Rendiconti del Circolo Matematico di Palermo Series 2 (2022) 71:225–231
https://doi.org/10.1007/s12215-021-00618-6

A lower bound for $\chi(\mathcal{O}_S)$

Vincenzo Di Gennaro[1]

Received: 17 February 2021 / Accepted: 11 May 2021 / Published online: 24 May 2021

Abstract
Let $(S, \mathcal{L})$ be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle $\mathcal{L}$ of degree $d > 25$. In this paper we prove that $\chi(\mathcal{O}_S) \geq -\frac{1}{8}d(d-6)$. The bound is sharp, and $\chi(\mathcal{O}_S) = -\frac{1}{8}d(d-6)$ if and only if d is even, the linear system $|H^0(S, \mathcal{L})|$ embeds S in a smooth rational normal scroll $T \subset \mathbb{P}^5$ of dimension 3, and here, as a divisor, S is linearly equivalent to $\frac{d}{2}Q$, where Q is a quadric on T. Moreover, this is equivalent to the fact that a general hyperplane section $H \in |H^0(S, \mathcal{L})|$ of S is the projection of a curve C contained in the Veronese surface $V \subseteq \mathbb{P}^5$, from a point $x \in V \backslash C$.

Keywords Projective surface · Castelnuovo–Halphen's Theory · Rational normal scroll · Veronese surface

Mathematics Subject Classification Primary 14J99 · Secondary 14M20 · 14N15 · 51N35

1 Introduction

In [6] D. Franco and the author prove a sharp lower bound for the self-intersection K_S^2 of the canonical bundle of a smooth, projective, complex surface S, polarized by a very ample line bundle $\mathcal{L}$, in terms of its degree $d = \deg \mathcal{L}$, assuming $d > 35$. Refining the line of the proof in [6], in the present paper we deduce a similar result for the Euler characteristic $\chi(\mathcal{O}_S)$ of S [1, p. 2], in the range $d > 25$. More precisely, we prove the following:

Theorem 1.1 *Let $(S, \mathcal{L})$ be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle $\mathcal{L}$ of degree $d > 25$. Then*:

$$\chi(\mathcal{O}_S) \geq -\frac{1}{8}d(d-6).$$

The bound is sharp, and the following properties are equivalent.

(i) $\chi(\mathcal{O}_S) = -\frac{1}{8}d(d-6)$;

Vincenzo Di Gennaro
digennar@axp.mat.uniroma2.it

1 Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Rome, Italy

(ii) $h^0(S,\mathcal{L})=6$, *and the linear system* $|H^0(S,\mathcal{L})|$ *embeds* S *in* $\mathbb{P}^5$ *as a scroll with sectional genus* $g=\frac{1}{8}d(d-6)+1$;

(iii) $h^0(S,\mathcal{L})=6$, d *is even, and the linear system* $|H^0(S,\mathcal{L})|$ *embeds* S *in a smooth rational normal scroll* $T\subset\mathbb{P}^5$ *of dimension* 3, *and here* S *is linearly equivalent to* $\frac{d}{2}(H_T-W_T)$, *where* H_T *is the hyperplane class of* T, *and* W_T *the ruling* (*i.e.* S *is linearly equivalent to an integer multiple of a smooth quadric* $Q\subset T$).

By Enriques' classification, one knows that if S is unruled or rational, then $\chi(\mathcal{O}_S)\geq 0$. Hence, Theorem 1.1 essentially concerns irrational ruled surfaces.

In the range $d>35$, the family of extremal surfaces for $\chi(\mathcal{O}_S)$ is exactly the same for K_S^2. We point out there is a relationship between this family and the Veronese surface. In fact one has the following:

Corollary 1.2 *Let* $S\subseteq\mathbb{P}^r$ *be a nondegenerate, smooth, irreducible, projective, complex surface, of degree* $d>25$. *Let* $L\subseteq\mathbb{P}^r$ *be a general hyperplane. Then* $\chi(\mathcal{O}_S)=-\frac{1}{8}d(d-6)$ *if and only* if $r=5$, *and there is a curve* C *in the Veronese surface* $V\subseteq\mathbb{P}^5$ *and a point* $x\in V\backslash C$ *such that a general hyperplane section* $S\cap L$ *of* S *is the projection* $p_x(C)\subseteq L$ *of* C *in* $L\cong\mathbb{P}^4$, *from the point* x.

