times(p-1)}$)

$$S = \mathbb{I} - \mu\left[\mathrm{diag}[d_{p-1}, \dots, d_1] + (T_p/c_p)bb^{\mathrm{T}}\right].$$

Observe that S, and therefore, $\mathcal{Y}$ is positive definite by Lemma A.1 in Appendix A, provided that μ is sufficiently small, namely

$$\mu \leq \left[\frac{Mp(p-1)}{2}\left(1 + \frac{T_p M(2p-1)}{3c_p}\right)\right]^{-1}.$$

Thus, from (21), we write for sufficiently small μ,

$$\mathcal{T}_1 + \mathcal{T}_2 \geq \lambda_m(\mathcal{Y})\left(\|\alpha_{\diamond 0}\|^2 + \sum_{i=1}^{p-1}\|\alpha_{\diamond i}\|^2\right) = \lambda_m(\mathcal{Y})\alpha_\diamond^{\mathrm{T}}\alpha_\diamond = \nu w_\diamond^{\mathrm{T}} B^{\mathrm{T}} B w_\diamond \geq \lambda_m(\mathcal{Y})\lambda_m(B^{\mathrm{T}}B)\|w_\diamond\|^2$$

in which the minimum eigenvalue $\lambda_m(\mathcal{Y})$ of $\mathcal{Y}$, in accordance with

$$0 < \lambda_m(\mathcal{Y}) = \min_{z\in\mathbb{C}^n}(z^{\mathrm{H}}\mathcal{Y}z)/(z^{\mathrm{H}}z) = \min_{y\in\mathbb{C}^n}(y^{\mathrm{H}}\mathcal{Y}_s y)/(y^{\mathrm{H}}L^{-1}(L^{\mathrm{T}})^{-1}y) \leq (y^{\mathrm{H}}\mathcal{Y}_s y)/(y^{\mathrm{H}}L^{-1}(L^{\mathrm{T}})^{-1}y)_{|y=[1,0,\dots,0]} = \mu c_p,$$

is a positive number that decreases as μ decreases, whereas $\lambda_m(B^{\mathrm{T}}B)$ denotes the minimum positive eigenvalue of the symmetric positive-definite $B^{\mathrm{T}}B$, with

$$B = \begin{bmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} & \dots & \dots & \dots & \mathbb{O} \\ \mathbb{I} & -\mathbb{I} & \mathbb{O} & \mathbb{O} & \dots & \dots & \mathbb{O} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & \mathbb{I} & -\mathbb{I} \end{bmatrix}$$

representing the matrix of the linear operator associating the new coordinate vector $[\alpha_{\diamond 0}^{\mathrm{T}}, \alpha_{\diamond 1}^{\mathrm{T}}, \dots, \alpha_{\diamond,p-1}^{\mathrm{T}}]^{\mathrm{T}}$ with $w_\diamond$, i.e., $\alpha_\diamond = Bw_\diamond$. Thus $T_{pG} = T_p$ and $c_{pG} = \lambda_m(\mathcal{Y})\lambda_m(B^{\mathrm{T}}B)$ in (17). □

Remark 3. The previous proof, in contrast to [18], does not rely on contradiction arguments. Instead, it originally provides an explicit characterization of the exponentially achieved consensus in terms of PE.

4. Identification of multi-periodic vectors

Assume that (1), (2), (3) are here replaced by

$$y_i(t) = \phi_i^{\mathrm{T}}\Theta(t), \quad i = 1, \dots, p, \tag{22}$$

in which the parameter vector $\Theta(t) = [\theta_1(t), \dots, \theta_m(t)]^{\mathrm{T}} \in \mathbb{R}^m$ is now composed by class C^1 periodic components $\theta_1(t), \dots, \theta_m(t)$ with known periods α_j, namely

$$\theta_j(t) = \theta_j(t - \alpha_j), \quad \forall\, t \geq -\alpha_j, \quad j = 1, \dots, m, \tag{23}$$

whereas the regressor vectors have constant components. The Cooperative PE Condition (5) for time-varying regressor components does not generally guarantee, in fact that the periodic $\Theta(t)$ is identifiable from the entire set of output measurements. A scalar example suffices to illustrate it. Consider $y(t) = \phi(t)\theta(t)$ in Fig. 2, with $\phi(t) = \max\{0, \sin(2\pi t)\}$, $\theta(t) = \cos(2\pi t)$: even though condition (5) is satisfied, $\theta(t)$-identification cannot be achieved. Notice how the convergence is just ruled, in accordance with the computations carried out in this section, by the non-positive term $-\tilde{y}^2(t) = -\phi(t)^2\tilde{\theta}(t)^2$ appearing in $\dot{\mathcal{V}}(t)$ and that nothing can be done in the repeated time intervals over which $\phi(t) \equiv 0$.

Let Φ be defined as in Section 3 and $y(t)$ denote the column vector with components $y_i(t)$, $i = 1, \dots, p$. In light of the previous developments, the aim of this section is to solve the multi-agent estimation problem with consensus by determining time-varying estimate vectors $\hat{\Theta}^{[i]}(t)$, $i = 1, \dots, p$, that asymptotically converge to the unknown multi-periodic parameter vector $\Theta \in \mathbb{R}^m$ involved in the set of the above linear time-varying equations, where $y_i(t)$ are the locally measured outputs and $\phi_i \in \mathbb{R}^m$ are the local (constant) regressor vectors, $i = 1, \dots, p$, each of them assumed to be available at each node of the considered graph. As in Section 3, under (5) – which here directly requires that $m \leq p$ with m rows of Φ^{T} being linearly independent –, a set of multi-periodic vectors $\hat{\Theta}^{[i]}(t) = [\hat{\Theta}_1^{[i]}(t), \dots, \hat{\Theta}_m^{[i]}(t)]^{\mathrm{T}}$ – all of them converging to the unknown $\Theta(t)$ – have to be (redundantly) designed at each node v_i, $i = 1, \dots, p$, of the considered graph. Again, it is assumed that each estimation scheme at the node can share information (namely, its own $\Theta(t)$-estimate) with the neighbours only. Even in this case, agents at the nodes have to face a local identification problem, in which they cannot consistently estimate the parameter vector in isolation, so they have to necessarily engage in communication with their neighbours. In particular, local measurements at each node do not allow for $\Theta(t)$-identification. In fact, $\Theta(t) + \Xi$ with $\phi_i^T\Xi = 0$ and $\Theta(t)$ are indistinguishable from $y_i(t)$ since any constant function preserves, by addition, any α_j periodicity on time (see [31]). Moreover, in the case of commensurable periods, any α_j function can be drawn from the related component and suitably incorporated into the component with the common period. On the other hand, $\Theta(t)$ could be easily caught by a centralized estimation scheme feeding back

