TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Dipartimento di Matematica

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SINGULARITIES OF LOW-DEGREE DIVISORS ON ABELIAN VARIETIES

Supervisor:
Prof. Giuseppe Pareschi
Coordinator:
Prof. Carlangelo Liverani

Candidate:
Flavio Blondeau

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“Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding.”

William Paul Thurston

Alla mia famiglia & a Enrica

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CHAPTER 1

Introduction

Abelian varieties have been a major topic for decades. Their double nature of algebraic variety (or scheme) and algebraic group gives them a rigid structure, and they are a central tool in serveral mathematical topics, like algebraic geometry, complex analysis and algebraic number theory. In order to describe their geometry, it is useful to study divisors on such varieties, with a particular focus on their singularities.

To do so, we will talk about *polarized* abelian variety. Thus, an important feature is the *polarization* of an abelian variety, which we can think of as a choice of an ample line bundle of the variety. Such polarization comes with a *degree*, which is simply the Euler characteristic of the associated line bundle.

In this thesis we present the evolution of some ideas and techniques behind the classification of singularities of divisors on low-degree polarizations, and we provide some results in the same framework, specifically for (indecomposable) polarizations of degree 3 and 4.

We start by giving a very general picture of the research and results about singularities of divisors on abelian varieties.

A first step in approaching the topic of singularities was made by looking at Jacobian varieties of compact Riemann surfaces, which are a particular class of principally polarized (that is, with a polarization of degree one) abelian varieties. In this case the polarization corresponds to a well-defined *theta divisor* Θ. The advantage of Jacobians is that their geometry can be described in terms of linear series of curves, and so it is well understood.

Using these tools, George Kempf [Kem73] proved that, for Jacobians, Θ is irreducible and normal, with *rational singularities.* We recall that a variety X has rational singularities if there exists a resolution of singularities $f\colon Y \to X$ such that $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$. Intuitively, this property means that on the fibres of f, the higher cohomology groups of the structure sheaf are zero.

What about an arbitrary principally polarized abelian variety? The difficulty here is that we do not have a good model for the theta divisor Θ. However, a leap forward was made by János Kollár [Kol95], who first had the intuition of using in this setting a class of important results known as *vanishing theorems*.

These theorems assert that some cohomology groups of certain sheaves are zero. The first one is the Kodaira vanishing theorem, which says that the higher cohomology groups of the sheaf $\omega_X \otimes L$ are zero when L is an ample line bundle on a variety X. Another important theorem is the Kawamata-Viehweg vanishing, which in its simplest formulation is a generalization of Kodaira vanishing, since it only requires that L is nef and big.

With this last theorem, Kollár proved that Θ has *log-canonical singularities*. Such definition of singularities comes from the Minimal Model Program (MMP) for the classification of higher dimensional algebraic varieties. Roughly, we can say that log-canonical singularities are the most general ones for which MMP results may be applied. They are then a mild kind of singularity, although different from rational singularities.

A couple of years after Kollár's work, Lawrence Ein and Robert Lazarsfeld [EL97] came up with a theorem which is the strongest general result about theta divisors so far. It states that if Θ is an irreducible theta divisor on an abelian variety, then it is normal with rational singularities.

By duality, Θ has rational singularities if $\omega_\Theta \simeq f_*\omega_X$ for every resolution of singularities $f\colon X \to \Theta$. The idea of the proof is to measure the difference between the two sides of the equivalence by a particular ideal sheaf, called *adjoint ideal*, then to prove that this sheaf is trivial. Here the crucial point is the use of a result by Mark Green and Lazarsfeld [GL87; GL91], known as the *generic vanishing theorem*.

The aim of the generic vanishing theorem is to see what happens to a specific kind of line bundles on a variety X, sometimes called topologically trivial line bundles, parametrized by a variety denoted as $\operatorname{Pic}^0(X)$. These line bundles are not ample or nef, and have some peculiar properties. As in classical vanishing theorems, we want to look at the cohomology of $\omega_X \otimes P$, where P is a topologically trivial bundle. So we define a locus

$$V^i(\omega_X) := \{P \in \operatorname{Pic}^0(X) \mid H^i(X, \omega_X \otimes P) \neq 0\}$$

for each $i \geq 0$. Then, if X has dimension n, the generic vanishing says that

$$\operatorname{codim} V^i(\omega_X) \geq i - (n-k)$$

where k is the dimension of the image of X via a particular map, called *Albanese map*. This in turn implies that, for a general $P \in \operatorname{Pic}^0(X)$, the cohomology groups $H^i(X, P)$ are zero for every $i \in [0, k]$ (and it is the reason why the theorem is called *generic vanishing*).

In the same spirit as their previous result, Ein and Lazarsfeld [EL97] have also proved a statement about divisors on $|m\Theta|$, where m is a positive integer. More precisely, if D is a divisor in $|m\Theta|$, then the $\mathbb{Q}$-divisor $(1/m)D$ (that is, a divisor with rational coefficients) has log-canonical singularities. This time, the goal is to prove that another ideal sheaf, the so-called *multiplier ideal*, is trivial; again, this fact follows by using some vanishing theorems.

Now, an obvious question arises: what can we say about polarizations that are not principal? A major issue is that, by rising up the degree of a polarization, there are more and more divisors representing such polarization. This means special cases can appear among them, and there may also exist reducible divisors.

Nonetheless, if we stick to a polarization of degree 2 which is *indecomposable* (so it is not the product of smaller polarizations), we end up with some important results. Indeed, Christopher Hacon and Olivier Debarre [Hac00; DH07] have shown the following statement: for an indecomposable polarization L of degree at most 2, every prime divisor in $|L|$ has rational singularities; also, for $m \in \mathbb{Z}_{>0}$, if $D \in |mL|$ and $\lfloor (1/m)D \rfloor = 0$, then $(1/m)D$ has *log-terminal singularities.*

This kind of singularities comes again from MMP, and it is a "better" one than the log-canonical, and it is the reason why there is the extra hypotesis $\lfloor (1/m)D \rfloor = 0$. Indeed, without this additional hypotesis, we can say that $(1/m)D$ has log-canonical singularities. Aside from the log-terminal question, the proof relies on the same basic idea of the preceeding ones: show that the adjoint or multiplier ideal is trivial by means of vanishing theorems and Green and Lazarsfeld theory.

An analogous investigation can be made for an abelian variety denoted as *simple*, which means it has no non-trivial abelian subvarieties. In their paper, Debarre and Hacon also proved that a simple abelian variety contains only prime divisors, and this fact has led them to a result which is in parallel with the one for polarizations of degree 2. This has been sharpened and improved lately by Giuseppe Pareschi [Par21], and we can state the final result as follows: given a simple abelian variety of dimension g with a polarization L of degree less than g, every divisor in $|L|$ is prime, normal, and has rational singularities; also, for $m \in \mathbb{Z}_{>0}$, if $D \in |mL|$ and $D \neq mE$ for any $E \in |L|$, then $(1/m)D$ has log-terminal singularities. Here again the extra hypotesis $D \neq mE$ is needed in order to guarantee log-terminal singularities instead of log-canonical ones.

Pareschi's proof uses all the ingredients of the other proofs, namely vanishing theorems for ideal sheaves and Green and Lazarsfeld theory, together with results from the Fourier-Mukai theory for regularity of abelian varieties, developed by Pareschi and Mihnea Popa [PP03; PP04; PP11b, and other papers].

The next natural step is to take into account (indecomposable) polarizations with degree greater than 2. The final part of this thesis shows a couple of results about polarizations with degree 3 and 4 that look for rational singularities for prime divisors and log-canonical ones for $\mathbb{Q}$-divisors, in the same spirit as the previous ones.

But, before moving ahead, it is worth defining concisely the notion of *gv*-index. We have seen the subset $V^i(\omega_X)$ of $\mathrm{Pic}^0(X)$. It is possible to define in the same way the subset $V^i(\mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on X; indeed, this is the set of all $P \in \mathrm{Pic}^0(X)$ such that the cohomology groups $H^i(X, \mathcal{F} \otimes P)$ are not zero. From it, we can define the gv-index of $\mathcal{F}$ as

$$gv(\mathcal{F}) := \min_{i>0}\{\operatorname{codim} V^i(\mathcal{F}) - i\},$$

which measure the size of loci V^i relatively to the index i (note that if all $V^i(\mathcal{F})$ are empty, then $gv(\mathcal{F})$ is ∞). If X is an abelian variety, gv-index can be used to classify the regularity of coherent sheaves.

Now we move on to a short introduction to the original results of this thesis.

Let us consider an abelian variety A with an indecomposable polarization L of degree d. We take a prime and normal divisor $D \in |L|$. In order to see if it has rational singularities, we look at its adjoint ideal $\operatorname{adj}(D)$ and find out when it becomes trivial. Following previous results, we will assume $\operatorname{adj}(D)$ to be non-trivial, and so its cosupport Z (that is, its zero locus) is not empty.

We have then two short exact sequences: the first one is

$$0 \longrightarrow \mathcal{O}_A \longrightarrow L \otimes \operatorname{adj}(D) \longrightarrow \mu_*\omega_X \longrightarrow 0$$

and comes with the definition of adjoint ideal (where $\mu\colon X \to D$ is a resolution of singularities of the divisor); the other is a standard one for ideal sheaves

$$0 \longrightarrow L \otimes \operatorname{adj}(D) \longrightarrow L \longrightarrow L_{|Z} \longrightarrow 0$$

with $L_{|Z} = L\otimes\mathcal{O}_Z$. By twisting these sequences for any $P \in \operatorname{Pic}^0(A)$ and taking the long exact sequences in cohomology, we can point out some properties of the sheaves $L_{|Z}$ and $L\otimes\operatorname{adj}(D)$. In particular, we end up having $gv(L_{|Z}) = gv(L\otimes\operatorname{adj}(D))+1$. This fact, assuming that $gv(L_{|Z})$ is finite, leads to the following upper bound of the degree of L:

$$d \geq 2\ gv(\omega_X) + 1,$$

recalling that X is a smooth model of D.

Thus, now the goal is to understand what values can have the gv-index of ω_X. We already know that, by our assumptions, $gv(\omega_X)$ cannot be 0, and has to be a non-negative number. It is possible to prove that this gv-index cannot be 1 either, by means of some powerful tools like the structure theorem by Green and Lazarsfeld and Kollár decomposition.

To sum up, we obtain the following statement:

Theorem A. *Let A be an abelian variety, and let L be an indecomposable polarization on A with degree $d \leq 4$. Let D be a prime and normal divisor in $|L|$. We assume that $gv(L_{|Z})$ is finite, where Z is the cosupport of the ideal sheaf $\operatorname{adj}(D)$. Then D has rational singularities.*

It turns out that we can remove the extra assumption about $gv(L_{|Z})$ being finite, at least for polarizations of degree $d \leq 3$. This argument uses intermediate results in the proofs of some lemmas by Debarre, Hacon [DH07] and Pareschi [Par21], obtaining again that if $d \leq 3$ the adjoint ideal must be trivial. Thus we have a complete result for polarizations of degree 3:

Theorem B. *Let A be an abelian variety, and let L be an indecomposable polarization on A with degree $d \leq 3$. Then every prime and normal divisor in $|L|$ has rational singularities.*

We then move on to $\mathbb{Q}$-divisors. So, we take a divisor $D \in |mL|$ with m positive integer. We want to say something about singularities of $(1/m)D$, so we look at multiplier ideals $\mathcal{J}_\varepsilon$ of the $\mathbb{Q}$-divisors $\frac{1-\varepsilon}{m}D$, for positive ε very close to zero. In fact, it is possible to prove that $(1/m)D$ has log-canonical singularities if (and only if) all such multiplier ideals are trivial.

So, as before, we make the assumption that at least one of them is not trivial, and denote it simply as $\mathcal{J}$. Again, we denote by Z its cosupport, which is not empty. With the help of the standard short exact sequence

$$0 \longrightarrow \mathcal{J} \otimes L \longrightarrow L \longrightarrow L_{|Z} \longrightarrow 0$$

and some vanishing theorems, we can prove that both sheaves $L_{|Z}$ and $\mathcal{J} \otimes L$ have some nice Fourier-Mukai properties and their Euler characteristic χ is positive. If we assume the degree d to be 3, the latter fact implies that the characteristic of the two sheaves must be 1 and 2 in some order.

If we assume $\chi(L_{|Z})$ to be 1, by using Fourier-Mukai theory and properties of abelian varieties we get a contradiction with the polarization L being indecomposable.

So, the only possible way to assign characteristics is $\chi(\mathcal{J} \otimes L) = 1$ and $\chi(L_{|Z}) = 2$. In this way, after some work, we will find out that our polarization L is the sum of a theta divisor Θ and an abelian subvariety B of codimension 1, and so it is of the form

$$L = \mathcal{O}_A(\Theta + B).$$

This particular kind of polarization, however, has been used by Debarre [Deb06b] as an example of polarized abelian variety containing non log-canonical $\mathbb{Q}$-divisors. Thus, in this case our divisor $(1/m)D$ has worse than log-canonical singularities.

Then, in conclusion, we can write the following result:

Theorem C. *Let A be an abelian variety, and let L be an indecomposable polarization on A with degree $d \leq 3$. Let $D \in |mL|$ with m positive integer. Then either*

1. *L is of the form $\mathcal{O}_A(\Theta + B)$ as seen above, or*
2. *$(1/m)D$ has log-canonical singularities.*

Structure of the thesis. In Chapter 2, we start with some preliminaries on abelian varieties and their properties, basic definitions about Fourier-Mukai transforms, and by recalling useful results by Kollár and David Mumford.

Then we explain the Green and Lazarsfeld theory, principally the generic vanishing theorem (focusing on Hacon version, more useful for our purposes) and the structure theorem.

Last section is dedicated to the work of Pareschi and Popa, namely definition and properties of GV-sheaves and M-regular ones. We also present the *gv*-index of a sheaf, together with some important notions coming from the work of Shigeru Mukai.

Chapter 3 is the core of the thesis. After some preliminaries on various types of singularities from MMP, a description of ideal sheaves and statements about useful classical vanishing theorems, we go through the important theorems about singularities of divisors in abelian varieties, in most part with proofs.

Eventually, the final part is dedicated to the detailed proof of the work on polarization of degree 3 and 4.

CHAPTER 2

Generic Vanishing and Mukai Regularity on abelian varieties

In this chapter we present the work of Green and Lazarsfeld about the generic vanishing of cohomology of topologically trivial line bundles, and we talk about the subsequent theory for irregular varieties, developed by Ein, Green, Lazarsfeld, Hacon, Pareschi, Popa and others.

In particular, we will describe a type of regularity for coherent sheaves following from the generic vanishing and some applications.

2.1 Preliminaries on abelian varieties and Fourier-Mukai transforms

Here we give some preliminary facts and some notations about abelian varieties, Fourier-Mukai transforms and cohomology base change.

Throughout all the chapters, we will work on $\mathbb{C}$; thus all the results work for an algebraically closed field of characteristic zero by Lefschetz principle.

2.1.1 Abelian varieties

We start with definitions and properties of abelian varieties. For a complete introduction to this subject, see for example [Mum08], [Mil08], [BL04a], [Deb99] or [GM07].

An *abelian variety* A is a complete algebraic variety over a field k with a group law $m\colon A \times A \to A$ such that m and the inverse map $\iota\colon A \to A$ are morphisms of varieties. We denote its dimension by g.

As an abstract group, A is commutative. Moreover, using the translation morphism $t_a\colon A \to A$ (with $a \in A$), it is easy to prove that A is everywhere non-singular. Being a complete algebraic group, A is also a projective variety, so it admits ample line bundles.

A *polarization* ℓ on A is a numerical equivalence class of ample line bundles on A. We write the polarized variety as (A, ℓ), or simply as (A, L), where L is in the class ℓ. For any ample line bundle L on A, we define the *degree* $d = H^0(A, L)$; it can also be seen as the degree of the associated polarization. A polarization of degree 1 is called *principal*, and a divisor representing it is called a *theta divisor*. A polarized abelian variety (A, L) is *indecomposable* if it is not the product of non-zero polarized abelian varieties.

Lemma 2.1 (Debarre, Hacon, [DH07, Lemma 1]). *Let (A, L) be an indecomposable polarized abelian variety. If the restriction of L to an abelian subvariety B of A is a principal polarization, then either $B = A$ or $B = 0$.*

$\mathrm{Pic}^0(A)$ **and the dual variety.** Given a line bundle L on A, we have the homomorphism

$$\varphi_L\colon A \longrightarrow \mathrm{Pic}(A) \tag{2.1}$$

which sends a point $a \in A$ to the (isomorphism class of) line bundle $t_a^*L \otimes L^\vee$, where t_a is the translation map. The kernel of the map φ_L is denoted by $K(L)$; we know that L is ample if and only if $K(L)$ is a finite group, and in that case its order is d^2, where d is the degree of L.

We define $\mathrm{Pic}^0(A)$ as the sugroup of $\mathrm{Pic}(A)$ consisting of all line bundles L such that the homomorphism φ_L is identically zero, that is, the line bundles that are translation invariant. By the Theorem of the square [Mum08, Corollary 4, p. 59], the image of φ_L is contained in $\mathrm{Pic}^0(A)$. A nice fact about non-trivial line bundles in $\mathrm{Pic}^0(A)$ is that they have trivial cohomology, that is, given $L \in \mathrm{Pic}^0(A)$ with L non-trivial, the groups $H^i(A, L)$ are zero for every $i \geq 0$ [Mum08, (vii), p. 76].

A key point in the theory of $\mathrm{Pic}^0(A)$ is the following:

Theorem 2.2 (Mumford, [Mum08, Theorem 1, p. 77]). *Let L be an ample line bundle on A and let $M \in \mathrm{Pic}^0(A)$. Then there exists a point $a \in A$ such that*

$$M = t_a^*L \otimes L^\vee.$$

This proves that the homomorphism $\varphi_L\colon A \to \mathrm{Pic}^0(A)$ is surjective. Thus, $\mathrm{Pic}^0(A)$ is isomorphic to the abelian variety $A/K(L)$, also denoted by $\widehat{A}$. We call it the *dual abelian variety* of A.

Consider the product $A \times \widehat{A}$. There exists a unique line bundle $\mathcal{P}$ on this product, called *Poincaré bundle* for A, such that

1. for all $\alpha \in \widehat{A}$, the restriction $P_\alpha := \mathcal{P}_{|A\times\{\alpha\}}$ corresponds to the line bundle on $\mathrm{Pic}^0(A)$ given by α under the isomorphism $\mathrm{Pic}^0(A) \simeq \widehat{A}$;
2. $\mathcal{P}_{|\{0\}\times\widehat{A}}$ is trivial.

Moreover, the pair $(\widehat{A}, \mathcal{P})$ satisfies a universal property: for every normal variety S and every line bundle $\mathcal{Q}$ on $A \times S$ such that $\mathcal{Q}_{|A\times\{s\}} \in \mathrm{Pic}^0(A)$ for every $s \in S$ and

$\mathcal{Q}_{|\{0\}\times S}$ is trivial, there is a unique morphism $f\colon S \to \widehat{A}$ such that $\mathcal{Q} \simeq (\mathrm{id}_A \times f)^*\mathcal{P}$ (for the construction of $\mathcal{P}$, see [Mum08, § 8]).

Thus we can see $\mathcal{P}$ as a family of line bundles parametrized by $\widehat{A}$, and these line bundles are exactly the ones in $\mathrm{Pic}^0(A)$, that is, the translation invariant line bundles of A.

Subvarieties of an abelian variety. Let Y be an irreducible subvariety of A. We say that Y is *geometrically non-degenerate* if, for every abelian subvariety $B \subseteq A$, we have

$$\dim(Y + B) = \min\{\dim Y + \dim B, \dim A\}.$$

In other terms, Y is geometrically non-degenerate if, for every abelian subvariety $B \subseteq A$ with canonical surjection $\pi\colon A \to A/B$, either $\pi(Y) = A/B$ or the restriction $\pi_{|Y}$ is generically finite onto its image.

Geometrically non-degenerate subvarieties behave well with all the subvarieties of A:

Proposition 2.3 (Debarre, [Deb99, VIII, Corollaire 2.6])**.** *Let A be an abelian variety, and let Y and Z be two subvarieties of A. Then*

1. *if Y is geometrically non-degenerate, we have*
$$\dim(Y + Z) = \min\{\dim Y + \dim Z, \dim A\};$$
2. *if Y and Z are both geometrically non-degenerate, also $Y+Z$ is geometrically non-degenerate.*

Geometrically non-degenerate subvarieties are interesting because they have a property which is a distinguished feature of subvarieties of $\mathbb{P}^n$, namely that they meet if the sum of their dimensions is at least n. More precisely:

Proposition 2.4 (Debarre, [Deb99, VIII, Corollaire 2.7])**.** *A subvariety Y of an abelian variety A is geometrically non-degenerate if and only if it has non-empty intersection with any subvariety $Z \subseteq A$ such that* $\dim Y + \dim Z \geq \dim A$*.*

Finally, we define A to be a *simple abelian variety* if it has no proper abelian subvarieties except 0. Thus, it follows immediately that every subvariety of A is geoemtrically non-degenerate.

2.1.2 Fourier-Mukai transform

We now talk about the theory developed by Mukai on (complexes of) coherent sheaves on abelian varieties.