In particular, $S\cap L$ is not linearly normal, even if S is.

2 Proof of Theorem 1.1

Remark 2.1 (i) We say that $S\subset\mathbb{P}^r$ is a *scroll* if S is a $\mathbb{P}^1$-bundle over a smooth curve, and the restriction of $\mathcal{O}_S(1)$ to a fibre is $\mathcal{O}_{\mathbb{P}^1}(1)$. In particular, S is a geometrically ruled surface, and therefore $\chi(\mathcal{O}_S)=\frac{1}{8}K_S^2$ [1, Proposition III.21].

(ii) By Enriques' classification [1, Theorem X.4 and Proposition III.21], one knows that if S is unruled or rational, then $\chi(\mathcal{O}_S)\geq 0$, and if S is ruled with irregularity >0, then $\chi(\mathcal{O}_S)\geq\frac{1}{8}K_S^2$. Therefore, taking into account previous remark, when $d>35$, Theorem 1.1 follows from [6, Theorem 1.1]. In order to examine the range $25<d\leq 35$, we are going to refine the line of the argument in the proof of [6, Theorem 1.1].

(iii) When $d=2\delta$ is even, then $\frac{1}{8}d(d-6)+1$ is the genus of a plane curve of degree δ, and the genus of a curve of degree d lying on the Veronese surface.

Put $r+1:=h^0(S,\mathcal{L})$. Therefore, $|H^0(S,\mathcal{L})|$ embeds S in $\mathbb{P}^r$. Let $H\subseteq\mathbb{P}^{r-1}$ be a general hyperplane section of S, so that $\mathcal{L}\cong\mathcal{O}_S(H)$. We denote by g the genus of H. If $2\leq r\leq 3$, then $\chi(\mathcal{O}_S)\geq 1$. Therefore, we may assume $r\geq 4$.

The case $r=4$.

We first examine the case $r=4$. In this case we only have to prove that, for $d>25$, one has $\chi(\mathcal{O}_S)>-\frac{1}{8}d(d-6)$. We may assume that S is an irrational ruled surface, so $K_S^2\leq 8\chi(\mathcal{O}_S)$ (compare with previous Remark 2.1, (ii)). We argue by contradiction, and assume also that

$$\chi(\mathcal{O}_S)\leq-\frac{1}{8}d(d-6). \tag{1}$$

We are going to prove that this assumption implies $d\leq 25$, in contrast with our hypothesis $d>25$.

By the double point formula:

$$d(d-5)-10(g-1)+12\chi(\mathcal{O}_S)=2K_S^2,$$

and $K_S^2 \leq 8\chi(\mathcal{O}_S)$, we get:

$$d(d-5)-10(g-1) \leq 4\chi(\mathcal{O}_S).$$

And from $\chi(\mathcal{O}_S) \leq -\frac{1}{8}d(d-6)$ we obtain

$$10g \geq \frac{3}{2}d^2-8d+10. \tag{2}$$

Now we distinguish two cases, according that S is not contained in a hypersurface of degree <5 or not.

First suppose that S is not contained in a hypersurface of $\mathbb{P}^4$ of degree <5. Since $d>16$, by Roth's Theorem ([12, p. 152], [8, p. 2, (C)]), H is not contained in a surface of $\mathbb{P}^3$ of degree <5. Using Halphen's bound [9], we deduce that

$$g \leq \frac{d^2}{10}+\frac{d}{2}+1-\frac{2}{5}(\epsilon+1)(4-\epsilon),$$

where $d-1=5m+\epsilon$, $0\leq\epsilon<5$. It follows that

$$\frac{3}{2}d^2-8d+10 \leq 10g \leq d^2+5d+10\Big(1-\frac{2}{5}(\epsilon+1)(4-\epsilon)\Big).$$

This implies that $d \leq 25$, in contrast with our hypothesis $d>25$.

