DISCOUNTED HAMILTON-JACOBI EQUATIONS ON NETWORKS AND ASYMPTOTIC ANALYSIS

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ABSTRACT. We study discounted Hamilton–Jacobi equations on networks, without putting any restriction on their geometry. Assuming the Hamiltonians are continuous and coercive, we establish a comparison principle and provide representation formulae for solutions. We follow the approach introduced in [11], namely we associate to the differential problem on the network, a discrete functional equation on an abstract underlying graph. We perform some qualitative analysis and single out a distinguished subset of vertices, called λ–Aubry set, which shares some properties of the Aubry set for Eikonal equations on compact manifolds. We finally study the asymptotic behavior of solutions and λ–Aubry sets as the discount factor λ becomes infinitesimal.

1. INTRODUCTION

We are concerned with discounted Hamilton–Jacobi equations on networks. We establish a comparison principle, provide representation formulae for solutions and perform some qualitative analysis. We emphasize that our results apply without any restriction on the geometry of the network. In particular multiple arcs connecti ng a given pair of vertices are allowed as well as loops or multiple loops based on a single vertex.

Given a finite family of Hamiltonians H_γ defined on $[0,1] \times \mathbb{R}$, indexed by a parameter γ, we consider the corresponding discounted equations

$$\lambda\, u + H_\gamma(s, u') = 0 \qquad \text{in } (0,1), \tag{1}$$

with discount factor λ independent of γ. The H_γ are assumed continuous in both arguments and coercive in the momentum variable, no convexity is required, see assumptions **(H1)**, **(H2)** in Section 2.

Since the Hamiltonians are unrelated and no boundary conditions are specified, these equations possess infinite viscosity solutions, when separately considered.

2010 *Mathematics Subject Classification.* 35F21, 35R02, 35B51, 49L25.

Key words and phrases. Hamilton-Jacobi equation, Embedded networks, Graphs, Viscosity solutions, Comparison principle, Discrete functional equation on graphs, Hopf–Lax formula, Discrete weak KAM theory.

A sort of geometric coupling is provided by setting each equation on an arc of a given network Γ immersed in $\mathbb{R}^N$, and combining them with additional conditions at the vertices, namely at the junction points of different arcs. In other terms boundary conditions are introduced in correspondence to endpoints 0 and 1 of the parametrization. The subtle point however is that these conditions are not required in the same way at all vertices for supersolutions, but are given taking into account the geometry of the network, as made precise in Definition 2.3 **iii)**.

The aim is to uniquely select distinguished solutions of all equations which piece together continuously at vertices, in other terms to uniquely determine a solution of the differential problem on Γ. Namely a continuous functions $u : \Gamma \longrightarrow \mathbb{R}$ satisfying

$$\lambda\, u \circ \gamma + H_\gamma(s, (u \circ \gamma)') = 0 \qquad \text{in } (0, 1)$$

in the viscosity sense for any arc γ, plus vertex conditions. Following [8], [11] we consider state constraint type boundary conditions which correspond to, at least on the arcs where these boundary conditions apply, so called maximal solutions of (1). By this we mean that fixing a number α, and considering the family S_α of all solutions taking the value α at 0, the element of S_α which also satisfies the state constraint boundary condition at 1 is maximal in S_α.

We attack the problem through the approach introduced in [11] for the Eikonal case. Namely we associate to the above described problem on Γ a discrete equation defined on an underlying graph, which has the same vertices of Γ and edges corresponding to the arcs of Γ.

The two problems are related by the fact that the trace on the vertices of a solution of the continuous equation solves the discrete one, and conversely any solution of the discrete equation can be uniquely extended, from vertices to the whole network, to a solution of the HJ discounted equation. See ($\mathbf{DFE}_\lambda$) and Theorem 4.2, Proposition 4.3 in Section 4.

We can therefore prove existence and comparison results for the discrete equation and then transfer it to the differential problem on the network. The advantage of this procedure is twofold. The comparison principles are obtained through simple combinatorial techniques bypassing Crandall–Lions doubling variables method, see Theorem 4.4. In addition, explicit representation formulae for solutions can be provided, see (17), which makes possible a qualitative analysis, using a suitable functional defined on the paths of the graph, see Definition 5.1.

In this way we can single out a special subset of vertices (and edges), called λ–Aubry set, which shares some properties of the Aubry set for Eikonal equations on networks with convex or quasiconvex Hamiltonians, see [11]. A similar entity has been found for discounted equations with regular Hamiltonians, (contact Hamiltonians) on compact manifolds in [9], [13] via dynamical techniques.

Assuming the H_γ convex, we study the link, as λ becomes infinitesimal, of λ–Aubry set with the Aubry set of the corresponding Eikonal equation ($\lambda = 0$). In particular we show, see Proposition 8.8, that, for λ suitably small, the λ–Aubry sets are contained in the Aubry set for the Eikonal equation. This should be compared with the convergence result established in [9].

The paper is organized as follows: The problem under investigation is presented in Section 2 together with the assumptions on the Hamiltonians and the main related definitions. In Section 3 we summarize the relevant properties of one–dimensional discounted HJ equations posed on an interval.

In Section 4 we introduce the discrete equation, prove the link with the differential problem on the network, and establish a comparison principle. Section 5 is devoted to the definition of a functional on the paths of the graph, which will play a major role in the representation formulae for solutions described in Section 6. In Section 7 we define the λ–Aubry sets via a condition on cycles. Section 8 provides in the first part a summary of the main properties of Eikonal equations on networks and then focus on the behavior of solutions and λ–Aubry set as $\lambda \longrightarrow 0$.

Finally Appendix A collects some basic material on graphs and networks , and in Appendix B we provide some proofs of results stated in Section 3.

2. Setting of the problem

We consider a network Γ immersed in $\mathbb{R}^N$. We denote by $\mathbf{V}$, $\mathcal{E}$ the set of vertices and arcs, respectively. We also consider the abstract graph $\mathbf{X}$ underlying Γ with the same vertices of Γ and edges that are, loosely speaking, an immaterial copy of the arcs of $\mathcal{E}$. See Appendix A for more detail and further terminology and notation on graphs and networks.

We are given a family of Hamiltonians

$$H_\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$$

indexed by the arcs of the network. They are unrelated for arcs of different support, and satisfy the compatibility condition

$$H_{-\gamma}(s,p) = H_\gamma(1-s,-p) \qquad \text{for any } \gamma \in \mathcal{E}. \tag{2}$$

We assume the H_γ to be:

(H1) continuous in (s,p);
(H2) coercive in p.

No convexity conditions are required for the discounted equation. Some additional assumptions will be introduced for the asymptotic results of Section 8 where an

Eikonal problem will appear at the limit, as the discount factor goes to 0. See hypotheses **(H3)**, **(H4)** in Section 8

For any given arc γ, we are concerned with the discounted equation

$$(\mathbf{HJ}\gamma_\lambda) \qquad \lambda\, w + H_\gamma(s, w') = 0 \qquad \text{in } (0,1).$$

The problem we are interested on is a combination of all the $(\mathbf{HJ}\gamma_\lambda)$. We look for continuous functions u defined on Γ such that

$$(\mathbf{HJ}\Gamma_\lambda) \qquad \lambda\, u \circ \gamma + H_\gamma(s, (u \circ \gamma)') = 0 \qquad \text{in } [0,1], \text{ for any } \gamma \in \mathcal{E}$$

in the viscosity sense, plus suitable conditions at the vertices, as made precise in the forthcoming Definition 2.3. We preliminarily recall some definition and terminology of viscosity solution theory.

2.1. **Definition.** Given a continuous function w in $[0,1]$, we say that a C^1 function φ is *supertangent* to w at $s \in (0,1)$ if

$$w = \varphi \ \text{ at } s \quad \text{and} \quad w \leq \varphi \ \text{ in } (s-\delta, s+\delta) \text{ for some } \delta > 0.$$

The notion of *subtangent* is given by just replacing $\leq$ by $\geq$ in the above formula.

Finally, φ is called *constrained subtangent* to w at 1 if

$$w = \varphi \ \text{ at } 1 \quad \text{and} \quad w \geq \varphi \ \text{ in } (1-\delta, 1) \text{ for some } \delta > 0.$$

A similar notion, with obvious adaptations, can be given at $t = 0$.

2.2. **Definition.** Given a continuous function w in $[0,1]$, a point $s_0 \in \{0,1\}$, we say that it satisfies *the state constraint boundary condition* for $(\mathbf{HJ}\gamma_\lambda)$ at s_0 if

$$\lambda\, \varphi(s_0) + H_\gamma(s_0, \varphi'(s_0)) \geq 0.$$

for any constrained C^1 subtangent φ to w at s_0.

2.3. **Definition.** We say that $u : \Gamma \longrightarrow \mathbb{R}$ is *subsolution* to $(\mathbf{HJ}\Gamma_\lambda)$ if

- **i)** it is continuous on Γ,
- **ii)** $s \mapsto u(\gamma(s))$ is subsolution to $(\mathbf{HJ}\gamma_\lambda)$ in $(0,1)$ for any $\gamma \in \mathcal{E}$.

We say that u is *supersolution* to $(\mathbf{HJ}\Gamma_\lambda)$ if

- **i)** it is continuous;
- **ii)** $s \mapsto u(\gamma(s))$ is supersolution of $(\mathbf{HJ}\gamma_\lambda)$ in $(0,1)$ for any $\gamma \in \mathcal{E}$;

iii) for every vertex x there is at least an arc γ, with x as terminal point, such that $u(\gamma(s))$ satisfies the state constraint boundary condition for $(\mathbf{HJ}\gamma_\lambda)$ at $s = 1$

A function u is said *solution* if it is at the same time super and subsolution.