To do this, we need a bit of derived categories. We only give a glimpse of what a derived category is; for a more exhaustive presentation, see for example [Huy06], [Tho01] or [Sch13], and for the second part see also the original paper [Muk81].

Derived categories. Derived categories were introduced to have a better understanding of derived functors. The main idea behind the derived category is to keep not just the cohomology modules or sheaves, but the complexes themselves. Because the same module or sheaf can be resolved in many different ways, keeping the complex only makes sense if we declare different complexes obtained in this way to be isomorphic. This leads to the notion of a *quasi-isomorphism*, that is, a morphism $f\colon A^\bullet \to B^\bullet$ between two complexes that induces isomorphisms on cohomology $H^i(f)\colon H^i(A^\bullet) \xrightarrow{\sim} H^i(B^\bullet)$ for every i.

So, we start with an abelian category $\mathcal{A}$. We first take the category $\mathrm{K}(\mathcal{A})$ of cochain complexes of elements in $\mathcal{A}$; we then form the homotopy category $\mathrm{H}(\mathcal{A})$, which has the same objects of $\mathrm{K}(\mathcal{A})$ but the morphisms between two complexes are taken up to homotopy. Finally, we form the *derived category* $\mathrm{D}(\mathcal{A})$ of $\mathcal{A}$ by inverting all quasi-isomorphisms; this means we have made a localization of $\mathrm{H}(\mathcal{A})$ with respect to quasi-isomorphisms.

From now on, we can look at the only category we are interested in, which is the category of coherent sheaves on a variety X, which we denote by $\mathrm{Coh}(X)$; its derived category is written as $\mathrm{D}(X)$. To be precise, we only consider the *bounded* derived category $\mathrm{D}^b(X)$, which is a subcategory of $\mathrm{D}(X)$ containing only the bounded complexes.

As we said before, we have introduced $\mathrm{D}^b(X)$ to study derived functors in a more natural framework. So, consider a functor $F\colon \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ which is left-exact (or right-exact). We can encapsulate all right derived functors R^nF in one functor $\mathbf{R}F\colon\ \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ (or the left derived functors L^nF in $\mathbf{L}F$). Examples of derived divisors we will use later are $\otimes^{\mathbf{L}}$, $\mathbf{R}f_*$ and $\mathbf{L}f^*$, where $f\colon X \to Y$ is a morphism between varieties.

The Fourier-Mukai functor. We can now define an important functor in the theory of abelian varieties, the Mukai version of Fourier transfrom. Take A an abelian variety of dimension g, and consider its Poincaré bundle $\mathcal{P}$ on $A \times \widehat{A}$. We have the product

$$\begin{tikzcd} A \times \widehat{A} \arrow[r, "q"] \arrow[d, "p"] & \widehat{A}. \\ A & \end{tikzcd}$$

Given a coherent sheaf $\mathcal{F}$ on A, we can pull it back to $A \times \widehat{A}$, tensor with $\mathcal{P}$ and then push it forward to $\widehat{A}$; the resulting sheaf $q_*(p^*\mathcal{F} \otimes \mathcal{P})$ is again a coherent sheaf because q is proper morphism. Since this is not an exact functor, we should shift to the derived category. We thus define the *Fourier-Mukai transform* as the exact derived functor

$$\mathbf{R}\Phi_{\mathcal{P}}\colon\ \mathrm{D}^b(A) \to \mathrm{D}^b(\widehat{A})$$

where, for $F \in \mathrm{D}^b(A)$, we have

$$\mathbf{R}\Phi_{\mathcal{P}}(F) = \mathbf{R}q_*(\mathbf{L}p^*F \otimes^{\mathbf{L}} \mathcal{P}) = \mathbf{R}q_*(p^*F \otimes \mathcal{P})$$

which is an object in $\mathrm{D}^b(\widehat{A})$. Note that the functors p^* and $\otimes\mathcal{P}$ are both already exact.

Sometimes we will also consider the analogous derived functor

$$\mathbf{R}\Phi_{\mathcal{P}^\vee}(F)\colon\ \mathrm{D}^b(A) \to \mathrm{D}^b(\widehat{A}),$$

and since $\mathcal{P}^\vee \simeq (\mathrm{id}_A \times -\mathrm{id}_{\widehat{A}})^*\mathcal{P}$, we only have a little difference, namely $\mathbf{R}\Phi_{\mathcal{P}^\vee}(F) = (-\mathrm{id}_{\widehat{A}})^*\mathbf{R}\Phi_{\mathcal{P}}(F)$.

Obviously, we can go in the opposite direction, just interchanging the role of A and $\widehat{A}$. We thus obtain the derived functor

$$\mathbf{R}\Psi_{\mathcal{P}}\colon\ \mathrm{D}^b(\widehat{A}) \to \mathrm{D}^b(A)$$

that sends an object $G \in \mathrm{D}^b(\widehat{A})$ to $\mathbf{R}p_*(q^*G \otimes \mathcal{P}) \in \mathrm{D}^b(A)$.

We give a couple examples to see how the Fourier-Mukai transform works. First, consider $\mathcal{O}_a$ the structure sheaf of a closed point $a \in A$. Then we have $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{O}_a) = P_a$, that is, the restriction of $\mathcal{P}$ to $\{a\} \times \widehat{A}$. Then we take the structure sheaf $\mathcal{O}_A$. Using the definition we have $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{O}_A) = \mathbf{R}q_*\mathcal{P}$, and by [Mum08, § 13] we have that $R^iq_*\mathcal{P} = 0$ for all $0 < i < g$, and $R^gq_*\mathcal{P} = \mathcal{O}_0$. Thus, we have $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{O}_A) = \mathcal{O}_0[-g]$ (here $[-g]$ is the shift of the complex to the right of g places). We can easily generalize the latter for an arbitrary element in $\mathrm{Pic}^0(A)$: we obtain $\mathbf{R}\Phi_{\mathcal{P}}(P_\alpha) = \mathcal{O}_{-\alpha}[-g]$.

These examples suggest the following important result

Theorem 2.5 (Mukai equivalence theorem, [Muk81, Theorem 2.2])**.** *There are natural isomorphisms of functors*

$$\mathbf{R}\Psi_{\mathcal{P}} \circ \mathbf{R}\Phi_{\mathcal{P}} \simeq (-\mathrm{id}_A)^*[-g] \qquad \text{and} \qquad \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}\Psi_{\mathcal{P}} \simeq (-\mathrm{id}_{\widehat{A}})^*[-g].$$

In particular, both $\mathbf{R}\Phi_{\mathcal{P}}$ *and* $\mathbf{R}\Psi_{\mathcal{P}}$ *are equivalences of categories.*

In order to use Mukai equivalence theorem, we provide two very useful definitions: we say that a coherent sheaf $\mathcal{F}$ on A satisfies the *Weak Index Theorem* with index j (or $\mathrm{WIT}(j)$ for short) if

$$R^i\Phi_{\mathcal{P}}(\mathcal{F}) = 0 \qquad \text{for all } \ i \neq j.$$

When $\mathcal{F}$ satisfies $\mathrm{WIT}(j)$, we denote $R^j\Phi_{\mathcal{P}}(\mathcal{F})$ on $\widehat{A}$ with $\widehat{\mathcal{F}}$ and we call it the *Fourier-Mukai transform* of $\mathcal{F}$. Note that then $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{F}) \simeq \widehat{\mathcal{F}}[-j]$.

Also, $\mathcal{F}$ satisfies the *Index Theorem* with index j (shortly, $\mathrm{IT}(j)$) if

$$H^i(A, \mathcal{F} \otimes P) = 0 \qquad \text{for all } \ P \in \mathrm{Pic}^0(A) \ \text{ and all } \ i \neq j.$$

Since $(p^*\mathcal{F} \otimes \mathcal{P})_{|A\times\{\alpha\}} = \mathcal{F} \otimes P_\alpha$, by base change we see that IT implies WIT. Moreover, if $\mathcal{F}$ satisfies $\mathrm{IT}(j)$, then $\widehat{\mathcal{F}}$ is locally free.

Generalization to irregular varieties. The definition of Fourier-Mukai transform can be generalized for an *irregular* variety X, that is, a smooth projective variety which has a non-trivial morphism $a\colon X \to A$ to an abelian variety A (actually, we can give a much more general definition, see [PP11a, §2]).

Let $\mathcal{P}$ be a Poincaré line bundle on $A \times \widehat{A}$. We will denote $\mathcal{P}_a = (a \times \mathrm{id}_{\widehat{A}})^*\mathcal{P}$. Again, we have projections

$$\begin{tikzcd} X \times \widehat{A} \arrow[r, "q"] \arrow[d, "p"] & \widehat{A} \\ X & \end{tikzcd}$$

and so we can define an *integral transformation*

$$\mathbf{R}\Phi_{\mathcal{P}_a}\colon \ \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(\widehat{A})$$

where, for $F \in \mathrm{D}^b(X)$, we have

$$\mathbf{R}\Phi_{\mathcal{P}_a}(F) = \mathbf{R}q_*(p^*F \otimes \mathcal{P}_a)$$

which is an object in $\mathrm{D}^b(\widehat{A})$. Mukai equivalence theorem works only in the setting of abelian varieties (that is, when the morphism a is the identity), but it can be useful also in the context of irregular varieties by noting that

$$\mathbf{R}\Phi_{\mathcal{P}_a} \simeq \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}a_*.$$

Also the definitions of sheaves satisfying WIT or IT can be easily generalized to the case of an irregular variety.

Throughout the paper, we will frequently need to apply duality to sheaves and functors. Given a variety X, the *derived dual* of an object F in $\mathrm{D}^b(X)$ is

$$F^\vee := \mathbf{R}\mathcal{H}om(F, \mathcal{O}_X);$$

moreover, the *Verdier-Grothendieck dual* of F is

$$\mathbf{R}\Delta F := \mathbf{R}\mathcal{H}om(F, \omega_X).$$

Note that if X is an abelian variety, the two duals coincide, since $\omega_X = \mathcal{O}_X$.

We have the following useful result about the skew-commutativity of functors $\mathbf{R}\Phi_{\mathcal{P}_a}$ and $\mathbf{R}\Delta$:

Lemma 2.6 (Mukai, [Muk81, (3.8)]). *Let X be an irregular variety of dimension n with a morphism $a\colon X \to A$ to an abelian variety. Then*

$$\mathbf{R}\Delta \circ \mathbf{R}\Phi_{\mathcal{P}_a} \simeq (-\mathrm{id}_{\widehat{A}})^*\mathbf{R}\Phi_{\mathcal{P}_a} \circ \mathbf{R}\Delta\ [n].$$

2.1.3 Kollár's theorem, cohomology and base change

We recall some results which will be widely used in the following sections. Those facts are not restricted to abelian or irregular varieties, but they hold for more general varieties (or schemes).

Higher direct images of ω_X. First, we present a result by Kollár about higher direct images of the dualizing sheaf of a projective variety:

Theorem 2.7 (Kollár, [Kol86a, Theorem 2.1] and [Kol86b, Theorem 3.1]). *Let $f\colon X \to Y$ be a surjective morphism between projective complex varieties. If X is smooth, then:*

1. *the sheaves $R^i f_* \omega_X$ are torsion-free sheaves in Y;*
2. *$R^i f_* \omega_X = 0$ for every $i > \dim X - \dim Y$;*
3. *there is a non-canonical isomorphism*

$$\mathbf{R} f_* \omega_X \simeq \bigoplus_i R^i f_* \omega_X[-i]$$

   in the derived category $\mathrm{D}^b(Y)$;

4. *if L is an ample line bundle on Y, then $H^j(Y, R^i f_* \omega_X \otimes L) = 0$ for all $j > 0$.*

This statement shows that the sheaves $R^i f_* \omega_X$ actually behave much like ω_X itself: indeed, ω_X is a torsion-free sheaf, and for an ample line bundle L on X, $H^j(X, \omega_X \otimes L) = 0$ for all $j > 0$ (see for example [Har77, § III.7]).

From the theorem, we have this useful result:

Corollary 2.8. *Under the same assumption of the theorem, let $k = \dim X - \dim Y$. Then*

$$H^j(X, \omega_X) \simeq \bigoplus_{i=0}^{k} H^{j-i}(Y, R^i f_* \omega_X).$$

Theorem 2.7 and Corollary 2.8 will be used later in the following version, which is also a particular case of a more general formulation (see [Kol86b, § 3]):

Proposition 2.9. *In the hypoteses and notations of Theorem 2.7 and Corollary 2.8, we can replace ω_X with $\omega_X \otimes P$, where $P \in \mathrm{Pic}^0(X)_{\mathrm{tors}}$ is a torsion point.*

Base change. Other important results, concerning cohomology groups and base change, were proved by Mumford (see [Mum08, § 5]). Here we state some corollaries:

Proposition 2.10 (Mumford, [Mum08, § 5, Corollary 2]). *Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes, with Y reduced and connected, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Then for all non-negative integers i, the following are equivalent:*

1. *$y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is a constant function;*
2. *$R^i f_* \mathcal{F}$ is a locally free sheaf $\mathcal{E}$ on Y, and for all $y \in Y$, the natural map*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} k(y) \longrightarrow H^i(X_y, \mathcal{F}_y)$$

   is an isomorphism, where $k(y)$ denotes the resiude field at y.

Moreover, if these conditions are fulfilled, we have that the natural map

$$R^{i-1}f_*\mathcal{F}\otimes_{\mathcal{O}_Y}k(y)\longrightarrow H^{i-1}(X_y,\mathcal{F}_y)$$

is an isomoprhism for all $y\in Y$.

Proposition 2.11 (Mumford, [Mum08, § 5, Corollary 3])**.** *Under the same assumpitons of the proposition above (except that* Y *is not needed to be reduced), assume that, for some* i, $H^i(X_y,\mathcal{F}_y)=0$ *for all* $y\in Y$. *Then the natural map*

$$R^{i-1}f_*\mathcal{F}\otimes_{\mathcal{O}_Y}k(y)\longrightarrow H^{i-1}(X_y,\mathcal{F}_y)$$

is an isomorphism for all $y\in Y$.

In the setting of integral transforms, let $\mathcal{G}$ be a coherent sheaf on an irregular variety X with a morphism $a\colon X\to A$ to an abelian variety. Given a point $\alpha\in\widehat{A}$, and its corresponding sheaf $P_\alpha\in\operatorname{Pic}^0(A)$, we say that the sheaf $R^i\Phi_{\mathcal{P}_a}(\mathcal{G})$ has *the base change property at* α if the natural map

$$R^i\Phi_{\mathcal{P}_a}(\mathcal{G})\otimes\mathbb{C}(\alpha)\longrightarrow H^i(X,\mathcal{G}\otimes a^*P_\alpha)$$

is an isomorphism.

From Propositions 2.10 and 2.11 we immediately have the following result:

Corollary 2.12 (Base change property)**.** *Let* X *be an irregular variety with a morphism* $a\colon X\to A$ *to an abelian variety, let* $\mathcal{G}$ *be a coherent sheaf on* X *and let* $\alpha\in\widehat{A}$, *corresponding to the sheaf* $P_\alpha\in\operatorname{Pic}^0(A)$. *If* $h^{i+1}(X,\mathcal{G}\otimes a^*P_\alpha)$ *is constant in a neighbourhood of* α, *then both* $R^{i+1}\Phi_{\mathcal{P}_a}(\mathcal{G})$ *and* $R^i\Phi_{\mathcal{P}_a}(\mathcal{G})$ *have the base change property at* α.

In particular, $R^i\Phi_{\mathcal{P}_a}(\mathcal{G})$ *and* $R^i\Phi_{\mathcal{P}_a}(\mathbf{R}\Delta\mathcal{G})$ *vanish for* $i>\dim X$, *and both* $R^{\dim X}\Phi_{\mathcal{P}_a}(\mathcal{G})$ *and* $R^{\dim X}\Phi_{\mathcal{P}_a}(\mathbf{R}\Delta\mathcal{G})$ *have the base change property at every point of* $\widehat{A}$.

2.2 The Generic Vanishing Theorem

Vanishing theorems play an important role in modern algebraic geometry. In fact, many questions can be phrased in terms of coherent sheaves and their cohomology; typical cases are lifting of sections or vanishing of obstructions.

The most famous one is due to Kodaira, and it states that, for an ample line bundle L on a smooth projective variety X, $H^i(X,\omega_X\otimes L)=0$ for $i>0$.

It turns out that we can relax the conditions on L to be nef and big. In that case, we have the Kawamata-Viehweg vanishing theorem.

All these vanishing theorems depend on the fact that the first Chern class of the line bundle L is in some way positive. The question that motivated the generic vanishing theorem is instead what happens if we consider line bundles with first Chern class $c_1(L)$ is zero. These line bundles, called *topologically trivial*, are parametrized by the points of $\operatorname{Pic}^0(X)$, which is the connected component of $\operatorname{Pic}(X)$

containing the trivial bundle; we can also define $\mathrm{Pic}^0(X)$ as $H^1(X, \mathcal{O}_X)/H_1(X, \mathbb{Z})$. Note that $\mathrm{Pic}^0(X)$ is an abelian variety.

We also define the *Albanese variety* $\mathrm{Alb}(X)$ of X to be the dual variety of $\mathrm{Pic}^0(X)$, or equivalently $\mathrm{Alb}(X) := H^0(X, \Omega^1_X)^\vee/H_1(X, \mathbb{Z})$. After choosing a base point $x_0 \in X$, we have the morphism

$$\begin{aligned} \mathrm{alb}\colon X &\longrightarrow \mathrm{Alb}(X) \\ x &\longmapsto (\omega \mapsto \int_{x_0}^{x} \omega) \end{aligned}$$

with the property that $\mathrm{alb}^*\colon \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$ is an isomorphism (see [GH78, p. 331]).

We can define a Poincaré bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^0(X)$ as the pull-back of a Poincaré bundle on $\mathrm{Alb}(X) \times \mathrm{Pic}^0(X)$ (see the construction in Section 2.1.2).

As we may expect, line bundles in $\mathrm{Pic}^0(X)$ do not behave like the ones with positive first Chern class. For example, given $P \in \mathrm{Pic}^0(X)$ such that $P \not\simeq \mathcal{O}_X$, we can show that $H^0(X, P) = 0$. Also, by Serre duality, we have $h^{\dim X}(X, P) = h^0(X, \omega_X \otimes P^\vee)$, and in many cases the cohomology group on the right is non-zero for every $P \in \mathrm{Pic}^0(X)$.

Thus, in general, we cannot expect to have a good vanishing theorem that works for all $P \in \mathrm{Pic}^0(X)$ because there can be line bundles whose cohomology does not vanish for geometric reasons. Also, typically the group $H^{\dim X}(X, P)$ is non-zero.

For these reasons, everytime we are dealing with topologically trivial line bundles, we focus on $\dim \mathrm{alb}(X)$, the *Albanese dimension* of X, rather than on the dimension $\dim(X)$ itself.

Also, we need to define the so-called *cohomological support loci*

$$S^i(X) = \{P \in \mathrm{Pic}^0(X) \mid H^i(X, P) \neq 0\}$$

which are closed subsets of $\mathrm{Pic}^0(X)$ for each i.

Now we can state the following result:

Theorem 2.13 (Green, Lazarsfeld, [GL87, Theorem 1])**.** *Let X be a smooth projective variety. Then*

$$\operatorname{codim}_{\mathrm{Pic}^0(X)} S^i(X) \geq \dim \mathrm{alb}(X) - i.$$

As a consequence we have a vanishing theorem for topologically trivial line bundles

Corollary 2.14 (Generic vanishing theorem)**.** *Let X be a smooth projective variety. For a general line bundle $P \in \mathrm{Pic}^0(X)$, we have*

$$H^i(X, P) = 0 \qquad \text{for every } 0 \leq i < \dim \mathrm{alb}(X).$$

Here general means on the complement of the proper closed subset $S^i(X)$.

In the original paper [GL87], the authors proved Theorem 2.13 for compact Kähler manifolds, relying heavily on Hodge theory. Indeed, the idea of the proof is to look at the deformation theory of cohomology groups of the form $H^i(X, P)$. More precisely, the aim is to understand how $H^i(X, P)$ changes when we move the line bundle P; this question is strictly related to the infinitesimal properties of the loci $S^i_m(X) = \{P \in \mathrm{Pic}^0(X) \mid h^i(X, P) \geq m\}$ (note that $S^i(X) = S^i_1(X)$).

There is another more recent proof of the generic vanishing theorem, discovered by Hacon, working for smooth projective varieties. This proof uses tools from algebraic geometry, namely derived categories and vanishing theorems, and gives a new perspective of the original result.

Let X be an irregular variety, that is, a smooth projective variety of dimension n which has a morphism $a\colon X \to A$ to a g-dimensional abelian variety. Given a coherent sheaf $\mathcal{F}$ on X, we can define the *cohomological support loci* in a more general way as

$$V^i_a(X, \mathcal{F}) = V^i_a(\mathcal{F}) = \{P_\alpha \in \mathrm{Pic}^0(A) \mid H^i(X, \mathcal{F} \otimes a^* P_\alpha) \neq 0\},$$

where P_α is the line bundle in $\mathrm{Pic}^0(A)$ represented by $\alpha \in \widehat{A}$. When the morphism a is just the Albanese map, we will drop the subscript, and simply write $V^i(\mathcal{F})$.