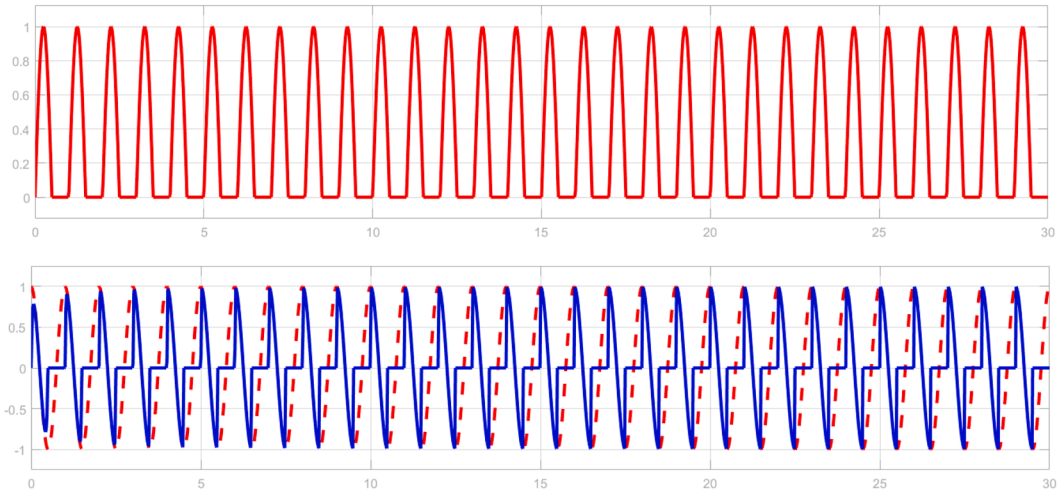


Fig. 2. *Cooperative PE Condition* (5) does not guarantee identification in the case of time-varying regressor and parameter vectors. Top: regressor $\phi(t) = \max\{0, \sin(2\pi t)\}$. Bottom: $\theta(t) = \cos(2\pi t)$ and its estimate $\hat{\theta}(t)$.

the measurable signal $(\Phi\Phi^{\mathrm{T}})^{-1}\Phi y(t) = \Theta(t)$ and designed by following the principles of [31]. Now, by mimicking the design steps of Section 3, define

$$\hat{y}_i(t) = \phi_i^{\mathrm{T}}\hat{\Theta}^{[i]}(t), \quad i = 1, \dots, p \tag{24}$$

leading to the equations

$$\tilde{y}_i(t) = \phi_i^{\mathrm{T}}\tilde{\Theta}^{[i]}(t), \quad i = 1, \dots, p \tag{25}$$

which involve the estimation errors $\tilde{y}_i = y_i - \hat{y}_i$ and $\tilde{\Theta}^{[i]} = \Theta - \hat{\Theta}^{[i]}$ with components $\tilde{\theta}_j^{[i]} = \theta_j - \hat{\theta}_j^{[i]}$, $i = 1, \dots, p$, $j = 1, \dots, m$. By transporting (8) into the multi-periodic learning estimation domain of [31], design[9] for $t \geq \max\{\alpha_1, \alpha_2, \dots, \alpha_p\}$

$$\hat{\Theta}^{[i]}(t) = \hat{\Theta}^{[i]}(t, \alpha) + \phi_i\tilde{y}_i(t) - \eta_i(t) \tag{26}$$

where: α is the vector $[\alpha_1, \dots, \alpha_m]^{\mathrm{T}}$; $\eta_i(t)$ is the same vector in (11) ruling the information exchange and

$$\hat{\Theta}^{[i]}(t, \alpha) = [\hat{\theta}_1^{[i]}(t - \alpha_1), \dots, \hat{\theta}_m^{[i]}(t - \alpha_m)]^{\mathrm{T}}. \tag{27}$$

Once the explicit expression for η_i of (11) is used in (26) and terms are suitably arranged, the non-implicit node-based form

$$(\mathbb{I} + \mathrm{diag}_i[T] + \phi_i\phi_i^{\mathrm{T}})\hat{\Theta}^{[i]}(t) = \hat{\Theta}^{[i]}(t, \alpha) + \phi_i y_i(t) - \overbrace{[\eta_i - \mathrm{diag}_i[T]\hat{\Theta}^{[i]}(t)]}^{\text{estimates from neighbours}}$$

is obtained[10] in terms of the i-th block $\mathrm{diag}_i[T]$ of the diagonal matrix $\mathrm{diag}[T]$. The compact expression

$$\begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \dots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix} = \begin{bmatrix} \tilde{\Theta}^{[1]}(t, \alpha) \\ \dots \\ \tilde{\Theta}^{[p]}(t, \alpha) \end{bmatrix} - (\Lambda + T)\begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \dots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix} \tag{28}$$

can be derived as well, in which

$$\tilde{\Theta}^{[i]}(t, \alpha) = [\tilde{\theta}_1^{[i]}(t - \alpha_1), \dots, \tilde{\theta}_m^{[i]}(t - \alpha_m)]^{\mathrm{T}}, \tag{29}$$

while Λ and T are the same matrices in (14) and (15), respectively. The following theorem then holds.