Let us observe that **iii)** for supersolutions is actually a partial boundary condition since it is given only at one endpoint. However when it is combined with a Dirichlet condition at the other endpoint, it gives the uniqueness of the solution as proved in Corollary 3.5. Also notice that in the definition of subsolution no conditions are required on vertices.

2.4. **Remark.** Passing from γ to $-\gamma$ and from H_γ to $H_{-\gamma}$, we see that the condition **iii)** in the definition of supersolution for a vertex x can be equivalently given at $s = 0$ considering the edges with initial vertex x.

2.5. **Remark.** The condition **iii)** in the above definition of supersolution is the same given in [8] at the junction point 0. In [8] the authors do not impose conditions for the test functions on the other vertices, but for the uniqueness principle they need considering some boundary condition at the other vertices of the junction. We assume condition **iii)** at any vertex but we get uniqueness of solutions without assuming any additional boundary condition, we do not even single out a boundary in our network.

3. Local analysis of HJ equations on arcs

We focus on an arc $\gamma \in \mathcal{E}$, our treatment is independent of whether or not γ is a cycle. We recall some basic facts about viscosity (sub)solutions to $(\mathbf{HJ}\gamma_\lambda)$, see for instance [1], [2].

3.1. **Theorem** (Comparison Principle)**.** *If u is an upper semicontinuous subsolution and v is a lower semicontinuous supersolution to $(\mathbf{HJ}\gamma_\lambda)$ with $u \le v$ in $\{0, 1\}$, then $u \le v$ in $[0, 1]$.*

Given $\alpha \in \mathbb{R}$, we define

$$u^\gamma_{\max}(s) = \sup\{u(s) \mid u \text{ subsolution to } (\mathbf{HJ}\gamma_\lambda)\} \tag{3}$$

$$u^\gamma_\alpha(s) = \sup\{u(s) \mid u \text{ subsolution to } (\mathbf{HJ}\gamma_\lambda) \text{ with } u(0) \le \alpha\} \tag{4}$$

3.2. **Lemma.** *The function $u^\gamma_{\max}$ is characterized by the property of being a Lipschitz continuous solution to* $(\mathbf{HJ}\gamma_\lambda)$ *in* $(0,1)$ *satisfying state constraints boundary conditions at* 0 *and* 1.

3.3. **Lemma.** *The function u^γ_α is a Lipschitz–continuous solution to* $(\mathbf{HJ}\gamma_\lambda)$ *in* $(0,1)$ *satisfying state constraint boundary conditions at* $s=1$. *In addition u^γ_α is equal to α at $s=0$ if and only if* $\alpha \leq u^\gamma_{\max}(0)$.

The proof of the two above lemmata is in Appendix B.

3.4. **Corollary.** *The identity* $u^\gamma_\alpha \equiv u^\gamma_{\max}$ *holds true in* $[0,1]$ *if and only if* $\alpha \geq u^\gamma_{\max}(0)$.

We deduce from the previous results the following characterization of u^γ_α:

3.5. **Corollary.** *The function u^γ_α, for $\alpha \leq u^\gamma_{\max}(0)$, is the unique solution to* $(\mathbf{HJ}\gamma_\lambda)$ *satisfying the Dirichlet boundary condition $u^\gamma_\alpha(0) = \alpha$ and the state constraint boundary condition at* $s=1$.

By slightly adapting the proof of Lemma 3.2, we also have:

3.6. **Corollary.** *Let w be a supersolution of* $(\mathbf{HJ}\gamma_\lambda)$ *with $w(0)=\alpha$ satisfying the state constraint boundary condition at $s=1$, then $w \geq u^\gamma_\alpha$ in* $[0,1]$.

We introduce the function $u^{-\gamma}_\alpha$ defined as u^γ_α, but with the Hamiltonian H_γ in equation $(\mathbf{HJ}\gamma_\lambda)$ replaced by $H_{-\gamma}$. This function is the analogue of u^γ_α on $-\gamma$ in the sense that it is the maximal subsolution to $\lambda\, u + H_{-\gamma}(s,u') = 0$ taking a value less than or equal to α at 0.

3.7. **Remark.** It is apparent that $w(s)$ is subsolution to $(\mathbf{HJ}\gamma_\lambda)$ with $H_{-\gamma}$ in place of H_γ if and only $s \mapsto w(1-s)$ has the same property for the original equations. This shows that $u^{-\gamma}_\alpha(1-s)$ is the maximal subsolution to $(\mathbf{HJ}\gamma_\lambda)$ taking value $\leq \alpha$ at $s=1$. In addition, $s \mapsto u^\gamma_{\max}(1-s)$ is the maximal subsolution to $(\mathbf{HJ}\gamma_\lambda)$ with $H_{-\gamma}$ in place of H_γ.

The next result is about Dirichlet boundary problems. It will be crucially used in the passage from the local problem on the arcs to the global problem on the network.

3.8. **Proposition.** *There exists an unique solution u of the equation* $(\mathbf{HJ}\gamma_\lambda)$ *with $u(0)=\alpha$, $u(1)=\beta$ if and only if*

$$\alpha \leq u^{-\gamma}_\beta(1), \quad \beta \leq u^\gamma_\alpha(1). \tag{5}$$

The proof is in Appendix B.

In the next result we show continuity of $u^\gamma_\alpha(1)$ with respect to α plus two monotonicity properties we will repeatedly exploit in what follows. We stress in particular that the strict monotonicity in item **ii)** will play a crucial role in the whole paper.

3.9. **Proposition.**

i) $\alpha \mapsto u^\gamma_\alpha(1)$ *is Lipschitz continuous and nondecreasing;*
ii) $\alpha \mapsto u^\gamma_\alpha(1) - \alpha$ *is strictly decreasing;*
iii) $\lim_{\alpha \to -\infty} u^\gamma_\alpha(1) - \alpha = +\infty$, $\quad \lim_{\alpha \to +\infty} u^\gamma_\alpha(1) - \alpha = -\infty$.

Proof: We start from **ii)**. We consider $\beta < \alpha$. The function $s \mapsto u^\gamma_\alpha + \beta - \alpha$ is a strict subsolution to ($\mathbf{HJ}\gamma_\lambda$) taking a value less than or equal to β at $s = 0$. This implies

$$u^\gamma_\alpha(1) + \beta - \alpha \le u^\gamma_\beta(1). \tag{6}$$

Arguing as in the proof of Lemma 3.2, we find a constrained subtangent to $u^\gamma_\alpha + \beta - \alpha$ at $s = 1$ of the form

$$\varphi(s) = u^\gamma_\alpha(1) + \beta - \alpha + q\,(s - 1)$$

for some $q > \max\{p \mid p \in \partial u^\gamma_\alpha(1)\}$ satisfying

$$\lambda\,(u^\gamma_\alpha(1) + \beta - \alpha) + H_\gamma(1, q) < 0. \tag{7}$$

If equality holds in (6) then φ is also a constrained subtangent to u^γ_β at $s = 1$ and (7) contradicts u^γ_β satisfying state constraint boundary condition at $s = 1$, see Corollary 3.5. Then a strict inequality must prevail. This shows item **ii)**.

We pass to **i)**. The nondecreasing character of $\alpha \mapsto u^\gamma_\alpha(1)$ is a direct consequence of the maximality of u^γ_α. From this and item **ii)** we derive for any $\alpha \ge \beta$,

$$0 \le u^\gamma_\alpha(1) - u^\gamma_\beta(1) \le \alpha - \beta,$$

which implies the claimed continuity.

To prove **iii)**, we recall that by Corollary 3.4

$$u^\gamma_\alpha(1) - \alpha = u^\gamma_{\max}(1) - \alpha \qquad \text{for } \alpha \text{ sufficently large },$$

which gives the claimed negative divergence as $\alpha \to +\infty$. Given any $p_0 > 0$, we consider α with

$$-\max_{s \in \mathbb{R}}\{H_\gamma(s, p_0)\} \ge \lambda\,(\alpha + p_0),$$

then $s \mapsto \alpha + s\,p_0$ is subsolution to ($\mathbf{HJ}\gamma_\lambda$), and consequently $u^\gamma_\alpha(1) - \alpha \ge p_0$. This implies the claimed positive divergence as $\alpha \to -\infty$. □

We derive:

3.10. **Corollary.** *There exists one and only one* α *such that* $u^\gamma_\alpha(1) = \alpha$*, and it satisfies* $\alpha \geq -\frac{1}{\lambda} \max_s H_\gamma(s,0)$.

Proof: If $\alpha = -\frac{1}{\lambda} \max_s H_\gamma(s,0)$ then the function constantly equal to α is subsolution to ($\mathbf{HJ}\gamma_\lambda$). Consequently $u^\gamma_\alpha(1) \geq \alpha$, and the conclusion follows from Proposition 3.9 **ii)**, **iii)**. □

3.11. **Remark.** According to Proposition 3.8, the equation ($\mathbf{HJ}\gamma_\lambda$) admits a periodic solution in $(0,1)$, namely attaining the same value at 0 and 1, if and only if the boundary value is less than or equal to the α appearing in the statement of Corollary 3.10.

We introduce the Eikonal equation

$$H_\gamma(s,u') = 0 \quad s \in (0,1) \tag{$\mathbf{HJ}_\gamma$}$$

under the additional assumptions **(H3)**, **(H4)**, see Section 8 for a precise statement of these conditions and a quick review of Eikonal equation on networks. We define

$$a_\gamma = \max_s \min_p H_\gamma(s,p)$$

3.12. **Lemma.** *If* $0 \geq a_\gamma$*, then there is a function* v *such that* $\alpha + v$ *is the maximal subsolution to* ($\mathbf{HJ}_\gamma$) *taking the value* α *at* 0*, for any* $\alpha \in \mathbb{R}$*. It is in addition a Lipschitz continuous solution of* ($\mathbf{HJ}_\gamma$).