By Serre duality, $H^i(X, \omega_X \otimes P)$ corresponds to $H^{n-i}(X, P^\vee)$; thus we can reformulate the statement of Theorem 2.13 using the new support loci as follows: for any n-dimensional smooth projective variety X, we have

$$\mathrm{codim}_{\mathrm{Pic}^0(X)} V^i(\omega_X) \geq i - (n - \dim \mathrm{alb}(X))$$

for every $i \geq 0$, where here $V^i(\omega_X)$ stands for $V^i_{\mathrm{alb}}(\omega_X)$.

As written before, we use the notation $\mathbf{R}\Delta\mathcal{F}$ for the Verdier-Grothendieck dual $\mathbf{R}\mathcal{H}om(\mathcal{F}, \omega_X)$; also, we denote by $\mathcal{F}^\vee$ the derived dual $\mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$.

We have the following result:

Theorem 2.15 (Hacon, [Hac04, Theorem 1.2])**.** *Let A be an abelian variety of dimension g, and let $\mathcal{F}$ be a coherent sheaf on A. Then the following are equivalent:*

1. *the Fourier-Mukai transform $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{F})$ satisfies, for all $i > 0$,*

$$\mathrm{codim}\,\mathrm{supp}\, R^i\Phi_{\mathcal{P}}(\mathcal{F}) \geq i;$$

2. *for any sufficiently ample line bundle L on $\widehat{A}$ and for all $i > 0$,*

$$H^i(A, \mathcal{F} \otimes \widehat{L}^\vee) = 0,$$

   where $\widehat{L} = \mathbf{R}\Psi_{\mathcal{P}}(L) = R^0\Psi_{\mathcal{P}}(L)$;

3. *the Fourier-Mukai transform of the dual sheaf $\mathcal{F}^\vee$ is a complex concentrated in the index g, that is*

$$\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{F}^\vee) \simeq R^g\Phi_{\mathcal{P}}(\mathcal{F}^\vee)[-g].$$

In [Hac04] we can also find a result about the vanishing of higher direct images of dualizing sheaves:

Proposition 2.16 (Hacon, [Hac04, Corollary 4.2])**.** *Let $f\colon X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then every irreducible component of $V^j(R^i f_*\omega_X)$ has codimension at least j in $\widehat{A}$.*

The original Green and Lazarsfeld Theorem 2.13 (for projective varieties) follows directly from Proposition 2.16 together with Kollár Theorem 2.7. Indeed, given the Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$, if we define $k = \dim X - \dim \mathrm{alb}(X)$, we have

$$\mathbf{R}\,\mathrm{alb}_*\,\omega_X = \bigoplus_{i=0}^{k} R^i\,\mathrm{alb}_*\,\omega_X[-i]$$

and using the projection formula we get

$$H^j(X, \omega_X \otimes \mathrm{alb}^* P_\alpha) = \bigoplus_{i=0}^{k} H^{j-i}(\mathrm{Alb}(X), R^i\,\mathrm{alb}_*\,\omega_X \otimes P_\alpha),$$

where $P_\alpha \in \mathrm{Pic}^0(X)$. Obviously, the left-hand side is non-zero if and only if at least one summand of the right-hand side is non-zero, and at the level of cohomology support loci this means that

$$V^j(X, \omega_X) = \bigcup_{i=0}^{k} V^{j-i}(\mathrm{Alb}(X), R^i\,\mathrm{alb}_*\,\omega_X).$$

To conclude, we use Proposition 2.16 and we have that

$$\mathrm{codim}\,V^{j-i}(\mathrm{Alb}(X), R^i\,\mathrm{alb}_*\,\omega_X) \geq j - i \geq j - k$$

for all $0 \leq i \leq k$, and thus

$$\mathrm{codim}_{\mathrm{Pic}^0(X)}\,V^j(\omega_X) \geq j - k = j - (\dim X - \dim \mathrm{alb}(X)),$$

obtaining then the generic vanishing theorem.

We can also use Proposition 2.16 in a variant form

Proposition 2.17 (Hacon, Pardini, [HP05, Theorem 2.2(a)])**.** *Let $f\colon X \to Y$ be a morphism between smooth proective varieties, and let $a\colon Y \to A$ be a morphism to an abelian variety A, and let $P_0 \in \mathrm{Pic}^0(X)_{\mathrm{tors}}$ be a torsion point. Then every irreducible component of $V^j_a(R^i f_*(\omega_X \otimes P_0))$ has codimension at least j in $\widehat{A}$.*

In practice, Proposition 2.16 and then also Theorem 2.13 still hold if we replace ω_X with $\omega_X \otimes P_0$, where P_0 is a torsion point in $\mathrm{Pic}^0(X)$.

2.2.1 The structure theorem

A very interesting topic is the geometric structure of cohomological support loci $V^i(\omega_X)$, where X is a smooth projective variety. Inspired by a result of Beauville about the locus $V^1(\omega_X)$ [Bea92, Theorem 1], Green and Lazarsfeld proved in [GL91] that any irreducible component of $V^i(\omega_X)$ is a translate of an abelian subvariety (or a subtorus if X is a compact Kähler manifold).

Around the same time, Beauville and Catanese [Cat83] conjectured that such irreducible components should be translates always by points of finite order. This idea has been clarified and proved some years later by Simpson in his paper [Sim93].

Eventually, we can state a quite complete structure theorem for cohomological support loci:

Theorem 2.18 (Structure Theorem; Green, Lazarsfeld, [GL91, Theorem 0.1]; Simpson, [Sim93, Theorem 4.2])**.** *Let X be a smooth projective variety, and let W be an irreducible component of $V^i(\omega_X)$ for some $i \geq 0$. Then*

1. *there exist an abelian subvariety B of $\mathrm{Pic}^0(X)$ and a torsion point $P_0 \in \mathrm{Pic}^0(X)$ such that $W = B + P_0$;*
2. *moreover, if $\pi\colon \mathrm{Alb}(X) \to \mathrm{Pic}^0(B)$ is the dual map of the embedding $B \hookrightarrow \mathrm{Pic}^0(X)$ and $f\colon X \to \mathrm{Pic}^0(B)$ is the composition map $f = \pi \circ \mathrm{alb}$, then*

$$\dim X - \dim f(X) \geq i.$$

As for the generic vanishing, we can use a variant form of Theorem 2.18:

Proposition 2.19 (Hacon, Pardini, [HP05, Theorem 2.2(b)])**.** *Let $f\colon X \to Y$ be a morphism between smooth proective varieties, and let $a\colon Y \to A$ be a morphism to an abelian variety A, and let $P_0 \in \mathrm{Pic}^0(X)_{\mathrm{tors}}$ be a torsion point. Then, for any $i, j \geq 0$, every irreducible component of $V^j_a(R^i f_*(\omega_X \otimes P_0))$ is a translate of an abelian subvariety by a torsion point.*

2.2.2 The maximal Albanese dimension case

The generic vanishing theorem is particularly useful when the variety X has *maximal Albanese dimension*, that is, when $\dim X = \dim \mathrm{alb}(X)$ (or equivalently, when the Albanese map of X is generically finite).

Indeed, if X has maximal Albanese dimension, then $H^i(X, P) = 0$ for $P \in \mathrm{Pic}^0(X)$ general and for any $i < \dim X$; in terms of cohomological support loci, we have that $\mathrm{codim}_{\mathrm{Pic}^0(X)} V^i(\omega_X) \geq i$. Using this fact, we have the following numerical consequence:

Proposition 2.20 (Green, Lazarsfeld, [GL87])**.** *If X is a smooth projective variety with maximal Albanese dimension, then the Euler characteristic $\chi(X, \omega_X)$ is greater or equal to zero.*

Proof. Let $n = \dim X$. By the Riemann-Roch theorem, the Euler characteristic of $\omega_X \otimes P$ is independent from $P \in \operatorname{Pic}^0(X)$. So we have

$$\chi(X, \omega_X) = \chi(X, \omega_X \otimes P) = \sum_{i=0}^{n} (-1)^n H^i(X, \omega_X \otimes P)$$

and, if P is sufficiently general, all the terms with $i > 0$ in the sum vanish. Thus, we have $\chi(X, \omega_X) = H^0(X, \omega_X \otimes P) \geq 0$. □

Next, we present a result of Ein and Lazarsfeld in the special case where the Euler characteristic of ω_X is zero:

Theorem 2.21 (Ein, Lazarsfeld, [EL97, Theorem 3]). *Let X be a smooth projective variety of maximal Albanese dimension, and assume that $\chi(X, \omega_X) = 0$. Then the image of the Albanese map of X is fibred by translates of abelian subvarieties of* $\operatorname{Alb}(X)$.

This result turns out to be very useful in the proof of some statements about singularities, as we will see in later sections.

2.3 GV-sheaves and M-regularity

The three equivalent conditions of Theorem 2.15 describe a particular class of coherent sheaves on an abelian variety A. More precisely, we say that a coherent sheaf $\mathcal{F}$ on A *satisfies the Generic Vanishing*, or is a *GV-sheaf*, if its cohomological support loci satisfiy the inequalities

$$\operatorname{codim} V^i(A, \mathcal{F}) \geq i$$

for every $i \geq 0$.

Most results connected with the generic vanishing theorem (with the remarkable exception of the structure theorem) are still true for GV-sheaves; this facts makes such sheaves very useful in several applications.

We can give a much more general definition of GV-sheaves. In their paper [PP11a], Pareschi and Popa refined the argument used by Hacon in the proof of Theorem 2.15 so that an analogous proof can work in greater generality, for practically any integral derived transform (not only the Fourier-Mukai one in the settings of abelian varieties).

We thus state the following result (for the very general result, see [PP11a], Theorem 3.7):

Theorem 2.22 (Pareschi, Popa, [PP11a, Theorem A] and [PP09, Theorem 2.2]). *Let X be a smooth projective variety with a morphism $a \colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a coherent sheaf on X. Given a positive integer k, the following are equivalent:*

1. $\operatorname{codim}_{\operatorname{Pic}^0(A)} V^i_a(\mathcal{F}) \geq i - k$ *for all* $i \geq 0$*;*

2. *$R^i\Phi_{\mathcal{P}_a}(\mathbf{R}\Delta\mathcal{F}) = 0$ for all $0 \leq i < \dim X - k$,*

where $\mathcal{P}_a$ is the pullback of the Poincaré bundle $\mathcal{P} \in A \times \operatorname{Pic}^0(A)$ by the morphism $a \times \mathrm{id}_{\operatorname{Pic}^0(A)} \colon X \times \operatorname{Pic}^0(A) \to A \times \operatorname{Pic}^0(A)$.

When one of the equivalent condition of Theorem 2.22 holds, the sheaf $\mathcal{F}$ is called a *GV_{-k}-sheaf* with respect to the morphism a. We will omit the reference to the morphism a when unnecessary.

Clearly, when $k = 0$, we recover the notion of GV-sheaves. In this case, we can read the second condition of Theorem 2.22 as $\mathbf{R}\Delta\mathcal{F}$ satisfies WIT with index $\dim X$, according to Mukai theory. Moreover, following the notation of a previous section, we will write $\widehat{\mathbf{R}\Delta\mathcal{F}}$ instead of $R^{\dim X}\Phi_{\mathcal{P}_a}(\mathbf{R}\Delta\mathcal{F})$.

These sheaves has some basic properties:

Proposition 2.23 (Pareschi, [Par12, Proposition 1.6])**.** *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a GV-sheaf on X with respect to a. Then*

1. *the rank of $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is equal to $\chi(\mathcal{F})$;*
2. *we have*

$$\mathcal{E}xt^i_{\mathcal{O}_{\operatorname{Pic}^0(A)}}(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\operatorname{Pic}^0(A)}) \simeq (-\mathrm{id}_{\operatorname{Pic}^0(A)})^* R^i\Phi_{\mathcal{P}_a}(\mathcal{F}).$$

2.3.1 Properties of GV-sheaves

We will look at properties of GV-sheaves. A first one was suggested by Green (see [EL97, Lemma 1.8]), and it holds for a variety X with maximal Albanese dimension: the cohomology support loci of ω_X satisfy:

$$\operatorname{Pic}^0(X) \supseteq V^0(\omega_X) \supseteq V^1(\omega_X) \supseteq \cdots \supseteq V^{\dim X}(\omega_X) = \{\mathcal{O}_X\}.$$

The exact same result is true for an arbitrary GV-sheaf:

Proposition 2.24 (Hacon, [Hac04, Corollary 3.2])**.** *Let $\mathcal{F}$ be a GV-sheaf on a smooth projective variety X of dimension n with respect to a morphism $a\colon X \to A$ to an abelian variety A. Then*

$$\operatorname{Pic}^0(A) \supseteq V^0_a(\mathcal{F}) \supseteq V^1_a(\mathcal{F}) \supseteq \cdots \supseteq V^{\dim X}_a(\mathcal{F}).$$

Proof. Fix $i > 0$ and let $P \in V^i_a(\mathcal{F})$. By Serre duality, we have $V^i_a(\mathcal{F}) = -V^{n-i}_a(\mathbf{R}\Delta\mathcal{F})$. Since $R^{n-i}\Phi_{\mathcal{P}_a}(\mathbf{R}\Delta\mathcal{F}) = 0$, it follows by the base change property (Corollary 2.12) that

$$P \in -V^{n-i+1}_a(\mathbf{R}\Delta\mathcal{F}) = V^{i-1}_a(\mathcal{F}),$$

where the equality follows again from Serre duality. □

The importance of GV-sheaves comes from the fact that some features of the cohomology groups $H^i(X, \mathcal{F} \otimes a^*P)$ and of the cohomological support loci $V^i_a(\mathcal{F})$ can be studied looking at local and sheaf-theoretic properties of the Fourier-Mukai transform $\widehat{\mathbf{R}\Delta\mathcal{F}}$. An example is provided by the following proposition, which is very useful in practice:

Proposition 2.25 (Pareschi, Popa, [PP11a, Proposition 3.15])**.** *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a GV-sheaf on X (with respect to a). If Z is an irreducible component of $V^0_a(\mathcal{F})$ of codimension k, then Z is actually an irreducible component of $V^k_a(\mathcal{F})$.*

Proof. We know that $\widehat{\mathbf{R}\Delta\mathcal{F}}$ has the base change property (see Corollary 2.12), and so it is supported at $V^{\dim X}_a(\mathbf{R}\Delta\mathcal{F}) = -V^0_a(\mathcal{F})$. Thus, $-Z$ is a component of the support of $\widehat{\mathbf{R}\Delta\mathcal{F}}$. Let now P be a general point of Z. By Proposition 2.23 (2), we have

$$R^i\Phi_{\mathcal{P}_a}(\mathcal{F}) = (-\operatorname{id}_{\operatorname{Pic}^0(A)})^* \mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\operatorname{Pic}^0(A)}).$$

But $\mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\operatorname{Pic}^0(A)})$ is zero if $\dim Z < g - i$ and non-zero if $\dim Z = g - i$ (see for example [Har77, § III.6 and 7]); so it follows that, in some neighbourhood of P, $R^i\Phi_{\mathcal{P}_a}(\mathcal{F})$ vanishes for $i < k$ and it is supported at Z for $i = k$. Hence, using again the base change property, we obtain that Z is contained in $V^k_a(\mathcal{F})$, and in fact it is a component since $\operatorname{codim} V^k_a(\mathcal{F}) \geq k$. $\square$

From this proposition it follows that, if $\mathcal{F}$ is a GV-sheaf, then either $V^0_a(\mathcal{F}) = \operatorname{Pic}^0(A)$ or there is a positive index i such that $\operatorname{codim} V^i_a(\mathcal{F}) = i$. We can rephrase this fact in the following way:

Corollary 2.26. *Let $\mathcal{F}$ be a non-zero GV-sheaf on X. Then* $\operatorname{codim} V^i_a(\mathcal{F}) = i$ *for at least one value of $i \geq 0$.*

Thus, the inequality $\operatorname{codim} V^i_a(\mathcal{F}) \geq i$ characterizing GV-sheaves cannot be strict for every $i \geq 0$.

What happens in Proposition 2.25 is a particular instance of a more general picture. Indeed, the fact that $V^0_a(\mathcal{F})$ is a proper subvariety of $\operatorname{Pic}^0(A)$ is equivalent to $\widehat{\mathbf{R}\Delta\mathcal{F}}$ having its generic rank zero, by Serre duality. This means that $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is a torsion sheaf on $\operatorname{Pic}^0(A)$. If this is true, then by Proposition 2.25, there is a positive index i such that $\operatorname{codim} V^i_a(\mathcal{F}) = i$.

It is possible to provide a converse to this statement, and so we obtain the following result:

Theorem 2.27 (Pareschi, Popa, [PP11b, Proposition 2.8] and [PP09, Corollary 3.2])**.** *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a GV-sheaf on X (with respect to a). Then the following are equivalent:*

1. *there is a positive index i such that* $\operatorname{codim} V^i_a(\mathcal{F}) = i$*;*

2. *the sheaf $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is a torsion sheaf on* $\operatorname{Pic}^0(A)$*.*

This characterization will be useful later, when we describe properties of more regular sheaves.

Using this new terminology, we can restate previous results we have seen. For example, here is the generic vanishing (Theorem 2.13) of Green and Lazarsfeld:

Theorem 2.28 (Generic vanishing theorem - GV-sheaves version)**.** *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, and let $k = \dim X - \dim a(X)$. Then ω_X is a GV$_{-k}$-sheaf with respect to a.*

In particular, if a is generically finite onto its image, then ω_X is a GV-sheaf.

Moreover, from Hacon's theorem (see Proposition 2.16), we have that also the higher direct image sheaves $R^i a_* \omega_X$ are GV-sheaves on A for all $i \geq 0$.

Another very clever example for the use of GV-sheaves is Theorem 2.21 for varieties of maximal Albanese dimension. Using the theory of GV-sheaves, we can give a proof which is partially different from the original one by Ein and Lazarsfeld in [EL97] (for references, see [Par12, Theorem 3.3]):

Alternative proof of Theorem 2.21. Since X has maximal Albanese dimension, the morphism alb is generically finite onto its image, and thus by Theorem 2.28, ω_X is a GV-sheaf. The condition $\chi(\omega_X) = 0$ says that $V^0(\omega_X)$ is a proper subvariety of $\mathrm{Pic}^0(X)$. By Proposition 2.25, there exists, for a positive index k, a locus $V^k(\omega_X)$ that has an irreducible component Z of codimension k.

Now, we can apply the structure theorem (Theorem 2.18): Z is a translate of an abelian subvariety $T \subset \mathrm{Pic}^0(X)$, and if we denote $A = \mathrm{Pic}^0(T)$, we have a diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{alb}"] \arrow[dr, bend right, "f"'] & \mathrm{Alb}(X) \arrow[d, "\pi"] \\
 & A
\end{tikzcd}$$

where $\dim f(X) \leq \dim X - k$. Moreover, π is surjective and its fibres are (finite unions of) translates of k-dimensional abelian subvarieties of $\mathrm{Alb}(X)$.

On the other hand, fibres of $\pi\colon\ \mathrm{alb}(X) \to f(X)$ have dimension at least

$$\dim \mathrm{alb}(X) - \dim f(X) = \dim X - \dim f(X) \geq k,$$

where the equality follows from the maximal Albanese dimension of X. Thus, we have $\dim X - \dim f(X) = k$, and this can only happen if $\mathrm{alb}(X)$ is a union of fibres of π, that is, a union of translates of abelian subvarieties of $\mathrm{Alb}(X)$. □

2.3.2 Mukai regularity

We now introduce a regularity condition on sheaves which is a stronger generic vanishing condition, in close relation to GV-sheaves. We call this condition *Mukai regularity*, or simply *M-regularity.*

This notion was introduced in a series of papers by Pareschi and Popa about regularity on abelian varieties. At first, it was used (in a particular form called *Theta regularity*) to give a result on abelian varieties analogous to Castelnuovo-Mumford regularity on projective spaces (see [PP03]).

M-regularity then turns out to have a large number of applications, from properties of linear series on abelian varieties, to study of special classes of vector bundles, to global generation properties.

We give a general definition, shown as two equivalent statements:

Proposition 2.29 (Pareschi, Popa, [PP11b, Proposition 2.7]). *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a coherent sheaf on X. Then the following are equivalent*

1. $\operatorname{codim} V^i_a(\mathcal{F}) > i$ *for all* $i > 0$;
2. $\operatorname{codim}\ \operatorname{supp}(R^i\Phi_{\mathcal{P}_a}(\mathcal{F})) > i$ *for all* $i > 0$.

A sheaf satisfying one condition is called an *M-regular sheaf.*

We note that this definition is very close to the one for GV-sheaves in Theorem 2.22 (1) (case $k = 0$) or in Theorem 2.15 (1), apart from inequalities being strict. Thus, every M-regular sheaf is also a GV-sheaf.

GV-sheaves which are not M-regular are those whose support loci have dimension as big as possible. By next result, this fact is equivalent to admitting torsion in the Fourier-Mukai transform of the dual.