In the second case, assume that S is contained in an irreducible and reduced hypersurface of degree $s\leq 4$. When $s\in\{2,3\}$, one knows that, for $d>12$, S is of general type [2, p. 213]. Therefore, we only have to examine the case $s=4$. In this case H is contained in a surface of $\mathbb{P}^3$ of degree 4. Since $d>12$, by Bezout's Theorem, H is not contained in a surface of $\mathbb{P}^3$ of degree <4. Using Halphen's bound [9], and [8, Lemme 1], we get:

$$\frac{d^2}{8}-\frac{9d}{8}+1 \leq g \leq \frac{d^2}{8}+1.$$

Hence, there exists a rational number $0\leq x\leq 9$ such that

$$g=\frac{d^2}{8}+d\Big(\frac{x-9}{8}\Big)+1.$$

If $0\leq x\leq\frac{15}{2}$, then $g\leq\frac{d^2}{8}-\frac{3}{16}d+1$, and from (2) we get

$$\frac{3}{20}d^2-\frac{4}{5}d+1 \leq g \leq \frac{d^2}{8}-\frac{3}{16}d+1.$$

It follows $d\leq 24$, in contrast with our hypothesis $d>25$.

Assume $\frac{15}{2}<x\leq 9$. Hence,

$$\Big(\frac{d^2}{8}+1\Big)-g=-d\Big(\frac{x-9}{8}\Big)<\frac{3}{16}d.$$

By [5, proof of Proposition 2, and formula (2.2)], we have

$$\chi(\mathcal{O}_S) \geq 1 + \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{349}{16} - (d-3)\left[\left(\frac{d^2}{8}+1\right) - g\right]$$
$$> 1 + \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{349}{16} - (d-3)\frac{3}{16}d = \frac{d^3}{96} - \frac{d^2}{4} - \frac{53}{48}d - \frac{333}{16}.$$

Combining with (1), we get

$$\frac{d^3}{96} - \frac{d^2}{4} - \frac{53}{48}d - \frac{333}{16} + \frac{1}{8}d(d-6) < 0,$$

i.e.

$$d^3 - 12d^2 - 178d - 1998 < 0.$$

It follows $d \leq 23$, in contrast with our hypothesis $d > 25$.

This concludes the analysis of the case $r = 4$.

The case $r \geq 5$.

When $r \geq 5$, by [6, Remark 2.1], we know that, for $d > 5$, one has $K_S^2 > -d(d-6)$, except when $r = 5$, and the surface S is a scroll, $K_S^2 = 8\chi(\mathcal{O}_S) = 8(1-g)$, and

$$g = \frac{1}{8}d^2 - \frac{3}{4}d + \frac{(5-\epsilon)(\epsilon+1)}{8}, \tag{3}$$

with $d - 1 = 4m + \epsilon$, $0 < \epsilon \leq 3$. In this case, by [6, pp. 73–76], we know that, for $d > 30$, S is contained in a smooth rational normal scroll of $\mathbb{P}^5$ of dimension 3. Taking into account that we may assume $K_S^2 \leq 8\chi(\mathcal{O}_S)$ (compare with Remark 2.1, (*i*) and (*ii*)), at this point Theorem 1.1 follows from [6, Proposition 2.2], when $d > 30$.

In order to examine the remaining cases $26 \leq d \leq 30$, we refine the analysis appearing in [6]. In fact assuming that $r = 5$ and S is a scroll, and assuming that (3) holds, then S is contained in a smooth rational normal scroll of $\mathbb{P}^5$ also in the range $26 \leq d \leq 30$. Then we may conclude as before, because [6, Proposition 2.2] holds true for $d \geq 18$.

First, observe that if S is contained in a threefold $T \subset \mathbb{P}^5$ of dimension 3 and minimal degree 3, then T is necessarily a *smooth* rational normal scroll [6, p. 76]. Moreover, observe that we may apply the same argument as in [6, pp. 75–76] in order to exclude the case S is contained in a threefold of degree 4. In fact the argument works for $d > 24$ [6, p. 76, first line after formula (13)].