Proof: See Proposition 5.6 in [11]. The solution $v + \alpha$ is given by formula (20) in [11] with α in place of $w(0)$ and 0 in place of a.

□

We study the asymptotic behavior of solutions to ($\mathbf{HJ}\gamma_\lambda$) as $\lambda \to 0$. Given a positive infinitesimal sequence λ_n, we indicate by $u^{\lambda_n}_{\max}$, $u^{\lambda_n}_\alpha$ the maximal solution to ($\mathbf{HJ}\gamma_\lambda$), with λ_n in place of λ, and the maximal solution among those taking the value α at $s = 0$, respectively. The function v is defined as in the statement of Lemma 3.12. The proof of the following result is in Appendix B.

3.13. **Lemma.** *Let* α_n *be a sequence converging to some* $\alpha \in \mathbb{R}$*. If* $u^{\lambda_n}_{\max}(0) \geq \alpha_n$ *for* n *sufficiently large, then* $u_n = u^{\lambda_n}_{\alpha_n}$ *uniformly converges in* $[0,1]$ *to* $\alpha + v$.

4. Discrete functional equations

We introduce a discrete functional equation on $\mathbf{V}$ suitably related to $(\mathbf{HJ}\Gamma_\lambda)$. The relation is made clear in Theorem 4.2, Proposition 4.3.

For $e = \Psi^{-1}(\gamma)$, we set

$$\begin{aligned} \rho(\alpha, e) &= u^\gamma_\alpha(1) \\ \underline{\alpha}(e) &= u^\gamma_{\max}(0) \\ \overline{\alpha}(e) &= u^\gamma_{\max}(1), \end{aligned}$$

We record for later use:

4.1. **Proposition.** *For any* $e \in \mathbf{E}$ *we have*

$$\overline{\alpha}(e) = \underline{\alpha}(-e) = \rho(\underline{\alpha}(e), e)$$

Proof: The equalities in the statement directly come from the definitions of $\underline{\alpha}$, $\overline{\alpha}$, ρ and Remark 3.7. □

The discrete functional equation in $\mathbf{V}$ is defined as follows:

$$(\mathbf{DFE}_\lambda) \qquad U(x) = \min_{e \in -\mathbf{E}_x} \rho(U(\mathrm{o}(e)), e).$$

We say that $U : \mathbf{V} \longrightarrow \mathbb{R}$ is a *subsolution* (resp. *supersolution*) to $(\mathbf{DFE}_\lambda)$ if

$$U(x) \leq \text{ (resp. } \geq \text{) } \min_{e \in -\mathbf{E}_x} \rho(U(\mathrm{o}(e)), e). \qquad \text{for any } x \in \mathbf{V}. \tag{8}$$

A solution is at the same time sub and supersolution. See (33) in the Appendix for the definition of $-\mathbf{E}_x$. Notice that in accordance with condition **iii)** in Definition 2.3 of the supersolution on the network, we have considered in $(\mathbf{DFE}_\lambda)$ only the edge ending at x. As pointed out in Remark 2.5, it is equivalent to instead consider arcs (in Definition 2.3) and edges (in $(\mathbf{DFE}_\lambda)$) starting at x

The following results provide the bridge linking $(\mathbf{DFE}_\lambda)$ to $(\mathbf{HJ}\Gamma_\lambda)$.

4.2. **Theorem.** *A solution* U *to* $(\mathbf{DFE}_\lambda)$ *can be uniquely extended to a solution* u *to* $(\mathbf{HJ}\Gamma_\lambda)$*. Conversely, given a solution* u *to* $(\mathbf{HJ}\Gamma_\lambda)$*,* $U = u|_{\mathbf{V}}$ *is a solution to* $(\mathbf{DFE}_\lambda)$*.*

Proof: Assume that U solves $(\mathbf{DFE}_\lambda)$. Let e be an edge in $\mathbf{E}$. We set, to ease notations, $\gamma = \Psi(e)$, $\alpha = U(\mathrm{o}(e))$, $\beta = U(\mathrm{t}(e))$. By the very definition of subsolution to $(\mathbf{DFE}_\lambda)$ and ρ we have

$$\beta \leq \rho(\alpha, e) = u_\alpha^\gamma(1)$$

$$\alpha \leq \rho(\beta, e) = u_\beta^{-\gamma}(1)$$

and this implies, thanks to Proposition 3.8, that there is a unique solution w to $(\mathbf{HJ}\gamma_\lambda)$ with $w(0) = \alpha$ and $w(1) = \beta$. We have in addition that for any $x \in V$ there exists $e_0 \in -\mathbf{E}_x$ with

$$U(x) = \rho(U(\mathrm{o}(e_0)), e_0).$$

This implies that U can be uniquely extended as the maximal subsolution to $(\mathbf{HJ}\gamma_\lambda)$ less than or equal to $U(\mathrm{o}(e_0))$ at $s = 0$. It is by Corollary 3.5 a solution to $(\mathbf{HJ}\gamma_\lambda)$ and satisfies the state constraint boundary condition at $s = 1$. This shows the first part of the assertion. Conversely, assume that u is a solution to $(\mathbf{HJ}\Gamma_\lambda)$, and set $U = u|_{\mathbf{V}}$. We deduce from the definition of ρ and Proposition 3.8

$$U(x) \leq \min_{e \in -\mathbf{E}_x} \rho(U(\mathrm{o}(e)), e) \qquad \text{for any } x \in \mathbf{V}. \tag{9}$$

Taking into account that u satisfies condition **iii)** in the definition of solution to $(\mathbf{HJ}\Gamma_\lambda)$, we find in force of Corollary 3.5 for any $x \in \mathbf{V}$ an $e_0 \in -\mathbf{E}_x$ for which formula (9) holds with equality. This shows that U solves $(\mathbf{DFE}_\lambda)$ and concludes the proof. □

As a consequence of the very definition of ρ and Corollary 3.6, we also have

4.3. **Proposition.** *The trace on* $\mathbf{V}$ *of any subsolution (resp. supersolution) to* $(\mathbf{HJ}\Gamma_\lambda)$ *is a subsolution (resp. supersolution) of* $(\mathbf{DFE}_\lambda)$

We establish a comparison principle for $(\mathbf{DFE}_\lambda)$.

4.4. **Theorem.** *Let* U, W *be a subsolution and a supersolution, respectively, to* $(\mathbf{DFE}_\lambda)$. *Then* $U \leq W$.

Proof: Assume by contradiction that $\max_{\mathbf{V}} U - W > 0$, and denote by x_0 a corresponding maximizer. In force of the very definition of subsolution and supersolution, there is $e_0 \in -\mathbf{E}_{x_0}$ with

$$\begin{aligned} U(x_0) &\leq \rho(U(\mathrm{o}(e_0)), e_0) \\ W(x_0) &\geq \rho(W(\mathrm{o}(e_0)), e_0). \end{aligned}$$

By subtracting the above relations, we obtain

$$U(x_0) - W(x_0) \leq \rho(U(\mathrm{o}(e_0)), e_0) - \rho(W(\mathrm{o}(e_0)), e_0), \tag{10}$$

and so, bearing in mind that $U(x_0) > W(x_0)$, we get

$$\rho(U(\mathrm{o}(e_0)), e_0) - \rho(W(\mathrm{o}(e_0)), e_0) > 0. \tag{11}$$

Since $\rho(\cdot, e_0)$ is nondecreasing by Proposition 3.9, we derive from (11) $U(\mathrm{o}(e_0)) > W(\mathrm{o}(e_0))$. Thus, exploiting the strictly decreasing character of $\alpha \mapsto \rho(\alpha, e_0) - \alpha$, we further get from (10)

$$U(\mathrm{o}(e_0)) - W(\mathrm{o}(e_0)) > \rho(U(\mathrm{o}(e_0)), e_0) - \rho(W(\mathrm{o}(e_0)), e_0) \geq U(x_0) - W(x_0)$$

which contradicts x_0 being a maximizer of $U - W$ in $\mathbf{V}$. □

We derive as a consequence:

4.5. **Theorem.** *The discounted discrete equation can have at most one solution.*

By combining Theorem 4.4 and Proposition 4.3, we finally state a comparison principle for $(\mathbf{HJ}\Gamma_\lambda)$.

4.6. **Theorem.** *Let u, w be sub and supersolution of $(\mathbf{HJ}\Gamma_\lambda)$, the $u \leq w$ in Γ.*

Proof: By Proposition 4.3 the traces of u, w on $\mathbf{V}$ are sub and supersolution to $(\mathbf{DFE}_\lambda)$, respectively. By Theorem 4.4 $u|_{\mathbf{V}} \leq w|_{\mathbf{V}}$. This gives the assertion in force of Theorem 3.1.

□

5. Analysis of the discrete equation

In this section we extend the definition of ρ from edges to general paths via an inductive procedure on the length of paths. We furthermore define some related quantities.

5.1. **Definition.** Given $\alpha \in \mathbb{R}$ and a path ξ, we define

$$\rho(\alpha, \xi) = \rho(\alpha, e) \qquad \text{if } \xi = e\ .$$

If $\xi = (e_i)_{i=1}^M$, for $M > 1$, we set $\bar{\xi} = (e_i)_{i=1}^{M-1}$ and define

$$\rho(\alpha, \xi) = \rho(\rho(\alpha, \bar{\xi}), e_M).$$

The following concatenation formula is inherent to the definition. Let ξ, η be paths with $\mathrm{t}(\xi) = \mathrm{o}(\eta)$ then

$$\rho(\alpha, \xi \cup \eta) = \rho(\rho(\alpha, \xi), \eta) \qquad \text{for any } \alpha. \tag{12}$$

Taking into account that the property of being continuous is stable for composition of functions, we get from Proposition 3.9:

5.2. **Proposition.** *Given any path ξ, the function*

$$\alpha \mapsto \rho(\alpha, \xi)$$

is continuous.