Theorem 3. *Consider system (28)-(29) with constant Φ satisfying A2, namely $\Phi\Phi^{\mathrm{T}}$ is positive definite in (5). Then the n-dimensional extended error vector $\tilde{\Theta}^{[e]}(t) = [\tilde{\Theta}^{[1]\mathrm{T}}(t), \dots, \tilde{\Theta}^{[p]\mathrm{T}}(t)]^{\mathrm{T}}$ ($n = pm$) globally asymptotically converges to zero. Furthermore, when $\alpha_1 = \dots = \alpha_m$, the convergence is exponential.*

[9] As mentioned in [31], $\varphi_i(t)$-functions are to be included here, for $t \in [0, \alpha_i]$, to make $\hat{\theta}_j^{[i]}(t)$, $i = 1, \dots, p$, belong to class C^1. On the other hand, as in Section 3, unitary learning gains are used for the sake of simplicity.

[10] Such an expression (it is valid also in [31]), resembles the Jacobi method for positive definite matrices ($\mathbb{I} + G$ shall be so in the proof of Theorem 3).

Proof. Recall the definition of (the symmetric) $G = \Lambda + T$ in Section 3, which, owing to its constant nature and Theorem 2, is positive definite under (5). Then introduce the following notations for the error vectors: $\tilde{\Theta}(t) = [\tilde{\Theta}^{[1]}(t)^{\mathrm{T}}, \dots, \tilde{\Theta}^{[p]}(t)^{\mathrm{T}}]^{\mathrm{T}}$ and $\tilde{\Theta}(t,\alpha) = [\tilde{\Theta}^{[1]}(t,\alpha)^{\mathrm{T}}, \dots, \tilde{\Theta}^{[p]}(t,\alpha)^{\mathrm{T}}]^{\mathrm{T}}$ so that the previous error equation can be rewritten in compact form as

$$(\mathbb{I} + G)\tilde{\Theta}(t) = \tilde{\Theta}(t,\alpha). \tag{30}$$

Now, for $t \geq 2\max\{\alpha_1, \alpha_2, \dots, \alpha_p\} \doteq \sigma$, consider the Lyapunov-like functional:

$$\mathcal{V}(t) = \frac{1}{2}\sum_{i=1}^{p}\left(\sum_{j=1}^{m}\int_{t-\alpha_j}^{t}\tilde{\theta}_j^{[i]2}(\tau)\mathrm{d}\tau\right), \tag{31}$$

whose time derivative, by direct computation, obeys

$$\dot{\mathcal{V}}(t) = \frac{1}{2}\left[\tilde{\Theta}(t)^{\mathrm{T}}\tilde{\Theta}(t) - \tilde{\Theta}(t,\alpha)^{\mathrm{T}}\tilde{\Theta}(t,\alpha)\right] = \frac{1}{2}\left[\tilde{\Theta}(t)^{\mathrm{T}}\tilde{\Theta}(t) - \tilde{\Theta}(t)^{\mathrm{T}}(\mathbb{I}+G)^2\tilde{\Theta}(t)\right]$$

$$\leq -\overbrace{\tilde{\Theta}(t)^{\mathrm{T}}G\tilde{\Theta}(t)}^{\tilde{\Theta}(t)^{\mathrm{T}}\Lambda\tilde{\Theta}(t)=\sum_{i=1}^{p}\tilde{y}_i^2(t)} \leq -\lambda_m(G)\|\tilde{\Theta}(t)\|^2.$$

Then, by integrating over $[\sigma, t]$, for $t \geq \sigma$, we have

$$\lambda_m(G)\int_{\sigma}^{t}\|\tilde{\Theta}(\tau)\|^2\mathrm{d}\tau \leq \mathcal{V}(\sigma). \tag{32}$$

By again drawing inspiration from [31], denote by $\tilde{\theta}_{j,D}^{[i]}(t)$ the time derivative of $\tilde{\theta}_j^{[i]}(t)$ and consider the Lyapunov-like functional ($t \geq \sigma$):

$$\mathcal{W}(t) = \frac{1}{2}\sum_{i=1}^{p}\left(\sum_{j=1}^{m}\int_{t-\alpha_j}^{t}\left(\tilde{\theta}_{j,D}^{[i]}(\tau)\right)^2\mathrm{d}\tau\right), \tag{33}$$

whose time derivative, by direct computation, obeys

$$\dot{\mathcal{W}}(t) = \frac{1}{2}\left[\dot{\tilde{\Theta}}(t)^{\mathrm{T}}\dot{\tilde{\Theta}}(t) - \dot{\tilde{\Theta}}(t,\alpha)^{\mathrm{T}}\dot{\tilde{\Theta}}(t,\alpha)\right] = \frac{1}{2}\left[\dot{\tilde{\Theta}}(t)^{\mathrm{T}}\dot{\tilde{\Theta}}(t) - \dot{\tilde{\Theta}}(t)^{\mathrm{T}}(\mathbb{I}+G)^2\dot{\tilde{\Theta}}(t)\right]$$

$$\leq -\dot{\tilde{\Theta}}(t)^{\mathrm{T}}G\dot{\tilde{\Theta}}(t) \leq -\lambda_m(G)\|\dot{\tilde{\Theta}}(t)\|^2.$$

Then, by integrating over $[\sigma, t]$, for $t \geq \sigma$, we get

$$\mathcal{W}(\sigma) \geq \mathcal{W}(t) + \lambda_m(G)\int_{\sigma}^{t}\|\dot{\tilde{\Theta}}(\tau)\|^2\mathrm{d}\tau, \tag{34}$$

which leads to

$$\mathcal{W}(\sigma) \geq \frac{1}{2}\sum_{i=1}^{p}\left(\sum_{j=1}^{m}\int_{t-\alpha_j}^{t}\left(\tilde{\theta}_{j,D}^{[i]}(\tau)\right)^2\mathrm{d}\tau\right)$$

$$\mathcal{W}(\sigma) \geq \lambda_m(G)\int_{\sigma}^{t}\|\dot{\tilde{\Theta}}(\tau)\|^2\mathrm{d}\tau.$$