In conclusion, assuming that $r = 5$ and S is a scroll, and assuming that (3) holds, it remains to exclude that S is not contained in a threefold of degree < 5 in the range $26 \leq d \leq 30$.

Assume S is not contained in a threefold of degree < 5. Denote by $\Gamma \subset \mathbb{P}^3$ a general hyperplane section of H. Recall that $26 \leq d \leq 30$.

- **Case I** $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) \geq 2$.

It is impossible. In fact, if $d > 4$, by monodromy [4, Proposition 2.1], Γ should be contained in a reduced and irreducible space curve of degree ≤ 4, and so, for $d > 20$, S should be contained in a threefold of degree ≤ 4 [3, Theorem (0.2)].

- **Case II** $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$ and $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) > 4$.

As before, if $d > 6$, by monodromy, Γ is contained in a reduced and irreducible space curve X of degree $\deg(X) \leq 6$. Again as before, if $\deg(X) \leq 4$, then S is contained in a threefold of degree ≤ 4. So we may assume $5 \leq \deg(X) \leq 6$.

Denote by h_Γ and h_X the Hilbert function of Γ and X. First notice that, since $\Gamma \subset \mathbb{P}^3$ is non degenerate, and $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$, we have:

$$h_\Gamma(1) = 4, \quad \text{and} \quad h_\Gamma(2) = 9. \tag{4}$$

Moreover, since $d \geq 26$, by Bezout's Theorem we have

$$h_\Gamma(i) = h_X(i) \quad \text{for every} \quad i \leq 4. \tag{5}$$

Let $X' \subset \mathbb{P}^2$ be a general plane section of X, and $h_{X'}$ its Hilbert function. By [7, Lemma (3.1), p. 83] we know that $h_X(i) - h_X(i-1) \geq h_{X'}(i)$ for every i. Therefore, for every i, we have:

$$h_X(i) \geq \sum_{j=0}^{i} h_{X'}(j). \tag{6}$$

On the other hand, by [7, Corollary (3.6), p. 87], we also know that

$$h_{X'}(j) \geq \min\{2j + 1, \deg(X)\}. \tag{7}$$

Therefore, by (5), (6), and (7) (recall that $5 \leq \deg(X) \leq 6$), we get:

$$h_\Gamma(3) \geq 14 \quad \text{and} \quad h_\Gamma(4) \geq 19. \tag{8}$$

By [7, Corollary (3.5), p. 86] we have:

$$h_\Gamma(i + j) \geq \min\{d, h_\Gamma(i) + h_\Gamma(j) - 1\} \quad \text{for every } i \text{ and } j. \tag{9}$$

Combining (9) with (4) and (8), we get:

$$h_\Gamma(5) \geq 22,\ h_\Gamma(6) \geq \min\{d, 27\},\ h_\Gamma(7) = d. \tag{10}$$

Since in general we have [7, Corollary (3.2) p. 84]

$$g \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i), \tag{11}$$

from (4), (8), and (10), taking into account that $26 \leq d \leq 30$, it follows that:

$$g \leq (d-4) + (d-9) + (d-14) + (d-19) + (d-22) + 3 = 5d - 65,$$

which is $< \frac{1}{8}d(d-6) + 1$ for $d \geq 26$. This is in contrast with (3).

• **Case III** $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$ and $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) = 4$.

Using these assumptions, by (4) and (9), we have:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 9,\ h_\Gamma(3) = 16,\ h_\Gamma(4) \geq 19, h_\Gamma(5) \geq 24,\ h_\Gamma(6) = d.$$

By (11) it follows that:

$$g \leq (d-4) + (d-9) + (d-16) + (d-19) + (d-24) = 5d - 72,$$

which is $< \frac{1}{8}d(d-6) + 1$ for $d \geq 26$. This is in contrast with (3).

• **Case IV** $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 0$.