The next Proposition is a direct consequence of Proposition 3.9 and will be repeatedly used in what follows.

5.3. **Proposition.** *The following monotonicity properties hold for any path ξ*

- **i)** *$\alpha \mapsto \rho(\alpha, \xi)$ is nondecreasing;*
- **ii)** *$\alpha \mapsto \rho(\alpha, \xi) - \alpha$ is strictly decreasing.*

Proof: We prove both items arguing by induction on the length of the path. If it is 1, and so the path reduces to an edge, the statement is a direct consequence of the definition of ρ and Proposition 3.9. We assume the assertion to be true for any path with length less than M and show it for $\xi := (e_i)_{i=1}^M$. By the very definition of ρ

$$\rho(\alpha, \xi) = \rho(\rho(\alpha, \bar{\xi}), e_M), \tag{13}$$

where $\bar{\xi} = (e_i)_{i=1}^{M-1}$. The functions $\alpha \mapsto \rho(\alpha, \bar{\xi})$ and $\alpha \mapsto \rho(\alpha, e_M)$ are nondecreasing by the inductive step, and $\rho(\cdot, \xi)$ is therefore nondecreasing as composition of nondecreasing functions. This concludes the proof of item **i)**. To show **ii)**, we argue again by induction. Given $\beta < \alpha$, we have by item **i)** $\rho(\beta, \overline{\xi}) \leq \rho(\alpha, \overline{\xi})$, exploiting this inequality, and the inductive step, we get

$$\begin{aligned} \rho(\alpha, \overline{\xi}) - \rho(\beta, \overline{\xi}) &< \alpha - \beta \\ \rho(\rho(\alpha, \bar{\xi}), e_M) - \rho(\rho(\beta, \bar{\xi}), e_M) &\leq \rho(\alpha, \overline{\xi}) - \rho(\beta, \overline{\xi}). \end{aligned}$$

By combining the above inequalities, we obtain

$$\rho(\alpha, \xi) - \rho(\beta, \xi) < \alpha - \beta$$

which gives **ii)**. □

The next result is a generalization to paths of Corollary 3.10. It has a crucial relevance since the fixed points of ρ will play a key role in our analysis.

5.4. **Corollary.** *For any path ξ there exists one and only one $\alpha \in \mathbb{R}$ with $\rho(\alpha, \xi) = \alpha$.*

Proof: We have by the definition of $\overline{\alpha}$ and ρ

$$\rho(\alpha, e) \leq \overline{\alpha}(e) \qquad \text{for any } e \in \mathbf{E},\ \alpha \in \mathbb{R}. \tag{14}$$

Let $\xi = (e_i)_{i=1}^M$ and $\bar\xi = (e_i)_{i=1}^{M-1}$. We get in force of (14) and the concatenation formula (12)

$$\rho(\alpha,\xi) = \rho(\rho(\alpha,\bar\xi),e_M) \le \overline{\alpha}(e_M) \qquad \text{for any } \alpha \in \mathbb{R}. \tag{15}$$

Taking into account Corollary 3.10, we set

$$\alpha_0 = \min\{\alpha \mid \rho(\alpha,e_i) = \alpha,\ i = 1,\cdots,M\}.$$

We claim that

$$\rho(\alpha,\xi) > \alpha \qquad \text{for } \alpha < \alpha_0. \tag{16}$$

We fix $\alpha > \alpha_0$ and prove the claim arguing by induction on the length of the curve. If the length is 1, say $\xi = e$, then (16) holds because of the strict monotonicity of $\alpha \mapsto \rho(\alpha,e) - \alpha$. Assuming the property true for curves of length less than M, we get $\rho(\alpha,\bar\xi) > \alpha$ and consequently by the nondecreasing character of $\rho(\cdot,e_M)$ and (12)

$$\rho(\alpha,\xi) = \rho(\rho(\alpha,\bar\xi),e_M) \ge \rho(\alpha,e_M) > \alpha,$$

proving the claim. Relations (15), (16) plus continuity and monotonicity of $\rho(\cdot,\xi)$, see Propositions 5.3, 5.2, give the assertion. □

In what follows, we will exploit the property highlighted by the above proposition solely for cycles.

5.5. **Definition.** Given a cycle ξ, we define $\beta(\xi)$ to be the unique fixed point of

$$\alpha \mapsto \rho(\alpha,\xi).$$

5.6. **Proposition.** *For any edge e, the cycle $\xi = (e,-e)$ satisfies*

$$\beta(\xi) = \underline{\alpha}(e).$$

Proof: We have

$$\rho(\underline{\alpha}(e),\xi) = \rho(\rho(\underline{\alpha}(e),e),-e)$$

and we derive, taking into account Lemma 4.1

$$\rho(\underline{\alpha}(e),\xi) = \rho(\underline{\alpha}(-e),-e) = \underline{\alpha}(e).$$

□

5.7. **Remark.** It is worth pointing out that $\beta(\xi)$, see Definition 5.5, also depends on the initial point of the cycle. In other terms, if we consider another cycle η with the same edges as ξ but different initial point then in general $\beta(\xi) \neq \beta(\eta)$. For example, if we define, for a given edge e, $\xi = \{e,-e\}$ and $\eta = \{-e,e\}$ then, according to Proposition 5.6, $\beta(\xi) = \underline{\alpha}(e)$ and $\beta(\eta) = \overline{\alpha}(e)$, which are clearly in general different.

In what follows when we will say that a cycle is a based on a certain vertex, we will mean that the vertex is the initial point of the cycle.

6. Existence of solutions of ($\mathbf{DFE}_\lambda$), ($\mathbf{HJ}\Gamma_\lambda$) and representation formulae

We show that a solution to ($\mathbf{DFE}_\lambda$) does exist providing a representation formula. We define a function $f : \mathbf{V} \to \mathbb{R}$ via

$$f(x) = \inf\{\beta(\xi) \mid \text{for some cycle } \xi \text{ based on } x\}.$$

The definition is well posed thanks to Corollary 5.4. We set for $x \in \mathbf{V}$

$$U(x) = \inf\{\rho(f(\mathrm{o}(\xi)), \xi) \mid \xi \text{ path with } \mathrm{t}(\xi) = x\}. \tag{17}$$

Since for any vertex x, any cycle based on x is an admissible path for (17), it is clear that

$$U(x) \le f(x).$$

We have

6.1. **Theorem.** *The function U defined in* (17) *is solution to* ($\mathbf{DFE}_\lambda$).

The rest of the section is devoted to the deduction of some properties of f and U, and to the proof of Theorem 6.1.

6.2. **Proposition.** *We have*

$$-\frac{1}{\lambda} \max_{e,s} H_{\Psi(e)}(s, 0) \le f(x) \le \min_{e \in \mathbf{E}_x} \underline{\alpha}(e) \qquad \textit{for any } x \in \mathbf{V}. \tag{18}$$

Proof: The rightmost inequality of the formula in the statement is a direct consequence of the definition of f and Proposition 5.6. We set $\overline{\alpha} = -\frac{1}{\lambda} \max\limits_{e,s} H_{\Psi(e)}(s, 0)$, and claim that

$$\rho(\overline{\alpha}, \xi) \ge \overline{\alpha} \qquad \text{for any path } \xi. \tag{19}$$

Were the claim true, we derive from it, because of the strict monotonicity of $\alpha \mapsto \rho(\alpha, \xi) - \alpha$, $\beta(\xi) \ge \overline{\alpha}$ for any cycle ξ. This in turn implies the leftmost inequality in (18). We prove (19) arguing inductively on the length of paths. It is true if the length is 1 in force of Corollary‘3.10. We take a general path $\xi = (e_i)_{i=1}^M$ and set $\bar{\xi} = (e_i)_{i=1}^{M-1}$. By inductive step $\rho(\overline{\alpha}, \bar{\xi}) \ge \overline{\alpha}$ and $\rho(\overline{\alpha}, e_M) \ge \overline{\alpha}$. Exploiting the monotonicity of $\rho(\cdot, e_M)$, we have

$$\rho(\overline{\alpha}, \xi) = \rho(\rho(\overline{\alpha}, \bar{\xi}), e_M) \ge \rho(\overline{\alpha}, e_M) \ge \overline{\alpha}.$$

This concludes the proof. $\square$

6.3. **Proposition.** *The infimum in the definition of U is realized by a simple path with terminal vertex x, for any $x \in \mathbf{V}$.*

Proof: We fix x and a path ξ with terminal vertex x, and set, to ease notation, $\overline{\alpha} = f(\mathrm{o}(\xi))$. Let us assume that there is a cycle η properly contained in ξ with

$$\mathrm{o}(\eta) \neq \mathrm{o}(\xi) \qquad \text{and} \qquad \mathrm{t}(\eta) \neq \mathrm{t}(\xi). \tag{20}$$

The path ξ can be consequently written in the form

$$\xi = \xi_1 \cup \eta \cup \xi_2$$

where ξ_1, ξ_2, η satisfy $\mathrm{t}(\xi_1) = \mathrm{o}(\xi_2) = \mathrm{o}(\eta)$. We have by the concatenation formula

$$\rho(\overline{\alpha}, \xi) = \rho(\rho(\rho(\overline{\alpha}), \xi_1), \eta), \xi_2). \tag{21}$$

If $\rho(\overline{\alpha}, \xi_1) \geq \beta(\eta)$ then by the usual monotonicity property

$$\rho(\rho(\overline{\alpha}, \xi_1), \eta) \geq \rho(\beta(\eta), \eta) = \beta(\eta)$$

which implies, taking also into account (21) and the definition of $\overline{\alpha}$

$$\rho(f(\mathrm{o}(\xi)), \xi) \geq \rho(\beta(\eta), \xi_2) \geq \rho(f(\mathrm{o}(\xi_2)), \xi_2)). \tag{22}$$

If instead $\rho(\overline{\alpha}, \xi_1) < \beta(\eta)$, then by the strict monotonicity of $\alpha \mapsto \rho(\alpha, \eta) - \alpha$, we have

$$\rho(\rho(\overline{\alpha}, \xi_1), \eta) > \rho(\overline{\alpha}, \xi_1)$$

and by (21) and the definition of $\overline{\alpha}$, we further get

$$\rho(f(\mathrm{o}(\xi)), \xi) > \rho(\rho(\overline{\alpha}, \xi_1), \xi_2) = \rho(\overline{\alpha}, \xi_1 \cup \xi_2) = \rho(f(\mathrm{o}(\xi_1 \cup \xi_2), \xi_1 \cup \xi_2). \tag{23}$$

Taking into account (22) (23), we realize that the cycle η can be removed without affecting the infimum in the definition of $U(x)$. By slightly adapting the argument, we reach the same conclusion getting rid of condition (20). The procedure can be repeated for all other cycle properly contained in ξ. We therefore see that

$$U(x) = \min\{\rho(f(\mathrm{o}(\zeta)), \zeta) \mid \zeta \text{ simple path with } \mathrm{t}(\zeta) = x\},$$

where the minimum in the above formula is justified by the fact that the simple paths are finite. This ends the proof.