As a consequence, it follows that $\tilde{\theta}_j^{[i]}(t)$, $i = 1, 2, \dots, p$, are uniformly continuous[11] over $[\sigma, +\infty)$. On the other hand, by using the Cauchy-Schwarz inequality on $\mathcal{L}_p$-spaces in the last step hereafter, we can write for any $t \geq \sigma$

$$\tilde{\theta}_j^{[i]2}(t) - \tilde{\theta}_j^{[i]2}(\sigma) = 2\int_{\sigma}^{t}\tilde{\theta}_j^{[i]}(\tau)\tilde{\theta}_{j,D}^{[i]}(\tau)\mathrm{d}\tau \leq 2\int_{\sigma}^{t}|\tilde{\theta}_j^{[i]}(\tau)||\tilde{\theta}_{j,D}^{[i]}(\tau)|\mathrm{d}\tau \leq 2\sqrt{\int_{\sigma}^{t}\tilde{\theta}_j^{[i]2}(\tau)\mathrm{d}\tau}\sqrt{\int_{\sigma}^{t}\left(\tilde{\theta}_{j,D}^{[i]}(\tau)\right)^2\mathrm{d}\tau} \leq \frac{2}{\lambda_m(G)}\sqrt{\mathcal{V}(\sigma)}\sqrt{\mathcal{W}(\sigma)}$$

showing that $\tilde{\theta}_j^{[i]}(t)$ is bounded over $[\sigma, +\infty)$. Thus $\tilde{\theta}_j^{[i]2}(t)$ is uniformly continuous over $[\sigma, +\infty)$ and therefore, by applying Barbalat Lemma (see [32] for instance), it follows that $\tilde{\theta}_j^{[i]}(t)$ tends asymptotically to zero. The last sentence regarding the special case of $\alpha_1 = \dots = \alpha_m$ can be proved from (30), by looking at the $(\mathbb{I}+G)$-eigenvalue-structure[12] while taking the supremum of the $\tilde{\Theta}(t)$-estimation error over $[k\alpha_1, (k+1)\alpha_1]$, $k \in \mathbb{N}$, and evaluating the solution to the resulting discrete-time equation. □

5. Simulation results

The first subsection refers to the content of Section 3 (constant parameters to be estimated), whereas the second subsection refers to the content of Section 4 (time-varying parameters to be estimated).

[11] Recall the definition of uniform continuity based on any $\varepsilon > 0$. Then recognize that, according to the Cauchy-Schwarz inequality on $\mathcal{L}_p$ spaces, $|\tilde{\theta}_j^{[i]}(t_1) - \tilde{\theta}_j^{[i]}(t_2)| = |\int_{t_1}^{t_2}\tilde{\theta}_{j,D}^{[i]}(t)\cdot 1\,\mathrm{d}t| \leq \sqrt{\int_{t_1}^{t_2}\left(\tilde{\theta}_{j,D}^{[i]}(t)\right)^2\mathrm{d}t}\sqrt{|t_1 - t_2|}$ and take $|t_1 - t_2| < \min\{\alpha_i, \frac{\varepsilon^2}{2\mathcal{W}(\sigma)}\}$.

[12] Under a restrictive condition such as the existence of uniform bounds on the G- positive eigenvalues, the time-varying Φ- case can be considered as well.

5.1. Application to vehicles with rear-wheel motors

Consider two induction motors [32] connected to the same mechanical system, as it happens in rear-wheel electric vehicles [33]. Each of such motors is: i) controlled by an algorithm that guarantees the same constant (positive) speed ω and flux modulus Ψ for both of them; ii) subject to the same unknown constant load torque T_L. Only one of the two motors is equipped with a speed sensor and the aim is to estimate the load torque in order to allow for the identification of the flux level minimizing power loss [34]. According to [35], the dynamics of the (constant positive) half a square modulus $w = (i_{sa}^2 + i_{sb}^2)/2 > 0$ and of $\xi = \sin(w_\theta)$, the sine of the angle w_θ of the stator current vector $[i_{sa}, i_{sb}]^{\mathrm{T}}$, can be computed as

$$\begin{aligned} \dot{w} &= -2\gamma w + P_e/\sigma + \phi_1^{\mathrm{T}}\Theta \\ \dot{\xi} &= \cos(w_\theta)Q_e/(2\sigma w) + \phi_2^{\mathrm{T}}\Theta \end{aligned}$$

where $\Theta = [\Psi_e, \omega T_L]^{\mathrm{T}}$ ($\Psi_e = \Psi^2/2$) is the vector of unknown constant parameters, $\phi_1^{\mathrm{T}} = [2\beta\alpha/M, -\beta L_r/M]$, $\phi_2^{\mathrm{T}} = -[\beta\omega\cos(w_\theta)/(Mw), \beta R_r\cos(w_\theta)/(2Mw\omega)]$ are the (constant) regressor vectors, (P_e, Q_e) are the active and reactive powers (depending on stator voltages and currents), and γ, σ, M, β, R_r, L_r are (known constant) motor parameters (motor resistances are preliminarly identified through the original procedure described in [36]). The first equation involves a regressor vector not depending on the motor speed and it is useful to design a preliminary estimation scheme

$$\dot{\hat{w}} = -2\gamma\hat{w} + P_e/\sigma + k_w(w - \hat{w})$$

for the sensorless motor (k_w is a positive design parameter), whereas the second one is useful to design a preliminary estimation scheme

$$\dot{\hat{\xi}} = \cos(w_\theta)Q_e/(2\sigma w) + k_\theta(\xi - \hat{\xi})$$

for the motor equipped with the speed sensor (k_θ is a positive design parameter). The equations in terms of the estimation errors $\tilde{w} = w - \hat{w}$, $\tilde{\xi} = \xi - \hat{\xi}$ read

$$\begin{aligned} \dot{\tilde{w}} &= -(2\gamma + k_w)\tilde{w} + \phi_1^{\mathrm{T}}\Theta \\ \dot{\tilde{\xi}} &= -k_\theta\tilde{\xi} + \phi_2^{\mathrm{T}}\Theta, \end{aligned}$$

whose solutions are

$$\begin{aligned} \tilde{w}(t) &= \phi_1^{\mathrm{T}}\Theta/(2\gamma + k_w) + \sigma_1(t) \\ \tilde{\xi}(t) &= \phi_2^{\mathrm{T}}\Theta/k_\theta + \sigma_2(t), \end{aligned}$$