Proposition 2.30 (Pareschi, Popa, [PP11b, Proposition 2.8]). *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a GV-sheaf on X. Then the following are equivalent*

1. $\mathcal{F}$ *is an M-regular sheaf;*
2. $\widehat{\mathbf{R}\Delta\mathcal{F}}$ *is a torsion-free sheaf.*

Proof. Since $\mathcal{F}$ is a GV-sheaf, we have $R^i\Phi_{\mathcal{P}_a}(\mathcal{F}) = \mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\widehat{A}})$ by Lemma 2.6 (see [PP11a, Remark 3.13]). Thus, $\mathcal{F}$ is M-regular if and only if for each $i > 0$

$$\operatorname{codim}\ \operatorname{supp}(\mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\widehat{A}})) > i.$$

To end the proof, we use the following algebraic fact (for the proof, see for example [PP11b, Lemma 2.9]): in our setting, a coherent sheaf $\mathcal{G}$ is torsion free if and only if $\operatorname{codim}\operatorname{supp}(\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\widehat{A}})) > i$ for all $i > 0$. $\square$

To have a first look at M-regular sheaves, we provide the following basic example. Let A be an abelian variety and let L be a line bundle on A. If L is ample, then by Kodaira vanishing we have $H^i(A, L \otimes P) = 0$ for all $i > 0$ and all $P \in \operatorname{Pic}^0(A)$. This means, in Mukai terminology, that L satisfies IT with index 0. Also, in this case $V^i(L) = \emptyset$ for all $i > 0$, and so L is also M-regular.

On the other hand, if L is M-regular, then by definition we have that, for a general $P \in \operatorname{Pic}^0(A)$, $H^i(A, L \otimes P) = 0$ for $i > 0$ but $H^0(A, L \otimes P) \neq 0$. By Mumford [Mum08, § 16], this means that $L \otimes P$ is a non-degenerate line bundle of index 0, which is equivalent of $L \otimes P$ being ample. In turn, this fact proves the ampleness of L itself.

Therefore, we have the following result:

Proposition 2.31. *Let A be an abelian variety, and let L a line bundle on A. Then the following are equivalent*

1. *L is ample;*
2. *L satisfies IT with index 0;*
3. *L is M-regular.*

It turns out that this statement can be generalized. In order to do so, we introduce a new notion. Let X be an irregular variety with morphism $a\colon X \to A$; a coherent sheaf $\mathcal{F}$ on X is *continuously globally generated* if for any non-empty set $U \subseteq \operatorname{Pic}^0(A)$ the sum of evaluation maps

$$\bigoplus_{P \in U} H^0(X, \mathcal{F} \otimes a^*P) \otimes a^*P^\vee \longrightarrow \mathcal{F}$$

is surjective. A key point for introducing this definition is the following:

Proposition 2.32 (Pareschi, Popa, [PP03, Proposition 2.13])**.** *Any M-regular sheaf on an irregular variety X is continuously globally generated.*

This result, together with a theorem of Debarre (see [Deb06a]) which says that any continuously globally generated sheaf on an abelian variety is ample, leads to the statement:

Theorem 2.33 (Debarre, [Deb06a, [Corollary 3.2]; Pareschi, Popa, [PP11b, Theorem 4.1])**.** *Let A be an abelian variety, and let $\mathcal{F}$ be a coherent sheaf on A. Then*

1. *if $\mathcal{F}$ is an M-regular sheaf, then it is ample.*
2. *if $\mathcal{F}$ is a GV-sheaf, then it is nef.*

The proof of the second part descends form the first one, looking at a GV-sheaf as "limit" of M-regular sheaves, as a nef bundle is the limit of ample bundles.

2.3.3 The generic vanishing index

Let X be an irregular variety with a morphism $a\colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a coherent sheaf on X. We define the *generic vanishing index* of $\mathcal{F}$, or shortly the *gv-index* of $\mathcal{F}$, as the quantity

$$gv_a(\mathcal{F}) := \min_{i>0} \{\operatorname{codim}_{\operatorname{Pic}^0(A)} V^i_a(\mathcal{F}) - i\}.$$

When the morphism a is the Albanese map, we just write $gv(\mathcal{F})$. If $V^i_a(\mathcal{F})$ is empty for all $i > 0$, we set $gv_a(\mathcal{F}) = \infty$. This index measures the size of cohomological support loci for positive indices, and reads a local algebraic property of the transform $\mathbf{R}\Phi_{\mathcal{P}_a}(\mathbf{R}\Delta\mathcal{F})$, namely the property of being a syzygy sheaf.

It is possible to define also a local version of the generic vanishing index: given any $\alpha \in \operatorname{Pic}^0(A)$, the *local generic vanishing index* of $\mathcal{F}$ at α is

$$gv_{a,\alpha}(\mathcal{F}) := \min_{i>0}\{\operatorname{codim}_\alpha V^i_a(\mathcal{F}) - i\},$$

where $\operatorname{codim}_\alpha V^i_a(\mathcal{F})$ denotes the codimension of the connected component of $V^i_a(\mathcal{F})$ which contains α. As before, we simply write $gv_\alpha(\mathcal{F})$ when the morphism a is the Albanese map.

As we can figure out by looking at the definitions, we have

$$gv_a(\mathcal{F}) = \inf_{\alpha\in\operatorname{Pic}^0(A)}\{gv_{a,\alpha}(\mathcal{F})\}.$$

In general, a coherent sheaf $\mathcal{F}$ on a variety X is called a *k-th syzygy sheaf* if locally there exists an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_k \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{E}_j$ are locally free sheaves for all j (for details, see for example [EG85] or [OSS11]). This sequence brings interesting properties: for example, a 1st syzygy sheaf is equivalent to be torsion-free, and a 2nd syzygy sheaf is equivalent to be reflexive. Any coherent sheaf is by definition a 0th syzygy sheaf, while at the opposite side a locally free sheaf is declared to be an ∞ syzygy sheaf.

An important property for our applications is the following:

Theorem 2.34 (Evans, Griffith, [EG81, Corollary 1.7]). *Let X be a smooth variety, and let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\mathcal{F}$ is a k-th syzygy sheaf and it is not locally free. Then* $\operatorname{rk}(\mathcal{F}) \geq k$.

An M-regular sheaf has gv-index 1 by definition, and in fact it is a 1st syzygy sheaf, that is, a torsion-free sheaf (see Proposition 2.30). For gv-indices greater or equal than 2, we have a characterization of a higher degree of regularity of sheaves on X. Indeed, we have the following result:

Proposition 2.35 (Pareschi, Popa, [PP09, Corollary 3.2]). *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, let $\mathcal{F}$ be a coherent sheaf on X, and let m be a non-negative integer or $m = \infty$. Let $\alpha \in \operatorname{Pic}^0(A)$. Then the following are equivalent*

1. $gv_{a,\alpha}(\mathcal{F}) \geq m$;

2. $\widehat{\mathbf{R}\Delta\mathcal{F}}$ *is an m-th syzygy sheaf on* $\operatorname{Pic}^0(A)$*, in a neighbourhood of* α.

This result shows that, if a sheaf $\mathcal{F}$ satisfies IT with index 0, and hence its gv-index is equal to ∞, then the transform of its dual $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is locally free.

The generic vanishing index of $\mathcal{F}$ can provide a lower bound for its Euler characteristic, assuming that there is $i > 0$ with $V_a^i(\mathcal{F}) \neq \emptyset$:

Theorem 2.36 (Pareschi, Popa, [PP09, Theorem 3.3])**.** *Let X be a smooth projective variety with a morphism $a\colon X \to A$ to an abelian variety, let $\mathcal{F}$ be a coherent sheaf on X and let $\alpha \in \mathrm{Pic}^0(A)$. Assume that $0 \leq gv_{a,\alpha}(\mathcal{F}) < \infty$. Then*

$$\chi(\mathcal{F}) \geq gv_{a,\alpha}(\mathcal{F}) \geq gv_a(\mathcal{F}).$$

Proof. By Grothendieck duality formula, we have that the sheaf $\mathcal{H}om(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\widehat{A}})$ is equivalent to $R^0\Phi_{\mathcal{P}_a}(\mathcal{F})$. The latter has rank equal to the generic value of $h^0(\mathcal{F} \otimes \alpha^{-1})$, which coincides with $\chi(\mathcal{F})$. Thus, we have that $\chi(\mathcal{F}) = \mathrm{rk}(\widehat{\mathbf{R}\Delta\mathcal{F}})$. So, by Proposition 2.35, $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is a $gv_{a,\alpha}(\mathcal{F})$ syzygy sheaf in the neighbourhood of α with rank $\chi(\mathcal{F})$, and moreover it is not locally free by hypotesis. Thus we can apply Theorem 2.34 to conclude that $\chi(\mathcal{F}) = \mathrm{rk}(\widehat{\mathbf{R}\Delta\mathcal{F}}) \geq gv_{a,\alpha}(\mathcal{F})$, and by definition we also have that $gv_{a,\alpha}(\mathcal{F}) \geq gv_a(\mathcal{F})$. $\square$

CHAPTER 3

Singularities of divisors on abelian varieties

In this chapter we show an important application of the generic vanishing theorem: the study of singularities of divisors on abelian varieties.

The aim is to prove that a (prime) divisor representing a polarization with a low degree, or a divisor representing a multiple of such polarization, have only mild singularities. Their descriptions are given using the terminology of log-singularities, introduced for studying singular varieties in higher dimensional geometry.

We will state results of Kollár (the first one using vanishing theorems for this purpose), Ein and Lazarsfeld about principal polarizations, that is, of degree 1. Then we will see the case of polarizations of degree 2, made by Debarre and Hacon, together with the case of a simple abelian variety, which has been stated in its strongest form by Pareschi.

Eventually, we will work out the proof for some cases of a polarization of degree 3 and 4.

3.1 Preliminaries on singularities of pairs and the multiplier ideal sheaf

Here we collect preliminary notations and definitions, along with basic properties and theorems, about singularities of pairs and special ideal sheaves which are useful to determine such singularities. There is also a review on different kinds of classical vanishing theorems that will be largely used throughout the chapter.

3.1.1 Singularities of pairs

We start talking about singularities of pairs. This terminology comes form the study of higher dimensional geometry, in particular from the minimal model program for studying singular varieties. For a more comprehensive description, see for example the surveys [Kol97] and [Fuj07].

We start with a couple of definitions and notations. A $\mathbb{Q}$-*divisor* $D = \sum d_i D_i$ on a normal variety X is a sum of prime divisors D_i with rational coefficients d_i. D is $\mathbb{Q}$-*Cartier* if there exists a positive integer m such that mD is a Cartier divisor. We define the round down of D as $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d_i \rfloor$ is the greatest integer smaller of equal to $d_i \in \mathbb{Q}$.

The *canonical divisor* of a variety X is denoted by K_X, and the corresponding line bundle (that is, the *canonical bundle*) is denoted by ω_X.

We say that X has *rational singularities* if there is a proper birational morphism $f\colon Y \to X$ from a non-singular variety such that $R^i f_* \mathcal{O}_Y = 0$; note that if such a resolution exists, then all resolutions have the same property.

Let X be a normal variety, and let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to X$ be a proper birational morphism from a non-singular variety Y. The we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E$$

where the sum runs over all the distinct prime divisors $E \subset Y$ and $a(E, X, D) \in \mathbb{Q}$. The coefficient $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D).

To get a global measure of the singularities of the pair (X, D), we define

$$\operatorname{discrep}(X, D) = \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\},$$

and we can show that either $\operatorname{discrep}(X, D) = -\infty$ or $-1 \leq \operatorname{discrep}(X, D) \leq 1$ (see [CKM88, 6.3, p. 39]).

We say that (X, D) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{(Kawamata) log terminal} \\ \text{purely log terminal} \\ \text{log canonical} \end{cases} \quad \text{if } \operatorname{discrep}(X, D) \begin{cases} > 0, \\ \geq 0, \\ > -1 \text{ and } \lfloor D \rfloor = 0, \\ > -1, \\ \geq -1. \end{cases}$$

We can relate this notions of canonical and log terminal singularities with the more classical one of rational singularities.

Theorem 3.1 (Elkik, [Elk81, Théorème 1]; Kollár, [Kol97, Theorem 11.1]). *Let X be a normal variety.*

1. *Assume that ω_X is locally free. Then X has rational singularities if and only if X has canonical singularities.*
2. *Assume that (X, D) has Kawamata log terminal singularities. Then X has rational singularities.*

3.1.2 Multiplier ideal sheaf

Multiplier ideals are an essential tool for working with vanishing theorems. Indeed, they were introduced (in the framework of complex analysis) by Demailly, Siu, Nadel and others to simplify the application of Kodaira-like vanishings. Then it became clear that such ideals can be define entirely in an algebro-geometric way (see [EV92, § 7]), and this leads to new applications. For a complete treatment and an overview of several applications, see [Laz04, Part Three].

Let X be a non-singular variety of dimension n. A divisor $D = \sum D_i$ on X has *simple normal crossing* (and D is a *SNC divisor*) if each D_i is smooth, and if D is defined in a neighbourhood of any point by an equation in local analytic coordinates of the type $z_1 \cdot \ldots \cdot z_k = 0$ for some $k \leq n$. A $\mathbb{Q}$-divisor $\sum d_i D_i$ has *simple normal crossing support* if $\sum D_i$ is a SNC divisor.

A *log resolution* of the pair (X, D) where X is a non-singular variety and D is a $\mathbb{Q}$-divisor on X, is a proper birational map $\mu\colon Y \to X$ whose exceptional locus is a divisor E, such that Y is non-singular and the divisor $\mu^* D + E$ has simple normal crossing support. The existence of such resolutions is due to the celebrated theorem of Hironaka on resolutions of singularities.

Given a log resolution $\mu\colon Y \to X$, we can also define the *relative canonical divisor* $K_{Y/X}$ of Y over X as the difference $K_Y - \mu^* K_X$.

Let X a non-singular variety and D and effective $\mathbb{Q}$-divisor on it, and fix a log resolution $\mu\colon Y \to X$ of (X, D). The *multiplier ideal sheaf* $\mathcal{J}(D) = \mathcal{J}(X, D) \subseteq \mathcal{O}_X$ associated to D is defined as

$$\mathcal{J}(D) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor).$$

$\mathcal{J}(D)$ is indeed a sheaf; moreover, using the fact that two resolutions can be dominated by a third one, we can show that the definition is independent of the log resolution μ (see [Laz04, Theorem 9.2.18]).

Multiplier ideals can be used for measuring singularities. In particular, there are some classes of singularities of pairs introduced before that can be defined in terms of triviality of a multiplier ideal: let X be a smooth variety and let D be a $\mathbb{Q}$-divisor on it. Then

1. the pair (X, D) is (Kawamata) log terminal if $\mathcal{J}(X, D) = \mathcal{O}_X$;
2. the pair (X, D) is log canonical if $\mathcal{J}(X, (1-\varepsilon)D) = \mathcal{O}_X$ for all $\varepsilon \in (0,1) \cap \mathbb{Q}$.

The equivalence between the previous definitions of log singularities and this one using multiplier ideals follows from the fact that $\mathcal{J}(X, D) = \mathcal{O}_X$ if and only if $\operatorname{ord}_E(K_{Y/X} - \lfloor \mu^* D \rfloor) > -1$ for every divisor E in the log resolution $\mu\colon Y \to X$.

Adjoint ideal. We now take an integral divisor D on X. We know that $\lfloor \mu^* D \rfloor = \mu^* D$, and using the projection formula we obtain $\mathcal{J}(D) = \mathcal{O}_X(-D)$; this fact seems to make the multiplier ideal of an integral divisor quite uninteresting. However, when D is reduced, we can define a variant of the multiplier ideal that carries some information about the singularities of D. So, take a reduced integral divisor D on

X, and fix a log resolution $\mu\colon Y \to X$ of (X, D) such that the proper transform D' of D is non-singular. Write $\mu^*D = D' + F$, where F is a μ-exceptional effective integral divisor on Y. The *adjoint ideal sheaf* $\operatorname{adj}(D) = \operatorname{adj}(X, D) \subseteq \mathcal{O}_X$ is defined as

$$\operatorname{adj}(D) = \mu_*\mathcal{O}_Y(K_{Y/X} - F).$$

Again, this definition is independent of the chosen log resolution.

The adjoint ideal has some basic properties, which are useful to give a measure of the singularities of the divisor associated to it:

Proposition 3.2 (Lazarsfeld, [Laz04, Proposition 9.3.48])**.** *Given (X, D) as before, let $\nu\colon E \to D$ be any resolution of singularities of D. Then*

1. *there exists an exact sequence*

$$0 \longrightarrow \omega_X \longrightarrow \mathcal{O}_X(K_X + D) \otimes \operatorname{adj}(D) \longrightarrow \nu^*\omega_E \longrightarrow 0; \qquad (3.1)$$

2. *the adjoint ideal* $\operatorname{adj}(D)$ *is trivial if and only if D is normal and has at worst rational singularities.*

Vanishing theorems. There are some basic vanishings for multiplier ideals coming from the theorem of Kawamata-Viehweg.

We first recall a couple of notions about positivity: given X a projective variety of dimension n and D a Cartier divisor on it, we say that D is *nef* (short for *numerically effective*) if, for every irreducible curve $C \subseteq X$, we have $(D.C) \geq 0$. Nefness means in effect that D is a limit of ample divisors. Also, D is said to be *big* if $h^0(X, \mathcal{O}_X(mD)) \sim m^n$ for $m \gg 0$, that is, the spaces of sections of mD grow maximally with m.

The first important vanishing theorem for divisors is the one of Kodaira, which proves that if D is an ample divisor, the cohomology groups $H^i(X, \mathcal{O}_X(K_X + D))$ vanish for every $i > 0$.

The condition of ampleness on D can be weakened, giving a much more general statement:

Theorem 3.3 (Kawamata-Viehweg Vanishing; [Laz04, Corollary 9.1.21])**.** *Let X be a smooth projective variety, and let D be an integral divisor and E be an effective $\mathbb{Q}$-divisor on X. Assume that E has SNC support and that $D - E$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(K_X + D - \lfloor E \rfloor)) = 0 \qquad \text{for all } i > 0.$$

The main difficulty in applying this theorem is that in practice the SNC condition is hardly satisfied. Given a $\mathbb{Q}$-divisor on X, a natural idea is to apply vanishing on a (log) resolution of singularities and then push down to get a statement in X. Doing so, multiplier ideals appear inevitably, leading to some basic vanishing theorems for these ideals.

A first local result is

Theorem 3.4 ([Laz04, Theorem 9.4.1])**.** *Let X be a smooth variety, and D a $\mathbb{Q}$-divisor on X. Take a log resolution $\mu\colon Y \to X$ of (X, D). Then*

$$R^j\mu_*\mathcal{O}_Y(K_{Y/X} - \lfloor \mu^*D \rfloor) = 0 \qquad \text{for all } j > 0.$$

Now we have the basic global vanishing for multiplier ideals:

Theorem 3.5 (Nadel Vanishing; [Laz04, Theorem 9.4.8]). *Let X be a smooth projective variety, and let D be an integral divisor and E be an effective $\mathbb{Q}$-divisor on X. Assume that $D - E$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{J}(E)) = 0 \qquad \text{for all } i > 0.$$

There are also Kodaira-type theorems for ample vector bundles with rank greater than 1. Here we state a fundamental one, which will be used later in the chapter.

Theorem 3.6 (Le Potier Vanishing; [Laz04, Theorem 7.3.5]). *Let X be a smooth projective variety, and let E be an ample vector bundle with rank r. Then*

$$H^i(X, \omega_X \otimes \Lambda^a E) = 0 \qquad \text{for all } a > 0 \text{ and all } i > r - a.$$

Moreover, we have that

$$H^i(X, \Omega_X^p \otimes E) = 0 \qquad \text{for } i + p \geq \dim X + r,$$

where Ω_X is the cotangent bundle on X.

3.2 Principal polarizations

Let A be an abelian variety of dimension g, and let L be a principal polarization on A, that is, a polarization with $h^0(A, L) = 1$. There is a well-defined effective divisor $\Theta \subset A$ such that $L \simeq \mathcal{O}_A(\Theta)$, which is called *theta divisor*. There has always been some interest in this divisor, with particular focus on understanding its singularities. A first important result is due to Kempf, in the special case of Jacobians of compact Riemann surfaces:

Theorem 3.7 (Kempf, [Kem73]). *Let A be the Jacobian variety of a compact Riemann surface of genus $g \geq 1$. Then its theta divisor is irreducible and normal, with rational singularities.*

While the geometry of Jacobians is well-understood, for arbitrary abelian varieties there are not such good models of Θ. Thus, it seems difficult to say something about the singularities of the theta divisor.

A general result in this direction was proved by Kollár. His approach uses for the first time vanishing theorems to analyze the geometry of singularities.

Here we give a variant of the original proof, using the notation of multiplier ideals introduced in the previous section. Also, we denote by

$$\Sigma_k(\Theta) = \{x \in A \mid \operatorname{mult}_x(\Theta) \geq k\}$$

the set of points where the multiplicity of Θ is at least k.