Using this assumption, by (4) and (9), we have:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 10,\ h_\Gamma(3) \geq 13,\ h_\Gamma(4) \geq 19,$$
$$h_\Gamma(5) \geq 22,\ h_\Gamma(6) \geq \min\{d, 28\},\ h_\Gamma(7) = d.$$

By (11) it follows that:

$$g \le (d-4)+(d-10)+(d-13)+(d-19)+(d-22)+2 = 5d-66,$$

which is $< \frac{1}{8}d(d-6)+1$ for $d \ge 26$. This is in contrast with (3).

This concludes the proof of Theorem 1.1.

Remark 2.2 (*i*) Let $Q \subseteq \mathbb{P}^3$ be a smooth quadric, and $H \in |\mathcal{O}_Q(1, d-1)|$ be a smooth rational curve of degree d [11, p. 231, Exercise 5.6]. Let $S \subseteq \mathbb{P}^4$ be the projective cone over H. A computation, which we omit, proves that

$$\chi(\mathcal{O}_S) = 1 - \binom{d-1}{3}.$$

Therefore, if S is singular, it may happen that $\chi(\mathcal{O}_S) < -\frac{1}{8}d(d-6)$. One may ask whether $1 - \binom{d-1}{3}$ is a lower bound for $\chi(\mathcal{O}_S)$ for every *integral* surface.

(*ii*) Let $(S, \mathcal{L})$ be a smooth surface, polarized by a very ample line bundle $\mathcal{L}$ of degree d. By Harris' bound for the geometric genus $p_g(S)$ of S [10], we see that $p_g(S) \le \binom{d-1}{3}$. Taking into account that for a smooth surface one has $\chi(\mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) \le 1 + h^2(S, \mathcal{O}_S) = 1 + p_g(S)$, from Theorem 1.1 we deduce (the first inequality only when $d > 25$):

$$-\binom{\frac{d}{2}-1}{2} \le \chi(\mathcal{O}_S) \le 1 + \binom{d-1}{3}.$$

3 Proof of Corollary 1.2

• First, assume that $\chi(\mathcal{O}_S) = -\frac{1}{8}d(d-6)$.

By Theorem 1.1, we know that $r = 5$. Moreover, S is contained in a nonsingular threefold $T \subseteq \mathbb{P}^5$ of minimal degree 3. Therefore, a general hyperplane section $H = S \cap L$ of S ($L \cong \mathbb{P}^4$ denotes a general hyperplane of $\mathbb{P}^5$) is contained in a smooth surface $\Sigma = T \cap L$ of $L \cong \mathbb{P}^4$, of minimal degree 3.

This surface Σ is isomorphic to the blowing-up of $\mathbb{P}^2$ at one point [1, p. 58]. Moreover, if V denotes the Veronese surface in $\mathbb{P}^5$, for a suitable point $x \in V \backslash L$, the projection of $\mathbb{P}^5 \backslash \{x\}$ on $L \cong \mathbb{P}^4$ from x restricts to an isomorphism

$$p_x : V \backslash \{x\} \to \Sigma \backslash E,$$

where E denotes the exceptional line of Σ [1, loc. cit.].

Since S is linearly equivalent on T to $\frac{d}{2}(H_T - W_T)$ (H_T denotes the hyperplane section of T, and W_T the ruling), it follows that H is linearly equivalent on Σ to $\frac{d}{2}(H_\Sigma - W_\Sigma)$ (now H_Σ denotes the hyperplane section of Σ, and W_Σ the ruling of Σ). Therefore, H does not meet the exceptional line $E = H_\Sigma - 2W_\Sigma$. In fact, since $H_\Sigma^2 = 3$, $H_\Sigma \cdot W_\Sigma = 1$, and $W_\Sigma^2 = 0$, one has:

$$(H_\Sigma - W_\Sigma) \cdot (H_\Sigma - 2W_\Sigma) = H_\Sigma^2 - 3H_\Sigma \cdot W_\Sigma + 2W_\Sigma^2 = 0.$$

This implies that H is contained in $\Sigma \backslash E$, and the assertion of Corollary 1.2 follows.