□

By following the same argument as in Proposition 6.3 we can also show

6.4. **Corollary.** *Assume that for a given x*

$$U(x) = \rho(f(\mathrm{o}(\xi)), \xi) \qquad \textit{for some path } \xi \textit{ with } \mathrm{t}(\xi) = x.$$

Then there exists a simple path ζ with $\mathrm{o}(\zeta) = \mathrm{o}(\xi)$, $\mathrm{t}(\zeta) = y$ such that

$$U(x) = \rho(f(\mathrm{o}(\zeta), \zeta).$$

Proof: (**of Theorem 6.1**) We fix $x \in \mathbf{V}$ and $e \in -\mathbf{E}_x$. By Proposition 6.3, there is a simple path ξ ending at $\mathrm{o}(e)$ with

$$U(\mathrm{o}(e)) = \rho(f(\mathrm{o}(\xi)), \xi).$$

By the very definition of U and the concatenation principle (12), we have

$$U(x) \leq \rho(f(\mathrm{o}(\xi)), \xi \cup e) = \rho(U(\mathrm{o}(e), e).$$

This shows that U is subsolution. Taking again into account Proposition 6.3, we proceed denoting by $\eta = (e_i)_{i=1}^M$ a simple path with terminal point x satisfying

$$U(x) = \rho(f(\mathrm{o}(\eta), \eta).$$

We set $\bar{\eta} = (e_i)_{i=1}^{M-1}$, and derive from concatenation formula, monotonicity and definition of U

$$U(x) = \rho(\rho(f(\mathrm{o}(\eta), \bar{\eta}), e_M) \geq \rho(U(\mathrm{o}(e_M)), e_M)$$

Knowing that U is subsolution and $e_M \in -\mathbf{E}_x$, equality must prevail in the above formula, showing that U is actually a solution, as was claimed.

□

6.5. **Remark.** If e is a loop with vertex x then clearly $U(x) \leq \beta(e) = \beta(-e)$, see Remark 3.11, if there is a strict inequality then the edge e (resp.the arc $\Psi(e)$) can be removed from the graph (resp. from the network) without affecting the solution of ($\mathbf{DFE}_\lambda$) (resp. the solution of ($\mathbf{HJ}\Gamma_\lambda$) on the arcs different from $\Psi(e)$). A similar phenomenon takes place for the Eikonal equation on graphs/networks, see Remark 6.17 in [11].

Combining the previous result with Theorems 4.2 and 4.5 we get

6.6. **Theorem.** *There is one and only one solution to* ($\mathbf{HJ}\Gamma_\lambda$)*, and its restriction to* $\mathbf{V}$ *coincide with the function* U *defined in* (17).

7. λ–Aubry sets

We define in this section the λ–Aubry sets, an analogue to the Aubry sets introduced for the Eikonal problem, see Section 8. These sets allow writing a new representation formula for solutions to ($\mathbf{DFE}_\lambda$), and will play a role in the asymptotic problem we will deal with in the next section.

7.1. **Definition.** The *(projected) λ–Aubry set* is given by

$$\mathcal{A}_\lambda = \{y \in \mathbf{V} \mid U(y) = \beta(\xi) \text{ for some cycle } \xi \text{ based on } y.\}$$

7.2. **Proposition.** *Given $y \in \mathcal{A}_\lambda$, then any cycle $\xi = (e_i)_{i=1}^M$ based on y with $U(y) = \beta(\xi)$ satisfies*

$$U(\mathrm{o}(e_j)) = \beta\big((e_i)_{i=j}^M \cup (e_i)_{i=1}^{j-1}\big) \tag{24}$$

$$U(\mathrm{o}(e_j)) = \rho\left(U(\mathrm{o}(e_k)), (e_i)_{i=k}^{j-1}\right) \tag{25}$$

for any j, $k = 1, \cdots, M$, $k \leq j$.

Proof: We start proving (24). Taking into account that U is solution to $(\mathbf{DFE}_\lambda)$, we have

$$U(y) \leq \rho(U(\mathrm{o}(e_M)), e_M). \tag{26}$$

We set $\eta = (e_i)_{i=1}^{M-1}$, $\zeta = e_M \cup \eta$, it is clear that ζ is a cycle based on $\mathrm{o}(e_M)$. By the concatenation formula

$$U(y) = \rho(U(y), \xi) = \rho(\rho(U(y), \eta), e_M). \tag{27}$$

We then derive from (26), (27) and the monotonicity of $\rho(\cdot, e_M)$

$$\rho(U(y), \eta) \leq U(\mathrm{o}(e_M))$$

which in turn implies, due to $U(y) = \beta(\xi) \geq f(y)$,

$$\rho(f(y), \eta) \leq \rho(U(y), \eta) \leq U(\mathrm{o}(e_M))$$

We then have by the very definition of U, and (27)

$$U(\mathrm{o}(e_M)) = \rho(U(y), \eta) \qquad \text{and} \qquad U(y) = \rho(U(\mathrm{o}(e_M)), e_M). \tag{28}$$

We finally get

$$\rho(U(\mathrm{o}(e_M)), \zeta) = \rho(\rho(U(\mathrm{o}(e_M)), e_M), \eta) = \rho(U(y), \eta) = U(\mathrm{o}(e_M))$$

and consequently

$$U(\mathrm{o}(e_M) = \beta(\zeta)$$

or, in other term, formula (24) with $j = M$. It can be extended to all j by iterating backward the above argument.

We proceed proving (25). We set

$$\alpha = \rho\left(U(\mathrm{o}(e_k)), (e_i)_{i=k}^{j-1}\right).$$

By (24) we have

$$\rho\left(\alpha, (e_i)_{i=j}^M \cup (e_i)_{i=1}^{k-1}\right) = U(\mathrm{o}(e_k))$$

and accordingly by the concatenation formula

$$\begin{aligned} \rho\left(\alpha, (e_i)_{i=j}^M \cup (e_i)_{i=1}^{j-1}\right) &= \rho\left(\rho\left(\alpha, (e_i)_{i=j}^M \cup (e_i)_{i=1}^{k-1}\right), (e_i)_{i=k}^{j-1}\right) \\ &= \rho\left(U(\mathrm{o}(e_k)), (e_i)_{i=k}^{j-1}\right) = \alpha. \end{aligned}$$

This implies by (24) that $\alpha = U(\mathrm{o}(e_j))$, as was claimed.

□

The above assertion can be slightly strengthen.

7.3. **Corollary.** *Given $y \in \mathcal{A}_\lambda$, there exists a circuit ζ based on y with $U(y) = \beta(\zeta)$. It therefore enjoys the same properties stated for ξ in Proposition 7.2.*

Proof: We adopt the same notation of Proposition 6.3. We denote by ξ a cycle based on y with $U(y) = \beta(\xi)$. We assume that there is a cycle η properly contained in ξ satisfying condition (20). Since $\mathrm{o}(\xi_2) = \mathrm{o}(\eta)$ we get thanks to the concatenation principle and (25)

$$U(y) = \rho(U(\mathrm{o}(\xi_2)), \xi_2) = \rho(U(\mathrm{o}(\eta)), \xi_2) = \rho(U(y), \xi_1 \cup \xi_2).$$

This shows that $U(y) = \beta(\xi_1 \cup \xi_2)$. By slightly adapting the argument, we reach the same conclusion getting rid of condition (20). This procedure can be repeated for all other cycles properly contained in ξ, and we end up with a circuit ζ satisfying the assertion. □

The next Proposition provide a further representation formula for the solution of ($\mathbf{DFE}_\lambda$) and shows that the λ–Aubry sets are nonempty. The argument is reminiscent of that of Proposition 6.15 in [11].