Theorem 3.8 (Kollár, [Kol95, Theorem 17.13]). *Let (A,Θ) be a principally polarized abelian variety. Then (A,Θ) is log canonical. In particular, we have, for all $k \geq 0$,*

$$\operatorname{codim}_A \Sigma_k(\Theta) \geq k.$$

Proof. In the multiplier ideal language, (A,Θ) being log-canonical means that $\mathcal{J}((1-\varepsilon)\Theta) = \mathcal{O}_A$ for all $\varepsilon \in (0,1) \cap \mathbb{Q}$. Suppose to the contrary that there is such an ε for which $\mathcal{J} := \mathcal{J}((1-\varepsilon)\Theta)$ is not trivial, and let Z be the closed subscheme defined by this multiplier ideal. We then consider the exact sequence

$$0 \longrightarrow \mathcal{O}_A(\Theta) \otimes \mathcal{J} \longrightarrow \mathcal{O}_A(\Theta) \longrightarrow \mathcal{O}_Z(\Theta) \longrightarrow 0,$$

and by twisting for any $P \in \operatorname{Pic}^0(A)$, we get the sequence

$$0 \longrightarrow \mathcal{O}_A(\Theta) \otimes \mathcal{J} \otimes P \longrightarrow \mathcal{O}_A(\Theta) \otimes P \longrightarrow \mathcal{O}_Z(\Theta) \otimes P \longrightarrow 0.$$

Since $H^i(A, \mathcal{O}_A(\Theta) \otimes P)$ is zero for all positive i, it follows that

$$H^i(A, \mathcal{O}_Z(\Theta) \otimes P) = H^{i+1}(A, \mathcal{O}_A(\Theta) \otimes \mathcal{J} \otimes P) \tag{3.2}$$

for all $i > 0$ and all $P \in \operatorname{Pic}^0(A)$.

Also, the divisor $\Theta - (1-\varepsilon)\Theta + P$ is ample, so we can apply Nadel vanishing (Theorem 3.5), and obtain that

$$H^i(A, \mathcal{O}_A(\Theta) \otimes \mathcal{J} \otimes P) = 0$$

for all $i > 0$ and all $P \in \operatorname{Pic}^0(A)$. By definition, this fact means that the sheaf $\mathcal{O}_A(\Theta) \otimes \mathcal{J}$ satisfies IT with index 0, and so we have that $\chi(\mathcal{O}_A(\Theta) \otimes \mathcal{J})$ is strictly positive. By means of equality (3.2), we obtain also that $\chi(\mathcal{O}_Z(\Theta)) > 0$. Hence we have

$$\chi(\mathcal{O}_A(\Theta)) = \chi(\mathcal{O}_A(\Theta) \otimes \mathcal{J}) + \chi(\mathcal{O}_Z(\Theta)) \geq 2,$$

which is a contradiction, since the degree of Θ is 1.

The result on codimension of $\Sigma_k(\Theta)$ follows from the triviality of the multiplier ideal. To show this fact, let us consider an irreducible component $Y \subseteq \Sigma_k(\Theta)$ of codimension m. We can construct a log resolution $\mu\colon X \to A$ of (A,Θ) by first blowing-up of Y. Let $E \subset X$ be the proper transform of the exceptional divisor of the initial blow-up. Since Y has codimension m, $K_{X/A}$ contains the divisor $(m-1)E$. Also $Y \subset \Sigma_k(\Theta)$, so $\mu^*\Theta$ contains kE. As a consequence

$$K_{X/A} - \lfloor (1-\varepsilon)\mu^*\Theta \rfloor \supseteq (m-1)E - \lfloor (1-\varepsilon)kE \rfloor = \big((m-1) - \lfloor (1-\varepsilon)k \rfloor\big)E,$$

and since the multiplier ideal $\mathcal{J} = \mu_*\mathcal{O}_X(K_{X/A} - \lfloor (1-\varepsilon)\mu^*\Theta \rfloor)$ is trivial, the coefficient of E must be non-negative. Thus $(m-1) - \lfloor (1-\varepsilon)k \rfloor \geq 0$, and for ε small enough, we have $m \geq k$. □

A straightforward consequence is that Θ cannot have points of multiplicity greater than the dimension of A.

Moreover, Smith and Varley in [SV96] proved that if Θ contains a point of multiplicity $g = \dim A$, then (A,Θ) splits as a product of g elliptic curves.

The strongest result about singularities of theta divisors is due to Ein and Lazarsfeld: indeed, they showed that the conclusion of Kempf Theorem 3.7 is true in a very general setting.

Theorem 3.9 (Ein, Lazarsfeld, [EL97, Theorem 1])**.** *Let* (A, Θ) *be a principally polarized abelian variety of dimension* g*. If* Θ *is irreducible, then it is normal and has rational singularities.*

Proof. According to Proposition 3.2, Θ is normal with rational singularities if and only if its adjoint ideal $\operatorname{adj}(\Theta)$ is trivial. Then we can construct a resolution of singularities $\nu\colon X \to \Theta$; note that X has maximal Albanese dimension, so $\chi(\omega_X) \geq 0$ by Proposition 2.20. Moreover, we have that $\chi(\omega_X) \geq 1$ by Theorem 2.21: indeed, $\operatorname{alb}(X)$ is exactly Θ, and since Θ is ample, it cannot be fibred by abelian subvarieties.

The adjoint sequence (3.1) in Proposition 3.2 has the form

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_A(\Theta) \otimes \operatorname{adj}(\Theta) \longrightarrow \nu_*\omega_X \longrightarrow 0,$$

and if we take $P \in \operatorname{Pic}^0(A)$, it can be twisted by P, obtaining

$$0 \longrightarrow P \longrightarrow \mathcal{O}_A(\Theta) \otimes P \otimes \operatorname{adj}(\Theta) \longrightarrow \nu_*\omega_X \otimes P \longrightarrow 0.$$

If $P \neq \mathcal{O}_A$, $H^0(A, P) = H^1(A, P) = 0$, and so we have

$$H^0(A, \mathcal{O}_A(\Theta) \otimes P \otimes \operatorname{adj}(\Theta)) = H^0(X, \omega_X \otimes \nu^*P).$$

Now we come to the crucial point: by means of the generic vanishing theorem, $H^i(X, \omega_X \otimes \nu^*P) = 0$ for all $i > 0$ and for P general. Then

$$H^0(X, \omega_X \otimes \nu^*P) = \chi(\omega_X)$$

for a general P. Since $\chi(\omega_X) \geq 1$, we obtain that $H^0(A, \mathcal{O}_A(\Theta) \otimes P \otimes \operatorname{adj}(\Theta)) \neq 0$ for general P.

We now recall that there is an isomorphism $\varphi_\Theta\colon A \to \operatorname{Pic}^0(A)$ determined by the principal polarization. So there is a point $a_P \in A$ corresponding to $P \in \operatorname{Pic}^0(A)$. Then

$$H^0(A, \mathcal{O}_A(\Theta + a_P) \otimes \operatorname{adj}(\Theta)) \neq 0$$

for general $a_P \in A$, and this fact means that the subscheme defined by $\operatorname{adj}(\Theta)$ is contained in any general translate of Θ. But this can happen only if such subscheme is empty. Therefore, $\operatorname{adj}(\Theta) = \mathcal{O}_A$, and this concludes the proof. □

In the same paper, the two authors showed also that any possible example on the boundary of Theorem 3.8 is split. Here is the statement (for the proof, see [EL97, Corollary 2]):

Corollary 3.10 (Ein, Lazarsfeld, [EL97, Corollary 2])**.** *Let* (A, Θ) *be a principally polarized abelian variety, and let* $k \geq 2$*. Then* $\Sigma_k(\Theta)$ *contains an irreducile component of codimension* k *in* A *if and only if* (A, Θ) *splits as a* k*-fold product of principally polarized abelian varieties.*

Ein and Lazarsfeld provided also an extension of Theorem 3.8 for pluri-theta divisors, as proposed in [Kol95, Problem 17.15]:

Theorem 3.11 (Ein, Lazarsfeld, [EL97, Proposition 3.5])**.** *Let (A,Θ) be a principally polarized abelian variety, and let $D \in |m\Theta|$ for some $m \geq 1$. Then the pair $(A, \frac{1}{m}D)$ is log-canonical. In particular, for all $k \geq 0$,*

$$\operatorname{codim}_A \Sigma_{mk}(D) \geq k.$$

Proof. For $\varepsilon \in (0,1) \cap \mathbb{Q}$, let us consider the divisor

$$E_\varepsilon := \frac{1-\varepsilon}{m} D \equiv (1-\varepsilon)\Theta.$$

We know that $(A, \frac{1}{m}D)$ is log-canonical if and only if all multiplier ideals $\mathcal{J}(E_\varepsilon)$ are trivial. In order to get a contradiction, suppose that there is such an ε for which $\mathcal{J} := \mathcal{J}(E_\varepsilon)$ is not trivial, and let Z be the closed subscheme defined by this multiplier ideal. The divisor $\Theta - E$ is ample, and also $\Theta - E + P$ is ample for all $P \in \operatorname{Pic}^0(A)$. Thus we can apply Nadel vanishing theorem, and obtain that

$$H^i(A, \mathcal{O}_A(\Theta) \otimes P \otimes \mathcal{J}) = 0 \qquad \text{for all } i > 0 \text{ and all } P \in \operatorname{Pic}^0(A). \tag{3.3}$$

This means that the sheaf $\mathcal{O}_A(\Theta) \otimes \mathcal{J}$ satisfies IT with index 0, and so its Euler characteristic must be positive.

Then, we consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_A(\Theta) \otimes \mathcal{J} \longrightarrow \mathcal{O}_A(\Theta) \longrightarrow \mathcal{O}_Z(\Theta) \longrightarrow 0.$$

By twisting by $P \in \operatorname{Pic}^0(A)$, and since $H^i(\mathcal{O}_A(\Theta) \otimes P) = 0$ for every $i > 0$, we have that

$$H^i(A, \mathcal{O}_Z(\Theta) \otimes P) = H^{i+1}(A, \mathcal{O}_A(\Theta) \otimes P \otimes \mathcal{J})$$

for all $i > 0$. This in turn implies that also the sheaf $\mathcal{O}_Z(\Theta)$ satisfies IT with index 0, and so $\chi(\mathcal{O}_Z(\Theta)) \geq 1$.

Hence, we end up having

$$\chi(\mathcal{O}_A(\Theta)) = \chi(\mathcal{O}_A(\Theta) \otimes \mathcal{J}) + \chi(\mathcal{O}_Z(\Theta)) \geq 2,$$

which is obviously impossible since $\chi(\mathcal{O}_A(\Theta)) = 1$. □

As a refinement of the previous results, Hacon in [Hac99] proved the following:

Theorem 3.12 (Hacon, [Hac99, Theorem 1])**.** *Let (A,Θ) be a principally polarized abelian variety, and given a positive integer m let $D \in |m\Theta|$ be a divisor such that $\lfloor \frac{1}{m}D \rfloor = 0$. Then the pair $(A, \frac{1}{m}D)$ is log-terminal.*

Note that the condition $\lfloor \frac{1}{m}D \rfloor = 0$ is equivalent to requiring that the multiplicity of each component of D is strictly smaller than m. With only this extra condition, it is possile to show that the pair $(A, \frac{1}{m}D)$ has better log singularities.

Sketch of the proof. We consider the decomposition into irreducible reduced components $D = \sum d_i D_i$, where $1 \leq d_i \leq m-1$. Let $f\colon X \to A$ be a log resolution of (A, D). We define the sheaf

$$\mathcal{L} = \mathcal{O}_X\Big(f^*(\Theta) - \Big\lfloor \frac{1}{m} f^*D \Big\rfloor\Big)$$

as shown in [EV92, § 4]. We have that $f_*(\omega_X \otimes \mathcal{L}) = \mathcal{J}(\frac{1}{m}D) \otimes \mathcal{O}_A(\Theta)$, where $\mathcal{J}(\frac{1}{m}D)$ is the multiplier ideal, and $R^i f_*(\omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$. We then obtain that

$$V^i := V^i(\omega_X \otimes \mathcal{L}) = V^i\Big(\mathcal{J}\Big(\frac{1}{m}D\Big) \otimes \mathcal{O}_A(\Theta)\Big).$$

The aim of the proof is to show that $\mathcal{J}(\frac{1}{m}D)$ is trivial.

The crucial fact is that the geometry of the loci $V^i(\omega_X \otimes \mathcal{L})$ behaves as the one of loci $V^i(\omega_X)$ described by Green and Lazarsfeld, and so the generic vanishing theorem and the structure theorem hold also for those new V^i (see [EV92], [EL97] and [Sim93] for details).

Let S be an irreducible component of V^0, which is not empty by the same argument at the end of proof of Theorem 3.11. Thus by structure theorem S is a transalate of a subtorus of $\mathrm{Pic}^0(X)$, and we can denote by C the dual of such subtorus. Thus we have a map $\pi\colon A \to C$.

Now, it is possible to prove that any component D_i of D such that $\pi(D_i) = C$ is contained in $\mathrm{Bs}|\mathcal{J}(\frac{1}{m}D) \otimes \mathcal{O}_A(\Theta) \otimes P|$ for a general $P \in \mathrm{Pic}^0(A)$. It follows that the divisor

$$\Theta - \sum_{\pi(D_i)=C} D_i$$

is algebraically equivalent to an effective divisor, and hence is nef. As a consequence, the divisor

$$H := D - \sum_{\pi(D_i)=C} d_i D_i,$$

turns out to be ample (this is the point where we need that $d_i < m$). Moreover, H is contained in the pullback of a divisor on C, and hence π has to be an isomorphism, so that $V^0 = \mathrm{Pic}^0(A)$. Therefore, $H^0(A, \mathcal{J}(\frac{1}{m}D) \otimes \mathcal{O}_A(\Theta) \otimes P) \neq 0$ for all $P \in \mathrm{Pic}^0(A)$, which means that all translates of Θ vanish along the cosupport of $\mathcal{J}(\frac{1}{m}D)$. Then $\mathcal{J}(\frac{1}{m}D)$ must be zero, concluding the proof. ☐

With this result, we can look at the loci of singularities of the divisor:

Corollary 3.13 (Hacon, [Hac99, Corollary 2])**.** *Let (A, Θ) be a principally polarized abelian variety, and given a positive integer m let $D \in |m\Theta|$. If k is the greatest integer such that the set $\Sigma_{mk}(D)$ contains an irreducible component of codimension k in A, then (A, Θ) splits as a product of at least k principally polarized abelian varieties.*

3.3 Polarizations of degree 2

We now focus our attention to abelian varieties with a polarization with degree greater than one. When the degree increases, many special cases start to appear, often due to the existence of reducible divisors that can represent the polarization.

A first result for an ample divisor of degree 2 is due to Hacon in [Hac00], where he proves that some non-splitting abelian varieties can form a log-canonical pair with a $\mathbb{Q}$-divisor representing their polarization, as a parallel with Theorem 3.11. Here is the statement:

Theorem 3.14 (Hacon, [Hac00, Theorem 4.1]). *Let (A, L) be a polarized abelian variety with polarization of degree 2, and given a positive integer m, let $D \in |mL|$. Then either*

1. *(A, L) splits as the product of a principally polarized abelian variety and an elliptic curve, or*
2. *$(A, \frac{1}{m}D)$ is a log-canonical pair.*

Proof. As in the proof of Theorem 3.11, we assume there is a non trivial multiplier ideal $\mathscr{J} := \mathscr{J}(\frac{1-\varepsilon}{m}D)$ for a rational $0 < \varepsilon \ll 1$, with cosupport Z. Since $L - \frac{1-\varepsilon}{m}D$ is ample, also $L - \frac{1-\varepsilon}{m}D + P$ is ample for all $P \in \operatorname{Pic}^0(A)$, and thus we can apply Nadel vanishing theorem. We obtain that

$$H^i(A, L \otimes \mathscr{J} \otimes P) = 0$$

for all $i > 0$ and all $P \in \operatorname{Pic}^0(A)$, showing that $\mathscr{J} \otimes L$ satisfies IT with index 0. By the short exact sequence

$$0 \longrightarrow \mathscr{J} \otimes L \longrightarrow L \longrightarrow L_{|Z} \longrightarrow 0,$$

we see that $H^i(A, L_{|Z} \otimes P) = H^{i+1}(A, L \otimes \mathscr{J} \otimes P)$ for all $i > 0$. This means that also $L_{|Z}$ satisfies IT with index 0, and thus both sheaves have positive Euler characteristic. But the short exact sequence says also that

$$2 = \chi(L) = \chi(\mathscr{J} \otimes L) + \chi(L_{|Z}),$$

and so we must have $\chi(\mathscr{J} \otimes L) = \chi(L_{|Z}) = 1$. In particular, it follows that $\mathscr{J} \otimes L$ is a principal polarization, and so a line bundle with one section. This means that $\mathscr{J} \simeq \mathscr{O}_A(-F)$, where F is a divisor, and there is a map $A \to E$ to an elliptic curve such that F is the pull-back of a principal polarization on E, namely $\mathscr{O}_E(p)$ for a point $p \in E$. Therefore, if $(A, \frac{1}{m}D)$ is not a log-canonical pair, (A, L) splits as a product of $(E, \mathscr{O}_E(p))$ and the principal polarized abelian variety with polarization $L \otimes \mathscr{O}_A(-F)$. □

A more complete result for polarizations of degree 2 has been proved by Debarre and Hacon in [DH07]. In that paper, they have stated some geometrical facts about cohomological support loci of particular sheaves, and from them they have obtained a theorem for divisors representing polarizations of degree 2. Indeed, those facts include and generalize some arguments already used in some proofs along this chapter.

We now state and prove the results:

Lemma 3.15 (Debarre, Hacon, [DH07, Lemma 5])**.** *Let A be an abelian variety of dimension g, and let L be an ample line bundle of degree d on A. Let Z be a closed subscheme of A with ideal sheaf $\mathcal{J}$. Set $h := h^0(A, \mathcal{J} \otimes L \otimes P)$ for a general $P \in \operatorname{Pic}^0(A)$. Then*

1. *for $i > \dim Z + 1$, the set $V^i(\mathcal{J} \otimes L)$ is empty;*
2. *if $h = 0$, the set $V_{>0} := \cup_{i>0} V^i(\mathcal{J} \otimes L)$ is non-empty;*
3. *we have $h \leq d$, and equality holds if and only if Z is empty;*
4. *if $h = d - 1$ and the polarized variety (A, L) is indecomposable, then either*
    - *(a) Z has finite length and $V^1(\mathcal{J} \otimes L) = \operatorname{Pic}^0(A)$, or*
    - *(b) Z is a single reduced point $\{z\}$ and $V^1(\mathcal{J} \otimes L) = \varphi_L(\mathrm{Bs}|L| - z)$, and so $\dim V^1(\mathcal{J} \otimes L) \geq g - d$;*
5. *if Z is geometrically non-degenerate and $0 < h < d$, then $\dim Z \leq d - 1 - h$; moreover, if A is simple, we have $\dim V_{>0} \geq g - \frac{(d+1)^2}{4}$.*

Proof. (1) Fix $i > \dim Z + 1$. Then $H^i(A, \mathcal{J} \otimes L \otimes P) = H^i(A, L \otimes P)$, which is zero by Kodaira vanishing. So $V^i(\mathcal{J} \otimes L) = \emptyset$.

(2) Suppose that $V_{>0}$ is empty, that is $H^i(A, \mathcal{J} \otimes L \otimes P) = 0$ for all $i > 0$ and all $P \in \operatorname{Pic}^0(A)$. Then $h = \chi(A, \mathcal{J} \otimes L)$, and if $h = 0$ we have

$$H^i(A, \mathcal{J} \otimes L \otimes P) = 0 \qquad \text{for all } i \geq 0 \text{ and all } P \in \operatorname{Pic}^0(A).$$

But by Mukai theory (see [Muk81]), this implies that the sheaf $\mathcal{J} \otimes L$ has to be zero, which is a contradiction.

(3) We have $h = d$ if and only if all global sections of L vanish on general translates of Z, but this is true if and only if Z is empty.

(4) We define the set

$$J := \{(D, a) \in |L| \times A \mid D \supseteq Z + a\} \subset |L| \times A$$

with two projections $p_1 : J \to |L| \simeq \mathbb{P}H^0(A, L)$ and $p_2 \colon J \to A$. For a point $a \in A$, the fibre $p_2^{-1}(a)$ is isomorphic to $\mathbb{P}H^0(A, \mathcal{J} \otimes L \otimes P_{\varphi_L(a)})$, where $\varphi_L \colon A \to \operatorname{Pic}^0(A)$ is the function (2.1) in the preliminaries.

So, assuming $h > 0$, there is a component $J' \subseteq J$ which dominates A, which can be written as

$$J' = \{(D, a) \in |\mathcal{J} \otimes L| \times A \mid D \supseteq Z + a\},$$

and with dimension $g + (h - 1)$.