with $\sigma_1(t)$ and $\sigma_2(t)$ exponentially converging to zero. By denoting $\tilde{w}$ by y_1 and $\tilde{\xi}$ by y_2, the estimation laws for the two estimate vectors $\hat{\Theta}^{[1]}$ and $\hat{\Theta}^{[2]}$ can be designed as in Section 3, leading to (δ is here introduced to weigh the contribution to the estimate generation)

$$\begin{aligned} \dot{\hat{\Theta}}^{[1]}(t) &= \phi_1(t)\Big(y_1(t) - \phi_1^{\mathrm{T}}(t)\hat{\Theta}^{[1]}(t)\Big) - \delta\eta_1(t) \\ \dot{\hat{\Theta}}^{[2]}(t) &= \phi_2(t)\Big(y_2(t) - \phi_2^{\mathrm{T}}(t)\hat{\Theta}^{[2]}(t)\Big) - \delta\eta_2(t) \\ \eta_1(t) &= \Big(\hat{\Theta}^{[1]}(t) - \hat{\Theta}^{[2]}(t)\Big) \\ \eta_2(t) &= \Big(\hat{\Theta}^{[2]}(t) - \hat{\Theta}^{[1]}(t)\Big). \end{aligned}$$

Both of such estimate vectors converge to the unknown Θ as Figs. 3 and 4 illustrate (motor parameter values are taken from [32], $\omega = 100$ rad/s, $\Psi = 1.16$ Wb ($\Psi_e = 0.6728$), $T_L = 5.8$ Nm; gains (in SI units) are $k_w = 1800$, $k_\theta = 900$, $\delta = 90$).

5.2. Illustrative example of Section 1

Simulation results are presented in this section to illustrate the theoretical derivations of Section 4. Consider a simplified version of the illustrative example reported in Section 1, in which two drones – knowing their own positions $X_1 = 1$ m, $Y_1 = 1$ m, $X_2 = 4$ m, $Y_2 = 2$ m – belong to the same plane of the target object ($x_0(t) = 2 + \sin(2t)$ m), i.e., $z_i = z_o = 3$ m, $i = 1, 2$, with the coordinate $Y_o = 4$ m of the target object being known. The two outputs

$$y_i(t) = D_i(t)^2 - (Y_i - Y_o)^2 - X_i^2 = \overbrace{[-2X_i, 1]}^{\phi_i^{\mathrm{T}}}\overbrace{[X_o(t), X_o(t)^2]^{\mathrm{T}}}^{\Theta(t)}$$

are expressed in terms of the Euclidean distance (in the space) between each drone and the target object. The scenario complies with the developments[13] of Section 4 with $\alpha_1 = \alpha_2 = \pi$ s (and in particular with the last sentence of Theorem 3). PE condition (5) is satisfied since the matrix $[-2, 1; -8, 1]$ is invertible. The learning estimation scheme is implemented with $\mu_1 = \mu_2 = 3200$ [SI

[13] Notice how α_i-periodic time functions, for $\theta_1(t) = X_o(t)$, $\theta_2(t) = Y_o(t)$, $\theta_3(t) = Z_o(t)$, are allowed in the general case of (at least) four drones, provided that $\alpha_i(t)$ (for $\theta_i(t)$) are pairwise-commensurable [31], $i = 1, 2, 3$, and $X_o(t)^2 + Y_o(t)^2 + Z_o(t)^2$ is thus periodic with a single period α_4.

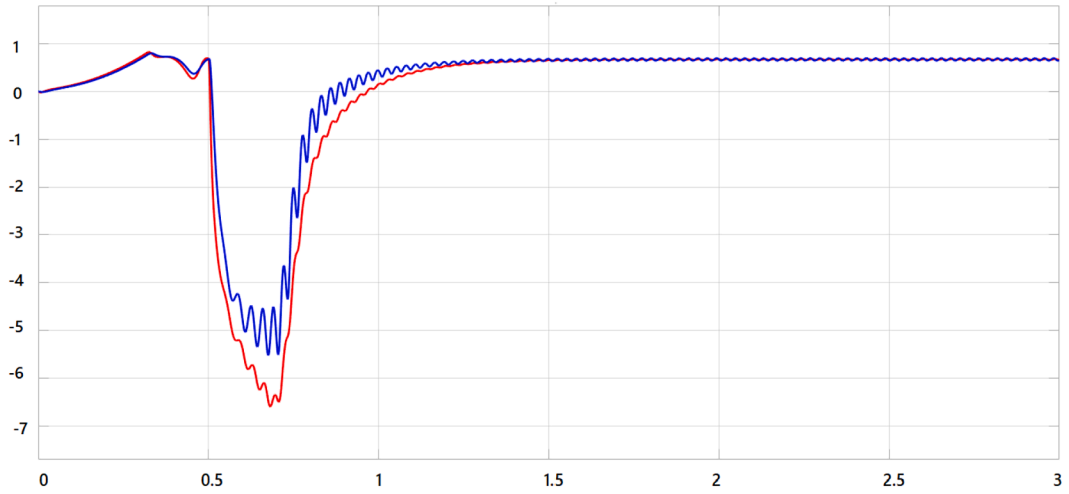


Fig. 3. *Consensus* achievement: first components of the estimate vectors $\hat{\Theta}^{[1]}$ [Wb^2] and $\hat{\Theta}^{[2]}$ [Wb^2], both of them converging to the first component of Θ, namely $\Psi_e = \Psi^2/2$ (time in seconds).