7.4. **Proposition.** *The λ–Aubry set is nonempty. Moreover, if U is the solution of* ($\mathbf{DFE}_\lambda$) *then the following formula holds true*

$$U(x) = \min\{\rho(U(y), \zeta) \mid y \in \mathcal{A}_\lambda,\ \zeta \text{ simple path that links } y \text{ to } x\}. \tag{29}$$

If y, $\zeta = (e_i)_{i=1}^M$ realize the minimum in (29)*,we in addition have*

$$U(\mathrm{o}(e_j)) = \rho\left(U(y), (e_i)_{i=1}^{j-1}\right) \qquad \text{for any } j = 2, \cdots, M. \tag{30}$$

Proof: Since U is solution, then there exists for any $x \in \mathbf{V}$ an edge $e \in -\mathbf{E}_x$ with

$$U(x) = \rho(U(\mathrm{o}(e)), e).$$

By iterating backward the previous procedure and using the concatenation formula, we can construct a path ξ of any possible length, with $\mathrm{t}(\xi) = x$ such that

$$U(x) = \rho(U(\mathrm{o}(\xi)), \xi).$$

Since the set $\mathbf{E}$ is finite, we will find, by going on in the iteration, a cycle η contained in ξ such that, by construction

$$U(\mathrm{o}(\eta)) = U(\mathrm{t}(\eta)) = \rho(U(\mathrm{o}(\eta)), \eta)$$

which implies $U(\mathrm{o}(\eta)) = \beta(\eta)$ and consequently that $y := \mathrm{o}(\eta) \in \mathcal{A}_\lambda$. We denote by ζ the portion of ξ after η. It is a simple path, up to suitable choice of the cycle η, joins y to x, and in addition

$$U(x) = \rho(U(y), \zeta).$$

This relation shows (29). Formula (30) is a direct consequence of the construction of ζ. □

7.5. **Remark.** If e is a loop with vertex x and $U(x) = \beta(e) = \beta(-e)$ then apparently $x \in \mathcal{A}_\lambda$. By combining it with Remark 6.5, we can say that if on the contrary $x \notin \mathcal{A}_\lambda$ then any loop based on x can be removed from the graph without affecting the solution U.

8. Asymptotic as $\lambda \longrightarrow 0$

In this section we will study the asymptotic behavior of the solutions to $(\mathbf{HJ}\Gamma_\lambda)$, $(\mathbf{DFE}_\lambda)$ and the corresponding λ–Aubry sets as λ tends to 0, assuming that the Hamiltonians H_γ satisfy, in addition to **(H1)** and **(H2)**, the conditions **(H3)** and **(H4)**, see Subsection 8.1. We plan to perform in a subsequent paper a more complete analysis of the issue with the aim of recovering in our setting the uniqueness of the limit established in [5].

8.1. **Eikonal equations on networks.** We summarize in this subsection some material taken from [11] needed for the forthcoming convergence results. We consider the Eikonal problem on Γ assuming, beside **(H1)**, **(H2)**, the following additional conditions

(H3) for any $x \in \Gamma$, $\gamma \in \mathcal{E}$, $H_\gamma(x, \cdot)$ is quasiconvex with

$$\mathrm{int}\{p \mid H_\gamma(x,p) \le a\} = \{p \mid H_\gamma(x,p) < a\} \quad \text{for any } a \in \mathbb{R},$$

where int stands for the interior.

(H4) given any $\gamma \in \mathcal{E}$, the map $s \mapsto \min_{p\in\mathbb{R}} H_\gamma(s,p)$ is constant in $[0,1]$.

8.1. **Remark.** Assumption **(H4)** can be actually formulated in a slightly weaker way, see [11], We have chosen the above version for simplicity.

We consider for any given arc γ the family of Eikonal equations

$$H_\gamma(s, w') = a \qquad \text{in } (0,1),$$

with $a \in \mathbb{R}$. We look for continuous functions v defined on Γ such that

$$H_\gamma(s, (v \circ \gamma)') = a \qquad \text{in } [0,1], \text{ for any } \gamma \in \mathcal{E}$$

The definition of (sub/super) solution is given as in Definition 2.3 with obvious adaptations.

8.2. **Proposition.** *There exists one and only one value of a, called* critical*, such that the above equation on Γ admits solutions.*

We assume throughout the paper, without any loss of generality, that the critical value is 0. It is then clear that

$$0 \geq \max_{\gamma \in \mathcal{E}} \min_{p \in \mathbb{R}} H_\gamma(0, p).$$

We focus on the critical equations

$$(\mathbf{HJ}\gamma) \qquad H_\gamma(s, w') = 0 \qquad \text{in } (0,1),$$

and

$$(\mathbf{HJ}\Gamma) \qquad H_\gamma(s, (v \circ \gamma)') = 0 \qquad \text{in } [0,1], \text{ for any } \gamma \in \mathcal{E}$$

We associate to $(\mathbf{HJ}\Gamma)$ the discrete equation on $\mathbf{V}$.

$$(\mathbf{DFE}) \qquad V(x) = \min_{e \in -\mathbf{E}_x} \big(V(\mathrm{o}(e)) + \sigma(e) \big)$$

where $\sigma(e) = v_{\Psi(e)}(1)$, and $v_{\Psi(e)}$ is the function appearing in Lemma 3.12 in relation with the equation $H_{\Psi(e)} = 0$. We define

$$\sigma(\xi) = \sum_{i=1}^{M} \sigma(e_i) \qquad \text{for any path } \xi = (e_i)_{i=1}^M.$$

8.3. **Proposition.** *A function $V : \mathbf{V} \to \mathbb{R}$ is subsolution to* $(\mathbf{DFE})$ *if and only if*

$$(31) \qquad V(y) - V(x) \leq \sigma(\xi) \qquad \text{for any path } \xi \text{ linking } x \text{ to } y.$$

There are results similar to Theorem 4.2, Proposition 4.3 linking $(\mathbf{HJ}\Gamma)$ and $(\mathbf{DFE})$. We recall in particular:

8.4. **Proposition.** *The trace on $\mathbf{V}$ of any solution to* $(\mathbf{HJ}\Gamma)$ *is solution of* $(\mathbf{DFE})$*. Conversely, any solution of* $(\mathbf{DFE})$ *can be uniquely extended to a solution of* $(\mathbf{HJ}\Gamma)$*.*

The Aubry set $\mathcal{A}$ is made up by vertices y such that there is a cycle ξ based on it with $\sigma(\xi) = 0$.

In general equation (**DFE**) has many solutions, not just differing by an additive constant. They are univocally determined, once a trace satifying (31) is assigned on $\mathcal{A}$. The Aubry set plays in a sense the role of a hidden boundary.

8.2. **Convergence results.** We denote by u_λ , for $\lambda > 0$, the solution to ($\mathbf{HJ}\Gamma_\lambda$). and set $U_\lambda = u_\lambda|_{\mathbf{V}}$. U_λ is then the solution of the corresponding discrete equation ($\mathbf{DFE}_\lambda$).

8.5. **Lemma.** *The functions $u_\lambda : \Gamma \to \mathbb{R}$ are equibounded with respect to $\lambda > 0$.*

Proof: Let v be a solution of the Eikonal equation on Γ. We can choose a large positive constant α such that that $v+\alpha$, $v-\alpha$ are super and subsolution of ($\mathbf{HJ}\Gamma_\lambda$) for any $\lambda > 0$. We derive from Theorem 4.6

$$v - \alpha \leq u_\lambda \leq v + \alpha.$$

□

8.6. **Proposition.** *The functions $u_\lambda : \Gamma \to \mathbb{R}$ converge to a solution of the Eikonal equation on Γ, up to subsequences.*

Proof: We have that

$$\lambda\, u_\lambda(x) \geq \min\{\lambda\, m, -\max_\gamma \max_s H_\gamma(s,0)\} \qquad \text{for any } x \in \Gamma,\ \lambda > 0,$$

where m is a lower bound for all the u_λ as x varies in Γ, see Lemma 8.5. We deduce, by the coercivity of the H_γ, that the functions u_λ are equi–Lipschitz continuous and equibounded. They are therefore convergent up to subsequences.

Assume, to fix ideas, that u_{λ_n}, for some infinitesimal sequence λ_n, converges to a function v. Then $v \circ \gamma$ is solution in $(0,1)$ of ($\mathbf{HJ}\gamma$), for any arc γ, by basic stability properties of viscosity solutions theory.

Given a vertex x, there is, by the very definition of solution to ($\mathbf{HJ}\Gamma_\lambda$), an arc γ_n with $\gamma_n(1) = x$ such that $u_{\lambda_n} \circ \gamma_n$ satisfies the state constraint boundary condition for ($\mathbf{HJ}\gamma_\lambda$), with $\lambda = \lambda_n$ at $s = 1$. The arcs being finite, we can extract a subsequence λ_{n_k} of λ_n and select γ with $\gamma(1) = x$ such that $u_{\lambda_{n_k}} \circ \gamma$ satisfies the state constraint boundary condition for ($\mathbf{HJ}\gamma_\lambda$), with $\lambda = \lambda_{n_k}$, for any k, at $s = 1$. By applying standard arguments, we derive that the limit function $v \circ \gamma$ satisfies the state constraint boundary condition for ($\mathbf{HJ}\gamma$). This concludes the proof, taking into account the definition of solution to ($\mathbf{HJ}\gamma$).

□

8.7. **Proposition.** *We have*

$$\rho_\lambda(\alpha_n,\xi) \longrightarrow \alpha+\sigma(\xi) \qquad \text{as } \lambda \longrightarrow 0,$$

for any path $\xi=(e_i)_{i=1}^M$ *and* $\alpha_n \longrightarrow \alpha \in \mathbb{R}$ *with* $\alpha_n \le \underline{\alpha}_\lambda(e_1)$ *and*

$$\rho_\lambda\left(\alpha_n,(e_i)_{i=1}^j\right) \le \underline{\alpha}(e_{j+1}) \qquad \text{for } j=1,\cdots,M-1, \quad n \text{ large.}$$

Proof: The argument proceeds by induction on the length of ξ. If $M=1$ then the assertion is a consequence of Lemma 3.13 . We assume it true for any path of length less than or equal to $M-1$ and deduce it for the length M. We write $\overline{\xi}=(e_i)_{i=1}^{M-1}$ and use the concatenation formula plus induction step, and Lemma 3.13 to get

$$\lim_{\lambda\to 0}\rho_\lambda(\alpha_n,\xi)=\rho_\lambda(\rho_\lambda(\alpha_n,\overline{\xi}),e_M)=\alpha+\sigma(\overline{\xi})+\sigma(e_M)=\alpha+\sigma(\xi).$$

□

As pointed out in the Introduction, the next proposition should be compared with the convergence result for Mather sets obtained in [9].