We consider then the map $p_1 \colon J' \to |L|$ (restriction of the first projection map) and, given $D \in |L|$, we define $F_D := p_2 \circ p_1^{-1}(D)$. By definition, it satisfies $Z + F_D \subseteq D$; we thus have

$$g - 1 \geq \dim F_D \geq \dim J' - \dim p_1(J') \geq (g + h - 1) - (d - 1) = g - (d - h).$$

We now assume that $d-h=1$. Then p_1 is surjective and $\dim F_D = g-1$. Taking a general (hence prime) divisor D, we have that $z+F_D = D$ for all $z \in Z$, and so Z has to be finite. In the case that $V^1(\mathcal{J}\otimes L) \neq \operatorname{Pic}^0(A)$, the length of Z is $d-h=1$, so we can write $Z=\{z\}$. Moreover, an element $P \in \operatorname{Pic}^0(A)$ is contained in $V^1(\mathcal{J}\otimes L)$ if and only if the restriction map

$$H^0(L\otimes P) \longrightarrow H^0(Z, \mathcal{O}_Z \otimes L \otimes P) \simeq \mathbb{C}(z)$$

is not surjective. That is, if all the sections of $L\otimes P$ vanish at z. This means that $z \in \operatorname{Bs}|L\otimes P| = \operatorname{Bs}|L| - a$, where $P = P_{\varphi_L(a)}$ for some $a \in A$. Therefore, $V^1(\mathcal{J}\otimes L) = \varphi_L(\operatorname{Bs}|L| - z)$.

(5) Since Z is geometrically non-degenerate, and by hypotesis, $Z \neq \emptyset$, we can use Corollary 2.3, so that the inclusion $Z + F_D \subseteq D$ implies that $\dim Z + \dim F_D \geq \dim D$. This means that

$$\dim Z \leq g-1-\dim F_D \leq d-h-1.$$

Let a be a general point of A. Then $p_1 \circ p_2^{-1}(a)$ is a linear subspace of $|L|$ of dimension $h-1$. Such a subspace must vary if we move a, otherwise we will have a divisor containing all the translates of Z. Thus the linear span of $p_1(J')$ has dimension at least h. So, we can take $D_1, \dots, D_{h+1}$ general elements of $p_1(J')$; we set $F_i := F_{D_i}$.

Since A is simple, using [Deb95, Corollaire 2.4], we have

$$\dim(F_1 \cap \cdots \cap F_{h+1}) \geq g-(h+1)(d-h) \geq g - \frac{(d+1)^2}{4}.$$

Now, let $\bar{a} \in F_1 \cap \cdots \cap F_{h+1}$. This means that the divisors $D_1, \dots, D_{h+1}$ all contain $Z+\bar{a}$. Then we have $h^0(A, \mathcal{J}\otimes L \otimes P_{\varphi_L(\bar{a})}) \geq h+1$, and since $\chi(\mathcal{J}\otimes L) = h$, there has to be an index $k>0$ such that $h^k(A, \mathcal{J}\otimes L\otimes P_{\varphi_L(\bar{a})}) > 0$. Thus, $P_{\varphi_L(\bar{a})} \in V_{>0}$, and hence $\dim V_{>0} \geq g - \frac{(d+1)^2}{4}$. ☐

Before stating the next fact, we need the following slight generalization of the Structure Theorem 2.18:

Proposition 3.16 (Hacon, Pardini, [HP05, Theorem 2.2]). *Let X be an irregular variety with a morphism $a\colon X \to A$ to an abelian variety, and let $\mathcal{F}$ be a sheaf on A which is a direct summand of $a_*\omega_X$. Let B be an abelian variety with a morphism $\pi\colon A \to B$. Then for every $P_0 \in \operatorname{Pic}^0(A)_{\text{tors}}$, and for all $i \geq 0$ and $j \geq 0$, every irreducible component of $V^i(R^j\pi_*(\mathcal{F}\otimes P_0))$ is a translate by a torsion point of an abelian subavariety of $\operatorname{Pic}^0(B)$ with codimension at least i.*

Now we are able to state and prove the following

Lemma 3.17 (Debarre, Hacon, [DH07, Lemma 6]). *In the same setting of Lemma 3.15, we assume moreover that there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_A^{\oplus\varepsilon} \longrightarrow \mathcal{J}\otimes L \longrightarrow \mathcal{F} \longrightarrow 0, \tag{3.4}$$

where $\varepsilon \in \mathbb{N}$ and $\mathcal{F}$ is a direct summand of a push-forward of a dualizing sheaf. Then

1. *every irreducible component of* $V^i(\mathcal{J}\otimes L)$ *is an abelian subvariety of* $\operatorname{Pic}^0(A)$ *translated by a torsion point and of codimension at least* i*;*
2. *if* $\operatorname{supp}\mathcal{F}$ *is not contained in any non-ample divisor of* A*, we have that*
$$\dim Z \geq i-1+\dim V^i(\mathcal{J}\otimes L)$$
for any $i>0$ *such that* $\dim V^i(\mathcal{J}\otimes L)$ *is not zero.*

Proof. (1) From Proposition 3.16, $\mathcal{F}$ satisfies the structure theorem. Also, by (3.4), we have $V^i(\mathcal{F})\setminus\{0\} = V^i(\mathcal{J}\otimes L)\setminus\{0\}$, and thus also $\mathcal{J}\otimes L$ satisfies the structure theorem. Analogously, since $\mathcal{F}$ is a GV-sheaf, also $\mathcal{J}\otimes L$ is a GV-sheaf.

(2) Fix $i>0$ such that $V^i(\mathcal{J}\otimes L)$ is not empty. If $V^i(\mathcal{J}\otimes L)$ is finite, then the assertion follows from Lemma 3.15 (1). So, we consider the common component W of $V^i(\mathcal{J}\otimes L)$ and $V^i(\mathcal{F})$ with maximal positive dimension. By structure theorem, W is a translate of an abelian subvariety T, and we set $B := \widehat{T}$. Being the induced morphism $\pi\colon A\to B$, we can write $W = \pi^*\operatorname{Pic}^0(B)+P_0$, where $P_0\in\operatorname{Pic}^0(A)_{\mathrm{tors}}$.

We know from Proposition 3.16 that $R^j\pi_*(\mathcal{F}\otimes P_0)$ is a GV-sheaf for every $j\geq 0$, so we have
$$H^k(B, R^j\pi_*(\mathcal{F}\otimes P_0)\otimes P) = 0$$
for all $k\geq 0$ and for general $P\in\operatorname{Pic}^0(B)$.

Thus, using Corollary 2.8, we get
$$H^0(B, R^i\pi_*(\mathcal{F}\otimes P_0)\otimes P) = H^i(A, \mathcal{F}\otimes P_0\otimes\pi^*P)$$
which is non-zero since $P_0\otimes\pi^*P\in V^i(\mathcal{F})$.

By twisting sequence (3.4) and doing a push-forward, we get
$$\cdots\to R^i\pi_*(P_0)^{\oplus\varepsilon}\longrightarrow R^i\pi_*(\mathcal{J}\otimes L\otimes P_0)\longrightarrow R^i\pi_*(\mathcal{F}\otimes P_0)\xrightarrow{\delta} R^{i+1}\pi_*(P_0)^{\oplus\varepsilon}\to\dots,$$
where sheaves $R^j\pi_*(P_0)^{\oplus\varepsilon}$ are direct sums of numerically trivial line bundles on B (see proof of Theorem 1 in [Mum70, Appendix by Kempf]). Also, by Theorem 2.1 in [HP05], the sheaf $R^i\pi_*(\mathcal{F}\otimes P_0)$ is torsion-free on $\pi(\operatorname{supp}\mathcal{F})$, which is B by our hypotesis.

Then, also $R^i\pi_*(\mathcal{J}\otimes L\otimes P_0)$ is supported on B. Otherwise, δ would be injective; since $R^{i+1}\pi_*(P_0)^{\oplus\varepsilon}\otimes P$ has no non-zero global sections for P general, we would have $H^0(B, R^i\pi_*(\mathcal{F}\otimes P_0)\otimes P)=0$, in contradiction with the previous fact.

We know that $R^i\pi_*(L\otimes P_0)=0$, and so the short exact sequence
$$0\longrightarrow\mathcal{J}\otimes L\otimes P_0\longrightarrow L\otimes P_0\longrightarrow L_{|Z}\otimes P_0\longrightarrow 0$$
yields a surjective map
$$R^{i-1}\pi_*(L_{|Z}\otimes P_0)\longrightarrow R^i\pi_*(\mathcal{J}\otimes L\otimes P_0).$$
Hence, also $R^{i-1}\pi_*(L_{|Z}\otimes P_0)$ is supported in B. This means that all the fibres of $\pi_Z\colon Z\to B$ have dimension at least $i-1$, and so $\dim Z\geq i-1+\dim B = i-1+\dim V^i(\mathcal{J}\otimes L)$. □

Using those lemmas, we can provide a result in the case of a polarization of degree 2, with the only request of such polarization being indecomposable:

Theorem 3.18 (Debarre, Hacon, [DH07, Theorem 8])**.** *Let (A, L) be a polarized abelian variety with indecomposable polarization of degree $d \leq 2$ and dimension $g > d$. Then*

1. *every prime divisor in $|L|$ is normal and has rational singularities;*
2. *for $m \geq 2$, given $D \in |mL|$ such that $\lfloor \frac{1}{m} D \rfloor = 0$, the pair $(A, \frac{1}{m} D)$ is log-terminal.*

Note that $|L|$ can contain reducible elements. Also, the pair $(A, \frac{1}{m} D)$ is log-canonical even if $\lfloor \frac{1}{m} D \rfloor \neq 0$, as shown in Theorem 3.14.

Proof. (1) Let $E \in |L|$ be a prime divisor. From Proposition 3.2, E is normal with rational singularities if and only if $\mathcal{J} := \operatorname{adj}(A, E)$ is trivial, or equivalently, if the cosupport Z of $\mathcal{J}$ is empty.

We start by setting $h := h^0(A, \mathcal{J} \otimes L \otimes P)$ for a general $P \in \operatorname{Pic}^0(A)$. Proposition 3.2 also provides the exact sequence

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{J} \otimes L \longrightarrow \mu_* \omega_X \longrightarrow 0,$$

where $\mu \colon X \to E$ is a resolution of singularities of E. We thus have the sequence of Lemma 3.17, with $\varepsilon = 1$ and $\mathcal{F} = \mu_* \omega_X$; it follows that $\mathcal{J} \otimes L$ is a GV-sheaf, and in particular

$$h = \chi(A, \mathcal{J} \otimes L \otimes P) = \chi(X, \omega_X).$$

Since E is ample, it is not fibred by abelian subvarieties, and so applying Theorem 2.21 we obtain that $h > 0$.

Suppose now that $h = 1$, that is $h = d - 1$. Then by Lemma 3.15, Z is a single point and $\dim V^1(\mathcal{J} \otimes L) \geq g - 2$. On the other hand, by Lemma 3.17, $\dim V^1(\mathcal{J} \otimes L)$ has to be zero, which is a contradiction.

Hence, we must have $h = 2 = d$, and so using again Lemma 3.15, we get that Z is empty.

(2) Let $D \in |mL|$ as in the hypotesis, and let $\mathcal{J} := \mathcal{J}(\frac{1}{m} D)$ be a multiplier ideal. The pair $(A, \frac{1}{m} D)$ is log-terminal if and only if $\mathcal{J}$ is trivial, or equivalently if its cosupport Z is empty. Again, we set $h := h^0(A, \mathcal{J} \otimes L \otimes P)$ for a general $P \in \operatorname{Pic}^0(A)$.

First, we want to prove that $\mathcal{J} \otimes L$ is a direct summand of a push-forward of a dualizing sheaf. So, let $\mu \colon A' \to A$ be a log resolution of (A, D); we define

$$L' := \mu^* L \otimes \mathcal{O}_{A'}\Big(- \Big\lfloor \frac{1}{m} \mu^* D \Big\rfloor \Big).$$

Using the divisor $\mu^* D - m \lfloor \frac{1}{m} \mu^* D \rfloor \in |mL'|$, we can define a cyclic cover $f \colon X \to A'$ of degree m, and X is normal with rational singularities. Let moreover $\nu \colon X' \to X$ be a resolution of singularities; we thus have a chain of maps

$$X' \xrightarrow{\nu} X \xrightarrow{f} A' \xrightarrow{\mu} A.$$

By [EV92, Lemma 3.22], the sheaf $\omega_{A'} \otimes L'$ is a direct summand of $f_*\omega_X = f_*\nu_*\omega_{X'}$. It follows that $\mu_*(f_*\nu_*\omega_{X'})$ splits as a direct sum of m torsion-free sheaves, and one of those sheaves is

$$\begin{aligned}\mu_*(\omega_{A'} \otimes L') &= \mu_*\Big(\omega_{A'} \otimes \mu^*L \otimes \mathcal{O}_{A'}\Big(-\Big\lfloor \frac{1}{m}\mu^*D\Big\rfloor\Big)\Big)\\ &= L \otimes \mu_*\Big(\omega_{A'/A} \otimes \mathcal{O}_{A'}\Big(-\Big\lfloor \frac{1}{m}\mu^*D\Big\rfloor\Big)\Big)\\ &= L \otimes \mu_*\mathcal{O}_{A'}\Big(K_{A'/A} - \Big\lfloor \frac{1}{m}\mu^*D\Big\rfloor\Big)\\ &= L \otimes \mathcal{J}.\end{aligned}$$

So $\mathcal{J} \otimes L$ is a direct summand of a push-forward of a dualizing sheaf, and we can apply again Lemma 3.17. Thus $\mathcal{J} \otimes L$ is a GV-sheaf and $V^i(\mathcal{J} \otimes L)$ are translates of abelian subvarieties of $\mathrm{Pic}^0(A)$. This fact, together with the hypoteses $g > d$ and $\lfloor \frac{1}{m}D\rfloor = 0$, allow us to conclude that $h > 0$, following the proof of [Hac99, Theorem 1].

Then, if $h = 1 = d - 1$, we can use the same argument of part (1) to have a contradiction. So again $h = 2 = d$, and we obtain that Z is empty. □

We now interpret this result in term of dimensions of loci of singularities. We get a statement analogous to Corollary 3.10 and Corollary 3.13:

Proposition 3.19 (Debarre, Hacon, [DH07, Theorem 10]). *Let (A, L) be an indecomposable polarized abelian variety of degree $d \leq 2$ and dimension $g > 2$, and let m and k be two positive integers. Then the following are equivalent:*

1. *given $D \in |mL|$, the set $\Sigma_{mk}(D)$ contains an irreducible component of codimension k in A;*
2. *(A, L) is a double étale cover of a product of k non-zero principally polarized abelian varieties.*

3.4 Simple abelian varieties

When we try to investigate these singularities on divisors, we see that a lot of prolems and exceptions arise, principally due to the presence of non-prime divisors in the class of the polarization. This phenomenon tends to be more and more evident when the degree of the polarization increases.

This problematic can be avoided by focusing on a special type of abelian variety. In fact, Debarre and Hacon showed in their paper [DH07, Proposition 2] that if a polarization of an abelian variety A contains a non-prime divisor, then A is not simple. Thus, in the same paper, they proved the following theorem for simple abelian varieties:

Theorem 3.20 (Debarre, Hacon, [DH07, Theorem 7]). *Let (A, L) be a polarized simple abelian variety of degree d and dimension $g > \frac{(d+1)^2}{4}$. Then*

1. *every divisor in $|L|$ is prime, normal and with rational singularities;*
2. *given $m \geq 2$, let $D \in |mL|$ be a divisor such that $D \neq mE$ for any $E \in |L|$; then the pair $(A, \frac{1}{m}D)$ is log-terminal.*

Note that the bound $g > \frac{(d+1)^2}{4}$ is not sharp, but it comes from a geometric fact proved in Lemma 3.15 (5). Also, in part (2) we have that $(A, \frac{1}{m}D)$ is always log-canonical.

As conjectured by Debarre and Hacon, the conclusions of Theorem 3.20 should hold under the weaker hypotesis that $g > d$. This conjecture has been settled by Pareschi in [Par21], using all the ingredients of the previous theorems about singularities together with the theory of GV-sheaves and M-regularity.

The technical core of his result is given by the following statement, which gives a slight more general outcome:

Lemma 3.21 (Pareschi, [Par21, Lemma B])**.** *Let (A, L) be a polarized abelian variety of dimension g and degree d, and let Z be a closed subscheme of A with geometrically non-degenerate support. Assume moreover that Z is not a divisor in $|L|$, and let $\mathcal{J}$ be its ideal sheaf. Set $V_{>0} := \bigcup_{i>0} V^i(A, \mathcal{J} \otimes L)$. Then*

1. *if $V_{>0}$ is empty, then $d \geq g + 1$;*
2. *if $\dim V_{>0} = 0$, then $d \geq g$.*

Proof. First, we consider the exact sequence

$$0 \longrightarrow \mathcal{J} \otimes L \longrightarrow L \longrightarrow L_{|Z} \longrightarrow 0. \tag{3.5}$$

From it we have that $H^i(A, L_{|Z} \otimes P) = H^{i+1}(A, \mathcal{J} \otimes L \otimes P)$ for all $i \geq 1$ and all $P \in \mathrm{Pic}^0(A)$, and then $V^i(L_{|Z}) = V^{i+1}(\mathcal{J} \otimes L)$ for all $i \geq 1$. In both cases (1) and (2), $L_{|Z}$ is M-regular, and so $\chi(L_{|Z}) > 0$. This means that $\chi(\mathcal{J} \otimes L) = d - \chi(L_{|Z}) < d$. On the other hand, if $\chi(\mathcal{J} \otimes L) = 0$, then $\mathcal{J} \otimes L$ would be a homogeneous vector bundle (see [Par21, Proposition 0.3]), and this can happen if and only if Z is a divisor in $|L|$. So, we have that $0 < \chi(\mathcal{J} \otimes L) < d$, and since, for a general P, $\chi(\mathcal{J} \otimes L) = h^0(A, \mathcal{J} \otimes L \otimes P)$, we can use Lemma 3.15 (5) to obtain

$$\chi(\mathcal{J} \otimes L) \leq d - 1 - \dim Z. \tag{3.6}$$

We look now to the single cases.

(1) By hypotesis, $\mathcal{J} \otimes L$ satisfies IT with index 0, and so its Fourier-Mukai transform $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{J} \otimes L) = R^0\Phi_{\mathcal{P}}(\mathcal{J} \otimes L)$ is a locally-free sheaf with rank $\chi(\mathcal{J} \otimes L)$ (that is, a vector bundle), and we denote it as $\mathcal{L}$. Using Mukai equivalene theorem, we get

$$\mathbf{R}\Phi_{\mathcal{P}^\vee}(\mathcal{L}) = \mathcal{J} \otimes L[-g] = R^g\Phi_{\mathcal{P}^\vee}(\mathcal{L})[-g]$$

and thus $\mathcal{L}^\vee$ is a GV-sheaf on $\widehat{A}$. Moreover, the transform of the dual of $\mathcal{L}^\vee$, which is $\mathcal{J} \otimes L$, is torsion-free, and thus by Proposition 2.30 $\mathcal{L}^\vee$ is M-regular. This also means that $\mathcal{L}^\vee$ is an ample vector bundle (see Theorem 2.33). So we can apply Le Potier vanishing (Theorem 3.6), and have

$$H^i(\widehat{A}, \mathcal{L}^\vee) = 0 \qquad \text{for all } i \geq \chi(\mathcal{J} \otimes L),$$

and then $V^i(\mathcal{L}^\vee)$ is empty for $i \geq \chi(\mathcal{J} \otimes L)$.

On the other hand, by Proposition 2.23 (2) and by base change, $V^i(\mathcal{L}^\vee)$ contains the support of $\mathcal{E}xt^i(\mathcal{J} \otimes L, \mathcal{O}_A)$, and using properties of the $\mathcal{E}xt$ sheaf and the sequence (3.5), we have that

$$V^i(\mathcal{L}^\vee) \supseteq \operatorname{supp} \mathcal{E}xt^{i+1}(\mathcal{O}_Z, \mathcal{O}_A).$$

This sheaf is non-zero when Z has a component of codimension $i+1$. Thus, we end up having

$$g - \dim Z \leq \chi(\mathcal{J} \otimes L).$$

Using this inequality together with (3.6), we have that $g + 1 \leq d$.

(2) Given a coherent sheaf $\mathcal{F}$ on A, we define the index

$$i_{\max}(\mathcal{F}) := \max\{i \geq 0 \mid V^i(A, \mathcal{F}) \neq 0\}.$$

We first assume that $i_{\max}(\mathcal{J} \otimes L) = 1$. This means that it gv-index can be measured only in $i = 1$, and so

$$gv(\mathcal{J} \otimes L) = \operatorname{codim} V^1(\mathcal{J} \otimes L) - 1 = g - 1$$

by the hypotesis on $V_{>0}$. Thus the sheaf $(\widehat{\mathcal{J} \otimes L})^\vee$ is not locally free and it is a $(g-1)$-syzygy sheaf, according to Proposition 2.35. So, by Theorem 2.34, we have $\chi(\mathcal{J} \otimes L) = \operatorname{rk}((\widehat{\mathcal{J} \otimes L})^\vee) \geq g - 1$. We know from the beginning of the proof that $\chi(\mathcal{J} \otimes L) < d$, and so $d \geq g$.