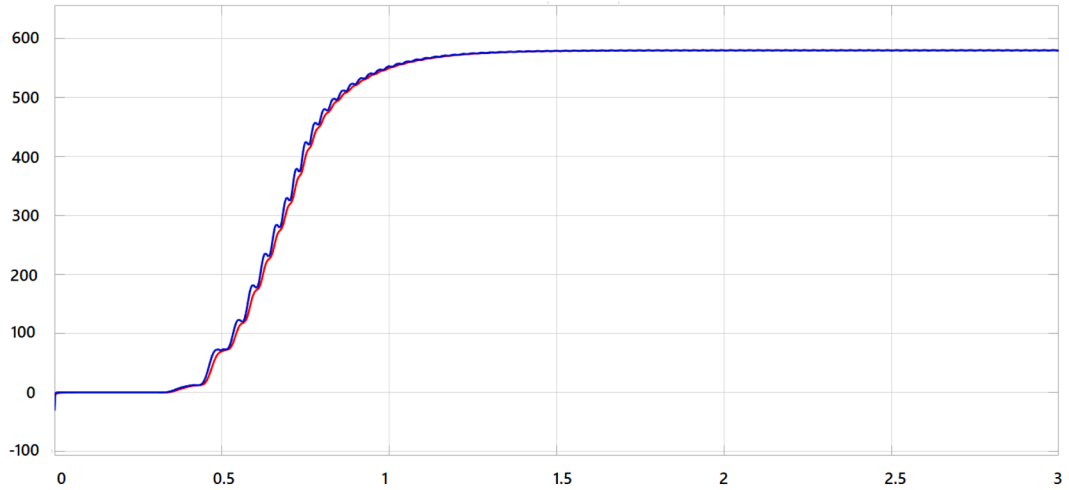


Fig. 4. *Consensus* achievement: second components of the estimate vectors $\hat{\Theta}^{[1]}$ [Nm] and $\hat{\Theta}^{[2]}$ [Nm], both of them converging to the second component of Θ, namely the product ωT_L between speed and load torque (time in seconds).

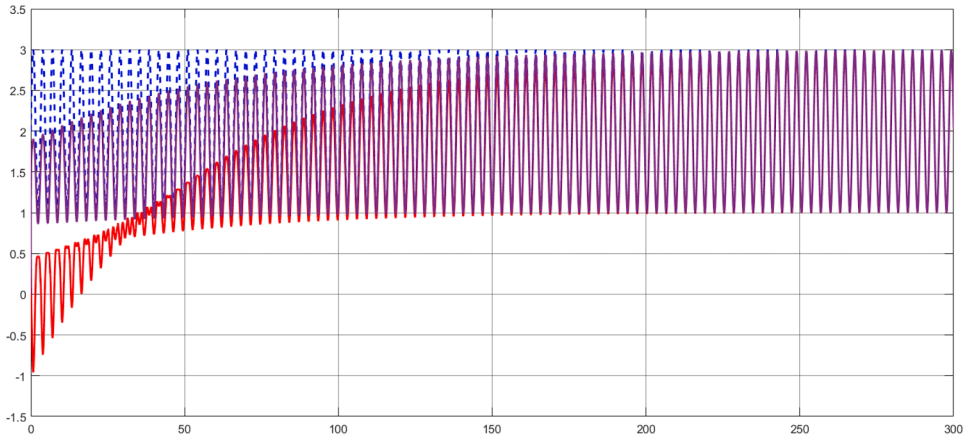


Fig. 5. Two drones with shared information. *Consensus* achievement: $\hat{\theta}_1^{[i]}(t)$ [m], $i = 1, 2$, converging to $\theta_1(t)$ [m] (time in seconds).

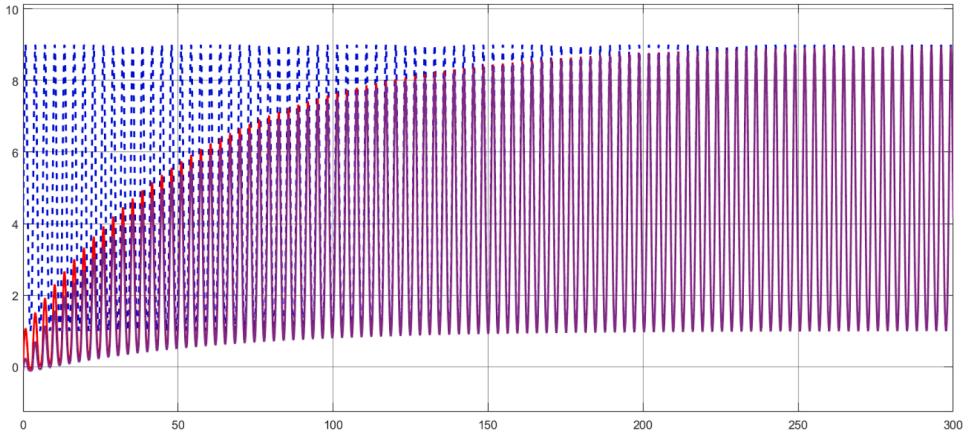


Fig. 6. Two drones with shared information. *Consensus* achievement: $\hat{\theta}_2^{[i]}(t)$ [m], $i = 1, 2$, converging to $\theta_2(t)$ [m] (time in seconds).

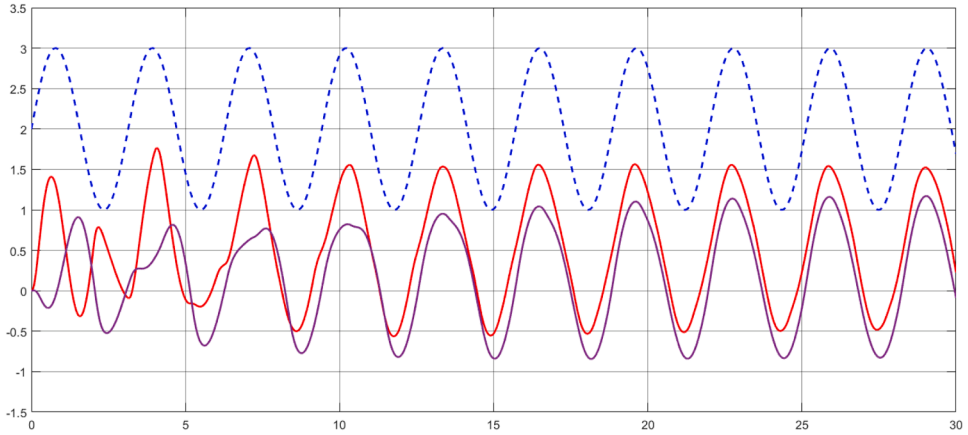


Fig. 7. Two drones with no information received from the neighbours: $\hat{\theta}_1^{[i]}(t)$ [m], $i = 1, 2$, converging, except for a constant, to $\theta_1(t)$ [m] (time in seconds).