• Conversely, assume there exists a curve C on the Veronese surface $V \subseteq \mathbb{P}^5$, and a point $x \in V \backslash C$, such that H is the projection $p_x(C)$ of C from the point x.

In particular, d is an even number, and H is contained in a smooth surface $\Sigma \subseteq L \cong \mathbb{P}^4$ of minimal degree, and is disjoint from the exceptional line $E \subseteq \Sigma$. By [3, Theorem (0.2)], S is contained in a threefold $T \subseteq \mathbb{P}^5$ of minimal degree. T is nonsingular. In fact, otherwise, H should be a Castelnuovo's curve in $\mathbb{P}^4$ [6, p. 76]. On the other hand, by our assumption, H is isomorphic to a plane curve of degree $\frac{d}{2}$. Hence, we should have:

$$g = \frac{d^2}{6} - \frac{2}{3}d + 1 = \frac{d^2}{8} - \frac{3}{4}d + 1$$

(the first equality because H is Castelnuovo's, the latter because H is isomorphic to a plane curve of degree $\frac{d}{2}$). This is impossible when $d > 0$.

Therefore, S is contained in a smooth threefold T of minimal degree in $\mathbb{P}^5$.

Now observe that in Σ there are only two families of curves of degree even d and genus $g = \frac{d^2}{8} - \frac{3}{4}d + 1$. These are the curves linearly equivalent on Σ to $\frac{d}{2}(H_\Sigma - W_\Sigma)$, and the curves equivalent to $\frac{d+2}{6}H_\Sigma + \frac{d-2}{2}W_\Sigma$. But only in the first family the curves do not meet E. Hence, H is linearly equivalent on Σ to $\frac{d}{2}(H_\Sigma - W_\Sigma)$. Since the restriction $\mathrm{Pic}(T) \to \mathrm{Pic}(\Sigma)$ is bijective, it follows that S is linearly equivalent on T to $\frac{d}{2}(H_T - W_T)$. By Theorem 1.1, S is a fortiori linearly normal, and of minimal Euler characteristic $\chi(\mathcal{O}_S) = -\frac{1}{8}d(d-6)$.

Funding Open access funding provided by Università degli Studi di Roma Tor Vergata within the CRUI-CARE Agreement.

References

1. Beauville, A.: Surfaces algébriques complexes. Astérisque 54, société mathématiques de france (1978)
2. Braun, R., Floystad, G.: A bound for the degree of smooth surfaces in $\mathbb{P}_4$ not of general type. Compos. Math. **93**(2), 211–229 (1994)
3. Chiantini, L., Ciliberto, C.: A few remarks on the lifting problem. Astérisque **218**, 95–109 (1993)
4. Chiantini, L., Ciliberto, C., Di Gennaro, V.: The genus of projective curves. Duke Math. J. **70**(2), 229–245 (1993)
5. Di Gennaro, V.: A note on smooth surfaces in $\mathbb{P}^4$. Geom. Dedicata **71**, 91–96 (1998)
6. Di Gennaro, V., Franco, D.: A lower bound for K_S^2. Rendiconti del Circolo Matematico di Palermo II Ser. **66**, 69–81 (2017)
7. Eisenbud, D., Harris, J.: Curves in Projective Space. Sém. Math. Sup. 85 Les Presses de l'Université de Montréal (1982)
8. Ellingsrud, G., Peskine, Ch.: Sur les surfaces lisses de $\mathbb{P}_4$. Invent. Math. **95**, 1–11 (1989)
9. Gruson, L., Peskine, Ch.: Genre des courbes dans l'espace projectif. In: Algebraic Geometry: Proceedings, Norway, 1977, Lecture Notes in Mathematics, vol. 687, pp. 31–59. Springer, New York (1978)
10. Harris, J.: A bound on the geometric genus of projective varieties. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **8**(4), 35–68 (1981)
11. Hartshorne, R.: Algebraic Geometry, GTM, vol. 52. Springer, New York (1983)
12. Roth, L.: On the projective classification of surfaces. Proc. Lond. Math. Soc. **42**, 142–170 (1937)