Assume now that $i_{\max}(\mathcal{J} \otimes L) > 1$. This time, the gv-index is given by the last non-zero locus $V^i(\mathcal{J} \otimes L)$, that is, $gv(\mathcal{J} \otimes L) = g - i_{\max}(\mathcal{J} \otimes L)$. With the same argument as before, we have that

$$\chi(\mathcal{J} \otimes L) \geq g - i_{\max}(\mathcal{J} \otimes L).$$

On the other hand, from the beginning of the proof we have that

$$i_{\max}(\mathcal{J} \otimes L) = i_{\max}(L_{|Z}) + 1 \geq \dim Z + 1,$$

and so $\chi(\mathcal{J} \otimes L) \geq g - \dim Z - 1$. Together with the inequality (3.6), we get $d \geq g$, and this concludes the proof. $\square$

Now we state the result, which is analogous to the one by Debarre and Hacon:

Theorem 3.22 (Pareschi, [Par21, Theorem A]). *Let (A, L) be a polarized simple abelian variety of degree d and dimension $g > d$. Then*

1. *every divisor in $|L|$ is prime, normal and has rational singularities;*

2. *given $m \geq 2$, let $D \in |mL|$ be a divisor such that $D \neq mE$ for any $E \in |L|$; then the pair $(A, \frac{1}{m}D)$ is log-terminal.*

Proof. We can prove both cases at once. Let $\mathcal{J}$ be either the adjoint ideal of a divisor in $|L|$ or the multiplier ideal of $\frac{1}{m}D$. Let also Z be the cosupport of $\mathcal{J}$; as before, we assume that Z is not empty (that is, $\mathcal{J}$ is not trivial) and we will look for a contradiction.

At first, we see that $\mathcal{J}$ cannot be of the form $\mathcal{O}_A(-E)$, where $E \in |L|$. This is obvious for the adjoint case, and it is true also in the multiplier case because we have required that $D \neq mE$, which implies that $\lfloor \frac{1}{m}D \rfloor = 0$ (see [DH07, Corollary 3]).

Furthermore, being A simple, Z must be geometrically non-degenerate.

Thus, in order to apply Lemma 3.21, we need to verify one of the possible requirement on $V_{>0}$, namely that it is empty or it has dimension zero.

As we have seen in the proof of Theorem 3.18, in both adjoint and multiplier cases $\mathcal{J} \otimes L$ is a GV-sheaf and all irreducible components of $V^i(\mathcal{J} \otimes L)$ are translates of abelian subvarieties of $\mathrm{Pic}^0(A)$. But in our case, the abelian variety is simple, and this means that all (finite unions of) proper non-empty abelian subvarieties are 0. Thus, $V^i(\mathcal{J} \otimes L)$ can only be empty or zero-dimensional for all $i > 0$.

So we can apply Lemma 3.21, obtaining a contradiction with the hypotesis that $g > d$. This proves the theorem. □

3.5 Polarizations of degrees 3 and 4

We are now at the core of the thesis. In this section we will give a proof of the results about polarizations of degree 3 and 4. In the first part, we have a result about prime divisors in the polarization, with an additional hypothesis. In the second part, we look at divisors on a multiple of the polarization, and we complete the result of the first part.

3.5.1 Divisors with rational singularities

We give here a detailed proof of Theorem A in the Introduction.

Theorem 3.23. *Let (A, L) be an indecomposable polarized abelian variety of degree $d < 5$ and dimension $g > d$. Let D be a prime and normal divisor in $|L|$, and let Z be the cosupport of* $\mathrm{adj}(A, D)$*. We assume that $gv(L_{|Z})$ is finite. Then D has rational singularities.*

Proof. Let D be a prime and normal divisor in $|L|$. From Proposition 3.2, D has rational singularities if and only if $\mathcal{J} := \mathrm{adj}(A, D)$ is a trivial sheaf, that is, if its cosupport Z is empty.

We assume that Z is not empty, and so that $\mathcal{J}$ is a proper ideal sheaf. Proposition 3.2 provides also the exact sequence

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{J} \otimes L \longrightarrow \mu_* \omega_X \longrightarrow 0, \tag{3.7}$$

where $\mu\colon X \to D \subset A$ is a resolution of singularities of D. Using this sequence in combination with Lemma 3.17, we get that $\mathcal{J} \otimes L$ is a GV-sheaf.

On the other hand, we have the exact sequence

$$0 \longrightarrow \mathcal{J} \otimes L \longrightarrow L \longrightarrow L_{|Z} \longrightarrow 0, \tag{3.8}$$

and for every $\alpha \in \mathrm{Pic}^0(A)$, we get the twisted sequence

$$0 \longrightarrow \mathcal{J} \otimes L \otimes P_\alpha \longrightarrow L \otimes P_\alpha \longrightarrow L_{|Z} \otimes P_\alpha \longrightarrow 0.$$

Since $H^i(A, L \otimes P_\alpha) = 0$ for every $i > 0$, by taking the cohomology of the last sequence, we have that $H^i(A, L_{|Z} \otimes P_\alpha) = H^{i+1}(A, \mathcal{J} \otimes L \otimes P_\alpha)$ for every $i > 0$. This means that

$$V^i(L_{|Z}) = V^{i+1}(\mathcal{J} \otimes L) \qquad \text{for all } i > 0.$$

Then it follows directly from the definition of gv-indices that

$$gv(L_{|Z}) \geq gv(\mathcal{J} \otimes L) + 1. \tag{3.9}$$

Since $\mathcal{J} \otimes L$ is a GV-sheaf, (3.9) implies that $L_{|Z}$ is a M-regular sheaf.

By hypothesis, $gv(L_{|Z})$ is finite, and so is $gv(\mathcal{J} \otimes L)$. Thus, we can apply Theorem 2.36, and together with (3.8) we obtain

$$\chi(L) = \chi(\mathcal{J} \otimes L) + \chi(L_{|Z}) \geq gv(\mathcal{J} \otimes L) + gv(L_{|Z}) \geq 2\ gv(\mathcal{J} \otimes L) + 1. \tag{3.10}$$

The polarization L has degree less than 5, and so we can focus only on values of $gv(\mathcal{J} \otimes L)$ smaller than 2. For these values, the idea is to compare the gv-index of $\mathcal{J} \otimes L$ with the gv-index of the push-forward of the dualizing sheaf $\mu_*\omega_X$ that appears in (3.7), and work with the latter.

We start by stating explicitely this comparison, which is in fact an equality:

Claim 1. If $gv(\mathcal{J} \otimes L) < 2$, then $gv(\mathcal{J} \otimes L) = gv(\mu_*\omega_X)$.

Proof of claim. If we twist the exact sequence (3.7) by P_α, where $\alpha \in \mathrm{Pic}^0(A)$, and we look at cohomology groups, we have that $H^i(A, \mathcal{O}_A \otimes P_\alpha) = H^i(A, P_\alpha) = 0$ for every $i \geq 0$ and for all α that are not $\hat{0}$. This means that

$$H^i(A, \mathcal{J} \otimes L \otimes P_\alpha) = H^i(A, \mu_*\omega_X \otimes P_\alpha)$$

for all $i \geq 0$ and all P_α different from $\mathcal{O}_A$, and consequently we obtain

$$V^i(\mathcal{J} \otimes L) \setminus \{\hat{0}\} = V^i(\mu_*\omega_X) \setminus \{\hat{0}\}.$$

Since the gv-index is measured using the codimension of cohomological support loci, the claim holds for every case but possibly one, namely when $V^{g-1}(\mathcal{J} \otimes L) = \{\hat{0}\}$ and this locus is used to measure the gv-index of $\mathcal{J} \otimes L$, which is then equal to 1. In this particular case, we would have $H^{g-1}(A, \mathcal{J} \otimes L) \neq 0$, but this is impossible.

Indeed, let us apply the Fourier-Mukai transform to the sequence (3.7): we will obtain a long exact sequence ending with

$$\cdots \to R^{g-1}\Phi_{\mathcal{P}}(\mathcal{O}_A) \longrightarrow R^{g-1}\Phi_{\mathcal{P}}(\mathcal{J}\otimes L) \longrightarrow R^{g-1}\Phi_{\mathcal{P}}(\mu_*\omega_X) \xrightarrow{\delta} R^g\Phi_{\mathcal{P}}(\mathcal{O}_A) \longrightarrow 0.$$

As we have seen in section 2.1.2, the Fourier-Mukai transform of the structure sheaf $\mathcal{O}_A$ is the sheaf $\mathcal{O}_{\hat{0}}$ concentrated in degree g, that is $R^g\Phi_{\mathcal{P}}(\mathcal{O}_A) \simeq \mathcal{O}_{\hat{0}}$ and $R^i\Phi_{\mathcal{P}}(\mathcal{O}_A) \simeq 0$ for every $i < g$.

On the other hand, $R^{g-1}\Phi_{\mathcal{P}}(\mu_*\omega_X) \simeq \mathcal{O}_{\hat{0}}$, as proved in [Bar+12, Proposition 6.1]. Thus, the previous sequence becomes

$$0 \longrightarrow R^{g-1}\Phi_{\mathcal{P}}(\mathcal{J}\otimes L) \longrightarrow R^{g-1}\Phi_{\mathcal{P}}(\mu_*\omega_X) \xrightarrow{\delta} R^g\Phi_{\mathcal{P}}(\mathcal{O}_A) \longrightarrow 0,$$

where δ is an isomorphism. This implies that $R^{g-1}\Phi_{\mathcal{P}}(\mathcal{J}\otimes L) = 0$, and so by the cohomology base change, also $H^{g-1}(\mathcal{J}\otimes L) = 0$. This proves the claim. ☐

Furthermore, by the Grauert-Riemenschneider vanishing theorem (see [Laz04, Theorem 4.3.9]) and Corollary 2.8, the cohomological support loci $V^i(A, \mu_*\omega_X)$ correspond to $V^i_\mu(X, \omega_X)$ and so $gv(A, \mu_*\omega_X) = gv_\mu(X, \omega_X)$.

Hence, the inequality (3.10) becomes simply

$$5 > \chi(L) \geq 2\ gv_\mu(X, \omega_X) + 1. \tag{3.11}$$

As this last inequality shows, with our hypotheses the only possible values for $gv_\mu(X, \omega_X)$ are 0 and 1.

Let us assume first that $gv_\mu(X, \omega_X) = 0$. Then, by Theorem 2.21, the image of X via the map $\mu\colon X \longrightarrow A$ is fibred by translates of abelian subvarieties of A. But this impossible, since the image of X is an ample divisor. Thus, $gv_\mu(X, \omega_X)$ cannot be 0.

Now, we assume that $gv_\mu(X, \omega_X) = 1$. From the definition of gv-index, there is an integer k between 1 and g such that $V^k_\mu(\omega_X)$ has a connected component W with codimension $k+1$.

By Theorem 2.18 (or, more precisely, by Proposition 2.19), W is an abelian subvariety $T \subseteq \mathrm{Pic}^0(A)$ translated by a torsion point $P_\eta \in \mathrm{Pic}^0(A)$. Note that T is also a subvariety of $\mathrm{Pic}^0(X)$, as showed in Lemma 3.24. We then denote by B the dual abelian variety of T, that is $B = \mathrm{Pic}^0(T)$.

Moreover, Theorem 2.18 says that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr, bend right, "f"'] & A \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

such that $\dim X - \dim f(X) \geq k$. Here f is a map with connected fibres (see Lemma 3.25).

We want to prove that the general fibre of f has exactly dimension k, or in other words:

Claim 2. We have $\dim X - \dim f(X) = k$.

Proof of claim. We denote by F a general fibre along f. F is then birational to a fibre along the map $\pi_{|D}\colon D \longrightarrow f(X)$. We know that $\dim B = g-k-1$, and so a general fibre of π has dimension $k+1$, and it is also a translate of the kernel of π, which we denote by K. Thus we have that

$$k \leq \dim X - \dim f(X) = \dim F \leq k+1.$$

We assume that $\dim F = k+1$. Then F is a translate of K, and this means that a general fibre of $\pi_{|D}$ is a translate of K. Thus, D is fibred by translates of abelian subvarieties of A, which is impossible because D is ample.

Hence, $\dim F = k$, and this proves the claim. □

Now, by definition, we have $W = f^* \operatorname{Pic}^0(B) + P_\eta$. This means that a point $P_\alpha \in \operatorname{Pic}^0(X)$ belongs to W if and only if it is of the form $f^*P_\beta \otimes P_\eta$ for some $P_\beta \in \operatorname{Pic}^0(B)$.

It follows that, for every $P_\beta \in \operatorname{Pic}^0(B)$, the group $H^k(X, \omega_X \otimes P_\eta \otimes f^*P_\beta)$ is not zero, and this implies that

$$f^* \operatorname{Pic}^0(B) = V^k_f(\omega_X \otimes P_\eta).$$

Our goal is to describe explicitly W, and so $f^* \operatorname{Pic}^0(B)$. To do that, we can expand the last equality using Kollár decomposition (we use the variant stated in Proposition 2.9). Indeed, at the level of cohomology groups we have

$$H^k(X, \omega_X \otimes P_\eta \otimes f^*P_\beta) \simeq \bigoplus_{i=0}^{k} H^{k-i}(B, R^i f_*(\omega_X \otimes P_\eta) \otimes P_\beta),$$

where the direct sum goes from 0 to $\dim X - \dim f(X) = k$ thanks to the previous claim. This equality clearly holds for every $P_\beta \in \operatorname{Pic}^0(B)$, and the left-hand side is non-zero if and only if at least one direct summand in the right-hand side is non-zero. So if we look now at the corresponding cohomological support loci, we obtain

$$V^k_f(X, \omega_X \otimes P_\eta) = \bigcup_{i=0}^{k} f^* V^{k-i}(B, R^i f_*(\omega_X \otimes P_\eta)). \tag{3.12}$$

By Hacon's version of the generic vanishing (see Proposition 2.17), every locus in the right-hand side of (3.12) is a proper subvariety of $\operatorname{Pic}^0(B)$, except the one with index equal to 0. Therefore, equality (3.12) becomes simply

$$V^k_f(X, \omega_X \otimes P_\eta) = f^* V^0(B, R^k f_*(\omega_X \otimes P_\eta)). \tag{3.13}$$

We focus on the right-hand side of (3.13). Following the notation of Beauville and Pareschi (see respectively [Bea92, p. 4] and [Par17, p. 138]), we define

$$\operatorname{Pic}^0(f) := \ker(\operatorname{Pic}^0(X) \longrightarrow \operatorname{Pic}^0(F))$$

where F is a general fibre along f and the map is a restriction. So a point $P \in \operatorname{Pic}^0(X)$ belongs to $\operatorname{Pic}^0(f)$ if and only if $H^0(F, P_{|F}) \neq 0$, that is, if and only if

$H^{\dim F}(F, (\omega_X \otimes P)_{|F}) \neq 0$. Since $\dim F = k$ by Claim 2, using Proposition 2.11 we obtain

$$\mathrm{Pic}^0(f) = \{P \in \mathrm{Pic}^0(X) \mid R^k f_*(\omega_X \otimes P) \neq \hat{0}\}.$$

Assume that P_η is not in $\mathrm{Pic}^0(f)$. Then $R^k f_*(\omega_X \otimes P_\eta)$ is zero, and in particular it is a point in $\mathrm{Pic}^0(B)$.

Assume instead that $P_\eta \in \mathrm{Pic}^0(f)$. By [Par17, p. 138], we have an exact sequence

$$0 \longrightarrow f^* \mathrm{Pic}^0(B) \longrightarrow \mathrm{Pic}^0(f) \longrightarrow \Gamma \longrightarrow 0,$$

where Γ is a finite subgroup of $\mathrm{Pic}^0(X)/f^* \mathrm{Pic}^0(B)$. In particular, this implies that $\mathrm{Pic}^0(f)$ contains $f^* \mathrm{Pic}^0(B)$; thus, by Lemma 3.26, we have that $\mathrm{Pic}^0(f) = f^* \mathrm{Pic}^0(B)$. So, there is a point $P_\beta \in \mathrm{Pic}^0(B)$ such that $P_\eta = f^* P_\beta$. Thus, we have

$$R^k f_*(\omega_X \otimes P_\eta) = R^k f_*(\omega_X \otimes f^* P_\beta) = R^k f_* \omega_X \otimes P_\beta = \mathcal{O}_B \otimes P_\beta,$$

where the last equality follows directly from [Kol86a, Proposition 7.6]. Again, $R^k f_*(\omega_X \otimes P_\eta)$ is a point in $\mathrm{Pic}^0(B)$.

Hence, being P_η in $\mathrm{Pic}^0(f)$ or not, we end up obtaining that $R^k f_*(\omega_X \otimes P_\eta)$ is a point in $\mathrm{Pic}^0(B)$, and this means that in both cases $f^* V^0(B, R^k f_*(\omega_X \otimes P_\eta))$ is zero-dimensional.

This in turn implies that the connected component W of $V^k_\mu(\omega_X)$ (and then also $V^k_\mu(\omega_X)$) is zero-dimensional itself.

Finally, we remember that k is the index of the cohomological support locus from which we have measured the *gv*-index of ω_X, that is

$$1 = gv_\mu(X, \omega_X) = \operatorname{codim} V^k_\mu(\omega_X) - k$$

and since $\dim V^k_\mu(\omega_X) = 0$, we have that $k = g - 1$. This means that we have measured the *gv*-index using $V^{g-1}_\mu(\omega_X)$, and it is easy to see that $V^{g-1}_\mu(\omega_X) = \{\hat{0}\}$. But this fact is in contradiction with what we have seen in the proof of Claim 1, where we have shown that the locus measuring the *gv*-index strictly contains $\hat{0}$.

Hence, we have proved the impossibility for $gv_\mu(\omega_X)$ to be neither 0 nor 1, in contradiction with the bound (3.11). So, Z has to be empty, that is, the adjoint ideal sheaf $\mathcal{J}$ is trivial. This means that the prime divisor $D \in |L|$ is normal with rational singularities. □

We now state and prove some lemmas needed in the previous proof:

Lemma 3.24. *Let (A, L) be a polarized abelian variety and let $D \in |L|$ be a prime and normal divisor. Let also $\mu\colon X \to D \subset A$ be a resolution of singularities of D.*

Then the map

$$\mu^*\colon \mathrm{Pic}^0(A) \longrightarrow \mathrm{Pic}^0(X)$$

is injective.

Proof. We choose a point $P_\alpha \in \operatorname{Pic}^0(A)$ such that $\mu^* P_\alpha = \mathcal{O}_X$. Then we have that $H^0(D, \mu_*\mu^* P_{\alpha|D})$ is non-zero. Since $\mu_*\mu^* P_{\alpha|D} = P_{\alpha|D} \otimes \mu_*\mathcal{O}_X = P_{\alpha|D} \otimes \mathcal{O}_D$, we have that $H^0(D, P_{\alpha|D}) \neq 0$.

Consider now the short exact sequence

$$0 \longrightarrow P_\alpha(-D) \longrightarrow P_\alpha \longrightarrow P_{\alpha|D} \longrightarrow 0,$$

and taking from this the correspondent long exact sequence in cohomology, we obtain in particular

$$\cdots \longrightarrow H^0(A, P_\alpha) \longrightarrow H^0(D, P_{\alpha|D}) \longrightarrow H^1(A, P_\alpha(-D)) \longrightarrow \cdots .$$

By hypothesis D is ample, so $H^1(A, P_\alpha(-D)) = 0$. Moreover, $H^0(D, P_{\alpha|D}) \neq 0$ as proved before, and this forces $H^0(A, P_\alpha)$ to be non-zero as well. But this in turns implies that $P_\alpha = \mathcal{O}_A$, which means that the map μ^* is injective. ☐

Lemma 3.25. *Let* (A, L) *be a polarized abelian variety and let* $D \in |L|$ *be a prime and normal divisor. Let also* $\mu\colon X \to D \subset A$ *be a resolution of singularities of* D*. Assume there is an abelian variety* B *and a surjective morphism* $\pi\colon A \to B$*.*

Then the composition map $f := \pi \circ \mu\colon X \to B$ *has connected fibres.*

Proof. We suppose there exists a fibre $F := f^{-1}(b)$ which is not connected, with $b \in B$. Let us denote by K the kernel of the map π; then K is an abelian subvariety of A. We know that the fibre F is birational to the divisor D', which is the restriction of D to $K + b$. Being D ample, also D' is ample. Then, if we consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_K(-D') \longrightarrow \mathcal{O}_K \longrightarrow \mathcal{O}_{D'} \longrightarrow 0$$

and we take its cohomology, we get the sequence

$$\cdots \longrightarrow H^0(K, \mathcal{O}_K) \longrightarrow H^0(D', \mathcal{O}_{D'}) \longrightarrow H^1(K, \mathcal{O}_K(-D')) \longrightarrow \cdots .$$

By ampleness of D', $H^1(K, \mathcal{O}_K(-D'))$ is zero, and so $h^0(D', \mathcal{O}_{D'}) = 1$, which means that D' is indeed connected. Thus we have a contradiction which proves the lemma. ☐

Lemma 3.26. *In the hypoteses of Lemma 3.25, let* $f := \pi \circ \mu\colon X \to B$ *be the composition map. We define the set*

$$\operatorname{Pic}^0(f) := \ker(\operatorname{Pic}^0(X) \longrightarrow \operatorname{Pic}^0(F))$$

where F *is a general fibre along* f *and the map is a restriction.*

Then $\operatorname{Pic}^0(f)$ *is equal to* $f^* \operatorname{Pic}^0(B)$*.*

Proof. Let K be the kernel of π. The map f induces a map $F \to K$ whose image spans K. Thus, the homomorphism $\nu\colon \operatorname{Pic}^0(K) \to \operatorname{Pic}^0(F)$ has finite kernel, which we denote by Γ. Hence, we have the following exact sequence

$$0 \longrightarrow f^* \operatorname{Pic}^0(B) \longrightarrow \operatorname{Pic}^0(f) \longrightarrow \Gamma \longrightarrow 0.$$

It is now sufficient to prove that Γ is trivial, that is, the map $\nu\colon \operatorname{Pic}^0(K) \to \operatorname{Pic}^0(F)$ is injective. This fact can be proven using the same argument showed in Lemma 3.24, replacing X and A respectively with F and K. ☐

3.5.2 Log-canonical pairs

We have already noted that when the degree of the polarization grows up, more and more exceptional cases arise. Indeed, for an indecomposable polarization $|L|$ of degree 3, we have an example by Debarre (see [Deb06b, p. 4]) showing the existence of a non-log-canonical pair $(A, \frac{1}{m}D)$, where $D \in |mL|$.