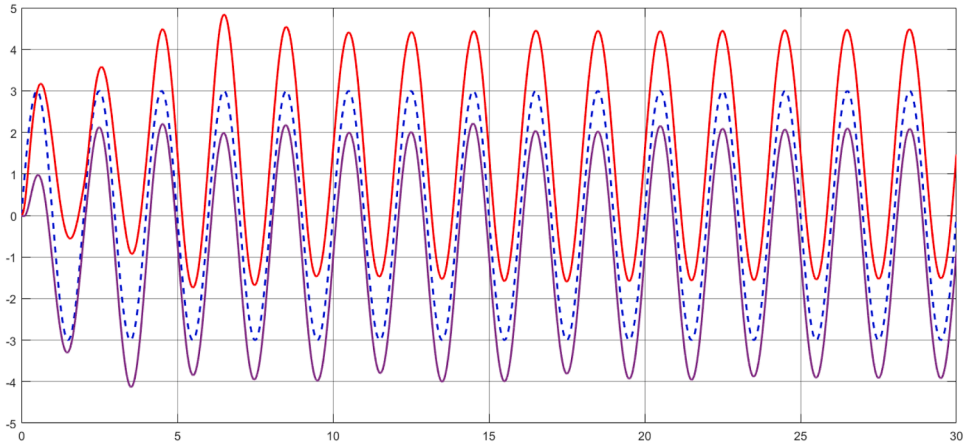


Fig. 8. Two drones with no information received from the neighbours: $\hat{\theta}_2^{[i]}(t)$ [m], $i = 1, 2$, converging, except for a constant, to $\theta_2(t)$ [m] (time in seconds).

units] (multiplying $\tilde{y}_i$ to shape the promptness of response and the robustness of the scheme, by modifying the relative weight of the estimation error within the estimation scheme, as mentioned in footnote 3) and with the C^1-class preserving functions: $\varphi_i(t) = -\frac{2}{\alpha_i^3}t^3 + \frac{3}{\alpha_i^2}t^2$ for $t \in [0, \alpha_i]$, $\varphi_i(t) = 1$ for $t > \alpha_i$. The corresponding estimates are reported in Figs. 5 and 6: identification is successfully performed. Identification fails in Figs. 7 and 8 as soon as no amount of information comes from the neighbours, since periodic errors in estimating $\theta_1(t)$, $\theta_2(t)$ appear, according to [31].

6. Conclusions

The exponential achievement of the unknown parameter estimation task on an undirected chain graph – within an originally designed symmetrical neighbourhood-based decentralized parallel architecture – has been guaranteed along with an explicit PE characterization. This has been obtained by exploiting the properties of the tridiagonal block matrix T in (15). The extension to the asymptotic identification of time-varying parameters – periodic with known periods – has been further considered: the corresponding open problem has been innovatively solved within the same theoretical framework through repetitive learning tools. Simulation results with reference to flying drones have been provided to illustrate the theoretical derivations. Even though the problem formulation given in Section 1 can be naturally generalized to different graph configurations in which strongly connected components can repeatedly play the role of (and replace) nodes in the graph (provided that a path connecting all the nodes of the graph is preserved and the zero column- (zero row-) sum structure for T is kept), the symmetric nature of T still keeps on being at the root of the proof of Theorems 1–3. Future work will consider different communication/graph configurations that might further reduce the amount of shared information but are still connected to structured matrix forms, for which the explicit characterization of the PE for consensus can be derived as well.

Acknowledgments

The work of Carmine Di Fiore has been partially supported by Gruppo Nazionale per il Calcolo Scientifico dell'Istituto Nazionale di Alta Matematica (INdAM-GNCS) and by the MUR Excellence Department Project MatMod@TOV 2023–2027 awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C23000330006.

Data availability

Data will be made available on request.

Appendix A. Lemma A.1

Let $v \in \mathbb{C}^r$, $v \neq 0$, and $\Lambda = \mathrm{diag}[\lambda_1, \dots, \lambda_r]$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, r$. Set $\lambda = \max\{0, \max_{i=1,\dots,r} \lambda_i\}$. If $\mu \in \mathbb{R}$ is such that

$$0 < \mu < (\lambda + v^{\mathrm{H}}v)^{-1}, \tag{A.1}$$

then $A = \mathbb{I} - \mu(\Lambda + vv^{\mathrm{H}})$ is positive definite.

Proof. Rewrite A as $(1 - \mu\lambda)\mathbb{I} - \mu vv^{\mathrm{H}} + \mu(\lambda\mathbb{I} - \Lambda)$. Denote by B the matrix $(1 - \mu\lambda)\mathbb{I} - \mu vv^{\mathrm{H}}$. It satisfies $B = B^{\mathrm{H}}$ and $\sigma(B) \subseteq \mathbb{R}^+$ if and only if (A.1) is satisfied. Thus B is positive definite. Moreover, for $z \in \mathbb{C}^r$, $z \neq 0$, we have

$$z^{\mathrm{H}}Az = z^{\mathrm{H}}Bz + \mu \sum_{i=1}^{r} (\lambda - \lambda_i)|z_i|^2 \geq z^{\mathrm{H}}Bz > 0.$$