8.8. **Proposition.** *The sets* $\mathcal{A}_\lambda$ *are contained in* $\mathcal{A}$ *for* λ *sufficiently small.*

Proof: The argument is by contradiction. Since the vertices are finite, we can therefore assume that there is $y \in \mathbf{V}$ and $\lambda_n \to 0$ with

$$y \in (\cap_n \mathcal{A}_{\lambda_n}) \setminus \mathcal{A}.$$

Taking into account the very definition of λ–Aubry set, Corollary 7.3, and the fact that the circuits are finite, we have, up to extracting a subsequence from λ_n, that there exists a circuit $\xi=(e_i)_{i=1}^M$ based on y satisfying $U_{\lambda_n}(y)=\beta_{\lambda_n}(\xi)$ for any n, and the conditions of Proposition 7.2. Taking into account Proposition 6.2, we then have

$$U_{\lambda_n}(y)=f_{\lambda_n}(y)\le \underline{\alpha}_{\lambda_n}(e_1)$$

and

$$\rho_{\lambda_n}\left(U_{\lambda_n}(y),(e_i)_{i=1}^j\right)=U_{\lambda_n}(\mathrm{o}(e_{j+1}))=f_{\lambda_n}(\mathrm{o}(e_{j+1}))\le \underline{\alpha}_{\lambda_n}(e_{j+1}) \qquad j=1,\cdots,M-1.$$

Since the sequence $U_{\lambda_n}(y)$ is bounded by Lemma 8.5, it is convergent to some α, up to subsequences, and we have by applying Proposition 8.7

$$\alpha=\lim_n U_{\lambda_n}(y)=\rho_{\lambda_n}(U_{\lambda_n}(y),\xi)=\alpha+\sigma(\xi).$$

This is impossible because $y \notin \mathcal{A}$, and consequently by the very definition of $\mathcal{A}$, $\sigma(\xi)>0$. □

We consider the limit set $\mathcal{B}$ defined as

$$\mathcal{B} = \{y \in \mathbf{V} \mid \exists\, \lambda_n \longrightarrow 0 \quad \text{with} \;\; y \in \mathcal{A}_{\lambda_n}\}$$

It comes from Proposition 8.8 that $\mathcal{B}$ is contained in the Aubry set $\mathcal{A}$. The next result shows that any limit of the U_λ is uniquely determined by its trace on $\mathcal{B}$.

8.9. **Proposition.** *Let U_{λ_n} be a sequence of solution to* $(\mathbf{DFE}_\lambda)$ *with $\lambda = \lambda_n$, converging to V. Then V is a solution of* $(\mathbf{DFE})$ *satisfying*

$$V(x) = \min\{V(y) + \sigma(\xi) \mid y \in \mathcal{B},\, \xi \text{ path joining } y \text{ to } x\}.$$

Proof: We set to ease notations

$$U_n = U_{\lambda_n}, \quad \rho_n = \rho_{\lambda_n}, \quad \mathcal{A}_n = \mathcal{A}_{\lambda_n}.$$

We know from Proposition 8.6 that V solves $(\mathbf{DFE})$. For any $x \in \mathbf{V}$, we have by Proposition 7.4 that

$$U_n(x) = \rho_n(U_n(y_n), \xi_n)$$

for some $y_n \in \mathcal{A}_n$, and some simple path ξ_n linking y_n to x. Since both vertices and simple paths are finite, we deduce that there is a subsequence λ_{n_k}, $y \in \cap_k \mathcal{A}_{n_k} \subset \mathcal{B}$, a simple path $\xi = (e_i)_{i=1}^M$ joining y to x such that

$$U_{n_k}(x) = \rho_{n_k}(U_{n_k}(y), \xi) \qquad \text{for any } k$$

and in addition

$$U_{n_k}(\mathrm{o}(e_j)) = \rho_{n_k}\left(U_{n_k}(y), (e_i)_{i=1}^{j-1}\right) \qquad \text{for any } j = 2, \cdots M-1.$$

Owing to Proposition 6.2 and to the inequality $f_{n_k} \geq U_{n_k}$, we are therefore in the position to apply Proposition 8.7 and get

$$\lim_k \rho_{n_k}(U_{n_k}(y), \xi) = V(y) + \sigma(\xi).$$

This implies

$$V(x) \geq \min\{V(y) + \sigma(\xi) \mid y \in \mathcal{B},\, \xi \text{ path joining } y \text{ to } x\}.$$

The converse inequality is a consequence of V being solution to $(\mathbf{DFE})$, see Proposition 8.3. □

Appendix A. Graphs and networks

An *immersed network* or *continuous graph* is a subset $\Gamma \subset \mathbb{R}^N$ of the form

$$\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma([0,1]) \subset \mathbb{R}^N,$$

where $\mathcal{E}$ is a finite collection of regular simple curves, called *arcs* of the network, we assume for simplicity parameterized in $[0,1]$. The main condition is

$$\gamma((0,1)) \cap \gamma'([0,1]) = \emptyset \qquad \text{whenever } \gamma \neq \pm\gamma', \tag{32}$$

where for any arc γ, the *inverse arc* $-\gamma$ defined as

$$-\gamma(s) = \gamma(1-s) \qquad \text{for } s \in [0,1].$$

We make precise that we consider throughout the paper γ, $-\gamma$ as distinct arcs. We call *vertices* initial and terminal points of the arcs, and denote by $\mathbf{V}$ the sets of all such vertices. Note that (32) implies that

$$\gamma((0,1)) \cap \mathbf{V} = \emptyset \qquad \text{for any } \gamma \in \mathcal{E}.$$

We assume that the network is connected, namely given two vertices there is a finite concatenation of arcs linking them.

As already pointed out, we do not put any restriction on the geometry of the network.

A graph $\mathbf{X} = (\mathbf{V}, \mathbf{E})$ is an ordered pair of sets $\mathbf{V}$ and $\mathbf{E}$, which are called, respectively, *vertices* and (directed) *edges*, plus two functions:

$$\mathrm{o} : \mathbf{E} \longrightarrow \mathbf{V}$$

which associates to each (oriented) edge its *origin* (initial vertex), and

$$\begin{array}{rrcl} - & : \mathbf{E} & \longrightarrow & \mathbf{E} \\ & e & \longmapsto & -e, \end{array}$$

which changes orientation, and is a fixed point free involution. We define the terminal vertex of e as

$$\mathrm{t}(e) = \mathrm{o}(-e)$$

We consider e and $-e$ as distinct edges. We call *loop* any edge e with $\mathrm{o}(e) = \mathrm{t}(e)$. We define *path* $\xi = (e_1, \cdots, e_M)$ any finite sequence of concatenated edges, namely satisfying

$$\mathrm{t}(e_j) = \mathrm{o}(e_{j+1}) \qquad \text{for any } j = 1, \cdots, M-1.$$

We define the *length of a path* as the number of its edges. We set $\mathrm{o}(\xi) = \mathrm{o}(e_1)$, $\mathrm{t}(\xi) = \mathrm{t}(e_M)$. We call a path *closed* or a *cycle* if $\mathrm{o}(\xi) = \mathrm{t}(\xi)$.

Given two paths ξ, η, we say that ξ is contained in η, mathematically $\xi \subset \eta$, if the edges of ξ make up a subset of the edges of η. If the condition $\mathrm{t}(\xi) = \mathrm{o}(\eta)$ holds true, we denote by $\xi \cup \eta$ the path obtained via concatenation of ξ and η.

We call *simple* a path without repetition of vertices, except possibly the initial and terminal vertex, in other terms $\xi = (e_i)_{i=1}^M$ is simple if

$$\mathrm{t}(e_i) = \mathrm{t}(e_j) \Rightarrow i = j.$$

A.1. **Remark.** There are finite many simple paths in a finite graph. In fact their number is estimated from above by that of the sum of the k–permutations of $|\mathbf{E}|$ objects for $2 \leq k \leq |\mathbf{E}|$.

A.2. **Proposition.** *A path is simple if and only there is no simple cycle properly contained in it.*

We define a *circuit* to be a simple cycle.

Given $x \in \mathbf{V}$, we set

$$-\mathbf{E}_x = \{e \in \mathbf{E} \mid \mathrm{t}(e) = x\}. \tag{33}$$

Starting from a network, a graph can be defined taking as vertices the same vertices of Γ and as edges the elements of any abstract set $\mathbf{E}$ equipotent to $\mathcal{E}$. We denote by Ψ a bijection from $\mathbf{E}$ to $\mathcal{E}$. The functions o, $-$ yielding the graph structure are given by

$$\begin{aligned} \mathrm{o}(e) &= \Psi(e)(0) \\ -e &= \Psi^{-1}(-\Psi(e)). \end{aligned}$$

A graph corresponding to a connected network is connected in the sense that any two vertices are linked by some path.

Appendix B. Basic material on HJ equations in $(0,1)$

Given a Lipschitz–continuous function w in $[0,1]$, we set for $s \in [0,1]$

$$\partial w(s) = \mathrm{co}\,\{p \mid p = \lim w'(s_i),\ w \text{ differentiable at } \ s_i,\ \ s_i \to s,\ s_i \in (0,1)\}, \tag{34}$$

where the symbol co stands for convex hull.