Here is the construction of such polarization and subsequent pair:

Example 3.27. We start with a general polarized abelian variety (B, M) of degree $d-1$, and an ellitpic curve E. We want to construct a polarized abelian variety from those, following Proposition 9.1 and 9.2 in [Deb88]. We pick an isomorphism

$$\psi \colon K(M) \longrightarrow E[d-1]$$

from the kernel of the map φ_M to the subgroup of $(d-1)$-torsion points in E; both these subgroups have order $(d-1)^2$. We then set the subgroup

$$G := \{(x, \psi(x)) \mid x \in K(M)\},$$

and we define the abelian variety A as the product $B \times E$ quotiented by G. As proved in [Deb88, (9.2)], there is a principal polarization Θ on A, and the canonical morphism $f \colon B \times E \to A$ is an isogeny, with the property that the polarization $f^*\mathcal{O}_A(\Theta)$ is the product of polarizations of B and E.

Thus, we can define a polarization

$$L = \mathcal{O}_A(\Theta + B),$$

which is of degree d and it is indecomposable for $d \geq 3$.

From this construction, we obtain that the linear series $|(d-1)\Theta - B|$ is not empty. So, if we take $m \geq d-1$ and $\frac{d}{d-1} \geq m' > m$, we have that $|m\Theta - (m'-m)B|$ is non-empty as well.

Then, picking $D' \in |m\Theta - (m'-m)B|$, we get a divisor

$$D := D' + m'B$$

which belongs to $|mL|$. But we note that $\frac{1}{m}D = \frac{1}{m}D' + \frac{m'}{m}B$, and since $m' > m$, we have a component with multiplicity greater than 1. Hence, the pair $(A, \frac{1}{m}D)$ cannot be log-canonical. □

However, if we consider an indecomposable polarization of degree 3 different from the one in the example above, we can get a result similar to previous theorems. Here are the statement and proof of Theorem C in the Introduction:

Theorem 3.28. *Let (A, L) be an indecomposable polarized abelian variety of degree $d \leq 3$ and dimension $g > d$, and given a positive integer m, let $D \in |mL|$.*

Then either

1. *the polarization L is of the form $\mathcal{O}_A(\Theta + B)$, where Θ is a theta divisor and B is an abelian subvariety of A with $\dim B = g-1$ and $\chi(L_{|B}) = d-1$, or*

2. the pair $(A, \frac{1}{m}D)$ is log-canonical.

Proof. The case of polarization of degree less or equal than 2 has been already proved in Theorem 3.14 and Theorem 3.18, so we will focus specifically on an indecomposable polarization of degree 3.

Let D be a divisor in $|mL|$ for $m \in \mathbb{Z}_{>0}$. We know from section 3.1.2 that the pair $(A, \frac{1}{m}D)$ is log-canonical if and only if the multiplier ideals $\mathcal{J}(\frac{1-\varepsilon}{m}D)$ are trivial for all $\varepsilon \in (0,1) \cap \mathbb{Q}$.

So, we assume that $(A, \frac{1}{m}D)$ is not log-canonical, that is, we say that there exist a $\bar{\varepsilon} \ll 1$ such that the multiplier ideal $\mathcal{J} := \mathcal{J}(\frac{1-\bar{\varepsilon}}{m}D)$ is a proper ideal sheaf of $\mathcal{O}_A$. Equivalently, we are assuming that the cosupport Z of $\mathcal{J}$ is not empty.

We start with the short exact sequence

$$0 \longrightarrow \mathcal{J} \otimes L \longrightarrow L \longrightarrow L_{|Z} \longrightarrow 0, \tag{3.14}$$

and for any $P_\alpha \in \operatorname{Pic}^0(A)$ we obtain the twisted sequence

$$0 \longrightarrow \mathcal{J} \otimes L \otimes P_\alpha \longrightarrow L \otimes P_\alpha \longrightarrow L_{|Z} \otimes P_\alpha \longrightarrow 0.$$

Then we take the cohomology long sequence from this short one. Since of course $H^i(A, L \otimes P_\alpha) = 0$ for all $i > 0$, we have that $H^i(A, L_{|Z} \otimes P_\alpha) = H^{i+1}(A, \mathcal{J} \otimes L \otimes P_\alpha)$ for $i > 0$. This means, from the cohomological support loci point of view, that

$$V^i(L_{|Z}) = V^{i+1}(\mathcal{J} \otimes L) \tag{3.15}$$

for all $i > 0$. More than this, we have (with a small abuse of notation) that the divisor $(L + P_\alpha) - \frac{1-\bar{\varepsilon}}{m}D$ is numerically equivalent to $\bar{\varepsilon}L$, which is ample. So we can apply Nadel vanishing theorem (Theorem 3.5) and obtain that

$$H^i(A, \mathcal{J} \otimes L \otimes P_\alpha) = 0$$

for every $i > 0$ and every $P_\alpha \in \operatorname{Pic}^0(A)$. This fact implies that $V^i(\mathcal{J} \otimes L)$ is empty for every $i > 0$, and so by (3.15), also $V^i(L_{|Z})$ is empty for all $i > 0$.

Hence, both $L_{|Z}$ and $\mathcal{J} \otimes L$ satisfies IT with index 0 (see section 2.1.2 for the precise definition of IT). In particular, they both have locally-free Fourier-Mukai transforms and positive Euler characteristic. We set $\mathcal{L} := \mathbf{R}\Phi_{\mathcal{P}}(L_{|Z})$ and $\mathcal{M} := \mathbf{R}\Phi_{\mathcal{P}}(\mathcal{J} \otimes L)$.

By sequence 3.14, we have that $\chi(L_{|Z}) + \chi(\mathcal{J} \otimes L) = \chi(L) = 3$, and since these characteristics are positive, they must have values 1 and 2.

We first assume that $\chi(L_{|Z}) = 1$. From Proposition 2.35, we know that $\mathcal{L}^\vee$ is a locally free sheaf. Also, by Proposition 2.23 (1), $\operatorname{rk}(\mathcal{L}^\vee) = \chi(L_{|Z}) = 1$. These two facts together show that $\mathcal{L}^\vee$ is indeed a line bundle on $\operatorname{Pic}^0(A)$.

We apply the (inverse) Fourier-Mukai transform on $\mathcal{L}$, and we get

$$\mathbf{R}\Psi_{\mathcal{P}^\vee}(\mathcal{L}) = \mathbf{R}\Psi_{\mathcal{P}^\vee} \circ \mathbf{R}\Phi_{\mathcal{P}}(L_{|Z}) = L_{|Z}[-g] \tag{3.16}$$

thanks to Mukai's equivalence theorem (Theorem 2.5) and the fact that $\mathbf{R}\Psi_{\mathcal{P}^\vee} = (-\operatorname{id}_A)^*\mathbf{R}\Psi_{\mathcal{P}}$. In particular, this means that the transform $\mathbf{R}\Psi_{\mathcal{P}^\vee}(\mathcal{L})$ is a sheaf concentrated in degree g, that is, $\mathcal{L}$ satisfies the WIT with index g. Thus, by definition, we have that $\mathcal{L}^\vee$ is a GV sheaf on $\mathrm{Pic}^0(A)$, and so by Theorem 2.33 $\mathcal{L}^\vee$ is nef.

On the other hand, $\mathcal{L}^\vee$ cannot be M-regular: indeed its characteristic $\chi(\mathcal{L}^\vee)$ is equal to the (generic) rank of $\widehat{(\mathcal{L}^\vee)^\vee}$ by Proposition 2.23, which is simply $\mathrm{rk}(L_{|Z})$; obviously, the latter is zero. But $\chi(\mathcal{L}^\vee) = 0$ means that $\mathcal{L}^\vee$ is not M-regular (see Proposition 2.30). Then by Proposition 2.31 $\mathcal{L}^\vee$ is not ample, and so it is a properly nef line bundle on $\widehat{A}$.

Hence, there is a proper quotient $\pi\colon \widehat{A} \to B$ to an abelian variety B with an ample line bundle N such that $\pi^*N = \mathcal{L}^\vee$ (see [BL04a, Lemma 3.3.2]).

From (3.16), we have that $H^g(\widehat{A}, \mathcal{L} \otimes P_a)$ is non-zero if and only if $a \in Z$, so $V^g(\mathcal{L}) = Z$; this means that also $V^0(\mathcal{L}^\vee) = Z$. Moreover, from the construction of the quotient π, it follows that $h^0(\widehat{A}, \mathcal{L}^\vee \otimes P_a) = h^0(B, N \otimes \pi_* P_a)$ for every $a \in Z$. Thus, $Z = V^0(\mathcal{L}^\vee) = \mathrm{Pic}^0(B)$, which implies that Z is an abelian subvariety of A.

So, $L_{|Z}$ is the restriction of the polarization L to an abelian subvariety, and it is principal by assumption. Using then Lemma 2.1, we conclude that Z has to be either zero or A, and both of these cases make a contradiction with the hypothesis.

So far, the only case left is $\chi(L_{|Z}) = 2$, which means $\chi(\mathcal{J} \otimes L) = 1$. As written above, we denote by $\mathcal{M}$ the transform $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{J} \otimes L)$. By the same argument used for $\mathcal{L}^\vee$, we can show that also $\mathcal{M}^\vee$ is a line bundle on $\widehat{A}$ and also a GV-sheaf.

This time, however, $\mathcal{M}^\vee$ happens to be M-regular: indeed, since $\mathcal{J} \otimes L$ has no torsion, we can use Proposition 2.30. Thus, by Proposition 2.31, the line bundle $\mathcal{M}^\vee$ is also ample and satisfies IT with index 0.

It follows from Proposition 2.35 that $\mathcal{J} \otimes L$ is a locally-free sheaf. Then, the cosupport Z of $\mathcal{J}$ has to be a Cartier divisor, and $\mathcal{J} \otimes L$ becomes $L \otimes \mathcal{O}_A(-Z)$.

Furthermore, by assumption $1 = \chi(\mathcal{J} \otimes L) = \chi(L \otimes \mathcal{O}_A(-Z))$, and so we have

$$L \otimes \mathcal{O}_A(-Z) = \mathcal{O}_A(\Theta),$$

where Θ is a theta divisor on A. Now, we can rewrite the previous equality in the following way:

$$L = \mathcal{O}_A(\Theta + Z), \tag{3.17}$$

obtaining that the polarization L is the sum of a theta divisor and a Cartier divisor.

The next step is to show that Z is somehow close to an abelian subvariety of A. We know that there exist a quotient map $\pi\colon A \to B$ to an abelian variety B and an ample divisor E in B such that $Z = \pi^*E$. If we denote by b the dimension of B, we have that the self-intersection (Z^b) is non-zero, while $(Z^{b+1}) = 0$. Also, for any topologically trivial line bundle $P \in \mathrm{Pic}^0(B)$, we have that $|Z - P|$ is non-empty.

So, we can define a map on linear series

$$\varphi_P\colon |\Theta + P| \times |Z - P| \longrightarrow |\Theta + Z|$$

for every $P \in \mathrm{Pic}^0(B)$. By (3.17), we have $|\Theta + Z| \simeq \mathbb{P}^2$, and so its dimension is 2. Also, we have that $\dim|\Theta + P| = 0$ and $n := \dim|Z - P| = h^0(\mathcal{O}_A(Z) \otimes P^\vee) - 1$.

Suppose first that $b = \dim B \geq 3$. Then, by varying $P \in \mathrm{Pic}^0(B)$, we have a family of (distinct) divisors in $|\Theta + P| \times |Z - P|$ of dimension $(b + n) \geq 3$. Thus, since $\dim |\Theta + Z| = 2$, there is a general element in $|\Theta + Z|$ that may be written as $\Theta_P + Z_P$ for infinitely many $P \in \mathrm{Pic}^0(B)$, where $\Theta_P \in |\Theta + P|$ and $Z_P \in |Z - P|$.

So, we have $\Theta_P + Z_P = \Theta_Q + Z_Q$ for some $P, Q \in \mathrm{Pic}^0(B)$, but since $\Theta_P \in |\Theta + P|$ is unique and different from Θ_Q, we must have $D_P \geq \Theta_Q$. Hence we get

$$\chi(L) = \chi(\Theta + Z) \geq \chi(2\Theta) > 3,$$

which is a contradiction.

Suppose now that $b = \dim B = 2$. In order to get a contradiction, we can use the same argument as before, unless $n = 0$, that is, $dim|Z - P| = 0$. In this case, the family of divisors in $|\Theta + P| \times |Z - P|$ obtained by varying $P \in \mathrm{Pic}^0(B)$ can be parmetrized by $\mathrm{Pic}^0(B)$ itself. So we get a map

$$\mathrm{Pic}^0(B) \longrightarrow |\Theta + Z| \simeq \mathbb{P}^2,$$

which is a generically finite rational map. Since $\mathrm{Pic}^0(B)$ is not rational (as an abelian surface), such a map must have degree greater than 1. This means that, if we take a general element $G \in |\Theta + Z|$, we have $G = \Theta_P + Z_P = \Theta_Q + Z_Q$ for distinct $P, Q \in \mathrm{Pic}^0(B)$. So, as before, we can conclude that $\chi(L) > 3$, obtaining a contradiction.

This argument shows that the divisor Z has self-intersection zero, in particular $(Z^2) = 0$. As proved in [Auf14, Propositions 3.2.2 and 3.2.4], this fact implies that Z is a translate of a so-called abelian divisor, that is, a Cartier divisor which is an abelian subvariety.

Hence, our polarization L is of the form $\mathcal{O}_A(\Theta + B + a)$, where B is an abelian subvariety of A of dimension $g - 1$ and $a \in A$. We can thus define Θ' as the translate of Θ by the point a, which is again a theta divisor, and so we end up obtaining

$$L = \mathcal{O}_A(\Theta' + B).$$

So, we are in the same setting of Example 3.27, and thus following the same argument we can find a pair $(A, \frac{1}{m}D)$ which is not log-canonical. This concludes the proof. $\square$

Using part of the proof above, we can give a more complete statement of Theorem 3.23 in the case where the degree is at most 3 (that is, Theorem B in the Introduction):

Theorem 3.29. *Let (A, L) be an indecomposable polarized abelian variety of degree $d \leq 3$ and dimension $g > d$. Then every prime and normal divisor in $|L|$ has rational singularities.*

Proof. Let D be a prime and normal divisor in $|L|$. As we have seen in previous proofs, from Proposition 3.2, D has rational singularities if and only if $\mathcal{J} := \mathrm{adj}(A, D)$ is a trivial sheaf, that is, if its cosupport Z is empty. We assume that Z is not empty, and so that $\mathcal{J}$ is a proper ideal sheaf.

We resume the proof of Theorem 3.23. We know that $\mathcal{J} \otimes L$ is a GV-sheaf and $L_{|Z}$ is a M-regular sheaf.

Since we have already proven the result for finite $gv(L_{|Z})$, we can assume that $gv(L_{|Z}) = \infty$, that is, $L_{|Z}$ satisfies IT with index 0.

Suppose first that $\chi(L_{|Z}) = 1$. In this case, by the same argument in the proof of Theorem 3.28, we obtain a contradiction.

So, we can assume that $\chi(L_{|Z}) = 2$. This clearly implies that $\chi(\mathcal{J} \otimes L) = 1$. It follows then that $gv(\mathcal{J} \otimes L)$ has to be infinite. Otherwise, by Claim 1 in the proof of Theorem 3.29, we would have $1 \geq gv(\mathcal{J} \otimes L) = gv_{\mu}(X, \omega_X)$, where $\mu\colon X \to D$ is the desingularization of D. But this leads to a contradiction, as we have already seen.

Thus, also $\mathcal{J} \otimes L$ satisfies IT with index 0. Hence, following the proof of Lemma 3.21, we get that

$$g - \dim Z \leq \chi(\mathcal{J} \otimes L),$$

and in our case, $\dim Z \geq g - 1$. But this is impossible, since Z is contained in the singular locus of a divisor of A, and so its dimension is at most $g - 2$.

This fact shows that Z has to be empty, and this concludes the proof. □

3.5.3 Suggestions for possible developments

In Theorem 3.23 and, consequently, in Theorem 3.29, we have made the assumption of normality for the prime divisor $D \in |L|$. This hypothesis is needed in the proof of Theorem 3.23 (and in particular for the Lemmas), and so it cannot be removed. Although, we believe that the result is true for every prime divisor in $|L|$, and so we present here an alternative (and incomplete) way that should avoid the normality hypothesis.

We are in the usual setting: let $D \in |L|$ be a prime divisor, and let Z be the cosupport of the adjoint ideal $\mathcal{J} = \mathrm{adj}(A, D)$. We assume that Z is not empty and we try to find a contradiction. In order to avoid the extra assumption of normality of D, we use the notion of local gv-index (see section 2.3.3 for the definition and properties). In particular, we will focus on the components of cohomological support loci containing $\hat{0}$.

So, we follow the path of the proof of Theorem 3.23, but with some significant changes. First, we have that

$$V^i(L_{|Z}) = V^{i+1}(\mathcal{J} \otimes L) \qquad \text{for all } i > 0,$$

and this implies locally that

$$gv_{\hat{0}}(L_{|Z}) \geq gv_{\hat{0}}(\mathcal{J} \otimes L) + 1.$$

This time, however, the hypothesis of finiteness of $gv(L_{|Z})$ is not sufficient to ensure that $gv_{\hat{0}}(\mathcal{J} \otimes L)$ and $gv_{\hat{0}}(L_{|Z})$ are finite. For now, we can assume that $gv_{\hat{0}}(L_{|Z})$

is finite, so that $gv_{\hat{0}}(\mathcal{J} \otimes L)$ is finite too. Then, using the same argument of the original proof, which is valid also in the local case, we end up with the inequality

$$5 > \chi(L) \geq 2\ gv_{\mu,\hat{0}}(X, \omega_X) + 1.$$

From here, we have two possibile values of $gv_{\mu,\hat{0}}(\omega_X)$. If its value is 0, then also $gv_{\mu}(\omega_X) = 0$, and we can look back to the original proof, where we proved the contradiction. Otherwise, $gv_{\mu,\hat{0}}(\omega_X) = 1$, which means by definition that there is an integer k between 1 and g such that the component $W_{\hat{0}}$ of $V^k_{\mu}(\omega_X)$ containing $\hat{0}$ has codimension $k + 1$.

We then proceed as in the proof of Theorem 3.23, knowing that this time $W_{\hat{0}}$ is an actual abelian subvariety of $\mathrm{Pic}^0(A)$, that is, there is no translation by some torsion point. This fact is the precise reason why we have decided to work locally in a neighbourhood of $\hat{0}$.

Using the same notation as in the original proof, we have that $W_{\hat{0}} = V^k_f(\omega_X)$, which is in turn equal to $f^*V^0(B, R^k f_*\omega_X)$. By [Kol86a, Proposition 7.6], $R^k f_*\omega_X = \mathcal{O}_B$, and thus $W_{\hat{0}} = \hat{0}$. We can now complete the proof in the same way as the original one.

Note that here we avoided using Lemma 3.26, which is the result where the normality hypothesis is needed.

Unfortunately, such an approach to the problem has some significant drawbacks. In fact, throughout the above discussion, we have left two possible issues, namely the cases when at least one between $gv_{\hat{0}}(\mathcal{J} \otimes L)$ and $gv_{\hat{0}}(L_{|Z})$ is not finite.

Let us start with the case of having $gv_{\hat{0}}(L_{|Z}) = \infty$, while $gv_{\hat{0}}(\mathcal{J} \otimes L)$ has a finite value. This one is still a manageable case, at least for degree $d = 3$.

We can assume that $gv_{\hat{0}}(\mathcal{J} \otimes L) \geq 2$, otherwise we are again in the non-exceptional setting above. From this assumption, we have that $gv(\mathcal{J} \otimes L) \geq 2$, and if we set $d = \chi(L) = 3$, we must have $\chi(L_{|Z}) = 1$. This implies that $gv(L_{|Z}) = 1$, because $L_{|Z}$ is an M-regular sheaf. But then, by (3.9) it follows that $gv(\mathcal{J} \otimes L) = 0$, and thus $gv_{\mu}(\omega_X) = 0$, which is a case covered by the original proof.

On the other hand, the case when both $gv_{\hat{0}}(\mathcal{J} \otimes L)$ and $gv_{\hat{0}}(L_{|Z})$ are infinite has for the moment no solution.

In conclusion, we have presented here an idea that seems very promising but rather incomplete. We hope that this will provide a starting point for further developments.

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