This and the hermitianity of A yield the thesis. □

References

[1] J. Wu, V. Ugrinovskii, F. Allgöwer, Cooperative estimation for synchronization of heterogeneous multi-agent systems using relative information, IFAC Proc. Vol. 47 (3) (2014) 4662–4667.
[2] A. Concas, L. Reichel, G. Rodriguez, Y. Zhang, Chained graphs and some applications, Appl. Netw. Sci. 6 (39) (2021).
[3] G. Parlangeli, G. Notarstefano, On the reachability and observability of path and cycle graphs, IEEE Trans. Autom. Control 57 (3) (2012) 743–748.
[4] N. Christofides, The shortest hamiltonian chain of a graph, SIAM J. Appl. Math. 19 (4) (1970) 689–696.
[5] P. Bobade, D. Panagou, A.J. Kurdila, Multi-agent adaptive estimation with consensus in reproducing kernel hilbert spaces, 2019 18Th Eur. Control Conf. (2019).
[6] M.P. Fanti, A.M. Mangini, F. Mazzia, W. Ukovich, A new class of consensus protocols for agent networks with discrete time dynamics, Automatica 54 (2015) 1–7.
[7] E. Estrada, Path laplacian matrices: introduction and application to the analysis of consensus in networks, Linear Algebra Appl. 436 (9) (2012) 3373–3391.
[8] I. Matei, J.S. Baras, C. Somarakis, Convergence results for the linear consensus problem under markovian random graphs, SIAM J. Control Optim. 51 (2) (2013) 1574–1591.
[9] P.-Y. Chevalier, J.M. Hendrickx, R.M. Jungers, Efficient algorithms for the consensus decision problem, SIAM J. Control Optim. 53 (5) (2015) 3104–3119.
[10] A. Sarlette, R. Sepulchre, Consensus optimization on manifolds, SIAM J. Control Optim. 48 (1) (2009) 56–76.
[11] Y. Chen, J. Lü, X. Yu, Z. Lin, Consensus of discrete-time second-order multiagent systems based on infinite products of general stochastic matrices, SIAM J. Control Optim. 51 (4) (2013) 3274–3301.
[12] J.J.P. Veerman, D.K. Hammond, Tridiagonal matrices and boundary conditions, SIAM J. Matrix Anal. Appl. 37 (1) (2016) 1–17.

[13] C. Yao, Y. Qi, Z. Qu, W. Wang, Z. Zheng, Consensus control for multi-agent systems with unknown fading channels and limited bandwidth, Appl. Math. Comput. 510 (2026).
[14] D. Yang, J. He, J. Sun, J. Zhang, Observer-based leader-follower bipartite consensus for linear multiagent systems with communication link faults, Appl. Math. Comput. 510 (2026).
[15] X. Song, S. Zhang, J. Li, S. Gao, The consensus performance of leader-follower multi-agent systems under reversing the directions of arcs, Appl. Math. Comput. 509 (2026).
[16] S. Miao, H. Su, Consensus of different-order switched multi-agent systems on matrix-weighted networks, Appl. Math. Comput. 508 (2026).
[17] B. Chen, Z. Tang, J. Feng, Matrix measure-based event-triggered consensus of multi-agent systems with hybrid time delays, Appl. Math. Comput. 463 (2024).
[18] W. Chen, C. Wen, S. Hua, C. Sun, Distributed cooperative adaptive identification and control for a group of continuous-time systems with a cooperative PE condition via consensus, IEEE Trans. Autom. Control 59 (1) (2014) 91–106.
[19] N.R. Chowdhury, S. Sukumar, M. Maghenem, A. Loria, On the estimation of the consensus rate of convergence in graphs with persistent interconnections, Int. J. Control 91 (1) (2018) 132–144.
[20] K. Rahnama Rad, A. Tahbaz-Salehi, Distributed parameter estimation in networks, 49Th IEEE Conf. Decis. Control (2010) 5050–5055.
[21] Q. Liu, S. Zheng, Z. Wang, N.D. Alotaibi, F.E. Alsaadi, Distributed estimation for multi-agent systems with relative measurements and quantized communication: a feedback quadratic framework, Int. J. Robust Nonlinear Control 33 (5) (2023) 3164–3184.
[22] H. Adjerid, A.R. Maouche, Multi-Agent system-based decentralized state estimation method for active distribution networks, Comput. Electr. Eng. 86 (2020).
[23] J. Yan, H. Ishii, Distributed parameter estimation under gaussian observation noises, IFAC-PapersOnLine 56 (2) (2023) 3960–3965.
[24] A.S. Matveev, M. Almodarresi, R. Ortega, A. Pyrkin, S. Xie, Diffusion-based distributed parameter estimation through directed graphs with switching topology: application of dynamic regressor extension and mixing, IEEE Trans. Autom. Control 67 (8) (2022) 4256–4263.
[25] C.M. Verrelli, P. Tomei, Non-anticipating lyapunov functions for persistently excited nonlinear systems, IEEE Trans. Autom. Control 65 (6) (2020) 2634–2639.
[26] M.A. Demetriou, S.S. Nestinger, Adaptive consensus estimation of multi-agent systems, 2011 50Th IEEE Conf. Decis. Control Eur. Control Conf. (2011) 354–359.
[27] P. Tomei, R. Marino, Adaptive nonlinear control with constrained parallel parameter estimates, Syst. Control Lett. 143 (2020).
[28] R. Bhatia, Matrix Analysis, Springer New York, NY, 1996.
[29] Z. Dostál, T. Kozubek, A. Markopoulos, M. Mensík, Cholesky decomposition of a positive semidefinite matrix with known kernel, Appl. Math. Comput. 217 (2011) 6067–6077.
[30] M. Schatzman, Numerical analysis: a mathematical introduction, Clarendon Press, Oxford (2022).
[31] C.M. Verrelli, Exponential output tracking via a saturated RLC for a class of nonlinear systems under PSD of periodic references, Syst. Control Lett. 19 (2024).
[32] R. Marino, P. Tomei, C.M. Verrelli, Induction Motor Control Design, Springer, 2010.
[33] C. Pomponi, S. Scalzi, L. Pasquale, C.M. Verrelli, R. Marino, Automatic motor speed reference generators for cruise and lateral control of electric vehicles with in-wheel motors, Control Eng. Pract. 79 (2018) 126–143.
[34] R. Marino, S. Scalzi, P. Tomei, C.M. Verrelli, Fault-tolerant cruise control of electric vehicles with induction motors, Control Eng. Pract. 21 (2013) 860–869.
[35] C.M. Verrelli, E. Lorenzani, R. Fornari, M. Mengoni, L. Zarri, Steady-state speed sensor fault detection in induction motors with uncertain parameters: a matter of algebraic equations, Control Eng. Pract. 80 (2018) 125–137.
[36] A. Accetta, S. Bifaretti, M. Pucci, P. Tomei, C.M. Verrelli, Further developments in state estimation and parameter identification for efficiency optimization of induction motor operations, 2024 IEEE 20th Int. Conf. Autom. Sci. Eng. (2024).