B.1. **Lemma.** *Given a Lipschitz–continuous function w in $[0,1]$, the function* $s \mapsto w(0) + q\,s$ *is a constrained subtangent to w at $s = 0$ if*

$$q < \min\{p \mid p \in \partial w(0)\}. \tag{35}$$

the function $s \mapsto w(1) + q\,(s-1)$ *is a constrained subtangent to w at $s = 1$ if*

$$q > \max\{p \mid p \in \partial w(1)\}. \tag{36}$$

Proof: We consider the case $s = 1$, the assertion at $s = 0$ can be proved similarly. We assume condition (36). By the very definition of $\partial w(1)$ there is an open interval I containing 1 with

$$q > p \qquad \text{for any } s \in I \cap (0,1),\ p \in \partial w(s). \tag{37}$$

Assume for purposes of contradiction that there is $s \in I \cap (0,1)$ with

$$w(s) < w(1) + q\,(s-1), \tag{38}$$

by Mean Value Theorem for generalized Clarke gradients (Theorem 2.3.7 in [4]), we find $\overline{s} \in (s,1) \subset I \cap (0,1)$ with

$$w(s) - w(1) = \overline{p}\,(s-1) \qquad \text{for some } \overline{p} \in \partial w(\overline{s}).$$

We derive, in the light of (38)

$$q\,(s-1) > \overline{p}\,(s-1)$$

which in turn implies $q < \overline{p}$, in contradiction with (37). □

Proof: (**of Lemma 3.2**) The function constantly equal to $c := -\frac{1}{\lambda} \max_s H_\gamma(s,0)$ is a subsolution to ($\mathbf{HJ}\gamma_\lambda$). By the coercivity of H_γ, the family of subsolutions greater than or equal to c is equi–Lipschitz continuous and is in addition dominated by

$$-\frac{1}{\lambda} \min\{H_\gamma(s,p) \mid s \in [0,1],\ p \in \mathbb{R}\}.$$

This shows that $u^\gamma_{\max}$ is finite valued and Lipschitz continuous. By standard arguments in viscosity solutions theory, the maximality of $u^\gamma_{\max}$ implies that it is a solution in $(0,1)$, and satisfies the state constraints boundary condition at $s = 0, 1$.

Assume now, for purposes of contradiction, that there is another solution w of the equation plus state constraints boundary conditions. We set

$$-\delta = \min_{[0,1]} (w - u^\gamma_{\max}) < 0.$$

We can use suitable sup–convolutions of $u^\gamma_{\max}$ as test functions from below to prove that the minimizers of $w - u^\gamma_{\max}$ cannot be interior points of the interval. To show that they cannot be boundary points, we exploit Lemma B.1. Assume, to fix ideas, that 1 is such a a minimizer. Therefore $u^\gamma_{\max} - \delta$ is a constrained subtangent to w at 1. We set

$$p_0 = \max\{p \mid p \in \partial u^\gamma_{\max}(1)\},$$

by the definition of $\partial u^\gamma_{\max}$, there is a sequence s_i of differentiability points of $u^\gamma_{\max}$ in $(0,1)$ converging to 1 with

$$(u^\gamma_{\max})'(s_i) \longrightarrow p_0.$$

Since

$$\lambda(u^\gamma_{\max}(s) - \delta) + H_\gamma(s, (u^\gamma_{\max})'(s)) = -\lambda\delta$$

at any differentiability point s of $u^\gamma_{\max}$, we derive by the continuity of H_γ

$$\lambda\,(u^\gamma_{\max}(1) - \delta) + H_\gamma(1, p_0) < 0, \tag{39}$$

and we can therefore find $q > p_0$ with

$$\lambda w(1) + H_\gamma(1, q) = \lambda\,(u^\gamma_{\max}(1) - \delta) + H_\gamma(1, q) < 0. \tag{40}$$

By Lemma B.1 the function $s \mapsto (u^\gamma_{\max}(1) - \delta) + q\,(s-1)$ is constrained subtangent to $(u^\gamma_{\max} - \delta)$ at 1 and consequently also to w at 1. Inequality (40) shows that w does not satisfy the state constraint boundary condition at 1, reaching a contradiction.

□

Proof: (**of Lemma 3.3**) The function $v \equiv c$ with

$$c = \min\left\{-\frac{1}{\lambda}\max_{s\in[0,1]} H_\gamma(s,0)\,,\,\alpha\right\},$$

is subsolution to ($\mathbf{HJ}\gamma_\lambda$) taking a value less than or equal to α at $s = 0$. We deduce that

$$u^\gamma_\alpha(s) = \sup\{v(s) \mid v \text{ subsolutions to } (\mathbf{HJ}\gamma_\lambda) \text{ with } v(0) \le \alpha,\, v \ge c\}$$

and by the coercivity of H_γ the functions of this family are equi–Lipschitz continuous and equibounded. This proves that u^γ_α is a Lipschitz continuous subsolution to ($\mathbf{HJ}\gamma_\lambda$). The supersolution property and the validity of the state constraint boundary condition at $s = 1$ are straightforward consequences of the maximality property.

If $\alpha \le u^\gamma_{\max}(0)$ then there is a subsolution taking the value α at 0 and consequently by maximality $u^\gamma_\alpha(0) = \alpha$. Conversely, if $u^\gamma_\alpha(0) = \alpha$ then $u^\gamma_{\max}(0) \ge u^\gamma_\alpha(0) = \alpha$. □

Proof: (**of Proposition 3.8**) Let u be a solution to ($\mathbf{HJ}\gamma_\lambda$) plus Dirichlet boundary conditions. The asserted uniqueness comes from Theorem 3.1 and (5) is a direct consequence of the definition of u^γ_α, $u^{-\gamma}_\beta$ and Remark 3.7. Conversely, let us assume (5), we define

$$\begin{aligned} \overline{u}(s) &= \min\{u^\gamma_\alpha(s), u^{-\gamma}_\beta(1-s)\} \\ \underline{u}(s) &= \max\{u^{-\gamma}_\beta(1-s) + \alpha - u^{-\gamma}_\beta(1)\,,\, u^\gamma_\alpha(s) + \beta - u^\gamma_\alpha(1)\}. \end{aligned}$$

The functions $\overline{u}$, $\underline{u}$ are super and subsolutions to ($\mathbf{HJ}\gamma_\lambda$), respectively. We derive from (5) and Remark 3.7 that

$$\alpha \leq u^{-\gamma}_\beta(1) \leq u^\gamma_{\max}(0)$$

and so $u^\gamma_\alpha(0) = \alpha$ by Lemma 3.3 and $\overline{u}(0) = \alpha$. We also have by (5)

$$\alpha = u^\gamma_\alpha(0) \geq u^\gamma_\alpha(0) + \beta - u^\gamma_\alpha(1)$$

which implies $\underline{u}(0) = \alpha$. Similarly

$$\beta \leq u^\gamma_\alpha(1) \leq u^\gamma_{\max}(1)$$

which implies $u^{-\gamma}_\beta(0) = \beta$ and $\overline{u}(1) = \beta$, in addition

$$\beta = u^{-\gamma}_\beta(0) \geq u^{-\gamma}_\beta(0) + \alpha - u^{-\gamma}_\beta(1)$$

which gives $\underline{u}(1) = \beta$. This shows that $\underline{u}$, $\overline{u}$ satisfy the same boundary Dirichlet conditions and are, in addition, both Lipschitz–continuous. Existence of the claimed solution then comes via a straightforward application of Perron Method, see [2]. □

Proof: of Lemma 3.13 We have that

$$\lambda_n\, u_n(s) \geq \min\{-\max_s H_\gamma(s,0),\, \alpha - 1\} \qquad \text{for any } s \in [0,1],\ n \text{ large.}$$

This implies that the u_n are equibounded and equi–Lipschitz continuous. They therefore converge, up to subsequences, to some function u with $u(0) = \alpha$. By stability properties of viscosity solutions u solves ($\mathbf{HJ}_\gamma$). Therefore

$$u \leq \alpha + v. \tag{41}$$

If $a_\gamma = 0$ then the above inequality must be an equality. If instead $a_\gamma < 0$ then there is a strict subsolution w of $H_\gamma = 0$ with

$$H_\gamma(s, w') \leq -\delta \quad \text{for a suitable } \delta > 0 \quad \text{and } w(s) \leq 0, \tag{42}$$

We consider a sequence of positive numbers μ_k converging to 1 and a subsequence λ_{n_k} of λ_n with

$$\lambda_{n_k} \leq \frac{(1-\mu_k)\delta}{\mu_k}\,\frac{1}{M}, \tag{43}$$

where M is an upper bound of $\alpha_n + v(s)$ for n large and s varying in $[0,1]$. We exploit (42), (43) and the convex character of H_γ to get

$$\begin{aligned} & \lambda_{n_k}(\mu_k\,(\alpha_{n_k}+v)+(1-\mu_k)\,w)+H_\gamma(s,\mu_k\,Dv+(1-\mu_k)\,Dw) \\ \leq\ & \lambda_{n_k}\,\mu_k\,(\alpha_{n_k}+v)-(1-\mu_k)\,\delta \leq \lambda_{n_k}\,\mu_k\,M-(1-\mu_k)\,\delta \\ \leq\ & \frac{(1-\mu_k)\,\delta}{\mu_k}\,\frac{1}{M}\,\mu_k\,M-(1-\mu_k)\,\delta = 0. \end{aligned}$$

We thus see that $\mu_k\,(\alpha_{n_k}+v)+(1-\mu_k)\,w$ is subsolution to $(\mathbf{HJ}\gamma_\lambda)$ with $\lambda = \lambda_{n_k}$ taking in addition, by (42), a value less than α_{n_k} at $s=0$, at least for k large. We infer by the maximality property of u_{n_k}

$$u_{n_k} \geq \mu_k\,(\alpha_{n_k}+v)+(1-\mu_k)\,w \qquad \text{in } [0,1],$$

so that

$$\liminf_k u_{n_k} \geq \lim_k \mu_k\,(\alpha_{n_k}+v)+(1-\mu_k)\,w = (\alpha+v).$$

The above relation, together with (41), shows the assertion.

□

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